Optimal asset allocation and capital adequacy management strategies for Basel III compliant banks

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Contents

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Declaration

I declare that

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is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.
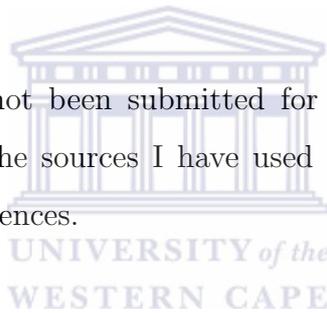
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Signed: ....

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Key words

Asset-to-debt ratio

Basel Regulatory Framework

Capital adequacy ratio

Deposit insurance

Hamilton-Jacobi-Bellman equation

Interest rate swap

Liquidity ratio

Monte Carlo simulation method

Optimal capital allocation strategy

Stochastic optimal control

Abstract

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Optimal asset allocation and capital adequacy management strategies for Basel III compliant banks

In this thesis we study a range of related commercial banking problems in discrete and continuous time settings. The first problem is about a capital allocation strategy that optimizes the expected future value of a commercial bank's total non-risk-weighted assets (TNRWAs) in terms of terminal time utility maximization. This entails finding optimal amounts of Total capital for investment in different bank assets. Based on the optimal capital allocation strategy derived for the first problem, we derive stochastic models for respectively the bank's capital adequacy and liquidity ratios in the second and third problems. The Basel Committee on Banking Supervision (BCBS) introduced these ratios in an attempt to improve the regulation of the international banking industry in terms of capital adequacy and liquidity management. As a fourth problem we derive a multi-period deposit insurance pricing model which incorporates the optimal capital allocation strategy, the BCBS' latest capital standard, capital forbearance and moral hazard. In the fifth and final problem we show how the values of LIBOR-in-arrears and vanilla interest rate swaps, typically used by commercial banks and other financial institutions to reduce risk, can be derived under a specialized version of the affine interest rate model originally considered by the bank in question.

More specifically, in the first problem we assume that the bank invests its Total capital in a stochastic interest rate financial market consisting of three assets, viz., a treasury security, a marketable security and a loan. We assume that the interest rate in the market is described by an affine model, and that the value of the loan follows a jump-diffusion process. We wish to find the optimal capital allocation strategy that maximizes an expected logarithmic utility of the bank's TNRWAs at a future date. Generally, analytical solutions to stochastic optimal control problems in the jump setting are very difficult to obtain. We propose an approximation method that exploits a similarity between the forms of the control problems of the jump-diffusion model and the diffusion model obtained by removing the jump. With the jump assumed sufficiently small, the analytical solution of the diffusion model then serves as a proxy to the solution of the control problem with the jump. In the second problem we construct models for the bank's capital adequacy ratios in terms of the proxy. We present numerical simulations to characterize the behaviour of the capital adequacy ratios. Furthermore, in this chapter, we consider the approximate optimal capital allocation strategy subject to a constant Leverage Ratio, which is a specific non-risk-based capital adequacy ratio, at the minimum prescribed level. We derive a formula for the bank's TNRWAs at constant (minimum) Leverage Ratio value and present numerical simulations based on the modified TNRWAs formula. In the third problem we model the bank's liquidity ratios and we monitor the levels of the liquidity ratios under the proxy numerically. In the fourth problem we derive a multi-period deposit insurance pricing model, the latest capital standard a la Basel III, capital forbearance and moral hazard behaviour. The deposit insurance pricing method utilizes an asset value reset rule comparable to the typical practice of insolvency resolution by insuring agencies. We perform numerical computations with our model to study its implications. In the final problem, we specialize the affine interest rate model considered previously to the Cox-Ingersoll-Ross (CIR) interest rate dynamic. We consider fixed-for-floating interest rate swaps under the CIR model. We show how analytical expressions for the values of both a LIBOR-in-arrears swap and a vanilla swap can be derived using a Green's function approach. We employ Monte Carlo simulation methods to compute the

values of the swaps for different scenarios.

We wish to make explicit the contributions of this project to the literature. A research article titled "An Optimal Portfolio and Capital Management Strategy for Basel III Compliant Commercial Banks" by Grant E. Muller and Peter J. Witbooi [1] has been published in an accredited scientific journal. In the aforementioned paper we solve an optimal capital allocation problem for diffusion banking models. We propose using the solution of the Brownian motions control problem of [1] as the proxy in problems two to four of this thesis. Furthermore, we wish to note that the methodology employed on the final problem of this study is actually from the paper [2] of Mallier and Alobaidi. In the paper [2] the authors did not present simulation studies to characterize their pricing models. We contribute a simulation study in which the values of the swaps are computed via Monte Carlo simulation methods.

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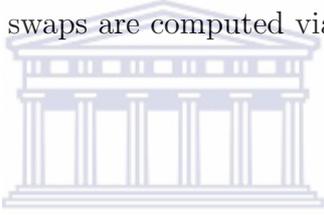
[1] G.E. Muller and P.J. Witbooi, "An optimal portfolio and capital management strategy for Basel III compliant commercial banks", *Journal of Applied Mathematics*, vol. 2014, Article ID 723873, 11 pages, 2014. doi:10.1155/2014/723873

[2] R. Mallier and G. Alobaidi, "Interest rate swaps under CIR", *Journal of Computational and Applied Mathematics*, 164-165, pp. 543-554, 2004.

List of Acronyms

Available Amount of Stable Funding (AASF)

Available Stable Factor (ASF)

Basel Committee on Banking Supervision (BCBS)

Countercyclical capital buffer (CCB)

Cox-Ingersoll-Ross (CIR)

Deposit insurance funds (DIFs)

Explicit deposit insurance fund (EDIF)

Federal Deposit Insurance Corporation (FDIC)

Forward rate agreement (FRA)

Hamilton-Jacobi-Bellman (HJB)

Implicit deposit insurance fund (IDIF)

Inverse Net Stable Funding Ratio (INSFR)

London Interbank Offer Rate (LIBOR)

Liquidity Coverage Ratio (LCR)

Net interest margins (NIMs)

Net Stable Funding Ratio (NSRF)

Ordinary differential equation (ODE)

Organization for Economic Co-operation and Development (OECD)

Over-the-counter (OTC)

Partial differential equation (PDE)

Required Amount of Stable Funding (RASF)
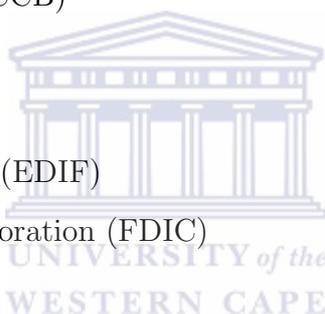
Required Stable Factor (RSF)

Shareholder Cash Flow Rights (SCFR)

Statutory Liquidity Ratio (SLR)

Stochastic differential equation (SDE)

Stock of High Quality Liquid Assets (SHQLAs)

Total Capital Ratio (CAR)

Total Net Cash Outflows (TNCOs)

Total non-risk-weighted assets (TNRWAs)

Total risk-weighted assets (TRWAs)

Value at risk (Var)

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List of Notations

C - Value of the Total capital

r - Value of the short-rate process

S_0 - Price of the treasury security

S - Price of the marketable security

L - Value of the loan

X - Value of the total non-risk-weighted assets (TNRWAs)

θ_S - Approximate optimal amount of Total capital invested in the marketable security

θ_L - Approximate optimal amount of Total capital invested in the loan

θ_r - Approximate optimal amount of Total capital invested in the treasury security

η_S - Approximate optimal proportion of Total capital invested in the marketable security

η_L - Approximate optimal proportion of Total capital invested in the loan

η_r - Approximate optimal proportion of Total capital invested in the treasury security

Y - Value of the total risk-weighted assets

Λ - Value of the Total Capital Ratio

C_{T1} - Value of the Tier 1 capital

C_{T2} - Value of the Tier 2 capital

Λ_{T1} - Value of the Tier 1 Ratio

Λ_l - Value of the Leverage Ratio

$\hat{X}$ - Value of the modified TNRWAs at constant Leverage Ratio value

H_Q - Value of the Stock of High Quality Liquid Assets

f_{S1} - Value of the secured funding backed by Level 1 assets

f_{S2} - Value of the secured funding backed by Level 2 assets

O_C - Value of the total expected cash outflows

l_{M1} - Value of the matured lending backed by Level 1 assets

l_{M2} - Value of the matured lending backed by Level 2 assets

I_C - Value of the total expected cash inflows

O_N - Value of the Total Net Cash Outflows

Λ_L - Value of the Liquidity Coverage Ratio

D - Value of the cash deposits

O - Value of the off balance sheet activities

F_A - Value of the Available Amount of Stable Funding

F_R - Value of the Required Amount of Stable Funding

Λ_N - Value of the Net Stable Funding Ratio

R - Annualized continuously compounded return of TNRWAs

μ_R - Annualized mean return of continuously compounded TNRWAs

σ_R - Annualized standard deviation of continuously compounded TNRWAs

$\bar{r}$ - Mean or expected value of the short-rate process r

q_l - Sets the lower bound for the value of the TNRWAs

q_u - Sets the upper bound for the value of the TNRWAs

ρ - Capital forbearance parameter

κ - Value of the cash payment corresponding to a put option

I - Value of the payment per dollar of deposits

δ_n - Fairly-priced premium rate per period in an n-period coverage horizon

ω - Risk-taking intensity parameter

V - Price of a bond

V_0 - Pay-off of a bond involving a single cash flow

List of Figures

List of Tables

Chapter 1

Introduction and scope of the thesis

By law, commercial banks are authorized to receive money from their customers and lend money to others. Commercial banks serve institutions and businesses and are also open to the general public. They fulfill many functions which include (1) receiving deposits from depositors, (2) making payments upon the direction of its depositors, (3) collecting funds from other banks payable to their customers, (4) investing funds in securities for a return, (5) safeguarding money, (6) maintaining and servicing savings and checking accounts of their depositors, (7) maintaining custodial accounts, i.e., accounts controlled by one person but for the benefit of another person and (8) lending money [42]. Due to these functions of commercial banks, it is not difficult to see their importance to economies. Commercial banks are corporations and are in business primarily to make a profit. However, due to their importance to the economies, and because the element of public trust is so crucial to their well-being, the regulation of the banking industry is very important. The Basel Committee on Banking Supervision (BCBS) regulates the international banking industry on behalf of governments [34, 94, 88, 53, 84, 86, 78].

In order for a commercial bank to make a profit, it must carefully manage its assets. This involves two factors, viz., the amount of resources (capital invested, retained earnings and deposits) it has available to invest, and its attitude towards risk and return. The bank must

decide how to allocate its resources optimally among its assets. An extremely useful tool to the banking industry is the theory of optimization.

In finance different approaches to stochastic optimization, from a methodological point of view, are exploited. A popular one is the stochastic control method. This method was used for the first time by Merton [70, 71]. The main feature of the stochastic optimal control methodology is to solve the Hamilton-Jacobi-Bellman (HJB) equation arising from dynamic programming under the real-world probability measure [94, 78]. A second method was developed by Cox and Huang [23] in the setting of complete markets. It relies on the theory of Lagrange multipliers. Also called the martingale method, this approach incorporates a risk-neutral measure and generally involves solving a partial differential equation (PDE) [94]. We will employ the stochastic optimal control approach in this study.

This thesis consists of two preliminary chapters and five main chapters. In the first preliminary chapter, we present some general commercial banking theory. This chapter also includes discussions on the regulation of the international banking industry, the importance of deposit insurance funds to the banking system and the usefulness of interest rate swaps. In the second preliminary chapter we cover all relevant mathematical ideas and concepts used in this thesis. The main chapters, i.e., Chapters 5-9, focus on five related commercial banking problems. We will now give a breakdown of each of these problems.

In a continuous time setting, the first problem involves deriving a capital allocation strategy that optimizes the expected future value of a commercial bank's total non-risk-weighted assets (TNRWAs) in terms of terminal time utility maximization. This entails finding optimal amounts of Total capital for investment in different bank assets. In particular, we consider a bank that invests its Total capital in a financial market consisting of three assets, viz., a treasury security, a marketable security and a loan. The dynamics of the loan is assumed to be described by a jump-diffusion process and we assume that the interest rate of the market

can be described by an affine model. We wish to find the capital allocation strategy that maximizes an expected logarithmic utility of the bank's TNRWAs at a future date. Generally analytical solutions to stochastic optimal control problems in the jump setting are very difficult to obtain. We propose an approximation method that exploits a similarity between the forms of the control problems of the jump-diffusion model and the diffusion model obtained by removing the jump. With the jump assumed sufficiently small, the approximation method replaces the jump-diffusion model with a diffusion model and solves the resulting control problem analytically. The analytical solution then serves as a proxy to the solution of the control problem with the jump.

In the second banking problem, which is also set in continuous time, we derive stochastic differential equations (SDEs) for the bank's capital adequacy ratios which incorporate the proxy derived in the first problem. The BCBS introduced these ratios in an attempt to improve the regulation of the international banking industry in terms of capital adequacy management. Since some of these ratios are computed from the total risk-weighted assets (TRWAs) of the bank, we also derive an SDE for this quantity. We monitor the performance of the capital adequacy ratios under the proxy numerically. In this chapter, we further consider the approximate optimal capital allocation strategy subject to specifically a constant Leverage Ratio, which is regarded as a non-risk-based capital adequacy ratio, at the minimum prescribed level. We derive a formula for the banks TNRWAs at constant (minimum) Leverage Ratio value and present numerical simulations based on the modified TNRWAs formula.

Still in continuous time, the third problem models the bank's liquidity ratios in terms of the proxy. These ratios were introduced in an attempt to improve the regulation of the international banking industry in terms of liquidity management. We simulate the behaviour of the liquidity ratios under the proxy numerically. In order to derive the models of the liquidity ratios, we require formulae for the Stock of High Quality Liquid Assets (SHQLAs), Total

3

Net Cash Outflows (TNCOs) and the Available and Required Amounts of Stable Funding (AASF and RASF). We also derive the SDEs describing these quantities here.

The fourth problem is set in a discrete time setting. In this problem we derive a multi-period deposit insurance pricing model which incorporates the proxy, the BCBS' latest capital standard, capital forbearance and moral hazard. The deposit insurance pricing method utilizes an asset value reset rule comparable to the typical practice of insolvency resolution by insuring agencies. We perform numerical analyses with our model to study its implications. In particular, we analyse the effect of the latest (Basel III) capital standard, capital forbearance and moral hazard on the fairly-priced premium rate under different coverage horizons and initial leverage (asset-to-debt) levels.

Lastly, in continuous time, we consider fixed-for-floating interest rate swaps under the Cox-Ingersoll-Ross or CIR [22] interest rate model, which is a special case of the affine model considered previously. Commercial banks, such as the one modelled in this thesis, and other financial institutions typically use interest rate swaps to reduce risk. We show how analytical expressions can be derived for the values of both a LIBOR-in-arrears interest rate swap and a vanilla interest rate swap. To price the swaps, we take a contingent claims approach. This means taking the common swap pricing approach of breaking each swap up into a series of forward rate agreements (FRAs) and then pricing each FRA using the CIR [22] model and a Green's function approach. The value of the swaps are then the sum of the values of these FRAs. By contrast, market practice is that instruments such as swaps and FRAs are commonly priced using a modification of the Black-Scholes formula, namely the Black-76 [13] formula. The Black-76 [13] formula was originally derived for commodities futures. In the latter the interest rate follows a lognormal random walk rather than the mean-reverting random walk CIR model. We wish to note that the methodology employed on this problem is actually from the paper [65] of Mallier and Alobaidi. As our own contribution to the analysis of this problem, we present numerical examples in which we compute the values of

4

the swaps for different scenarios with Monte Carlo simulation methods.

Chapter 2

Basics of commercial banking

In this chapter we give an overview of the commercial banking concept. We also discuss the regulation of banks, the importance of deposit insurance to the banking industry and the usefulness of interest rate swaps. Firstly, we present the general commercial banking model. In particular, we explain the balance sheet of commercial banks and define the items appearing thereon. These include the banks' assets, liabilities and capital. Secondly, we give a background on the Basel Accords, which the BCBS introduced in an attempt to improve the regulation of internationally active banks. Specifically, we highlight the differences and improvements on the accords over one another, but our main focus will be the current set of banking regulatory rules known as Basel III. We will present the discussions on deposit insurance and interest rate swaps thereafter.

In this chapter, the main references on commercial banking and the regulation thereof are the Basel documents [9, 10, 11, 12], the research articles [90, 92, 34, 76, 77, 53] and the book Mukuddem-Petersen and Petersen [84]. We refer to the papers [29, 56, 58] and the reference [19] when highlighting the importance of deposit insurance pricing, while we mainly reference the paper Mallier and Alobaidi [65] when discussing swaps.

2.1 The commercial banking concept

To understand the operation and management of a commercial bank, for a practical problem we study its stylized balance sheet, which records the assets (uses of funds) and liabilities (sources of funds) of the bank.

The role of bank capital is to balance the assets and liabilities of the bank. A useful way, for our analysis, of representing the balance sheet of the bank is as follows:

$$R + S + L = D + B + C, \tag{2.1}$$

where R, S, L, D, B and C represent the values of reserves, securities, loans, deposits, borrowings and capital respectively. Each of the variables above is regarded as a stochastic process.

In order for a commercial bank to make a profit, it is important that the bank manages the asset side of its balance sheet properly. The latter is determined by the amount of capital and other resources (retained earnings and deposits) it has available to invest and the attitude it has toward risk and return. The bank must therefore allocate its capital and other resources optimally among its assets. Below we explain each of the items on the balance sheet of a commercial bank.

The term *reserves* refer to the sum of the vault cash of the bank and the compulsory amount of its money deposited at the central bank. The bank uses its vault cash to meet the day-to-day currency withdrawals by its customers.

Securities consist of treasury securities (treasuries) and marketable securities. *Treasuries* are bonds issued by national treasuries in most countries as a means of borrowing money to meet government expenditures not covered by tax revenues, while *marketable securities* are stocks and bonds that can be converted to cash quickly and easily.

The types of *loans* granted by the bank include business loans, mortgage loans (land loans) and consumer loans. *Consumer loans* include credit extended by the bank for credit card purchases. *Mortgages* are long term loans used to buy a house or land, where the house or land acts as collateral. *Business loans* are taken out by firms that borrow funds to finance their inventories, which act as collateral for the loan. A loan which has collateral (secured loan) has a lower interest rate associated with it compared to a loan which has no collateral (unsecured loan).

In order to raise *bank capital*, banks sell new equity, retain earnings and issue debt or build up loan-loss reserves. It is usually the responsibility of a bank's risk management department to calculate its capital requirements. Calculated risk capital is then approved by the bank's top executive management. Furthermore, the structure of bank capital are proposed by the Finance Department and subsequently approved by the bank's top executive management. The dynamics of bank capital is stochastic in nature as it depends in part on the uncertainty related to debt- and shareholder contributions. Further uncertainty are from the general economic environment. In theory, the bank can decide on the rate at which debt and equity is raised.

Under Basel III the banks' Total capital C has the form

$$C = C_{\mathrm{T1}} + C_{\mathrm{T2}},$$

where C_{T1} and C_{T2} are Tier 1 and Tier 2 capital respectively (see [10, 53] for instance). *Tier 1 capital* consists of the sum of Common Equity Tier 1 capital and Additional Tier 1 capital. *Common Equity Tier 1 capital* is defined as the sum of the following elements [10]:

- Common shares issued by the bank that meet the criteria for classification as common shares for regulatory purposes (or the equivalent for non-joint stock companies);

- Stock surplus (share premium) resulting from the issue of instruments included Common Equity Tier 1;

- Retained earnings;

- Accumulated other comprehensive income and other disclosed reserves;

- Common shares issued by consolidated subsidiaries of the bank and held by third parties (i.e., minority interest) that meet the criteria for inclusion in Common Equity Tier 1 capital;

- Regulatory adjustments applied in the calculation of Common Equity Tier 1.

The sum of the following elements make up the *Additional Tier 1 capital* [10]:
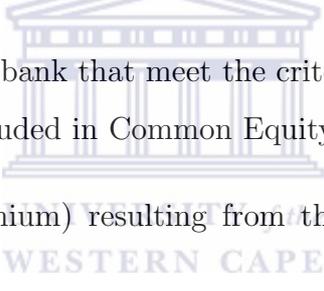
- Instruments issued by the bank that meet the criteria for inclusion in Additional Tier 1 capital (and are not included in Common Equity Tier 1);

- Stock surplus (share premium) resulting from the issue of instruments included in Additional Tier 1 capital;

- Instruments issued by consolidated subsidiaries of the bank and held by third parties that meet the criteria for inclusion in Additional Tier 1 capital and are not included in Common Equity Tier 1;

- Regulatory adjustments applied in the calculation of Additional Tier 1 Capital.

Tier 2 capital consists of the sum of [10]

- Instruments issued by the bank that meet the criteria for inclusion in Tier 2 capital (and are not included in Tier 1 capital);

- Stock surplus (share premium) resulting from the issue of instruments included in Tier 2 capital;

9

- Instruments issued by consolidated subsidiaries of the bank and held by third parties that meet the criteria for inclusion in Tier 2 capital and are not included in Tier 1 capital;

- Certain loan loss provisions;

- Regulatory adjustments applied in the calculation of Tier 2 Capital.

Deposits are considered to be the main liability of banks and refer to the money that the banks' customers place in the banking institution for safekeeping. Deposits are made to deposit accounts at a banking institution, such as savings accounts, checking accounts and money market accounts. The holder of a deposit account has the right to withdraw any deposited funds, as set forth in the terms and conditions of the account.

2.2 The regulation of the international banking industry
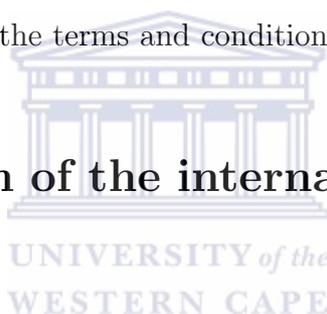
The BCBS administers the regulation and supervision of the international banking industry by imposing minimum capital requirements and other measures on the aforementioned industry. The BCBS introduced the Basel Accords which provide recommendations on international banking regulations in regard to market risk, capital risk and operational risk. The purpose of the Basel Accords is to ensure that internationally active banks hold enough capital to meet obligations and to absorb unexpected losses [2].

In 1988 the BCBS issued the 1988 Basel Capital Accord also known as the Basel I Accord. With Basel I the BCBS aimed to assess the banks' capital in relation to its credit risk, or the risk of a loss occurring if a party does not fulfil its obligations. Basel I resulted in the trend toward increasing risk modelling research by creating a bank asset classification system that grouped banks' assets into the following risk categories [1]:

- 0% - cash, central bank and government debt and any Organization for Economic Co-operation and Development (OECD) government debt;

- 0%, 10%, 20% or 50% - public sector debt;

- 20% - development bank debt, OECD bank debt, OECD securities firm debt, non-OECD bank debt (under one year maturity) and non-OECD public sector debt, cash in collection;

- 50% - residential mortgages;

- 100% - private sector debt, non-OECD bank debt (maturity over a year), real estate, plant and equipment, capital instruments issued at other banks.

Banks were to maintain Total capital (calculated as the sum of Tier 1 and Tier 2 capital) equal to at least 8% of its total-risk-weighted assets under Basel I [1]. However, Basel I was based on simplified calculations and classifications, which have simultaneously called for its disappearance. As a result the BCBS introduced the Basel II Capital Accord and further agreements as the symbol of the continuous refinement of risk and capital [97].

With the 2004 (revised) framework of the Basel II Capital Accord (see [9]), the BCBS layed down regulations seeking to provide incentives for greater awareness of differences in risk through more risk-sensitive minimum capital requirements based on numerical formulas. The *capital adequacy ratios* (see for instance [90, 92, 34, 76, 77]) measure the amount of the bank's capital relative to its amount of credit exposures. Internationally, a standard has been adopted that requires banks to adhere to minimum levels of capital requirements. Banks complying with minimum capital requirements are guaranteed the ability to absorb reasonable levels of losses before becoming insolvent. Thus, capital adequacy ratios ensure the safety and stability of the banking system.

11

Mathematically, capital adequacy ratios are defined as

$$\text{Capital Adequacy Ratio} = \frac{\text{Indicator of Absolute Amount of Bank Capital}}{\text{Indicator of Absolute Level of Bank Risk}}.$$

The bank or the regulator can use this equation to determine whether an absolute amount of bank capital is adequate when compared to a measure of absolute risk [34].

According to the Federal Insurance Deposit Corporation (FDIC), capital adequacy ratios can be divided into *risk-based capital adequacy ratios* and *non-risk-based capital adequacy ratios* [34]. Examples of risk-based capital adequacy ratios can be the *Total Capital Ratio* and *Tier 1 Ratio* [9, 34]. The Total Capital Ratio or CAR is a comparison between banks' Total capital and *total risk-weighted assets* (TRWAs), where TRWAs are constituted by the capital charges for credit, market and operational risk. Similarly, the Tier 1 Ratio is a comparison between the banks' Tier 1 capital and TRWAs. Under Basel II banks were considered to be adequately capitalized if they maintained a CAR of at least 8% and a Tier 1 Ratio of at least 4%. An example of the non-risk-based capital adequacy ratios can be the *Equity Ratio*, which compares banks' Equity capital to its TNRWAs. Under Basel II it was recommended that banks maintain a minimum Equity Ratio of 2%.

In 2010, the BCBS released the Basel III Accord. Globally, Basel III is the latest regulatory standard on bank capital adequacy, stress testing and market liquidity risk. Basel III is more stringent than the Basel I and II Accords and has two main objectives [10, 11, 12, 53, 84]:

- To strengthen global regulation of capital and liquidity with the goal of promoting a more resilient banking sector;

- To improve the banking sector's ability to absorb shocks arising from the financial and economic stress.

The enhancements of Basel III over Basel II come primarily in the following areas: (i) augmentation in the level and quantity of capital; (ii) introduction of a leverage ratio; and (iii)

introduction of liquidity standards [10, 11, 12, 53, 84]. We discuss these enhancements below.

Basel III contains various measures aimed at improving the quantity and quality of capital. In this regard, the ultimate aim of Basel III is to improve the loss-absorption capacity in both going concerns and liquidation scenarios. Basel III proposes that banks retain the minimum CAR of 8% while the minimum Tier 1 Ratio should be increased to 6%. The equity component of the latter is stipulated at 4.5% under Basel III. Basel III introduced the new concepts of *capital conversion buffer* and *countercyclical capital buffer* (CCB). Generally the term "countercyclical" is used when there is a negative correlation between an economic quantity and the overall state of the economy. The capital conversion buffer ensures that banks are able to absorb losses without breaching the minimum capital requirement, and are able to carry on business even in a downturn without deleveraging. This does not form part of the regulatory minimum. Thus while the 8% minimum capital requirement remains unchanged under Basel III, there is now an added 2.5% as capital cushion buffer. The CCB is a pre-emptive measure that requires banks to build up capital gradually as imbalances in the credit market develop. The CCB may be in the range of 0-2.5% of TRWAs which could be imposed on banks during periods of excess credit growth. There is also a provision for a higher capital surcharge on systemically important banks. Basel III strengthens the counterparty credit risk framework in market risk instruments. This includes the use of stressed input parameters to determine the capital requirement for counterparty credit default risk. Basel III introduced a new capital requirement known as credit valuation adjustment risk capital charge for over-the-counter (OTC) derivatives. Its purpose is to protect banks against the risk of decline in the credit quality of the counterparty [10, 53, 84].

Basel III's new *Leverage Ratio* can be considered as another example of non-risk-based capital adequacy ratios. It acts as a non-risk-sensitive backstop measure to reduce the risk of a buildup of excessive leverage at the institution level and in the financial system as a whole. The Leverage Ratio requirement would hence set an all-encompassing floor to minimum cap-

ital requirements. This would limit the potential erosive effects of gaming and model risk on capital against true risks. Basel III recommends a 3% minimum Leverage Ratio [10, 53, 84]. The Leverage Ratio is defined as the comparison between banks' Tier 1 capital and TNRWAs [10, 83].

With the aim of further strengthening the liquidity framework the BCBS developed two minimum standards for quantifying funding liquidity. These are the *Liquidity Coverage Ratio* or LCR and *Net Stable Funding Ratio* or NSRF. The LCR standard aims at a bank having an adequate SHQLAs (recall, the Stock of High Quality Liquid Assets). SHQLAs consist of cash or assets that can be converted into cash at little or no loss of value in private markets to meet its liquidity requirements in a 30 calendar day liquidity stress scenario. The LCR consists of the two components SHQLAs and the TNCOs (recall, the Total Net Cash Outflows) over the next 30 calendar days. By design the NSRF encourages and incentivises banks to use stable sources to fund their activities. The NSRF aids in reducing the dependence on short term wholesale funding during times of buoyant market liquidity while it encourages better assessment of liquidity risk across all on- and off-balance sheet items. NSFR requires a minimum amount of stable sources of funding at a bank relative to the liquidity profiles of the assets, as well as the potential for contingent liquidity needs arising from off-balance sheet commitments, over a one-year horizon. The implications here would pertain to the type of current short term markets available for banks to provide liquidity, the type of long term markets needed, the cost of deposit, and the impact on the profitability of banks. One issue with reference to liquidity is how the regulator would consider the *Statutory Liquidity Ratio* (SLR) securities. The SLR is defined as the amount that commercial banks are required to maintain in the form of cash, or gold or government approved securities before providing credit to their customers. Banks are already investing a substantial amount (around 25%) of their deposits in the SLR securities. The relevance of the cash reserve ratio has also come into question. All these have implementation implications for deposit pricing, cost of funds, and profitability [53].

