

# Delay-dependent robust stability analysis for systems with interval delays

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**Abstract**—This paper is concerned with the stability analysis for uncertain systems with interval time-varying delays. An appropriate class of Lyapunov-Krasovskii functionals is proposed and, splitting the known bounds of the delay interval in two subintervals, a new delay-dependent criterion is derived. In addition, we resort to the use of a polytope to introduce a novel treatment of the time-varying delays. A number of different examples are given to demonstrate the reduced conservatism of this method.

## I. INTRODUCTION

During the last two decades, the stability of time-delay systems have been widely investigated by the control community. Some examples of time-delay systems include networked control systems, chemical processing systems, transportation systems, and power systems, see e.g. [1].

In the time-domain there are two main approaches in order to study the stability of time-delay systems: Lyapunov-Razumikhin and Lyapunov-Krasovskii theorems. Both approaches can handle with time-varying delays, but the obtained results using Lyapunov-Krasovskii functionals are usually less conservative, since they include additional information on the derivative of the time-varying delay, [2]. Stability conditions in terms of Linear Matrix Inequalities (LMIs) are usually derived by applying either of the two aforementioned theorems.

In this framework, most of recent works address the problem of finding sufficient delay-dependent conditions to ensure the stability of linear time-delay systems. These delay-dependent conditions introduce information on the bounds of the delay and get better results than the delay-independent approaches. The first papers on this topic supposed that the delay was constant and unknown, [3], [4].

However, there are a number of practical applications in which the delay is in general time-varying. In such cases, some authors have derived delay-dependent conditions using the upper bound on the delay, [5], [6]. In practice, the delay may vary in a range for which the lower bound is not restricted to be zero. Recently, a growing number of works have proposed the use of the information on the lower bound of the delay, [7], [8]. They show that it is possible to improve the results if this information is taken into account.

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As mentioned above, the Lyapunov-Krasovskii approach can include the bound of the derivative of the time-varying delay. The derivative was restricted to be less than one in the first publications, see for instance [5], [9]. Nonetheless, the restriction was recently relaxed allowing for both fast and slow time-varying delay profiles, [6], [8], [10].

However, the criteria to guarantee asymptotic stability of time-delay systems suffered from excessive conservatism since its inception. In order to reduce the conservatism, there have been several research directions. We can enumerate some of them. In [3] and [4] they found new bounds for the inner product of two vectors. In the derivative of some Lyapunov-Krasovskii functionals these products appear and, in most cases it is necessary to bound them. Moreover, Fridman introduced a new system description which brings a reduction in the conservatism, [5]. Nowadays, a fairly standard technique, that was introduced in [9], is the use of free weighting matrices (also called *slack matrices*). The mathematical argument consists in adding null terms to the derivative of the Lyapunov-Krasovskii functional, by using the Leibniz-Newton formula. These null terms include free matrices to provide additional degrees of freedom. However, this carries additional computational burden. Recently, several authors have worked to improve the bounds of some integral terms appearing frequently in this context. In [10], they estimated the upper bound of an appropriate Lyapunov-Krasovskii functional without ignoring some useful terms as  $-\int_{t-h}^{t-d(t)} \dot{x}^T(s)Z\dot{x}(s)ds$ , as had been done up until then. In [8], a modification of Jensen's inequality is used to bound this class of terms.

In practical cases, the plant models are always subject to uncertainties due to unknown dynamics and modeling errors. In the study of robust stability of time-delay systems, there are two main uncertainty descriptions: polytopic and norm-bounded. In the former, the LMI conditions must be feasible for all the vertices of a polytope to ensure the stability of the uncertain system. In the case of norm-bounded uncertainties, most researchers have used the *S-procedure* in [11] to introduce them in the LMI-based criteria.

We propose in this paper a new delay-dependent condition to ensure robust stability of time-delay systems. The delay is supposed to be lower and upper bounded, and the information on the bound of its derivative is also considered. LMI robust stability criteria are obtained for norm-bounded uncertainties. In order to reduce the conservatism we introduce some new ideas. First, we choose an appropriate Lyapunov-Krasovskii functional and then, dividing the time delay range into two subintervals, less restrictive bounds for some fairly standard

terms are found separately in each subinterval. Although two LMI conditions (one for each subinterval) need to be solved now, we will show that this idea reduces the conservatism of our criterion, specially if the value which divides the interval is chosen adequately. Next, we combine this idea with the use of an unidimensional polytope to handle the time-varying delay. In this manner, we retain it instead of utilizing its bounds, and we derive LMI conditions which need to be feasible for the vertices of the polytope. Using a polytopic covering combined with the division of the time-delay range, the results are significantly improved.

