

Stochastic Modelling in Bank Management and Optimization of Bank Asset Allocation

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Abstract

The Basel Committee published its proposals for a revised capital adequacy framework (the Basel II Capital Accord) in June 2006. One of the main objectives of this framework is to improve the incentives for state-of-the-art risk management in banking, especially in the area of credit risk in view of Basel II. The new regulation seeks to provide incentives for greater awareness of differences in risk through more risk-sensitive minimum capital requirements based on numerical formulas. This attempt to control bank behaviour has a heavy reliance on regulatory ratios like the risk-based capital adequacy ratio (CAR). In essence, such ratios compare the capital that a bank holds to the level of credit, market and operational risk that it bears. Due to this fact the objectives in this dissertation are as follows. Firstly, in an attempt to address these problems and under assumptions about retained earnings, loan-loss reserves, the market and shareholder-bank owner relationships, we construct continuous-time models of the risk-based CAR which is computed from credit and market risk-weighted assets (RWAs) and bank regulatory capital (BRC) in a stochastic setting. Secondly, we demonstrate how the CAR can be optimized in terms of equity allocation. Here, we employ dynamic programming for stochastic optimization, to obtain and verify the results. Thirdly, an important feature of this study is that we apply the mean-variance approach to obtain an optimal strategy that diversifies a portfolio consisting of three assets. In particular, chapter 5 is an original piece of work by the author of this dissertation where we demonstrate how to employ a mean-variance optimization approach to equity allocation under certain conditions.

Key words: Bank management, Stochastic optimization, Optimal asset allocation, Amor-

tizations, Capital adequacy ratio, Bank regulatory capital, Stochastic banking model, Mean-Variance approach, Credit and market risk-weighted assets.



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Furthermore, I am grateful to the National Research Foundation (NRF) for providing me with financial support.

Declaration

I declare that *Stochastic modelling in Bank Management and Optimization of Bank Asset Allocation* is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Garth Van Schalkwyk

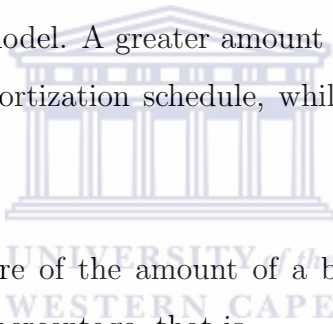


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Key Definitions

Amortization is the distribution of a single lump-sum cash flow into many smaller cash flow installments, as determined by an amortization schedule. Unlike other repayment models, each repayment installment consists of both principal and interest. Amortization is chiefly used in loan repayments (a common example being a mortgage loan) and in sinking funds. Payments are divided into equal amounts for the duration of the loan, making it the simplest repayment model. A greater amount of the payment is applied to interest at the beginning of the amortization schedule, while more money is applied to principal at the end.



Capital Adequacy Ratio is a measure of the amount of a bank's capital relative to its risk weighted assets expressed as a percentage, that is,

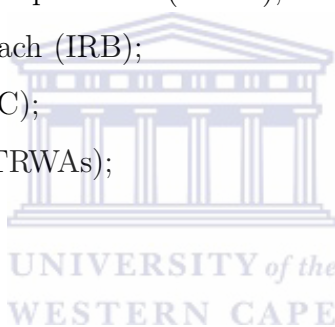
$$\text{CAR} = \frac{\text{Indicator of Absolute Amount of Bank Capital}}{\text{Indicator of Absolute Level of Bank Risk}}.$$

Credit risk is defined as the potential of a bank borrower or counter party failing to meet its obligations in accordance with agreed terms.

Market risk as the risk of losses in on- and off-balance sheet positions arising from movements in market prices.

Index of Abbreviations

Adjustable Rate Mortgages (ARMs);
Demand Deposits (DD);
Negotiable Order of Withdrawal Accounts (NOWA);
Money Market Deposit Accounts (MMDAs);
Basel Committee on Banking Supervision (BCBS);
Internal Ratings Based Approach (IRB);
Bank Regulatory Capital (BRC);
Total Risk-Weighted Assets (TRWAs);
Project Finance (PF);
Object Finance (OF);
Commodities Finance (CF);
Income Producing Real Estate (IPRE);
Specialized Lending High Volatility Commercial Real Estate (SLHVCRE);
Specialized Lending Not Including High Volatility Commercially Real Estate (SLNI-
HVCRE);
Bank Exposure (BE);
Sovereign Exposure (SE);
Retail Residential Mortgage (RRM);
Home Equity Line of Credit (HELOC) ;
Other Retail Exposure (ORE);
Qualifying Revolving Retail Exposure (QRRE) ;
Small to Medium Size Enterprises with Corporate Treatment (SMECT);



Small to Medium Size Enterprises with Retail Treatment (SMERT);
Equity Exposure Not Held in the Trading Book (EENHTB);
Probability of Default (PD);
Loss Given Default (LGD);
Exposure At Default (EAD);
Effective Maturity (EM);
Expected Losses (EL);
Unexpected Losses (UL);
Risk-Weighted Function (RWF);
Value-at-Risk (VaR);
Federal Deposit Insurance Corporation (FDIC);
Federal Deposit Insurance Corporation Improvement Act (FDICIA);



Index of symbols

Bank Reserves at time t - $R(t)$;

Bank Loans at time t - $L(t)$;

Bank Securities at time t - $S(t)$;

Marketable securities at time t - $M(t)$;

Treasury securities at time t - $T(t)$;

Bank Deposits at time t - $D(t)$;

Bank Borrowing at time t - $B(t)$;

Bank Regulatory Capital at time t - $C(t)$;

Volatility matrix - $\Psi = (\sigma)_{i,j=1}^n$;

Brownian motion - $dX_0(t)$;

n -dimensional Brownian Motion - $dX(t)$;

Brownian Motion under the risk-neutral measure - $d\tilde{X}(t)$;

Market prices of risk - ζ' ;

Proportions of bank funds invested in the different risky assets at time t - $\pi'(t)$;

Cumulative distribution function for a standard normal random variable - $N(x)$;

Inverse cumulative function for a standard normal random variable - $G(z)$;

Short risk-free rate of interest at time t - $r_0(t)$;

Fixed short risk-free of interest - r ;

State of the economy - Ω ;

Real-world probability measure - \mathbb{P} ;

Risk-neutral probability measure - \mathbb{Q} ;

Natural filtration - $\{\mathcal{F}\}_{t \geq 0}$;

Expectation Operator under the real world probability measure - \mathbb{E} ;
 Expectation Operator under the risk-neutral measure - $\mathbb{E}_{\mathbb{Q}}$;
 Risk premium in the i -th asset - γ_i ;
 Tier 1 Capital at time t - $C_{T1}(t)$;
 Tier 2 Capital at time t - $C_{T2}(t)$;
 Tier 3 Capital at time t - $C_{T3}(t)$;
 Book value of bank stock at time t - $E(t)$;
 Retained earnings at time t - $E_r(t)$;
 Subordinate debt at time t - $S_D(t)$;
 Loan-loss reserves at time t - $R_L(t)$;
 Total risk-weighted assets at time t - $a(t)$;
 Amortization function at time t - $A(t, r(t))$;
 Loan repayment function at time t - $F(t, r(t))$;
 Interest function at time t - $I(t, r(t))$;
 Capital Adequacy Ratio at time t - $z(t)$;
 Capital Adequacy Ratio threshold process at time t - $z_p(t)$;
 Value function at time t - $V(t, r, z)$;
 Utility function at expiration date - $U(r(T), z(T))$;
 Price of a bond - $B(t, T)$;
 Reciprocal of Capital Adequacy Ratio at time t - $z^{-1}(t)$;
 Yield of investment at time t - $Y_A(t)$;

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Table (7.1) : Risk Categories, Risk-Weights and Representative Items.

Chapter 1

Introduction

Bank management mainly involves four operational concerns namely liquidity management, investment management, liability management and capital adequacy management. Liquidity management involves managing reserves to meet inflows and outflows and varying levels of loan commitments. These deposit flows are affected by interest rate movements that are relative to other financial instruments. Deposit flows are also affected by competitive rates determined by banks in their respective geographic markets. Two types of liquidity are available to meet potential liquidity requirements, that is, bank asset management and liability management. Bank asset management mainly involves achieving profit maximization through high return on loans and securities, reducing risk and providing for liquidity needs. In terms of meeting liquidity needs, banks use near-cash assets, including net funds sold to other banks and money market securities. Banks endeavour to grant loans to credit worthy entities that are willing to pay high interest rates and are unlikely to default on their loan contracts. Furthermore, banks are likely to purchase high return securities with low risk. Banks try to lower the risk associated with these securities by diversifying their investment portfolio.

Liability management supports lending activities and achieve balanced growth in earnings and bank assets without excessive liquidity risk. It involves accepting funding from depositors, and securing additional funds from other financial institutions, for use in lend-

ing and investing. Other tools of liability management are interest rate hedging against unexpected market moves, and maintaining a controlled gap between asset and liability maturities for controlled speculation on interest rate shifts.

The bank item that can be considered as the second largest asset on a bank's balance sheet is investments (securities held by the bank). Investment management mainly involves securities that are purchased by the bank to produce income in the form of interest paid, capital gains and it can also fulfill the role for liquidity needs. Investment securities are an alternative source of income during recession periods when the demand for commercial loans is relatively low. As the economic environment recovers and loan demands increase, these securities can be converted into loans or may be sold to fund higher-earning loans and other investment opportunities. Investment securities may be pledged as collateral on public deposits of federal, state and local governments borrowing from the federal reserve bank. The investment securities can be categorized into two types of securities namely government securities (these are treasury notes and treasury bonds purchased by the bank having maturities ranging from 1 – 5 years) and municipal securities (these are bonds issued by the state and local governments to finance various public works such as bridges, schools and roads). Purchasing municipal securities may be used to reduce income taxes. Moreover investment securities can increase the diversification of the bank's total asset portfolio or in certain cases take advantage of interest movements that can increase capital gains.

Capital adequacy management involves the decision of how much a bank should hold and how it should be accessed. From a shareholder's perspective, using less capital is one way to increase asset earnings and so earn higher return on equities. From the regulator's perspective, banks should increase their capital to ensure the safety and soundness in the case where earnings may become negative. Bank regulators are also concerned about financial risk that could increase the probability of bank failure. In the event where the variability of earnings after taxes increases, the interest and non-interest expenses may

exceed bank earnings and bank capital should absorb such potential losses. Although requiring a bank to maintain a higher capital level lowers the financial risk, such a requirement disrupts the efficiency and competitiveness of the banking system meaning that the aforementioned requirement acts as a constraint on the lending activities of a bank. It may also constrain the rate at which bank assets may be expanded. A more detailed discussion on these different management topics can be found in Fraser, Gup, Kolari [49] and Mishkin [71].

The bank is assumed to engage in unrestricted borrowing, short-selling and capitalization activities. The study of the dynamics of portfolio and capital structure (see the review papers Bhattacharya, Thakor [25]; Freixas, Rochet [50] and Santos [85]) has always been an important issue in risk management for banks. In this regard, Dangl, Lehar [36] and Decamps, Rochet, Roger [38] construct continuous-time models which permit optimal control problems to be solved in the context of capital requirements and portfolio selection. With regard to the former, the driving force behind bank capital stipulations is the risk shifting incentive due to the deposit insurance guarantee. Also, bank portfolio choice is important for a number of reasons. Firstly, it may contribute to an increase in the bank's charter value that directly benefits depositors (or providers of deposit insurance), shareholders and creditors. Also, it assists regulators in taking corrective action when confronted with related market information.

1.1 Main problems and Outline of the Dissertation

This study has connections with each of the areas of importance mentioned in the previous sections (that is sections (1.3.3), (1.3.4) and (1.3.5)). In this regard, a key assumption is that the underlying market is complete so that the complete set of possible gambles on future bank states can be constructed with existing assets. Also, we assume that every debtholder is a shareholder and vice versa with their philosophies being perfectly aligned with that of the bank owners. The main problems addressed in this dissertation can be

formulated as follows.

Problem 1.1.1 (Stochastic Dynamic Modelling of TRWAs and BRC): *How can we model the dynamics of Risk-Weighted Assets (RWAs) and Bank Regulatory Capital (BRC) stochastically ? (see Sections (2.1) and (2.2)).*

Problem 1.1.2 (Amortization Function): *Can we find an amortization function that provides an improved model for loan repayments by bank debtors ? (see Proposition 3.3.1 in Section 3.3).*

Problem 1.1.3 (Stochastic Dynamic Modelling of Risk-Based CARs): *Under Basel II, can we find a stochastic differential equation (SDE) for the dynamics of the risk-based Capital Adequacy Ratio (CAR) that takes the stochastic features of the BRC and RWAs into account ? (Theorem 4.1.1 in Section 4.1).*

Problem 1.1.4 (Capital adequacy ratio threshold process): *Can we find an explicit formula for the capital adequacy ratio threshold process $z_p(t)$? (see Theorem 4.2.1 in Section 4.2).*

Problem 1.1.5 (Optimal Bank Equity Allocation for Risk-Based CARs): *Under Basel II, can we find an optimal equity allocation strategy that will optimize a portfolio consisting of three assets via the dynamic programming algorithm for stochastic optimization? (Theorem 4.3.1 in Section 4.3.1).*

Problem 1.1.6 (An Optimal equity allocation strategy): *Under Basel II, can we find an optimal equity allocation strategy that will optimize a portfolio consisting of three assets under the mean-variance approach? (Proposition 5.1.4 in Chapter 5).*

The study is organized in the following manner. In chapter 2 we explore the asset-liability management of a commercial bank. In particular, we explore bank regulatory capital and total risk-weighted assets. In both cases we propose a continuous-time model for each of the aforementioned banking items. Chapter 3 discusses the importance of an alternative

form of amortization function that may describe how loan repayments by bank debtors can be improved. In obtaining the amortization function we first derive a partial differential equation that the amortization function must satisfy through a traditional approach and a martingale approach under the assumption that the interest rate is modelled as a diffusion process. Furthermore we decompose the amortization function into a loan repayment and interest function. Under this scenario we provide explicit formulas for a fixed loan, a series loan and an annuity loan, under the assumption that the interest rate is fixed. We also discuss and simulate bank loan-issuing rate. Chapter 4 discusses certain types of capital adequacy ratios (CARs), that is, core, equity, risk-based Tier 1 and total CAR. In this chapter we derive an explicit formula for the capital adequacy ratio of a commercial bank under the Basel II CAR paradigm and provide simulations over a certain period. We also discuss threshold processes and banking benchmarks (see for instance Mikkelsen-Petersen, Petersen [76]). Furthermore we explore an optimal asset allocation strategy for a commercial bank and provide a numerical example that illustrates key results. In particular, we make an optimal asset allocation decision (choice of how much of each asset to hold) where the weight in risky assets is equivalent to investing in a combination of bank assets consisting of cash, bond and equity funds. Chapter 5 discusses an optimal strategy in bank management where we derive explicit formulae associated with the capital adequacy ratio, bank capital and total risk-weighted assets respectively. In doing so we provide simulations for these banking items to capture its behaviour under certain assumptions. Furthermore we obtain an optimal strategy via a mean-variance approach that diversify a portfolio consisting of three assets. We point out that this chapter constitutes a new contribution. Chapter 6 discusses the main issues encountered in this study and point out shortcomings that need further investigation.

1.2 Preliminaries

In this section we introduce some basic elementary concepts and properties of probability and measure theory that is used throughout this study. We provide now definitions

and concepts relevant to brownian motion (see for instance Bhattacharya, Waymire [24], Etheridge [42] and Øksendal [80]).

Definition 1.2.1 (see Ash [5] or Cohn [34])

Let \mathcal{F} be a collection of subsets of a set Ω . Then \mathcal{F} is called a *field* (algebra) if and only if $\Omega \in \mathcal{F}$ and \mathcal{F} is closed under complementation and finite union, that is,

1. $\Omega \in \mathcal{F}$.
2. For a set A , if $A \in \mathcal{F}$, then also $A^c \in \mathcal{F}$ (A^c is the complement of A).
3. if $A_1, A_2, A_3, \dots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^n A_i \in \mathcal{F}$.

Remark: It follows that \mathcal{F} is closed under finite intersection. For if $A_1, A_2, A_3, \dots, A_n \in \mathcal{F}$, then

$$\bigcap_{i=1}^n A_i = \left(\bigcup_{i=1}^n A_i^c \right)^c \in \mathcal{F}.$$

Definition 1.2.2 (see Ash [5] or Cohn [34])

Let Ω be an arbitrary set and let \mathcal{F} be a collection of subsets of a set Ω . Then \mathcal{F} is called a σ -field (σ -algebra) if and only if \mathcal{F} is a field and \mathcal{F} is closed under countable intersection.

For a further discussion on infinite sequences, algebras and σ -algebra we refer the reader to Ash [5] or Cohn [34].

Definition 1.2.3 (see for instance Grimmett, Stirzaker [53])

A probability measure \mathbb{P} on (Ω, \mathcal{F}) is a function $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ satisfying

1. $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$;
2. If A_1, A_2, A_3, \dots is a collection of disjoint members of \mathcal{F} , so that $A_i \cap A_j = \emptyset$ for all pairs i, j satisfying $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The following definition is fundamental since it explains the idea behind a stochastic process.

Definition 1.2.4 (see Bhattacharya, Waymire [24]):

Given an indexed set I , a stochastic process indexed by I is a collection of random variables $\{B_\lambda : \lambda \in I\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a set S . The set S is called the state space of the process.

For a more detailed description of stochastic processes the reader is referred to Bhattacharya, Waymire [24].

Definition 1.2.5 (see Etheridge [42])

A real-valued stochastic process $\{X(t)\}_{t \geq 0}$ is a \mathbb{P} -Brownian motion if for some real constant σ , under \mathbb{P} ,

1. for each $s \geq 0$ and $t > 0$ the random variable $X(t + s) - X(s)$ has the normal distribution with mean zero and variance $\sigma^2 t$,
2. for each $n \geq 1$ and any times $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$, the random variables $\{X(t_r) - X(t_{r-1})\}$ are independent,
3. $X(0) = 0$,
4. $X(t)$ is continuous in $t \geq 0$.

Consider an n -dimensional process $X(t) = (X_1(t), X_2(t), \dots, X_n(t))'$. If each of the $X_i(t)$ is a standard one-dimensional brownian motion and if each $X_i(t)$ are independent of each other, then $X(t)$ is said to be standard n -dimensional brownian motion (see Cairns [31]).

Definition 1.2.6 (see Hunt, Kennedy [56])

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}$ be a filtration of \mathcal{F} . A stochastic process $M(t)$ is a $\{\mathcal{F}_t\}_{t \geq 0}$ -martingale (or just a martingale when the filtration is clear) if:

1. $M(t)$ is adapted to $\{\mathcal{F}(t)\}$ (that is, for every $t > 0$, $M(t)$ is $\{\mathcal{F}(t)\}$ -measurable);

2. $\mathbb{E}[|M(t)|] < \infty$ for all $t \geq 0$;

3. the conditional expectation $\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s)$ almost surely for all $s \in [0, t]$.

Theorem 1.2.7 (The Integrations by parts (stochastic product rule)): (see Etheridge [42])

If $K(t) = M^K(t) + A^K(t)$ and $P(t) = M^P(t) + A^P(t)$ where $\{M^K(t)\}$ and $\{M^P(t)\}$ are continuous $(\mathbb{P}, \{\mathcal{F}\}_{t \geq 0})$ -martingales and $A^K(t)$ and $A^P(t)$ are continuous processes of bounded variation, then

$$d(K(t)P(t)) = K(t)dP(t) + P(t)dK(t) + d[M^K(t), M^P(t)]. \quad (1.1)$$

Theorem 1.2.8 (The Feynman-Kac stochastic representation): (see for instance Cairns [31] and Etheridge [42])

Assume that the function F solves the boundary value problem

$$\begin{aligned} \frac{\partial F}{\partial t}(t, x) + \mu(t, x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2}(t, x) &= 0, \quad 0 \leq t \leq T, \\ F(t, x) &= \Phi(x). \end{aligned} \quad (1.2)$$

Define $\{H(t)\}_{t \leq 0 \leq T}$ to be the solution of the stochastic differential equation

$$dH(t) = \mu(t, H(t)) + \sigma(t, H(t))dX(t), \quad 0 \leq t \leq T,$$

where $\{X(t)\}_{t \geq 0}$ is a standard Brownian motion under the measure \mathbb{P} . If

$$\int_0^T \mathbb{E} \left[\left(\mu(t, H(t)) \frac{\partial F}{\partial x}(t, H(t)) \right)^2 \right] ds < \infty,$$

then $F(t, x) = \mathbb{E}^{\mathbb{P}}[\Phi(H(T)) | H(t) = x]$.

Theorem 1.2.9 (The n -dimensional Itô formula): (see Etheridge [42])

Let $\{J(t)\}_{t \geq 0} = \{J^1(t), J^2(t), \dots, J^{1n}(t)\}_{t \geq 0}$ solve

$$dJ^i(t) = \mu_i(t) + \sum_{j=1}^n \sigma_{ij}(t) dX^j(t), \quad i = 1, 2, \dots, n,$$

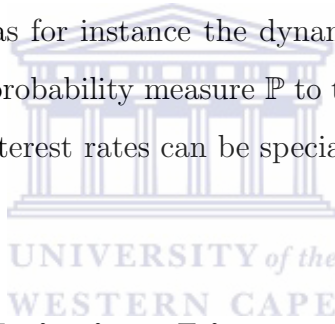
where $\{X^j(t)\}_{t \geq 0}$ are independent \mathbb{P} -Brownian motions. Further suppose that the real-valued functions $f(t, x)$ on $\mathbb{R}_+ \times \mathbb{R}^n$ are continuously differentiable with respect to t and twice continuously differentiable in the x -variables. Then defining $Q(t) = f(t, J(t))$ we have

$$dQ(t) = \frac{\partial f}{\partial t}(t, J(t)) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, J(t)) dJ^i(t) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t, J(t)) C_{ij}(t) dt \quad (1.3)$$

where $C_{ij}(t) = \sum_{k=1}^n \sigma_{ik}(t) \sigma_{jk}(t)$.

For detailed descriptions on brownian motion and Itô integrals and its properties, *Martingale Representation Theorem*, *Girsanov Theorem* and *Radon-Nikodym* derivatives, we refer the reader to Etheridge [42] and Øksendal [80].

The introduction of a new measure \mathbb{Q} provides a useful computational tool to determine an alternative equation such as for instance the dynamics of the stochastic risk-free interest rate. Moving from the probability measure \mathbb{P} to the new measure \mathbb{Q} enables one to illustrate how results about interest rates can be specialized to real-world scenarios.



1.3 Relation to Existing Literature

In this section we consider the connection between this study and previous banking literature.

1.3.1 A Discussion and Brief Literature Review about Stochastic Banking models

Bank Securities

Treasury securities or *treasuries* are bonds issued by national treasuries in most countries as a means of borrowing money to meet government expenditures not covered by tax revenues. As a result, they are the debt finance instruments of the federal government. Also,

they act as an index that is used to establish interest rates for adjustable rate mortgages (ARMs). On the other hand, *marketable securities* are stocks and bonds that can easily and quickly be converted into cash. A marketable security has a readily determined fair market value and will generally have highly liquid markets allowing the security to be sold at a reasonable price. The banking institution comprises of treasury securities (illiquid assets) as well as marketable securities (liquid assets). Marketable securities are used to combat expected and unexpected fluctuations on the bank's balance sheet. Commercial banks also hold certain amounts to protect against the large volatile transaction deposits. The need to hold large amounts of marketable securities may be reduced by means of growth and sustainability of financial markets and the diversity of financial derivative products such as forwards, options and futures contracts which enhances the flexibility in bank liquidity management. In certain countries such as Japan, Germany and United States of America where the banking environment and financial markets are well developed, banks have been forced (obligated) to purchase government bonds with the purpose to meet deposit demands. Van Greuning, Brajovic Bratanovic [90] states that the main purpose of such asset requirements is to allow the flow of finance to customers (recipients) in a predictable manner. The following paragraph discusses the financial market in which the commercial bank operates.

We allow a commercial bank to invest in a financial market with $(n + 1)$ assets (that is a market with n risky assets and 1 riskless asset). One of these assets is riskless (representing the treasuries with a return rate $r(t)$) while the assets $1, 2, \dots, n$ are risky (representing the market shares). In the paper of Mukkudem-Petersen, Petersen [73] (see also Fouche, Mukkudem-Petersen, Petersen [46]) the dynamics of the riskless asset (denoted by $P_0(t)$) and risky assets ($P_i(t)$) are represented by stochastic differential equations. The dynamics of the riskless asset are represented by:

$$dP_0(t) = P_0(t) \left[r_0 dt + \sigma_0 dX_0(t) \right], \quad P_0(0) = 1. \quad (1.4)$$

Considering the case where the short risk-free rate of interest $r_0(t) > 0$ is constant, we assume that the volatility parameter $\sigma_0 = 0$. The dynamics of the riskless asset (1.4) reduces to

$$dP_0(t) = P_0(t)r_0 dt, \quad P_0(0) = 1$$

and the value of the monetary units in the bank account at time t is given by (see for instance, Korn [62])

$$P_0(t) = \exp \left\{ \int_0^t r_0(s) ds \right\}.$$

The evolution of the risky assets follow a geometric brownian motion and is given by (see for instance, Korn [62]):

$$\begin{aligned} dP_i(t) &= P_i(t) \left[b_i dt + \sum_{j=1}^n \sigma_{ij} dX_j(t) \right], \quad P_i(0) = P_0, \quad 1 \leq i \leq n, \\ P_i(t) &= P_0 \exp \left(\int_0^t \left(b_i - \frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2 \right) ds + \sum_{j=1}^m \int_0^t \sigma_{ij}(s) dX_j(s) \right), \end{aligned} \quad (1.5)$$

where b_i and σ_{ij} , $1 \leq i, j \leq n$ are considered as positive constants and the vector

$$(X_0(t), X_1(t), X_2(t), \dots, X_n(t))'$$

is an $(n+1)$ -dimensional brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The completion of the filtration $\{\mathcal{F}\}_{t \geq 0}$ is defined by

$$\sigma\{(X_0(t), X_1(t), X_2(t), \dots, X_n(t))' : 0 \leq s \leq t\}.$$

The coefficient b_i is the mean rate of return of the i -th risky asset and $\sigma_{ij} \geq 0$ represents the covariance between asset i and asset j for all $i, j = 1, 2, \dots, n$. The explicit representation of the risky assets $P_i(t)$ is obtained from *Itô's formula* (1.3).