The LCR is calculated as the ratio between SHQLAs and TNCOs over a 30-day stress period. The Basel III framework requires the LCR to be above or equal to 100%. SHQLAs are calculated as the market value of assets multiplied by an asset factor for individual levels of assets. A mathematical expression for the LCR is

$$\text{Liquidity Coverage Ratio} = \frac{\text{Stock of High Quality Assets}}{\text{Total Net Cash Outflows}}.$$

The quantity TNCOs in the denominator of the above equation is defined as the total expected cash outflows minus total expected cash inflows in the specified stress scenario for the subsequent 30 calendar days. *Total expected cash outflows* are calculated by multiplying the outstanding balances of various categories or types of liabilities and off-balance sheet commitments by the rates at which they are expected to run off or be drawn down. *Total expected cash inflows* are calculated by multiplying the outstanding balances of various categories of contractual receivables by the rates at which they are expected to flow in under the scenario up to an aggregate cap of 75% of total expected cash outflows (see [11]). The formula for calculating TNCOs is [11]

$$
\begin{aligned}
\text{Total Net Cash Outflows} \quad = \quad & \text{total expected cash outflows} \\
- \quad & \text{Min(total expected cash inflows}, 75\% \text{ of total} \\
& \text{expected cash outflows).}
\end{aligned}
$$

The NSFR requires a minimum amount of stable sources of funding at a bank relative to the liquidity of the assets and the potential for contingent liquidity needs from off-balance sheet activities over a one-year horizon. The aim of this ratio is to promote medium to long term resiliency [12]. The NSFR is calculated as

$$\text{Net Stable Funding Ratio} = \frac{\text{Available Amount of Stable Funding}}{\text{Required Amount of Stable Funding}},$$

and this ratio is also required to be equal to at least 100%. The RASF or *Required Amount of Stable Funding* depends solely on the characteristics of the respective instrument's liquidity,

which in turn determine the *Available Stable Factor* (ASF) or *Required Stable Factor* (RSF). ASF factors define the amount of assets that would be expected to stay with the bank for an extended period in an idiosyncratic stress event. RSF factors approximate the amount of a particular asset that could not be monetised during a liquidity event lasting one year.

2.3   The importance of deposit insurance to banks

Deposit insurance funds (DIFs) are a form of protection to depositors of banks against risk of loss arising from failure of banks and other depository institutions. DIFs are usually provided by a government agency. Deposit insurance is mandatory. It claims from a pool of funds to which every depository institution regularly contributes. Deposit insurance only covers a fixed maximum amount per depositor or deposit account holder.

Deposit account holders at banks certainly feel more secure if their deposits are insured. This feeling of security reduces the type of fear that has caused bank runs in the 1930s. The DIF number of a bank is commonly used to compare the value of its TNRWAs to those of problematic banks appearing on the FDIC's quarterly issued "Problem Banks List". Since the FDIC could borrow from the Treasury Department it could not run out of funds. However, large losses would mean increased premiums for the remaining banks in the years to come.

A country's DIF can be either an *explicit deposit insurance fund* (EDIF) or an *implicit deposit insurance fund* (IDIF). It has been argued that EDIF coverages are contractual obligations while IDIF coverages are only conjectural. IDIF coverage exist to the extent that political incentives which influence a government's reaction to large or widespread banking problems make taxpayer bailouts of insolvent banks seem inevitable. Banking crises pressurize governments to rescue at least some banks. This amounts to a sort of implicit deposit insurance being applicable in every country. Despite some differences between EDIFs and IDIFs, both

aim to protect depositors and enhance the stability of the financial system [19].

Under IDIFs, the government's protection of depositors is discretionary [39]. IDIFs do not have any formal laws or regulations relating to the compensation of depositors in the event of a bank failure. The reimbursement amount and the form of protection is based on an ad hoc decision solely made by the government and is responsible for the financing of depositors. Under an IDIF system the government can make payments directly to depositors. Alternatively the government could either arrange for the failed bank's deposits to be assumed by another bank, or arrange and facilitate the merger of a problem bank with a healthy bank, or bail out the troubled bank through direct capital injection. IDIFs have drawbacks in that they create uncertainty about how and when depositors will be compensated. Funding depends on a government's ability to access funds after a bank failure. As a result, in some countries depositors have not been reimbursed at all [19].

EDIFs, on the other hand, have laws that provide for bank deposit guarantees and establish basic aspects of the deposit insurance system. Such basics are coverage limits, if and how the system will be funded, how depositors are to be paid in the event of a bank failure, types of institutions and deposits eligible for protection and whether membership is voluntary or compulsory. EDIFs, normally created by an Act of Parliament, can have 100% or limited depositor coverage. The latter is currently more popular than IDIFs. When a country adopts an EDIF it does not mean that implicit guarantees by government are eliminated, especially during a systemic crisis. They can be privately or publicly administered. Its merits are that it helps the governments to meet its obligations to depositors, limits the scope for discretionary decisions and enhances public confidence, enhances financial stability by establishing a framework for the resolution of failing or failed banks and help to contain the costs of resolving bank failures [19]. It also has its drawbacks, on which we dwell below [19]:

- During a financial crisis limited coverage deposit protection will not prevent bank runs;

17

- Moral hazard-explicitly protected depositors may have less incentives to monitor their banks;

- When depositors are protected banks have more incentive to take excessive risks.

According to Duan and Yu [29] the majority of defaulting banks continue to operate with deposit insurance after reorganization. These banks can be regarded as receiving an at-the-money put option at the point of insolvency resolution. In light of this, deposit insurance can be viewed as a stream of one-period Merton-type put options with occasional TNRWAs value resets. Banks are assumed to pay out cash dividends whenever the value of their TNR-WAs exceeds the level required by a threshold asset-to-debt ratio. The *asset-to-debt ratio* is regarded as the maximum level of paid-in capital above which the bank's equity holders would consider to be excessive and start distributing cash dividends. The threshold level is dictated by the dividend policy of the bank [29].

The insuring agent levies a premium rate which is assumed to be constant over a particular coverage horizon. The fixed premium rate coverage horizon can be one year or several years. Charging a fixed premium rate over a period of several years is, in reality, standard practice of most insuring agents. A *fairly-priced deposit insurance premium rate* can be determined by equating the present value of premium proceeds to that of the puts until the terminal point of the coverage horizon. The risk-neutral valuation technique can be used to price the stream of one-period Merton-type put options. A closed-form solution can not be derived, but the present value can be computed by means of Monte Carlo simulation methods [29].

According to [29] the Federal Savings and Loan Deposit Insurance Corporation and the FDIC allowed the troubled depository institutions to remain open. They believed that these institutions were suffering temporary financial setbacks and would later return to sound financial conditions. According to the paper [8] of Bartholomew, the thrift regulator took an average of 38 months to close and resolve the failed savings and loans institutions over the period 1980-1990. Kane [56] and Kaufman [58] criticized this practice of capital forbearance.

The aforementioned authors argue that capital forbearance encourages "zombie" institutions to engage in excessive risk-taking. Capital forbearance effectively postpones recognizing and realizing losses in a multi-period setting. The postponement of a timely closure simply substitutes an immediate cash settlement with future liabilities. Without excessive risk-taking by troubled institutions, it is not clear whether capital forbearance is a bad practice.

2.4 The usefulness of interest rate swaps to the banking industry

An *interest rate swap* or *swap contract* is an agreement between two parties, known as counterparties, to exchange a series of cash flows according to some pre-specified terms. Swap contracts are OTC, meaning that they are private arrangements. They can be directly between two parties or facilitated by a swap dealer, rather than exchange-traded. The cash flows are usually based on some underlying asset, such as an exchange rate, an interest rate, a commodity price, an equity, etc. [65]

In interest rate swaps, the two parties exchange cash flows that constitute the interest on a *notional principal*. The term "notional principal" refers to the value of the underlying asset on which the cash flows are based. The notional principal is relevant for determining contingent liabilities and capital market requirements. Typically, only the cash flows are exchanged in such a swap, not the principal. An example of such an interest rate swap can be a *vanilla interest rate swap*, or a *fixed-for-floating swap*. In such a swap one party agrees to pay the other a fixed interest rate and receives a floating rate [65].

In swaps that are arranged by a swap dealer, the dealer typically charges a fee for arranging the swap. The fee could be either in the form of an up-front fee, or more usually in the form of a spread on the interest payment. Hull [47] estimates the spread at 3-4 basis points or 0.03-0.04% per annum. A major advantage of a swap directly between the end-users is

that the costs involved are less. However, not every institution has the resources to arrange a swap without using a dealer. Typically, when a swap dealer is involved, the swap will consist of two separate contracts between the dealer and the two parties. The dealer often warehouses a swap. That is, the dealer enters into one side of the swap without having found a counterparty for the other side of the swap. The advantages of using a dealer are firstly that it makes a swap easier to arrange. Secondly, the dealer assumes the credit and default risk. This means that even if one party defaults, the dealer will honour his/her agreement with the other party, and the spread earned by the dealer is partly as compensation for assuming this risk: typically, the fee charged to a swap participant by the dealer will depend upon the credit rating of the participant, with low credit ratings meaning higher fees and vice versa. For swaps arranged without a dealer, a difference between the credit ratings of the two counterparties would typically be reflected in the fixed interest rate [65]

One of the most common floating rates used in an interest rate swap agreement is *London Interbank Offer Rate* (LIBOR). It is considered as a benchmark rate that some of the world's leading banks charge each other for short-term loans. Interest rate swaps are typically used to reduce risk by institutions whose assets and liabilities have a different structure, such as a bank having assets in the form of fixed rate mortgages but short-term liabilities in the form of deposits on which a competitive rate of interest must be paid to attract depositors [65]. Typically, the other party is more interested in increasing profit potential and is willing to take on added risk by swapping a fixed rate interest stream for one that is variable. Both parties benefit by better matching financial positions to bank needs.

Chapter 3

Literature review

We now briefly discuss the works of some of the authors who have contributed to the study and analysis of optimal asset allocation and capital adequacy management problems of commercial banks under the Basel II and III regimes. We also discuss some research papers on the modelling and analyses of Basel III's new liquidity ratios. After these discussions, we summarize the works of some of the authors who have developed deposit insurance pricing models. Finally, we discuss research articles on the pricing of interest rate swaps.

3.1 The increasing popularity of optimization theory under Basel II

We now describe how the popularity of the application of optimization theory in commercial banking problems under the Basel II regime came about. In addition, we describe some examples of optimization problems that were studied in Basel II settings. The latter include the optimization problems of the papers Mukuddem-Petersen and Petersen [75], Fouche et al. [34], Mulaudzi et al. [77], Mukuddem-Petersen and Petersen [76] and Witbooi et al. [94].

Theoretical evidence on the Basel I Accord suggests that the revised version of the Accord may have had an influence on the structure of commercial banks' balance sheets. According

to Berger and Udell [14], for instance, Basel I assigned higher risk weights to commercial loans than securities, where the risk-based capital requirements operates as a regulatory tax. As a result there was a reduction in profitability of commercial loans relative to securities. This of course allowed banks to reallocate their funds to other assets. Further evidence supporting this theory is Jones [54], who undertook regulatory capital arbitrage as in incentive to adjust their on and off balance sheet activities to the Basel I capital requirements. In addition, the empirical evidence by Hall [43], Haubrich and Wachtel [46], Brinkmann and Horvitz [17], Thakor [90] and Furfine [36, 37], provided more clues as to how the revised risk-based capital requirements may have had significant impact on commercial bank balance sheets. The aforementioned authors found that the risk-based capital requirements indeed caused a reduction in bank lending. They concluded that the risk-based capital requirements may have been one of the factors responsible for the credit crunch in the early 90's, where banks decreased their investments in commercial lending and simultaneously shifted their funding towards government securities. Furthermore, authors such as Haubrich and Wachtel [46], Keeton [59] and Jacques and Nigro [51] observed that capital-constraint banks responded to the revised requirements by shifting away from high risk-weighted assets, such as commercial loans, and towards low risk-weighted assets such as government securities.

The rise of Basel II saw an increase in the popularity of the application of optimization theory in banking optimization problems. Classes of optimization problems encountered in the banking literature are on the optimal management of bank asset portfolios and capital adequacy. These generally involve bank asset portfolio diversification as was taking place under the revised Basel Accord. The most common optimization technique used in this field is the method of stochastic optimal control. As stated earlier, this method is generally a tedious one to apply, as it involves solving the HJB equation arising from dynamic programming under the real world probability measure. Examples of the application of the aforementioned methodology in optimal bank asset and capital management problems in banking can for instance be observed in the work of Mukuddem-Petersen and Petersen [75],

Fouche et al. [34], Mulaudzi et al. [77] and Mukuddem-Petersen and Petersen [76]. Another method called the Martingale methodology was used, to our knowledge, for the first time in an optimal asset and capital management problem in banking by Witbooi et al. [94]. The martingale method relies on the theory of Lagrange multipliers and involves solving a PDE under a risk-neutral measure.

The paper [75] studies a banking problem related to the optimal risk management of banks in a stochastic dynamic setting. The authors of paper [75] particularly minimize market and capital adequacy risk. These respectively involve the safety of the securities held and the stability of sources of funds. In this regard, Mukuddem-Petersen and Petersen [75] suggest an optimal portfolio choice and rate of bank capital inflow that will keep the loan level as close as possible to an actuarially determined reference process. This set-up leads to a non-linear stochastic optimal control problem whose solution may be determined by means of the dynamic programming algorithm. The analysis of Mukuddem-Petersen and Petersen [75] relies on the construction of continuous-time stochastic models for bank behaviour upon which a spread method for loan capitalization is imposed. The main novelty of paper [75] is the solution of an optimal stochastic control problem that minimizes bank market and capital adequacy risks by making choices about security allocation and capital requirements, respectively. The former is measured by the deviations of the banks securities from the loan issuing process and is an indicator of the bank's safety. The latter provides information about the size of the deviation of bank capital requirements from the bank capital reference process and is related to the financial stability of the bank.

In their paper [34], Fouche et al. model non-risk-based and risk-based capital adequacy ratios. More specifically, the authors of [34] construct continuous-time stochastic models for the dynamics of the Leverage, Equity and Tier 1 ratios with the aim of deriving the CAR. The aforementioned authors show how their result is relevant to the banking sector by studying an optimal control problem in which an optimal asset allocation strategy is derived

23

for the Leverage Ratio on a given time interval. In particular, Fouche et al. [34] determine the optimal expected terminal utility of the Leverage Ratio and derive the asset allocation strategy that makes it possible to maximize the expected terminal utility of the Leverage Ratio on the given time interval.

Mulaudzi et al. [77] investigate the investment of bank funds in loans and treasuries with the aim of generating an optimal final fund level. The study of [77] considers a bank that takes behavioural aspects such as risk and *regret* into account. Regret is the disutility a bank experiences from the gap in value between an actual asset return and the best possible return that the bank could have attained in a particular economic state. Mulaudzi et al. [77] apply a branch of optimization theory that enables them to consider a regret attribute alongside a risk component as an integral part of the utility function. In this case, regret-aversion corresponds to the convexity of the regret function and the bank's preference is assumed to be representable by optimization subject to the utility. Furthermore, they compare risk- and regret-averse banks in terms of optimal asset allocation between loans and treasuries. One of their main results implies that banks with regret-averse attributes will select optimal asset allocations that are less extreme than those predicted by conventional expected utility. In the case of a very risky portfolio being selected by a purely risk-averse bank, its regret-averse counterpart would select a less risky portfolio. Conversely, should the purely risk-averse bank choose a non-risky portfolio, the regret-averse bank would prefer a riskier portfolio. In essence, banks that are regret-averse will tend to hedge their bets, taking into account the possibility that their preferences may turn out to be suboptimal after the expiry of the loan period. The paper [77] also relates the aforementioned conclusions to the credit crunch phenomenon.

Mukuddem-Petersen and Petersen [76] consider the application of stochastic optimization theory to asset portfolio and capital adequacy management in banking. Their study is largely motivated by the Basel II banking regulation that emphasizes risk minimization practices

associated with assets and capital. The analysis of [76] depends on the dynamics of the CAR which they compute in a stochastic setting, by dividing *regulatory capital* (RC) by the credit risk charge. By definition, RC is the amount of risk capital held by banks and other financial institutions which enable them to survive difficulties such as market or credit risk. This amount is determined by legislation or the regulator. Mukuddem-Petersen and Petersen [76] further demonstrate how the CAR can be optimized in terms of bank equity allocation and the rate at which additional debt and equity is raised. In their analysis, Mukuddem-Petersen and Petersen [76] employ the dynamic programming algorithm for stochastic optimization. Moreover, the authors of [76] contribute to the debate about a major shortcoming of the Basel II regulation associated with reference processes for capital adequacy ratios. Their analysis includes an illustration of aspects of bank management practice in relation to this regulation. Another feature of the paper Mukuddem-Petersen and Petersen [76] is that the authors consider historical data from OECD countries in order to characterize the cyclicality of capital ratios.

In the paper Witbooi et al. [94], the authors apply the Cox-Huang methodology in a continuous-time banking problem where the term structure of the interest rate is affine. The problem addressed in paper [94] particularly involves obtaining an optimal capital allocation strategy that optimizes the bank's TNRWAs consisting of three assets namely a treasury, a marketable security and a loan. The optimal capital allocation strategy is derived by constructing SDEs for the dynamics of the assets in the financial market, the dynamics of the TNRWAs of the bank and developing an allocation strategy that maximizes the TNRWAs of the bank by means of power utility maximization. At the same time, the authors of [94] derive an explicit SDE for the dynamics of the CAR which is calculated by dividing the Total bank capital by the TRWAs. Witbooi et al. [94] observe the behaviour of the CAR under the diversification of the bank's TNRWAs. Their main result is a numerical simulation study in which their CAR resembles a mean-reverting process whose level prevails above the required minimum level of 8%.

3.2 Analyses of Basel III related commercial banking problems

This section presents a summary of commercial banking problems studied under Basel III. In particular, we discuss the works of Muller and Witbooi [78], Jarrow [52], Petersen et al. [83], Gideon et al. [40], De Waal et al. [27] and King [60] here.

To the best of our knowledge, the research article by Muller and Witbooi [78] on optimal capital allocation and capital adequacy management strategies of commercial banks, is currently the only paper in the literature to address these issues in a Basel III setting. The paper [78] models the CAR in terms of optimal capital allocation. In particular, Muller and Witbooi [78] present a model for a bank's CAR in terms of the optimal capital allocation strategy which maximizes an expected logarithmic utility of the bank's TNRWAs at a future date. Furthermore, the paper [78] derives a modified version of the formula for the bank's TNRWAs corresponding to a constant CAR at the 8% level. It presents simulations of the performances of the CAR and modified TNRWAs. For the set of simulation parameters considered, the CAR value persists above the 8% level for the entire investment horizon considered, while the value of the modified TNRWAs is improved if the CAR is at its constant minimal value.

The paper [52] of Jarrow studies the economic foundations for maximum Leverage Ratio rules. In paper [52], Jarrow makes three contributions to the banking literature. First, the author shows how to determine the maximum Leverage Ratio such that the probability of insolvency is less than some predetermined quantity. Secondly, he shows that as an alternative to Value-at-Risk (VaR) rules, Leverage Ratio rules can also be used as a tool for controlling insolvency risk. Lastly, Jarrow [52] argues that Leverage Ratio rules are better than VaR rules because they are more intuitive and easier to compare across firms.

In the paper [83], Petersen et al. study the new Basel III Leverage Ratio. The paper [83] makes use of BankScope data to study internationally active Class I banks that have Tier 1 capital and TNRWAs in excess of US $4 and 100 billion respectively. The authors of [83] also consider Class II banks which do not satisfy the aforementioned conditions. Their study reveals the following. Under both Basel II and Basel III regimes Class I banks are more leveraged than Class II banks. A larger proportion of TNRWAs are made up of off-balance sheet items for Class I banks than for their Class II counterparts. Both types of banks are more leveraged under Basel III leverage calculations than under the Basel II dispensation. It appears that in isolation, high Basel leverage does not appear to be a reliable predictor of subsequent bank distress [83]. An increase in regulation restrictiveness from Basel II to Basel III will however significantly influence leverage. More restrictive regulation is particularly associated with relatively higher leverage. Basel III has to adopt a more than one-size-fits-all approach with respect to leverage.

Gideon et al. [40] provide a framework for the liquidity management of banks. The authors of [40] provide a description for the Inverse Net Stable Funding Ratio (INSFR) dynamics which promote resilience over a longer time horizon by creating additional incentives with more stable funding sources. The paper [40] also makes a clear connection between liquidity and financial crises in a numerical-quantitative frameworks. In addition, Gideon et al. [40] derive a stochastic model for the dynamics of the INSFR that depends mainly on required stable funding, available stable funding as well as the liquidity provisioning rate. Furthermore, the authors of the paper [40] obtain an analytic solution to an optimal bank INSFR problem with a quadratic objective function. This solution can in principle assist in the management of the INSFRs of banks. Liquidity provisioning and bank asset allocation are expressed in terms of a reference process here. Furthermore, Gideon et al. [40] provide a numerical example in order to describe the interplay between the amount of net stable funding and liquidity demands. Gideon et al. [40] find that the INSFR has some limitations regarding the characterization of banks' liquidity positions. For a more complete analysis

27

complementary Basel III ratios such as the NSFR should be considered. According to [40] the latter should take the structure of the short-term assets and liabilities of residual maturities into account.

In the research article [27], De Waal et al. study a numerical problem based on the new Basel III liquidity regulation. More specifically, De Waal et al. [27] explore the relationships between *Shareholder Cash Flow Rights* (SCFR), i.e., shareholders' claims on cash payouts or dividends, and capital stability and liquidity via the NSFR and LCR, respectively. Their results suggest that, as SCFR concentration rises, banks liquidity increases in a statistically and economically significant manner. Furthermore, De Waal et al. [27] hypothetically explore the impact of Basel III via the NSFR for sample banks. The evidence suggests that capital stability will be related to SCFR concentration. At lower levels of SCFR concentration, concentrated SCFR diminishes capital stability. On the other hand, at higher levels, concentrated SCFR enhances capital stability. Their results provide insights into how Basel III liquidity regulation might be applied in future.
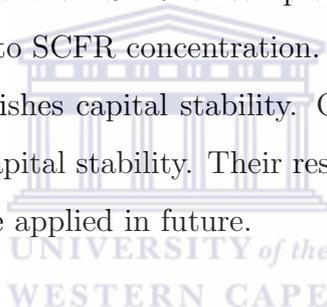
King [60] presents the first comprehensive assessment of the NSFR. The paper [60] outlines how the NSFR is calculated and estimates the ratio for the representative bank in 15 countries. For banks that are below the minimum threshold, King [60] examines different strategies to meet the NSFR and estimates the impact of these changes on bank *net interest margins* (NIMs). NIMs measure the difference between the interest income generated by banks and other financial institutions and the amount of interest paid out to their lenders relative to the amount of their (interest-earning) assets. King [60] highlights the trade-offs between liquidity regulation, bank risk and profitability. The NSFR is designed to encourage banks to hold more high-quality, unencumbered, liquid assets and to increase funding from stable sources such as deposits, longer maturity debt, and equity. These changes should increase the resilience of banks during stressful periods. De-risking the bank in this way should bring some benefits, such as increasing capital ratios, lowering the cost of capital and

increasing charter value. The tradeoff, however, is lower profitability during normal times. Holding fewer illiquid assets and more high-quality assets that cannot be pledged as collateral will lower interest income. Funding assets with longer maturity liabilities will increase interest expense. The resulting decline in net interest income combined with the increase in interest earning assets will cause NIMs to decline. Bank submissions to the BCBS suggest the liquidity requirements may dramatically and adversely impact bank business models and profitability. Concerned about potential unintended consequences, regulators have delayed implementation of the LCR until 2015 and the NSFR until 2018.

3.3   Deposit insurance pricing via put options

Below we summarize the contributions of Merton [72, 73], Marcus and Shaked [68], McCulloch [69], Ronn and Verma [85], Pennacchi [82] and Duan et al. [28], Allen and Saunders [6], and Duan and Yu [29, 30]. All of the aforementioned authors have modelled deposit insurance as some form of put option.

Merton [72] first suggested an analogy between deposit insurance and a put option to value deposit insurance contracts. Since Merton's analogy, there has been a tradition of modelling deposit insurance as a one-period European put option. Examples in the literature of modelling deposit insurance in this way can for instance be observed in the research papers Merton [73], Marcus and Shaked [68], McCulloch [69], Ronn and Verma [85], Pennacchi [82] and Duan et al. [28]. The aforementioned authors derived the formula for the put option under the assumption that, at the time of the audit, which could be either deterministic or stochastic, the put option is exercised if the insured institution is found to be insolvent. The deposit insuring agent renegotiates the terms for the next period if the insured institution is solvent.

Allen and Saunders [6] were the first to depart from the tradition of modelling deposit in-

surance in this way. They argue that deposit insurance can be described as a callable put in the sense that deposit insurance is a perpetual put option with the insuring agent holding the right to terminate the put prematurely. In the paper [29] of Duan and Yu, the authors propose an alternative way of interpreting deposit insurance in a multi-period framework. The defaulting banks in the model of Duan and Yu [29] are assumed to have their assets reset to the level of the outstanding deposits plus accrued interests when an insolvency resolution takes place. According to the deposit insurance contract, the amount required to reset the assets is the legal liability of the insuring agent. Historical data on deposit insurance from the U.S. supports this set-up. The majority of defaulting depository institutions were resolved through either purchase-and-assumption or the government-assisted merger method. Bartholomew [8] reported data for 1730 thrifts that were resolved during the period 1980-1990. Of the 1730 thrifts, 1478 or 85.4% were resolved through this form of reorganization. According to Table 125 of the FDIC 1990 Annual Report, 1813 banks were closed during the period from 1945 through 1990. Among these, a total of 1261 or 69.6% of banks were resolved through this form of reorganization. Duan and Yu [29] found their fairly-priced premium rate to be substantially different from that of Merton [72]. Duan and Yu [29] consider several interesting aspects of deposit insurance, which include varying the fixed coverage premium rate, capital forbearance and the accompanying risk-taking behaviour. Their results show that varying the fixed premium rate coverage horizon affects the fairly-priced deposit insurance premium rate; and that the fairly-priced premium rate is not neutral to capital forbearance. The risk-taking intensity determines how the fairly-priced premium rate responds to forbearance policy.

In their paper [30], Duan and Yu propose a multi-period deposit insurance pricing model that simultaneously incorporates the capital standard and the possibility of capital forbearance. Their model employs the GARCH option pricing technique in determining the deposit insurance value. Their GARCH pricing model offers two distinctive advantages. It explicitly considers the implications of the strict enforcement on capital standard as stipulated in

FDIC Improvement Act of 1991. Additionally, the use of the GARCH model allows them to capture many robust features exhibited by financial asset returns. By the GARCH option pricing theory, the value of a contingent claim is a function of the asset risk premium. This unique feature is found to be prominent in determining the bank's deposit insurance value. Duan and Yu [30] further examine the effects of capital forbearance and moral hazard behaviour in the multi-period deposit insurance setting. They report that their fairly priced premium rate shows an increase with the asset-to-debt ratio. Increasing the coverage horizon in their model leads to a rise or fall in the value of the premium depending on the initial leverage (asset-to-debt) position. For a high initial leverage, an increase in the coverage horizon reduces the fairly priced premium rates. The reverse is true for a low initial leverage. A longer run deposit insurance coverage has the effect of lowering the fairly priced premium rate. If the capital standard is low relative to the current asset-to-debt ratio, the fairly priced premium rate tends to increase with the coverage horizon.
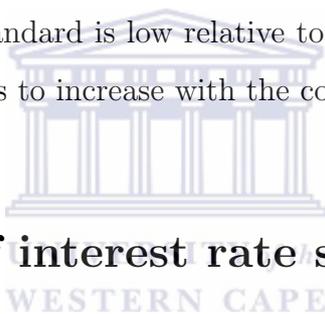
3.4 The pricing of interest rate swaps

We now discuss the interest rate swap pricing models of Mallier and Alobaidi [65], Xiaofeng et al. [96] and Mitra et al. [74].

Mallier and Alobaidi [65] derive expressions for the value of a vanilla fixed-for-floating interest rate swap and an in-arrears swap by treating the swaps as a series of FRAs. Their analysis can be applied both to swaps arranged directly between two counterparties and to swaps arranged by a dealer. In addition, their analysis also accommodates cases where the two counterparties have different credit ratings. In deriving these expressions, Mallier and Alobaidi [65] assume that the floating interest rate follows the mean-reverting CIR model [21, 22]. In contrast to their use of the CIR model, many market practitioners use the Black-76 formula [13], which is a modification of the Black-Scholes model and was originally intended for pricing commodities futures. Under Black-76, the underlying forward rates in

the FRAs which comprise the swap are assumed to be lognormal, and Mallier and Alobaidi [65] feel that the mean-reverting CIR is a better model to use for interest rate derivatives than the Black-76 formula. Market practitioners commonly also take the approach whereby they use a more realistic interest rate model, such as the CIR model used in Mallier and Alobaidi [65], but to price swaps numerically, usually with a binomial tree or Monte Carlo simulation technique. The formulae of Mallier and Alobaidi [65] for the swaps under the CIR model are comparatively simple and could be evaluated numerically both quickly and accurately, making their formulae extremely competitive with other methods for practitioners who want to accurately and quickly price a swap using the CIR model.

Xiaofeng et al. [96], under the foundation of Duffie and Huang [32], integrates the reduced form model and the structure model for a default risk measure, giving rise to a new pricing model of interest rate swap with a bilateral default risk. The swap pricing model of [96] avoids the shortcomings of ignoring the dynamic movements of the firms assets of the reduced form model. When compared to Li [64], their model adds only a little complexity and simplifies the pricing formula significantly. By employing a Crank-Nicholson difference method, Xiaofeng et al. [96] give numerical solutions of their model in a study of the default risk effects on the swap rate. Their results are that for a one year interest rate swap with the coupon paid per quarter, the variance of the default fixed rate payer decreases from 0.1 to 0.01, causing only about a 1.35‰s increase in the swap rate. Their finding is consistent with previous results. With the valuation model of [96], the institutes wanting to enter into a swap contract can consult the swap rate calculated by their model. Contract holders can find the prices of the swap at any time, having an intuition on the value of it.