The paper is organized as follows. Section II is devoted to the problem statement. The main results are obtained in Section III. Sections III-A and III-B contains a detailed exposition of the stability result for nominal and uncertain systems, respectively. Several numerical examples are given in Section IV. Conclusions and future works are summarized in Section V.

## II. PROBLEM STATEMENT

Consider the following uncertain linear system with time-varying delay:

$$\begin{aligned} \dot{x}(t) &= [A + \Delta A(t)]x(t) + [B + \Delta B(t)]x(t - d(t)), \quad t > 0, \quad (1) \\ x(t) &= \phi(t), \quad t \in [-h_2, 0], \quad (2) \end{aligned}$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $A$  and  $B$  are constant matrices of appropriate dimensions.  $\Delta A(t)$  and  $\Delta B(t)$  denote the parametric uncertainties, which satisfy the following conditions:

$$\Delta A(t) = GF(t)E_1, \quad (3)$$

$$\Delta B(t) = GF(t)E_2. \quad (4)$$

where  $G$ ,  $E_i$  ( $i = 1, 2$ ) are constant matrices of appropriate dimensions and  $F(t)$  is an unknown time-varying matrix, which is Lebesgue measurable in  $t$  and satisfies  $F^T(t)F(t) \leq I$ .

The time delay,  $d(t)$ , is a time-varying continuous function that satisfies

$$h_1 \leq d(t) \leq h_2, \quad (5)$$

$$\dot{d}(t) \leq \mu, \quad (6)$$

where  $0 < h_1 < h_2$  and  $\mu$  are constants. The initial condition,  $\phi(t)$ , is a continuous vector-valued function of  $t \in [h_2, 0]$ .

In order to derive a less restrictive stability criterion for system (1)-(2), the time delay range will be divided into two subintervals. To proceed,  $h_m$  is defined as follows,

$$h_1 < h_m < h_2. \quad (7)$$

This way, the complete delay range:  $d(t) \in [h_1, h_2]$  is divided in two disjoint subintervals,  $d(t) \in [h_1, h_m) \cup [h_m, h_2]$ .

## III. MAIN RESULTS

In this section the main ideas are the following. First, if the delay range is divided into smaller subintervals, it will be possible to reduce the uncertainty of the time-varying delay in each one of them, obtaining less restrictive conditions. In order to prove system stability for the time delay varying in the whole interval, an unique Lyapunov-Krasovskii functional will be required to be continuous in  $t$  and strictly decreasing in both subintervals.

Secondly, we propose the use of techniques based on polytopic covering in order to improve the results. The key idea is the following. When deriving stability conditions based on some adequate Lyapunov-Krasovskii functionals, terms as  $d(t)x^T(t)Zx(t)$ ,  $(d(t) - h_1)x^T(t)Zx(t)$  or  $(h_2 - d(t))x^T(t)Zx(t)$ , are usually substituted for others in which  $d(t)$  does not appear. To make this substitution,  $d(t)$  is replaced by the worst-case taking its lower or upper bound. However, this brings conservatism since the delay can not take his maximum and minimum value at the same time.

One possibility to overcome this problem is the use of polytopes. If the idea of splitting the delay known bounds in two subintervals is combined with a polytopic description of  $d(t)$ , then the vertices of the polytopes are  $\{h_1, h_m\}$  and  $\{h_m, h_2\}$  in the first and second subinterval, respectively. In the following, instead of substituting  $d(t)$  by its bounds, it will be retained and the LMIs will be solved simultaneously for all the vertices of the polytope.

### A. Stability analysis for nominal time-delay systems

Consider the following linear system with time-varying delay,

$$\dot{x}(t) = Ax(t) + Bx(t - d(t)), \quad t > 0, \quad (8)$$

$$x(t) = \phi(t), \quad t \in [-h_2, 0]. \quad (9)$$

The following theorem presents a novel delay-dependent stability criterion for system (8)-(9).