The loan-issuing rate, $l(t)$ (described in section (3.4)), is conditioned on the increase in the returns on securities. Furthermore we assume there exist a correlation $-1 \leq p_i \leq 1$ between the Brownian motions X_l and X_i for $i = 1, \dots, n$. This implies that

$$\mathbb{E} \left[X_l(t), X_i(s) \right] = p_i \min(t, s)$$

for $i = 1, \dots, n$ and

$$X_i(t) = \sqrt{1 - \tilde{p}'\tilde{p}}X_0(t) + \tilde{p}'\tilde{X}(t),$$

where \tilde{p} is defined as

$$\tilde{p} = (p_1(t), \dots, p_n(t))'$$

and

$$\tilde{X}(t) = (X_1(t), \dots, X_n(t))'.$$

$\tilde{p}'\tilde{p} \neq 1$ implies that the risk in loan issuing cannot be eliminated by trading in the financial market (see Mukkudem-Petersen, Petersen [73]).

The market price of risk, $\tilde{\zeta}$, is defined as the expected excess return, or risk premium, that investors (shareholders) are prepared to absorb due to the investment in risky assets.

The market price of risk (also known as the Sharp Ratio) is expressed as



$$\begin{aligned}\tilde{\zeta} &= \frac{\tilde{b} - r_0\bar{1}}{\sigma} \\ \tilde{b} &= r_0\bar{1} + \sigma\tilde{\zeta}\end{aligned}\tag{1.6}$$

where $\tilde{b} = (b_1, b_2, \dots, b_n)'$, $\bar{1}$ represents a column vector of ones 1's and the matrix σ is assumed to be non-singular. In this study we assume the market price of risk to be constant which reflects an economy without business cycles. If the market price of risk is modelled as a stochastic model such as the mean-reverting process then it will reflect an economy with business cycles. The risk premium, γ_i , on a risky asset i is defined by

$$\gamma_i = \sum_{j=1}^n \sigma_{ij}\tilde{\zeta}_j,\tag{1.7}$$

where $\tilde{\zeta} = (\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_n)'$. Deducing from (1.6) and (1.7) it follows that

$$b_i = r_0 + \gamma_i.\tag{1.8}$$

Expression (1.8) suggests that the return on investments from the risky assets is generally higher than the return on the riskless asset therefore we have $b_i > r_0$ for each $1 \leq i \leq n$.

This means that banks have incentives to invest with risk. The stochastic differential equation (1.5) may now be expressed as

$$dP_i(t) = P_i(t) \left[(r_0 + \sum_{j=1}^n \sigma_{ij} \tilde{\zeta}_j) dt + \sum_{j=1}^n \sigma_{ij} dX_j(t) \right], \quad P_i(0) = P_0, \quad 1 \leq i \leq n.$$

$P_i(t)$ is defined as the total return, that is, the amount of a single premium investment in risky asset i with reinvestment of dividend income. The volatility matrix denoted by $\Psi = (\sigma_{ij})_{i,j=1}^n$ is invertible which allows the symmetric matrix $\Psi = \sigma\sigma'$ being positive definite. The value of the marketable securities invested at time t in the risky asset P_i is denoted by $\pi_i(t)$ for $i = 1, \dots, n$. The remainder $S - \sum_{i=1}^n \pi_i(t)$, is invested into the riskless asset. No bounds are placed on any of these variables. Borrowing as well as short-selling are allowed. A negative value of $\pi_i(t) < 0$, means that the bank is selling part of its risky asset, $P_i(t)$ short. On the other hand if $\pi_i(t) > S(t)$ then the bank gets into debt to purchase the stock, borrowing at a riskless rate of interest r_0 . We assume that the portfolio process or strategy $\{\tilde{\Pi}(t) : t \geq 0\}$, with $\tilde{\Pi}(t) = (\pi_1(t), \pi_2(t), \pi_3(t), \dots, \pi_n(t))'$. The portfolio strategy or portfolio process is an \mathbb{R}^n -measurable process adapted to the filtration $\{\mathcal{F}\}_{t \geq 0}$ such that

$$\int_0^\infty \tilde{\Pi}(s)' \tilde{\Pi}(s) ds < \infty.$$

Bank Reserves

Bank reserves refer to the amount of money a bank sets aside, and does not lend, to meet day-to-day currency withdrawals by its customers. Since it is uncommon for a bank to have all its depositors withdraw all of their funds simultaneously, only a portion of total deposits is needed as reserves. The bank uses the remaining deposits to earn profit, either by issuing loans or by investing in assets such as bonds and stocks (see Gideon, Mukkudem-Petersen, Petersen [52] and Mukkudem-Petersen, Petersen, Schoeman, Tau [75]).

1.3.2 Bank liabilities

Liabilities in general provide a good indication as to which types of risk a financial institution is exposed too. Bank liabilities such as deposits, borrowing and bank capital constitute the sources of funds. The decomposition of bank liabilities depends greatly on a bank's business operation and market orientation. In general, the bank's liability structure also has an impact on the risk management policies of a bank.

Borrowing

According to Mishkin [71] bank borrowing constitutes the second largest proportion of a bank's total liabilities. Banks borrow a certain amount from other banks (known as interbank funding) as well as from the central bank. We denote this transaction (that is bank borrowing) by $B : \Omega \times T \rightarrow \mathbb{R}_+$ from other banks and the central bank. The reason why banks participate in interbank funding is due to the temporary loan requirements and large withdrawals of customer deposits. These amounts due include deposits and loans which are considered as volatile sources of funding.

Commercial banks that participate in international borrowing are exposed to currency risk. Van Greuning, Brajovic Bratanovic [90] distinguishes between two types of international borrowing namely direct and indirect borrowing. Direct borrowing consists of loans from foreign banks, export promotion agencies in different countries and international lending agencies. Examples of indirect borrowing includes bank notes and guarantees. The main reason why banks borrow from the central is that the changes in the required reserves are effected by the uncertain behaviour of deposit withdrawals. Due to this fact, the value of bank borrowing has randomness associated with it and therefore it can be considered stochastic. Fouche, Mukkudem-Petersen, Petersen [46] provides a continuous-time model for borrowing which plays a key role in deriving an explicit formula for non-risk weighted assets. They further state that there exist a connection between the return on bank investments and the dynamics of bank borrowing (see Mukkudem-Petersen, Petersen

[73]).

Bank Deposits

Bank deposits represents money accepted by banks from the general public such as demand deposits and savings deposits. According to van Greuning, Brajovic Bratanovic [90] bank deposits constitutes the largest liability of a bank's balance sheet. In this study bank deposits is categorized into two types of deposits namely chequeable deposits and nontransaction deposits. Chequeable deposits are deposit accounts that permit the holder of the account to write cheques to third parties. Chequeable deposits also includes Demand Deposits, Negotiable Order of Withdrawal Accounts (NOW) and Money Market Deposit Accounts (MMDAs). Demand Deposits are accounts that pay no interest. Negotiable Order of Withdrawal Accounts (NOW) are accounts that pay interest. Money Market Deposit Accounts (MMDAs) are high-yielding deposit accounts that restrict the cheque-writing privileges of the account holder. The advantages of chequeable deposits is that it allows withdrawals on demand and it is the lowest-cost source of funds. Chequeable deposits are costly for banks to maintain since it has to go through the procedure of setting up monthly statements, processing check and maintain other bank branches.

Nontransaction deposits are the major source of funds for banks. Mishkin [71] states that nontransaction deposits can be categorized into two types of deposits namely savings accounts and time deposits. Savings accounts are accounts that pay interest and it can be withdrawn at any time. Time deposits are deposit accounts held for a fixed-term with the understanding that the depositor can only withdraw by giving written notice.

Mishkin [71] states that when a bank receives additional deposits, it gains an equal amount of reserves and when it loses deposits, it loses an equal amount of reserves. Therefore, for all t we have:

$$dD(t) = dR(t). \quad (1.9)$$

The banking principle (1.9) plays a key role in the analysis and derivation of certain banking items such as non risk-weighted assets (see Mukkudem-Petersen, Petersen [46])

and Mikkelsen-Petersen, Petersen [73]).

1.3.3 A Discussion and Brief Literature Review of Bank Regulatory Capital

The value of a bank is determined by bank equity and long-term debt. Bank equity is the difference between the total assets and total liabilities of the balance sheet. Bank capital includes reserves that protects banks against the losses from loans and securities. Fraser, Gup, Kolari [49] defines bank equity as common stock, surplus and undivided profits. The value of the common stock and preferred stock is equal to the number of shares outstanding multiplied by their par value per share. The undivided profits is equal to the retained earnings which are not paid out as dividends to bank investors. The sum of these components are collectively known as the book value of equity.

The availability of bank capital influences the daily operations of banks. Bank capital also plays an important role when it comes to the safety and soundness of the banking system. The amount of bank capital determines a bank's lending capacity. The amount of capital held by banks is costly therefore it has an impact on a bank's competitive position in financial markets. In the case where banks experience a shortage of capital or the cost of holding capital is too high, banks stand the chance of losing business to its competitors. Van Greuning, Brajovic Bratanovic [90] states that the characteristics of bank capital is that:

- it should be permanent;
- it should not impose fixed charges against earnings;
- it should allow legal subordination to depositors and creditors.

The key purposes of bank capital is that it acts as a safeguard and stabilizer, thereby protecting banks against unexpected losses and it addresses the question of capital requirements. From the viewpoint of van Greuning, Brajovic Bratanovic [90] in terms of a

source of funding, banks require funds to finance the cost of capital investment in land, plant and equipment. Well established banks require funds to maintain its operations and growth in the financial markets. The fact that bank capital acts as a buffer against possible losses provides a basis for maintaining confidence in the general public. Bank regulators and bank shareholders have different viewpoints about the adequacy of capital. Bank regulators expect banks to increase its capital to ensure the stability and soundness in the event that return on investments are negative. Shareholders expect banks to decrease its capital so that it can earn higher rates of return on investments.

A bank's available capital can be modelled stochastically (see, for instance, Berger, Herring, Szego [21]; Decamps, Rochet, Roger [38]; Dangl, Lehar [36]; Dangl, Zechner [37]; Diamond, Rajan [41], Fouche, Mukkudem-Petersen, Petersen [46]; Hancock, Laing, Wilcox [54]; Hellmann, Murdock, Stiglitz [55]; Leland [63]; Mukkudem-Petersen, Petersen [73]; Mukkudem-Petersen, Petersen [76]; Mukkudem-Petersen, Petersen, Schoeman, Tau [75]; Mukkudem-Petersen, Petersen, Schoeman, Tau [74] and Repullo [84]) with its evolution being affected by disruptive and unexpected events that are related to the investment philosophy of shareholders, general state of the economy and profitability of the bank. In the paper of Fouche, Mukkudem-Petersen, Petersen [46] the equity capital is modelled by means of a geometric brownian motion. The model will for instance reflect positive values and its increments will follow a log-normal distribution. In Fouche, Mukkudem-Petersen, Petersen [46] paper the dynamics of Tier 1 Capital is analogous to the description of the equity capital. The dynamics of the supplementary capital capital is also analogous to that described for Tier 1 capital. The dynamics of the eligible regulatory capital (bank capital) is expressed in terms of these different types of capital. In the paper of Mukkudem-Petersen, Petersen [73] the evolution of bank capital is modelled as a diffusion process.

1.3.4 A Discussion and Brief Literature Review about Total Risk-weighted Assets

Credit Risk-Weighted Assets

The *Basel Committee on Banking Supervision* (BCBS) (see [10]), observed that over the last number of years, the world's largest banks have developed sophisticated systems in an attempt to model the credit risk arising from important aspects of their business lines. *Credit risk* is defined as the potential event of a bank borrower or counter party failing to meet its obligations in accordance with agreed terms. Banks need to manage the credit risk inherent in the entire portfolio as well as the risk in individual credits or transactions. The efficient management of credit risk forms an essential part to the long-term success of any banking organization. Credit exposures arise when a bank lends money to a customer, or buys a financial asset (for example a commercial bill issued by a company or another bank), or has any other arrangement with another party that requires that party to pay money to the bank (for example under a foreign exchange contract). The risks inherent in a credit exposure are affected by the financial strength of the party owing money to the bank. The greater this is, the more likely it is that the debt will be paid or that the bank can, if necessary, enforce repayment. Credit risk is also affected by market factors that impact on the value or cash flow of assets that are used as security for loans. For example, if a bank has made a loan to a person to buy a house, and taken a mortgage on the house as security, movements in the property market have an influence on the likelihood of the bank recovering all money owed to it. Even for unsecured loans or contracts, market factors which affect the debtor's ability to pay the bank can impact on credit risk.

The BCBS (see [10], [14], [18]) proposed two types of broad methodologies for banks to calculate their capital requirement for credit risk, namely the standardized approach and the internal ratings based approach (IRB). The standardized and internal ratings based approaches have been used for the evolution of credit risk management. Under the standardized approach, banks are required to use ratings from external credit rating agencies to quantify required capital for credit risk. The internal ratings based approach

to measure credit risk, requires that changes must be made to the asset values appearing on the bank's balance sheet. This implies that the different categories of the issuing of bank loans are weighted according to the degree of riskiness it carries. Off-balance sheet activities such as foreign exchange trades, servicing a mortgage back-security and guaranteeing back securities carries credit risk as well. These exposures are converted to credit equivalent amounts which are also weighted in the same manner as on-balance sheet credit exposures (for a detail discussion on these methodologies we refer the reader to [18]).

The BCBS (see [10], [14], [18]) states that banks should categorize banking-book exposures into broad classes of assets with different underlying risk characteristics. The classes of assets consist of corporate exposures, sovereign exposures, bank exposures, retail and equity exposures. The corporate classes are further categorized into 5 sub-classes of specialized lending that are separately identified. The retail asset class is again categorized into three sub-classes separately identified. According to the BCBS the classification of these exposures in such a manner is globally consistent with established bank practices. The BCBS (see [10]) represents the 15 credit risk exposure types in the following manner:

- $j = 1$: Project Finance (PF);
- $j = 2$: Object Finance (OF);
- $j = 3$: Commodities Finance (CF);
- $j = 4$: Income Producing Real Estate (IPRE);
- $j = 5$: Specialized Lending High Volatility Commercial Real Estate (SLHVCRE);
- $j = 6$: Specialized Lending Not Including
High Volatility Commercially Real Estate (SLNIHVCRE);
- $j = 7$: Bank Exposure (BE);
- $j = 8$: Sovereign Exposure (SE);
- $j = 9$: Retail Residential Mortgage (RRM);
- $j = 10$: Home Equity Line of Credit (HELOC) ;
- $j = 11$: Other Retail Exposure (ORE);

- $j = 12$: Qualifying Revolving Retail Exposure (QRRE) ;
- $j = 13$: Small to Medium Size Enterprises with Corporate Treatment (SMECT);
- $j = 14$: Small to Medium Size Enterprises with Retail Treatment (SMERT);
- $j = 15$: Equity Exposure Not Held in the Trading Book (EENHTB).

Here $1 \leq j \leq 6$ and $9 \leq j \leq 12$ constitute corporate and retail exposures, respectively. **Corporate exposure** is defined as a debt obligation of a corporation, partnership, or proprietorship. The BCBS ([8], see also [9]) argues that an exposure is retail if it satisfies all of the following criteria:

- Exposures to individuals - such as revolving credits and lines of credit (for example credit cards, overdrafts, and retail facilities secured by financial instruments) as well as personal term loans and leases (for example instalment loans, auto loans and leases, student and educational loans, personal finance, and other exposures with similar characteristics) - are generally eligible for retail treatment regardless of exposure size, although supervisors may wish to establish exposure thresholds to distinguish between retail and corporate exposures.
- Residential mortgage loans (including first and subsequent clients, term loans and revolving home equity lines of credit) are eligible for retail treatment regardless of exposure size as long as the credit is extended to an individual that is an owner occupier of the property (with the understanding that supervisors exercise reasonable flexibility regarding buildings containing only a few rental units - otherwise they are treated as corporate). Loans secured by a single or small number of condominium or co-operative residential housing units in a single building or complex also fall within the scope of the residential mortgage category. National supervisors may set limits on the maximum number of housing units per exposure.
- Loans extended to small businesses and managed as retail exposures are eligible for retail treatment provided the total exposure of the banking group to a small business

borrower (on a consolidated basis where applicable) is less than € 1 million. Small business loans extended through or guaranteed by an individual are subject to the same exposure threshold.

- It is expected that supervisors provide flexibility in the practical application of such thresholds such that banks are not forced to develop extensive new information systems simply for the purpose of ensuring perfect compliance. It is, however, important for supervisors to ensure that such flexibility is not being abused.

Precise definitions for the other credit risk exposures are provided in [18]. The Basel Committee on Banking Supervision (see [18]) further states that banks that have received approval for using the internal ratings based (IRB) approach subjected to certain conditions and disclosure requirements, may use their own internal approximation method for risk components in determining the capital requirement for a given exposure. The risk components for the credit risk categories consists of probability of default (PD) (likelihood that a loan will not be repaid and fall into default), loss given default (LGD) (it represents the magnitude of likely loss on the exposure and it is expressed as a percentage), exposure at default (EAD) (it is a measure of potential exposure expressed as a currency and calculated by a Basel Credit Risk Model for the period of 1 year or until maturity) and effective maturity (EM) (effective maturity is measured in years). The derivation of risk-weighted assets are dependent on the aforementioned risk components. The values for the risk components PD, LGD, EAD and effective maturity will be denoted by pd , lgd , ead and em respectively. Probability and loss given default are measured as decimals, therefore they will take on the values:

$$0 \leq pd \leq 1, \quad 0 \leq lgd \leq 1.$$

Unexpected and Expected Losses for Credit Risk Exposure

The Basel Committee on Banking Supervision (see [17]) released a document in which it was describing its movement towards the new capital accord. The BCBS particularly focused on the possible modification and enhancements to the third consultative paper

based on the public's comments. The third consultative paper [17] incorporates both the expected losses (EL) and unexpected losses (UL) into the internal ratings based capital requirement. The *BCBS* suggested that a separate treatment of the expected losses and unexpected losses within the internal ratings based approach (IRB) will result in an improved, superior and consistent framework. The BCBS expected that under this new modified approach, the measurement of the risk-weighted assets would be based only on the unexpected losses (UL) portion of the IRB calculation. Under this approach a *risk-weighted function* (RWF) will transform risk components into risk-weighted assets and ultimately into capital requirements. *Credit risk exposure not in default* are categorized into 7 unexpected loss risk-weighted functions (RWF) for which the calculated risk-weighted assets can be distinguished. The *weighted correlation for the given exposure* is represented as follows:

$$R = d_1 w + d_2(1 - w), \quad (1.10)$$

where the *weight of the given exposure*, w , is expressed as follows:

$$w = \frac{1 - \exp\{Jpd\}}{1 - \exp\{J\}}.$$

The *maturity adjustment for the exposure* has the form

$$b = (p_A + p_B \times \ln(pd))^2.$$

Following from equation (1.10), a *firm-size adjustment* can be made by subtracting the following quantity for Small to Medium Size Enterprises with Corporate Treatment (SMECT) and Equity Exposure Not Held in the Trading Book (EENHTB):

$$0,04 \left[1 - \frac{S - 5}{45} \right]$$

provided with the constraint that $S_1 = 5 \leq S \leq S_2 = 50$. We rewrite equation (1.10) as

$$R = d_1 w + d_2(1 - w) - 0,04 \left[1 - \frac{S - 5}{45} \right],$$

where S denotes the total annual sales expressed in millions of euros (€) with values of S falling in the range of € 5 million up to € 50 million . The total annual sales that are less

than 5 million euros (€) will be treated as if it were equivalent to the 5 million euros (€). The reason why this adjustment is made is to offset the corporate exposure to the Small to Medium Size Enterprises with Corporate Treatment (SMECT) and Equity Exposure Not Held in the Trading Book (EENHTB) borrowers. Under the internal ratings based framework, banks are allowed to distinguish exposure to the aforementioned borrowers. After considering these components, the *capital requirement for the credit risk exposure* may be expressed as follows:

$$K = \text{lgd} \left[N \left[\sqrt{\frac{1}{1-R}} G(pd) + G(0, 9999) \sqrt{\frac{R}{1-R}} - pd \right] \left[\frac{1 + (m-2, 5)b}{1-1, 5b} \right] \right],$$

where $N(x)$ denotes the cumulative distribution function for a standard normal random variable, that is the probability that a normal random variable X with a mean $(\mu) = 0$ and a variance $(\sigma) = 1$ is less than or equal to x . The value x is expressed as follows:

$$x = \sqrt{\frac{1}{1-R}} G(pd) + G(0, 9999) \sqrt{\frac{R}{1-R}},$$

where $G(z)$ denotes the inverse cumulative function for a standard normal random variable, that is the value of x such that $G(z) = x$. On the next page we provide a schematic representation of credit risk under the Basel II Accord.

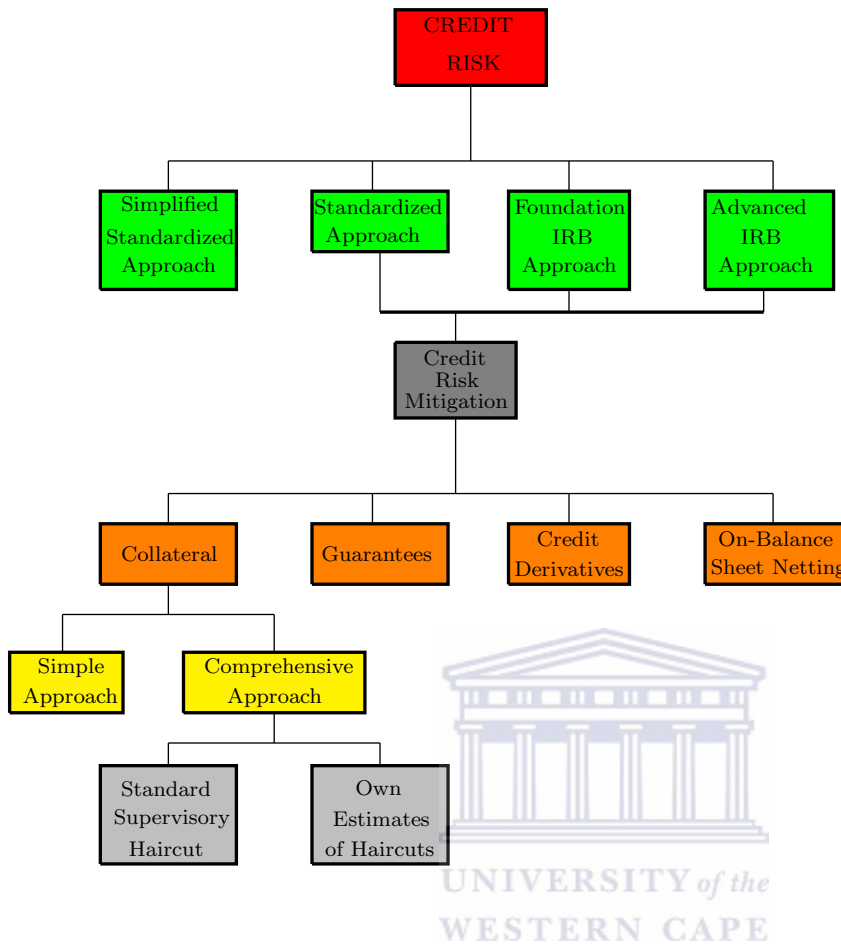


Figure 1.1: Diagrammatic Overview of Basel II Credit Risk

Market Risk-Weighted Assets

The *1988 Basel Accord*, proposed by the BCBS, imposed international capital minimum requirement guidelines that connects banks' capital to their credit exposures. This accord was developed to raise capital ratios, which were generally perceived as being too low and it was also intended to standardize capital ratios. However, regulators have focused much on the measurement of credit risk capital charge and ignored market risk as well as other types of risks. Due to this fact, the BCBS proposed a so-called 1996 Amendment that extended the 1988 Basel Accord to incorporate risk-based capital requirements for market

risk that banks are exposed to in their trading accounts. Under this capital accord, banks are subjected to three capital adequacy requirements namely

- a maximum ratio of assets to capital multiple of 20;
- secondly an 8% minimum ratio of regulatory capital to risk-weighted assets;
- and thirdly a minimum capital charge to make provision for market risk of traded financial derivatives on-and off-balance sheet activities.