Under the assumption of stochastic interest rates, the paper of Mitra et al. [74] reformulates the valuation of interest rate swaps, swap leg payments and swap risk measures as a problem of solving a system of linear equations with random perturbations. The aforementioned authors develop a sequence of uniform approximations which solves this system and allows

for fast and accurate computation. The method proposed by Mitra et al. [74] provides a computationally efficient alternative to Monte Carlo based valuations and risk measurement of swaps. Mitra et al. [74] demonstrate this by conducting numerical experiments. Their method provides a potentially important real-time application for analysis and calculation in markets. For swap valuation and risk management, their paper offers potential avenues for exploring accelerations of Monte Carlo techniques. This may be achieved by combining the methods with variance reduction and importance sampling techniques for Monte Carlo simulations. The linear formulation of equations may offer significant potential benefits for computational optimization of portfolios, whereby powerful optimization techniques can be applied from stochastic linear programming methods. Their method may possibly be adapted to investigate exotic derivatives, which pose many non-trivial analytical and computational challenges. Mitra et al. [74] believe that their contribution offers computational advantages of significance to academic researchers as well as industry, where it is important to calculate swap and risk measures in short time periods. With the growing trend of computerised and high frequency trading in industry, this requirement is becoming increasingly important.
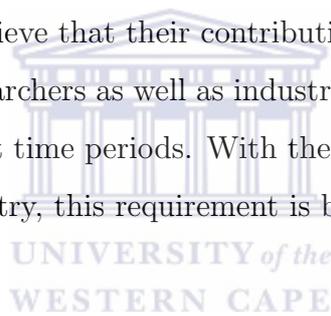
Chapter 4

Mathematical preliminaries

In this chapter, we present mathematical concepts that are used in the commercial banking problems that follow. Here, our main references are the books Abramowitz and Stegun [4], Bracewell [16], Etheridge [33], Hartmann [45], Kanwal [57], Nielsen [79], Øksendal [80] and Øksendal and Sulem [81].

This chapter is split into two sections. In the first section we present concepts that are required to formulate and solve all of the banking problems studied in this thesis. In the second section we present additional theory needed to formulate and solve the interest rate swap pricing problem.

4.1 Mathematical concepts relevant to all the banking problems of the thesis

We now introduce the concepts that are useful for the formulation of all the commercial banking problems of this thesis. These include concepts such as the Legendre transform, for the optimal control problem specifically, and basic ideas and definitions from stochastic calculus, which apply to all of the problems.

The Legendre transform defined below will be used to transform the non-linear second order PDE arising from the optimal control problem associated with the proxy of our optimal control problem.

Definition 4.1. *(see [55]) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function. For $z > 0$, define the Legendre transform, or the Legendre dual of the function $f(x)$ by the formula:*

$$L(z) = \max_x \{f(x) - zx\}.$$

If $f(x)$ is strictly convex, the maximum in the equation above is attained at a unique point, which we denote by x_0. The maximum is in fact attained at the unique solution to the first-order condition, namely,

$$\frac{df(x)}{dx} - z = 0.$$

The ideas and concepts presented in Definition 4.2 to Remark 4.24 are the basics that allow us to formulate our banking model, derive the capital adequacy and liquidity ratios, etc. For instance, Itô's formula without jumps in Remark 4.11 will be used to derive the formula for the LCR, as this model does not include a jump. The Itô formula with the jump in Remark 4.24, on the other hand, will be employed when deriving the capital adequacy ratios (recall, the CAR, Tier 1 and Leverage ratios) and NSFR liquidity ratio.

Definition 4.2. *(see [79, p.317]) Let Ω be any non-empty set. A σ-algebra or σ-field on Ω is a class $\mathcal{F}$ of subsets of Ω with the following three properties:*

1. $\Omega \in \mathcal{F}$;

2. If $\{A(t)\}$ is a finite or infinite sequence of sets in $\mathcal{F}$, then $\bigcup A(t) \in \mathcal{F}$;

3. If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$.

Definition 4.3. *(see [79, p.14]) A filtration is a family $\{\mathcal{F}(t)\}_{t \in J}$ of σ-algebras $\mathcal{F}(t) \subset \mathcal{F}$ which is increasing in the sense that whenever $s, t \in J$ and $s \leq t$, then $\mathcal{F}(s) \subset \mathcal{F}(t)$.*

Definition 4.4. *(see [33, p.29]) A probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *consists of a set* Ω *(sample space), a collection of subsets* $\mathcal{F}$ *of* Ω *(events) and a probability measure* $\mathbb{P}$, *which specifies the probability of each event* $A \in \mathcal{F}$. *The collection* $\mathcal{F}$ *is assumed closed under the operations of countable union and taking complements (σ-field).* $\mathbb{P}$ *must of course satisfy the following axioms:*

1. $0 \leq \mathbb{P}(A) \leq 1$ *for all* $A \in \mathcal{F}$;

2. $\mathbb{P}[\Omega] = 1$;

3. $\mathbb{P}[A \bigcup B] = \mathbb{P}[A] + \mathbb{P}[B]$ *for any disjoint* A *and* B *in* $\mathcal{F}$;

4. *If* $A(n) \in \mathcal{F}$ *for all* $n \in N$ *and* $A(1) \subseteq A(2) \subseteq \ldots$ *then* $\mathbb{P}[A(n)] \uparrow \mathbb{P}[\bigcup_n A(n)]$ *as* $n \uparrow \infty$.

Definition 4.5. *(see [79, p.2]) Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, and let* J *be a time interval. Specifically, assume that* $J = [0, \infty)$ *or* $J = [0, T]$ *for some* T. *A* k-*dimensional stochastic process is a mapping* $X : \Omega \times J \to \mathbb{R}^k$ *such that for each fixed* $t \in J$, *the mapping*

$$X(t) : \omega \mapsto X(\omega, t) = X(t)(w) : \Omega \to \mathbb{R}^k$$

is measurable. A stochastic process is said to be adapted *to a filtration* $\{\mathcal{F}(t)\}_{t \in J}$ *if for each* $t \in J$, *the random variable or vector resulting from the latter mapping is measurable with respect to* $\mathcal{F}(t)$. *This means that the value* $X(t)$ *of* X *at* t *depends only on information available at time* t.

Definition 4.6. *(see [79, p.5]) A* k-*dimensional standard Brownian motion is a* k-*dimensional process* $\{W(t)\}_{t \geq 0}$ *such that:*

1. $W(0) = 0$ *with probability one;*

2. W *is continuous;*

3. *if* $0 \leq t(0) \leq \cdots \leq t(n)$, *then the increments* $W(t(1)) - W(t(0)), \ldots, W(t(n)) - W(t(n-1))$ *are independent;*

36

4. if $0 \leq s < t$, then the increment $W(t) - W(s)$ is normally distributed with mean zero and covariance matrix $(t - s)I$, where I is the $k \times k$ identity matrix.

If W is a one-dimensional standard Brownian motion, and if $0 \leq s < t$, then the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$. A one-dimensional process is called a geometric Brownian motion if it has the form e^Z, where Z is a one-dimensional generalized Brownian motion with constant initial value $Z(0)$.

Definition 4.7. *(see [79, p.16]) Let $\{\mathcal{F}(t)\}_{t \geq 0}$ be a filtration. A process X is a martingale if it is integrable and adapted and whenever $s, t \in J$ and $0 \leq s \leq t$*

$$\mathbb{E}[X(t)| \ \mathcal{F}(s)] = X(s).$$

Definition 4.8. *(see [80, p.8]) Let $\mathcal{H}_U$ denote the σ-algebra generated by the collection of all open subsets, U, of a topological space Ω. Then $\mathcal{H}_U$ is called the Borel σ-algebra on Ω and the members $B \in \mathcal{H}_U$ are called the Borel sets.*

Definition 4.9. *(see[80, p.35]) $\mathcal{W}_{\mathcal{H}}(S, T)$ denotes the class of processes $f(t, \omega) \in \mathbb{R}$ satisfying:*

1. *$(t, \omega) \to f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0, \infty)$;*

2. *There exists an increasing family of σ-algebras $\mathcal{H}(t)$ with $t \geq 0$, such that $W(t)$ is a martingale with respect to $\mathcal{H}(t)$ and that $f(t)$ is $\mathcal{H}(t)$-adapted;*

3.
$$\mathbb{P}\left[\int_S^T f(s, \omega)^2 ds < \infty\right] = 1.$$

Definition 4.10. *(see [80, p.44]) Let $W(t)$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. A (one-dimensional) Itô process (or stochastic integral) is a stochastic process $X(t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form*

$$X(t) = X(0) + \int_0^t u(s, \omega)ds + \int_0^t v(s, \omega)dW(s), \tag{4.1}$$

where $v \in \mathcal{W}_\mathcal{H}$, so that

$$\mathbb{P}\left[\int_0^t v(s,\omega)^2 ds < \infty \ \ \forall \ t \geq 0\right] = 1.$$

We also assume that u is $\mathcal{H}(t)$-adapted, where $\mathcal{H}(t)$ is an increasing family of σ-algebras, $\{\mathcal{H}(t)\}_{t \geq 0}$, such that $W(t)$ is a martingale with respect to $\mathcal{H}(t)$, and

$$\mathbb{P}\left[\int_0^t |u(s,\omega)| ds < \infty \ \ \forall \ t \geq 0\right] = 1.$$

If $X(t)$ is an Itô process of the form (4.1), Eq.(4.1) is sometimes written in the shorter differential form

$$dX(t) = u dt + v dW(t). \tag{4.2}$$

Remark 4.11. *(see [80, p.44]) Let $X(t)$ be an Itô process given by*

$$dX(t) = u dt + v dW(t).$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Then $Y(t) = g(t, X(t))$ is again an Itô process, and

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t)) dt + \frac{\partial g}{\partial x}(t, X(t)) dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(dX(t))^2,$$

where $(dX(t))^2 = dX(t) dX(t)$ is computed according to the rules

$$dt dt = dt dW(t) = dW(t) dt = 0; \qquad dW(t) dW(t) = dt.$$

Remark 4.12. *(see [80, p.55]) For Itô processes $X(t)$ and $Y(t)$ in $\mathbb{R}$, Itô's product rule gives*

$$d(X(t)Y(t)) = X(t) dY(t) + Y(t) dX(t) + dX(t) dY(t).$$

Definition 4.13. *(see [81, p.1]) Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space. An $\mathcal{F}(t)$-adapted process $\{\eta(t)\}_{t \geq 0} \subset \mathbb{R}$ with $\eta(0) = 0$ a.s. is called a Lévy process if $\eta(t)$ is continuous in probability and has stationary and independent increments.*

Remark 4.14. *(see [81, p.1]) Let $\{\eta(t)\}$ be a Lévy process. Then $\eta(t)$ has a càdlàg version (right continuous with left limits) which is also a Lévy process.*

In this thesis we will assume that the Lévy processes we work with are càdlàg. The jump of $\eta(t)$ at time $t \geq 0$ is defined by

$$\Delta\eta(t) = \eta(t) - \eta(t-). \tag{4.3}$$

Let $\mathcal{B}_0$ be the family of Borel sets $U \subset \mathbb{R}$ whose closure $\bar{U}$ does not contain 0. For $U \in \mathcal{B}_0$ we define

$$N(t, U) = N(t, U, \omega) = \sum_{s:0<s\leq t} \chi_U(\Delta\eta(s)). \tag{4.4}$$

In other words, $N(t, U)$ is the number of jumps of size $\Delta\eta(s) \in U$ which occur before or at time t. Here $N(t, U)$ is called the Poisson random measure (or jump measure) of $\eta(\cdot)$.

Remark 4.15. *(see [81, p.2]) Note that $N(t, U)$ is finite for all $U \in \mathcal{B}_0$.*

To see why Remark 4.15 is true, define

$$T_1(\omega) = \inf\{t > 0; \eta(t) \in U\}.$$

We claim that $T_1(\omega) > 0$ a.s. To prove this, note that by right continuity of paths we have

$$\lim_{t\to 0^+} \eta(t) = \eta(0) = 0 \quad \text{a.s.}$$

Therefore, for all $\epsilon > 0$ there exists $t(\epsilon) > 0$ such that $|\eta(t)| < \epsilon$ for all $t < t(\epsilon)$. This implies that $\eta(t) \notin U$ for all $t < t(\epsilon)$, if $\epsilon < \text{dist}(0, U)$.

Next we define inductively

$$T_{n+1}(\omega) = \inf\{t > T_n(\omega); \Delta\eta(t) \in U\}.$$

Then by the above argument $T_{n+1} > T_n$ a.s. We claim that

$$T_n \to \infty \quad \text{as} \quad n \to \infty, \quad \text{a.s.}$$

Assume not, then $T_n \to T < \infty$. But then

$$\lim_{s\to T^-} \eta(s)$$

39

can not exist, contradicting the existence of left limits of the paths.

It is well known that Brownian motion $\{W(t)\}_{t \geq 0}$ has stationary and independent increments. Thus $W(t)$ is a Lévy process.

Remark 4.16. *(see [81, p.2]) The Poisson process $\pi(t)$ of intensity $\lambda > 0$ is a Lévy process taking values in $\mathbb{N} \cup 0$ and such that*

$$\mathbb{P}[\pi(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}; \quad n = 0, 1, 2, \ldots$$

Remark 4.17. *(see [81, p.2]) The set function*

1. *$U \to N(t, U, \omega)$ defines a σ-finite measure on $\mathcal{B}_0$ for each fixed t, ω. The differential form of this measure is written $N(t, dz)$;*

2. *$[a, b) \times U \to N(b, U, \omega) - N(a, U, \omega)$; $[a, b) \subset [0, \infty)$, $U \in \mathcal{B}_0$ defines a σ-finite measure for each fixed ω. The differential form of this measure is written $N(dt, dz)$;*

3. *$\nu(U) = \mathbb{E}[N(1, U)]$, where $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ denotes expectation with respect to $\mathbb{P}$, also defines a σ-finite measure on $\mathcal{B}_0$, called the Lévy measure of $\{\eta(t)\}$;*

4. *Fix $U \in \mathcal{B}_0$. Then the process*

$$\pi_U(t) := \pi_U(t, \omega) := N(t, U, \omega)$$

is a Poisson process of intensity $\lambda = \nu(U)$.

To find the Lévy measure ν of $Y(t)$ note that if $U \in \mathcal{B}_0$ then

$$
\begin{aligned}
\nu(U) &= \mathbb{E}[N(1, U)] = \mathbb{E}\left[\sum_{s; 0 < s \leq 1} \chi_U(\Delta Y(s)) \right] \\
&= \mathbb{E}[(\text{number of jumps}) \times \chi_U(\text{jump})] = \mathbb{E}[\pi(1) \chi_U(X)] = \lambda \mu_X(U),
\end{aligned}
$$

by independence. We conclude that

$$\nu = \lambda \mu_X. \tag{4.5}$$

40

This shows that a Lévy process can be represented by a compound Poisson process if and only if its Lévy measure is finite.

Remark 4.18. *(see [81, p.3]) Let $\{\eta(t)\}$ be a Lévy process. Then $\eta(t)$ has the decomposition*

$$\eta(t) = \alpha t + \sigma W(t) + \int_{|z|<R} z\tilde{N}(t,dz) + \int_{|z|\geq R} zN(t,dz), \qquad (4.6)$$

for some constants $\alpha, \sigma \in \mathbb{R}$ and $R \in [0,\infty]$. Here

$$\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt \qquad (4.7)$$

is the compensated Poisson random measure of $\eta(\cdot)$, and $W(t)$ is a Brownian motion independent of $\tilde{N}(dt,dz)$. For each $A \in \mathcal{B}_0$ the process

$$M(t) := \tilde{N}(t,A) \qquad (4.8)$$

is a martingale. If $\alpha = 0$ and $R = \infty$, we call $\eta(t)$ a Lévy martingale. We can always choose $R = 1$.

Remark 4.19. *(see [81, p.4]) If $\mathbb{E}|\eta(t)| < \infty$ for all $t \geq 0$, then*

$$\int_{|z|\geq 1} |z|\nu(dz) < \infty$$

and hence we may choose $R = \infty$ and write

$$\eta(t) = \alpha_1 t + \sigma W(t) + \int_{\mathbb{R}} z\tilde{N}(t,dz),$$

where

$$\alpha_1 = \alpha + \int_{|z|\geq 1} z\nu(dz).$$

Remark 4.20. *(see [81, p.4]) A Lévy process is a strong Markov process.*

Remark 4.21. *(see [81, p.5]) A Lévy process is a semimartingale.*

Definition 4.22. *(see [81, p.5]) Let* $\mathrm{D_{ucp}}$ *denote the space of cádlág adapted processes, equipped with the topology of uniform convergence on compacts in probability (ucp) :* $H_n \rightarrow H$ *ucp if for all* $t > 0$ $\sup_{0 \le s \le t} |H_n(s) - H(s)| \rightarrow 0$ *in probability* $(A_n \rightarrow A$ *in probability if for all* $\epsilon > 0$ *there exists* $n_\epsilon \in \mathbb{N}$ *such that* $n \ge n_\epsilon \implies \mathbb{P}[|A_n - A| > \epsilon] < \epsilon)$.

Let $\mathrm{L_{ucp}}$ *denote the space of adapted cáglád processes (left continuous with right limits), equipped with the ucp topology. If* $H(t)$ *is a step function of the form*

$$H(t) = H_0 \chi_{\{0\}}(t) + \sum_i H_i \chi_{(T_i, T_{i+1}]}(t),$$

where $H_i \in \mathcal{F}(T_i)$ *and* $0 = T_0 \le T_1 \le \cdots \le T_{n+1} < \infty$ *are* $\mathcal{F}(t)$*-stopping times and* X *is cáglád, we define*

$$J_X H(t) := \int_0^t H_s dX(s) := H_0 X(0) + \sum_i H_i(X(T_{i+1} \wedge t) - X(T_i \wedge t)), \quad t \ge 0.$$

Remark 4.23. *(see [81, p.5]) Let* X *be a semi-martingale. Then the mapping* J_X *can be extended to a continuous linear map*

$$J_X : \mathrm{L_{ucp}} \rightarrow \mathrm{D_{ucp}}.$$

This construction allows us to define stochastic integrals of the form

$$\int_0^t H(s) d\eta(s)$$

for all $H \in \mathrm{L_{ucp}}$. In view of the decomposition (4.6) this integral can be split into integrals with respect to ds, $dW(s)$, $\tilde{N}(ds, dz)$ and $N(ds, dz)$. This makes it natural to consider the more general stochastic integrals of the form

$$X(t) = X(0) + \int_0^t \alpha(s, \omega) ds + \int_0^t \beta(s, \omega) dW(s) + \int_0^t \int_{\mathbb{R}} \gamma(s, z, \omega) \bar{N}(ds, dz), \qquad (4.9)$$

where the integrands are $\mathcal{F}(t)$-predictable and satisfy the growth condition

$$\int_0^t \left\{ |\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}} \gamma^2(s, z) \nu(dz) \right\} ds < \infty$$

42

a.s. for all $t > 0$.

For simplicity we have put

$$\bar{N}(ds, dz) = \begin{cases} N(ds, dz) - \nu(dz)ds & \text{if } |z| < R \\ N(ds, dz) & \text{if } |z| \geq R, \end{cases}$$

with R as in Remark 4.18.

The following shorthand differential notation for the process $X(t)$ satisfying Eq.(4.9) will be used:

$$dX(t) = \alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}} \gamma(t, z)\bar{N}(dt, dz). \tag{4.10}$$

Processes such as in Eq.(4.10) are called Itô-Lévy processes.

Recall that a semi-martingale $M(t)$ is called a local martingale up to time T (with respect to $\mathbb{P}$) if there exists an increasing sequence of $\mathcal{F}(t)$-stopping times τ_n such that $\lim_{n \to \infty} \tau_n = T$ a.s. and $M(t \wedge \tau_n)$ is a martingale with respect to $\mathbb{P}$ for all n.

Note that if

1.
$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \gamma^2(t, z)\nu(dz)dt\right] < \infty, \tag{4.11}$$

then the process
$$M(t) := \int_0^t \int_{\mathbb{R}} \gamma(s, z)\tilde{N}(ds, dz), \quad 0 \leq t \leq T$$

is a martingale;

2.
$$\int_0^T \int_{\mathbb{R}} \gamma^2(t, z)\nu(dz)dt < \infty \quad \text{a.s.}, \tag{4.12}$$

then $M(t)$ is a local martingale, $0 \leq t \leq T$.

43

Remark 4.24. *(see [81, p.7]) Suppose $X(t) \in \mathbb{R}$ is an Itô-Lévy process of the form*

$$dX(t) = \alpha(t, \omega)dt + \sigma(t, \omega)dW(t) + \int_{\mathbb{R}} \gamma(t, z, \omega)\bar{N}(dt, dz), \tag{4.13}$$

where

$$\bar{N}(dt, dz) = \begin{cases} N(dt, dz) - \nu(dz)ds & \text{if } |z| < R \\ N(dt, dz) & \text{if } |z| \geq R, \end{cases} \tag{4.14}$$

for some $R \in [0, \infty]$.

Let $f \in C^2(\mathbb{R}^2)$ and define $Y(t) = f(t, X(t))$. Then $Y(t)$ is again an Itô-Lévy process and Itô's formula applied to $Y(t)$ gives

$$
\begin{aligned}
dY(t) & = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))[\alpha(t, \omega)dt + \sigma(t, \omega)dW(t)] \\
& + \frac{1}{2}\sigma^2(t, \omega)\frac{\partial^2 f}{\partial x^2}(t, X(t))dt \\
& + \int_{|z|<R} \left\{ f(t, X(t-) + \gamma(t, z, \omega)) - f(t, X(t-)) \right. \\
& - \left. \frac{\partial f}{\partial x}(t, X(t-))\gamma(t, z, \omega) \right\}\nu(dz)dt \\
& + \int_{\mathbb{R}} \left\{ f(t, X(t-) + \gamma(t, z, \omega)) - f(t, X(t-)) \right\}\bar{N}(dt, dz). \tag{4.15}
\end{aligned}
$$

4.2 Additional theory for pricing interest rate swaps

In this section we introduce additional concepts necessary for deriving the swap pricing models. These include concepts such as Kummer's functions, Laguerre polynomials, Bessel functions and Green's function. Applying these, it is possible to interpret the value of a swap as the sum of a series of FRAs.

Definition 4.25. *(see [89, p.2] or [98, p.124]) The confluent hypergeometric differential equation is the second-order ordinary differential equation (ODE)*

$$x\frac{d^2y}{dx^2} + (c - x)\frac{dy}{dx} - ay = 0.$$

It is also known as Kummer's differential equation. It has a regular singular point at zero and irregular singularity at ∞. The solutions

$$y = b_1 F_1(a, c, x) + b_2 U(a, c, x)$$

are called confluent hypergeometric functions of the first and second kinds respectively.

Definition 4.26. *The confluent hypergeometric function of the first kind $F_1(a, b, z)$, also known as Kummer's function of the first kind, is a degenerate form of the hypergeometric function $F_2(a, b, c, z)$ which arises as a solution to the confluent hypergeometric differential equation. Some notations used for this function include $F(\alpha, \beta, x)$ (see [63]), $M(a, b, z)$ (see [5]) and $\Phi(a, b, z)$ (see [48]).*

Kummer's function of the first kind has a hypergeometric series given by

$$F_1(a, b, z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{z^k}{k!},$$

where $(a)_k$ and $(b)_k$ are Pochhammer symbols. If a and b are integers, $a < 0$, and either $b < 0$ or $b < a$, then the series yields a polynomial with a finite number of terms. If b is an integer such that $b \leq 0$, then $F_1(a, b, z)$ is undefined. Kummer's function of the first kind is given in terms of the Laguerre polynomial by

$$L_n^m(x) = \frac{(m+n)!}{m!n!}F_1(-n, m+1, x),$$

(see [7]). It has the following integral representation (see [4]):

$$F_1(a, b, z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}dt.$$

Definition 4.27. *(see [50, p.1481] or [98, p.124]) The Laguerre differential equation is given by*

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0.$$

This equation is a special case of the more general associated Laguerre differential equation, defined by

$$x\frac{d^2y}{dx^2} + (v+1-x)\frac{dy}{dx} + \lambda y = 0,$$

where λ and v are real numbers with $v = 0$. The general solution to the associated equation is

$$t = C_1 U(-\lambda, 1 + v, x) + C_2 L_\lambda^v(x),$$

where $U(a, b, x)$ is a confluent hypergeometric function of the first kind and $L_\lambda^v(x)$ is a generalized Laguerre polynomial.

Definition 4.28. *(see [7, p.726]) Solutions to the associated Laguerre differential equation with $v \neq 0$ and k an integer are called associated Laguerre polynomial $L_v^k(x)$.*

Definition 4.29. *(see [4]) A Bessel function $I_n(x)$ is defined by the recurrence relations*

$$I_{n+1} + I_{n-1} = \frac{2n}{x} I_n$$

and

$$I_{n+1} - I_{n-1} = -2\frac{dI_n}{dx}.$$

The Bessel functions are frequently defined as the solutions to the differential equation (see [4, p.358])

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0.$$

Definition 4.30. *(see [4, p.1020]) The Heaviside step function is a mathematical function denoted $H(x)$, or sometimes $\theta(x)$ or $u(x)$, and also known as the "unit step function". The term "Heaviside step function" and its symbol can represent either a piecewise constant function or a generalized function.*

When defined as a piecewise constant function, the Heaviside step function is given by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(see [4, p.1020]) or [16, p.61]). When defined as a generalized function, it can be defined as a function $\theta(x)$ such that

$$\int \theta(x)\phi'(x)dx = -\phi(0)$$

46

for $\phi'(x)$ the derivative of a sufficiently smooth function $\phi(x)$ that decays sufficiently quickly (see [57]).

Definition 4.31. *(see [4, p.1020]) The Laplace transform, denoted $\mathcal{L}$, is an integral transform defined as*

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t) dt,$$

where $F(t)$ is a function of the real variable t and s is a complex variable.

Definition 4.32. *The delta function δ is a linear functional from a space of test functions f. The action of δ on f, commonly denoted $\delta[f]$ or $\langle \delta, f \rangle$, gives the value at zero of f for any function f.*

The delta function can be viewed as the derivative of the Heaviside step function (see [15, p.94])

$$\frac{d}{dx}[H(x)] = \delta(x).$$

The delta function has the fundamental property that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

and, in fact,

$$\int_{a-\epsilon}^{a+\epsilon} f(x)\delta(x-a)dx = f(a)$$

for $\epsilon > 0$.

Additional identities include $\delta(x-a) = 0$ for $x \neq a$, and

$$\delta(ax) = \frac{1}{|a|}\delta(x);$$

$$\delta(x^2 - a^2) = \frac{1}{2|a|}\left[\delta(x+a) + \delta(x-a)\right].$$

More generally, the delta function of a function of x is given by

$$\delta[g(x)] = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|},$$

where the x_i's are the roots of g.

Definition 4.33. *(see [45]) Given a linear differential operator $\mathcal{L} = \mathcal{L}(x)$ acting on the collection of distributions over a subset Ω of some Euclidean space $\mathbb{R}^n$, a Green's Function $G = G(x, s)$ at the point $s \in \Omega$ corresponding to $\mathcal{L}$ is any solution of*

$$\mathcal{L}G(x, s) = \delta(x - s),$$

where δ denotes the delta function.

The motivation for defining Green's function is widespread, but by multiplying the above identity by a function $f(s)$ and integrating with respect to s yields (see [15, p.94])

$$\int \mathcal{L}G(x, s)f(s)ds = \int \delta(x - s)f(s)ds.$$

The right hand side reduces to $f(x)$ due to the properties of the delta function, and hence because $\mathcal{L}$ is a linear operator acting only on x, and not s, the left hand side can be rewritten as

$$\mathcal{L}\left(\int G(x, s)f(s)ds \right).$$

This reduction is particularly useful when solving for $u = u(x)$ in differential equations of the form

$$\mathcal{L}u(x) = f(x),$$

where the above arithmetic confirms that

$$\mathcal{L}u(x) = \mathcal{L}\left(\int G(x, s)f(s)ds \right)$$

and whereby it follows that u has the specific integral form

$$u(x) = \int G(x, s)f(s)ds.$$

Chapter 5

The jump-diffusion banking model and optimal control problem
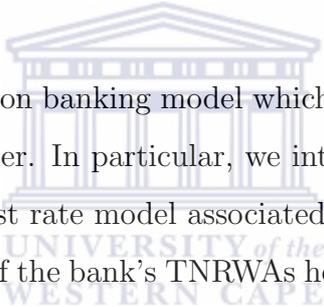
We now present the jump-diffusion banking model which will be used to formulate problems two through four described earlier. In particular, we introduce models for the bank's Total capital, its assets and the interest rate model associated with the financial market. We also construct an SDE for the value of the bank's TNRWAs here. Further, we present the optimal control problem and show, using the methodology of Gao [38], how the proxy (which is the optimal solution of [38, 78]) to the solution of the control problem with the jump can be derived via the Legendre transform and dual theory. We will on occasion directly quote from the methodology of Gao [38].

The main references of this chapter are Vasiček [91], Cox et al. [22], Deelstra et al. [26], Kramkov and Schachermayer [62], Jonsson and Sircar [55], Choulli and Hurd [20], Xiao et al. [95], and Cox and Huang [24].

5.1 Introducing the financial market and formulating the asset portfolio

We assume throughout this chapter that we are working with a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(t)_{t \geq 0}, \mathbb{P})$, which satisfies the usual hypotheses of completeness and right continuity. The filtration $(\mathcal{F}(t))_{t \geq 0}$ is assumed to be generated by the Brownian motions appearing in the dynamics of the bank items which we will introduce throughout.

We now introduce the financial market in which the bank operates. We assume that the bank invests its Total capital in a market which allows for at least two investment opportunities, viz., a riskless treasury security and risky marketable security. It is assumed that the aforementioned assets can be bought and sold without incurring any transaction cost or restriction on short sales. The market is also assumed to allow the bank the opportunity to invest in a loan. In the dynamics of the bank items introduced below, W_r and W_S denote two independent one-dimensional standard Brownian motions.

In our optimization problem we assume the bank to continuously raise Total capital at the rate

$$
\begin{aligned}
dC(t) &= c(t)dt, \\
C(0) &> 0.
\end{aligned}
\tag{5.1}
$$

The first asset in the financial market is a riskless treasury. We denote its price at time t by $S_0(t)$ and assume that its dynamics evolve according to the ODE

$$
\begin{aligned}
\frac{dS_0(t)}{S_0(t)} &= r(t)dt, \\
S_0(0) &= 1.
\end{aligned}
\tag{5.2}
$$

The dynamics of the short-rate process, $r(t)$, are given by the SDE

$$
dr(t) = (a - br(t))dt - \sigma_r dW_r(t),
\tag{5.3}
$$

for $t \geq 0$ and where $\sigma_r = \sqrt{k_1 r(t) + k_2}$. The coefficients a, b, k_1 and k_2, as well as the initial value $r(0)$ are all positive real constants. The above dynamics recover, as a special case, the Vasiček [91] (resp. Cox et al. [22]) dynamics when k_1 (resp. k_2) is equal to zero. The term structure of the interest rates is affine under the aforementioned dynamics.