**Theorem 1.** *Given scalars  $0 < h_1 < h_m < h_2$ ,  $\mu$  and  $\epsilon > 0$ , the linear system (8)-(9) with time-varying delay  $d(t)$  satisfying (5) and (6) is asymptotically stable if there exist matrices  $P, Q_1, Q_2, Q_3, Q_4, Z_1, Z_2 > 0$  and matrices  $N_{ij}, M_{ij}, R_{ij}$ ,  $i = 1, 2$ ,  $j = 1, 2$ , of appropriate dimensions such that the LMIs (10)-(11) are satisfied for the vertices of  $d(t)$ .*

**Proof.** Choose the following Lyapunov-Krasovskii functional candidate:

$$\begin{aligned} V(t) &= x^T(t)Px(t) + \int_{t-h_1}^t x^T(s)Q_1x(s)ds \\ &+ \int_{t-h_2}^t x^T(s)Q_2x(s)ds + \int_{t-h_m}^t x^T(s)Q_3x(s)ds \\ &+ \int_{t-d(t)}^t x^T(s)Q_4x(s)ds + \int_{-h_2}^0 \int_{t+\theta}^t \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta \\ &+ \int_{-h_2}^{-h_1} \int_{t+\theta}^t \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta, \quad (12) \end{aligned}$$

$$\begin{bmatrix} \Gamma_1 & (h_m - d(t) + \epsilon)\bar{M}_1 & h_1\bar{N}_1 & (d(t) + \epsilon - h_1)\bar{R}_1 & \bar{A}U \\ * & -(h_m - d(t) + \epsilon)(Z_1 + Z_2) & 0 & 0 & 0 \\ * & * & -h_1Z_1 & 0 & 0 \\ * & * & * & -(d(t) + \epsilon - h_1)(Z_1 + Z_2) & 0 \\ * & * & * & * & -U \end{bmatrix} < 0, \quad h_1 \leq d(t) < h_m; \quad (10)$$

$$\begin{bmatrix} \Gamma_2 & (h_2 - d(t) + \epsilon)\bar{M}_2 & h_1\bar{N}_2 & (d(t) + \epsilon - h_m)\bar{R}_2 & \bar{A}U \\ * & -(h_2 - d(t) + \epsilon)(Z_1 + Z_2) & 0 & 0 & 0 \\ * & * & -h_1Z_1 & 0 & 0 \\ * & * & * & -(d(t) + \epsilon - h_m)(Z_1 + Z_2) & 0 \\ * & * & * & * & -U \end{bmatrix} < 0, \quad h_m \leq d(t) \leq h_2; \quad (11)$$

where,

$$\Gamma_1 = \begin{bmatrix} \theta_{1,11} & \theta_{1,12} & -M_{11} & R_{11} - N_{11} & 0 \\ * & \theta_{1,22} & -M_{12} & R_{12} - N_{12} & 0 \\ * & * & -Q_3 - \frac{Z_1+Z_2}{h_2-h_m} & 0 & \frac{Z_1+Z_2}{h_2-h_m} \\ * & * & * & -Q_1 & 0 \\ * & * & * & * & -Q_2 - \frac{Z_1+Z_2}{h_2-h_m} \end{bmatrix}; \quad \Gamma_2 = \begin{bmatrix} \theta_{2,11} & \theta_{2,12} & R_{21} & -N_{21} & -M_{21} \\ * & \theta_{2,22} & R_{22} & -N_{22} & -M_{22} \\ * & * & -Q_3 - \frac{Z_1+Z_2}{(h_m-h_1)} & \frac{Z_1+Z_2}{(h_m-h_1)} & 0 \\ * & * & * & -Q_1 - \frac{Z_1+Z_2}{(h_m-h_1)} & 0 \\ * & * & * & * & -Q_2 \end{bmatrix};$$

$$\begin{aligned} \theta_{i,11} &= PA + A^T P + Q_1 + Q_2 + Q_3 + Q_4 + N_{i1} + N_{i1}^T, \quad i = 1, 2; \\ \theta_{i,12} &= PB + M_{i1} - R_{i1} + N_{i2}^T, \quad i = 1, 2; \\ \theta_{i,22} &= -(1 - \mu)Q_4 + M_{i2} + M_{i2}^T - R_{i2} - R_{i2}^T, \quad i = 1, 2; \\ U &= h_2 Z_1 + (h_2 - h_1)Z_2; \end{aligned}$$

$$\begin{aligned} \bar{A}^T &= [A \ B \ 0 \ 0 \ 0]; \\ \bar{M}_i^T &= [M_{i1}^T \ M_{i2}^T \ 0 \ 0 \ 0], \quad i = 1, 2; \\ \bar{N}_i^T &= [N_{i1}^T \ N_{i2}^T \ 0 \ 0 \ 0], \quad i = 1, 2; \\ \bar{R}_i^T &= [R_{i1}^T \ R_{i2}^T \ 0 \ 0 \ 0], \quad i = 1, 2. \end{aligned}$$

where  $P > 0$ ,  $Q_i > 0$ , ( $i = 1, \dots, 4$ ) and  $Z_i > 0$ , ( $i = 1, 2$ ).

