# Perturbations of Dark Energy models

**By**

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# Keywords

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# Abstract

The growth of structure in the Universe proceeds via the collapse of dark matter and baryons. This process is retarded by dark energy which drives an accelerated expansion of the late Universe. In this thesis we use cosmological perturbation theory to investigate structure formation for a particular class of dark energy models, i.e. interacting dark energy models. In these models there is a non-gravitational interaction between dark energy and dark matter, which alters the standard evolution (with non-interacting dark energy) of the Universe. We consider a simple form of the interaction where the energy exchange in the background is proportional to the dark energy density. We analyse the background dynamics to uncover the effect of the interaction. Then we develop the perturbation equations that govern the evolution of density perturbations, peculiar velocities and the gravitational potential. We carefully account for the complex nature of the perturbed interaction, in particular for the momentum transfer in the dark sector. This leads to two different types of model, where the momentum exchange vanishes either in the dark matter rest-frame or the dark energy rest-frame. The evolution equations for the perturbations are solved numerically, to show how structure formation is altered by the interaction.

# Declaration

I declare that *Perturbations of Dark Energy Models* is my own work, that it has not been submitted for any degree or examination in any other university, and that all sources I have used or quoted have been indicated and acknowledged by complete references.

Mohammed A. M. A. Elmufti 29 February 2012.

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# Chapter 1

# Introduction

Perturbation theory is one of the most important tools in cosmology. It consists of three major parts: scalar, vector and tensor perturbations. Scalar perturbations are the most important ones, where they include density and velocity perturbations. They produce structures in the Universe starting with small initial perturbations. They experience gravitational instability, so that overdense regions grow more overdense. Vector perturbations couple to the perturbations of the rotational velocity in the cosmic fluids and they decay in the expanding Universe. Tensor perturbations are gravitational waves in an expanding Universe. If they are strong enough, then with observations we can detect their effect on the anisotropy of the cosmic microwave background (CMB).



The current cosmological observations indicate the need for non-baryonic cold dark matter (CDM) to grow the galaxies, and for dark energy (DE) to drive a late-time acceleration of the Universe [3]. Models of interacting DE assume only an interaction with CDM. There could be an interaction between CDM and DE that would not show itself in any laboratory experiments but would show an effect in the dynamics and structure formation of the Universe.

In our work we consider an interaction between CDM and a DE fluid model, in which energy transfers from CDM to the DE.

This thesis is constructed as follows:

In Chapter 1 we introduce the Friedmann-Lemaître-Robertson-Walker (FLRW) background spacetime. We give a summary of the main components of the Universe and connect that with the epochs of radiation, matter and dark energy domination. We conclude by defining basic cosmological concepts, some of which have been used in our work.

In Chapter 2 we give a general review of scalar perturbations in cosmology. Based on Einstein's theory of general relativity, we introduce the mathematical formulas for the perturbed metric, curvature and energy-momentum tensors. We show how to define a gauge transformation rule to find a correspondence formula between scalar quantities in the background and the perturbed spacetimes. We provide perturbed conservation equations for the energy and momentum and also the perturbed field equations. Then we use Fourier expansion to expand the perturbations in the background spacetime.

In Chapter 3 we introduce the dynamics of interacting fluids that will be used in the following chapters. Then we define their governing equations that produce background and perturbed spacetime equations.

In Chapter 4 we study the dynamics of the background in a simple interacting dark energy model. We show how that model can produce significant changes to the background dynamics.

In Chapter 5 we provide a full analysis of the perturbations in the interacting dark energy models. We use that to derive evolution equations in terms of velocity and density perturbations. We connect that with our background part in Chapter 4 to investigate structure formation in the current Universe.

Conclusions are provided in Chapter 6.

## 1.1 A Homogeneous and Isotropic Universe

Theoretical cosmology aims to find out the origin, composition and evolution of the entire Universe as a single system. In modern cosmology, all the fundamental assumptions are the cosmological principle and the theory of general relativity (GR). The cosmological prinicple embodies isotropy and homogeneity of the Universe.

Isotropy indicates that there is no preferred direction in the Universe in a way that, if the Universe is viewed from any specific point it looks the same regardless of the direction. Homogeneity implies that the Universe appears the same everywhere at any given cosmic time in the spacetime.

Based on GR, the FLRW spacetime model provides these properties in which the line element is described as

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big], \tag{1.1}$$

where $t$ is the proper time, $a$ is the scale factor, $K$ is the curvature, $H$ is the Hubble parameter and $(r, \theta, \phi)$ are the coordinates of the 3-spaces.
To describe the matter, an ideal fluid will be used, where

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{1.2}$$

where $u^\mu = \delta_0^\mu$ is the fluid four-velocity and $\rho, p$ are the fluid energy density and pressure.
The equation of state

$$w = \frac{p}{\rho}, \tag{1.3}$$

relates pressure and density, with $w$ often assumed constant. Then the GR field equations (1.1) and (1.2) give [1],[2]

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \tag{1.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{1.5}$$

know as the Friedmann and Raychaudhuri equations respectively.