Since banks participate in many trading activities such as swaps and foreign exchange contracts, they are exposed to the risks resulting from these activities. If the risk that they are exposed to exceeds 10% of their capital then it needs to be reported on their trading book. Banks are not allowed to take positions that exceeds 25% of the bank's capital without receiving explicit approval from their national or provisional regulator. Since the incorporation of market risk, the so called 1996 Amendment officially allowed banks to use their internal models based on Value-At-Risk models (VaR) methodology to assess market risk exposure. Value-At-Risk is a numerical procedure to assess the possible loss that can be incurred by a bank over a given time period and for a given portfolio of assets. The BCBS [8], defined market risk as the risk of losses in on-and off-balance sheet positions arising from movements in market prices.

The BCBS [8] released a consultative document to the amendment to incorporate market risk. The two broad methodologies proposed by the BCBS would be allowed to use only if it is subjected to the approval of national authorities. The standardized method uses a so-called building block approach in which the capital charge for each different risk category, that is interest rate, equity, foreign exchange and commodity risk, is determined separately. These four measures are then added together to obtain a total capital charge for market risk. According to the BCBS (see [8]) the capital charge for interest rate and equity risk applies to current market value of items in a bank's trading book and the capital charges for foreign exchange and commodity risk applies to a bank's total currency and commodity positions. Financial institutions require a wide variety of

advanced mathematical and computational tools to measure influence of risk. These institutions also analyze strategic ways in which they can control and allocate the risk. The consultative document on incorporating market risk released by the Basel committee on banking supervision [8] permits these sophisticated financial institutions to use their internal (VaR) models to assess the regulatory capital to protect against the movement of market prices. The implementation of the internal (VaR) model which is subjected to certain conditions requires an explicit approval of the national authorities. This alternative method are subjected to the following conditions:

- certain general criteria concerning the adequacy of the risk management system;
- qualitative standards for internal oversight of the use of models, notably by management;
- guidelines for specifying an appropriate set of market risk factors (that is, the market rates and prices that affect the value of the banks' positions);
- quantitative standards setting out the use of common minimum statistical parameters for measuring risk;
- guidelines for stress testing and validation procedures for external oversight of the use of models;
- rules for banks which use a mixture of models and the standardised approach.

The general criteria for using internal Value-At-Risk models are outlined as follows:

- risk management practices in banking should be efficient and conceptually sound and the banking system should be well organised and well structured;
- the bank should have skillful employees that can implement the sophisticated models not only in the trading area but also in risk control and auditing;
- the sophisticated models should have a great history of generating repeated accurate reasonable results of measuring risk;

- the bank will conduct stress testing on a regular basis;
- supervisory authorities will closely monitor and do testing on a bank's internal models before it will use it for supervisory capital purposes.

In addition to these general conditions outlined above, banks that want to use their internal models for capital purposes will be subjected to the conditions outlined in [8].

Fouche, Mukkudem-Petersen, Petersen [46] presents a well-known VaR model used to describe the capital requirement for market risk. The VaR model that the aforementioned authors used is presented in the following way:

$$\hat{a}_{m_p}(t) = \max[\text{VaR}(t_-) + d(t)\text{ASR}^{\text{VaR}}(t_-), M(t)\frac{1}{60}\sum_{k=1}^{60}\text{VaR}((t-k)_-) + d(t)\frac{1}{60}\sum_{k=1}^{60}\text{ASR}^{\text{VaR}}((t-k)_-)], \quad (1.11)$$

where

VaR(s) : Value-at-Risk at Time s;

VaR(s₋): Value-at-Risk 24 hours before Time s;

$d(t)$: 0-1 Indicator Function Related to Estimation of Specific Risk Measured Through Additional Specific Risk (ASR) Measure from VaR;

$M(t)$: Multiplier for Stress Factor, $M(t) \geq 3$;

p : Days, $1 \leq p \leq 60$.

This type of specific model is commonly used among many banks in the Group-Ten (G-10) countries. The reason why Fouche, Mukkudem-Petersen, Petersen [46] chose this model is that it satisfied the qualitative standards for the model approach to market risk set out in [8]. In the sequel, Mukkudem-Petersen, Petersen [72] makes a technical contribution whereby the aforementioned authors evaluate the total risk-weighted assets using the internal ratings approach that incorporates Value-At-Risk (VaR) models. Mukkudem-Petersen, Petersen [72] further provides a description of the capital charge for operational risk from the viewpoint of the standardised approach (see [15]).

1.3.5 Discussion and Brief Literature Review about Bank Loans

Merton [69] states that the important functions of banks are money lending to financial institutions and individuals. The bank provides a service to depositors in exchange for the use of their funds and charges interest on loans. The individual or firm together with the bank enters into a financial contract and both parties respect the conditions attached to it. An important and common type of contract, is a loan agreement. In this section we discuss the process of repaying a loan to a bank. Bank loans constitute the largest asset in a bank's balance sheet. Bank loans can be categorized into three types of loans namely commercial and industrial loans, real estate loans and consumer loans. Commercial and industrial loans are used by businesses to purchase new equipment, acquiring a variety of goods and raw material. Real estate loans are used for the purchasing of homes, apartments and office buildings. Consumer loans are loans made to customers. The consumer loan can be considered as a credit account that is granted to customers and not a business. Customers use these loans for own personal needs such as car loans, home loans and credit cards. Before banks can make loans to customers, they first need to evaluate certain information on the client pertaining to credit worthiness. Obtaining this information can be costly.

Loan contracts are less complex because the obligation to repay the amount of loans and the interest on the debt are specified over the whole duration, that is $0 \leq t \leq T$, of the contract. Loan contracts might be less complex but they certainly lack flexibility, and for instance, they require costly auditing. Freixas, Rochet [50] mentioned that models such as Townsend's costly state verification model, further developed by Gale and Hellwig, develops the idea of how to design an optimal loan contract efficiently. In the aforementioned model asymmetric information are taken into consideration. Asymmetric information occurs when the one party (the borrower) has more or better information than the other party (the bank). Due to this fact, banks normally charge a higher interest that reflects the average rate of all risk borrowers (see Fraser, Gup and Kolari [49]).

In the Townsend model the lender cannot observe the investment made by the borrower unless a costly audit is performed. Thakor [89] also investigated the reasons that led to a decline in loans relative to security holdings (government bonds). Thakor [89] developed a model that explores the aforementioned phenomena by considering two key lending functions namely, the prelending screening of loan creditors and postlending monitoring (the supervision of borrowing's management on an asset). The Thakor [89] model assumes that each borrower can approach simultaneous multiple banks. Each bank knows how many banks the borrower has approached which leads to the idea of symmetric information. Based on this available information the bank can decide whether it will screen applicants and then extend the loan. The Thakor [89] model is set up in such a way that the bank will not lend to a borrower that has not been screened. According to Thakor [89] the idea behind screening is similar to credit worthiness. His model generates three key results that are relevant to this study. Thakor [89] states that a small increase in bank risk-based capital requirements promote the probability of a borrower being denied credit by the banking system which minimizes aggregate bank lending. Secondly, if a bank agrees to lend then this can cause an abnormal behaviour in the borrower's stock price. If a bank is capital-constrained then this abnormal behaviour will be greater. Thirdly the effect of monetary policy on bank lending depends on its effects on the term structure of interest rates. Thakor [89] further explains the third stage whereby he states that if you increase the money supply, then the short-term interest rates will reduce more than the long-term interest rates, the probability of credit-denial by banks will increase which will lead to the reduction of aggregate bank lending.

Kashyap, Rajan, Stein [61] defines bank lending as the involvement of acquiring important information about borrowers and extend credit based on this information. The model of Kashyap, Rajan, Stein [61] is designed under a framework whereby their model catch the important activities of a bank. The Kashyap, Rajan, Stein [61] model incorporates the bank's participation in providing funds to its customers; raising external finance (sources

of funds) unexpectedly is expensive and this implies that the bank should hold a buffer stock of liquid assets to protect themselves from such unpredictable events. The holding of these liquid assets is also costly.

Loans have the following distinguishing characteristics (see [64]):

- Time to maturity refers to the length of the loan contract. Loans can be categorized according to their maturity into short-term debt, intermediate-term debt, and long-term debt. Revolving credit and perpetual debt have no fixed date for retirement. Revolving credit is a type of credit that does not have a fixed number of payments (for instance a credit card). Banks allow entities (customers and institutions) to continuously borrow money up to a certain credit limit whereas a perpetual loan requires only regular interest payments.
- In the case of a repayment schedule, the payments are made either at the end of the contract or at set intervals, usually on a monthly, quarterly or semi-annual basis. This payment is decomposed into a portion of the outstanding principal and the interest costs. During the loan contract the principal amount of the loan is amortized. As the principal balance reduces, the interest on the remaining balance also declines. Interest-only loans do not pay down the principal.
- Interest refers to the cost of borrowing money. Interest rates charged by lending institutions must be sufficient to cover certain costs such as operating costs, administrative costs, and an acceptable rate of return. Banking interest rates may be fixed on a loan contract, or adjusted to reflect changing market conditions. An example of the latter is for instance credit contracts where the rates maybe adjusted daily, annually, or at certain intervals of 2, 6, and 10 years.

Common types of loans

Consumers and small businesses obtain loans with different maturity periods to finance purchases of real estate, transportation, equipment, supplies, and other needs. These

entities may acquire these loans from external sources, including friends and relatives, banks, credit unions, finance companies, insurance companies. Small businesses acquire funds from the the state and federal governments. Here are examples of some common types of loans.

- *Short term loans* are loans with a maturity of less than one year ($0 < T < 1$) and its purpose is to cover cash shortages resulting from a one-time increase in current assets, such as a special inventory purchase and an unexpected increase in accounts receivable. Trade credit is an example of a short term loan.
- *Intermediate term loans* are loans that are used to finance the purchase of furniture, fixtures, vehicles, plant and office equipment. The maturity of these type of loans generally runs more than one year but less than five years, that is $1 \leq T \leq 5$. An example of an intermediate term loan are consumer loans for autos, boats and home repairs.
- *Long term loans* are loans to be used to for purchasing real estate and are secured by the asset itself. The maturity on this type of contract generally run between ten and forty years, that is $10 \leq T \leq 40$. Mortgage loans are an example of long term loans.

Companies with good credit and a stable history of revenues, earnings, and cash flow may use borrowing as a useful strategy, but small businesses should be careful before committing to large loans in order to avoid cash flow problems and reduced flexibility. Therefore, in general, small businesses should consider a combination of loans and other types of financing strategies.

The main disadvantage of loans is that they expect, for instance, a small business to make regular monthly payments of principal and interest. Small companies that are in the beginning stages of building, in general, experience shortages in cash flow that may make such regular payments difficult. Therefore, most financial institutions provide severe penalties for late or missed payments, which may for instance

include charging late fees and taking possession of collateral. In the case of small businesses, failing to meet the loan requirements may have an adverse effect on the company's credit rating and its ability to obtain external funds. Another disadvantage of loans is that it is often limited to companies that are creditworthy or well established.



Chapter 2

Stochastic Banking Model

To understand the operation and management of banks, we have to study its balance sheet, which records the bank assets (uses of funds) and bank liabilities (sources of funds). The items on the balance sheet behave in an unpredictable manner which is consistent with the uncertain behaviour of the activities related to the evolution of reserves, loan demand, risky and riskless investments, deposits, loan repayments, borrowings and eligible regulatory capital. Bank capital plays an important role because it balances assets and liabilities by the relation

$$\text{Total Assets} = \text{Total Liabilities} + \text{Bank Capital}.$$

As in Mikkelsen-Petersen, Petersen ([73]), a commercial bank's balance sheet at time t can be represented as

$$R(t) + L(t) + M(t) + T(t) = D(t) + B(t) + C(t), \quad (2.1)$$

where R , L , M , T , D , B and C are reserves, loans to private agents, marketable securities, treasury securities, deposits, borrowings and bank capital respectively. The Basel II capital accord allows internal models to be used by banks to measure, for example, the riskiness of their portfolios and the regulatory capital requirement. Following in this manner, continuous-time stochastic models have been developed by Diamond and Dybvig [40], whereby they constructed a model that allows illiquid assets (assets that cannot be

exchanged to cash) into liquid liabilities. One of the characteristics of bank capital is that it reduces the probability of a financial crisis but reduces the liquidity creation. Diamond and Rajan [41] constructed a model whereby bank assets and liabilities are closely related. They further argue that bank capital affects three areas namely bank safety, to refinance at a minimum cost and the ability to liquidate assets. Previous research on describing stochastic modelling of bank assets has been done by Hancock, Laing, Wilcox [54] whereby they use VaR techniques to estimate banks' responses to capital shocks. The objective in the following section is to provide dynamic continuous-time models for bank capital and total risk-weighted assets respectively. At the outset we assume that we work in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on a time interval $0 \leq t \leq T$. Here we have that $\{\mathcal{F}\}_{t \geq 0}$ is a complete, right continuous filtration generated by the n -dimensional brownian motion $\{X(t)\}_{t \geq 0}$. The filtration represent the information available up to a certain time t . Also \mathbb{P} is a probability measure on Ω . We define the aforementioned bank items:

$$\begin{aligned}
 R : \Omega \times T \rightarrow \mathbb{R}_+ & \quad \text{: - Reserves;} & D : \Omega \times T \rightarrow \mathbb{R}_+ & \quad \text{: - Deposits;} \\
 L : \Omega \times T \rightarrow \mathbb{R}_+ & \quad \text{: - Loans;} & B : \Omega \times T \rightarrow \mathbb{R}_+ & \quad \text{: - Borrowings;} \\
 S : \Omega \times T \rightarrow \mathbb{R}_+ & \quad \text{: - Securities;} & C : \Omega \times T \rightarrow \mathbb{R}_+ & \quad \text{: - Bank Capital.}
 \end{aligned}$$

2.1 Bank Regulatory Capital

Bank capital are decomposed into *Tier 1*, *Tier 2* and *Tier 3* capital. Tier 1 capital consists of equity shares, retained earnings, and non-redeemable, non-cumulative preference shares. A more detailed discussion on bank regulatory capital can be found on the website [33]. However for this study we only provide a brief description of the aforementioned types of capital using the website [33] as our source.

Tier 1 capital is freely available and safeguard banks against unexpected losses. It also measures a bank's financial strength in the financial system. Tier 1 capital or *Core* capital is considered the most important because it is common in all banking systems and it is reported in any bank's published financial statements. Tier 2 capital can be divided into lower Tier 2 capital and upper Tier 2 capital. Upper Tier 2 capital has no fixed

maturity, while lower Tier 2 capital has a limited life span, which makes it less effective in providing a buffer against losses by the bank. The upper Tier 2 capital comprises of unaudited retained earnings, revaluation reserves, general provisions for bad debts, perpetual cumulative preference shares (that is preference shares with no maturity date whose dividends accrue for future payment even if the bank's financial condition does not support immediate payment), perpetual subordinated debt (that is debt with no maturity date which ranks in priority behind all creditors except shareholders). The lower Tier 2 capital includes subordinated debt with a term of at least 5 years, redeemable preference shares which may not be redeemed for at least 5 years. Tier 2 capital absorbs losses only in the event of a winding-up of a bank, and so provides a lower level of protection for depositors and other creditors. Tier 2 capital plays a major role in the case where Tier 1 capital has been lost by the banks. The Basel committee on banking supervision [8] introduced the concept of Tier 3 capital. Tier 3 capital comprises of short term subordinated debt. Tier 3 capital is used to protect banks against the unexpected losses that arise from market risk if Tier 1 and Tier 2 capital is insufficient for this.

The total of bank capital, $C(t)$, can be expressed as the sum of Tier 1 capital ($C_{T1}(t)$), Tier 2 Capital ($C_{T2}(t)$) and Tier 3 capital ($C_{T3}(t)$) that is,

$$C(t) = C_{T1}(t) + C_{T2}(t) + C_{T3}(t) \quad (2.2)$$

Tier 1 capital is the book value of its stock, $E(t)$, plus retained earnings, $E_r(t)$. Tier 2 and Tier 3 capital (collectively known as *supplementary capital*) is the sum of subordinate debt, $S_D(t)$, and loan-loss reserves, $R_L(t)$. As a result, we may set

$$C_{T1}(t) = E(t) + E_r(t) \quad (2.3)$$

and

$$C_{T2}(t) + C_{T3}(t) = S_D(t) + R_L(t). \quad (2.4)$$

We assume that the bank holds capital in $n + 1$ categories of which n are related to bank equity. In this case, the market value of subordinate debt at t may be given by

$$S_D(t) = S_D(0) \exp \left\{ \int_0^t r_0(u) du \right\}.$$

For the return on the i -th bank equity we have

$$de_i(t) = e_i(t) \left[\left(r_0(t) + \sum_{j=1}^n \sigma_{ij} \tilde{\zeta}_j \right) dt + \sum_{j=1}^n \sigma_{ij} dX_j(t) \right], \quad i = 1, 2, \dots, n.$$

Here the co-variance matrix and the market prices of risk, given by

$$\Psi = (\sigma_{ij})_{i,j=1}^n \text{ and } \tilde{\zeta} = (\zeta_1, \dots, \zeta_n)',$$

respectively, are assumed to be constant.



2.1.1 Dynamics of Bank Regulatory Capital

At time t , we assume that the bank capital is continuously being consumed by loans to private agents and marketable securities at the rate of $p_a(t) = p\bar{a}(t)dt$, so that loan consumption is a constant proportion, p , of such assets. Assuming no transaction costs, we may compute the total bank capital as

$$C(t) = E(t) + E_r(t) + S_D(t) + R_L(t). \tag{2.5}$$

Because of their non-dynamic nature we do not consider retained earnings and loan-loss reserves to be active constituents of bank capital (see Mukkudem-Petersen, Petersen [76]).

Therefore, in the case where

$$dE_r(t) = dR_L(t) = 0, \text{ for all } t,$$

the C -dynamics may be expressed as

$$\begin{aligned}
dC(t) &= C(t) \sum_{i=1}^n \pi_i(t) \frac{de_i(t)}{e_i(t)} + \left(1 - \sum_{i=1}^n \pi_i(t)\right) C(t) \frac{dS_D(t)}{S_D(t)} - p\bar{a}(t)dt \\
&= C(t) \sum_{i=1}^n \pi_i(t) \left[(r_0(t) + \sum_{j=1}^n \sigma_{ij} \tilde{\zeta}_j) dt + \sum_{j=1}^n \sigma_{ij} dX_j(t) \right] \\
&\quad + \left(1 - \sum_{i=1}^n \pi_i(t)\right) C(t) \left[r_0(t) dt \right] - p\bar{a}(t)dt \\
&= C(t) \sum_{i=1}^n \pi_i(t) \left[r_0(t) dt + \sum_{j=1}^n \sigma_{ij} \tilde{\zeta}_j dt + \sum_{j=1}^n \sigma_{ij} dX_j(t) \right] \\
&\quad + C(t)r_0(t)dt - C(t) \sum_{i=1}^n \pi_i(t)r_0(t)dt - p\bar{a}(t)dt \\
&= C(t) \sum_{i=1}^n \pi_i(t)r_0(t)dt + C(t) \sum_{i=1}^n \pi_i(t) \sum_{j=1}^n \sigma_{ij} \tilde{\zeta}_j dt + \\
&\quad C(t) \sum_{i=1}^n \pi_i(t) \sum_{j=1}^n \sigma_{ij} dX_j(t) + C(t)r_0(t)dt - C(t) \sum_{i=1}^n \pi_i(t)r_0(t)dt - p\bar{a}(t)dt \\
&= C(t) \left[\left(\sum_{i=1}^n \pi_i(t) \sum_{j=1}^n \sigma_{ij} \tilde{\zeta}_j + r_0(t) \right) dt + \sum_{i=1}^n \pi_i(t) \sum_{j=1}^n \sigma_{ij} dX_j(t) \right] - p\bar{a}(t)dt \\
&= C(t) \left[\left(r_0(t) + \pi'(t)\Psi\tilde{\zeta} \right) dt + \pi'(t)\Psi dX(t) \right] - p\bar{a}(t)dt, \tag{2.6}
\end{aligned}$$

where $\pi'(t)$ are the proportions invested in the risky assets (bank equities). The diffusion term $\pi'(t)\Psi dX(t)$ in (2.6) establishes a correlation between bank capital and total risk-weighted assets.

2.2 Total-Risk Weighted Assets

2.2.1 Continuous-time model of the Total Risk-Weighted Assets

Continuous-time models for total risk-weighted assets have been proposed by Fouche, Mukkudem-Petersen, Petersen [46] (see also Mukkudem-Petersen, Petersen [76]). In the said paper the total risk-weighted assets is calculated by multiplying the capital charges for market and operational risk by 12,5 (the percentage of the reciprocal of capital adequacy ratio) and adding the result to the sum of risk-weighted assets for credit risk. Section (1.3.4) suggests that risk-weighted assets (RWAs) are defined by placing each on- and off-balance item into a risk category. In this regard, the riskier the asset the higher the risk-weight. It is clear that RWAs are a weighted average of the various assets of the bank. In the sequel, our primary objective is to provide a coherent analysis of these issues in a simplified framework. In this regard, we introduce table (7.1) (see Appendix A) that provides illustrative risk categories, their risk-weights and representative items in an on-balance sheet context. For sake of argument, in the ensuing discussion, we restrict ourselves to the information contained in table (7.1). According to this table, the balance sheet assets, viz., reserves, $R(t)$, and treasuries, $T(t)$, have a zero risk-weighting so that, in our case, the TRWAs are solely constituted by loans to private agents, L , and 20% of marketable securities, M . This means that we are mainly concerned about the effect of credit and market risk in the formulation of the risk-based CAR. Suppose that L is the credit RWAs for which

$$dL(t) = L(t) \left\{ \frac{a(t)}{L(t)} \left[\left(r(t) + \mu_L(t) \right) dt + \sum_{j=1}^n \sigma_j dX_j(t) \right] \right\}$$

and a_0 is the market RWAs with dynamics

$$da_0(t) = a_0(t) \left\{ \frac{a(t)}{a_0(t)} \left[\mu_0(t) dt + \sigma_0 dX_0(t) \right] \right\}, \quad a_0 = 0, 2M.$$

In this case, the TRWAs, a , are expressible as

$$a(t) = L(t) + a_0(t)$$

with dynamics given by

$$da(t) = a(t) \left[\left(r(t) + \mu(t) \right) dt + \sigma_0 dX_0(t) + \sum_{j=1}^n \sigma_j dX_j(t) \right], \quad a_0 = a(t_0), \quad (2.7)$$

where $\mu(t) = \mu_L(t) + \mu_0(t)$ is a deterministic function of time. Here $X_0(t)$ is standard Brownian motion that is independent of $X_1(t), \dots, X_n(t)$ and we define $\sigma = (\sigma_1, \dots, \sigma_n)'$, where the σ_j 's are constants. The volatility σ allows for a possible correlation between the TRWAs (consisting of credit and market RWAs) and bank regulatory capital. Assuming that a is completely hedgeable, under the risk-neutral measure \mathbb{Q} , the a -dynamics previously given by (2.7), may be rewritten as

$$da(t) = a(t) \left[\left(r(t) + \mu(t) - \sum_{j=1}^n \sigma_j \theta_j \right) dt + \sum_{j=1}^n \sigma_j d\widehat{X}_j(t) \right], \quad j = 1, \dots, n. \quad (2.8)$$

This implies that

$$a(\tau) = a(t) \exp \left\{ \int_t^\tau \left(r(s) + \mu(s) \right) ds - \sum_{j=1}^n \sigma_j \left(\theta_j + \frac{1}{2} \sigma_j \right) (\tau - t) + \sum_{j=1}^n \sigma_j \left(\widehat{X}_j(\tau) - \widehat{X}_j(t) \right) \right\}. \quad (2.9)$$

Chapter 3

Amortizations

The objective of this chapter is to derive an explicit amortization function via a general pricing equation and a martingale approach. The amortization function has to satisfy the partial differential equation under certain conditions. The main reason for this approach is to illustrate that this should be seen as a possible starting point for obtaining more sophisticated and complex alternative amortization functions.

Amortization is the process of paying off a debt with interest over a period of time. The choice of amortization functions determines the overall loan structure of a bank.

A *bank loan* is a financial contract between two parties, a lender (the bank) and borrower (the debtor) that has the features outlined below.

At time $t = 0$, the bank pays an amount of money to the borrower $S(0)$, called the *principal*, and the borrower pays back or amortizes the loan. We let $A(t)$ denote the total amount of money paid back and call it the amortization function, with t as the variable (see Norberg [79]). An amortization function is a finite-valued, right-continuous, non-decreasing function and $A(0) = 0$. The type of amortization function considered here is a non-negative function defined on the real line, that is:

$$A(t) : [0, \infty) \rightarrow [0, \infty)$$

where $[0, \infty)$ denotes the set of non-negative real numbers. The borrower (debtor) pays

amortizations back to the bank at an *interest rate* of r , which is influenced by the uncertainty of the market conditions. In a scenario like this, the total amount of amortization repayments at time $t \geq 0$ at rate r will be denoted by, $A(t, r(t))$. When accounting and taxation are taken into account, then the loan contract needs to be designed in such a way that the amortization function, $A(t, r(t))$, is decomposed into repayments and interest (see Norberg [79] for instance). In this case, $A(t, r(t))$ may be presented as

$$A(t, r(t)) = F(t, r(t)) + I(t, r(t)), \quad (3.1)$$

where F is the *loan repayment* and I is the *interest function* of time, both being non-negative, right-continuous and non-decreasing with the constraint conditions $F(0) = I(0, r(0)) = 0$ and $F(n) = 1$. The repayments should be fractions of the principal, that is, $F(t) \leq 1$, and a finite term loan should be repaid in full, that is, $F(T) = 1$. The term of the loan contract is defined as

$$T = \inf\{t; A(t) = A_\infty\}. \quad (3.2)$$

The bank loan is said to be perpetual if $T = \infty$. The bank may choose to fix the interest rate or adapt it to the market conditions. In this study, we will let the interest rate be of a stochastic nature, that is, the interest rate will be influenced by unexpected events. The interest rate is specified in such a way as to be non-negative and usually the term of the contract is finite, that is, $0 \leq t \leq T$. However, the situation where the borrower (debtor) pays the loan of the principal $S(0)$ forever may occur, and is represented by

$$\int_0^\infty r(s) ds = \infty. \quad (3.3)$$

The amortizations are designed in a such a way that its present value at time 0 will be equal to the principal. The principal will be set to one monetary unit so that $S(0) = 1$. The loan that will be paid back therefore satisfies the condition

$$\int_0^T \exp\left(-\int_0^\tau r(s) ds\right) dA(\tau) = S(0) = 1. \quad (3.4)$$

The following observation come from Norberg [79]. We present it formally and in more detail.