The second asset in the market is a risky marketable security whose price is denoted by $S(t)$, $t \geq 0$. Its dynamics are given by the equation

$$
\begin{aligned}
\frac{dS(t)}{S(t)} &= r(t)dt + \sigma_1(dW_s(t) + \lambda_1 dt) + \sigma_2\sigma_r(dW_r(t) + \lambda_2\sigma_r dt), \\
S(0) &= 1,
\end{aligned}
\tag{5.4}
$$

with λ_1 and λ_2 (resp σ_1, σ_2) being constants (resp. positive constants) as in Deelstra et al. [26].

The third asset is a loan to be amortized over a period $[0, T]$, whose value at time $t \geq 0$ is denoted by $L(t)$. We assume that its dynamics can be described by the following SDE with Lévy noise

$$
\frac{dL(t)}{L(t-)} = r(t)dt + \sigma_L(T - t, r(t))(dW_r(t) + \lambda_2\sigma_r dt) + dK(t).
\tag{5.5}
$$

Here $K(t)$ is given by

$$
K(t) = \int_0^t \int_{\mathbb{R}} \gamma(t, z, \omega) \bar{N}(dt, dz).
$$

In the above dynamics,

$$
\bar{N}(dt, dz) = \begin{cases} N(dt, dz) - \nu(dz)dt & \text{if } |z| < R \\ N(dt, dz) & \text{if } |z| \geq R, \end{cases}
\tag{5.6}
$$

for some $R \in [0, \infty)$, denotes a compensated Poisson random measure independent of W_r and W_S. In Eq.(5.6), N denotes an $\mathcal{F}(t)$-adapted Poisson random measure, while ν denotes an intensity measure assumed to be a Lévy measure.

We now model the TNRWAs of the bank. Let $X(t)$ denote the value of the TNRWAs at time $t \in [0, T]$. The dynamics of the TNRWAs are described by the formula

$$
\begin{aligned}
dX(t) &= \theta_r(t)\frac{dS_0(t)}{S_0(t)} + \theta_S(t)\frac{dS(t)}{S(t)} + \theta_L(t)\frac{dL(t)}{L(t-)} + dC(t) \\
&= [X(t)r(t) + \theta_S(t)(\lambda_1\sigma_1 + \lambda_2\sigma_2\sigma_r^2) + \theta_L(t)\lambda_2\sigma_L(T-t,r(t))\sigma_r + c]dt \\
&\quad + \theta_S(t)\sigma_1 dW_S(t) + (\theta_L(t)\sigma_L(T-t,r(t)) + \theta_S(t)\sigma_2\sigma_r)dW_r(t) \\
&\quad + \theta_L(t)dK(t),
\end{aligned}
\tag{5.7}
$$

where $\theta_S(t)$, $\theta_L(t)$ and $\theta_r(t)$ denote the amounts of Total capital invested in the two risky assets (marketable security and loan) and in the riskless asset (treasury) respectively.

5.2 Formulating the control problem and deriving the proxy

In this section we formulate the optimization problem and derive the proxy to its solution. We wish to choose a capital allocation strategy in order to maximize the expected utility of the bank's TNRWAs at a future date $T > 0$. Mathematically, the stochastic optimal control problem can be stated as follows:

Problem 5.1. *Our objective is to maximize the expected utility of the bank's TNRWAs at a future date $T > 0$. Thus we must*

$$
\text{maximize} \quad J(\theta_S, \theta_L) = \mathbb{E}[u(X(T))]
$$

$$
\text{subject to} \quad
\begin{cases}
dr(t) = (a - br(t))dt - \sigma_r dW_r(t), \\
dX(t) = [X(t)r(t) + \theta_S(t)(\lambda_1\sigma_1 + \lambda_2\sigma_2\sigma_r^2) + \theta_L(t)\lambda_2\sigma_L(T-t,r(t))\sigma_r + c(t)]dt \\
\quad + \theta_S(t)\sigma_1 dW_S(t) + (\theta_L(t)\sigma_L(T-t,r(t)) + \theta_S(t)\sigma_2\sigma_r)dW_r(t) + \theta_L(t)dK(t), \\
X(0) = x_0, \quad r(0) = r_0,
\end{cases}
$$

with $0 \leq t \leq T$, and where $X(0) = x_0$ and $r(0) = r_0$ denote the initial conditions of the optimal control problem.

If we discard the effect of the jump associated with the loan and, in addition, describe the bank's objective with the logarithmic utility function $u(x) = \ln x$ for $x > 0$, then Problem 5.1 becomes identical to the one solved in the papers Gao [38] and Muller and Witbooi [78]. Since we assume small jumps in the value of the loan, we propose using the optimal solution of the control problem of [38, 78] as a proxy to the optimal capital allocation strategy that solves Problem 5.1. Based on the methodology of Gao [38], we will now show how the Legendre transform and dual theory can be used to derive the proxy.

We note that the utility function $u(\cdot)$ is strictly concave up and satisfies the Inada conditions $u'(+\infty) = 0$ and $u'(0) = +\infty$. By using the classical tools of stochastic optimal control, we define the value function:

$$H(t, r, x) = \sup_{\theta_S, \theta_L} \mathbb{E}(u(X(T) | \ r(t) = r, X(t) = x)), \qquad 0 < t < T. \tag{5.8}$$

The value function can be considered as a kind of utility function. The marginal utility of the value function is a constant, while the marginal utility of the original utility function $u(\cdot)$ decreases to zero as $x \to \infty$ (see Kramkov and Schachermayer [62]). The value function also inherits the convexity of the utility function (see Jonsson and Sircar [55]). Moreover, it is strictly convex for $t < T$ even if $u(\cdot)$ is not.

The maximum principle leads to the HJB equation below (see also [38]):

$$
\begin{aligned}
H_t \ + \ &\sup_{\theta_S, \theta_L} \Bigg[a(b - r)H_r + [xr + (\lambda_1 \sigma_1 + \lambda_2 \sigma_2 \sigma_r^2)\theta_S + \lambda_2 \sigma_L \sigma_r \theta_L + c]H_x \\
+ \ &\frac{1}{2}[\sigma_1^2 \theta_S^2 + (\sigma_L \theta_L + \sigma_2 \sigma_r \theta_S)^2]H_{xx} + \frac{\sigma_r^2}{2}H_{rr} \\
- \ &(\sigma_L \sigma_r \theta_L + \sigma_2 \sigma_r^2 \theta_S)H_{rx} \Bigg] = 0,
\end{aligned}
\tag{5.9}
$$

where the time variable t has been suppressed. Above H_t, H_r, H_x, H_{rr}, H_{xx} and H_{rx} denote

53

partial derivatives of first and second orders with respect to time, interest rate and TNRWAs.

The first-order maximizing conditions for the optimal strategies θ_S and θ_L of the proxy are:

$$\theta_S = -\frac{\lambda_1}{\sigma_1}\frac{H_x}{H_{xx}} \tag{5.10}$$

and

$$\theta_L = \frac{\sigma_r(\lambda_1\sigma_2 - \lambda_2\sigma_1)H_x + \sigma_1\sigma_r H_{rx}}{\sigma_1\sigma_L H_{xx}}. \tag{5.11}$$

If we put Eqs. (5.10) and (5.11) into Eq.(5.9), we obtain a PDE for the value function H:

$$
\begin{aligned}
H_t &+ a(b-r)H_r + \frac{\sigma_r^2}{2}H_{rr} + (xr+c)H_x \\
&- \frac{\lambda_1^2}{2}\frac{H_x^2}{H_{xx}} - \frac{(\lambda_2\sigma_r H_x - \sigma_r H_{rx})^2}{2H_{xx}} = 0.
\end{aligned}
\tag{5.12}
$$

We must now solve Eq.(5.12) for the value function H and replace it in Eq.(5.10) and Eq.(5.11). The non-linear second order PDE above is very difficult to solve.

At this point we shall specify a particular candidate for the function σ_L appearing in Eq.(5.5). We assume σ_L to take the form

$$\sigma_L(T - t, r(t)) = h(T - t)\sigma_r,$$

with

$$h(t) = \frac{2(e^{mt} - 1)}{m - (b - k_1\lambda_2) + e^{mt}(m + b - k_1\lambda_2)}$$

and

$$m = \sqrt{(b - k_1\lambda_2)^2 + 2k_1}.$$

We now transform the non-linear second order PDE into a linear PDE via the Legendre transform and dual theory. We try to find an explicit solution to the transformed PDE

54

under the logarithm utility function:

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function. For $z > 0$, we define the Legendre dual of the function $f(x)$ as

$$L(z) = \max_{x}\{f(x) - zx\}. \tag{5.13}$$

We assume that $f(x)$ is strictly convex. Then the maximum in the equation above is attained at a unique point, which we denote by x_0. In fact, the maximum is attained at the unique solution to the first-order condition

$$\frac{df(x)}{dx} - z = 0. \tag{5.14}$$

We may thus write

$$L(z) = f(x_0) - z(x_0). \tag{5.15}$$

According to Definition 4.1, we may take advantage of the assumed convexity of the value function $H(t, r, x)$ to define the Legendre transform:

$$\hat{H}(t, r, x) := \sup_{x>0}\{H(t, r, x) - zx \mid 0 < x < \infty\}, \quad 0 < t < T, \tag{5.16}$$

where $z > 0$ denotes the dual variable to x. The value of x where this optimum occurs is denoted by $g(t, r, z)$, so that

$$g(t, r, z) := \inf_{x>0}\{x \mid H(t, r, x) \geq zx + \hat{H}(t, r, z)\}, \quad 0 < t < T. \tag{5.17}$$

This leads to

$$\begin{aligned} \hat{H}(t, r, z) &= H(t, r, g) - zg, \\ g(t, r, z) &= x. \end{aligned} \tag{5.18}$$

According to Eqs. (5.15) and (5.16), we have

$$H_x = z. \tag{5.19}$$

Based on Eqs. (5.18) and (5.19), the function $\hat{H}$ related to g is given by

$$g = -\hat{H}_z. \tag{5.20}$$

We can therefore take either one of the two functions g and $\hat{H}$ as the dual of H. We choose to work with the function g, as it is easier to compute numerically and suffices for the purpose of computing the proxy.

If we differentiate Eqs. (5.18) and (5.19) with respect to t, r and z, the transformation rules for the derivatives of the value function H and the dual function $\hat{H}$ are according to Choulli and Hurd [20], Jonsson and Sircar [55] and Xiao et al. [95]:

$$
\begin{aligned}
H_x &= z, & H_t &= \hat{H}_t, & H_r &= \hat{H}_r, \\
H_{rr} &= \hat{H}_{rr} - \frac{\hat{H}_{rz}^2}{\hat{H}_{zz}}, & H_{xr} &= -\frac{\hat{H}_{rz}}{\hat{H}_{zz}}, & H_{xx} &= -\frac{1}{\hat{H}_{zz}}.
\end{aligned} \tag{5.21}
$$

At time T, we define

$$\hat{u}(z) := \sup_{x>0}\{u(x) - zx\},$$

and

$$G(z) := \inf_{x>0}\{x \mid u(x) \geq zx - \hat{u}(z)\}.$$

Kramkov and Schachermayer [62] and Cox and Huang [24] showed that the function $\hat{u}(z)$ and $u(x)$ can themselves be obtained from each other by using Legendre transforms:

$$
\begin{aligned}
\hat{u}(z) &= \sup_{x>0}\{u(x) - zx\}, \\
u(x) &= \inf_{z>0}\{x \mid \hat{u}(z) + zx\}.
\end{aligned} \tag{5.22}
$$

The primary problem can thus be turned into a dual problem. By substituting expression (5.21), we rewrite Eq.(5.12) as

$$\hat{H}_t + a(b-r)\hat{H}_r + \frac{\sigma_r^2}{2}\Big(\hat{H}_{rr} - \frac{\hat{H}_{rz}^2}{\hat{H}_{zz}}\Big) + (rx+c)z$$

56

$$+ \quad \frac{\lambda_1^2}{2} z^2 \hat{H}_{zz} - \frac{(\lambda_2 z \sigma_r \hat{H}_{zz} + \hat{H}_{rz})^2}{2\hat{H}_{zz}} = 0, \tag{5.23}$$

namely, as the following PDE:

$$
\begin{aligned}
\hat{H}_t &+ a(b-r)\hat{H}_r + \frac{\sigma_r^2}{2}\hat{H}_{rr} + (rx+c)z \\
&+ \frac{1}{2}(\lambda_1^2 - \lambda_2^2 \sigma_r^2)z^2 \hat{H}_{zz} - \lambda_2 \sigma_r^2 z \hat{H}_{rz} = 0.
\end{aligned}
\tag{5.24}
$$

If we combine the above equation with Eq.(5.20) and differentiate the result for $\hat{H}$ with respect to z, we get

$$
\begin{aligned}
g_t &+ a(b-r)g_r - rg - c - \lambda_2^2 \sigma_r^2 g_r - rzg_z + \frac{\sigma_r^2}{2} g_{rr} \\
&- (\lambda_2 \sigma_r^2 - \lambda_1^2)zg_z - \frac{1}{2}(\lambda_2^2 \sigma_r^2 - \lambda_1^2)z^2 g_{zz} - \lambda_2^2 \sigma_r^2 z g_{rz} = 0.
\end{aligned}
\tag{5.25}
$$

We notice that the non-linear second order PDE Eq.(5.12) has been transformed into a linear PDE Eq.(5.25) by using a Legendre transform and dual theory. For Eq.(5.25) a solution can easily be found under a given utility function via the classical variable decomposition approach.

From Eqs. (5.10), (5.11) and (5.18)-(5.21), the proxy is computed as the feedback formulas in terms of the derivatives of the value function. In terms of the dual function g, it is given by

$$
\begin{aligned}
\theta_L &= \frac{\sigma_r(\lambda_1 \sigma_2 - \lambda_2 \sigma_1)H_x + \sigma_1 \sigma_r H_{rx}}{\sigma_1 \sigma_L H_{xx}} \\
&= \frac{\sigma_r[(\lambda_1 \sigma_2 - \lambda_2 \sigma_1)z - \sigma_1 \frac{\hat{H}_{rz}}{\hat{H}_{zz}}]}{\sigma_1 \sigma_L(-\frac{1}{\hat{H}_{zz}})} \\
&= \frac{\sigma_r[(\lambda_1 \sigma_2 - \lambda_2 \sigma_1)zg_z - \sigma_1 g_r]}{\sigma_1 \sigma_L},
\end{aligned}
\tag{5.26}
$$

and

$$\theta_S = -\frac{\lambda_1}{\sigma_1}\frac{H_x}{H_{xx}} = \frac{\lambda_1}{\sigma_1}z\hat{H}_{zz} = -\frac{\lambda_1}{\sigma_1}zg_z. \tag{5.27}$$

We now solve the linear PDE Eq.(5.25) for g and replace these solutions in Eqs. (5.26) and (5.27). From Eq.(5.14), we derive the dual of the logarithm utility function : $G(z) = \frac{1}{z}$ and $\hat{H}(z) = -\ln z - 1$. We try to find a solution of Eq.(5.25) in the following way:

$$g(t, r, z) = \frac{1}{z}A(r_t) + B(t), \tag{5.28}$$

where the boundary conditions are given by $A(r_T) = 1$ and $B(T) = 0$. Substituting Eq.(5.28) into Eq.(5.25), we obtain:

$$B'(t) - rB(t) - c(t) + \frac{1}{z}[a(b-r)A'(r) - \lambda_2^2\sigma_r^2 A'(r) + \frac{\sigma_r^2}{2}A''(r)] = 0.$$

This equation can be decomposed into two conditions in order to eliminate the dependence in z:

$$
\begin{aligned}
B'(t) - rB(t) - c(t) &= 0, \\
B(T) &= 0,
\end{aligned}
\tag{5.29}
$$

$$
\begin{aligned}
a(b-r)A'(r) - \lambda_2^2\sigma_r^2 A'(r) + \frac{\sigma_r^2}{2}A''(r) &= 0, \\
A(r_T) &= 1.
\end{aligned}
\tag{5.30}
$$

The solutions to Eqs. (5.29) and (5.30) which take the boundary conditions into account are

$$A(r_T) = 1. \tag{5.31}$$

$$B(t) = -c(t)\left\{\frac{1 - e^{-r(T-t)}}{r}\right\}, \tag{5.32}$$

or $B(t) = -c(t)\bar{a}_{\overline{T-t}|}$, where $\bar{a}_{\overline{T-t}|}$ is an annuity of duration $T - t$. This leads to

$$g = \frac{1}{z} - c(t)\bar{a}_{\overline{T-t}|}. \tag{5.33}$$

Introducing Eq.(5.33) into Eqs. (5.26) and (5.27), we obtain the approximate optimal allocation strategy of Total capital in the loan under a logarithm utility as

$$\theta_L = \frac{\sigma_r[(\lambda_1\sigma_2 - \lambda_2\sigma_1)zg_z - \sigma_1 g_r]}{\sigma_1\sigma_L}$$

$$= \frac{\sigma_r[(\lambda_1\sigma_2 - \lambda_2\sigma_1)(-\frac{1}{z}) - \sigma_1\bar{a}_{\overline{T-t}|}\frac{c}{r} + \sigma_1 e^{-r(T-t)}\frac{(T-t)c}{r}]}{\sigma_1\sigma_L}$$

$$= \frac{\sigma_r[((\lambda_2\sigma_1 - \lambda_1\sigma_2)(c\bar{a}_{\overline{T-t}|} + x)) - \sigma_1\bar{a}_{\overline{T-t}|}\frac{c}{r} + \sigma_1 e^{-r(T-t)}\frac{(T-t)c}{r}]}{\sigma_1\sigma_L}$$

$$= \frac{\sigma_r(\lambda_2\sigma_1 - \lambda_1\sigma_2)x}{\sigma_1\sigma_L} - \sigma_r c \left[\frac{(\lambda_1\sigma_2 - \lambda_2\sigma_1)\bar{a}_{\overline{T-t}|r_t}}{\sigma_1\sigma_L} + \frac{\bar{a}_{\overline{T-t}|r_t} - (T-t)(1 - r\bar{a}_{\overline{T-t}|r_t})}{r\sigma_L}\right],$$

or if we denote the approximate optimal proportion of Total capital invested in the loan by η_L, then we can write

$$\eta_L = \frac{\sigma_r(\lambda_2\sigma_1 - \lambda_1\sigma_2)}{\sigma_1\sigma_L}$$
$$- \frac{\sigma_r c}{x}\left[\frac{(\lambda_1\sigma_2 - \lambda_2\sigma_1)\bar{a}_{\overline{T-t}|r_t}}{\sigma_1\sigma_L} + \frac{\bar{a}_{\overline{T-t}|r_t} - (T-t)(1 - r\bar{a}_{\overline{T-t}|r_t})}{r\sigma_L}\right]. \tag{5.34}$$

Furthermore, the approximate optimal amount of Total capital invested in the marketable security is given by

$$\theta_S = -\frac{\lambda_1}{\sigma_1}zg_z = \frac{\lambda_1}{\sigma_1}\frac{1}{z} = \frac{\lambda_1}{\sigma_1}(x + c\bar{a}_{\overline{T-t}|r_t}),$$

or

$$\eta_S = \frac{\lambda_1}{\sigma_1} + \frac{c\bar{a}_{\overline{T-t}|r_t}}{x}, \tag{5.35}$$

where η_S denotes the approximate optimal proportion of Total capital invested in the marketable security. According to the above models, we may write the approximate optimal amount of Total capital invested in the treasury as

$$\theta_r = x - \theta_L - \theta_S,$$

or

$$\eta_r = 1 - \eta_L - \eta_S, \tag{5.36}$$

where η_r is the approximate optimal proportion of Total capital invested in the treasury.

Below we present a simulation of the optimal solution to the control problems of [38, 78]. We assume that the interest rate follows the CIR [22] dynamics ($k_2 = 0$). We consider an investment horizon of $T = 10$ years and assume that Total capital is raised at the fixed rate of $c = 0.415$. The rest of the parameters of the simulation are

$$a = 0.0112,\ b = 0.0332,\ k_1 = 0.00112,\ \sigma_1 = 0.11,\ \lambda_1 = 0.05,\ \sigma_2 = 0.22\ \text{and}\ \lambda_2 = 0.1$$

with initial conditions

$$r(0) = 0.09\ \text{and}\ X(0) = 2.95.$$
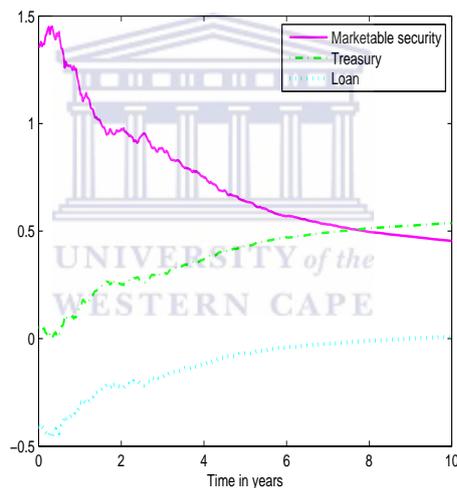
Figure 5.1: A simulation of the optimal proportions η_S, η_r and η_L of Total capital invested respectively in the marketable security, treasury and loan of the diffusion banking model.

For the simulation parameters considered, the optimal capital allocation strategy depicted in Figure 5.1 is to diversify the TNRWAs away from the risky assets (marketable security and loan) and towards the riskless treasury security.

60

Chapter 6

The capital adequacy ratios of the jump-diffusion banking model

In this chapter we derive the formulae of the capital adequacy ratios in terms of the proxy to the solution of Problem 5.1. We present a numerical example illustrating the performance of the ratios under the proxy. In the example we compare the levels of the capital adequacy ratios of our jump-diffusion banking model to that corresponding to the diffusion banking model of Muller and Witbooi [78].

In this chapter we will mainly reference Oksendal and Sulem [81], Mukuddem-Petersen and Petersen [84], Muller and Witbooi [78] and the Basel document [10].

6.1   Modelling the capital adequacy ratios

In Propositions 6.1-6.3 we derive the formulae of the bank's capital adequacy ratios in terms of the proxy. First we derive the dynamics of the TRWAs of the bank. We will also introduce the Tier 1 capital model which is needed to derive the Tier 1 and Leverage ratios. In the proofs of the propositions, we apply the general Itô formulae (with jumps), for which we refer to the book [81] of Oksendal and Sulem.

We assume that the TRWAs of the bank can at time t be described by the SDE

$$
\begin{aligned}
dY(t) &= 0 \times \theta_r(t)\frac{dS_0(t)}{S_0(t)} + 0.2 \times \theta_S(t)\frac{dS(t)}{S(t)} + 0.5 \times \theta_L(t)\frac{dL(t)}{L(t-)} + dC(t) \\
&= [0.2\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2) + 0.5\theta_L(t)(r(t) + \sigma_L(T - t, r(t))\lambda_2\sigma_r) \\
&\quad + c(t)]dt + 0.2\theta_S(t)\sigma_1 dW_S(t) \\
&\quad + (0.2\theta_S(t)\sigma_2\sigma_r + 0.5\theta_L(t)\sigma_L(T - t, r(t)))dW_r(t) + 0.5\theta_L(t)dK(t), \quad (6.1)
\end{aligned}
$$

where 0, 0.2 and 0.5 are the risk-weights associated with respectively the treasury, marketable security and loan under the Basel III dispensation (see [10, 84, 78]).

Proposition 6.1. *With the dynamics of the Total capital, $C(t)$, given by the ODE in Eq.(5.1), and with the dynamics of the TRWAs, $Y(t)$, given by Eq.(6.1), we can write the dynamics of the CAR at time t as:*

$$
\begin{aligned}
d\Lambda(t) &= \frac{c(t)}{Y(t)}dt + C(t)\Bigg\{ \Bigg\{ -\frac{1}{Y^2(t)}\Big[0.2\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2) \\
&\quad + 0.5\theta_L(t)(r(t) + \sigma_L(T - t, r(t))\lambda_2\sigma_r) + c(t)\Big] \\
&\quad + \frac{1}{Y^3(t)}\Big[(0.2\theta_S(t)\sigma_1)^2 + (0.2\theta_S(t)\sigma_2\sigma_r + 0.5\theta_L(t)\sigma_L(T - t, r(t)))^2\Big]\Bigg\}dt \\
&\quad - \frac{1}{Y^2(t)}\Big[0.2\theta_S(t)\sigma_1 dW_S(t) + (0.2\theta_S(t)\sigma_2\sigma_r + 0.5\theta_L(t)\sigma_L(T - t, r(t)))dW_r(t)\Big] \\
&\quad + \int_{|z|<R}\left\{\frac{1}{Y(t-) + \gamma(t, z, \omega)} - \frac{1}{Y(t-)} + \frac{\gamma(t, z, \omega)}{Y^2(t-)}\right\}\nu(dz)dt \\
&\quad + \int_{\mathbb{R}}\left\{\frac{1}{Y(t-) + \gamma(t, z, \omega)} - \frac{1}{Y(t-)}\right\}\bar{N}(dt, dz)\Bigg\}. \quad (6.2)
\end{aligned}
$$

Proof of Proposition 6.1: Let $dY^c(t)$ denote the continuous part of $dY(t)$. Then

$$
\begin{aligned}
dY^c(t) &= [0.2\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2) + 0.5\theta_L(t)(r(t) + \sigma_L(T - t, r(t))\lambda_2\sigma_r) \\
&\quad + c(t)]dt + 0.2\theta_S(t)\sigma_1 dW_S(t) + (0.2\theta_S(t)\sigma_2\sigma_r + 0.5\theta_L(t)\sigma_L(T - t, r(t)))dW_r(t).
\end{aligned}
$$

By applying Itô's Formula to $\Phi(Y(t)) = \bar{g}(t, Y(t)) = \frac{1}{Y(t)}$, we get

$$
d\Phi(Y(t)) = \frac{\partial \bar{g}}{\partial t}(t, Y(t))dt + \frac{\partial \bar{g}}{\partial y}(t, Y(t))dY^c(t) + \frac{1}{2}\frac{\partial^2 \bar{g}}{\partial y^2}(t, Y(t))\Big[(0.2\theta_S(t)\sigma_1)^2 dt
$$

$$+ \quad (0.2\theta_S(t)\sigma_2\sigma_r + 0.5\theta_L(t)\sigma_L(T-t,r(t)))^2 dt\Big]$$

$$+ \quad \int_{|z|<R}\left\{\bar{g}(t, Y(t-)+\gamma(t,z,\omega)) - \bar{g}(t,Y(t-)) - \frac{\partial\bar{g}}{\partial y}(t,Y(t-))\gamma(t,z,\omega)\right\}\nu(dz)dt$$

$$+ \quad \int_{\mathbb{R}}\left\{\bar{g}(t, Y(t-)+\gamma(t,z,\omega)) - \bar{g}(t,Y(t-))\right\}\bar{N}(dt,dz)$$

$$= \quad \left\{-\frac{1}{Y^2(t)}\Big[0.2\theta_S(t)(r(t)+\sigma_1\lambda_1+\sigma_2\sigma_r^2\lambda_2) + 0.5\theta_L(t)(r(t)+\sigma_L(T-t,r(t))\lambda_2\sigma_r)\right.$$

$$+ \quad c(t)\Big] + \frac{1}{Y^3(t)}\Big[(0.2\theta_S(t)\sigma_1)^2 + (0.2\theta_S(t)\sigma_2\sigma_r + 0.5\theta_L(t)\sigma_L(T-t,r(t)))^2\Big]\bigg\}dt$$

$$- \quad \frac{1}{Y^2(t)}\Big[0.2\theta_S(t)\sigma_1 dW_S(t) + (0.2\theta_S(t)\sigma_2\sigma_r + 0.5\theta_L(t)\sigma_L(T-t,r(t)))dW_r(t)\Big]$$

$$+ \quad \int_{|z|<R}\left\{\frac{1}{Y(t-)+\gamma(t,z,\omega)} - \frac{1}{Y(t-)} + \frac{\gamma(t,z,\omega)}{Y^2(t-)}\right\}\nu(dz)dt$$

$$+ \quad \int_{\mathbb{R}}\left\{\frac{1}{Y(t-)+\gamma(t,z,\omega)} - \frac{1}{Y(t-)}\right\}\bar{N}(dt,dz).$$

Let $\Lambda(t)$ denote the CAR at time t for $t \in [0, T]$. Then by definition, we can write $\Lambda(t)$ as

$$\Lambda(t) = \frac{C(t)}{Y(t)} = C(t)\Phi(Y(t)).$$

We apply Itô's Product Rule to $\Lambda(t) = C(t)\Phi(Y(t))$ to find an expression for $d\Lambda(t)$:

$$d\Lambda(t) \quad = \quad dC(t)\Phi(Y(t)) + C(t)d\Phi(Y(t)) + dC(t)d\Phi(Y(t))$$

$$= \quad \frac{c(t)}{Y(t)}dt + C(t)\Bigg\{\left\{-\frac{1}{Y^2(t)}\Big[0.2\theta_S(t)(r(t)+\sigma_1\lambda_1+\sigma_2\sigma_r^2\lambda_2)\right.$$

$$+ \quad 0.5\theta_L(t)(r(t)+\sigma_L(T-t,r(t))\lambda_2\sigma_r) + c(t)\Big]$$

$$+ \quad \frac{1}{Y^3(t)}\Big[(0.2\theta_S(t)\sigma_1)^2 + (0.2\theta_S(t)\sigma_2\sigma_r + 0.5\theta_L(t)\sigma_L(T-t,r(t)))^2\Big]\bigg\}dt$$

$$- \quad \frac{1}{Y^2(t)}\Big[0.2\theta_S(t)\sigma_1 dW_S(t) + (0.2\theta_S(t)\sigma_2\sigma_r + 0.5\theta_L(t)\sigma_L(T-t,r(t)))dW_r(t)\Big]$$

$$+ \quad \int_{|z|<R}\left\{\frac{1}{Y(t-)+\gamma(t,z,\omega)} - \frac{1}{Y(t-)} + \frac{\gamma(t,z,\omega)}{Y^2(t-)}\right\}\nu(dz)dt$$

$$+ \quad \int_{\mathbb{R}}\left\{\frac{1}{Y(t-)+\gamma(t,z,\omega)} - \frac{1}{Y(t-)}\right\}\bar{N}(dt,dz)\Bigg\}.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