## 1.2 Present Composition of the Universe

In terms of the current Hubble rate $H_0$ we can define a critical density $\rho_c$

$$\rho_c = \frac{3H_0^2}{8\pi G}. \tag{1.6}$$

Since the sum of different contributions represents the total density of those contributions, then

$$\rho = \sum_A \rho_A. \tag{1.7}$$

Then we define the dimensionless quantities

$$\Omega_{I0} = \frac{\rho_{I0}}{\rho_c} = \frac{8\pi G\rho_{I0}}{3H_0^2}, \tag{1.8}$$

$$\sum_{I0} \Omega_{I0} = 1, \tag{1.9}$$

where the curvature term is $\Omega_{K0} = -K/(a_0^2 H_0^2)$.

In the current Universe, there are four components contributing to this sum [3] :

## Radiation

including the photons of the Cosmic Microwave Background (CMB) and massless neutrinos, with equation of state

$$p_r = \frac{1}{3}\rho_r \quad (w_r = \frac{1}{3}), \tag{1.10}$$

and $\Omega_{r0} \approx 10^{-4}$.

## Baryonic Matter

respresented by the current non-relativistic atoms and nuclei that constitute the ordinary matter, with equation

$$p_b = 0 \quad (w_b = 0), \tag{1.11}$$



and $\Omega_b^{(0)} \approx 0.05$.

## Cold Dark Matter (CDM)

is related to theories beyond the standard model of particle physics and not yet detected. It is introduced to offer a more consistent explanation for the growth and properties of galaxies, with equation of state

$$p_c = 0 \quad (w_c = 0), \tag{1.12}$$

and $\Omega_{c0} \approx 0.2$.

### Dark Energy

Observations of supernovae and other observational evidence motivate the idea of introducing dark energy (DE) which is a field that causes the late-time acceleration of the Universe. The simplest model of DE is a cosmological constant with

$$p_\Lambda = -\rho_\Lambda \quad (w_\Lambda = -1)\ , \quad \rho_\Lambda = \frac{\Lambda}{8\pi G}, \tag{1.13}$$

and $\Omega_\Lambda \approx 0.75$.

## 1.3 The History of the Universe

From the energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$, we can extract a continuity equation in the form

$$\dot{\rho} = -3H(\rho + p). \tag{1.14}$$

This equation is correct for each $\rho_I$ in case the interaction is purely gravitational between the fluids. The solution of equation (1.14) when $w =$ constant is given by

$$\rho_I = \rho_{I0} \left(\frac{a_0}{a}\right)^{3(1+w_I)}. \tag{1.15}$$



Then the Friedmann equation (1.4) for $K = 0$ may be written as [4]

$$\frac{H^2(a)}{H_0^2} = \Omega_{r0} \left(\frac{a_0}{a}\right)^4 + (\Omega_{b0} + \Omega_{c0}) \left(\frac{a_0}{a}\right)^3 + \Omega_{\Lambda 0}. \tag{1.16}$$

The right hand side of the above equation varies with the scale factor $a$ in a different negative power as $a$ grows with time. That means each term on the right dominates at different times during the history of the present Universe :

### Dark Energy Dominated Era

At recent times, $a \lesssim a_0, \Lambda$ dominates the right of (1.16). This leads to an approximate solution of (1.4): $a \propto \exp \sqrt{\frac{\Lambda}{3}} t$, resulting in a period of exponential expansion with constant $H$ known as de Sitter phase.

### Matter Dominated Era

Before DE domination, matter dominated. The approximation solution of (1.4) gives $a \propto t^{2/3}$.

### Radiation Dominated Era

When we go back far enough into the past history of the Universe, radiation dominates the total energy density. The apprximate solution of (1.4) gives $a \propto t^{1/2}$.

The radiation energy density dilates more quickly compared to the non-relativistic matter. That is caused by the expansion of the Universe, in which the photons faced additional energy decrease.
In the observed redshift $z$ (see below) DE dominates for $z \lesssim 1$, matter dominates for $1 \lesssim z \lesssim 10^4$, and radiation dominates for $z \gtrsim 10^4$.

## 1.4 Basic Cosmological Concepts



**Comoving Coordinates:** a system of coordinates fixed with respect to the overall Hubble flow of the Universe, so that any given location of a galaxy in comoving coordinates does not change during the expansion of the Universe.
**Comoving distance:** the comoving distance $d$ from an observer to a distant object (e.g. a galaxy) can be computed by

$$d = \int_{te}^{t_0} \frac{dt'}{a(t')} \quad , \tag{1.17}$$

where $a(t')$ is the scale factor, $t_e$ is the cosmic time of emission of photons detected by the observer and $t_0$ is the cosmic time at the observer. The comoving distance between two points remains constant during the evolution of the Universe, while the physical distance between two points does, however, change as the Universe expands: $d_{phys} = ad$ [5].
**Hubble Scale:** The Hubble parameter defines a length scale $H^{-1}$ known as the Hubble radius. At each cosmic time $t$, this defines a sphere of radius $H^{-1}(t)$.