In order to prove Theorem 1, the delay range is divided in two subintervals taking into account  $h_m$ , i.e.,  $[h_1, h_2] \equiv [h_1, h_m] \cup [h_m, h_2]$ . The proof will be divided in two parts corresponding to each of the defined intervals.

*Interval 1)  $h_1 \leq d(t) < h_m$ .*

Taking the derivative of (12) along the solutions of (8)-(9) yields,

$$\begin{aligned} \dot{V}(t) &= 2x^T(t)P\dot{x}(t) + x^T(t)\left(\sum_{i=1}^4 Q_i\right)x(t) \\ &- (1 - \dot{d}(t))x^T(t-d(t))Q_4x(t-d(t)) \\ &- x^T(t-h_1)Q_1x(t-h_1) - x^T(t-h_2)Q_2x(t-h_2) \\ &- x^T(t-h_m)Q_3x(t-h_m) + \dot{x}^T(t)U\dot{x}(t) \\ &- \int_{t-h_2}^t \dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-h_2}^{t-h_1} \dot{x}^T(s)Z_2\dot{x}(s)ds. \end{aligned} \quad (13)$$

We can rewrite the integral terms employing the following equality:  $[-h_2, 0] \equiv [-h_2, -h_m] \cup [-h_m, -d(t)] \cup [-d(t), -h_1] \cup [-h_1, 0]$ .

Consider the addition of the following terms to  $\dot{V}(t)$ :

$$\begin{aligned} 0 &= 2\xi^T(t)\bar{M}_1 \left[ x(t-d(t)) - x(t-h_m) - \int_{t-h_m}^{t-d(t)} \dot{x}(s)ds \right], \\ 0 &= 2\xi^T(t)\bar{R}_1 \left[ x(t-h_1) - x(t-d(t)) - \int_{t-d(t)}^{t-h_1} \dot{x}(s)ds \right], \\ 0 &= 2\xi^T(t)\bar{N}_1 \left[ x(t) - x(t-h_1) - \int_{t-h_1}^t \dot{x}(s)ds \right], \end{aligned}$$

being  $\xi^T(t) = [x^T(t) \ x^T(t-d(t)) \ x^T(t-h_m) \ x^T(t-h_1) \ x^T(t-h_2)]$  the augmented state. Equation

(13) can be rewritten as:

$$\begin{aligned} \dot{V}(t) &\leq \xi^T(t)\tilde{\Gamma}_1\xi(t) + \dot{x}^T(t)U\dot{x}(t) \\ &- \int_{t-h_2}^{t-h_m} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds - \\ &- \int_{t-h_m}^{t-d(t)} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds - \\ &- \int_{t-d(t)}^{t-h_1} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds - \\ &- \int_{t-h_1}^t \dot{x}^T(s)Z_1\dot{x}(s)ds - 2\xi^T(t)\bar{N}_1 \int_{t-h_1}^t \dot{x}(s)ds - \\ &- 2\xi^T(t)\bar{M}_1 \int_{t-h_m}^{t-d(t)} \dot{x}(s)ds - 2\xi^T(t)\bar{R}_1 \int_{t-d(t)}^{t-h_1} \dot{x}(s)ds, \end{aligned} \quad (14)$$

where,

$$\tilde{\Gamma}_1 = \begin{bmatrix} \theta_{1,11} & \theta_{1,12} & -M_{11} & R_{11} - N_{11} & 0 \\ * & \theta_{1,22} & -M_{12} & R_{12} - N_{12} & 0 \\ * & * & -Q_3 & 0 & 0 \\ * & * & * & -Q_1 & 0 \\ * & * & * & * & -Q_2 \end{bmatrix}.$$

Up to this point, no conservatism has been introduced in the expressions, as no bounding terms have been required for  $\dot{V}(t)$ . Now, using the well-known upper bound for the inner product of two vectors:

$$-2b^T a - a^T X a \leq b^T X^{-1} b, \quad X > 0, \quad (15)$$

the following upper bounds for the integral terms in (14) can be found:

$$\begin{aligned} &- \int_{t-d(t)}^{t-h_1} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds - 2\xi^T(t)\bar{R}_1 \int_{t-d(t)}^{t-h_1} \dot{x}(s)ds \leq \\ &\quad (d(t) + \epsilon - h_1)\xi^T(t)\bar{R}_1(Z_1 + Z_2)^{-1}\bar{R}_1^T\xi(t), \\ &- \int_{t-h_1}^t \dot{x}^T(s)Z_1\dot{x}(s)ds - 2\xi^T(t)\bar{N}_1 \int_{t-h_1}^t \dot{x}(s)ds \leq \\ &\quad h_1\xi^T(t)\bar{N}_1 Z_1^{-1}\bar{N}_1^T\xi(t), \end{aligned}$$

$$\begin{aligned}
& - \int_{t-h_m}^{t-d(t)} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds - \\
& \quad - 2\xi^T(t)\bar{M}_1 \int_{t-h_m}^{t-d(t)} \dot{x}(s)ds \leq \\
& (h_m - d(t) + \epsilon)\xi^T(t)\bar{M}_1(Z_1 + Z_2)^{-1}\bar{M}_1^T\xi(t). \quad (16)
\end{aligned}$$

Please note that, in order to find upper bounds for the integral terms in (16),  $a$  and  $b$  in (15) are adequately chosen and the resulting inequalities are integrated in  $s$ . The terms which finally bound the integral terms do not depend on  $s$  and their integrations result in the presence of  $d(t)$  in the final bounds. However, the time-varying delay is not substituted for the worst cases, which are  $h_1$  and  $h_m$ .

It is also necessary to use the Jensen's inequality to obtain:

$$\begin{aligned}
& - \int_{t-h_2}^{t-h_m} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds \leq \\
& \quad - \left[ \int_{t-h_2}^{t-h_m} \dot{x}(s)ds \right]^T \frac{Z_1 + Z_2}{h_2 - h_m} \left[ \int_{t-h_2}^{t-h_m} \dot{x}(s)ds \right]. \quad (17)
\end{aligned}$$

Then, combining (14) with (16) and (17), it can be shown that, for  $h_1 \leq d(t) < h_m$ ,

$$\begin{aligned}
\dot{V}(t) \leq & \xi^T(t) (\Gamma_1 + (h_m - d(t) + \epsilon)\bar{M}_1(Z_1 + Z_2)^{-1}\bar{M}_1^T \\
& + h_1\bar{N}_1Z_1^{-1}\bar{N}_1^T + (d(t) + \epsilon - h_1)\bar{R}_1(Z_1 + Z_2)^{-1}\bar{R}_1^T \\
& + \bar{A}U\bar{A}^T) \xi(t). \quad (18)
\end{aligned}$$

Finally, applying Schur complement to equation (10), it can be proved from (18) that  $V(t)$  decreases for  $h_1 \leq d(t) < h_m$ .

*Interval 2)*  $h_m \leq d(t) \leq h_2$ .

The derivative of (12) for  $h_m \leq d(t) \leq h_2$  is the same as before, i.e. (13). However, in this case the integral terms are decomposed in a different manner, using  $[-h_2, 0] \equiv [-h_2, -d(t)] \cup [-d(t), h_m] \cup [-h_m, -h_1] \cup [-h_1, 0]$ .

Different null terms are added to  $\dot{V}(t)$ :

$$\begin{aligned}
0 & = 2\xi^T(t)\bar{M}_2 \left[ x(t-d(t)) - x(t-h_2) - \int_{t-h_2}^{t-d(t)} \dot{x}(s)ds \right], \\
0 & = 2\xi^T(t)\bar{R}_2 \left[ x(t-h_m) - x(t-d(t)) - \int_{t-d(t)}^{t-h_m} \dot{x}(s)ds \right], \\
0 & = 2\xi^T(t)\bar{N}_2 \left[ x(t) - x(t-h_1) - \int_{t-h_1}^t \dot{x}(s)ds \right].
\end{aligned}$$

Defining the same augmented state, (13) can be rewritten for this case as:

$$\begin{aligned}
\dot{V}(t) \leq & \xi^T(t)\tilde{\Gamma}_2\xi(t) + \dot{x}^T(t)U\dot{x}(t) - \\
& - \int_{t-h_m}^{t-h_1} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds - \\
& - \int_{t-h_2}^{t-d(t)} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds - \\
& - \int_{t-d(t)}^{t-h_m} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds - \\
& - \int_{t-h_1}^t \dot{x}^T(s)Z_1\dot{x}(s)ds - 2\xi^T(t)\bar{N}_2 \int_{t-h_1}^t \dot{x}(s)ds - \\
& - 2\xi^T(t)\bar{M}_2 \int_{t-h_2}^{t-d(t)} \dot{x}(s)ds - 2\xi^T(t)\bar{R}_2 \int_{t-d(t)}^{t-h_m} \dot{x}(s)ds, \quad (19)
\end{aligned}$$

where,

$$\tilde{\Gamma}_2 = \begin{bmatrix} \theta_{2,11} & \theta_{2,12} & R_{21} & -N_{21} & -M_{21} \\ * & \theta_{2,22} & R_{22} & -N_{22} & -M_{22} \\ * & * & -Q_3 & 0 & 0 \\ * & * & * & -Q_1 & 0 \\ * & * & * & * & -Q_2 \end{bmatrix}.$$