Proposition 3.0.1 *In the given scenario, at any time t the remaining principal is*

$$1 - F(t) = \int_t^T \exp\left(-\int_t^\tau r(s) ds\right) dA(\tau). \quad (3.5)$$

Proof.

Inserting $A(t, r(t)) = F(t, r(t)) + I(t, r(t))$ into $\int_0^T \exp\left(-\int_0^\tau r(s) ds\right) dA(\tau)$ yields the following

$$\int_0^T \exp\left(-\int_0^\tau r(s) ds\right) dI(\tau) + \int_0^T \exp\left(-\int_0^\tau r(s) ds\right) dF(\tau) = 1.$$

Now by applying the technique of stochastic integration by parts, on the term $\exp\left(-\int_0^\tau r(s) ds\right) dF(\tau)$, we obtain the following result:

$$\exp\left(-\int_0^\tau r(s) ds\right)(1 - F(t)) = 1 - \int_0^t \exp\left(-\int_0^\tau r(s) ds\right)(1 - F(\tau))r(s)d\tau - \int_0^t \exp\left(-\int_0^\tau r(s) ds\right)dF(\tau).$$

Analysing the first case where it is at the end of the term of the contract, that is, $t = T$, and using the condition that the loan is repaid, $F(T) = 1$, we have the following

$$\exp\left(-\int_0^\tau r(s) ds\right)(1 - 1) = 1 - \int_0^t \exp\left(-\int_0^\tau r(s) ds\right)(1 - F(\tau))r(s)d\tau - \int_0^t \exp\left(-\int_0^\tau r(s) ds\right)dF(\tau).$$

Rearranging the terms in the following manner:

$$\int_0^t \exp\left(-\int_0^\tau r(s) ds\right)(1 - F(\tau))r(s)d\tau + \int_0^t \exp\left(-\int_0^\tau r(s) ds\right)dF(\tau) = 1.$$

Analysing the case when $T = \infty$, that is, a perpetual loan and using the fact that $\int_0^\tau r(s) ds = \infty$, the following is derived:

$$\exp\left(-\int_0^\tau r(s) ds\right)(1 - F(t)) = 1 - A \quad (3.6)$$

where A is expressed as:

$$A = 1 - \int_0^t \exp\left(-\int_0^\tau r(s)ds\right)(1 - F(\tau))r(s)d\tau - \int_0^t \exp\left(-\int_0^\tau r(s)ds\right)dF(\tau).$$

Now also re-arranging the terms in a similar manner as above, the following is obtained:

$$\int_0^t \exp\left(-\int_0^\tau r(s)ds\right)(1 - F(\tau))r(s)d\tau + \int_0^t \exp\left(-\int_0^\tau r(s)ds\right)dF(\tau) = 1.$$

Comparing by inspection the term for the interest function is identical to the term for the outstanding loan

$$\int_0^T \exp\left(-\int_0^\tau r(s)ds\right)dI(\tau) = \int_0^t \exp\left(-\int_0^\tau r(s)ds\right)(1 - F(\tau))r(s)d\tau. \quad (3.7)$$

The equality (3.7) states that the discounted value of all interest payments is identical to the discounted value of all interest amounts arising from the outstanding balance. The above equality is only satisfied if

$$dI(t) = (1 - F(t))r(t)dt. \quad (3.8)$$

Differential equation (3.8) above states that interest is paid currently and instantaneously on the outstanding balance, $1 - F(t)$, on the interval $[t, t + \delta t)$. Under the natural interest rate scheme the differential equation

$$\begin{aligned} dA(t, r(t)) &= dF(t, r(t)) + dI(t, r(t)) \\ &= dF(t, r(t)) + (1 - F(t))r(t)dt \end{aligned} \quad (3.9)$$

establishes a one-to-one relationship between the amortizations and the repayments. Integrating the equation (3.9) over $(0, t]$ to obtain

$$A(t, r(t)) = F(t, r(t)) + \int_0^t (1 - F(\tau))r(\tau)d\tau. \quad (3.10)$$

Multiplying $\exp\left(-\int_0^\tau r(s)ds\right)$ with equation (3.6) yields:

$$\exp\left(-\int_0^\tau r(s)ds\right)(1 - F(t)) = \exp\left(-\int_0^\tau r(s)ds\right) \times \{A\}. \quad (3.11)$$

This implies that equation (3.7) takes the form

$$1 - F(t) = \exp\left(-\int_0^t r(s)ds\right) \left\{ 1 - \int_0^t \exp\left(-\int_0^\tau r(s)ds\right) dA(\tau) \right\}. \quad (3.12)$$

The equality (3.12) implies that the remaining principal is the value of the difference between the principal and the paid amortizations provided that all the amounts are compounded with interest. Therefore the remaining principal may be represented as follows:

$$1 - F(t) = \int_t^T \exp\left(-\int_t^\tau r(s)ds\right) dA(\tau). \quad (3.13)$$

□



3.1 The Partial Differential Equation Approach for Amortization

In order to derive an alternative amortization function, we provide a systematic procedure whereby we use the partial differential equation approach to obtain bond prices. This approach has been proposed by Vasicek [91] (see also Baz, Chacko [20] and Cairns [31]). The following *assumptions* are made for deriving an amortization function.

3.1.1 Conditions for deriving the amortization function

1. $r(t)$ should be a markovian process with normally distributed increments and should be a continuous function of time;
2. At a certain time t , the value of a amortization function, $A(t, r(t), T)$, which matures at the end of a contract is fully determined by the time assessment of $\{r(s) : t \leq s \leq T\}$;
3. the market is efficient, that is, we assume no transaction cost and all investors are rational.

Under these assumptions, we may express the amortization function as follows:

$$A(t, r(t)) = A(t, r(t), T).$$

Baz, Chacko [20] derives the general pricing equation but it is not discussed in detail. Therefore we follow a similar discussion from Cairns [31]; Mamon [68] and Wilmott, Howison, Dewynne [93]. We derive the general pricing equation using two alternative methods namely the traditional approach (see Wilmott, Howison, Dewynne [93]) and via the martingale approach (see Mamon [68]).

Applying Itô's formula to the amortization function $A(t, r(t), T)$ and making use of the

stochastic differential equation $dr(t) = \mu(r, t)dt + \sigma(r, t)dX(t)$ we have

$$\begin{aligned} dA(t, r(t), T) &= \frac{\partial A(t, r(t), T)}{\partial t}dt + \frac{\partial A(t, r(t), T)}{\partial r}dr + \frac{1}{2} \frac{\partial^2 A(t, r(t), T)}{\partial r^2}d\langle r \rangle \\ &= \frac{\partial A}{\partial t}dt + \frac{\partial A}{\partial r} \left(\mu(r, t)dt + \sigma(r, t)dX \right) + \frac{1}{2} \frac{\partial^2 A}{\partial r^2}dt \\ &= \left[\frac{\partial A}{\partial t} + \mu \frac{\partial A}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 A}{\partial r^2} \right] dt + \sigma \frac{\partial A}{\partial r} dX(t). \end{aligned}$$

So the dynamics of the amortization function can be expressed as

$$dA(t, r(t), T) = A(t, r(t), T) \left[a(t, r(t), T)dt + b(t, r(t), T)dX(t) \right] \quad (3.14)$$

where

$$a(t, r(t), T) = \frac{1}{A} \left[\frac{\partial A}{\partial t} + \mu \frac{\partial A}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 A}{\partial r^2} \right] \quad (3.15)$$

and

$$b(t, r(t), T) = \frac{1}{A} \sigma \frac{\partial A}{\partial r}. \quad (3.16)$$

In practice, the pricing of bonds is more difficult than the pricing of option contracts since there are no underlying assets to hedge it with. For example you cannot go and "buy" an interest rate of 4% or 6%. Due to this reason one way of hedging is to construct a self-financing portfolio containing two bonds with different maturity dates, T_1 and T_2 . In order to do this we follow a method to the analysis by Wilmott, Howison, Dewynne [93] in constructing the hedging portfolio and we obtain the following result.

Proposition 3.1.1 *Under the conditions (3.1.1) above, the amortization function satisfies the following partial differential equation:*

$$\begin{aligned} \frac{\partial A(t, r(t), T)}{\partial t} + \frac{\partial A(t, r(t), T)}{\partial r} [\mu(r, t) - \lambda(r, t)\sigma(r, t)] \\ + \frac{1}{2} \frac{\partial^2 A(t, r(t), T)}{\partial r^2} \sigma^2(r, t) = r(t)A(t, r(t), T) \end{aligned} \quad (3.17)$$

Proof.

Consider two loans where one loan maturing at T_1 has a price $A_1(r(t), t)$ and the other loan maturing at T_2 has a price $A_2(r(t), t)$, respectively. Holding the first loan and a multiple $-\Delta$ of the other loan, the portfolio Λ has the form

$$\Lambda = A_1(r(t), t) - \Delta A_2(r(t), t).$$

Applying Itô's formula (1.3) to the functions $A_1(r(t), t)$ and $A_2(r(t), t)$, the change in the portfolio over the interval $(t, t + dt]$ is

$$\begin{aligned} d\Lambda = & \frac{\partial A_1(r(t), t)}{\partial t} dt + \frac{\partial A_1(r(t), t)}{\partial r} dr + \frac{1}{2}\sigma^2 \frac{\partial^2 A_1(r(t), t)}{\partial r^2} dt \\ & - \Delta \left(\frac{\partial A_2(r(t), t)}{\partial t} dt + \frac{\partial A_2(r(t), t)}{\partial r} dr + \frac{1}{2}\sigma^2 \frac{\partial^2 A_2(r(t), t)}{\partial r^2} dt \right) dt. \end{aligned} \quad (3.18)$$

Choosing $\Delta = \frac{\partial A_1(r(t), t)}{\partial r} / \frac{\partial A_2(r(t), t)}{\partial r}$ we see that the random term vanishes from the dynamics of the portfolio (3.18), thus we have

$$\begin{aligned} d\Lambda = & \left(\frac{\partial A_1(r(t), t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 A_1(r(t), t)}{\partial r^2} \right. \\ & \left. - \frac{\partial A_1(r(t), t)}{\partial r} / \frac{\partial A_2(r(t), t)}{\partial r} \left(\frac{\partial A_2(r(t), t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 A_2(r(t), t)}{\partial r^2} \right) \right) dt \\ = & r \left(A_1(r(t), t) - \frac{\partial A_1(r(t), t)}{\partial r} / \frac{\partial A_2(r(t), t)}{\partial r} A_2(r(t), t) \right) dt \\ = & r\Lambda dt \end{aligned}$$

where we have use arbitrage arguments to set the return on the portfolio equal to the risk-free interest rate. Grouping all $A_1(r(t), t)$ terms on the left-hand side and all the $A_2(r(t), t)$ terms on the right-hand side we obtain

$$\begin{aligned} & \left(\frac{\partial A_1(r(t), t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 A_1(r(t), t)}{\partial r^2} - rA_1(r(t), t) \right) / \frac{\partial A_1(r(t), t)}{\partial r} = \\ & \left(\frac{\partial A_2(r(t), t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 A_2(r(t), t)}{\partial r^2} - rA_2(r(t), t) \right) / \frac{\partial A_2(r(t), t)}{\partial r}. \end{aligned} \quad (3.19)$$

This is an equation in two unknowns namely $A_1(r(t), t)$ and $A_2(r(t), t)$. The left-hand side of equation (3.19) is a function of T_1 but not of T_2 and the right-hand side of equation (3.19) is a function of T_2 but not of T_1 . The only way for equation (3.19) to hold is for

both sides to be independent of the expiry date. Therefore eliminating the subscripts from $A_1(r(t), t)$ and $A_2(r(t), t)$ we have

$$\left(\frac{\partial A(r(t), t)}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 A(r(t), t)}{\partial r^2} - rA(r(t), t) \right) / \frac{\partial A(r(t), t)}{\partial r} = a(r(t), t)$$

for some function $a(r(t), t)$. We write

$$a(r(t), t) = \sigma(r(t), t)\lambda(r(t), t) - \mu(r(t), t).$$

For a given $\sigma(r(t), t)$ (not identically zero) and $\mu(r(t), t)$ this is always possible. The function $\lambda(r(t), t)$ is known as the market price of risk.

The general pricing equation for determining an amortization function is as follows:

$$\begin{aligned} \frac{\partial A(t, r(t), T)}{\partial t} + \frac{\partial A(t, r(t), T)}{\partial r} \left[\mu(r, t) - \lambda(r, t)\sigma(r, t) \right] \\ + \frac{1}{2} \frac{\partial^2 A(t, r(t), T)}{\partial r^2} \sigma^2(r, t) = r(t)A(t, r(t), T). \end{aligned}$$

□

We are now in a position to interpret the market price of risk $\lambda(r(t), t)$. Suppose we choose to hold just one loan with maturity date T instead of holding the hedged portfolio constructed above. Then the value of the loan changes over the interval $(t, t + dt]$ by

$$dA = \sigma \frac{\partial A(r(t), t)}{\partial r} dX(t) + \left(\frac{\partial A(r(t), t)}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 A(r(t), t)}{\partial r^2} + \mu \frac{\partial A(r(t), t)}{\partial r} \right) dt. \quad (3.20)$$

From the general pricing equation (3.17) the value of the loan changes to

$$\begin{aligned} dA &= \sigma \frac{\partial A(r(t), t)}{\partial r} dX(t) + \left(\sigma \lambda \frac{\partial A(r(t), t)}{\partial r} + rA(r(t), t) \right) dt, \\ dA - rAdt &= \sigma \frac{\partial A(r(t), t)}{\partial r} \left(dX(t) + \lambda dt \right). \end{aligned} \quad (3.21)$$

The existence of the random term dX (a Wiener process) shows that the hedge portfolio is not riskless. The right-hand side of expression (3.21) is the excess return above the risk-free interest rate for accepting a certain level of risk. Wilmott, Howison, Dewynne [93] states that by taking on the extra risk the portfolio profits by an extra λdt per unit

of extra risk.

Obtaining the amortization function from the general pricing equation (3.17) we will be following a similar procedure proposed by Mamon [68]. Mamon [68] propose three ways to derive bond prices (in our case an amortization function) in the Vasicek interest rate model. In the first case the bond price is derived based on the short rate distribution. In the second case the bond price is obtained by solving the general pricing equation (3.17) and thirdly the price of the bond is obtained within the Heath-Jarrow-Morton framework (HJM).

Vasicek [91] proposed the dynamics of the risk free interest rate as follows:

$$dr(t) = \alpha(\gamma - r(t))dt + \sigma d\tilde{X}(t), \quad (3.22)$$

where $\tilde{X}(t)$ is a standard brownian motion under the risk-neutral measure \mathbb{Q} and α, γ and σ are all constants with $\alpha > 0$. The drift term $\alpha(\gamma - r(t))$ has the property of being mean reverting, that is, the short term interest rate is pulled back to its long-term mean, γ , and σ represents the volatility of the interest rate. The adjustment parameter, α , determines how quickly the interest rate $r(t)$ converge to its long-term mean γ . Therefore the higher the value of α , the closer the interest rate $r(t)$ will be to the average mean. The interest rate process (3.22) is characterised as an Ornstein-Uhlenbeck process which means that it is characterised by stationary distribution. The aforementioned model is defined as a term structure model having the characteristics that the interest rate $r(t)$ is autoregressive, that is $r(t)$ cannot drift off to $+\infty$ or $-\infty$ or to 0, but will eventually be pulled back to some long-term target and by deriving simple formulae for bond prices (in this case an amortization function). The explicit solution to the stochastic differential equation (3.22) is given by

$$r(t) = \exp(-\alpha t) \left[r(0) + \int_0^t \alpha \gamma \exp(\alpha u) du + \sigma \int_0^t \exp(\alpha u) dX(u) \right]. \quad (3.23)$$

Expression (3.23) can be obtained by using Itô's formula.

3.2 The Martingale Approach

Under the martingale-oriented approach (see Mamon [68]) the derivation is based on the assumption that r_u is a markovian process. In practice this means that determining the future value of $r(u) = r_u$ solely depends on the current value $r(t) = r_t$ where $t \leq u \leq T$ and the knowledge of its past is irrelevant. However, we point out that in the following proposition we only derive the partial differential equation under a martingale approach. We derive the amortization function in section 3.3. Therefore we avoid the computation of the function in the following proposition since both proposition 3.2.1 and proposition 3.3.1 will derive the resulting amortization function in a similar fashion.

Proposition 3.2.1 *Suppose that r_u is a markov diffusion process and under the measure \mathbb{Q} , r_u satisfies the stochastic differential equation $dr(r_u, u) = \mu(r_u, u)du + \sigma(r_u, u)d\tilde{X}_u$ where \tilde{X}_u is a standard brownian motion and by the Feynman-Kac formula the amortization function is expressed as*

$$A(t, r_t, T) = \tilde{\mathbb{E}}_{\mathbb{Q}} \left[\exp \left(- \int_t^T r_u du \right) \middle| r_t \right]$$

and r_u is given by

$$r_u = \exp \left(- \alpha(u - t) \right) \left[r_t + \gamma \left(\exp \left(\alpha(u - t) \right) - 1 \right) + \sigma \int_t^u \exp \left(\alpha(s - t) \right) dX_s \right].$$

Then the amortization function satisfies the following partial differential equation

$$\begin{aligned} \frac{\partial A(t, r(t), T)}{\partial t} + \frac{\partial A(t, r(t), T)}{\partial r} [\mu(r, t) - \lambda(r, t)\sigma(r, t)] \\ + \frac{1}{2} \frac{\partial^2 A(t, r(t), T)}{\partial r^2} \sigma^2(r, t) = r(t)A(t, r(t), T). \end{aligned} \quad (3.24)$$

Proof.

The symbol $\tilde{\mathbb{E}}_{\mathbb{Q}}$ represents the expectation operator under the risk neutral measure \mathbb{Q} . Since r_t is a parameter we can obtain the partial derivative of r_u with respect to r_t as follows

$$\frac{\partial r_u}{\partial r_t} = \exp \left(- \alpha(u - t) \right). \quad (3.25)$$

Integrating equation (3.25) yields

$$\begin{aligned}\int_t^T \frac{\partial r_u}{\partial r_t} du &= \int_t^T \exp\left(-\alpha(u-t)\right) du \\ &= \frac{1}{\alpha} \left(1 - \exp\left(-\alpha(T-t)\right)\right),\end{aligned}$$

which is deterministic. Taking the partial derivative of the Amortization function $A(t, r_t, T)$ with respect to the interest rate r_t we obtain

$$\begin{aligned}\frac{\partial A(t, r_t, T)}{\partial r_t} &= \tilde{\mathbb{E}}_{\mathbb{Q}} \left[- \left(\int_t^T \frac{\partial r_u}{\partial r_t} du \right) \exp \left(- \int_t^T r_u du \right) \right] \\ &= -\frac{1}{\alpha} \left(1 - \exp\left(-\alpha(T-t)\right)\right) \tilde{\mathbb{E}}_{\mathbb{Q}} \left[\exp \left(- \int_t^T r_u du \right) \right] \\ &= -D(t, T)A(t, r_t, T).\end{aligned}$$

Thus, $\frac{\partial A}{\partial r_t} = -D(t, T)A(t, r_t, T)$. So we have,

$$A(t, r_t, T) = C(t, T) \exp\left(-D(t, T)r_t\right),$$

for some unknown function $C(t, T)$ independent of r_t . Consider

$$\begin{aligned}\exp\left(-\int_0^t r_u du\right) A(t, r(t), T) &= \exp\left(-\int_0^t r_u du\right) \tilde{\mathbb{E}}_{\mathbb{Q}} \left[\exp\left(-\int_t^T r_u du\right) \middle| \mathcal{F}_t \right] \\ &= \tilde{\mathbb{E}}_{\mathbb{Q}} \left[\exp\left(-\int_0^t r_u du\right) \exp\left(-\int_t^T r_u du\right) \middle| \mathcal{F}_t \right] \\ &= \tilde{\mathbb{E}}_{\mathbb{Q}} \left[\exp\left(-\int_0^t r_u du - \int_t^T r_u du\right) \middle| \mathcal{F}_t \right] \\ &= \tilde{\mathbb{E}}_{\mathbb{Q}} \left[\exp\left(-\int_0^T r_u du\right) \middle| \mathcal{F}_t \right].\end{aligned}$$

The expression above is a \mathbb{Q} -martingale by the tower property. Applying Itô's formula (1.3) and writing it in integral form, we obtain the following

$$\begin{aligned}\exp\left(-\int_0^t r_u du\right) A(t, r_t, T) &= A(0, r_0, T) + \int_0^t -r_u \exp\left(-\int_0^u r_s ds\right) A(u, r_u, T) du \\ &\quad + \int_0^t \exp\left(-\int_0^u r_s ds\right) \frac{\partial}{\partial u} A(u, r_u, T) du \\ &\quad + \int_0^t \exp\left(-\int_0^u r_s ds\right) \frac{\partial}{\partial r_u} A(u, r_u, T) (dr) \\ &\quad + \frac{1}{2} \int_0^t \exp\left(-\int_0^u r_s ds\right) \frac{\partial^2}{\partial r_u^2} A(u, r_u, T) \sigma^2 du\end{aligned}$$

where the expression $dr = \mu(r_u, u)du + \sigma(r_u, u)dX_u$ is the general form for continuous-time interest rate models. In terms of the Vasicek model $\mu(r_u, u) = \alpha(\gamma - r_u)$ and $\sigma(r_u, u) = \sigma$. Since the expression above is a martingale, all the du terms must sum to zero. Therefore,

$$\begin{aligned}
 -r_t A(t, r_t, T) + \frac{\partial}{\partial t} A(t, r_t, T) + \frac{\partial}{\partial r_t} A(t, r_t, T) (\mu(r_u, u)) \\
 + \frac{\sigma^2}{2} \frac{\partial^2}{\partial r_t^2} A(t, r_t, T) = 0. \quad (3.26)
 \end{aligned}$$

□

3.3 Analytical solution for the amortization function

We are now in the position to formulate a proposition for obtaining an explicit formula for the amortization function (see Cairns [31], Mamon [68] for deriving the partial differential equation and the martingale approach).

Proposition 3.3.1 *Suppose that the interest rate is modelled by the following stochastic differential equation*

$$dr(t) = \alpha(\gamma - r(t))dt + \sigma d\tilde{X}(t),$$

the market price of risk $\lambda(\mu, \sigma) = \lambda$ is constant and given the condition $A(t, r(t), T) = S(0) = 1$ where the amortization function satisfies the partial differential equation (3.17) then the amortization function has the form

$$\begin{aligned}
 A(t, r(t), T) &= \exp \left\{ C(t, T) - D(t, T)r(t) \right\}, \quad (3.27) \\
 &= \exp \left\{ \left(\gamma + \frac{\sigma\lambda}{\alpha} - \frac{\sigma^2}{2\alpha^2} \right) \left[\frac{1}{\alpha} \left(1 - \exp \left(-\alpha(T-t) \right) \right) - (T-t) \right] \right. \\
 &\quad \left. - \frac{\sigma^2}{4\alpha^3} \left(1 - \exp \left(-\alpha(T-t) \right) \right)^2 - \frac{1}{\alpha} \left(1 - \exp \left(-\alpha(T-t) \right) \right) r(t) \right\}.
 \end{aligned}$$

Proof.

Solving equation (3.17) we need to specify the parameters of the Vasicek interest rate

model, define the market price of risk, $\lambda(r, t)$ and apply the condition $A(t, r(t), T) = S(0) = 1$.