The next step is to derive the dynamics of the Leverage and Tier 1 ratios based on the proxy. At this point, we introduce a model for the Tier 1 capital of the bank. Suppose that the Tier 2 capital of the bank is described by the ODE $dC_{T2}(t) = c_2(t)dt$ with $C_{T2}(0) > 0$ and $c_2(t) < c(t)$ for $t \in [0, T]$. Then the dynamics of the Tier 1 capital can be written as

$$
\begin{aligned}
dC_{T1}(t) &= (c(t) - c_2(t))dt, \\
C_{T1}(0) &> 0.
\end{aligned}
\tag{6.3}
$$

Proposition 6.2. *With the dynamics of the Tier 1 capital, $C_{T1}(t)$, given by the ODE in Eq.(6.3) and with the TRWAS, $Y(t)$, given by Eq.(6.1), the dynamics of the Tier 1 Ratio at time t can be expressed in the following way:*

$$
\begin{aligned}
d\Lambda_{T1}(t) &= \frac{c(t) - c_2(t)}{Y(t)}dt + C_{T1}(t)\Bigg\{\Bigg\{ -\frac{1}{Y^2(t)}\Big[0.2\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2) \\
&\quad + 0.5\theta_L(t)(r(t) + \sigma_L(T - t, r(t))\lambda_2\sigma_r) + c(t)\Big] \\
&\quad + \frac{1}{Y^3(t)}\Big[(0.2\theta_S(t)\sigma_1)^2 + (0.2\theta_S(t)\sigma_2\sigma_r + 0.5\theta_L(t)\sigma_L(T - t, r(t)))^2\Big]\Bigg\}dt \\
&\quad - \frac{1}{Y^2(t)}\Big[0.2\theta_S(t)\sigma_1 dW_S(t) + (0.2\theta_S(t)\sigma_2\sigma_r + 0.5\theta_L(t)\sigma_L(T - t, r(t)))dW_r(t)\Big] \\
&\quad + \int_{|z|<R}\Bigg\{\frac{1}{Y(t-) + \gamma(t, z, \omega)} - \frac{1}{Y(t-)} + \frac{\gamma(t, z, \omega)}{Y^2(t-)}\Bigg\}\nu(dz)dt \\
&\quad + \int_{\mathbb{R}}\Bigg\{\frac{1}{Y(t-) + \gamma(t, z, \omega)} - \frac{1}{Y(t-)}\Bigg\}\bar{N}(dt, dz)\Bigg\}.
\end{aligned}
\tag{6.4}
$$

Proof of Proposition 6.2: If we let $\Lambda_{T1}(t)$ denote the Tier 1 Ratio at time t for $t \in [0, T]$, then by definition we can write $\Lambda_{T1}(t)$ as

$$
\Lambda_{T1}(t) = \frac{C_{T1}(t)}{Y(t)} = C_{T1}(t)\Phi(Y(t)).
$$

By applying Itô's Product Rule to $\Lambda_{T1}(t) = C_{T1}(t)\Phi(Y(t))$, we calculate $d\Lambda_{T1}(t)$ as follows:

$$
\begin{aligned}
d\Lambda_{T1}(t) &= dC_{T1}(t)\Phi(Y(t)) + C_{T1}(t)d\Phi(Y(t)) + dC_{T1}(t)d\Phi(Y(t)) \\
&= \frac{c(t) - c_2(t)}{Y(t)}dt + C_{T1}(t)\Bigg\{\Bigg\{ -\frac{1}{Y^2(t)}\Big[0.2\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\quad 0.5\theta_L(t)(r(t)+\sigma_L(T-t,r(t))\lambda_2\sigma_r)+c(t)\Bigg] \\
&+\quad \frac{1}{Y^3(t)}\Big[(0.2\theta_S(t)\sigma_1)^2+(0.2\theta_S(t)\sigma_2\sigma_r+0.5\theta_L(t)\sigma_L(T-t,r(t)))^2\Big]\Bigg\}dt \\
&-\quad \frac{1}{Y^2(t)}\Big[0.2\theta_S(t)\sigma_1dW_S(t)+(0.2\theta_S(t)\sigma_2\sigma_r+0.5\theta_L(t)\sigma_L(T-t,r(t)))dW_r(t)\Big] \\
&+\quad \int_{|z|<R}\left\{\frac{1}{Y(t-)+\gamma(t,z,\omega)}-\frac{1}{Y(t-)}+\frac{\gamma(t,z,\omega)}{Y^2(t-)}\right\}\nu(dz)dt \\
&+\quad \int_{\mathbb{R}}\left\{\frac{1}{Y(t-)+\gamma(t,z,\omega)}-\frac{1}{Y(t-)}\right\}\bar{N}(dt,dz)\Bigg\}.
\end{aligned}
$$

This concludes the proof. $\qquad\square$

Proposition 6.3. *For the simplified version of the TNRWAs appearing in Problem 5.1 and with the dynamics of the Tier 1 capital, $C_{\mathrm{T1}}(t)$, given by Eq.(6.3), the dynamics of the bank's Leverage Ratio follows the SDE*

$$
\begin{aligned}
d\Lambda_l(t) \;=\;& \frac{c(t)-c_2(t)}{X(t)}dt+C_{\mathrm{T1}}(t)\Bigg\{\left\{-\frac{1}{X^2(t)}\Big[X(t)r(t)+\theta_S(t)(\lambda_1\sigma_1+\lambda_2\sigma_2\sigma_r^2)\right. \\
&+\;\theta_L(t)\lambda_2\sigma_L(T-t,r(t))\sigma_r+c(t)\Big]+\frac{1}{X^3(t)}\Big[(\theta_S(t)\sigma_1)^2+(\theta_L(t)\sigma_L(T-t,r(t)) \\
&+\;\theta_S(t)\sigma_2\sigma_r)^2\Big]\Bigg\}dt-\frac{1}{X^2(t)}\Big[\theta_S(t)\sigma_1dW_S(t)+(\theta_L(t)\sigma_L(T-t,r(t))+\theta_S(t)\sigma_2\sigma_r)dW_r(t)\Big] \\
&+\;\int_{|z|<R}\left\{\frac{1}{X(t-)+\gamma(t,z,\omega)}-\frac{1}{X(t-)}+\frac{\gamma(t,z,\omega)}{X^2(t-)}\right\}\nu(dz)dt \\
&+\;\int_{\mathbb{R}}\left\{\frac{1}{X(t-)+\gamma(t,z,\omega)}-\frac{1}{X(t-)}\right\}\bar{N}(dt,dz)\Bigg\}.
\end{aligned}
\tag{6.5}
$$

Proof of Proposition 6.3: Let $dX^c(t)$ denote the continuous part of $dX(t)$, i.e.,

$$
\begin{aligned}
dX^c(t) \;=\;& \{X(t)r(t)+\theta_S(t)[\lambda_1\sigma_1+\lambda_2\sigma_2\sigma_r^2]+\theta_L(t)\lambda_2\sigma_L(T-t,r(t))\sigma_r+c\}dt \\
&+\;\theta_S(t)\sigma_1dW_S(t)+(\theta_L(t)\sigma_L(T-t,r(t))+\theta_S(t)\sigma_2\sigma_r)dW_r(t).
\end{aligned}
$$

Itô's Lemma applied to $\Phi(X(t))=\bar{g}(t,X(t))=\frac{1}{X(t)}$ yields

$$
d\Phi(X(t)) \;=\; \frac{\partial\bar{g}}{\partial t}(t,X(t))dt+\frac{\partial\bar{g}}{\partial x}(t,X(t))dX^c(t)+\frac{1}{2}\frac{\partial^2\bar{g}}{\partial x^2}(t,X(t))\Big[(\theta_S(t)\sigma_1)^2dt
$$

65

$$+ \quad (\theta_S(t)\sigma_2\sigma_r + \theta_L(t)\sigma_L(T-t,r(t)))^2 dt \Big]$$

$$+ \quad \int_{|z|<R} \left\{ \bar{g}(t, X(t-) + \gamma(t,z,\omega)) - \bar{g}(t, X(t-)) - \frac{\partial \bar{g}}{\partial x}(t, X(t-))\gamma(t,z,\omega) \right\} \nu(dz)dt$$

$$+ \quad \int_{\mathbb{R}} \left\{ \bar{g}(t, X(t-) + \gamma(t,z,\omega)) - \bar{g}(t, X(t-)) \right\} \bar{N}(dt,dz)$$

$$= \quad \left\{ -\frac{1}{X^2(t)} \left[X(t)r(t) + \theta_S(t)(\lambda_1\sigma_1 + \lambda_2\sigma_2\sigma_r^2) + \theta_L(t)\lambda_2\sigma_L(T-t,r(t))\sigma_r + c(t) \right] \right.$$

$$+ \quad \left. \frac{1}{X^3(t)} \left[(\theta_S(t)\sigma_1)^2 + (\theta_L(t)\sigma_L(T-t,r(t)) + \theta_S(t)\sigma_2\sigma_r)^2 \right] \right\} dt$$

$$- \quad \frac{1}{X^2(t)} \left[\theta_S(t)\sigma_1 dW_S(t) + (\theta_L(t)\sigma_L(T-t,r(t)) + \theta_S(t)\sigma_2\sigma_r)dW_r(t) \right]$$

$$+ \quad \int_{|z|<R} \left\{ \frac{1}{X(t-) + \gamma(t,z,\omega)} - \frac{1}{X(t-)} + \frac{\gamma(t,z,\omega)}{X^2(t-)} \right\} \nu(dz)dt$$

$$+ \quad \int_{\mathbb{R}} \left\{ \frac{1}{X(t-) + \gamma(t,z,\omega)} - \frac{1}{X(t-)} \right\} \bar{N}(dt,dz).$$

Let $\Lambda_l(t)$ denote the Leverage Ratio at time t for $t \in [0, T]$. By definition,

$$\Lambda_l(t) := \frac{C_{\mathrm{T1}}(t)}{X(t)} = C_{\mathrm{T1}}(t)\Phi(X(t)).$$

We apply Itô's Product Rule to $\Lambda_l(t) = C_{\mathrm{T1}}(t)\Phi(X(t))$ to find an expression for $d\Lambda_l(t)$ as follows:

$$d\Lambda_l(t) \quad = \quad dC_{\mathrm{T1}}(t)\Phi(X(t)) + C_{\mathrm{T1}}(t)d\Phi(X(t)) + dC_{\mathrm{T1}}(t)d\Phi(X(t))$$

$$= \quad \frac{c(t) - c_2(t)}{X(t)} dt + C_{\mathrm{T1}}(t)\left\{ \left\{ -\frac{1}{X^2(t)} \left[X(t)r(t) + \theta_S(t)(\lambda_1\sigma_1 + \lambda_2\sigma_2\sigma_r^2) \right. \right. \right.$$

$$+ \quad \theta_L(t)\lambda_2\sigma_L(T-t,r(t))\sigma_r + c(t) \Big] + \frac{1}{X^3(t)} \left[(\theta_S(t)\sigma_1)^2 + (\theta_L(t)\sigma_L(T-t,r(t)) \right.$$

$$+ \quad \theta_S(t)\sigma_2\sigma_r)^2 \Big] \bigg\} dt - \frac{1}{X^2(t)} \left[\theta_S(t)\sigma_1 dW_S(t) + (\theta_L(t)\sigma_L(T-t,r(t)) + \theta_S(t)\sigma_2\sigma_r)dW_r(t) \right]$$

$$+ \quad \int_{|z|<R} \left\{ \frac{1}{X(t-) + \gamma(t,z,\omega)} - \frac{1}{X(t-)} + \frac{\gamma(t,z,\omega)}{X^2(t-)} \right\} \nu(dz)dt$$

$$+ \quad \int_{\mathbb{R}} \left\{ \frac{1}{X(t-) + \gamma(t,z,\omega)} - \frac{1}{X(t-)} \right\} \bar{N}(dt,dz) \bigg\}.$$

This concludes the proof. $\qquad\square$

6.2 Simulating the capital adequacy ratios numerically

We now present a numerical simulation in order to characterize the behaviour of the capital adequacy ratios. The simulation is based on the assumption that the interest rate follows the CIR [22] dynamics ($k_2 = 0$) and that the financial market consists of a treasury, a marketable security and a loan (with a jump). Furthermore, we consider an investment horizon of $T = 10$ years and assume that Total capital is raised at the fixed rate of $c = 0.415$. We assume that the intensity of the Poisson process, which counts the number of jumps of size ± 0.05, is $\nu = \lambda_k = 0.4$. The rest of the parameters of the simulation are

$$c_2 = 0.25,\ a = 0.0112,\ b = 0.0332,\ k_1 = 0.00112,$$

$$\sigma_1 = 0.11,\ \lambda_1 = 0.05,\ \sigma_2 = 0.22 \text{ and } \lambda_2 = 0.1$$

with initial conditions

$$C(0) = 1,\ C_{\text{T2}}(0) = 0.45,\ r(0) = 0.09,\ X(0) = 2.95,\ Y(0) = 2.8,$$

$$\Lambda(0) = 0.08,\ \Lambda_{\text{T1}}(0) = 0.06 \text{ and } \Lambda_l(0) = 0.03.$$

In Figure 6.1 we present an approximate solution of Problem 5.1 by simulating the approximate optimal proportions of capital to invest in the bank's assets. The approximate optimal capital allocation strategy depicted in Figure 6.1 leads to the capital adequacy ratios (represented by the solid curves) in Figures 6.3-6.5. We note that by diversifying its TNRWAs according to the approximate optimal capital allocation strategy illustrated by Figure 6.1, the bank maintains its CAR and Tier 1 Ratio in such a manner that they are above their minimum Basel III-prescribed levels. By Basel III standards, the bank is strongly capitalized, and guaranteed the ability to sustain unexpected losses. However, the value of the Leverage Ratio falls below its minimum predescribed level. This high leverage or low level

of the Leverage Ratio can be remedied by increasing the rate at which Tier 1 capital is raised.

We note that compared to the diffusion model of Muller and Witbooi [78], the levels of our jump model's CAR and Tier 1 Ratio are improved by the jump. This is also the case for the Leverage Ratio of the jump model.

In the next section we will derive a formula for the TNRWAs at constant (minimum) Leverage Ratio value. This formula ensures that the bank's Leverage Ratio satisfies the Basel III minimum requirement on the entire investment period $[0, T]$. A similar approach was followed by Muller and Witbooi [78]. In [78] the authors derive a TNRWAs formula at constant (minimum) CAR value. The TNRWAs formula at constant (minimum) CAR value of [78] ensures that the bank is adequately capitilized to absorb unexpected losses at all times.
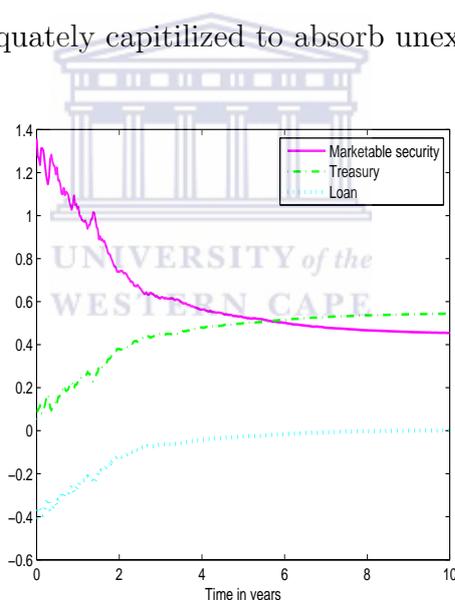
Figure 6.1: A simulation of the approximate optimal proportions η_S, η_r and η_L of Total capital invested respectively in the marketable security, treasury and loan.
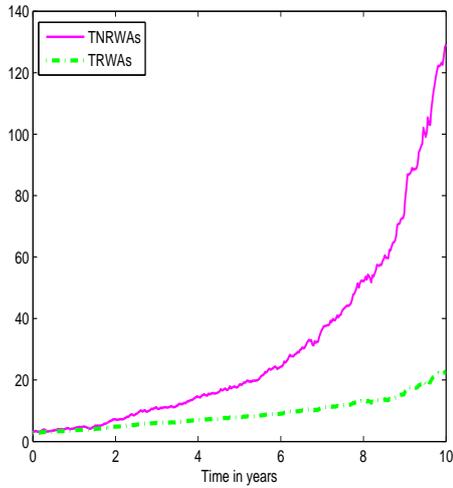
Figure 6.2: A simulation of the total non-risk-weighted and risk-weighted assets X and Y, given a constant stream of capital inflow.
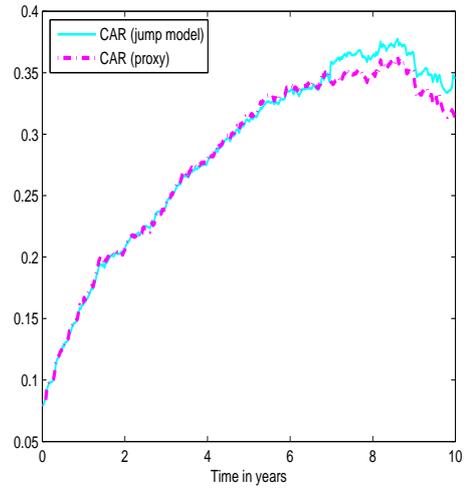
Figure 6.3: A simulation of the Total Capital Ratio Λ, given a constant stream of capital inflow.
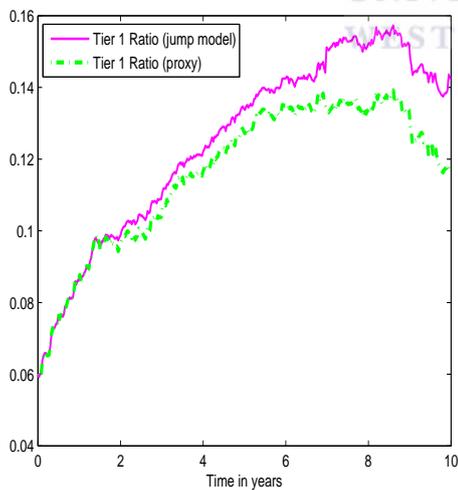
Figure 6.4: A simulation of the Tier 1 Ratio Λ_{T1}, given a constant stream of capital inflow.
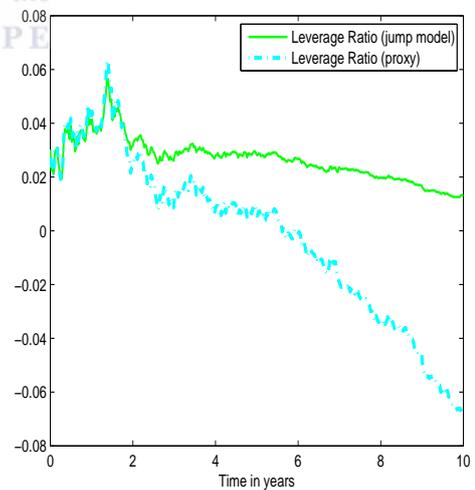
Figure 6.5: A simulation of the Leverage Ratio Λ_l, given a constant stream of capital inflow.

69
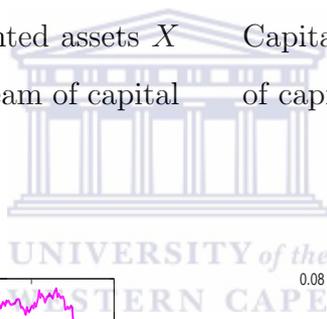
6.3 Deriving and simulating the asset portfolio at constant (minimum) Leverage Ratio value

We now set out to modify the TNRWAs formula of Problem 5.1 in such a way as to maintain the Leverage Ratio at a constant rate of 3%. To this end we need to have the Tier 1 capital model $C_{T1}(t)$ to be stochastic, and in fact include a jump. We assume that both the stochastic term and jump are sufficiently small in order to use the solution of Problem 5.1 as a reasonable approximation. The actual form of $C_{T1}(t)$ is deduced from the identity $C_{T1}(t) = 0.03X(t)$. The formula for the TNRWAs is presented in the remark below.

Remark 6.1. *At time t the dynamics of the TNRWAs, $\hat{X}(t)$, of the bank investing its capital according to the optimal investment strategy from Problem 5.1 and, in addition, maintains its Leverage Ratio at 3%, can be written as*

$$
\begin{aligned}
d\hat{X}(t) = {} & \{1.03[X(t)r(t) + \theta_S(t)(\sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2) + \theta_L(t)\sigma_L(T-t, r(t))\lambda_2\sigma_r] \\
& + 0.03c(t) + c_2(t)\}dt + 1.03\theta_S(t)\sigma_1 dW_S(t) + 1.03(\theta_S(t)\sigma_2\sigma_r \\
& + \theta_L(t)\sigma_L(T-t, r(t)))dW_r(t) + 1.03\theta_L(t)dK(t). \tag{6.6}
\end{aligned}
$$

To obtain the modified TNRWAs formula in Eq.(6.6), we differentiate both sides of the identity $C_{T1}(t) = 0.03X(t)$ and get $dC_{T1}(t) = 0.03dX(t)$, which is equivalent to

$$
dC_{T1}(t) = 0.03\theta_r(t)\frac{dS_0(t)}{S_0(t)} + 0.03\theta_S(t)\frac{dS(t)}{S(t)} + 0.03\theta_L(t)\frac{dL(t)}{L(t-)} + 0.03dC(t). \tag{6.7}
$$

Replacing the left hand side of Eq.(6.7) by the right hand side of Eq.(6.3), i.e., by $(c(t) - c_2(t))dt$, we can write

$$
(c(t) - c_2(t))dt = 0.03\theta_r(t)\frac{dS_0(t)}{S_0(t)} + 0.03\theta_S(t)\frac{dS(t)}{S(t)} + 0.03\theta_L(t)\frac{dL(t)}{L(t-)} + 0.03dC(t)
$$

and obtain the following form for $c(t)dt$:

$$
c(t)dt = 0.03\theta_r(t)\frac{dS_0(t)}{S_0(t)} + 0.03\theta_S(t)\frac{dS(t)}{S(t)} + 0.03\theta_L(t)\frac{dL(t)}{L(t-)} + 0.03dC(t) + c_2(t)dt.
$$

Substituting this expression as the $dC(t)$ term in Eq.(5.7), we obtain

$$
\begin{aligned}
d\hat{X}(t) &= \theta_r(t)\frac{dS_0(t)}{S_0(t)} + \theta_S(t)\frac{dS(t)}{S(t)} + \theta_L(t)\frac{dL(t)}{L(t-)} \\
&+ 0.03\theta_r(t)\frac{dS_0(t)}{S_0(t)} + 0.03\theta_S(t)\frac{dS(t)}{S(t)} + 0.03\theta_L(t)\frac{dL(t)}{L(t-)} + 0.03dC(t) + c_2(t)dt \\
&= 1.03\theta_r(t)\frac{dS_0(t)}{S_0(t)} + 1.03\theta_S(t)\frac{dS(t)}{S(t)} + 1.03\theta_L(t)\frac{dL(t)}{L(t-)} + 0.03dC(t) + c_2(t)dt.
\end{aligned}
$$

This expression can be simplified to take the form

$$
\begin{aligned}
d\hat{X}(t) &= \{1.03[\theta_r(t)r(t) + \theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2) + \theta_L(t)r(t) + \sigma_L(T-t, r(t))\lambda_2\sigma_r] \\
&+ 0.03c(t) + c_2(t)\}dt + 1.03\theta_S(t)\sigma_1 dW_S(t) + 1.03(\theta_S(t)\sigma_2\sigma_r \\
&+ \theta_L(t)\sigma_L(T-t, r(t)))dW_r(t) + 1.03\theta_L(t)dK(t).
\end{aligned}
$$

Since $\theta_r(t) + \theta_S(t) + \theta_L(t) = X(t)$, the above expression can be rewritten to take the form of the asserted expression.

We now characterize the behaviours of the modified TNRWAs formula and the controlled version of the Tier 1 capital needed to maintain the Leverage Ratio at 3%. The simulation is still based on the parameters and initial of the simulation study of the previous section. We consider the additional initial condition $\hat{X}(0) = 2.95$ for the modified TNRWAs.

We note that in order to maintain the Leverage Ratio at 3%, the value of the modified TNRWAs must be slightly lower than when the Leverage Ratio is not maintained at Basel III's minimum prescribed level. The amounts of Tier 1 capital needed to maintain the Leverage Ratio at 3% is considerably higher than for the Leverage Ratio corresponding to the original deterministic Tier 1 capital model.
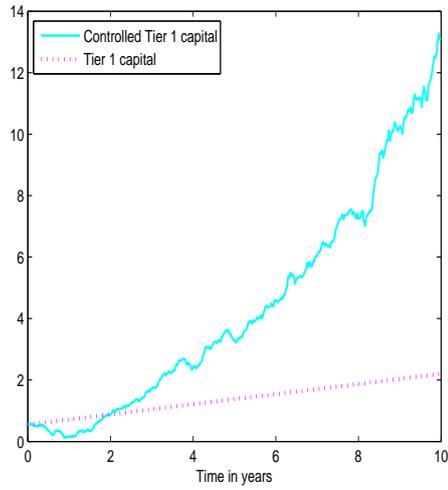
Figure 6.6: A simulation of the Tier 1 capital C_{T1}, required to maintain the Leverage Ratio at 3%.
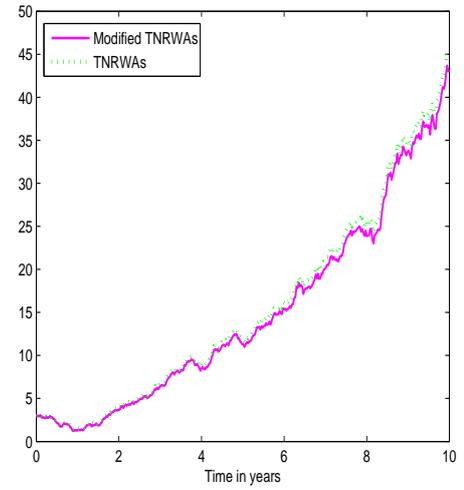
Figure 6.7: A simulation of the modified total non-risk-weighted assets $\hat{X}$, required to maintain the Leverage Ratio at 3%.

Chapter 7

The liquidity ratios of the jump-diffusion banking model

We now derive the formulae of the liquidity ratios in terms of the proxy to the solution of Problem 5.1. We present a numerical example of the behaviour of the ratios under the proxy. As in the previous chapter, we compare numerically the levels of our jump-diffusion banking model with the Brownian motions banking model of Muller and Witbooi [78] in terms of liquidity ratio performance where applicable. We will refer to the Basel documents [11, 12] in this chapter.

7.1   Deriving the liquidity ratios

In this section we derive the formulae for the bank's liquidity ratios which incorporate the proxy to the solution of Problem 5.1. In order to derive the model of the LCR, we require formulae for the SHQLAs and TNCOs, while we need formulae for the AASF and RASF when deriving the NSFR. We will derive expressions for these quantities here. We give detailed proofs of Propositions 7.1-7.2, for which we refer to the books Oksendal [80] and Oksendal and Sulem [81].

We assume that the dynamics of the SHQLAs can at time t be described by the SDE

$$
\begin{aligned}
dH_Q(t) &= 1.0 \times \theta_r(t)\frac{dS_0(t)}{S_0(t)} + 0.85 \times \theta_S(t)\frac{dS(t)}{S(t)} \\
&= [\theta_r(t)r(t) + 0.85\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2)]dt + 0.85\theta_S(t)\sigma_1 dW_S(t) \\
&\quad + 0.85\theta_S(t)\sigma_2\sigma_r dW_r(t). \tag{7.1}
\end{aligned}
$$

Above, 1.0 and 0.85 are the risk factors associated with Level 1 (the treasury) and Level 2 (the marketable security) assets respectively under Basel III (see [11]).

If we assume that the total expected cash outflows of the bank are comprised of secured funding backed by Level 1 and Level 2 assets, with the dynamics of the secured funding backed by Level 1 and Level 2 assets respectively given by the equations

$$
\frac{df_{S_1}(t)}{f_{S_1}(t)} = \mu_{S_1}dt + \sigma_{S_1}dW_{S_1}(t) \tag{7.2}
$$

and

$$
\frac{df_{S_2}(t)}{f_{S_2}(t)} = \mu_{S_2}dt + \sigma_{S_2}dW_{S_2}(t), \tag{7.3}
$$

then the total expected cash outflows can be modelled by the equation

$$
\begin{aligned}
dO_C(t) &= 0 \times \frac{df_{S_1}(t)}{f_{S_1}(t)} + 0.15 \times \frac{df_{S_2}(t)}{f_{S_2}(t)} \\
&= 0.15\mu_{S_2}dt + 0.15\sigma_{S_2}dW_{S_2}(t). \tag{7.4}
\end{aligned}
$$

In the above dynamics μ_{S_1}, σ_{S_1}, μ_{S_2} and σ_{S_2} are positive constants while W_{S_1} and W_{S_2} denote two independent one-dimensional standard Brownian motions. The weights 0 and 0.15 represent run-off factors associated with secured funding backed by respectively Level 1 and 2 assets under Basel III (see [11]).