Now, using (15), the following upper bounds for the integral terms in (19) can be found:

$$\begin{aligned}
& - \int_{t-h_2}^{t-d(t)} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds - \\
& \quad - 2\xi^T(t)\bar{M}_2 \int_{t-h_2}^{t-d(t)} \dot{x}(s)ds \leq \\
& (h_2 - d(t) + \epsilon)\xi^T(t)\bar{M}_2(Z_1 + Z_2)^{-1}\bar{M}_2^T\xi(t), \\
& - \int_{t-d(t)}^{t-h_m} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds - 2\xi^T(t)\bar{R}_2 \int_{t-d(t)}^{t-h_m} \dot{x}(s)ds \leq \\
& (d(t) + \epsilon - h_m)\xi^T(t)\bar{R}_2(Z_1 + Z_2)^{-1}\bar{R}_2^T\xi(t). \\
& - \int_{t-h_1}^t \dot{x}^T(s)Z_1\dot{x}(s)ds - 2\xi^T(t)\bar{N}_2 \int_{t-h_1}^t \dot{x}(s)ds \leq \\
& h_1\xi^T(t)\bar{N}_2Z_1^{-1}\bar{N}_2^T\xi(t), \quad (20)
\end{aligned}$$

By applying Jensen's inequality again,

$$\begin{aligned}
& - \int_{t-h_m}^{t-h_1} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)ds \leq \\
& \quad - \left[ \int_{t-h_m}^{t-h_1} \dot{x}(s)ds \right]^T \frac{Z_1 + Z_2}{h_m - h_1} \left[ \int_{t-h_m}^{t-h_1} \dot{x}(s)ds \right]. \quad (21)
\end{aligned}$$

Then, combining (19) with (20)-(21), it can be shown that, for  $h_m \leq d(t) \leq h_2$ ,

$$\begin{aligned}
\dot{V}(t) \leq & \xi^T(t) (\Gamma_2 + (h_2 - d(t) + \epsilon)\bar{M}_2(Z_1 + Z_2)^{-1}\bar{M}_2^T \\
& + h_1\bar{N}_2Z_1^{-1}\bar{N}_2^T + (d(t) + \epsilon - h_m)\bar{R}_2(Z_1 + Z_2)^{-1}\bar{R}_2^T \\
& + \bar{A}U\bar{A}^T) \xi(t). \quad (22)
\end{aligned}$$

By Schur complement it can be seen from (22) that, if (11) holds, then  $V(t)$  decreases for  $h_m \leq d(t) \leq h_2$ .

Hence, it has been proved that  $V(t)$  decreases for all  $d(t) \in [h_1, h_2]$ . Obviously,  $V(t)$  is continuous in  $t$  since  $x(t)$  is continuous in  $t$ . Therefore,  $\dot{V}(t) < \varrho\|x(t)\|^2$ ,  $\forall t$ , for a sufficient small  $\varrho > 0$  and the asymptotic stability of system (8)-(9) can be ensured, see e.g. [12].  $\square$

**Remark 1.** In order to study the stability of system (8)-(9) using Theorem 1, it is necessary to solve the LMIs (10) and (11) simultaneously for the vertices of  $d(t)$  in each subinterval. Therefore, four LMIs need to be feasible.

**Remark 2.** The scalar parameter  $\epsilon > 0$  needs to be introduced in order to make strictly feasible the LMIs. Otherwise, some null matrices appear in the diagonal of the LMIs. It is worth mentioning that this modification does not introduce any conservatism, since  $\epsilon > 0$  can be chosen as small as necessary, i.e.  $\epsilon \rightarrow 0^+$ .

**Remark 3.** The value of  $h_m$  that divides the delay range is a design parameter. Selecting  $h_m = \frac{h_1+h_2}{2}$ , the

expressions (10)-(11) simplify. However this choice is not always adequate, as it will be shown in the examples.