In terms of the Vasicek model we define, $\mu(r, t) = \alpha(\gamma - r)$; $\sigma(r, t) = \sigma$ and $\lambda(r, t) = \lambda$. Therefore the general pricing equation becomes:

$$\begin{aligned} \frac{\partial A(t, r(t), T)}{\partial t} + \frac{\partial A(t, r(t), T)}{\partial r} [\alpha(\gamma - r) - \lambda\sigma] \\ + \frac{1}{2} \frac{\partial^2 A(t, r(t), T)}{\partial r^2} \sigma^2(r, t) = r(t)A(t, r(t), T) \end{aligned}$$

subject to the condition $A(r(t), T, T) = 1$ for a loan with the principal $S(0) = 1$ (Baz, Chacko [20]). We guess an amortization function having the form:

$$A(t, r(t), T) = C(t, T) \exp \left\{ -D(t, T)r(t) \right\} \quad (3.28)$$

for some unknown functions $C(t, T)$ and $D(t, T)$. Obtaining partial derivatives of equation (3.28) yields:

$$\begin{aligned} \frac{\partial A(t, r(t), T)}{\partial t} &= \frac{dC(t, T)}{dt} \exp \left\{ -D(t, T)r(t) \right\} \\ &\quad - rC(t, T) \frac{dD(t, T)}{dt} \exp \left\{ -D(t, T)r(t) \right\}; \\ \frac{\partial A(t, r(t), T)}{\partial r} &= -C(t, T)D(t, T) \exp \left\{ -D(t, T)r(t) \right\}; \\ \frac{\partial^2 A(t, r(t), T)}{\partial r^2} &= C(t, T)D(t, T)^2 \exp \left\{ -D(t, T)r(t) \right\}. \end{aligned}$$

Substituting the partial derivatives into the general pricing equation (3.17) yields:

$$\begin{aligned} \frac{dC(t, T)}{dt} \exp \left\{ -D(t, T)r(t) \right\} - rC \frac{dD(t, T)}{dt} \exp \left\{ -D(t, T)r(t) \right\} \\ - (\alpha(\gamma - r) + \lambda\sigma)CD \exp \left\{ -D(t, T)r(t) \right\} \\ + \frac{1}{2} \sigma^2 CD^2 \exp \left\{ -D(t, T)r(t) \right\} - rC \exp \left\{ -D(t, T)r(t) \right\} = 0. \end{aligned}$$

Cancelling the factor $\exp \left\{ -D(t, T)r(t) \right\}$ and rearranging the terms we have

$$\begin{aligned} \frac{dC(t, T)}{dt} - rC \frac{dD(t, T)}{dt} - (\alpha(\gamma - r) + \lambda\sigma)CD + \frac{1}{2}\sigma^2CD^2 - rC &= 0 \\ \frac{dC(t, T)}{dt} - (\alpha\gamma + \sigma\lambda)CD + \frac{1}{2}\sigma^2CD^2 &= rC + rC \frac{dD(t, T)}{dt} - \alpha rCD \\ &= rC \left(1 + \frac{dD(t, T)}{dt} - \alpha D \right), \end{aligned} \quad (3.29)$$

where the notation C and D represents $C(t, T)$ and $D(t, T)$ respectively. Since the right-hand side of expression (3.29) is a function of the interest rate r and the left-hand side is a function of t and T only then the following must hold:

$$\frac{dC(t, T)}{dt} - (\alpha\gamma + \sigma\lambda)CD + \frac{1}{2}\sigma^2CD^2 = 0 \quad (3.30)$$

and

$$\left(1 + \frac{dD(t, T)}{dt} - \alpha D \right) = 0. \quad (3.31)$$

Equations (3.30) and (3.31) are both separable ordinary differential equations. Solving equation (3.31) with boundary condition $D(T, T) = 0$ we obtain the following

$$D(t, T) = \frac{1}{\alpha} \left(1 - \exp \left(-\alpha(T - t) \right) \right).$$

The reason for specifying the conditions in such a manner follows from the fact that $A(T, r(t), T) = 1$, therefore:

$$C(T, T) \exp \left\{ -D(t, T)r(t) \right\} = 1 \quad \forall t. \quad (3.32)$$

Solving for $C(t, T)$ with boundary condition $C(T, T) = 1$ yields the following:

$$\begin{aligned} \frac{dC(t, T)}{dt} - (\alpha\gamma + \sigma\lambda)CD + \frac{1}{2}\sigma^2CD^2 &= \frac{dC(t, T)}{dt} - CD\alpha\gamma - \sigma\lambda CD + \frac{1}{2}\sigma^2CD^2 \\ &= \frac{dC(t, T)}{dt} + CD \left(\frac{1}{2}\sigma^2D - \alpha\gamma - \sigma\lambda \right) \\ &= \frac{dC(t, T)}{dt} + \frac{\sigma^2}{2}CD^2 - (\alpha\gamma + \sigma\lambda)CD \\ &= \frac{dC(t, T)}{C} + \frac{\sigma^2}{2}D^2 - (\alpha\gamma + \sigma\lambda)Ddt = 0. \end{aligned}$$

Integrating over a time interval $(t, T]$ and let $d\zeta = dt$ observe the following:

$$\begin{aligned}
& \int_t^T \frac{1}{C} dC + \frac{\sigma^2}{2\alpha^2} \int_t^T \left(1 - \exp\left(-\alpha(T - \zeta)\right)\right)^2 d\zeta - \\
& \quad \left(\gamma + \frac{\sigma\lambda}{\alpha}\right) \int_t^T \left(1 - \exp\left(-\alpha(T - \zeta)\right)\right) d\zeta = 0 \\
\Rightarrow & \ln C(T, T) - \ln C(t, T) + \frac{\sigma^2}{2\alpha^2} \left[\zeta - \frac{2}{\alpha} \exp\left(-\alpha(T - \zeta)\right) \right. \\
& \quad \left. + \frac{1}{2\alpha} \exp\left(-2\alpha(T - \zeta)\right) \right]_{\zeta=t}^{\zeta=T} \\
& \quad - \left(\gamma + \frac{\sigma\lambda}{\alpha}\right) \left[\zeta - \frac{1}{\alpha} \exp\left(-\alpha(T - \zeta)\right) \right]_{\zeta=t}^{\zeta=T} = 0 \\
\Rightarrow & \ln C(T, T) - \ln C(t, T) + \frac{\sigma^2}{2\alpha^2} \left[T - \frac{2}{\alpha} + \frac{1}{2\alpha} - t + \frac{2}{\alpha} \exp\left(-\alpha(T - t)\right) \right. \\
& \quad \left. - \frac{1}{2\alpha} \exp\left(-2\alpha(T - t)\right) \right] \\
& \quad - \left(\gamma + \frac{\sigma\lambda}{\alpha}\right) \left[T - t - \frac{1}{\alpha} + \frac{1}{\alpha} \exp\left(-\alpha(T - t)\right) \right] = 0 \\
\Rightarrow & \ln C(T, T) - \ln C(t, T) + \frac{\sigma^2}{2\alpha^2} \left[T - t - \frac{2}{\alpha} \left(1 - \exp\left(-\alpha(T - t)\right)\right) \right. \\
& \quad \left. + \frac{1}{2\alpha} \left(1 - \exp\left(-2\alpha(T - t)\right)\right) \right] \\
& \quad - \left(\gamma + \frac{\sigma\lambda}{\alpha}\right) \left[T - t - \frac{1}{\alpha} \left(1 - \exp\left(-\alpha(T - t)\right)\right) \right] = 0
\end{aligned}$$

since $C(T, T) = 1$:

$$\begin{aligned}
\ln C(t, T) &= \frac{\sigma^2}{2\alpha^2} \left[T - t - \frac{2}{\alpha} \left(1 - \exp\left(-\alpha(T - t)\right)\right) + \frac{1}{2\alpha} \left(1 - \exp\left(-2\alpha(T - t)\right)\right) \right] \\
& \quad - \left(\gamma + \frac{\sigma\lambda}{\alpha}\right) \left[T - t - \frac{1}{\alpha} \left(1 - \exp\left(-\alpha(T - t)\right)\right) \right] \\
&= \left(\gamma + \frac{\sigma\lambda}{\alpha} - \frac{\sigma^2}{2\alpha^2}\right) \left[\frac{1}{\alpha} \left(1 - \exp\left(-\alpha(T - t)\right)\right) - (T - t) \right] \\
& \quad - \frac{\sigma^2}{4\alpha^3} \left(1 - \exp\left(-\alpha(T - t)\right)\right)^2. \tag{3.33}
\end{aligned}$$

The amortization function may now be represented as follows:

$$\begin{aligned}
A(r(t), t, T) &= \exp \left\{ \left(\gamma + \frac{\sigma\lambda}{\alpha} - \frac{\sigma^2}{2\alpha^2} \right) \left[\frac{1}{\alpha} \left(1 - \exp \left(-\alpha(T-t) \right) \right) - (T-t) \right] \right. \\
&\quad \left. - \frac{\sigma^2}{4\alpha^3} \left(1 - \exp \left(-\alpha(T-t) \right) \right)^2 \right\} \times \\
&\quad \exp \left\{ -\frac{1}{\alpha} \left(1 - \exp \left(-\alpha(T-t) \right) \right) r(t) \right\}.
\end{aligned}$$

The amortization function may also be represented as:

$$\begin{aligned}
A(t, r(t), T) &= \exp \left\{ C(t, T) - D(t, T)r(t) \right\}, \tag{3.34} \\
&= \exp \left\{ \left(\gamma + \frac{\sigma\lambda}{\alpha} - \frac{\sigma^2}{2\alpha^2} \right) \left[\frac{1}{\alpha} \left(1 - \exp \left(-\alpha(T-t) \right) \right) - (T-t) \right] \right. \\
&\quad \left. - \frac{\sigma^2}{4\alpha^3} \left(1 - \exp \left(-\alpha(T-t) \right) \right)^2 - \frac{1}{\alpha} \left(1 - \exp \left(-\alpha(T-t) \right) \right) r(t) \right\}.
\end{aligned}$$

□

Here $C(t, T)$ and $D(t, T)$ provide information about the principal and outstanding amounts, credit rating of the debtor, the amortization rate and the effect of default and debtor bankruptcy.



3.3.1 Types of Loans

There exists three types of loans namely a fixed rate loan, series loan and an annuity loan (Norberg [78]). The interest rate on the contract is fixed and the duration of the contract is only valid on the interval $0 \leq t \leq n$. A fixed rate loan is entirely repaid at the end of the contract, that is, $F(t, r(t)) = \epsilon_n(t)$ where

$$\epsilon_n(t, r(t)) = \begin{cases} 0, & \text{if } 0 \leq t < n \\ 1, & \text{if } t \geq n. \end{cases}$$

The amortization function is obtained directly from (3.10):

$$A(t, r(t)) = \epsilon_n(t, r(t)) + rt. \quad (3.35)$$

A series loan has repayments of an annuity form. Therefore it is as a recurring periodic series of payments over a specified period of time. The continuous version of the series loan is given by $F(t, r(t)) = \frac{t}{n}$. The amortization function is obtained from (3.10):

$$\begin{aligned} A(t, r(t)) &= F(t, r(t)) + \int_0^t (1 - F(\tau))r(\tau)d\tau \\ &= \frac{t}{n} + rt \left(1 - \frac{t}{2n}\right). \end{aligned} \quad (3.36)$$

We also obtain the differential equation (3.9):

$$\begin{aligned} \frac{dA(t, r(t))}{dt} &= \frac{dF(t, r(t))}{dt} + \frac{dI(t, r(t))}{dt} \\ &= \frac{1}{n} + r \left(1 - \frac{t}{n}\right), \end{aligned} \quad (3.37)$$

where $\frac{dF(t, r(t))}{dt}$ is fixed and $\frac{dI(t, r(t))}{dt}$ is linear decreasing.

An annuity loan is called so because the instalments are the same amount throughout the repayment period, assuming the interest rate remains the same. To start with the interest portion of the instalment is high and the repayment portion of the instalment is low. As the loan is repaid, the interest portion decreases and the loan repayment portion increases. The continuous version of this type of loan is given by $A(t, r(t)) = \frac{t}{\bar{a}_n}$ where

\bar{a}_n is given by:

$$\begin{aligned}\bar{a}_n &= \int_0^n \exp(-r\tau) d\tau \\ &= \frac{1 - \exp(-rn)}{r}.\end{aligned}\tag{3.38}$$

We obtain the repayment function $F(t, r(t))$ from equation (3.13):

$$\begin{aligned}F(t) &= 1 - \frac{\bar{a}_{n-t}}{\bar{a}_n} \\ &= 1 - \frac{1 - \exp(-r(n-t))}{1 - \exp(-rn)}.\end{aligned}\tag{3.39}$$

We apply the differential equation (3.9) to obtain:

$$\begin{aligned}\frac{dA(t, r(t))}{dt} &= \frac{dF(t, r(t))}{dt} + \frac{dI(t, r(t))}{dt} \\ &= \frac{\exp(-r(n-t))}{\bar{a}} + \frac{1 - \exp(-r(n-t))}{\bar{a}}.\end{aligned}\tag{3.40}$$

In the case where $n = \infty$ both the fixed and series loan contracts specializes to an infinite loan without complete repayment.

3.4 Dynamic modelling of the loan-issuing rate

One of the biggest assumptions to make about interest rates is to consider it to be constant. In reality, this is becoming less fashionable since financial markets have become more sophisticated and complex. There are many securities with a longer duration that are influenced by trading in these markets. Therefore short-term stochastic interest rate models (such as Vasicek, Cox-Ingersoll, Ho-Lee) have been developed (see for instance Cairns [31] and Baz, Chacko [20] for a description on interest rate models).

The rate of return $r(t)$ may behave in an unpredictable manner and therefore it can be modelled as a one-factor diffusion process (see, for instance, Cairns, Blake, Dowd [32]) by the following stochastic differential equation:

$$\begin{aligned} dr(t) &= \mu_r(r(t))dt + \sum_{k=1}^N \sigma_{rk}(r(t))dX_k(t), \\ &= \mu_r(r(t))dt + \sigma'_r(r(t))dX(t). \end{aligned} \quad (3.41)$$

where $X(t) = (X_1(t), \dots, X_N(t))'$ are independent standard brownian motions. We further define $\sigma_r(r) = (\sigma_{r1}(r), \dots, \sigma_{rN}(r))'$, where σ_r is the r -th row of the $n \times n$ volatility matrix $(\sigma_{rk})_{r,k=1}^N$ (see Fouche, Mukkudem-Petersen, Petersen [46]).

Fouche, Mukkudem-Petersen, Petersen [46] model the loans applied exogeneously which can be expressed by a stochastic integral formula:

$$l(t) = l(0) + \int_0^T r_l(s)ds + \int_0^T \sigma_l(s)dX_l(s). \quad (3.42)$$

The lending model can be expressed in differential form by the dynamics:

$$dl(t) = r_l(t)dt + \sigma_l(t)dX_l(t), \quad (3.43)$$

where $l : \Omega \times T \rightarrow \mathbb{R}$ is a stochastic process denoting the loan issuing rate whose value at time t is represented by $l(t)$, $\sigma_l(t)$ denotes the volatility (unpredictable movement of the process) and $X_l : \Omega \times T \rightarrow \mathbb{R}$ is a standard Brownian motion satisfying the properties of (1.2.6). Under these characteristics, the loan issuing rate is described by the following

dynamics,

$$dl(t) = l(t) \left[\phi_l \left(\mu - \ln l(t) \right) dt + \sigma_l dX_l(t) \right], \quad (3.44)$$

where ϕ_l represents the rate of mean reversion to the long run mean denoted by μ . This model is proposed in this study and is distinct from those in the aforementioned bank lending literature. This mean-reversion model has been employed by Fouche, Mukkudem-Petersen, Petersen [46] whereby they model the loan issuing rate as a stochastic process. Applying Itô's formula (1.3) to the log-normal diffusion process, $S_t = \ln l(t)$, yields:

$$\begin{aligned} d(S_t) &= \frac{1}{S_t} \left[\phi_l \left(\mu - S_t \right) S_t dt + \sigma_l S_t dX_l(t) \right] + 0 \\ &\quad + \frac{1}{2} \left(\frac{-1}{S_t^2} \right) \sigma_l^2 S_t^2 dt \\ &= \phi_l \left(\mu - S_t \right) dt + \sigma_l dX_l(t) - \frac{1}{2} \sigma_l^2 dt. \end{aligned}$$

Grouping similar terms together, yields:

$$\begin{aligned} d(S_t) &= \left[\phi_l \left(\mu - S_t \right) - \frac{1}{2} \sigma_l^2 \right] dt + \sigma_l dX_l(t) \\ &= \phi_l \left(\hat{\alpha} - S_t \right) dt + \sigma_l dX_l(t), \end{aligned} \quad (3.45)$$

where $\hat{\alpha} = \mu - \frac{\sigma^2}{2\phi_l}$. The loan issuing rate can now be characterized as an Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process is expressed in the following manner:

$$dl(t) = \left[\phi_l \left(\mu - l(t) \right) dt + \sigma_l dX_l(t) \right]. \quad (3.46)$$

According to [87] the Ornstein-Uhlenbeck process is used to model commodities such as agricultural products, metals, petroleum, foreign currencies, financial instruments, indexes and physical items such as oil and gold. The Ornstein-Uhlenbeck process, $l : \Omega \times T \rightarrow \mathbb{R}$ can be modelled as a path-continuous scalar Itô process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and it can be represented by the stochastic integral formula:

$$\begin{aligned} l(t) &= l(0) \exp(-\phi_l t) + \mu(1 - \exp(-\phi_l t)) \\ &\quad + \sigma_l \exp(-\phi_l t) \int_0^t \exp(\phi_l s) dX_l(s). \end{aligned} \quad (3.47)$$

The stochastic variable S_t has a normal distribution with an expected value, denoted by $\mathbb{E}[l(t)]$, and a variance denoted by $Var[l(t)]$, respectively under an equivalent martingale measure as follows:

$$\begin{aligned}\mathbb{E}[l(t)] &= \mu + (X(0) - \mu) \exp(-\phi_l t) \\ &= X(0) \exp(-\phi_l t) + \mu(1 - \exp(-\phi_l t)).\end{aligned}$$

where $\mathbb{E} \int_0^t \exp(-\phi_l(t-s)) dX_l(s) = 0$ and for the variance:

$$\begin{aligned}Var[l(t)] &= \mathbb{E}\{l(t) - \mathbb{E}[l(t)]\}^2 \\ &= \sigma^2 \mathbb{E} \left\{ \int_0^t \exp(-\phi_l(t-s)) dX_l(s) \right\}^2 \\ &= \sigma^2 \int_0^t \mathbb{E} \{ \exp(-\phi_l(t-s)) dX_l(s) \}^2 \\ &= \sigma^2 \int_0^t \exp(-2\phi_l(t-s)) ds \\ &= \frac{\sigma^2(1 - \exp(-2\phi_l t))}{2\phi_l}.\end{aligned}$$

The parameters for the mean reversion, $\hat{\alpha}$ and ϕ_l , can be estimated by regressional changes using a time series technique called differencing. The Ornstein-Uhlenbeck process is the continuous-time version of the first-order autoregressive time series process AR(1). Applying the difference method on equation (3.47) (see [87]) yields:

$$\begin{aligned}dl(t) &= l(t) - l(t-1) \\ &= \mu(1 - \exp(-\phi_l \Delta t)) + (\exp(-\phi_l \Delta t) - 1)l(t-1) + \varepsilon(t),\end{aligned}\tag{3.48}$$

where $\varepsilon(t)$ are independent, identically distributed normal random variables with mean zero and standard deviation, σ_ε , that is $\varepsilon_t \sim N(0, \sigma_\varepsilon)$. In order to estimate the regression for the loan issuing rate, we run the regression:

$$l(t) - l(t-1) = b + cl(t-1) + \varepsilon(t),$$

where $b = \mu(1 - \exp(-\phi_l \Delta t))$ and $c = (\exp(-\phi_l \Delta t) - 1)$. Estimating the parameters $\hat{\alpha}$ and ϕ yields:

$$\begin{aligned}\mu &= \frac{b}{c}; \\ \phi_l &= -\ln(1 + b); \\ \sigma &= \sigma_\varepsilon \sqrt{\frac{2 \ln(1 + b)}{(1 + b)^2 - 1}}.\end{aligned}\tag{3.49}$$

In the paper of Mikkudem-Petersen, Petersen [73], the loan-issuing rate is represented by means of a geometric brownian motion. The advantages of this type of model is that it makes the problem more analytically tractable and it provides a closed form solution which could be simulated. The geometric brownian motion is one of the most widely used continuous stochastic processes in economic theory with applications to option pricing, equities, commodities and stock prices. The increments of the loan-issuing rate will follow a lognormal distribution. The behaviour of the loan-issuing rate can be represented by a path-continuous scalar Ito process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as

$$\delta(t) = \delta_0 + \int_0^t \mu_\delta(\delta(s), s) ds + \int_0^t \sigma_\delta(\delta(s), s) dX_\delta(s), t \geq 0.\tag{3.50}$$

The integral equation (3.50) are represented by the following stochastic differential equation:

$$d\delta(t) = \delta(t) \left[\mu_\delta dt + \sigma_\delta dX_\delta(t) \right],\tag{3.51}$$

where the drift parameter is denoted by $\mu_\delta \in \mathbb{R}$ and the volatility parameter is denoted by $\sigma_\delta \in \mathbb{R}_+$ in the loan-issuing rate, respectively. The differential $dX_\delta(t)$ represents the economic shocks that the loan-issuing rate is exposed to. Applying Itô's formula (1.3) to the log-normal diffusion process, $F(\delta(t), t) = \log \delta(t)$, yields:

$$\begin{aligned}dF(\delta(t), t) &= \left(\frac{1}{\log \delta(t)} \mu_\delta \log \delta(t) + F(t) + \frac{1}{2} \left(\frac{-1}{\log^2 \delta(t)} \right) \sigma_\delta^2 \log^2 \delta(t) \right) dt \\ &+ \frac{1}{\log \delta(t)} \sigma_\delta \log \delta(t) dX_\delta(t) \\ &= \left\{ \mu_\delta - \frac{1}{2} \sigma_\delta^2 \right\} dt + \sigma_\delta dX_\delta(t).\end{aligned}$$

Integrating over the interval from $(0, T]$ yields the following:

$$\begin{aligned}
 \log \delta(t) &= \log \delta(0) + \int_0^t dF(\delta(s), s) \\
 &= \log \delta(0) + \int_0^t \left\{ \mu - \frac{1}{2}\sigma_\delta^2 \right\} ds + \int_0^t \sigma_\delta dX_\delta(s) \\
 &= \log \delta(0) + \left\{ \mu - \frac{1}{2}\sigma_\delta^2 \right\} t + \sigma_\delta X_\delta(t) \\
 \delta(t) &= \delta_0 \exp\left(\left(\mu - \frac{1}{2}\sigma_\delta^2\right)t + \sigma_\delta X_\delta(t)\right), \tag{3.52}
 \end{aligned}$$

where we assume $\delta(0) = \delta_0$.

Therefore the change in the loan-issuing rate will grow at an exponential rate. Modelling the loans by a geometric brownian process may have its advantages but it also has its drawbacks such as that the loan issuing rate may explode towards infinity which is not realistic. We illustrate by producing two graphs of the disadvantages of modelling the loan-issuing rate as a geometric brownian motion over period of 10 years and 20 years respectively.

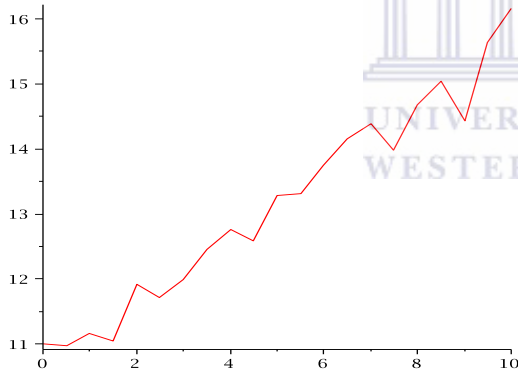


Figure 3.1: Modelling the loan-issuing rate as a geometric brownian motion over a period of 10 years, that is, $0 \leq t \leq 10$;

Parameters are $\sigma = 0.04$ and $\mu = 0.04$.

The initial interest rate is $\delta_0 = 11\%$.

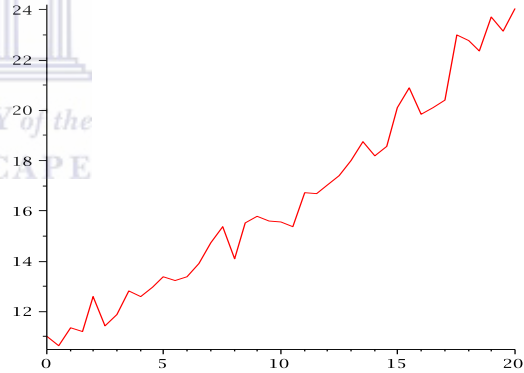


Figure 3.2: Modelling the loan-issuing rate as a geometric brownian motion over a period of 20 years, that is, $0 \leq t \leq 20$;

Parameters are $\sigma = 0.04$ and $\mu = 0.04$.

The initial interest rate is $\delta_0 = 11\%$.

Instead the dynamics of the loan-issuing rate may be represented by means of an Ornstein-

Uhlenbeck process. In this study we choose to model the loan-issuing rate via the square root process. The mean reverting square root process (also known as the Cox-Ingersoll-Ross model) (see Cox, Ingersoll, Ross [35]) is a stochastic differential equation that has been widely applied to forecast interest rates and other financial quantities (see for instance Adkins, Krehbiel [1], Bhanot [23]). It is an alternative model to that of the Vasicek model because of its desirable property of positivity and its richness of behaviour. The aforementioned model does not allow the variability of the interest rates to grow too large as interest rates rises. The mean reverting square root process has the form

$$dl(t) = \phi_l \left(\mu - l(t) \right) dt + \sigma_l \sqrt{l(t)} dX_l(t), \quad (3.53)$$

where ϕ , μ and σ_l are positive constants and dX_l is a scalar brownian motion.

We illustrate the model by producing two graphs where the bank for instance charges 11% on their loans contracts to business and other financial institutions over period of 20 years and 30 years respectively.

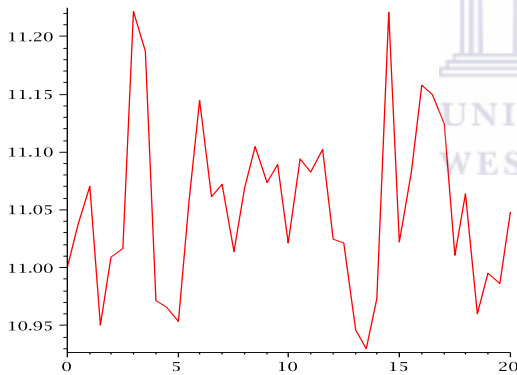


Figure 3.3: Modelling the loan-issuing rate as a square root process over a period of 20 years, that is, $0 \leq t \leq 20$; Parameters are $\sigma = 0.04$ and $\mu = 0.04$. The initial interest rate is $\delta_0 = 11\%$.