If we now assume that the bank's total expected cash inflows are comprised of maturing secured lending backed by Level 1 and Level 2 assets as collateral, with the dynamics of the

maturing secured lending backed Level 1 and Level 2 assets being described respectively by the equations

$$\frac{dl_{M_1}(t)}{l_{M_1}(t)} = \mu_{M_1} dt + \sigma_{M_1} dW_{M_1}(t) \tag{7.5}$$

and

$$\frac{dl_{M_2}(t)}{l_{M_2}(t)} = \mu_{M_2} dt + \sigma_{M_2} dW_{M_2}(t), \tag{7.6}$$

then the total expected cash inflows can be modelled by the equation

$$
\begin{aligned}
dI_C(t) &= 0 \times \frac{dl_{M_1}(t)}{l_{M_1}(t)} + 0.15 \times \frac{dl_{M_2}(t)}{l_{M_2}(t)} \\
&= 0.15\mu_{M_2} dt + 0.15\sigma_{M_2} dW_{M_2}(t).
\end{aligned}
\tag{7.7}
$$

The coefficients μ_{M_1}, σ_{M_1}, μ_{M_2} and σ_{M_2} are positive constants while W_{M_1} and W_{M_2} denote two independent one-dimensional standard Brownian motions. The weights 0 and 0.15 represent inflow rates associated with maturing secured lending backed by Level 1 and Level 2 assets under Basel III (see [11]).

According to the formula for the denominator of the LCR, i.e., the TNCOs, we must consider two cases when deriving the model of TNCOs. First, when $I_C(t) < 0.75O_C(t)$, we have $O_N(t) = O_C(t) - I_C(t)$. Then the equation

$$
\begin{aligned}
dO_N(t) &= dO_C(t) - dI_C(t) \\
&= 0.15(\mu_{S_2} - \mu_{M_2})dt + 0.15\sigma_{S_2} dW_{S_2} - 0.15\sigma_{M_2} dW_{M_2}
\end{aligned}
\tag{7.8}
$$

describes the dynamics of the TNCOs of the bank. Alternatively, when $I_C(t) \geq 0.75O_C(t)$, we have $O_N(t) = O_C(t) - 0.75O_C(t) = 0.25O_C(t)$ and the TNCOs dynamics follow the SDE

$$
\begin{aligned}
dO_N(t) &= 0.25dO_C(t) \\
&= 0.0375\mu_{S_2} dt + 0.0375\sigma_{S_2} dW_{S_2}.
\end{aligned}
\tag{7.9}
$$

75

Proposition 7.1. *With the dynamics of the SHQLAs, $H_Q(t)$, given by the SDE in Eq.(7.1), and with the dynamics of the TNCOs, $O_N(t)$, given by either Eq.(7.8) or Eq.(7.9), we can write the dynamics of the LCR, $\Lambda_L(t)$, at time t as*

$$
\begin{aligned}
d\Lambda_L(t) \;=\; & \frac{1}{O_N(t)}\bigg\{ [\theta_r(t)r(t) + 0.85\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2)] + 0.85\theta_S(t)\sigma_1 dW_S(t) \\
& + \; 0.85\theta_S(t)\sigma_2\sigma_r dW_r(t)\bigg\} + H_Q(t)\bigg\{\bigg\{ -\frac{1}{O_N^2(t)}0.15(\mu_{S_2} - \mu_{M_2}) \\
& + \; \frac{1}{O_N^3(t)}\Big[(0.15\sigma_{S_2})^2 + (0.15\sigma_{M_2})^2 \Big]\bigg\}dt \\
& - \; \frac{1}{O_N^2(t)}\Big(0.15\sigma_{S_2}dW_{S_2} - 0.15\sigma_{M_2}dW_{M_2} \Big)\bigg\}
\end{aligned}
\tag{7.10}
$$

when $I_C(t) < 0.75O_C(t)$, or

$$
\begin{aligned}
d\Lambda_L(t) \;=\; & \frac{1}{O_N(t)}\bigg\{ [\theta_r(t)r(t) + 0.85\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2)] + 0.85\theta_S(t)\sigma_1 dW_S(t) \\
& + \; 0.85\theta_S(t)\sigma_2\sigma_r dW_r(t)\bigg\} \\
& + \; H_Q(t)\bigg\{\bigg[-\frac{1}{O_N^2(t)}0.0375\mu_{S_2} + \frac{1}{O_N^3(t)}\Big(0.0375\sigma_{S_2} \Big)^2 \bigg]dt \\
& - \; \frac{1}{O_N^2(t)}0.0375\sigma_{S_2}dW_{S_2}\bigg\}
\end{aligned}
\tag{7.11}
$$

otherwise.

Proof of Proposition 7.1: In the case $I_C(t) < 0.75O_C(t)$, Itô's Lemma applied to $\Phi(O_N(t)) = \bar{g}(t, O_N(t)) = \frac{1}{O_N(t)}$ gives

$$
\begin{aligned}
d\Phi(O_N(t)) \;=\; & \frac{\partial \bar{g}}{\partial t}(t, O_N(t))dt + \frac{\partial \bar{g}}{\partial o_n}(t, O_N(t))dO_N(t) + \frac{1}{2}\frac{\partial^2 \bar{g}}{\partial o_n^2}(t, O_N(t))(dO_N(t))^2 \\
\;=\; & \bigg\{ -\frac{1}{O_N^2(t)}0.15\Big(\mu_{S_2} - \mu_{M_2} \Big) + \frac{1}{O_N^3(t)}\Big[(0.15\sigma_{S_2})^2 + (0.15\sigma_{M_2})^2 \Big]\bigg\}dt \\
& - \; \frac{1}{O_N^2(t)}\Big(0.15\sigma_{S_2}dW_{S_2} - 0.15\sigma_{M_2}dW_{M_2} \Big).
\end{aligned}
$$

Let $\Lambda_L(t)$ denote the LCR at time t for $t \in [0, T]$. Then by definition, we can write $\Lambda_L(t)$ as

$$
\Lambda_L(t) = \frac{H_Q(t)}{O_N(t)} = H_Q(t)\Phi(O_N(t)).
$$

We apply Itô's Product Rule to $\Lambda_L(t) = H_Q(t)\Phi(O_N(t))$ to find an expression for $d\Lambda_L(t)$ as follows:

$$
\begin{aligned}
d\Lambda_L(t) &= \Phi(O_N(t))dH_Q(t) + H_Q(t)d\Phi(O_N(t)) + dH_Q(t)d\Phi(O_N(t)) \\
&= \frac{1}{O_N(t)}\left\{ [\theta_r(t)r(t) + 0.85\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2)] + 0.85\theta_S(t)\sigma_1 dW_S(t) \right. \\
&\quad + \left. 0.85\theta_S(t)\sigma_2\sigma_r dW_r(t) \right\} + H_Q(t)\left\{ \left\{ -\frac{1}{O_N^2(t)}0.15(\mu_{S_2} - \mu_{M_2}) \right.\right. \\
&\quad + \left.\left. \frac{1}{O_N^3(t)}\left[(0.15\sigma_{S_2})^2 + (0.15\sigma_{M_2})^2 \right] \right\}dt - \frac{1}{O_N^2(t)}\left(0.15\sigma_{S_2}dW_{S_2} - 0.15\sigma_{M_2}dW_{M_2} \right) \right\}.
\end{aligned}
$$

Alternatively, in the case $I_C(t) \geq 0.75 O_C(t)$,

$$
\begin{aligned}
d\Phi(O_N(t)) &= \frac{\partial \bar{g}}{\partial t}(t, O_N(t))dt + \frac{\partial \bar{g}}{\partial o_n}(t, O_N(t))dO_N(t) + \frac{1}{2}\frac{\partial^2 \bar{g}}{\partial o_n^2}(t, O_N(t))(dO_N(t))^2 \\
&= \left[-\frac{1}{O_N^2(t)}0.0375\mu_{S_2} + \frac{1}{O_N^3(t)}\left(0.0375\sigma_{S_2} \right)^2 \right]dt - \frac{1}{O_N^2(t)}0.0375\sigma_{S_2}dW_{S_2}.
\end{aligned}
$$

Let $\Lambda_L(t)$ denote the LCR at time t for $t \in [0, T]$. Then by definition, we can write $\Lambda_L(t)$ as

$$
\Lambda_L(t) = \frac{H_Q(t)}{O_N(t)} = H_Q(t)\Phi(O_N(t)).
$$

We apply Itô's Product Rule to $\Lambda_L(t) = H_Q(t)\Phi(O_N(t))$ to find an expression for $d\Lambda_L(t)$ as follows:

$$
\begin{aligned}
d\Lambda_L(t) &= \Phi(O_N(t))dH_Q(t) + H_Q(t)d\Phi(O_N(t)) + dH_Q(t)d\Phi(O_N(t)) \\
&= \frac{1}{O_N(t)}\left\{ [\theta_r(t)r(t) + 0.85\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2)] + 0.85\theta_S(t)\sigma_1 dW_S(t) \right. \\
&\quad + \left. 0.85\theta_S(t)\sigma_2\sigma_r dW_r(t) \right\} + H_Q(t)\left\{ \left[-\frac{1}{O_N^2(t)}0.0375\mu_{S_2} + \frac{1}{O_N^3(t)}\left(0.0375\sigma_{S_2} \right)^2 \right]dt \right. \\
&\quad - \left. \frac{1}{O_N^2(t)}0.0375\sigma_{S_2}dW_{S_2} \right\}.
\end{aligned}
$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The next step is to derive the model of the NSFR. Let us at this point introduce models for the bank's deposits and off-balance sheet activities. We assume that the bank's deposits

evolve according to the SDE

$$\frac{dD(t)}{D(t)} = \mu_D dt + \sigma_D dW_D(t), \tag{7.12}$$

where μ_D and σ_D are assumed to be positive constants and W_D is a one-dimensional standard Brownian motion.

We will further assume that the bank's off-balance sheet activities can be modelled by the equation

$$\frac{dO(t)}{O(t)} = \mu_O dt + \sigma_O dW_O(t). \tag{7.13}$$

In the above dynamics μ_O and σ_O are positive constants and W_O is another one-dimensional standard Brownian motion.

We assume that the AASF can at time t be described by the SDE

$$\begin{aligned} dF_A(t) &= 1.0 \times dC(t) + 0.95 \times dD(t) \\ &= (c + 0.95D(t)\mu_D)dt + 0.95D(t)\sigma_D dW_D(t), \end{aligned} \tag{7.14}$$

where 1.0 and 0.95 are the ASF factors associated with the Total capital and stable deposits under the Basel III Accord.

Next we assume that the RASF can at time t be described by the SDE

$$\begin{aligned} dF_R(t) &= 0.05 \times \theta_r(t)\frac{dS_0(t)}{S_0(t)} + 0.15 \times \theta_S(t)\frac{dS(t)}{S(t)} + 0.85 \times \theta_L(t)\frac{dL(t)}{L(t-)} \\ &\quad + 0.05 \times \frac{dO(t)}{O(t)} + dC(t) \\ &= [0.05\theta_r(t)r(t) + 0.15\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2) + 0.85\theta_L(t)(r(t) \\ &\quad + \sigma_L(T - t, r(t))\lambda_2\sigma_r) + 0.05\mu_O + c(t)]dt + 0.15\theta_S(t)\sigma_1 dW_S(t) \\ &\quad + (0.15\theta_S(t)\sigma_2\sigma_r + 0.85\theta_L(t)\sigma_L(T - t, r(t)))dW_r(t) + 0.05\sigma_O dW_O(t) \\ &\quad + 0.85\theta_L(t)dK(t), \end{aligned} \tag{7.15}$$

78

where the weights 0.05, 0.15 and 0.85 are the RSF factors associated with respectively the treasury and off-balance sheet activities, marketable security and loan under Basel III (see [12]).

Proposition 7.2. *With the dynamics of the AASF, $F_A(t)$, given by the SDE in Eq.(7.14), and with the dynamics of the RASF, $F_R(t)$, given by Eq.(7.15), we can write the dynamics of the NSFR, $\Lambda_N(t)$, at time t as:*

$$
\begin{aligned}
d\Lambda_N(t) \; = \;& \frac{1}{F_R(t)}\Big[(c + 0.95D(t)\mu_D)dt + 0.95D(t)\sigma_D dW_D(t)\Big] \\
&+ \; F_A(t)\Bigg\{\Bigg\{ -\frac{1}{F_R^2(t)}\Big[0.05\theta_r(t)r(t) + 0.15\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2) \\
&+ \; 0.85\theta_L(t)(r(t) + \sigma_L(T - t, r(t))\lambda_2\sigma_r) + 0.05\mu_O + c(t)\Big] \\
&+ \; \frac{1}{F_R^3(t)}\Big[(0.15\theta_S(t)\sigma_1)^2 + (0.15\theta_S(t)\sigma_2\sigma_r + 0.85\theta_L(t)\sigma_L(T - t, r(t)))^2 + (0.05\sigma_O)^2\Big]\Bigg\}dt \\
&- \; \frac{1}{F_R^2(t)}\Big[0.15\theta_S(t)\sigma_1 dW_S(t) + (0.15\theta_S(t)\sigma_2\sigma_r + 0.85\theta_L(t)\sigma_L(T - t, r(t)))dW_r(t) \\
&+ \; 0.05\sigma_O dW_O(t)\Big] + \int_{|z|<R}\Big\{\frac{1}{F_R(t-) + \gamma(t, z, \omega)} - \frac{1}{F_R(t-)} + \frac{\gamma(t, z, \omega)}{F_R^2(t-)}\Big\}\nu(dz)dt \\
&+ \; \int_{\mathbb{R}}\Big\{\frac{1}{F_R(t-) + \gamma(t, z, \omega)} - \frac{1}{F_R(t-)}\Big\}\bar{N}(dt, dz)\Bigg\}. \tag{7.16}
\end{aligned}
$$

Proof of Proposition 7.2: Let $dF_R^c(t)$ denote the continuous part of $dF_R(t)$. Then

$$
\begin{aligned}
dF_R^c(t) \; = \;& [0.05\theta_r(t)r(t) + 0.15\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2) + 0.85\theta_L(t)(r(t) \\
&+ \; \sigma_L(T - t, r(t))\lambda_2\sigma_r) + 0.05\mu_O + c(t)]dt + 0.15\theta_S(t)\sigma_1 dW_S(t) \\
&+ \; (0.15\theta_S(t)\sigma_2\sigma_r + 0.85\theta_L(t)\sigma_L(T - t, r(t)))dW_r(t) + 0.05\sigma_O dW_O(t).
\end{aligned}
$$

By applying Itô's Formula to $\Phi(F_R(t)) = \bar{g}(t, F_R(t)) = \frac{1}{F_R(t)}$, we get

$$
\begin{aligned}
&d\Phi(F_R(t)) \\
&= \; \frac{\partial\bar{g}}{\partial t}(t, F_R(t))dt + \frac{\partial\bar{g}}{\partial f_r}(t, F_R(t))dF_R^c(t) + \frac{1}{2}\frac{\partial^2\bar{g}}{\partial f_r^2}(t, F_R(t))\Big[(0.15\theta_S(t)\sigma_1)^2 dt \\
&+ \; (0.15\theta_S(t)\sigma_2\sigma_r + 0.85\theta_L(t)\sigma_L(T - t, r(t)))^2 dt + (0.05\sigma_O)^2 dt\Big]
\end{aligned}
$$

79

$$
+ \quad \int_{|z|<R} \left\{ \bar{g}(t, F_R(t-) + \gamma(t,z,\omega)) - \bar{g}(t, F_R(t-)) - \frac{\partial \bar{g}}{\partial f_r}(t, F_R(t-))\gamma(t,z,\omega) \right\} \nu(dz)dt
$$

$$
+ \quad \int_{\mathbb{R}} \left\{ \bar{g}(t, F_R(t-) + \gamma(t,z,\omega)) - \bar{g}(t, F_R(t-)) \right\} \bar{N}(dt, dz)
$$

$$
= \quad \left\{ -\frac{1}{F_R^2(t)} \left[0.05\theta_r(t)r(t) + 0.15\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2) \right. \right.
$$

$$
+ \quad \left. 0.85\theta_L(t)(r(t) + \sigma_L(T-t, r(t))\lambda_2\sigma_r) + 0.05\mu_O + c(t) \right]
$$

$$
+ \quad \frac{1}{F_R^3(t)} \left[(0.15\theta_S(t)\sigma_1)^2 + (0.15\theta_S(t)\sigma_2\sigma_r + 0.85\theta_L(t)\sigma_L(T-t, r(t)))^2 + (0.05\sigma_O)^2 \right] \left. \right\} dt
$$

$$
- \quad \frac{1}{F_R^2(t)} \left[0.15\theta_S(t)\sigma_1 dW_S(t) + (0.15\theta_S(t)\sigma_2\sigma_r + 0.85\theta_L(t)\sigma_L(T-t, r(t)))dW_r(t) \right.
$$

$$
+ \quad \left. 0.05\sigma_O dW_O(t) \right] + \int_{|z|<R} \left\{ \frac{1}{F_R(t-) + \gamma(t,z,\omega)} - \frac{1}{F_R(t-)} + \frac{\gamma(t,z,\omega)}{F_R^2(t-)} \right\} \nu(dz)dt
$$

$$
+ \quad \int_{\mathbb{R}} \left\{ \frac{1}{F_R(t-) + \gamma(t,z,\omega)} - \frac{1}{F_R(t-)} \right\} \bar{N}(dt, dz).
$$

Let $\Lambda_N(t)$ denote the NSFR at time t for $t \in [0, T]$. Then by definition, we can write $\Lambda_N(t)$ as

$$
\Lambda_N(t) = \frac{F_A(t)}{F_R(t)} = F_A(t)\Phi(F_R(t)).
$$

We apply Itô's Product Rule to $\Lambda_N(t) = F_A(t)\Phi(F_R(t))$ to find an expression for $d\Lambda_N(t)$:

$$
d\Lambda_N(t)
$$

$$
= \quad \Phi(F_R(t))dF_A(t) + F_A(t)d\Phi(F_R(t)) + dF_A(t)d\Phi(F_R(t))
$$

$$
= \quad \frac{1}{F_R(t)} \left[(c + 0.95D(t)\mu_D)dt + 0.95D(t)\sigma_D dW_D(t) \right]
$$

$$
+ \quad F_A(t) \left\{ \left\{ -\frac{1}{F_R^2(t)} \left[0.05\theta_r(t)r(t) + 0.15\theta_S(t)(r(t) + \sigma_1\lambda_1 + \sigma_2\sigma_r^2\lambda_2) \right. \right. \right.
$$

$$
+ \quad \left. 0.85\theta_L(t)(r(t) + \sigma_L(T-t, r(t))\lambda_2\sigma_r) + 0.05\mu_O + c(t) \right]
$$

$$
+ \quad \frac{1}{F_R^3(t)} \left[(0.15\theta_S(t)\sigma_1)^2 + (0.15\theta_S(t)\sigma_2\sigma_r + 0.85\theta_L(t)\sigma_L(T-t, r(t)))^2 + (0.05\sigma_O)^2 \right] \left. \right\} dt
$$

$$
- \quad \frac{1}{F_R^2(t)} \left[0.15\theta_S(t)\sigma_1 dW_S(t) + (0.15\theta_S(t)\sigma_2\sigma_r + 0.85\theta_L(t)\sigma_L(T-t, r(t)))dW_r(t) \right.
$$

$$
+ \quad \left. 0.05\sigma_O dW_O(t) \right] + \int_{|z|<R} \left\{ \frac{1}{F_R(t-) + \gamma(t,z,\omega)} - \frac{1}{F_R(t-)} + \frac{\gamma(t,z,\omega)}{F_R^2(t-)} \right\} \nu(dz)dt
$$

$$+ \int_{\mathbb{R}} \left\{ \frac{1}{F_R(t-) + \gamma(t, z, \omega)} - \frac{1}{F_R(t-)} \right\} \bar{N}(dt, dz) \Bigg\}.$$

This concludes the proof. □

7.2   A simulation study of the liquidity ratios

We now characterize the behaviour of the liquidity ratios under the proxy by means of a numerical simulation. The simulation is based on the simulation parameters of Chapter 6. In addition, we consider the parameters

$$\mu_{S_2} = 0.02, \ \mu_{M_2} = 0.03, \ \sigma_{S_2} = 0.12, \ \sigma_{M_2} = 0.15,$$

$$\mu_D = 0.06, \ \sigma_D = 0.08, \ \mu_O = 0.05 \text{ and } \sigma_O = 0.07$$

and initial conditions

$$H_Q(0) = 1, \ O_C(0) = 1.95, \ I_C(0) = 1, \ O_N(0) = 0.95,$$

$$\Lambda_L(0) = 1.05, \ D(0) = 2.71, \ F_A(0) = 2.95, \ F_R(0) = 2.95 \text{ and } \Lambda_N(0) = 1.$$

The approximate optimal capital allocation strategy depicted in Figure 6.1 leads to the liquidity ratios in Figures 7.2 and 7.4. By following the approximate optimal strategy illustrated by Figure 6.1, the bank maintains its LCR and NSFR well above their minimum Basel III-prescribed levels over the 10-year horizon. By Basel III standards, the bank holds enough high quality liquid assets to withstand short term stress periods over the duration of the investment period, as the LCR satisfies its minimum requirement. Since the bank meets the minimum NSFR requirement, it is also able to withstand medium-long term stress periods as it has adequate funding to support its investment practices. The jump improves the level of the NSFR over that of its proxy much like for the capital adequacy ratios.
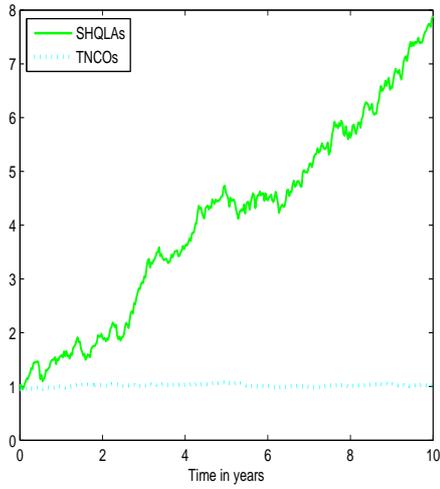
Figure 7.1: A simulation of the Stock of High Quality Liquid Assets and Total Net Cash Outflows H_Q and O_N, given a constant stream of capital inflow.
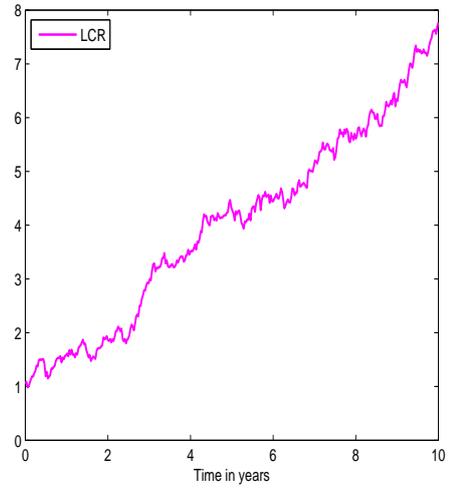
Figure 7.2: A simulation of the Liquidity Coverage Ratio Λ_L, given a constant stream of capital inflow.
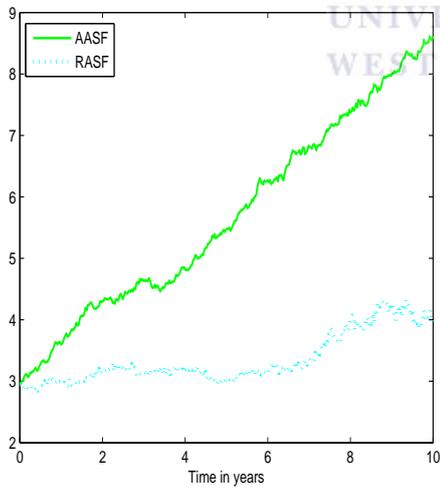
Figure 7.3: A simulation of the Available and Required Amounts of Stable Funding F_A and F_R, given a constant stream of capital inflow.
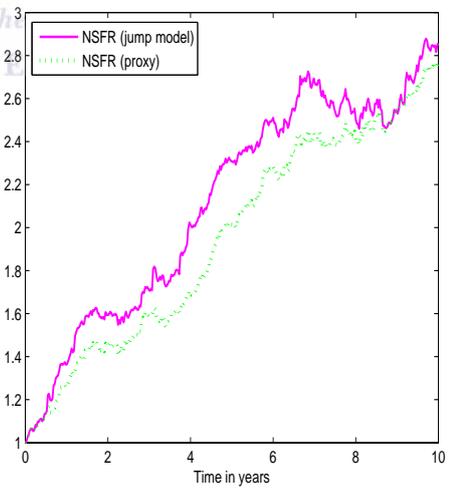
Figure 7.4: A simulation of the Net Stable Funding Ratio Λ_N, given a constant stream of capital inflow.

82

Chapter 8

The jump-diffusion deposit insurance pricing model

In this chapter we derive the multi-period deposit insurance pricing method in terms of the proxy. Our pricing method, which is based on the methodologies [29, 30] of Duan and Yu, utilizes an asset value reset rule comparable to the typical practice of insolvency resolution by insuring agencies. In deriving the pricing method, we will on occasion directly quote the results from the papers [29, 30] for application. Furthermore, we employ a Monte Carlo simulation method and examine the effects of Basel III's capital standard, capital forbearance and moral hazard on the fairly-priced deposit insurance premium rate of our model under the same values for the forbearance parameter considered by Duan and Yu [30]. We compare our findings with those of Duan and Yu [30].

In addition to Duan and Yu [29, 30], we will also be referencing the papers Merton [72], Cox and Ross [25] and Harrison and Krepps [44].

8.1 Deriving the multi-period deposit insurance pricing method

We now derive the multi-period deposit insurance pricing model. Since we want to incorporate insolvency resolution into the model, we must allow for some discrete adjustments to the level of the TNRWAs at the points of auditing. Thus, the evolution of the TNRWAs can only be described by Eq.(5.7) during periods between any two consecutive auditing times. Let us denote the sequence of auditing points by $t(i), i = 1 \ldots n$, where n is some larger integer. We still take the assumption that the jump associated with the loan is sufficiently small, and that its effect on the value of the bank's TNRWAs value can be approximated in a simple manner.

We assume that the annualized continuously compounded return of the TNRWAs over the interval $t(i-1)$ to $t(i)$, can be approximated by

$$R(t(i)) \sim N[\mu_R(t(i-1))(t(i) - t(i-1)), \sigma_R^2(t(i-1))(t(i) - t(i-1))]. \tag{8.1}$$

The variables $\mu_R(t(i-1))$ and $\sigma_R(t(i-1))$, as in Duan and Yu [29], are the annualized mean return and standard deviation of the TNRWAs returns assumed to be known at time $t(i-1)$. These variables are assumed to be measurable with respect to the information set generated by the continuously compounded returns up to and including time $t(i-1)$, which means that they can be stochastic by being functions of past returns.

For the remainder of this chapter, we will assume that $\mu_R(t(i-1))$ and $\sigma_R(t(i-1))$ can be approximated by the following expressions:

$$
\begin{aligned}
\mu_R(t(i-1)) &= X(t(i-1))r(t(i-1)) + \theta_S(t(i-1))(\lambda_1\sigma_1 + \lambda_2\sigma_2\sigma_r^2) \\
&+ \theta_L(t(i-1))\lambda_2\sigma_L(T - t(i-1), r(t(i-1)))\sigma_r + c
\end{aligned}
$$

and

$$
\sigma_R(t(i-1)) = \frac{1}{2}\Big(\theta_S(t(i-1))\sigma_1 + \theta_L(t(i-1))\sigma_L(T - t(i-1), r(t(i-1)))
$$

$$+ \quad \theta_S(t(i-1))\sigma_2\sigma_r \bigg).$$

We denote the initial face value of the bank's deposits by $D(0)$, and we assume that earned interest is ploughed back into the deposit base. The deposits are insured and we consider $\bar{r}$ as an applicable risk-free rate of return, with $\bar{r}$ denoting the mean or expected value of the short-rate process r given by Eq.(5.3) over the interval $[0, T]$. We assume that the level of the bank's TNRWAs is subject to reset at the time of the audit. The insuring agent typically arranges for a reorganization of the failing bank in the event of a failure resolution, and then continues to provide deposit insurance coverage. The value of the TNRWAs of the defaulting bank are reset to the level required under the Basel III capital standard. After the TNRWAs reset, the newly reorganized bank continues to operate with deposit insurance. This set-up is supported by the historical failure resolution experience using either purchase-and-assumption or government-assisted-merger methods in the U.S [29, 30]. From this perspective, the deposit insurance contract is automatically renewed to cover a new period. It can thus be viewed as a stream of single-period put options with occasional TNRWAs value resets. The value of the TNRWAs is subject to another type of reset. Since the shareholders of profitable banks may consider withdrawing excessive capital, a ceiling is placed on the bank's TNRWAs value. Specifically, at the auditing time $t(i)$, the TNRWAs value reset rule can, according to Duan and Yu [30], be described by

$$X(t(i)) = \begin{cases} q_u D(0)e^{\bar{r}t(i)} & \text{if} \quad X^*(t(i)) \geq q_u D(0)e^{\bar{r}t(i)} \\ X^*(t(i)) & \text{if} \quad q_u D(0)e^{\bar{r}t(i)} > X^*(t(i)) \geq \rho D(0)e^{\bar{r}t(i)} \\ q_l D(0)e^{\bar{r}t(i)} & \text{if} \quad \text{otherwise.} \end{cases} \quad (8.2)$$

In the TNRWAs value reset rule, we define $X^*(t(i))$ as $X^*(t(i)) = X(t(i-1))e^{R(t(i))(t(i)-t(i-1))}$ as in [29], and throughout this chapter $t(0)$ is defined as $t(0) = 0$. As in [30] the parameters q_l and q_u $(1 \leq q_l < q_u)$ set the upper and lower bounds for the value of the TNRWAs. The parameter q_l reflects the capital standard set by the regulatory authority. The parameter q_u is a threshold level of asset-to-debt ratio. It determines the extent to which the profitable bank equity holders are willing to leave the capital with the bank before paying themselves

cash dividends. The parameter ρ $(0 < \rho \leq q_l)$ models capital forbearance [30].

The new Basel III capital standard calls for the Total capital in the amount equal to or exceeding 8% of the TRWAs [10, 53, 84, 78]. This capital standard can be translated into $q_l = 1.087$ [30]. When the forbearance parameter ρ is smaller than one, the bank, if insolvent, will not be forced to face an immediate intervention from the insuring agent provided that it remains within the capital forbearance range [30]. Under such circumstances a bank in financial distress is still considered operational as the insuring agent guarantees the performance of its deposit liabilities [30]. An interesting feature of the regulated deposit-taking industry is failure to mark-to-market the bank's assets and liabilities immediately. An insured bank faces a failure resolution only when the value of its TNRWAs falls below $\rho D(0)e^{\bar{r}t(i)}$ [29, 30]. Even though the parameter ρ alters the condition for triggering an TNRWAs value reset, the reset will, if taking place, fully restore the TNRWAs value to the level dictated by the capital standard. When capital forbearance occurs, it amounts to a breach of the capital standard. The scenario $1 \leq \rho < q_l$, also a breach of the capital standard, should according to [30] not be considered as capital forbearance since the bank still remains solvent.