It is straightforward to derive some equivalent results for the cases in which the information on the derivative of the time delay is not available and the lower bound of the delay is strictly zero. However, because of lack of space, these results have not been included as corollaries.

### B. Robust stability for uncertain systems

In this section the system with parametric uncertainties described by (1)-(2) is considered. The following theorem presents a new delay-dependent result for uncertain systems.

**Theorem 2.** *Given scalars  $0 < h_1 < h_m < h_2, \mu$  and  $\epsilon > 0$ , the linear system (1)-(2) with time-varying delay  $d(t)$  satisfying (5) and (6), and uncertainties described by (3)-(4) is asymptotically stable if there exist matrices  $P, Q_1, Q_2, Q_3, Q_4, Z_1, Z_2 > 0$ , any matrices  $N_{ij}, M_{ij}, R_{ij}$ ,  $i = 1, 2, j = 1, 2$ , of appropriate dimensions and a scalar  $e > 0$ , such that the following LMIs satisfy,*

$$\begin{bmatrix} \Xi_i & \alpha & e\beta \\ * & -eI & 0 \\ * & * & -eI \end{bmatrix} < 0, \quad i = 1, 2, \quad (23)$$

where  $\Xi_i, i = 1, 2$ , are the matrices required to be negative definite by (10)-(11) and,

$$\begin{aligned} \alpha^T &= [G^T P \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G^T U]^T; \\ \beta^T &= [E_1 \quad E_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T. \end{aligned}$$

**Proof.** Substituting  $A$  and  $B$  for  $A + \Delta A(t)$  and  $B + \Delta B(t)$  respectively in (10)-(11) and taking into account equations (3)-(4),  $\Xi_i < 0$  for  $i = 1, 2$  in (10)-(11) can be written as,

$$\Xi_i + \alpha F(t) \beta^T + \beta F^T(t) \alpha^T < 0, \quad i = 1, 2.$$

By Lemma 2.4 in [13], the conditions above hold if and only if there exists a scalar  $\lambda > 0$  such that,

$$\Xi_i + \lambda \alpha \alpha^T + \frac{1}{\lambda} \beta \beta^T < 0, \quad i = 1, 2.$$

Using Schur complements and naming  $e = \frac{1}{\lambda}$ , yields (23).  $\square$

In the following section, we use these results in different fairly standard examples to find the upper bound of the delay so that the system can withstand without losing its stability. We will compare our results with others in the literature. Also, some simulations are included to show that the results obtained using deterministic Lyapunov-Krasovskii techniques are, still, quite conservative in some situations.

## IV. NUMERICAL EXAMPLES

### A. Example 1

Consider the system,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t - d(t)). \quad (24)$$

The maximum admissible upper bound of the delay  $h_2$ , for recent methods in the literature, are listed in Table I and Table II with given  $h_1$ .

TABLE I  
ADMISSIBLE UPPER BOUND  $h_2$  FOR VARIOUS  $h_1$  AND  $\mu = 0.3$ .

Method	$h_1$	1	2	3	4	5
He et al. [10]	$h_2$	2.2125	2.4091	3.3342	4.2799	5.2393
Shao [8]	$h_2$	2.2474	2.4798	3.3893	4.325	5.2773
Theorem 1	$h_2$	2.3527	2.6087	3.4897	4.4063	5.3451
	$h_m$	1.91	2.32	3.25	4.21	5.19

TABLE II  
ADMISSIBLE UPPER BOUND  $h_2$  FOR VARIOUS  $h_1$  AND UNKNOWN  $\mu$ .

Method	$h_1$	0.3	0.5	0.8	1	2
Jiang et al. [14]	$h_2$	0.91	1.07	1.33	1.50	2.39
He et al. [10]	$h_2$	0.9431	1.0991	1.3476	1.5187	2.4000
Shao [8]	$h_2$	1.0715	1.2191	1.4539	1.6169	2.4798
Theorem 1	$h_2$	1.2986	1.4289	1.6415	1.7919	2.6087
	$h_m$	0.87	1.01	1.26	1.43	2.33

From Table I and II, it can be seen that the results are better than those in [8], [10] and [14].

### B. Example 2

Consider now the following system,

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - d(t)). \quad (25)$$

For unknown  $\mu$ , the admissible upper bound of the delay  $h_2$  is listed in Table III, with given  $h_1$ .

TABLE III  
ADMISSIBLE UPPER BOUND  $h_2$  WITH GIVEN  $h_1$  AND UNKNOWN  $\mu$ .