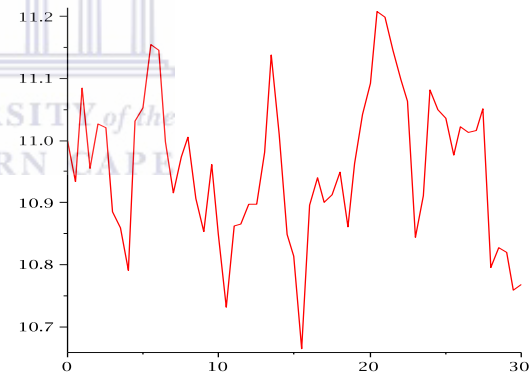


Figure 3.4: Modelling the loan-issuing rate as a square root process over a period of 30 years, that is, $0 \leq t \leq 30$; Parameters are $\sigma = 0.04$ and $\mu = 0.07$. The initial interest is $\delta_0 = 11\%$.

Chapter 4

Capital Adequacy Ratios

The Basel committee on banking supervision (BCBS) drafted a document, the *1988 Basel Accord*, that was aimed at how banks should manage and regulate their capital requirements. This accord was an attempt to develop regulatory requirements of the banking industry with four objectives in mind:

- to protect depositors and deposit insurance from the ravages of reckless portfolio management by banks;
- to prevent system instabilities arising from bank failures;
- to strengthen the soundness and stability of the international banking system;
- to be applied with a high degree of consistency with a view to remove any source of undesirable competitive behaviour among internationally active banks.

The 1988 Basel Accord consolidated capital requirements as the cornerstone of bank regulation. The 1988 Basel Accord required banks to hold a minimum capital-to-risk-weighted assets ratio of at least 8% (see for instance Berger, Herring, Szego [21] and Dewatripont, Tirole [39]). According to von Thadden [92] this ratio is used to protect depositors and deposit insurance schemes from the ravages of inadequate or reckless portfolio management and promote the stability and efficiency of the banking structure. However, the 1988 Basel Accord, received widespread criticism for being too crude and oversimplified with

the ever-changing standards set for the management and assessment of banking performances. The 1988 Basel Accord, also known as the *Basel I Accord*, was further criticized for treating all credit risk-types alike which potentially could lead to regulatory arbitrage and it also seems to neglect contemporary credit risk management techniques. Moreover the 1988 Basel Accord also failed to take into account the dynamic distortions of capital regulation and complementary regulatory instruments such as supervisory monitoring or prompt corrective regulatory action (see for instance Altmann [3] and Jackson, Perraudin [57]). Reacting to these criticisms, the BCBS made several adjustments to the 1988 Basel Accord document which led to the existence of the first consultative paper (see [11]). Von Thadden [92] further states that experiments carried out from the first consultative paper in the banking sector has resulted in a second and third consultative papers in January 2001 (see [12]) and April 2003 (see [17]) respectively. These three consultative papers were conducted in an attempt to finalize the new accord. This new capital adequacy framework will be formally known as the *Basel II Capital Accord* (see [12] and [18]) and was to be implemented by all the major international banks globally from the end of year 2007. A cornerstone of the minimum capital requirement related to this accord is the *capital adequacy ratio (CAR)* given by

$$\text{CAR} = \frac{\text{Indicator of Absolute Amount of Bank Capital}}{\text{Indicator of Absolute Level of Bank Risk}}. \quad (4.1)$$

The capital adequacy ratio is a measure of the amount of a bank's capital relative to the amount of its credit exposures. This ratio is normally expressed as a percentage for example a capital adequacy ratio of 8% means that a bank's capital is 8% of the size of its credit exposures. An international standard has been cultivated that requires banks to hold minimum capital requirements. In the case where the *CAR* drops below a certain minimum level due to the exposure to the risks (such as credit or market risk), the bank might go bankrupt or the regulatory body may take certain actions on the bank. This may result in the ultimate closure of the bank, thus affecting the socio-economic development or financial status of a country. The aim of having minimum capital adequacy ratios is to guarantee that banks are prepared to absorb a reasonable level of losses before becoming

insolvent. Applying minimum capital adequacy ratios helps to promote the stability and effectiveness of the banking system by reducing the likelihood of banks becoming insolvent. When a financial institution, in this case a bank, becomes insolvent then this may lead to a loss of confidence in the financial system, causing financial problems for other banks and it might even threaten to distort the smooth functioning of financial markets. Determining capital adequacy ratios requires some adjustments to be made to the amount of capital shown on the balance sheet.

On the other hand, CARs depend on the ratio of bank capital to the risk-weighted assets. The numerator of (4.1) relies on the market values of all on- and -off-balance sheet assets and liabilities. The denominator of (4.1) should measure the bank's risk exposure or the fluctuation of its wealth or charter value. In principle, it should be possible for the aforementioned components of the CAR to be used to resolve the trade-off between flexibility and regulatory standardization in the banking industry. In this study, we concentrate our efforts on the Basel II risk-based capital adequacy ratio (*Basel II CAR*) given by

$$\text{Basel II CAR } (z) = \frac{\text{BRC } (C)}{\text{TRWAs } (a)} \quad (4.2)$$

The main objectives of the Basel II CAR are to:

- make capital allocation of banks more risk sensitive;
- separate operation rational risk from credit risk and calculate separate charges for each;
- ensure that regulatory capital requirements are more in line with economic capital requirements of banks;
- encourage banks to use their own internal systems for arriving at levels of regulatory capital.

We provide a diagrammatic overview of the Basel II capital accord.

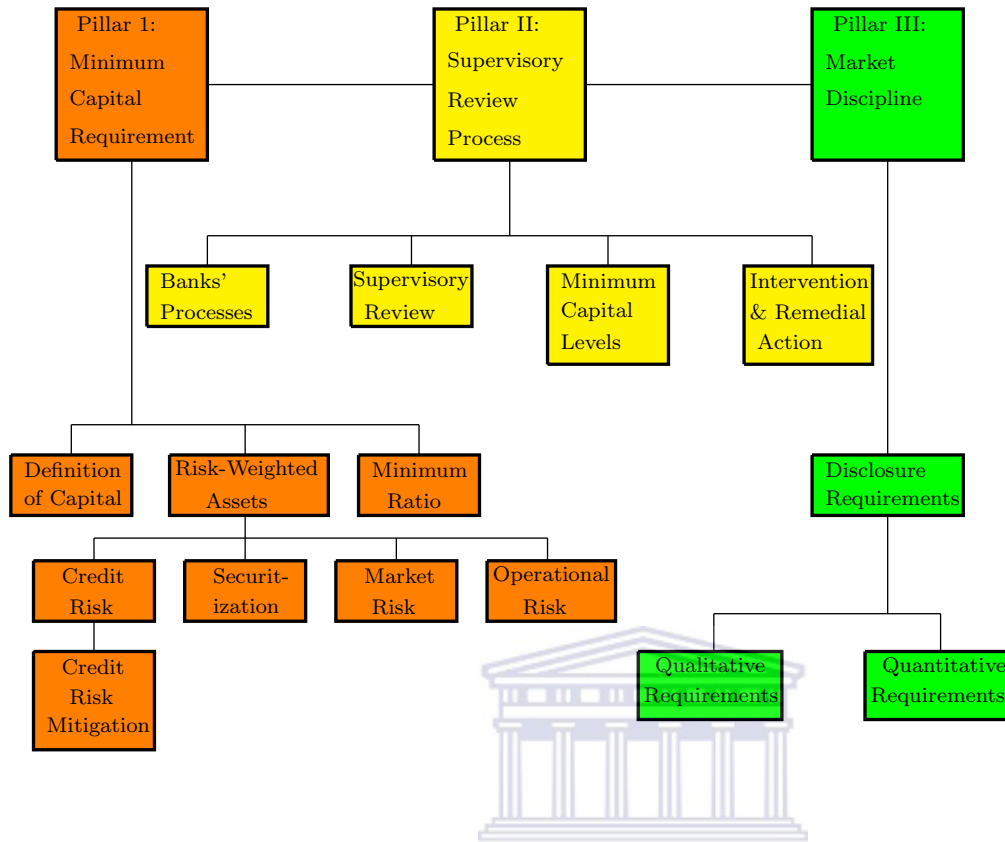


Figure 4.1: Diagrammatic Overview of the Basel II Capital Accord

There exist different types of common capital adequacy ratios, such as the Tier 1 risk-based capital ratio, total risk-based capital ratio (Tier 1 + Tier 2 + Tier 3), leverage ratio and common stockholders' equity ratio.

The Tier 1 capital ratio is defined as

$$\text{Tier 1 Capital ratio} = \frac{\text{Tier 1 Capital}}{\text{Risk-adjusted Assets}}. \quad (4.3)$$

Internationally active banks are expected to meet a minimum tier 1 risk-based capital ratio of at least 4%.

The total risk-based capital adequacy ratio under the Basel II Accord is defined as

$$\text{Total risk-based Capital ratio} = \frac{\text{Bank Regulatory Capital}}{\text{Risk-adjusted Assets}}. \quad (4.4)$$

Commercial banks are expected to meet a minimum total risk-based capital ratio of at least 8 %.

The leverage ratio is expressed as

$$\text{Leverage ratio} = \frac{\text{Tier 1 capital}}{\text{Average total consolidated assets}}. \quad (4.5)$$

The average total consolidated assets is defined as the quarterly average assets from a bank's most recent Call Report less goodwill and other intangible assets.

The Common stockholders' equity ratio is defined as

$$\text{Common stockholders' Equity ratio} = \frac{\text{Common stockholders equity}}{\text{Balance sheet assets}}. \quad (4.6)$$

Fouche, Mukuddem-Petersen, Petersen (see [46]) provides continuous-time stochastic models for each of the aforementioned capital adequacy ratios. In each case they derive explicit formulae separately and ultimately express them in the ratio forms (4.3), (4.4), (4.5) and (4.6) respectively.



4.1 Dynamics of Capital Adequacy Ratio

In this study we concentrate our efforts on deriving an explicit formula for the total risk-based capital adequacy ratio. In computing the total risk-based capital adequacy ratio (CAR) we introduce a new state variable

$$\text{CAR} (z(t)) = \frac{\text{Bank Regulatory Capital } (C(t))}{\text{Total Risk-Weighted Assets } (a(t))}. \quad (4.7)$$

Theorem 4.1.1 (Explicit SDE for the Capital Adequacy Ratio of a Bank):

Suppose that the dynamics of bank regulatory capital $C(t)$ and total risk-weighted assets $a(t)$ are described by (2.6) and (2.7), respectively, and $p_a(t) = pa(t)dt$. Then the dynamics of total risk-based capital adequacy ratio z of a bank may be represented by

$$\begin{aligned} dz(t) = z(t) & \left\{ \left(-\mu(t) + \pi'(t)\Psi \left[\zeta - \sigma \right] + \sigma_0^2 + \sigma' \sigma \right) dt \right. \\ & \left. - \sigma_0 dX_0(t) + \left(\pi'(t)\Psi - \sigma' \right) dX(t) \right\} - pa(t)dt. \end{aligned} \quad (4.8)$$

Proof.

In this proof we derive (4.8) by mainly using the general Itô formula. Let $U(t) = \frac{1}{a(t)}$ then

$$\begin{aligned}
dU(t) &= \frac{-1}{a^2(t)}da(t) + \frac{1}{2}\frac{2}{a^3(t)}a^2(t)\sigma'\sigma dt \\
&= \frac{\sigma'\sigma}{a(t)}dt - \frac{1}{a^2(t)}\left[a(t)\left[\left(r(t) + \mu(t)\right)dt + \sigma_0dX_0(t) + \sigma'dX(t)\right]\right] \\
&= \frac{\sigma'\sigma}{a(t)}dt - \frac{1}{a(t)}\left[\left(r(t) + \mu(t)\right)dt + \sigma_0dX_0(t) + \sigma'dX(t)\right] \\
&= \frac{1}{a(t)}\left(\sigma'\sigma - r(t) - \mu(t) + \sigma_0^2\right)dt - \frac{1}{a(t)}\sigma_0dX(t) - \frac{1}{a(t)}\sigma'dX(t).
\end{aligned}$$

Now we apply the Itô stochastic product rule:

$$\begin{aligned}
dz(t) &= d(C(t)U(t)) \\
&= C(t)dU(t) + U(t)dC(t) + \left(C(t)\pi'(t)\sigma\Psi\frac{1}{a(t)}\right)dt.
\end{aligned}$$

Since we defined the state variable (4.7) we have

$$\begin{aligned}
dz(t) &= z(t)\left\{\left(\sigma'\sigma - r(t) - \mu(t) + \sigma_0^2\right)dt - \sigma_0dX_0(t) - \sigma'dX(t)\right\} \\
&+ z(t)\left\{r(t) + \pi'(t)\Psi\zeta\right\}dt + z(t)\pi'(t)\Psi dX(t) - pa(t)dt + z\Psi\pi'(t)\sigma dt \\
&= z(t)\left\{\sigma'\sigma - \mu(t) + \sigma_0^2 + \pi'(t)\Psi\left[\zeta - \sigma\right]\right\}dt - pa(t)dt \\
&- z(t)\sigma_0dX_0(t) + z\left(\Psi\pi'(t) - \sigma'\right)dX(t) \\
&= z(t)\left\{\left(-\mu(t) + \pi'(t)\Psi\left[\zeta - \sigma\right] + \sigma_0^2 + \sigma'\sigma\right)dt\right. \\
&\quad \left.- \sigma_0dX_0(t) + \left(\pi'(t)\Psi - \sigma'\right)dX(t)\right\} - pa(t)dt.
\end{aligned}$$

□

We note that $p = 0$ corresponds to the situation where we have a once-off TRWAs outflow from bank regulatory capital at time t_0 . On the other hand, $p \neq 0$ implies that there is a continuous outflow of TRWAs from bank regulatory capital at a rate of $pa(t)$ throughout the interval $T = [t_0, t_1]$.

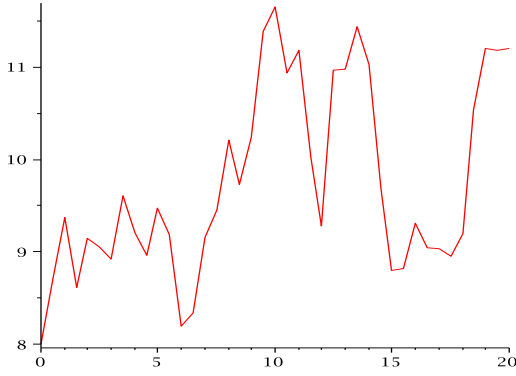


Figure 4.2: Trajectory of the capital adequacy ratio over a period of 20 years, that is, $0 \leq t \leq 20$.

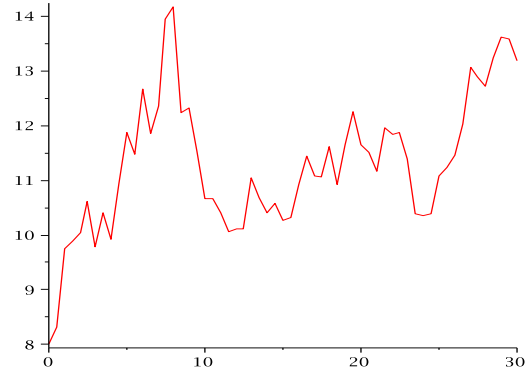


Figure 4.3: Trajectory of capital adequacy ratio over a period of 30 years, that is, $0 \leq t \leq 30$.

4.2 Threshold Processes and Benchmarks

In situations where $z(t)$ exceeds a certain CAR reference process, $z_r(t)$, or a banking benchmark, b , regulators may pressurize banks to increase CARs. This process may involve the withdrawal of insurance coverage, cease-and-desist orders, limits on asset growth and brokered deposits, prohibition of dividend payments and even bank closure. However, these measures are sometimes not very effective and may only be applicable to a small minority of banks. In an attempt to address this problem, in the USA, the prompt corrective action feature of the Federal Deposit Insurance Corporation Improvement Act (FDICIA) was implemented to improve capital-based incentives by making some of the aforementioned regulatory actions mandatory when CARs fall into certain capitalization categories. The CAR reference process, $z_r(t)$, may be a deterministic function of time and largely depends on the rate of inflow and variability of bank capital. How to choose the constant benchmark, b , in terms of the optimal operation and regulation of the bank (see for instance Berger, Herring, Szego [21]; Bhattacharya, Thakor [25]; Freixas, Rochet [50] and Santos [85]) offers another interesting challenge. The CAR, $z(t)$, at which moral hazard incentives become important relies more on the difference between $z(t)$ and its benchmark, b , than on the actual level of $z(t)$. Thus, for problematic banks, one of the

objectives should at least be to keep $z(t)$ as close as possible to $z_r(t)$ and ultimately to b . The next result provides an explicit formula for the CAR threshold, $z_p(t)$, and considers its relationship with an associated TRWA threshold, $a_p(t)$, and a CAR regulatory benchmark, b . Although many approaches can be adopted to characterize the aforementioned concepts, we consider $z_p(t)$ to be a deterministic function of time with a dependence on the rate of change and variability of TRWAs between t and T .

Theorem 4.2.1 (CAR and TRWA Threshold Processes): *Suppose that the dynamics of bank regulatory capital, $C(t)$, total risk-weighted assets $a(t)$ and capital adequacy ratio $z(t)$ are described by (2.6), (2.7), and (4.8), respectively. Then there exists an explicit formula for the CAR threshold process, $z_p(t)$, of the form*

$$z_p(t) := \frac{C_p(t)}{a_p(t)} = b + pv(\tau), \quad v(\tau) = \int_t^T \exp \left\{ M(t, \tau) - \sigma' \theta(\tau - t) \right\} d\tau, \quad (4.9)$$

where C_p and a_p are the threshold values of the bank regulatory capital (BRC) and total risk-weighted assets (TRWAs), respectively, and b is a CAR regulatory benchmark. Here a_p and b may be expressed as

$$a_p(t) = \frac{C_p(t)}{b + pv(\tau)} \text{ and } b = \frac{C_p(t) - a_p(t)pv(\tau)}{a_p(t)}, \quad (4.10)$$

respectively.

Proof. Since we work in a complete market, we have that the TRWAs are completely hedgeable. Also we suppose that \mathbb{Q} is a risk-neutral pricing measure under which the n risky equities have the dynamics

$$dx_i(t) = x_i(t) \left[r(t)dt + \sum_{j=1}^n \sigma_{ij} d\widehat{X}_j(t) \right], \quad i = 1, 2, \dots, n,$$

where the \widehat{X}_j 's are independent standard \mathbb{Q} -Brownian motions. In this case, with the help of (2.9), we can price future Tier 1 capital inflows uniquely as

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[\int_t^T \exp \left\{ - \int_t^\tau r(s) ds \right\} pa(\tau) d\tau \middle| \mathcal{F}_t \right] \\ &= p \mathbb{E}_{\mathbb{Q}} \left[\int_t^T a(t) \exp \left\{ \int_t^\tau \mu(s) ds - \sigma' \theta(\tau - t) - \frac{1}{2} |\sigma|^2 (\tau - t) \right. \right. \\ & \quad \left. \left. + \sigma' \left(\widehat{X}(\tau) - \widehat{X}(t) \right) \right\} d\tau \middle| \mathcal{F}_t \right] \\ &= a(t) p \int_t^T \exp \left\{ M(t, \tau) - \sigma' \theta(\tau - t) \right\} d\tau = a(t) \left\{ z_p(t) - b \right\}, \end{aligned}$$

where

$$M(t, \tau) = \int_t^\tau \mu(s) ds - \frac{1}{2} |\sigma|^2 (\tau - t) + \sigma' \left(\widehat{X}(\tau) - \widehat{X}(t) \right).$$

□

We present the categories of banking benchmark regulatory ratios (see Mukkudem-Petersen, Petersen [76]).

Categories	b	T1CAR	TCAR	TE
Well-Capitalized	≥ 0.1	and ≥ 0.06	and ≥ 0.06	–
Adequately Capitalized	≥ 0.08	and ≥ 0.04	and ≥ 0.04	–
Undercapitalized	≥ 0.06	and ≥ 0.03	and ≥ 0.03	–
Significantly Undercapitalized	< 0.06	or ≥ 0.03	or ≥ 0.03	and > 0.02
Critically Undercapitalized				≤ 0.02

Figure 4.4: Categories of banking benchmark regulatory ratios

In figure 4.4, we have that TCAR , T1CAR and TE are the abbreviations for total CAR (also known as the leverage ratio), Tier 1 CAR and tangible equity, respectively. Also, the

CAR column in the said figure gives an indication of possible values for the benchmark, b . In reality, the vast majority of banks fit into the "well-capitalized" category.

In practice, capital adequacy regulation stipulates a uniform b below which banks are subject to regulatory intervention. These minimums remain relatively stable over a period of years, although regulators have the discretion to set higher requirements for banks that are perceived to pose higher risks. How the CAR benchmark, b , is chosen in relation to the critical values presented in figure 4.4, is dependent on several factors. Among these are the type and size of the bank in question. Virtually every bank failure theory postulates that a higher CAR is associated with a lower future probability of failure. Despite this, the relationship between the CAR and bank safety is often relatively weak. A higher CAR does not always predict a lower probability of failure in the immediate future and explains little of bank performance variation.

The precise choice of a CAR regulatory benchmark, b , in terms of the optimal risk profile and regulation of the bank (see for instance Berger, Herring, Szego [21]; Mukkudem-Petersen, Petersen [76]), offers an interesting challenge. The CAR, $z(t)$, at which moral hazard incentives become important relies more on the difference between $z(t)$ and its benchmark, b , than on the actual level of $z(t)$. Thus, for problematic banks, one of the objectives should at least be to keep $z(t)$ as close as possible to $z_p(t)$ and ultimately to b . Another important issue is related to the impact of $z_p(t) \leq b$ or $b < z_p(t)$ over time. In the former case, we conjecture that $z_p(t)$ will act as a threshold for corrective action or even bank closure. As far as the other inequality is concerned, $z_p(t)$ may be a threshold which indicates that supervisory intervention may be relaxed. We note that at $t = T$ the CAR threshold, $z_p(t)$, corresponds to the industry benchmark, b . A similar discussion as the one for $z(t)$, $z_p(t)$ and b , can be undertaken for $a(t)$, $a_p(t)$ and some TRWAs regulatory benchmark, b_a .

4.3 An Optimal Asset Allocation Strategy in Bank Management

We present and solve an optimization problem using the asset allocation strategy π as the control variable. The optimal allocation strategy π will be split over three assets. The objective is to maximize a terminal utility function of the capital adequacy ratio z . Numerical simulations will be done on π_0^* , π_1^* and π_2^* . Here π_0^* represents the proportion invested in the cash funds, π_1^* represent the proportion invested in the bond fund and π_2^* represent the proportion invested in the equities fund.

4.3.1 Optimal Bank Asset Allocation

In this section, we make use of the outcomes of theorem 4.1.1 to solve an optimal bank asset allocation problem. Our analysis of CARs considers the stochastic differential equation (4.8) from theorem 4.1.1 on a given time interval $[0, T]$.

An Optimal Asset Allocation Problem

In the sequel, we concentrate our efforts on maximising a terminal utility function, $u(r(T), z(T))$, where u is expressed as

$$u(r(T), z(T)) = \frac{1}{\alpha} f(T, r(T)) g(T, z(T))^\alpha, \quad \text{for } \alpha < 1, \alpha \neq 0; \quad (4.11)$$

We express the function g as follows

$$g(t, z(t)) = z(t) + (z_p(t) - \rho), \quad (4.12)$$

where for z_p defined by (4.9), we have $z_p(T) = \rho$ so that

$$g(T, z(T)) = z(T).$$

The choice of (4.12) for g is motivated by the fact that throughout $[0, T]$ the risk of bank failure is described in terms of $z(t)$, $z_p(t)$ and ρ . We assume that a commercial

bank's terminal utility depends on both its bank regulatory capital and its total risk-weighted assets. In this study we focus on two special cases: (a) the capital adequacy ratio, $z(t) = \frac{C(t)}{a(t)}$. The bank's terminal utility has the form

$$u(r(T), z_\pi(T)) \equiv u(z_\pi(T)).$$

For the asset allocation strategy, π , we choose the expected terminal utility as

$$J(t, r, z; \pi) = \mathbb{E} \left[u(r(T), z_\pi(T)) : r(t) = r, z_\pi(t) = z \right], \quad (4.13)$$

where z_π is the trajectory of z given π . We determine a control law that maximizes the expected terminal utility $J : \mathcal{G} \rightarrow \mathbb{R}_+$ given by (4.13), where \mathcal{G} is the class of admissible control laws

$$\mathcal{G} = \left\{ (\pi(\cdot, z) : \pi \text{ bounded, adapted so that } z \geq 0 \text{ a.s.}) \right\}. \quad (4.14)$$

We are now in a position to state the stochastic optimal control problem for bank asset allocation that we solve in the sequel assuming $\sigma_0 = 0$, that is, only hedgeable market risk. The condition $p > 0$ indicates positive ongoing contribution to the bank wealth and zero non-hedgeable market risk. Now let

$$V(t, r, z) = \sup_{\pi \in \mathcal{G}} J(t, r, z; \pi),$$

Theorem 4.3.1 (Optimal strategy): *Suppose that $\mathcal{G} \neq \emptyset$, where the admissible class of control laws, \mathcal{G} is given by (4.14). Also, consider the stochastic differential equation for the z -dynamics from (4.8) and the expected terminal utility, $J : \mathcal{G} \rightarrow \mathbb{R}_+$, given by (4.13). For the optimization problem above, the optimal solution,*

$$\pi^* = \arg \sup_{\pi \in \mathcal{G}} J(t, r, z; \pi) \in \mathcal{G}$$

if it exists, is given by:

$$\pi^* = C'^{-1} \left\{ \sigma - \left[\theta - \sigma \right] \frac{V_z}{zV_{zz}} - \sigma_r(r) \frac{V_{zr}}{zV_{zz}} \right\}. \quad (4.15)$$

Proof.