The deposit insuring agent is required to implement a tight capital standard. This implies an early closure of any troubled bank even if the bank is technically still solvent. Strict enforcement of the capital standard implies $\rho = q_l$ [30]. In the single-period setting, traditionally, the decision of closing early or granting capital forbearance is irrelevant, as depository institutions will be liquidated at the end of the period anyway. The typical adjustment made to the deposit insurance pay-off in the single-period setting is somewhat artificial and unrealistic [30].

The insuring agent is exposed to a stream of put option-like liabilities. The put option at

86

time $t(i)$ gives rise to a cash payment $\kappa(t(i))$, in an amount equal to

$$\kappa(t(i)) = \begin{cases} 0 & \text{if} \quad X^*(t(i)) \geq \min(\rho, 1)D(0)e^{\bar{r}t(i)} \\ D(0)e^{\bar{r}t(i)} - X^*(t(i)) & \text{if} \quad \text{otherwise.} \end{cases} \tag{8.3}$$

According to Duan and Yu [30] we must use $\min(\rho, 1)$ to reflect the fact that even if $\rho > 1$, the cash liability facing the insuring agent in the event of settlement is unaltered. The chances of incurring cash payments due to the bank's future insolvency is nevertheless reduced through the TNRWAs value reset rule. At the termination point of this multiperiod coverage, ρ must by definition be set to one, regardless of its original value. Therefore, the last liability can be written as the familiar put option pay-off expression

$$\kappa(T) = \text{Max}(D(0)e^{\bar{r}T} - X^*(T), 0). \tag{8.4}$$

We assume that the time $t(i-1)$ value of the payment at time $t(i) < T$ per dollar of deposits can be priced, similar to those of the models of Merton [72] and Duan and Yu [29], to yield

$$\begin{aligned} I(t(i), \rho) &= N[\sigma_R(t(i-1))\sqrt{t(i) - t(i-1)} - d(t(i-1), \rho)] \\ &\quad - \frac{X(t(i-1))}{D(0)e^{\bar{r}t(i-1)}} N[-d(t(i-1), \rho)], \end{aligned} \tag{8.5}$$

where $N(\cdot)$ denotes the cumulative standard normal distribution function; and

$$d(t(i-1), \rho) = \frac{\ln \frac{X(t(i-1))}{\rho D(0)e^{\bar{r}t(i-1)}} + \frac{\sigma_R^2(t(i-1))}{2}(t(i) - t(i-1))}{\sigma_R(t(i-1))\sqrt{t(i) - t(i-1)}}.$$

For the cash payment at the terminal time $T = t(n)$, its value at the preceding time point, $t(n-1)$, can then be computed by simply letting $\rho = 1$. Specifically,

$$\begin{aligned} I(t(n-1), 1) &= N[\sigma_R(t(n-1))\sqrt{t(n) - t(n-1)} - d(t(n-1), 1)] \\ &\quad - \frac{X(t(n-1))}{D(0)e^{\bar{r}t(n-1)}} N[-d(t(n-1), 1)]. \end{aligned} \tag{8.6}$$

The formula in Eq.(8.6) is the same as that of Merton [72], whereas the formula in Eq.(8.5) is also that of Merton [72] if $\rho = 1$ [29].

The time $t(0)$ value of an individual put option at time $t(i)$ is the present value of the product of $I(t(i))$ and $D(0)e^{\bar{r}t(i)}$. The present value operator can be derived by using the risk-neutral valuation technique. Based on the findings of Cox and Ross [25] and Harrison and Kreps [44], we assume that the continuously compounded return on the bank's TNRWAs, under the risk-neutralized pricing measure, distributes according to:

$$R(t(i)) \sim N[(\bar{r} - \frac{\sigma_R^2(t(i-1))}{2})(t(i) - t(i-1)), \sigma_R^2(t(i-1))(t(i) - t(i-1))]. \qquad (8.7)$$

We now present the fairly-priced deposit insurance premium rate. We let δ_n denote the fairly-priced premium rate per period of an n-period deposit insurance coverage. The fairly-priced premium rate, a theoretical entity, is a risk-based rate which equates the present value of the entire stream of deposit insurance liabilities with the present value of the total insurance levies at this premium rate [30]. According to [30] the global practise of rate-setting by deposit insurance agents can hardly be considered as setting a fair premium rate. Nevertheless, the fairly-priced premium rate serves as a convenient measure for the intrinsic value of the deposit insurance coverage [30]. We follow the approach of Duan and Yu [30] and calculate the fairly-priced premium rate per period in an n-period coverage horizon as follows:

$$\delta_n = \frac{1}{nD(0)} \sum_{i=1}^{n} e^{-\bar{r}t(i)} \mathbb{E}_{t(0)}^*[I(t(i))], \qquad (8.8)$$

where $\mathbb{E}_{t(0)}^*[\cdot]$ denotes expectation taken at time $t(0)$ with respect to the distribution specified in relation (8.7).

We assume that risk-taking behaviour (or moral hazard behaviour) is governed by the outcomes of the bank's TNRWAs value, which is classified into three categories. First, if the value of the TNRWAs is greater than the level required by the Basel III capital standard, the bank functions normally and its portfolio risk characteristics remain unchanged. In other words, $\sigma_R(t(i)) = \sigma_R(t(i-1))$. Second, if the bank's TNRWAs value breaches the capital standard but is tolerated by the regulatory authority, then the moral hazard behaviour occurs; i.e., the bank starts to take on more risk in its portfolio. A simple way of

modelling this effect is to force an increase on σ_R. According to Duan and Yu [30], this action increases the stationary standard deviation of its portfolio by $100\omega\%$. Hence we have $\sigma_R(t(i)) = (1+\omega)^2\sigma_R(t(i-1))$. Lastly, once the troubled bank breaks the threshold level, we assume that the situation becomes intolerable and the insuring agent steps in to reorganize the bank. This results in the bank's original risk level being restored, i.e., $\sigma_R(t(i)) = \sigma_R(t(0))$. The adjustment process can be formulated as follows [30]:

$$
\sigma_R(t(i)) = \begin{cases} \sigma_R(t(i-1)) & \text{if} \quad X^*(t(i)) \geq q_l D(0)e^{\bar{r}t(i)} \\ (1+\omega)^2\sigma_R(t(i-1)) & \text{if} \quad q_l D(0)e^{\bar{r}t(i)} > X^*(t(i)) \geq \rho D(0)e^{\bar{r}t(i)} \\ \sigma_R(t(0)) & \text{if} \quad \text{otherwise.} \end{cases} \tag{8.9}
$$

In the above dynamic $\sigma_R(\cdot)$ is indexed by time to reflect its time-varying nature.

8.2 Studying the deposit insurance pricing model numerically
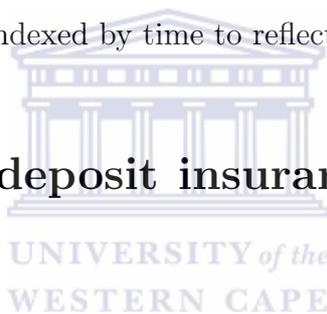
We now perform numerical simulations with our deposit insurance pricing model to study its implications. Using a Monte Carlo simulation method, where ten thousand sample paths are used in every Monte Carlo calculation, we compute the fairly-priced premium rate for different scenarios. In particular, we study the impact of capital forbearance and moral hazard on the fairly-priced premium rate for both the scenario where the capital standard is strictly enforced by the regulatory authority, and the scenario where the bank faces a looser capital standard. The computations are based on the simulation parameters of Chapters 6 and 7.

We assume that auditing takes place once a year, at the end of the year, and we consider coverage horizons of duration 5, 10 and 15 years. For the scenario where the capital standard is strictly enforced, the parameters q_u and q_l must be set to 1.15 and 1.087 respectively (see [30]). In this case $\rho = q_l = 1.087$. To study the impact of a looser capital standard we

consider the case that $q_l = 1.05$. Three initial values of the asset-to-debt ratio are considered. These are 1.09, 1.11 and 1.13, all of which fall inside the range established by q_l and q_u.

Table 8.1 presents the fairly-priced premium rates corresponding to different coverage horizons and leverage positions. The values in this table are based on the assumption that the capital standard is strictly enforced by the regulatory authority. For a fixed asset-to-debt ratio an increase in the coverage horizon causes the premium to rise. By keeping the coverage horizon fixed and decreasing the level of the initial leverage (increase in the value of the initial asset-to-debt ratio $X(0)/D(0)$), the value of the fairly-priced premium rate drops.

Table 8.1: A comparison of the fairly-priced deposit insurance premium rates under different model assumptions when the capital standard is strictly enforced, i.e., $\rho = q_l$.

ρ	q_l	q_u	$X(0)$	$D(0)$	$X(0)/D(0)$	δ_5	δ_{10}	δ_{15}
1.087	1.087	1.15	2.95	2.71	1.09	0.00429	0.00637	0.00666
				2.66	1.11	0.00386	0.00610	0.00646
				2.61	1.13	0.00347	0.00585	0.00627

When the capital standard is not strictly enforced by the regulatory authority, the insured bank effectively faces a looser capital requirement. Failure to enforce a higher capital standard is not exactly the same as setting a lower capital standard, because capital forbearance is likely to encourage the risk-taking behaviour on the part of an insured bank under financial distress [30].

Table 8.2 highlights the effect of capital forbearance, with the forbearance parameter ρ equal to 0.97, on the fairly-priced deposit insurance premium. In the computations the risk-taking intensity parameter ω is assumed to be 0.2. For a fixed initial asset-to-debt ratio an increase in the coverage horizon leads to a rise in the premium rate. In this situation, keeping the coverage horizon fixed and reducing the level of the initial leverage (increase in $X(0)/D(0)$

causes the value of the fairly-priced premium rate to fall. Recall that this behaviour was also observed for the scenario where the capital standard is strictly enforced.

Table 8.2: A comparison of the fairly-priced deposit insurance premium rates under different model assumptions when capital forbearance is present, i.e., $\rho < 1$.

ρ	q_l	q_u	ω	$X(0)$	$D(0)$	$X(0)/D(0)$	δ_5	δ_{10}	δ_{15}
0.97	1.05	1.15	0.20	2.95	2.71	1.09	0.00953	0.01204	0.01239
					2.66	1.11	0.00920	0.01190	0.01231
					2.61	1.13	0.00890	0.01177	0.01224
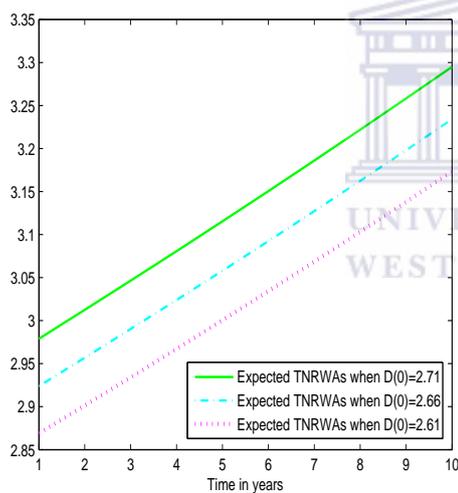
Figure 8.1: A simulation of the expected TNRWAs at the auditing times when the initial asset-to-debt ratio $X(0)/D(0)$ is respectively 1.09, 1.11 and 1.13, and the capital forbearance parameter is $\rho = 1.087$.
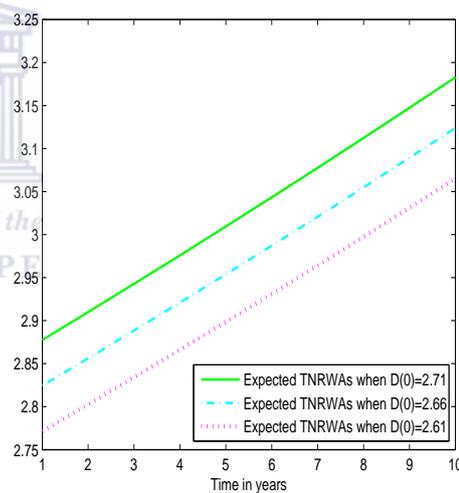
Figure 8.2: A simulation of the expected TNRWAs at the auditing times when the initial asset-to-debt ratio $X(0)/D(0)$ is respectively 1.09, 1.11 and 1.13, and the capital forbearance parameter is $\rho = 0.97$.

The behaviour of our fairly-priced premium rate differs substantially from that of Duan

and Yu [30]. The aforementioned authors employed a GARCH option pricing technique to determine the fairly-priced deposit insurance premium rate under different conditions. Duan and Yu [30] reports that their fairly priced premium rate increases with the asset-to-debt ratio. An increase in the coverage horizon in their model causes the premium to rise or fall depending on the initial leverage (asset-to-debt) position. When the initial leverage is high, an increase in the coverage horizon reduces the fairly priced premium rates. The reverse is true when the leverage is low. A longer run deposit insurance coverage has the effect of lowering the fairly priced premium rate. If the capital standard is low relative to the current asset-to-debt ratio, the fairly priced premium rate tends to increase with the coverage horizon.

Chapter 9

Pricing interest rate swaps under the CIR dynamic

This chapter presents methods for pricing LIBOR-in-arrears and vanilla interest rate swaps under the CIR [22] dynamic. We employ the methodology of Mallier and Alobaidi [65], who used a Green's function approach to derive analytical expressions for the values of the aforementioned swaps. We quote directly from the methodology of Mallier and Alobaidi [65] here. To characterize the pricing models of [65], we contribute numerical examples based on Monte Carlo simulation methods. In particular, we examine the effect of the value of the fixed interest rate on the prices of the LIBOR-in-arrears swap and the vanilla swap. Besides the reference [65], other key references of this chapter are: Cox et al. [21, 22], Wilmott [93], Duffie [31], Klugman [61], Mallier and Mansi [66, 67], Abramowitz and Stegun [3], Gradshteyn and Ryzhik [41], and Büttler and Waldvogel [18].

9.1   Deriving the swap pricing methods

We now proceed to demonstrate how the swap pricing models can be derived. We consider a general stochastic interest rate r which obeys the SDE

$$dr(t) = u(r,t)dt + w(r,t)dW_r(t), \tag{9.1}$$

93

where the coefficient $u(r,t)$ represents the drift of the interest rate process and the expression $w(r,t)$ in the second term of the above SDE can be thought of as its volatility. Above, W_r still denotes the one-dimensional standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(t)_{t\geq 0}, \mathbb{P})$ of Chapter 5.

We assume that the bank's TNRWAs consist of a risk-free hedged portfolio consisting of two bonds with different maturities, each of which are derivatives of the interest rate model described by Eq.(9.1). According to Wilmott [93] the price $V(r,t)$ of such a bond, regardless of its maturity, follows the PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0. \tag{9.2}$$

The coefficient $u - \lambda w$ in Eq.(9.2) represents the risk-adjusted drift while $\lambda(r,t)$ is known as the market price of risk. The functional forms of $u - \lambda w$ and w will depend on the specifics of the interest rate model chosen. Many of the popular one-factor interest rate models are special cases of the general affine mode, for which $u - \lambda w = a(t) - b(t)r$ and $w = \sqrt{c(t)r - d(t)}$. One such case is the CIR model (see [21, 22]), which is the model we will be using in this analysis. For the aforementioned model (hereafter the CIR [22] model), $u - \lambda w = \varrho\theta - \zeta r$ and $w = \sigma\sqrt{r}$, where the coefficients are constants, opposed to functions as in the general affine model.

For a bond involving a single cash flow at time $t = T$, Eq.(9.2) must be solved together with the pay-off at time T. Let us denote the value of this pay-off by $V_0(r)$. If we specialize to the CIR [22] model, and further make the transformation $t = T - \tau$, so that τ is the remaining life of the bond, then Eq.(9.2) may be written as

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2 r}{2}\frac{\partial^2 V}{\partial r^2} + (\varrho\theta - \zeta r)\frac{\partial V}{\partial r} - rV, \tag{9.3}$$

together with the condition that the pay-off at maturity $V_0(r) = V(r,0)$ is specified at $\tau = 0$ [65]. Several authors have solved this problem using various techniques. A popular technique is to assume that the solution has the form $V(r,t) = \exp[A(r,t) - rB(r,t)]$, which Duffie [31]

94

and Klugman [61] have shown is a solution for the general affine model. A slightly different approach, which Mallier and Mansi [66, 67] have taken, is to take the Laplace transform in time of Eq.(9.3),

$$\hat{V}(p) = \int_0^\infty V(\tau)e^{-p\tau}d\tau, \tag{9.4}$$

and arrive at the following non-homogeneous ODE for the transform of the bond price,

$$\left[\frac{\sigma^2 r}{2}\frac{\partial^2}{\partial r^2} + (\varrho\theta - \zeta r)\frac{\partial}{\partial r} + (p - r)\right]\hat{V} = V_0(r). \tag{9.5}$$

Two linearly independent homogeneuous solutions to Eq.(9.5) are

$$\hat{V}_1 = \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2 + 2}} - \frac{1}{2}\right)\tilde{r}\right]M[2\tilde{\varrho} + \tilde{p} - 1, 2\tilde{\varrho}, \tilde{r}], \tag{9.6}$$

and

$$\hat{V}_2 = \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2 + 2}} - \frac{1}{2}\right)\tilde{r}\right]U[2\tilde{\varrho} + \tilde{p} - 1, 2\tilde{\varrho}, \tilde{r}], \tag{9.7}$$

where the transformations $\tilde{\zeta} = \frac{\zeta}{\sigma}$, $\tilde{\varrho} = \frac{\varrho\theta}{\sigma^2}$, $\tilde{r} = \frac{2r\sqrt{\tilde{\zeta}^2+2}}{\sigma}$ and $\tilde{p} = (\frac{p}{\sigma} + \tilde{\zeta}\tilde{\varrho})(\tilde{\zeta}^2 + 2)^{-1/2} - \tilde{\varrho}$ have been applied, and $M(a, b, \tilde{r})$ and $U(a, b, \tilde{r})$ are Kummer functions [3, 41]. Let us write $r^{\#} = \frac{\tilde{r}}{\sqrt{\tilde{\zeta}^2+2}}$. Then using these homogeneous solutions we can construct a solution to Eq.(9.5),

$$
\begin{aligned}
\hat{V} = {} & \frac{\Gamma(2\tilde{\varrho} + \tilde{p} - 1)}{\Gamma(2\tilde{\varrho})\sigma\sqrt{\tilde{\zeta}^2 + 2}}, \\
& \times \left[\hat{V}_1 \int_{\tilde{r}}^\infty \exp(-\zeta\tilde{r^{\#}})\tilde{r}^{4\tilde{\varrho}-1}U[2\tilde{\varrho} + \tilde{p} - 1, 2\tilde{\varrho}, \tilde{r}]V_0\left(\frac{\sigma r^{\#}}{2}\right)d\tilde{r}\right. \\
& \left.+ \hat{V}_2 \int_{\tilde{0}}^{\tilde{r}} \exp(-\zeta\tilde{r^{\#}})\tilde{r}^{4\tilde{\varrho}-1}M[2\tilde{\varrho} + \tilde{p} - 1, 2\tilde{\varrho}, \tilde{r}]V_0\left(\frac{\sigma r^{\#}}{2}\right)d\tilde{r}\right], \tag{9.8}
\end{aligned}
$$

where the boundary conditions that we require $\hat{V} \to 0$ as $\tilde{r} \to \infty$ and $\hat{V}_{\tilde{r}}$ bounded as $\tilde{r} \to 0$ have been imposed. The value of the option can be recovered by inverting the transform with

$$V(r, \tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{V}(r, p)e^{p\tau}dp. \tag{9.9}$$

Here c lies to the right of all the singularities of $\hat{V}(r,p)$. This integral can be evaluated by closing the contour to the left, and the value of the contour integral is $2\pi i$ times the sum of the residues contained inside the loop. Recalling that $\Gamma(cz)$ is single-valued and analytic over the entire complex plane, except for simple poles with residue $\frac{(-1)^n c^{-1}}{n!}$ at the points $z = -\frac{n}{c}$ $(n = 0, 1, 2, \dots)$, we deduce that $\hat{V}$ has simple poles at the points $\tilde{p} = 1 - n - 2\tilde{\varrho}$, or at $p = \sigma(1 - n - \tilde{\varrho})\sqrt{\tilde{\zeta}^2 + 2} - \sigma\tilde{\zeta}\tilde{\varrho}$, and it follows that the inverse is

$$
\begin{aligned}
V \;=\;& \frac{1}{\Gamma(2\tilde{\varrho})} \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2 + 2}} - \frac{1}{2}\right)\tilde{r} + (\sigma(1 - \tilde{\varrho})\sqrt{\tilde{\zeta}^2 + 2} - \sigma\tilde{\zeta}\tilde{\varrho})\tau\right] \\
&\times \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{-n\sigma\sqrt{\tilde{\zeta}^2 + 2}\tau} \\
&\times \left[M[-n, 2\tilde{\varrho}, \tilde{r}]\int_{\tilde{r}}^{\infty} \exp(-\tilde{\zeta} r^{\#})\tilde{r}^{4\tilde{\varrho}-1}U[-n, 2\tilde{\varrho}, \tilde{r}]V_0\left(\frac{\sigma r^{\#}}{2}\right)d\tilde{r}\right. \\
&\left.+\; U[-n, 2\tilde{\varrho}, \tilde{r}]\int_{0}^{\tilde{r}} \exp(-\tilde{\zeta} r^{\#})\tilde{r}^{4\tilde{\varrho}-1}M[-n, 2\tilde{\varrho}, \tilde{r}]V_0\left(\frac{\sigma r^{\#}}{2}\right)d\tilde{r}\right].
\end{aligned}
\tag{9.10}
$$

The above expression can be rewritten in terms of the Laguerre polynomials using the relations [3]

$$
M[-n, 2\tilde{\varrho}, \tilde{r}] = \frac{n!\Gamma(2\tilde{\varrho})}{\Gamma(2\tilde{\varrho} + n)} L_n^{2\tilde{\varrho}-1}(\tilde{r}),
\tag{9.11}
$$

$$
U[-n, 2\tilde{\varrho}, \tilde{r}] = (-1)^n n! L_n^{2\tilde{\varrho}-1}(\tilde{r}),
\tag{9.12}
$$

and we arrive at the simplified expression

$$
\begin{aligned}
V \;=\;& \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2 + 2}} - \frac{1}{2}\right)\tilde{r} + (\sigma(1 - \tilde{k})\sqrt{\tilde{\zeta}^2 + 2} - \sigma\tilde{\zeta}\tilde{\varrho})\tau\right] \\
&\times \sum_{n=0}^{\infty} \frac{n! e^{-n\sigma\sqrt{\tilde{\zeta}^2 + 2}\tau}}{\Gamma(2\tilde{\varrho} + n)} L_n^{2\tilde{\varrho}-1}(\tilde{r})\int_{0}^{\infty} \exp(-\tilde{\zeta} r^{\#})\tilde{r}^{4\tilde{\varrho}-1}L_n^{2\tilde{\varrho}-1}(\tilde{r}) \\
&\times V_0\left(\frac{\sigma r^{\#}}{2}\right)d\tilde{r}.
\end{aligned}
\tag{9.13}
$$

This expression can be further simplified using the identity [41]

$$
\sum_{n=0}^{\infty} \frac{n! z^n L_n^{\alpha}(x)L_n^{\alpha}(y)}{\Gamma(n + \alpha + 1)} = \frac{(xyz)^{-\frac{\alpha}{2}}}{1 - z} \exp\left[-\frac{z(x + y)}{1 - z}\right]I_{\alpha}\left[\frac{2\sqrt{xyz}}{1 - z}\right],
\tag{9.14}
$$

where I_α is a Bessel function so that

$$
\begin{aligned}
V &= \frac{1}{2} r^{-\tilde{\varrho}+\frac{1}{2}} \operatorname{csch}\left(\frac{\sigma\sqrt{\tilde{\zeta}^2+2\tau}}{2}\right) \\
&\times \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2+2}}-\frac{1}{2}-\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2\tau}}-1}\right)\tilde{r}+(2\sqrt{\tilde{\zeta}^2+2}-\tilde{\zeta}\tilde{\varrho})\sigma\tau\right] \\
&\times \int_0^\infty \tilde{r}'^{3\tilde{\varrho}-\frac{1}{2}}\exp\left[-\left(\frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}+\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2\tau}}-1}\right)\tilde{r}'\right] \\
&\times I_{2\tilde{\varrho}-1}\left[\frac{2\sqrt{\tilde{r}\tilde{r}'}}{\sinh\sigma\sqrt{\tilde{\zeta}^2+2\tau}}\right]V_0\left(\frac{\sigma\tilde{r}'}{2\sqrt{\tilde{\zeta}^2+2}}\right)d\tilde{r}'. \qquad(9.15)
\end{aligned}
$$

In Eq.(9.15) we have an expression for the value of a single payment bond. It can be written using a Green's function in the form

$$
V = \int_0^\infty G(\tilde{r},\tilde{r}',\tau)\tilde{V}_0(\tilde{r}')d\tilde{r}', \qquad(9.16)
$$

with

$$
\begin{aligned}
G(\tilde{r},\tilde{r}',\tau) &= \frac{1}{2}\tilde{r}^{-\tilde{\varrho}+\frac{1}{2}}\exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2+2}}-\frac{1}{2}-\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2\tau}}-1}\right)\tilde{r}\right] \\
&\times \exp\left[\left(2\sqrt{\tilde{\zeta}^2+2}-\tilde{\zeta}\tilde{\varrho}\right)\sigma\tau\right]\operatorname{csch}\frac{\sigma\sqrt{\tilde{\zeta}^2+2\tau}}{2} \\
&\times \tilde{r}'^{3\tilde{\varrho}-\frac{1}{2}}\exp\left[-\left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2\tau}}-1}+\frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)\tilde{r}'\right] \\
&\times I_{2\tilde{\varrho}-1}\left[\frac{2\sqrt{\tilde{r}\tilde{r}'}}{\sinh\sigma\sqrt{\tilde{\zeta}^2+2\tau}}\right]
\end{aligned}
$$

and where $\tilde{V}_0(\tilde{r}) = V_0(r)$. In Büttler and Waldvogel [18] this Green's function solution was presented in a slightly different but equivalent form in the context of the valuation of callable bonds.

In an interest rate swap, typically a payment is either made or received every six months, with each of the payments being the same as that of an FRA. To value the swap, we can

apply Eq.(9.16) to each of these FRAs and then sum them to arrive at a value for the swap. In what follows, a swap will be priced from the viewpoint of a receiver, i.e., an investor that receives fixed and pays floating; the price from the viewpoint of a payer, i.e., someone who pays fixed and receives floating, is the negative of the value found here. The fixed rate is assumed to be specified a priori, and we will denote it by r_f. The floating rate for each payment is determined at the "reset time". The reset time is usually earlier than the payment time, which is the moment at which payments exchange hands. In fact, the floating rate for each payment is usually determined at the previous payment date. One instrument where the payment time and reset time coincide is the LIBOR-in-arrears swap.

For a LIBOR-in-arrears swap, the reset and payment dates coincide. For each payment, the cash flow at the payment date is simply the difference between the fixed interest rate r_f and the value of the floating rate r at the time of the payment, multiplied by $\frac{1}{2}$ since payments are made every six months. Thus $V_0(r) = \frac{r_f - r}{2}$, or

$$\tilde{V}_0(\tilde{r}') = \frac{1}{2}\left(r_f - \frac{\sigma \tilde{r}'}{2\sqrt{\tilde{\zeta}^2 + 2}} \right). \tag{9.17}$$

If V_0 is negative, the receiver has to pay the balance to the payer, while it is negative, the payer must pay the receiver. Using this pay-off in the Green's function solution Eq.(9.16) gives the following value for each of the cash flows, where it is assumed that cash flow number i is received at a time τ_i later:

$$
\begin{aligned}
V_i \;=\; & \frac{1}{4}\tilde{r}^{-\tilde{\varrho}+\frac{1}{2}} \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2 + 2}} - \frac{1}{2} - \frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2 + 2\tau_i}} - 1} \right) \tilde{r} \right] \\
& \times\; \exp\left[\left(2\sqrt{\tilde{\zeta}^2 + 2} - \tilde{\zeta}\tilde{\varrho} \right)\sigma\tau_i \right] \operatorname{csch} \frac{\sigma\sqrt{\tilde{\zeta}^2 + 2\tau_i}}{2} \\
& \times\; \int_0^\infty \tilde{r}'^{3\tilde{\varrho}-\frac{1}{2}} \left(r_f - \frac{\sigma \tilde{r}'}{2\sqrt{\tilde{\zeta}^2 + 2}} \right) I_{2\tilde{\varrho}-1}\left[\frac{2\sqrt{\tilde{r}\tilde{r}'}}{\sinh \sigma\sqrt{\tilde{\zeta}^2 + 2\tau_i}} \right] \\
& \times\; \exp\left[-\left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2 + 2\tau_i}} - 1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2 + 2}} \right) \tilde{r}' \right] d\tilde{r}'. \tag{9.18}
\end{aligned}
$$

98

The integral in the above expression can be evaluated using the relations

$$
\int_0^\infty \tilde{r}'^b \exp[-a\tilde{r}'] I_c\left[d\sqrt{\tilde{r}'}\right] d\tilde{r}' = \frac{\Gamma(1+b+\frac{c}{2})}{\Gamma(1+c)}\left(\frac{d}{2\sqrt{a}}\right)^c a^{-1-b}
$$
$$
\times \quad M\left[1+b+\frac{c}{2}, 1+c, \frac{d^2}{4a}\right], \tag{9.19}
$$

where M is Kummer's function and Γ the gamma function (see [3, 41]), giving the following closed form expression for the value of the cash flow at time τ_i:

$$
\begin{aligned}
V_i &= \frac{\Gamma(4\tilde{\varrho})}{2^{2\tilde{\varrho}+1}\Gamma(2\tilde{\varrho})} \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2+2}} - \frac{1}{2} - \frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2\tau_i}}-1}\right)\tilde{r}\right] \\
&\times \left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2\tau_i}}-1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)^{-4\tilde{\varrho}-1} \exp\left[\left(2\sqrt{\tilde{\zeta}^2+2} - \tilde{\zeta}\tilde{\varrho}\right)\sigma\tau_i\right] \operatorname{csch}^{2\tilde{\varrho}}\frac{\sigma\sqrt{\tilde{\zeta}^2+2\tau_i}}{2} \\
&\times \left\{ M\left[4\tilde{\varrho}+1, 2\tilde{\varrho}, \frac{\tilde{r}}{4}\left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2\tau_i}}-1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)^{-1} \operatorname{csch}^2\frac{\sigma\sqrt{\tilde{\zeta}^2+2\tau_i}}{2}\right] \right. \\
&\times \left(-\frac{2\sigma\tilde{\varrho}}{\sqrt{\tilde{\zeta}^2+2}}+\right) + M\left[4\tilde{\varrho}, 2\tilde{\varrho}, \frac{\tilde{r}}{4}\left(\frac{1}{e^{-\sigma\sqrt{\tilde{\zeta}^2+2\tau_i}}-1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)^{-1} \operatorname{csch}^2\frac{\sigma\sqrt{\tilde{\zeta}^2+2\tau_i}}{2}\right] \\
&\times \left. r_f\left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2\tau_i}}-1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)\right\}. \tag{9.20}
\end{aligned}
$$

The expression above is the value of an FRA. The value of the swap is then simply the sum of the FRA values, i.e.,

$$
V = \sum_i V_i, \tag{9.21}
$$

where the value of the FRA i which occurs at time τ_i is given by Eq.(9.20), and the sum is over all FRAs in the swap.