Method	$h_1$	1	2	3	4
Jiang et al. [7]	$h_2$	1.64	2.39	3.20	4.06
He et al. [10]	$h_2$	1.7424	2.4328	3.2234	4.0644
Shao [8]	$h_2$	1.8737	2.5049	3.2591	4.0744
Theorem 1	$h_2$	2.1684	2.6461	3.3215	4.0905
	$h_m$	1.68	2.35	3.17	4.05

As shown Table III, the obtained results are less conservative than the existing ones for this example.

Next, some simulations (using MATLAB-SIMULINK) are shown. The evolution of the system (25) is drawn for different configurations of the delay. First, we study a configuration which is proved to be stable, that is,  $h_1 = 4$  and  $h_2 = 4.0905$ . Figure 1.a shows it. Figures 1.b, 1.c and 1.d show the evolution of the states for three different situations.

It can be seen that depending on the statistical distribution of the delay, the results are more or less conservative. By simulation, we can check that the system is stochastically stable for bigger bounds of the delay, specially if the delay has a wide probability density function. However, using deterministic techniques it is not possible to prove the stability of the system for situations in which the upper bound of the delay is bigger than the value for which the system is unstable with constant delay. Note that this sentence does not mean that our result is conservative. We are only saying that we can not ensure, for all the possibilities, that the system is stable. Using an statistical point of view the results could be improved for some particular probability density function.

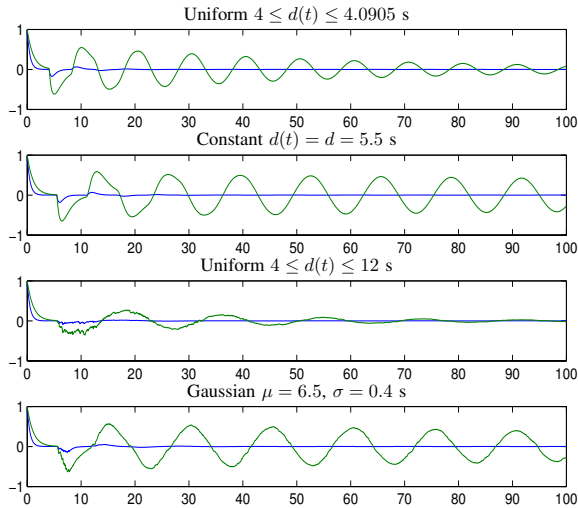


Fig. 1. Evolution of the system for different configurations

### C. Example 3

Consider the following uncertain system,

$$\dot{x}(t) = \begin{bmatrix} -2 + \delta_1 & 0 \\ 0 & -1 + \delta_2 \end{bmatrix} x(t) + \begin{bmatrix} -1 + \gamma_1 & 0 \\ -1 & -1 + \gamma_2 \end{bmatrix} x(t - d(t)), \quad (26)$$

where  $|\delta_1| \leq 1.6$ ,  $|\delta_2| \leq 0.05$ ,  $|\gamma_1| \leq 0.1$ ,  $|\gamma_2| \leq 0.3$ , and suppose that  $\bar{d}(t)$  is a continuous function. Choosing

$$G = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.04 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 160 & 0 \\ 0 & 1.25 \end{bmatrix}, \\ E_2 = \begin{bmatrix} 10 & 0 \\ 0 & 7.5 \end{bmatrix},$$

Table V lists the maximum admissible upper bound  $h_2$  for unknown  $\mu$  and  $h_1 = 0$ .

TABLE IV  
ADMISSIBLE UPPER BOUND  $h_2$  UNKNOWN  $\mu$ .

Method	$h_2$
Fridman et al. [15]	0.7692
Jiang et al. [7]	0.8654
Jiang et al. [16]	0.9442
Theorem 2 ( $h_m = 0.9$ )	1.1603

One can see that the proposed criteria can provide better results than other methods existing in the literature. The maximum admissible delay is improved in more than a 10%. Please note that the value of  $h_m$  is far from the middle of the interval. So an adequately choice of  $h_m$  let us get improvements in the criterion.

## V. CONCLUSIONS

In this paper, a new stability criterion is derived based on the division of the time delay range and a polytopic covering of the time-varying delay. A number of different examples are given to demonstrate the reduced conservatism of this method compared with other works in the literature.

However, these examples also show some limitations which need to be solved in future works.

If the probability density function (pdf) of the delay is known, it is possible that some new criteria can be studied (taking into account this information) such that they obtain less conservative results in statistical sense.

Another stimulating continuation of this work includes the investigation of potential improvements from multiple divisions of the time-varying delay range. The optimal election of the values that divide the intervals can also be studied.

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