In order to prove the theorem, we follow the stochastic optimization procedure via dynamic programming (see, for instance, Bjork [26]; Korn [62] and Øksendal [80]). We provide a systematic procedure to obtain the optimal asset allocation strategy. The Hamilton-Jacobi-Bellman Equation (HJBE) for this problem is

$$V_t + \sup_{\pi \in \mathcal{G}} \left\{ \mathcal{A}^\pi V \right\} = 0,$$

with

$$\mathcal{A}^\pi = \mu_r(r) \frac{\partial}{\partial r} + \mu_z^\pi \frac{\partial}{\partial z} + \frac{1}{2} \nu_{rr} \frac{\partial^2}{\partial r^2} + \nu_{rz}^\pi \frac{\partial^2}{\partial r \partial z} + \frac{1}{2} \nu_{zz}^\pi \frac{\partial^2}{\partial z^2},$$

where

$$\begin{aligned} \mu_z^\pi &= z \left\{ -\mu(t) + \pi'(t)C \left[\theta - \sigma \right] + \sigma_0^2 + \sigma^T \sigma \right\} + p; \\ \nu_{rr} &= \sigma_r(r)^T \sigma_r(r); \quad \nu_{rz}^\pi = \left(\pi'(t)C - \sigma' \right) \sigma_r(r) z; \\ \nu_{zz}^\pi &= \sigma_0^2 z^2 + \left(\pi(t)'C - \sigma' \right) \left(C' \pi(t) - \sigma \right) z^2. \end{aligned}$$

This is a partial differential equation (PDE) with the value function, V , being the unknown. The solution of the optimization problem is an optimal path

$$\pi^* = \pi^*(t, r, z; V). \quad (4.16)$$

In order to maximize $\mathcal{A}^\pi V$ we differentiate the expression $\mathcal{A}^\pi V$ with respect to π and equate to zero. Thus

$$zC(\theta - \sigma)V_z + (D\pi - C\sigma)z^2V_{zz} + C\sigma_r(r)zV_{zr} = 0,$$

where $D = CC'$. Solving for π we find that the optimal asset allocation strategy has the form

$$\pi^* = C'^{-1} \left\{ \sigma - \left[\theta - \sigma \right] \frac{V_z}{zV_{zz}} - \sigma_r(r) \frac{V_{zr}}{zV_{zz}} \right\}. \quad (4.17)$$

□

We need to know the partial derivatives of value function $V(t, z, r)$ when making simulations of π^* . We shall avoid giving the detail of the computations and instead refer to the paper of Cairns, Blake, Dowd [32], where the value function for a similar utility maximisation was found. Our function is similar to the function appearing in Cairns, Blake, Dowd [32]. We note that the interest rate used in Cairns, Blake, Dowd [32] for this analysis is the Vasicek model. Nevertheless the value function in this way is at least an approximation for the value function that we require. Thus we take $V(t, z, r)$ to be as follows

$$V(t, z, r) = \frac{1}{\alpha} \exp \left[f_1(\alpha, T - t) + \alpha q(\alpha)(T - t) \right] \times \exp \left[f_2(\alpha, T - t)r(t) \right] \left(z(t) + (z_p(t) - \rho) \right)^\alpha \quad (4.18)$$

where $f_1(\alpha, T - t)$, $f_2(\alpha, T - t)$ and $q(\alpha)$ are defined as

$$f_1(\alpha, T - t) = -\alpha C(t, T) + \alpha D(t, T) \hat{\mu}_r (1 - \exp\{-\kappa_r(T - t)\}) + \frac{\alpha^2 D(t, T)^2 \sigma_r^2 (1 - \exp\{-2\kappa_r(T - t)\})}{4\kappa_r(1 - \alpha)}, \quad (4.19)$$

$$f_2(\alpha, T - t) = \alpha D(t, T) \exp\{-\kappa_r(T - t)\} \quad (4.20)$$

and

$$q(\alpha) = \sigma^T \theta - \frac{1}{2(\alpha - 1)} (\theta - \sigma)^T (\theta - \sigma)$$

respectively. In particular, at $t = T$, we have that

$$f_1(\alpha, T - T) = -\alpha C(t, T), \quad f_2(\alpha, T - T) = \alpha D(t, T).$$

We now provide 2 different graphs of the portfolio consisting of three assets over a period of 20 and 30 years respectively.

The optimal weight in risky assets is equivalent to investing in a portfolio consisting of three efficient mutual funds namely cash (π_0^*), bonds (π_1^*) and equities (π_2^*). The three mutual funds can be interpret as follows:

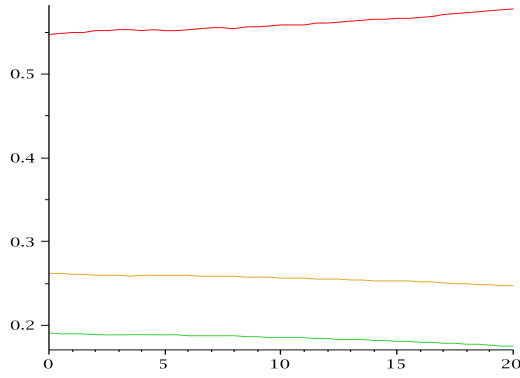


Figure 4.5: Trajectories for the optimal strategies π_0^* , π_1^* and π_2^* over a period of 20 years, that is, $0 \leq t \leq 20$.

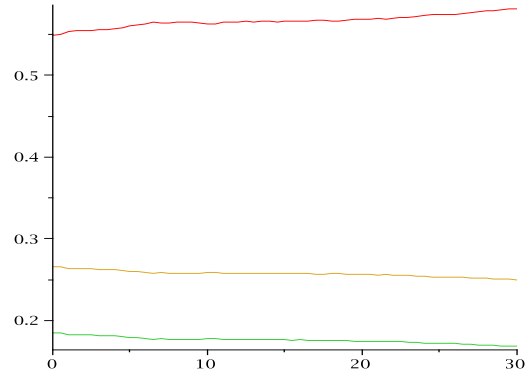


Figure 4.6: Trajectories for the optimal strategies π_0^* , π_1^* and π_2^* over a period of 30 years, that is, $0 \leq t \leq 30$.

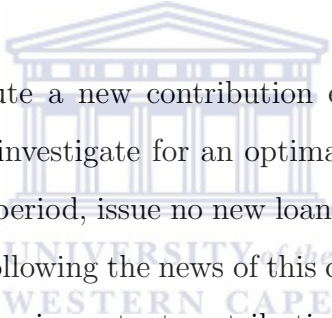
- The cash fund (Top Line) is the minimum-risk portfolio measured relative to the total-risk weighted assets, $a(t)$, and its purpose is to hedge against market risk. Asset proportions are represent by the vector π_0^* . This fund can contain 100 % cash if bank regulatory and total-risk weighted assets is uncorrelated however if bank regulatory capital and total risk-weighted assets is correlated then this fund contains also other assets.
- The bond fund (Middle Line) is the minimum-risk portfolio measured relative to $a(t)/A(t, r(t))$ its purpose is to hedge credit risk. Asset proportions are represent by the vector π_1^* . The returns on the bond fund is highly correlated with the amortization yields.
- The equities fund (Bottom Line) is a risky portfolio which is efficient when measured relative to both $a(t)$ and $a(t)/A(t, r(t))$. Asset proportions are given by the vector (π_2^*) . and its purpose is to satisfy the risk appetite of the bank.

We observe in figure (4.5) and figure (4.6) that the proportions $(\pi_0^*, \pi_1^*$ and $\pi_2^*)$ invested into each asset respectively tends to remain consistent over time.

Chapter 5

Modelling of a Equity Allocation

Problem in Bank Management



The results in this chapter constitute a new contribution except where references are explicitly given. In this chapter we investigate for an optimal portfolio composition in a case where a bank will, over a short period, issue no new loans. This could well happen in the current world economic crisis. Following the news of this catastrophe during the latter part of 2008, there has since been some important contributions in the academic literature providing explanations for the causes of the subprime mortgage crisis. An example of such a paper is the one of Fouche, Mukuddem-Petersen, Petersen, Senosi (see [47]). The said points out that the so called *procyclicality* has become a buzzword in discussions about banking regulation. In essence, the movement in a financial variable is said to be procyclical if it tends to amplify business cycle fluctuations. As such, procyclicality is an inherent property of any financial system. A feature of procyclicality is that banks tend to restrict their lending activity during economic downturns because of their concern about loan quality and the probability of loan defaults. This exacerbates the recession since credit constrained businesses and individuals cut back on their investment activity. In our contribution we give direction as to how to find strategies for a bank towards recovery but we first provide a brief discussion on mean-variance portfolio approach. Mean-variance

portfolio selection is concerned with the allocation of wealth among a variety of securities so as to achieve the optimal trade-off between the expected return of the investment and its risk over a fixed planning horizon. Here we mean the risk of a portfolio measured by the variance of its return. In this spirit, Markowitz (see [67]) designed a model whereby he showed how to formulate the problem of minimizing a portfolio's variance subject to the constraint that its expected return equals a prescribed level as a quadratic program. In this framework such an optimal portfolio is said to be variance minimizing, and if it also achieves the maximum expected return among all portfolios having the same variance of return then it is said to be efficient. From an optimization point of view, in the problem of portfolio selection it is desired to attain the highest possible expected return with the lowest possible variance.

The mean-variance methodology has been surfacing in the literature from the static case to the dynamic setting in banking (see for instance Alexander, Baptista, Yan [2]; Barber, Chang, Thurston [6] and Leippold, Trojani, Vanini [66]). In the pension fund context, there is a growing amount of papers that solves certain problems under the mean-variance framework (see for instance Josa-Fombellida, Rincon-Zapatero [59] and the references contained in it).



5.1 Optimizing the equity allocation

At time $t = 0$ we decompose the total-risk weighted assets $a(0)$ into two components. The first component comprises of loans made to private agents and the second component assembles the rest of the assets which are invested into marketable securities

$$a(0) = L(0) + M(0).$$

The marketable securities at time $t = 0$ will continue to evolve as $M(t)$, which we assume to follow a geometric brownian motion. In time, $L(0)$ will be reduced at a rate $dF(t)$. Here we consider that the amortization is decomposed into a loan repayment and an interest payment on the principle $A(t, r(t)) = F(t, r(t)) + I(t, r(t))$. Furthermore, the inflow of the

amortizations $dA(t, r(t))$ will be split over three assets. The following proposition gives an expression for $\frac{dF(t, r(t))}{dt}$, which will be required later on.

Proposition 5.1.1 *For a given rate of amortization $\frac{dA(t, r(t))}{dt}$, the rate $\frac{dF(t, r(t))}{dt}$ is given by :*

$$\frac{dF(t, r(t))}{dt} = \frac{dA(t, r(t))}{dt} - L(t)r(t). \quad (5.1)$$

Proof.

Note that in the absence of amortizations, the interest accumulated over a period dt by a loan of value $L(t)$ will be:

$$dL(t) = L(t)r(t)dt. \quad (5.2)$$

When $dA(t, r(t))$ exceeds this amount, the remainder is directed to reducing the principal debt. Hence the claim of the proposition follows. \square

Now notice that on its own, cash is fixed (it does not grow in time). The bond and bank equity have randomness associated with it and we denote it by $B(t)$ and $e_1(t)$ respectively. The bond is assumed to evolve as in (5.3) and equity evolves as $de_1(t)$. We use $B(t, T)$ (see Boulier, Huang, Taillard [27]), to denote the price of this bond at time $t \in [0, T]$, the diffusion equation of $B(t, T)$ is

$$\frac{dB(t, T)}{B(t, T)} = r(t)dt + \sigma_B(T - t)(dX(t) + \lambda_r dt), \quad (5.3)$$

where the premium λ_r is assumed to be constant. Recall that we opted to divide $\Delta A(t, r(t))$ into 3 parts. Now this will be done according to the fractions: $f_c(t)$, $f_B(t)$ and $f_{e_1}(t)$. Hereby we mean that the amount $f_c(t)\Delta A(t, r(t))$ is invested into cash, $f_B(t)\Delta A(t, r(t))$ is invested into a bond and $f_{e_1}(t)\Delta A(t, r(t))$ is invested into one equity. Then $\Delta A(t, r(t))$ contributes to the risk-weighted assets the amount:

$$\Delta \bar{a} = f_c(t)w_c \frac{dA(t, r(t))}{dt} \Delta t + f_B(t)w_B \frac{dA(t, r(t))}{dt} \Delta B(t, s) + f_{e_1}(t)w_{e_1} \frac{dA(t, r(t))}{dt} \Delta e_1(t).$$

Considering table (7.1) in the Appendix, we note that the weight in cash and bonds have a zero risk-weighting, that is $w_c = w_B = 0$. In this case the total risk-weighted assets may then be expressed as

$$\bar{a}(t) = M(t) + L(0) - F(t, r(t)) + \int_0^t f_{e_1}(\tau) w_{e_1} \frac{dA(\tau, r(\tau))}{d\tau} de_1(\tau).$$

In differential form

$$\begin{aligned} d\bar{a}(t) &= dM(t) - \frac{dF(t)}{dt} dt + f_{e_1}(t) w_{e_1} \frac{dA(t, r(t))}{dt} de_1(t) \\ &= M(t) \left[r_m dt + \sigma_m dX(t) \right] - \left(\frac{dF(t)}{dt} \right) dt \\ &\quad + f_{e_1}(t) w_{e_1} \frac{dA(t, r(t))}{dt} \left[e_1(t) \left(r_0(t) + \sigma_{11} \zeta_1 \right) dt + e_1(t) \sigma_{11} dX(t) \right] \\ &= \left[M(t) r_m - \frac{dF(t)}{dt} + e_1(t) f_{e_1}(t) w_{e_1} \frac{dA(t, r(t))}{dt} \left(r_0(t) + \sigma_{11} \zeta_1 \right) \right] dt \\ &\quad + \left[M(t) \sigma_m + e_1(t) f_{e_1}(t) w_{e_1} \frac{dA(t, r(t))}{dt} \sigma_{11} \right] dX(t). \end{aligned} \tag{5.4}$$

The next step is to obtain the reciprocal of capital adequacy ratio.

Proposition 5.1.2 (Explicit SDE for the Reciprocal of the capital adequacy ratio): *Suppose that the dynamics of bank regulatory capital $C(t)$ and total risk-weighted assets $\bar{a}(t)$ are described by (2.6) and (5.4), respectively. Then the dynamics of the reciprocal of capital adequacy ratio $z^{-1}(t)$ of a bank satisfies the following SDE:*

$$\begin{aligned} dz^{-1}(t) &= \left[z^{-1}(t) \frac{\beta^2(t)}{C^2(t)} - z^{-1}(t) \rho(t) + p z^{-2}(t) \right. \\ &\quad + \frac{1}{C(t)} \left(M(t) r_m - \left(\frac{dF(t, r(t))}{dt} \right) + e_1(t) f_{e_1}(t) w_{e_1} \frac{dA(t, r(t))}{dt} \left(r_0(t) + \sigma_{11} \zeta_1 \right) \right) \\ &\quad + \left. \frac{\beta(t)}{C(t)} \left(M(t) \sigma_m + e_1(t) f_{e_1}(t) w_{e_1} \frac{dA(t, r(t))}{dt} \sigma_{11} \right) \right] dt \\ &\quad + \left[- z^{-1}(t) \beta(t) + \frac{1}{C(t)} \left(M(t) \sigma_m + e_1(t) f_{e_1}(t) w_{e_1} \frac{dA(t, r(t))}{dt} \sigma_{11} \right) \right] dX(t), \end{aligned} \tag{5.5}$$

where $\rho(t) = r_0(t) + \pi'(t) \Psi \zeta$ and $\beta(t) = \pi'(t) \Psi$ are the drift and diffusion term of bank regulatory capital respectively.

Proof.

In this proof we derive (5.5) by mainly using the general Itô formula. Let $f(C) = \frac{1}{C(t)}$.

Then

$$\begin{aligned}
df(C) &= \dot{f}(t) + f'(t)dC(t) + \frac{1}{2}f''(t)\beta^2(t)dt \\
&= 0dt - \frac{1}{C^2(t)}dC(t) + \frac{1}{2}\left(\frac{2}{C^3(t)}\beta^2(t)\right)dt \\
&= \frac{\beta^2(t)}{C^3(t)}dt - \frac{1}{C^2(t)}\left[C(t)\left[\rho(t)dt + \beta(t)dX(t)\right] - p\bar{a}(t)dt\right] \\
&= \frac{\beta^2(t)}{C^3(t)}dt - \frac{1}{C(t)}\left[\rho(t)dt + \beta(t)dX(t)\right] + \frac{p\bar{a}(t)}{C^2(t)}dt \\
&= \left(\frac{\beta^2(t)}{C^3(t)} - \frac{\rho(t)}{C(t)} + \frac{p\bar{a}(t)}{C^2(t)}\right)dt - \frac{\beta(t)}{C(t)}dX(t).
\end{aligned}$$

Now we apply the Itô stochastic product rule:

$$\begin{aligned}
dz^{-1}(t) &= d(\bar{a}(t)C^{-1}(t)) = \bar{a}(t)dC^{-1}(t) + C^{-1}(t)d\bar{a}(t) + d\bar{a}(t)dC^{-1}(t) \\
&= \bar{a}(t)\left[\left(\frac{\beta^2(t)}{C^3(t)} - \frac{\rho(t)}{C(t)} + \frac{p\bar{a}(t)}{C^2(t)}\right)dt - \frac{\beta(t)}{C(t)}dX(t)\right] \\
&+ \frac{1}{C(t)}\left[\left(M(t)r_m - \frac{dF(t,r(t))}{dt} + e_1(t)f_{e_1}(t)\omega_{e_1}\frac{dA(t,r(t))}{dt}\right)\left(r_0(t) + \sigma_{11}\zeta_1\right)\right]dt \\
&+ \left(M(t)\sigma_m + e_1(t)f_{e_1}(t)\omega_{e_1}\frac{dA(t,r(t))}{dt}\sigma_{11}\right)dX(t) \\
&+ \frac{\beta(t)}{C(t)}\left(M(t)\sigma_m + e_1(t)f_{e_1}(t)\omega_{e_1}\frac{dA(t,r(t))}{dt}\sigma_{11}\right)dt.
\end{aligned}$$

Continuing in this fashion we have

$$\begin{aligned}
dz^{-1}(t) &= \left[z^{-1}(t)\frac{\beta^2(t)}{C^2(t)} - z^{-1}(t)\rho(t) + pz^{-2}(t) + \frac{M(t)r_m}{C(t)} - \frac{1}{C(t)}\left(\frac{dF(t,r(t))}{dt}\right)\right. \\
&+ \frac{e_1(t)f_{e_1}(t)\omega_{e_1}\frac{dA(t,r(t))}{dt}\left(r_0(t) + \sigma_{11}\zeta_1\right)}{C(t)} \\
&+ \left.\frac{\beta(t)}{C(t)}\left(M(t)\sigma_m + e_1(t)f_{e_1}(t)\omega_{e_1}\frac{dA(t,r(t))}{dt}\sigma_{11}\right)\right]dt \\
&+ \left(-z^{-1}(t)\beta(t) + \frac{1}{C(t)}\left(M(t)\sigma_m + e_1(t)f_{e_1}(t)\omega_{e_1}\frac{dA(t,r(t))}{dt}\sigma_{11}\right)\right)dX(t).
\end{aligned}$$

Grouping the drift and diffusion terms for $dz^{-1}(t)$ yields

$$\begin{aligned}
dz^{-1}(t) &= \left[z^{-1}(t) \frac{\beta^2(t)}{C^2(t)} - z^{-1}(t) \rho(t) + pz^{-2}(t) + \frac{1}{C(t)} \left(M(t)r_m - \left(\frac{dF(t, r(t))}{dt} \right) \right. \right. \\
&\quad \left. \left. + e_1(t)f_{e_1}(t)\omega_{e_1} \frac{dA(t, r(t))}{dt} \left(r_0(t) + \sigma_{11}\zeta_1 \right) \right) \right. \\
&\quad \left. + \frac{\beta(t)}{C(t)} \left(M(t)\sigma_m + e_1(t)f_{e_1}(t)\omega_{e_1} \frac{dA(t, r(t))}{dt} \sigma_{11} \right) \right] dt \\
&\quad + \left[-z^{-1}(t)\beta(t) + \frac{1}{C(t)} \left(M(t)\sigma_m + e_1(t)f_{e_1}(t)\omega_{e_1} \frac{dA(t, r(t))}{dt} \sigma_{11} \right) \right] dX(t).
\end{aligned}$$

□

The yield of this investment is

$$\begin{aligned}
dY_A(t) &= f_c(t) \frac{dA(t, r(t))}{dt} \Delta t + f_B(t) \frac{dA(t, r(t))}{dt} dB(t) + f_{e_1}(t) \frac{dA(t, r(t))}{dt} de_1(t) \\
&= f_c(t) \frac{dA(t, r(t))}{dt} + f_B(t) \frac{dA(t, r(t))}{dt} \left[B(t)r(t)dt + B(t)\sigma_B(T-t)(dX(t) + \lambda_r dt) \right] \\
&\quad + f_{e_1}(t) \frac{dA(t, r(t))}{dt} \left[e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right) dt + e_1(t)\sigma_{11}dX(t) \right]
\end{aligned}$$

which further simplifies to:

$$\begin{aligned}
dY_A(t) &= \left[f_c(t) \frac{dA(t, r(t))}{dt} + B(t)f_B(t) \frac{dA(t, r(t))}{dt} \left(r(t) + \sigma_B(T-t)\lambda_r \right) \right. \\
&\quad \left. + f_{e_1}(t) \frac{dA(t, r(t))}{dt} e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right) \right] dt \\
&\quad + \left[B(t)f_B(t) \frac{dA(t, r(t))}{dt} \sigma_B(T-t) + f_{e_1}(t) \frac{dA(t, r(t))}{dt} e_1(t)\sigma_{11} \right] dX(t). \quad (5.6)
\end{aligned}$$

Let $D_{Y_A}(t)$ denote the drift coefficient of expression (5.6) and let $D_{z^{-1}(t)}$ denote the diffusion coefficient of $dz^{-1}(t)$. Then

$$\begin{aligned}
D_{Y_A}(t) &= \left[f_c(t) \frac{dA(t, r(t))}{dt} + B(t)f_B(t) \frac{dA(t, r(t))}{dt} \left(r(t) + \sigma_B(T-t)\lambda_r \right) \right. \\
&\quad \left. + f_{e_1}(t) \frac{dA(t, r(t))}{dt} e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right) \right] \quad (5.7)
\end{aligned}$$

and

$$D_{z^{-1}(t)} = \left[-z^{-1}(t)\beta(t) + \frac{1}{C(t)} \left(M(t)\sigma_m + e_1(t)f(t)\omega_{e_1} \frac{dA(t, r(t))}{dt} \sigma_{11} \right) \right]. \quad (5.8)$$

We now pursue the following problem, and thither we allow for a function $\alpha(t)$ yet to be determined. Our aim is now to maximize $D_{Y_A(t)}$ while minimizing $D_{z^{-1}(t)}$. More precisely we formulate the above situation as follows.

Problem 5.1.3 (Optimal Bank Equity Allocation Problem): *Given the coefficients $D_{Y_A(t)}$ and $D_{z^{-1}(t)}$ as above, we want to maximize the quantity*

$$Q = D_{Y_A(t)} - \alpha(t)D_{z^{-1}(t)}^2 \quad (5.9)$$

for some $\alpha(t) > 0$ with respect to the proportions $f_c(t)$, $f_B(t)$ and $f_{e_1}(t)$.

In order to obtain a solution for Problem 5.1.3 we shall determine the analytical solution for the optimal $f(t)$ and then run simulations on $f(t)$, $dz^{-1}(t)$ and $dY_A(t)$. For the analysis, we consider two different types of amortization functions, having the linear form $A(t) = Kt$ where K is a constant and a quadratic form $A(t, r(t)) = Kt + \frac{rt^2}{2}$. The problem as it stands will not have a unique solution, unless we impose certain limitations. Thus we shall assume that $f_c(t) = f_B(t)$ and write $g(t) = f_c(t) = f_B(t)$. We also write $f(t) = f_{e_1}(t)$. In this case $g(t) = \frac{1 - f(t)}{2}$ and $\frac{dg(t)}{f(t)} = \frac{1}{2}$.

Theorem 5.1.4 (Solution to Optimal Bank Equity Allocation Problem): *Suppose that the $z^{-1}(t)$ -dynamics is described by the stochastic differential equation (5.5) and we consider only the drift coefficient of $dY_A(t)$ given by $D_{Y_A(t)}$ and the diffusion coefficient of $dz^{-1}(t)$ given by $D_{z^{-1}(t)}$. In this case, a solution $f(t)$ to the optimal bank equity allocation problem stated in Problem 5.1.3 is of the form*

$$f(t) = \frac{C^2(t)}{2\alpha(t)e_1^2(t)\omega_{e_1}^2 \frac{dA(t, r(t))}{dt} \sigma_{11}^2} \left[-\frac{1}{2} - \frac{1}{2}B(t) \left(r(t) + \sigma_B(T-t)\lambda_r \right) + e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right) + \frac{\Gamma(t)z^{-1}(t)\beta(t)}{C(t)} - \frac{\Gamma(t)M(t)\sigma_m}{C^2(t)} \right], \quad (5.10)$$

where $\Gamma(t) = 2\alpha(t)e_1(t)\omega_{e_1}\sigma_{11}$.