We now show how to price a vanilla swap under the CIR [22] dynamic. This requires a slightly different approach. Typically for such a swap, the floating rate for one payment date is determined at the previous payment date, so that the payment and reset dates do not

coincide. To value a vanilla swap using the CIR [22] model, we must distinguish between the first FRA and subsequent FRAs. For the first payment, which we assume takes place at a future date τ_1 and which has a present value of V_1, the reset date has already occurred. Hence, we know the floating rate which will be used for the first payment. If we denote this rate by r_1, the cash flow at τ_1 will be $V_0 = \frac{r_f - r_1}{2}$, so that the value of the first FRA is simply that of a zero coupon bond with principal $\frac{r_f - r_1}{2}$ and time until maturity of τ_1, which is given by [67],

$$
\begin{aligned}
V_1 \;=\; & \frac{(r_f - r_1)\Gamma(4\tilde{\varrho})}{2^{2\tilde{\varrho}+1}\Gamma(2\tilde{\varrho})} \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2+2}} - \frac{1}{2} - \frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}\tau_1} - 1}\right)\tilde{r}\right] \\
& \times \left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}\tau_1} - 1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)^{-4\tilde{\varrho}} \exp\left[\left(2\sqrt{\tilde{\zeta}^2+2} - \tilde{\zeta}\tilde{\varrho}\right)\sigma\tau_1\right] \operatorname{csch}^{2\tilde{\varrho}}\frac{\sigma\sqrt{\tilde{\zeta}^2+2\tau_1}}{2} \\
& \times M\left[4\tilde{\varrho}, 2\tilde{\varrho}, \frac{\tilde{r}_1}{4}\left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}\tau_1} - 1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)^{-1}\operatorname{csch}^2\frac{\sigma\sqrt{\tilde{\zeta}^2+2\tau_1}}{2}\right]. \quad (9.22)
\end{aligned}
$$

For subsequent cash flows occuring at times τ_i, for $i > 1$, we will assume that the reset date occurs at the previous payment date, so that it occurs a time $\frac{1}{2}$ before the payment. The floating rate r_i for the payment at τ_i is fixed at this reset date, and the cash flow at the payment date will be $V_0 = \frac{r_f - r_i}{2}$. For these FRAs, we consider the fixed and floating legs separately. For the fixed leg, we know that the present value is once again that of a zero coupon bond, with time to expiration of τ_i and principal $\frac{r_f}{2}$, given by

$$
\begin{aligned}
V_i^{(a)} \;=\; & \frac{r_f\Gamma(4\tilde{\varrho})}{2^{2\tilde{\varrho}+1}\Gamma(2\tilde{\varrho})} \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2+2}} - \frac{1}{2} - \frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}\tau_i} - 1}\right)\tilde{r}\right] \\
& \times \left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}\tau_i} - 1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)^{-4\tilde{\varrho}} \exp\left[\left(2\sqrt{\tilde{\zeta}^2+2} - \tilde{\zeta}\tilde{\varrho}\right)\sigma\tau_i\right] \operatorname{csch}^{2\tilde{\varrho}}\frac{\sigma\sqrt{\tilde{\zeta}^2+2\tau_i}}{2} \\
& \times M\left[4\tilde{\varrho}, 2\tilde{\varrho}, \frac{\tilde{r}_1}{4}\left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}\tau_i} - 1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)^{-1}\operatorname{csch}^2\frac{\sigma\sqrt{\tilde{\zeta}^2+2\tau_i}}{2}\right]. \quad (9.23)
\end{aligned}
$$

A little more work is required for the floating leg at time τ_i. The value of this leg at the time of the reset date, rather than at the present time, is given by a zero coupon bond, this

time with principal $-\frac{r_i}{2}$ and time until expiry of $\frac{1}{2}$, which is

$$
\begin{aligned}
U_i(r_i) &= \tilde{U}_i(\tilde{r}_i) \\
&= -\frac{r_i \Gamma(4\tilde{\varrho})}{2^{2\tilde{\varrho}+1}\Gamma(2\tilde{\varrho})} \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2+2}} - \frac{1}{2} - \frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}/2}-1}\right)\tilde{r}_i\right] \\
&\quad \times \left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}/2}-1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)^{-4\tilde{\varrho}} \exp\left[\left(2\sqrt{\tilde{\zeta}^2+2} - \tilde{\zeta}\tilde{\varrho}\right)\frac{\sigma}{2}\right] \operatorname{csch}^{2\tilde{\varrho}}\frac{\sigma\sqrt{\tilde{\zeta}^2+2}}{4} \\
&\quad \times M\left[4\tilde{\varrho}, 2\tilde{\varrho}, \frac{\tilde{r}_i}{4}\left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}/2}-1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)^{-1} \operatorname{csch}^2\frac{\sigma\sqrt{\tilde{\zeta}^2+2}}{4}\right].
\end{aligned}
\tag{9.24}
$$

Next we find the expected value of the floating leg at the present time. We know that if the interest rate at a time $\tau_i - \frac{1}{2}$ in the future is r_i, then this leg has a value $U_i(r_i)$ at that time, but of course the interest rate r_i is unknown at the present time. To value the floating leg, we can again use the Green's function formula (9.16), this time with τ replaced by $\tau_i - \frac{1}{2}$ and V_0 by U_i given by Eq.(9.24), so that the present value of this leg is

$$
\begin{aligned}
V_i^{(b)} &= \frac{1}{2}\tilde{r}^{-\tilde{\varrho}+\frac{1}{2}} \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2+2}} - \frac{1}{2} - \frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}(\tau_i-\frac{1}{2})}-1}\right)\tilde{r}\right] \\
&\quad \times \exp\left[\left(2\sqrt{\tilde{\zeta}^2+2} - \tilde{\zeta}\tilde{\varrho}\right)\sigma\left(\tau_i - \frac{1}{2}\right)\right] \operatorname{csch}\frac{\sigma\sqrt{\tilde{\zeta}^2+2}\left(\tau_i - \frac{1}{2}\right)}{2} \\
&\quad \times \int_0^\infty \tilde{r}'^{3\tilde{\varrho}-\frac{1}{2}} \exp\left[-\left(\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}(\tau_i-\frac{1}{2})}-1} + \frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}\right)\tilde{r}'\right] \\
&\quad \times I_{2\tilde{\varrho}-1}\left[\frac{\sqrt{\tilde{r}\tilde{r}'}}{\sinh\sigma\sqrt{\tilde{\zeta}^2+2}\frac{\tau_i-\frac{1}{2}}{2}}\right] \tilde{U}_i(\tilde{r}')d\tilde{r}',
\end{aligned}
\tag{9.25}
$$

or

$$
\begin{aligned}
V_i^{(b)} &= -\frac{\sigma\Gamma(4\tilde{\varrho})}{2^{2\tilde{\varrho}+3}\Gamma(2\tilde{\varrho})\sqrt{\tilde{\zeta}^2+2}}\left(\frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}} + \frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}/2}-1}\right)^{-4\varrho} \\
&\quad \times \tilde{r}^{-\tilde{\varrho}+\frac{1}{2}} \exp\left[\left(\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2+2}} - \frac{1}{2} - \frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}(\tau_i-\frac{1}{2})}-1}\right)\tilde{r}\right]
\end{aligned}
$$

$$
\times \quad \exp\left[\left(2\sqrt{\tilde{\zeta}^2+2}-\tilde{\zeta}\tilde{\varrho}\right)\sigma\tau_i\right] \operatorname{csch}^{2\tilde{\varrho}} \frac{\sigma\sqrt{\tilde{\zeta}^2+2}}{4} \operatorname{csch} \frac{\sigma\sqrt{\tilde{\zeta}^2+2}(\tau_i-\frac{1}{2})}{2}
$$

$$
\times \quad \int_0^\infty \tilde{r}'^{3\tilde{\varrho}+\frac{1}{2}} I_{2\tilde{\varrho}-1}\left[\frac{\sqrt{\tilde{r}\tilde{r}'}}{\sinh\sigma\sqrt{\tilde{\zeta}^2+2}\frac{\tau_i-\frac{1}{2}}{2}}\right]
$$

$$
\times \quad \exp\left[-\left(\frac{1}{2}+\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}/2}-1}+\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}(\tau_i-\frac{1}{2})}-1}+\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2+2}}\right)\tilde{r}'\right]
$$

$$
\times \quad M\left[4\tilde{\varrho},2\tilde{\varrho},\frac{\tilde{r}'}{4}\left(\frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}+\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}/2}-1}\right)^{-1}\operatorname{csch}^2\frac{\sigma\sqrt{\tilde{\zeta}^2+2}}{4}\right]d\tilde{r}'. \qquad (9.26)
$$

The present value of the cash flow to be received at time τ_i is then the sum of the fixed and floating legs and is given by

$$
V_i^{(a)}+V_i^{(b)}, \qquad (9.27)
$$

where $V_i^{(a)}$ came from the fixed leg and $V_i^{(b)}$ came from the floating leg. The total V_i can be thought of as the value of an FRA. As with the LIBOR-in-arrears swap, the value of the vanilla swap is now simply the sum of the present values of the future cash flows, namely

$$
V = \sum_i V_i, \qquad (9.28)
$$

where the present value of cash flow i occuring at time τ_i is given by Eq.(9.22) for the first cash flow at τ_1 and by Eq.(9.27) for the subsequent cash flows at $\tau_2, \tau_3, \ldots$ and the sum is over all future cash flows in the swap.

9.2 Computing the value of the swaps numerically

Using Monte Carlo simulation methods, we now perform numerical simulations with the swap pricing models. In the simulations we use five thousand sample paths in every Monte Carlo calculation to compute the values of the swaps for different scenarios. In particular, we study the impact of the fixed interest rate r_f on the value of both the LIBOR-in-arrears

swap and the vanilla swap by considering different values of r_f.

In what follows, we assume that the Basel III compliant commercial bank which we modelled in the previous banking problems enters into a LIBOR-in-arrears interest rate swap with another bank, and a vanilla swap with a company. The bank and the company pay fixed interest rates to our bank, and in return receive floating interest rates. Our bank is the receiver in this instance, as it receives fixed rates (from the bank and company) and pays floating rates in return. Recall that the price of a swap from the viewpoint of the payer is the negative value of that for the receiver. We will present both of these values in the simulations.

The computations presented are based on the simulation parameters

$\varrho = 0.08$, $\theta = 0.45$, $\zeta = 0.25$ and $\sigma = 0.10$.
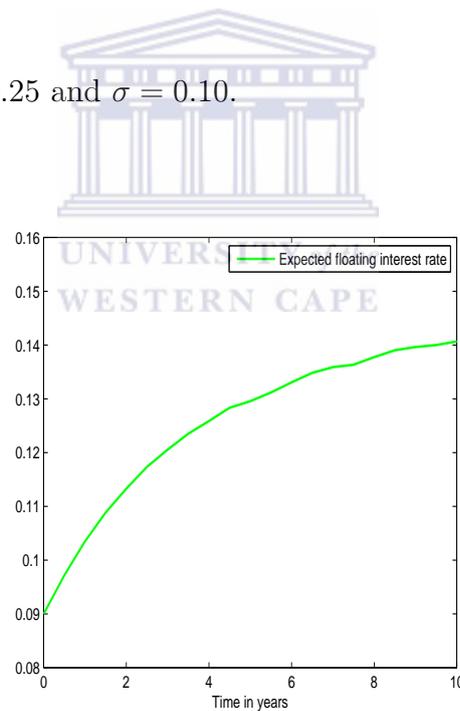
Figure 9.1: A simulation of the expected floating interest rate r for $\varrho = 0.08$, $\theta = 0.45$, $\zeta = 0.25$ and $\sigma = 0.10$.

We first compute the value of the LIBOR-in-arrears swap. We consider the following values

for the fixed interest rate r_f in the simulation: 0.07, 0.08, 0.09, 0.10 and 0.11. The initial value of the floating interest rate $r(0)$ will remain constant at 0.09 in all the calculations. The length or duration of the swap is assumed to be 10 years.

For a drop in the value of the fixed interest rate r_f, Table 9.1 reports an increase in the price of the LIBOR-in-arrears swap from the viewpoint of the payer. From the viewpoint of the receiver on the other hand, Table 9.1 reports a decrease in the value of the swap with a drop in the fixed interest rate r_f.

Table 9.1: A comparison of the price of the LIBOR-in-arrears swap from the viewpoints of the payer and receiver under decreasing values of the fixed interest rate r_f.

$r(0)$	r_f	V_{payer}	V_{receiver}
0.09	0.09	9.906633286E+009	-9.906633286E+009
	0.08	9.906633289E+009	-9.906633289E+009
	0.07	9.906633291E+009	-9.906633291E+009

For a rise in the value of the fixed interest rate r_f, Table 9.2 reports a fall in the price of the LIBOR-in-arrears swap from the viewpoint of the payer. From the viewpoint of the receiver on the other hand, Table 9.2 reports an increase in the value of the swap with a rise in r_f.

Table 9.2: A comparison of the price of the LIBOR-in-arrears swap from the viewpoints of the payer and receiver under increasing values for the fixed interest rate r_f.

$r(0)$	r_f	V_{payer}	V_{receiver}
0.09	0.09	9.906633286E+009	-9.906633286E+009
	0.10	9.906633284E+009	-9.906633284E+009
	0.11	9.906633281E+009	-9.906633281E+009

104

We now proceed to compute the value of the vanilla swap for different scenarios. In the computations we consider fixed interest rate (r_f) values of 0.07, 0.08, 0.09, 0.10, 0.11, as well as a fixed initial floating rate of $r(0) = 0.09$. The duration of the swap is also assumed to be 10 years. In this simulation we assume that the first payment takes place at time τ_1, which is one year after our bank enters into the swap with the company. Thus the floating interest rate r_1 for the first payment will be the value of the floating rate r at time $1/2$.

Our results reveal the following. For a fall in the value of the fixed interest rate r_f, Table 9.3 reports a rise in the price of the vanilla swap from the viewpoint of the payer. From the viewpoint of the receiver on the other hand, Table 9.3 reports a fall in the value of the vanilla swap with a fall in the value of r_f.

Table 9.3: A comparison of the price of the vanilla swap from the viewpoints of the payer and receiver under decreasing values for the fixed interest rate r_f.
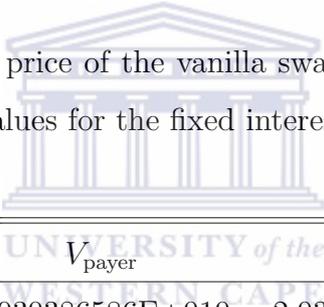
$r(0)$	r_f	V_{payer}	V_{receiver}
0.09	0.09	2.030386586E+010	-2.030386586E+010
	0.08	2.030386701E+010	-2.030386701E+010
	0.07	2.030386817E+010	-2.030386817E+010

For a rise in the value of the fixed interest rate r_f, Table 9.4 reports a drop in the price of the vanilla swap from the viewpoint of the payer. From the viewpoint of the receiver on the other hand, Table 9.4 reports a rise in the value of the vanilla swap with a rise in r_f.

Table 9.4: A comparison of the price of the vanilla swap from the viewpoints of the payer and receiver under increasing values for the fixed interest rate r_f.

$r(0)$	r_f	V_{payer}	V_{receiver}
0.09	0.09	2.030386586E+010	-2.030386586E+010
	0.10	2.030386470E+010	-2.030386470E+010
	0.11	2.030386354E+010	-2.030386354E+010

We illustrate graphically the expected cash flows of the LIBOR-in-arrears swap for fixed interest rate r_f values of 0.07, 0.08, 0.09, 0.10 and 0.11 in Figure 9.2. We also present the graphs of the expected cash flows of the vanilla swap for these r_f values in Figure 9.3.
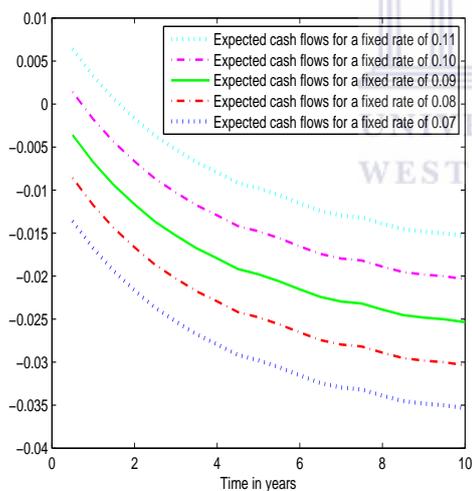


Figure 9.2: A simulation of the expected cash flows of the LIBOR-in-arrears swap for fixed interest rate r_f values of 0.11, 0.10, 0.09, 0.08 and 0.07.
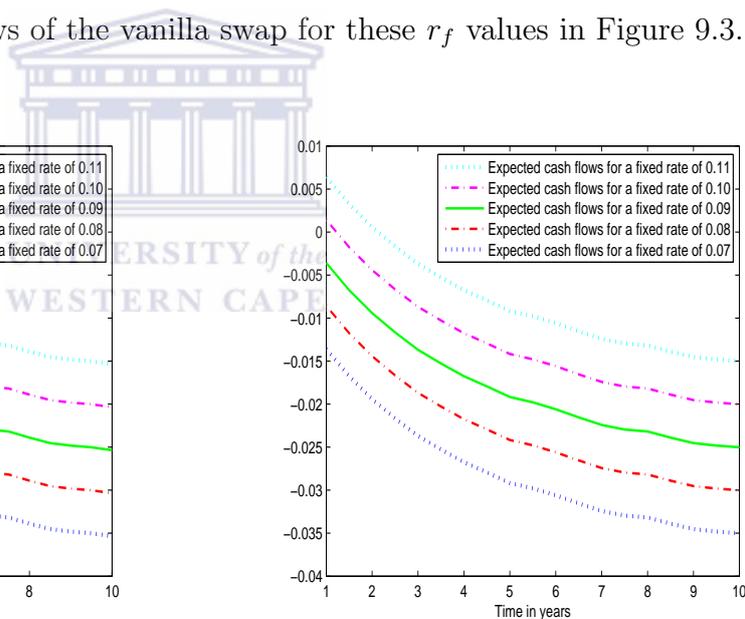
Figure 9.3: A simulation of the expected cash flows of the vanilla swap for fixed interest rate r_f values of 0.11, 0.10, 0.09, 0.08 and 0.07.

In order to perform five thousand iterations of the vanilla swap pricing method, with the

106

computing power available to us, we resort to approximating the value of the integral

$$\int_0^\infty \tilde{r}'^{3\tilde{\varrho}+\frac{1}{2}} I_{2\tilde{\varrho}-1}\left[\frac{\sqrt{\tilde{r}\tilde{r}'}}{\sinh\sigma\sqrt{\tilde{\zeta}^2+2}\frac{\tau_i-\frac{1}{2}}{2}}\right]$$

$$\times\quad \exp\left[-\left(\frac{1}{2}+\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}/2}-1}+\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}(\tau_i-\frac{1}{2})}-1}+\frac{\tilde{\zeta}}{2\sqrt{\tilde{\zeta}^2+2}}\right)\tilde{r}'\right]$$

$$\times\quad M\left[4\tilde{\varrho},2\tilde{\varrho},\frac{\tilde{r}'}{4}\left(\frac{\tilde{\zeta}}{\sqrt{\tilde{\zeta}^2+2}}+\frac{1}{e^{\sigma\sqrt{\tilde{\zeta}^2+2}/2}-1}\right)^{-1}\operatorname{csch}^2\frac{\sigma\sqrt{\tilde{\zeta}^2+2}}{4}\right]d\tilde{r}'\qquad (9.29)$$

appearing in the formula for the value of the vanilla swap with the trapezoidal rule, where the upper limit of integration has a value of 10. Alternatively our resources allow us to use a bigger upper limit of integration of 100. However, this means using fewer iterations of the Monte Carlo simulation method. For two thousand iterations of the Monte Carlo method, with an upper limit of integration of 100, we present the results pertaining to the vanilla swap below.

Table 9.5: Another comparison of the price of the vanilla swap from the viewpoints of the payer and receiver under decreasing values for the fixed interest rate r_f.

$r(0)$	r_f	V_{payer}	V_{receiver}
0.09	0.09	6.313511169E+007	-6.313511169E+007
	0.08	8.261722896E+007	-8.261722896E+007
	0.07	1.020993462E+008	-1.020993462E+008

We note that for the improved approximation of the integral (9.29) the value of the vanilla swap still rises under decreasing r_f values from the viewpoint of the payer, while it decreases from the viewpoint of the receiver. Under increasing r_f values we also note that the value of the vanilla swap still drops from the point of view of the payer, while from the viewpoint of the receiver it rises.

107

Table 9.6: Another comparison of the price of the vanilla swap from the viewpoints of the payer and receiver under increasing values for the fixed interest rate r_f.

$r(0)$	r_f	V_{payer}	V_{receiver}
0.09	0.09	6.313511169E+007	-6.313511169E+007
	0.10	4.365299443E+007	-4.365299443E+007
	0.11	2.417087716E+007	-2.417087716E+007

At the present time the computations presented in the simulations above were obtained by exhausting the computing power available to us. The results, in principle, could be refined further by applying more iterations on more powerful computing machines and by also considering higher values of the upper limit of integration in the integral (9.29).
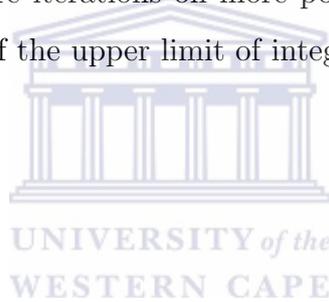
Chapter 10

Conclusion

This thesis presents a study of a range of related commercial banking problems in discrete and continuous time settings. Firstly, in a continuous time setting, we study an optimization problem that involves deriving a capital allocation strategy that maximizes the expected logarithmic utility of the future value of a Basel III compliant commercial bank's TNRWAs. The bank is assumed to invest its Total capital in a stochastic interest rate financial market consisting of three assets, viz., a treasury, a marketable security and a loan. The loan dynamic is assumed to be described by a jump-diffusion process. Generally analytical solutions to stochastic optimal control problems in the jump setting are not easily obtainable. We propose an approximation method that exploits a similarity between the forms of the control problems of the jump-diffusion model and the diffusion model obtained by removing the jump. With the jump assumed sufficiently small, the approximation method replaces the jump-diffusion model with a diffusion model and solves the resulting control problem analytically. The analytical solution then serves as a proxy to the solution of the control problem with the jump. We study a second banking problem, which is also set in continuous time. In this problem we derive SDEs for the bank's capital adequacy ratios which incorporate the proxy to the solution of the jump control problem. Using numerical simulations, we monitor the performance of the capital adequacy ratios under the proxy. The third problem is also in continuous time. Here we derive models for the bank's liquidity ratios in

terms of the proxy, and we simulate and observe the behaviour of the liquidity ratios under the proxy numerically. The fourth problem of this study is set in a discrete time setting. In this particular problem we derive a multi-period deposit insurance pricing model which incorporates the proxy, the BCBS' latest capital standard, capital forbearance and moral hazard. The deposit insurance pricing method utilizes an asset value reset rule comparable to the typical practice of insolvency resolution by insuring agencies. We perform numerical analyses with our model to study the effect of the Basel III capital standard, capital forbearance and moral hazard behaviour on the model's fairly-priced premium rate under different coverage horizons and initial leverage (asset-to-debt) levels. In the final problem, which is set in continuous time, we consider fixed-for-floating interest rate swaps under the CIR [22] model. We show how analytical expressions for the values of both a LIBOR-in-arrears swap and a vanilla swap can be derived using a Green's function approach. We present numerical studies where we employ Monte Carlo simulation methods to compute the value of the swaps for different scenarios.
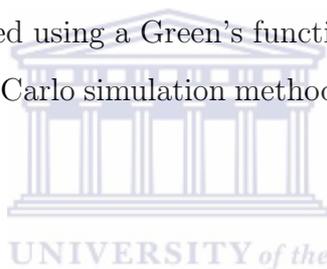
We now summarize the main findings of our study. Under the proxy, which is to diversify the bank's TNRWAs away from the marketable security and loan and towards the treasury, the bank maintains its CAR and Tier 1 Ratio, as well as both of the liquidity ratios well above their Basel III prescribed minimum values. By Basel III standards the bank is considered to be strongly capitalized and guaranteed the ability to sustain unexpected losses since both the CAR and Tier 1 Ratio prevail above their respective minimum prescribed levels. Since the bank also maintains its LCR well above the Basel III-prescribed minimum level, the bank holds enough high quality liquid assets to withstand short term stress periods over the duration of the investment period. Since the bank meets the minimum NSFR requirement, it is classified as able to withstand medium to long term stress periods as it has adequate funding to support its investment practices. However, the value of the Leverage Ratio falls below its minimum predescribed level. This can be remedied by maintaining higher levels of Tier 1 capital, which can be achieved if the rate at which Tier 1 capital is raised is increased.

110

To this end, we consider the approximate optimal capital allocation strategy subject to a constant Leverage Ratio at the minimum prescribed level. We derive a formula for the bank's TNRWAs at constant (minimum) Leverage Ratio value and present numerical simulations based on the modified TNRWAs formula. This TNRWAs formula ensures that the value of the bank's Leverage Ratio always meets the Basel III minimum requirement. To construct such a TNRWAs formula, the Tier 1 capital model is also modified. In fact, the modified Tier 1 capital model follows an SDE with a jump. We further note that the levels of the jump model's capital adequacy ratios and the NSFR are improved over that of the diffusion model for the set of simulation parameters considered in the thesis, and many others for which the simulations are not presented here. Introducing a jump into a banking model thus seems like a viable method for improving these ratios.

The deposit insurance pricing method reveals the following behaviour. When the capital standard is strictly enforced by the regulatory authority and we fix the level of the initial leverage (asset-to-debt), an increase in the coverage horizon causes the fairly-priced premium rate to rise. By keeping the coverage horizon fixed and decreasing the level of the initial leverage (asset-to-debt), the value of the fairly-priced premium rate drops. For the scenario in which the bank faces a looser capital standard, we observe the same behaviour as when the capital standard is strictly enforced.

The swap pricing methods analyzed in the final problem of this thesis behave as follow. For a drop in the value of the fixed interest rate our simulations report a rise in the value of the LIBOR-in-arrears swap, as well as for the value of vanilla swap, from the viewpoint of the payer. From the viewpoint of the receiver on the other hand, the simulations report a drop in the values of the LIBOR-in-arrears and vanilla swaps with a drop in the fixed interest rate. For a rise in the value of the fixed interest rate, we observe a drop in the values of both swaps from the viewpoint of the payer. From the viewpoint of the receiver on the other hand, we note a rise in the values of the swaps with a rise in the fixed interest rate.

111

In this thesis we rely on the simulated data approach in order to model Basel III's capital adequacy and liquidity ratios together with a multi-period deposit insurance pricing mehod. We derive the aforementioned models based on an approximate capital allocation strategy for optimizing an expected logarithmic utility of a future value of the bank's TNRWAs in a jump market. This provides a means for us to monitor the bahaviour of these models under the approximate capital allocation strategy. Other avenues related to our topic worth exploring include optimizing the capital adequacy and liquidity ratios themselves, as well as going beyond the simulated data approach by modelling the optimal capital adequacy and liquidity ratios using real data sourced from e.g., the US Federal Deposit Insurance Corporation (FDIC). Both such possibilities are currently being explored independently from this study. In addition, we are pursuing a study in which we aim to address the deposit insurance pricing issue with explicit consideration of bankruptcy costs and closure policies, similar to what was done in the research article Hwang et al [49]. We are considering extending the analysis of [49] to jump markets and then calculating the deposit insurance price numerically via Monte Carlo simulation techniques.

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Research articles emanating from this study

The following articles have already been published:

- P.J. Witbooi, G.J. van Schalkwyk, and G.E. Muller, "An optimal investment strategy in bank management". *Mathematical Methods in the Applied Sciences*, vol. 34, no. 13 , pp. 1606-1617, 2011.

- G.E. Muller and P.J. Witbooi, "An optimal portfolio and capital management strategy for Basel III compliant commercial banks", *Journal of Applied Mathematics*, vol. 2014, Article ID 723873, 11 pages, 2014. doi:10.1155/2014/723873

I am working on preparing further material from the thesis for publication. The following one is near completion:

G.E. Muller and P.J. Witbooi, "Optimal investment and capital adequacy management of Basel III compliant banks in Lévy markets".