Proof.

In order to maximise Q with respect to $f(t)$, a necessary condition is that $\frac{dQ}{df(t)} = 0$.

$$\begin{aligned} \frac{dQ}{df(t)} &= -\frac{1}{2} \frac{dA(t, r(t))}{dt} - \frac{1}{2} B(t) \frac{dA(t, r(t))}{dt} \left(r(t) + \sigma_B(T-t)\lambda_r \right) \\ &+ \frac{dA(t, r(t))}{dt} e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right) - \frac{\Gamma(t) \frac{dA(t, r(t))}{dt}}{C(t)} D_{z^{-1}(t)} = 0. \end{aligned}$$

We obtain the common factor $\frac{dA(t, r(t))}{dt}$:

$$\frac{dA(t, r(t))}{dt} \left[-\frac{1}{2} - \frac{1}{2} B(t) \left(r(t) + \sigma_B(T-t)\lambda_r \right) + e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right) - \frac{\Gamma(t)}{C(t)} D_{z^{-1}(t)} \right] = 0$$

and divide by $\frac{dA(t, r(t))}{dt}$:

$$-\frac{1}{2} - \frac{1}{2} B(t) \left(r(t) + \sigma_B(T-t)\lambda_r \right) + e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right) - \frac{2\alpha(t)e_1(t)\omega_{e_1}\sigma_{11}}{C(t)} D_{z^{-1}(t)} = 0.$$

Simplifying it further:

$$\begin{aligned} -\frac{1}{2} - \frac{1}{2} B(t) \left(r(t) + \sigma_B(T-t)\lambda_r \right) + e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right) + \frac{\Gamma(t)z^{-1}(t)\beta(t)}{C(t)} - \frac{\Gamma(t)M(t)\sigma_m}{C^2(t)} \\ - \frac{2\alpha(t)e_1^2(t)f(t)\omega_{e_1}^2 \frac{dA(t, r(t))}{dt} \sigma_{11}^2}{C^2(t)} = 0. \end{aligned}$$

Rearranging the terms, we obtain:

$$\begin{aligned} -\frac{2\alpha(t)e_1^2(t)f(t)\omega_{e_1}^2 \frac{dA(t, r(t))}{dt} \sigma_{11}^2}{C^2(t)} &= \frac{1}{2} + \frac{1}{2} B(t) \left(r(t) + \sigma_B(T-t)\lambda_r \right) \\ &- e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right) - \frac{\Gamma(t)z^{-1}(t)\beta(t)}{C(t)} + \frac{\Gamma(t)M(t)\sigma_m}{C^2(t)} \\ \frac{2\alpha(t)e_1^2(t)f(t)\omega_{e_1}^2 \frac{dA(t, r(t))}{dt} \sigma_{11}^2}{C^2(t)} &= -\frac{1}{2} - \frac{1}{2} B(t) \left(r(t) + \sigma_B(T-t)\lambda_r \right) \\ &+ e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right) + \frac{\Gamma(t)z^{-1}(t)\beta(t)}{C(t)} - \frac{\Gamma(t)M(t)\sigma_m}{C^2(t)}. \end{aligned}$$

Solving for $f(t)$ we obtain:

$$f(t) = \frac{C^2(t)}{2\alpha(t)e_1^2(t)\omega_{e_1}^2 \frac{dA(t,r(t))}{dt} \sigma_{11}^2} \left[-\frac{1}{2} - \frac{1}{2}B(t) \left(r(t) + \sigma_B(T-t)\lambda_r \right) \right. \\ \left. + e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right) + \frac{\Gamma(t)z^{-1}(t)\beta(t)}{C(t)} - \frac{\Gamma(t)M(t)\sigma_m}{C^2(t)} \right].$$

□

Simplifying it further and substituting $\Gamma(t) = 2\alpha(t)e_1(t)\omega_{e_1}\sigma_{11}$ into expression (5.10), we obtain a particular form for $f(t)$:

$$f(t) = -\frac{C^2(t)}{4\alpha(t)e_1^2(t)\omega_{e_1}^2 \frac{dA(t,r(t))}{dt} \sigma_{11}^2} - \frac{C^2(t)B(t) \left(r(t) + \sigma_B(T-t)\lambda_r \right)}{4\alpha(t)e_1^2(t)\omega_{e_1}^2 \frac{dA(t,r(t))}{dt} \sigma_{11}^2} \\ + \frac{C^2(t)e_1(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right)}{2\alpha(t)e_1^2(t)\omega_{e_1}^2 \frac{dA(t,r(t))}{dt} \sigma_{11}^2} + \frac{C^2(t)2\alpha(t)e_1(t)\omega_{e_1}\sigma_{11}z^{-1}(t)\beta(t)}{2\alpha(t)e_1^2(t)\omega_{e_1}^2 \frac{dA(t,r(t))}{dt} \sigma_{11}^2 C(t)} \\ - \frac{C^2(t)2\alpha(t)e_1(t)\omega_{e_1}\sigma_{11}M(t)\sigma_m}{2\alpha(t)e_1^2(t)\omega_{e_1}^2 \frac{dA(t,r(t))}{dt} \sigma_{11}^2 C^2(t)} \\ = -\frac{C^2(t)}{4\alpha(t)e_1^2(t)\omega_{e_1}^2 \frac{dA(t,r(t))}{dt} \sigma_{11}^2} - \frac{C^2(t)B(t) \left(r(t) + \sigma_B(T-t)\lambda_r \right)}{4\alpha(t)e_1^2(t)\omega_{e_1}^2 \frac{dA(t,r(t))}{dt} \sigma_{11}^2} \\ + \frac{C^2(t) \left(r_0(t) + \sigma_{11}\zeta_1 \right)}{2\alpha(t)e_1(t)\omega_{e_1}^2 \frac{dA(t,r(t))}{dt} \sigma_{11}^2} + \frac{C(t)z^{-1}(t)\beta(t)}{e_1(t)\omega_{e_1} \frac{dA(t,r(t))}{dt} \sigma_{11} C(t)} - \frac{M(t)\sigma_m}{e_1(t)\omega_{e_1} \frac{dA(t,r(t))}{dt} \sigma_{11}}.$$

With an explicit formula for the optimal allocation proportions we are now in a position to run simulations.

5.2 Simulations

Based on the foregoing theory, in this section we present simulations on some of the more important variables. The specific ones considered here are bank capital, total risk-weighted assets, capital adequacy ratio and optimal bank equity allocation. We consider two different forms of the amortization function namely of a linear form and of a quadratic form.

5.2.1 Linear Amortization function

In this section we only provide the graphs based on the assumption that an amortization function has a linear form for bank capital, total risk-weighted assets, capital adequacy ratio and a loan repayment. We will jointly interpret the aforementioned items in the following section, that is, the quadratic amortization function and linear amortization function together. Figure 5.1, figure 5.2, figure 5.3 and figure 5.4 are obtained from an amortization function having a linear form $A(t) = Kt$, where K is a constant repayment of the loan from an entity.

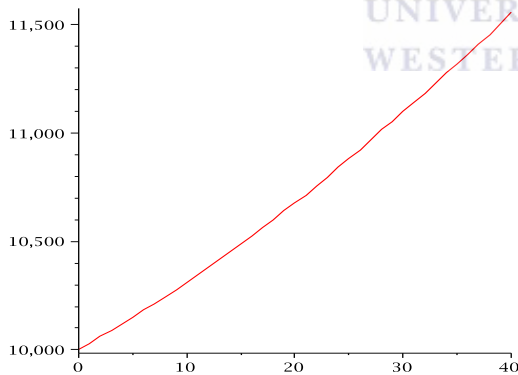


Figure 5.1: Trajectory of the total risk-weighted assets over a period of 40 months, that is, $0 \leq t \leq 40$.

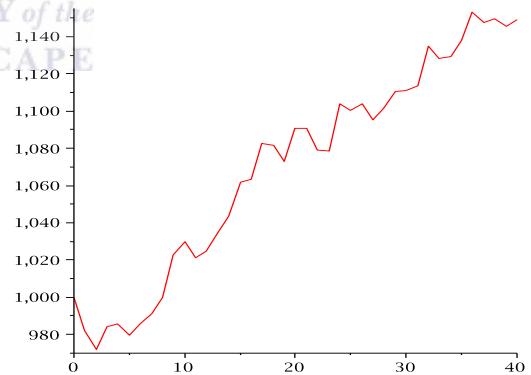


Figure 5.2: Trajectory of bank regulatory capital over a period of 40 months, that is, $0 \leq t \leq 40$.

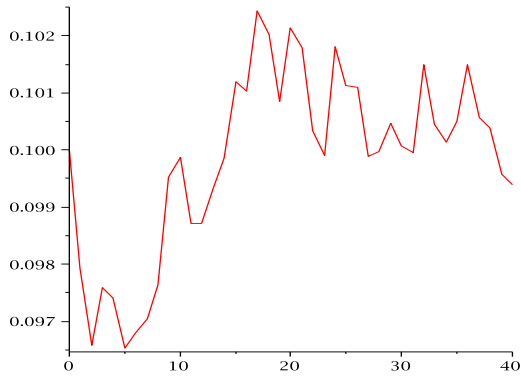


Figure 5.3: Trajectory of the capital adequacy ratio over a period of 40 months, that is, $0 \leq t \leq 40$.

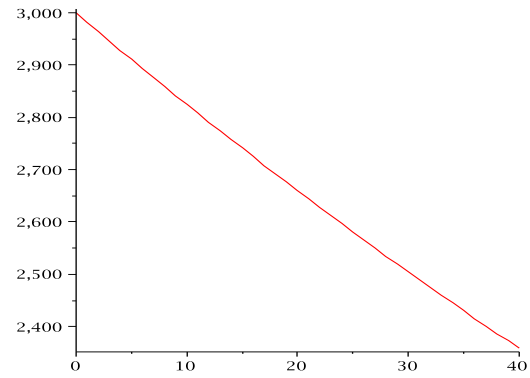


Figure 5.4: Repayment of a loan over a period of 40 months, that is, $0 \leq t \leq 40$.

5.2.2 Quadratic Amortization function

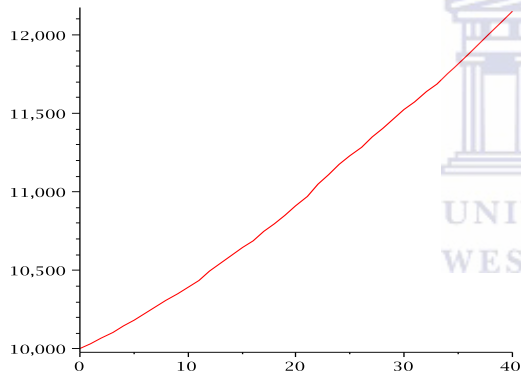


Figure 5.5: Simulation of the total risk-weighted assets over a period of 40 months, that is, $0 \leq t \leq 40$.

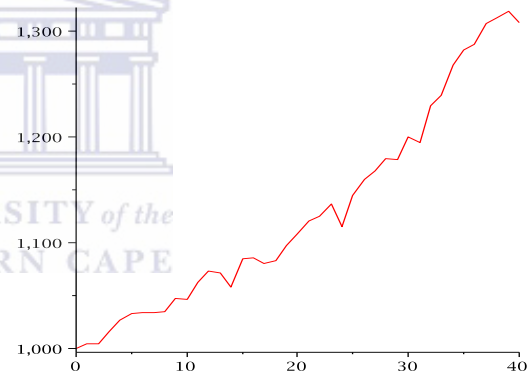
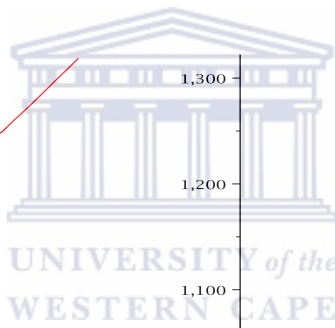


Figure 5.6: Trajectory of bank regulatory capital over a period of 40 months, that is, $0 \leq t \leq 40$.

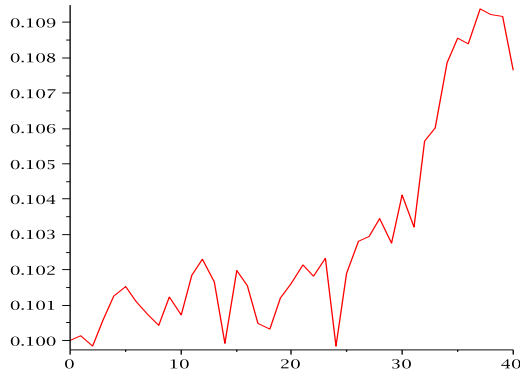


Figure 5.7: Trajectory of the capital adequacy ratio over a period of 40 months, that is, $0 \leq t \leq 40$.

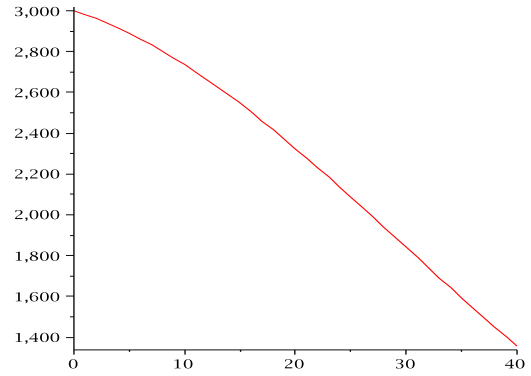


Figure 5.8: Repayment of a loan over a period of 40 months, that is, $0 \leq t \leq 40$.

Figure 5.5, figure 5.6, figure 5.7 and figure 5.8 are all obtained from an amortization function having a form, $A(t, r(t)) = Kt + \frac{rt^2}{2}$. Figure 5.1, figure 5.2, figure 5.5 and figure 5.6 shows the trajectories for bank capital and the bank's total risk-weighted assets respectively. We observe that both items increase over the forty month period. The total assets increases at a steady rate due to loans that are paid off by consumers or other financial institutions. Also loans represent the majority of a bank's asset and a bank can earn a higher interest on a loan contract than securities. Another factor contributing to this increase is that banks do not like putting their assets into fixed-income securities because the the yield is not that great. However, investment-grade securities are liquid, and they have higher yields than cash, so it is always prudent for a bank to keep securities on hand in case they need to free up some liquidity. On the other hand bank capital can effect the lending behaviour of a bank. The regulatory capital requirements are explicitly taken into account. Here the regulatory requirement depends on the loans granted which establishes a relationship between bank capital and bank lending (we refer the reader to Gambacorta, Mistrulli [51] and the references contained within for literature on the relationship). Gambacorta, Mistrulli [51] explores how bank capital influences bank lending by considering the effects of two economic disturbances namely monetary policy and Gross Domestic Products (GDP) shocks. The impact of monetary policy effects lending

in two ways, both based on adverse selection problems that affect banks fund-raising: the bank lending channel, which relies on imperfections in the market for bank debt and the bank capital channel, which concentrates on an imperfect market for bank equity (we refer the reader to Gambacorta, Mistrulli [51] for a more detailed discussion on the bank capital channel and bank lending channel). The bank capital channel depends on three assumptions. Firstly, there is an imperfect market for bank equity: banks cannot easily issue new equity because of the presence of agency costs and tax disadvantages. Secondly, banks are subject to interest rate risk due to the fact that their assets have a longer maturity than their liabilities and thirdly, banks have to meet regulatory capital requirements linked to credit supply.

Figure 5.3 and figure 5.7 shows trajectories for the capital adequacy ratios of a commercial bank. In our case the trajectories represents a bank that is well-capitalized (a bank whose capital-to-asset ratio is more than 10%). Well capitalized banks are in a better position than less-capitalized banks to absorb economic disturbances such as the monetary policies. Because they hold more capital in excess, well-capitalized banks need to adjust their lending activities during economic downturns in order to avoid regulatory capital shortfalls. Another reason could also be that their profits are less sensitive to the business cycle, as their portfolio choices may differ from those taken by less-capitalized banks. The way in which we constructed the continuous-time models ensures that the capital adequacy ratio remains always above the minimum requirement of 8%.

Figure 5.4 and figure 5.8 show the trajectories for a loan that is paid-off by an amortization rate having a linear form and a quadratic form respectively. Figure 5.4 shows how a debt, in our case, is repaid over time by regular instalments. Figure 5.8 shows how the debt is repaid where the amortization rate has the form $\frac{dA(t, r(t))}{dt} = K + rt$ where K is the principal and rt is the accrued interest.

In figure 5.8, there is substantial distinct allocation of the monthly payments toward the interest, especially during the first few months of the loan. Payment 1 allocates about

80 – 90% of the total payment towards interest and only 10 – 20% toward the principal balance. The percentage allocation towards payment of the principal solely depends on the interest rate. Only after a certain payment into the loan does the payment allocation towards principal and interest even out. After that, the majority of the monthly payment is towards the principal balance pay down.

Secondly, the repetitive nature of an amortized loan, even in cases of decreasing interest rates and principal balance decrease, can cause the borrower to pay a high percentage of the original loan amount. This creates a situation that is economically unfavorable because it is often mitigated by monthly decreasing payments and interest rate of refinance.

Thirdly, the payment made on an amortized loan remains fixed for the entire loan contract, regardless of principal balance owed. Paying down a large amount of the principal balance in no way affects the monthly payment, it simply reduces the term of the loan contract and reduces the amount of interest that can be charged by the lender resulting in a quicker payoff. To avoid these obstacles many borrowers may prefer to choose an interest-only loan to satisfy their financing needs.

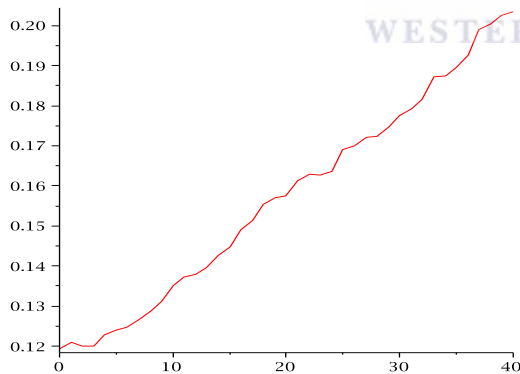


Figure 5.9: Trajectory of the fraction $f(t)$ over a period of 40 months, that is, $0 \leq t \leq 40$.

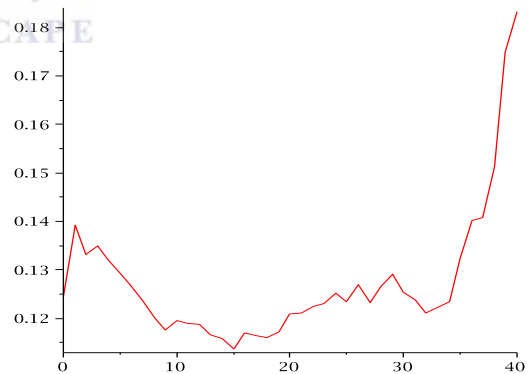


Figure 5.10: Simulation of the fraction $f(t)$ over a period of 40 months, that is, $0 \leq t \leq 40$.

Figure 5.9 and figure 5.10 represents the trajectories of the optimal allocation strategy to

maximize the quantity

$$Q = D_{Y_A(t)} - \alpha(t)D_z^{-1(t)}.$$

Figure 5.9 is obtained from the a linear amortization function whereas figure 5.10 is obtained from an amortization function having a quadratic form. We observe that in both cases the fraction remains between 0 and 1. The aim of this investment strategy is to balance risk and reward by apportioning a portfolio's assets according to an institutions goals, risk tolerance and investment horizon. The bank fund represents the proportions of the fund invested in the portfolio in order to minimize the terminal solvency risk. The mutual fund that provides investors with a portfolio of a fixed or variable mix of the three main asset classes - stocks, bonds and cash equivalents - in a variety of securities. In the first years where debt is large, the optimal strategy is to take more risk, borrowing money to invest in the equity. The optimal strategy also has a prominent role whereby it contributes to the value of the bank's total risk-weighted assets.



Chapter 6

Conclusion

In this section, we interpret the main results encountered in this dissertation. In accordance with the objectives of the Basel II capital accord, the models of banking items constructed in this study are related to the methods currently being used to assess the riskiness of bank portfolios and their minimum capital requirement (see [12] and [18]). The assessment procedure mainly involves a consideration of the capital adequacy and perceived supervisory risk. In particular, chapter 4 of this dissertation is devoted to the description of the capital adequacy ratio and as well in chapter 5. Here we constructed continuous-time models for the capital adequacy ratio in a stochastic setting. We observe in figure 4.2 and 4.3 that the trajectories of the capital adequacy ratio always remain above the stipulated minimum requirement of 8% suggested by Basel II capital accord. In figures 5.3 and 5.7 we observe that the trajectories of the capital adequacy ratio for a well-capitalized bank ($CAR \geq 10\%$) always remain above the minimum requirement (see figure 4.4 for the categories of the ratios).

Sections 4.2 and 4.3 are entirely devoted to the demonstration of how the capital adequacy ratio can be optimized in terms of equity allocation. We observed that in figure 4.5 and figure 4.6 that the optimal allocation strategies (π_0^* , π_1^* and π_2^*) which is split over the portfolio consisting of three assets (cash, bond and equity) remained consistent over time.

Chapter 5 is an original piece of work by the author of this dissertation where we demonstrate how to employ a mean-variance optimization approach to equity allocation under certain conditions. Determining the optimal investment strategy employed by the investor, in other words the decision on exactly how to distribute the total investment over the different possible assets in order to maximize their profit from the final contribution in the planning horizon, is known as portfolio-optimization. In particular figures 5.9 and 5.10 illustrates the aforementioned concept.

The main thrust of future research may involve models of bank items driven by Lévy processes. These processes have an advantage over the more traditional modelling tools such as Brownian motion in that they describe the non-continuous evolution of the value of economic and financial items more accurately. For instance, because the behavior of bank loans, securities, capital and CARs are characterized by jumps, the representation of the dynamics of these items by means of Lévy processes is more realistic. As a result of this, recent research (see Gideon, Mukuddem-Petersen, Petersen [52]) has strived to replace the existing Brownian motion-based bank models (see for instance Decamps, Rochet, Roger [38], Leland [63], Fouche, Mukuddem-Petersen, Petersen [46]) by systems driven by more general processes.

Chapter 7

Appendix A

7.0.3 Table Containing Risk Categories, Risk-Weights and Representative On-Balance Sheet Items

In this section, we provide a table of risk categories, risk-weights and representative on-balance sheet items and verify the main results obtained in the previous sections.

Risk Category	Risk Weight	Representative On-Balance Sheet Items
1	0%	Cash, Reserves, Bonds
2	20%	Marketable Securities, equities
3	50%	Home Mortgages
4	100%	Loans to Private Agents

Table 7.1: Risk Categories, Risk-Weights and Representative On-Balance Sheet Items.

We first provide a tabel of values for the following trajectories of each of the aforementioned items.

Symbol	Value	Parameter
n	40	period
h	0.1	increment
$C(0)$	1000	Bank capital at time 0
$L(0)$	3000	Loans in dollar (\$) at time 0
ζ_1	0.1	Market price of risk at time 0
λ_r	0.04	constant premium for bond
w_{e_1}	0.2	weight in equity
σ_m	0.04	constant volatility for marketable securities
σ_{11}	0.04	constant volatility for equity
$\alpha(t)$	3	value for the notation
r	0.04	constant interest rate
r_m	0.03	constant interest rate for marketable securities
$\frac{dA(t,r(t))}{dt}$	0.1	rate at which the loan is paid off (rate of amortization).
$\beta(t)$	0.03	diffusion term for bank capital
$\rho(t)$	0.2	drift term for bank capital
p	0.02	constant proportion of assets
$M(0)$	7000	marketable securities at time 0
$e_1(t)$	1000	value of the first bank equity at time 0
$a(0)$	10000	value of total assets at time 0
σ_B	0.04	volatility for the price of the bond
$B(0)$	100	value of the bond at time 0

Table 7.2: Parameter values for the constructed models and the amortization function $A(t) = Kt$.

Symbol	Value	Parameter
n	40	period
h	0.1	increment
$C(0)$	1000	Bank capital at time 0
$L(0)$	3000	Loan in dollar (\$) at time 0
ζ_1	0.02	Market price of risk at time 0
λ_r	0.10	constant premium for bond
w_{e_1}	0.2	weight in equity
σ_m	0.03	constant volatility for marketable securities
σ_{11}	0.04	constant volatility for equity
$\alpha(t)$	1	value for the notation
r	0.04	constant interest rate
r_m	0.03	constant interest rate for marketable securities
$\frac{dA(t, r(t))}{dt}$	0.1	rate at which the loan is paid off (rate of amortization).
$\beta(t)$	0.03	diffusion term for bank capital
$\rho(t)$	0.2	drift term for bank capital
p	0.02	constant proportion of assets
$M(0)$	7000	marketable securities at time 0
$e_1(t)$	1000	value for the first bank equity at time 0
$a(0)$	10000	value of total assets at time 0
σ_B	0.03	volatility for the price of the bond
$B(0)$	1000	value of the bond at time 0
r_A	0.07	constant interest rate

Table 7.3: Parameter values for the constructed models and the amortization function

$$A(t, r(t)) = Kt + \frac{rt^2}{2}.$$

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