**Redshift:** the expansion of the Universe alters the wavelength of the light emitted by distant objects. The ratio of the observed wavelength to the emitted rest wavelength defines the redshift $z$

$$1 + z = \frac{\lambda_0}{\lambda_e}, \tag{1.18}$$

where $\lambda_0$ and $\lambda_e$ are the observed and emitted wavelengths respectively. In terms of the scale factor

$$1 + z = \frac{a_0}{a(t_e)}. \tag{1.19}$$

**Fourier decomposition:** is given by

$$\phi(x) = \int d^3\vec{k} e^{-\vec{k}\cdot\vec{x}} \phi_k. \tag{1.20}$$

The comoving wave number $k$ of Fourier mode is related to the physicsal wavelength $\lambda$ of the mode by

$$k = \frac{a}{\lambda}. \tag{1.21}$$

**Super-Hubble:** modes have wavelength greater than the Hubble scale

$$\lambda > H^{-1} \qquad \Leftrightarrow \qquad \frac{a}{k} > H^{-1} \qquad \Leftrightarrow \qquad k < aH. \tag{1.22}$$

On super-Hubble scales the wavelength is greater than the Hubble scale which is the scale defining speed of light sphere. On super-Hubble scales there can be no causal contacts, so any two physical conditions in two events A and B at the left and right hand sides outside of the light sphere cannot influence each other.

So super-Hubble modes do not respond to physical influence like pressure.

**Sub-Hubble:** modes satisfy

$$\lambda < H^{-1} \qquad \Leftrightarrow \qquad \frac{a}{k} < H^{-1} \qquad \Leftrightarrow \qquad k > aH. \tag{1.23}$$

For any two physical conditions in two events A and B at the left and right hand sides inside the light sphere all physical influences are felt, including pressure.

# Chapter 2

# Review of cosmological scalar perturbations

## 2.1 Perturbations in General Relativity



The idea of the perturbation theory of general relativity is to consider a perturbed spacetime close to another symmetric and simple spacetime known as the background spacetime, as shown in Figure (2.1) below

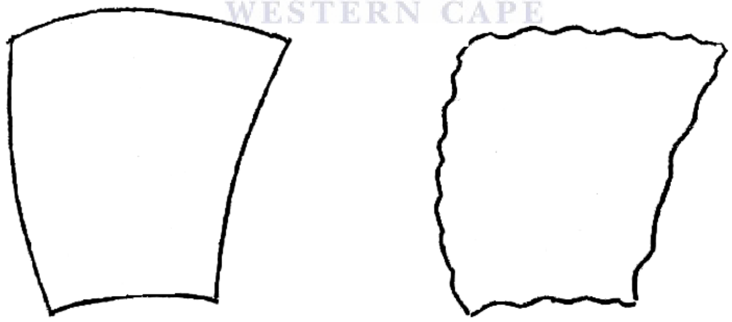

Figure 2.1: Comparison between the background (left) and the perturbed (right) spacetimes (adapted from [2]).

Physically, that means there is a coordinate system on that perturbed spacetime, and its metric will be in the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{2.1}$$

where $\bar{g}_{\mu\nu}$ represents the metric of that background spacetime and $\delta g_{\mu\nu}$ is small.

For a linear perturbation theory, the product of small quantities such as $\delta g_{\mu\nu}$, $\delta g_{\mu\nu,\rho}$ and $\delta g_{\mu\nu,\rho\sigma}$ will be neglected. By contrast, for a second order perturbaion theory, the product of each two small quantities will survive.

Consequently, the curvature and energy-momentum tensors of that perturbed space-time should be expressed in the form

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + \delta G_{\mu\nu}, \tag{2.2}$$

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}. \tag{2.3}$$

Then the field equations for the perturbation are

$$\delta G^{\mu}_{\nu} = 8\pi G \delta T^{\mu}_{\nu}. \tag{2.4}$$

The background Universe is homogeneous and isotropic, and, $t =$ constant time slices have Euclidean geometry.

## 2.2 The Background Universe

Since the background spacetime is a FLRW Universe in comoving coordinates $x^\mu$ the background metric is given by



$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right], \tag{2.5}$$

where $x^\mu = (t, x^i)$ and $a$ is the scale factor.

The field equations (1.4) in a flat Universe are

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho}, \tag{2.6}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p}), \tag{2.7}$$

where we use overbars to indicate the background density and pressure.

The energy conservation equation (or continuity equation) is

$$\dot{\bar{\rho}} = -3H(\bar{\rho} + \bar{p}). \tag{2.8}$$

Equations (2.6) and (2.7) also imply

$$\dot{H} \equiv \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G(\bar{\rho} + \bar{p}). \tag{2.9}$$

We can also use conformal time ($\eta$) defined as

$$d\eta = \frac{dt}{a(t)}, \tag{2.10}$$

accordingly to which the background metric can be written in the form

$$d\bar{s}^2 = a^2(\eta)\Big[-d\eta^2 + \delta_{ij}dx^i dx^j\Big] = a^2(\eta)\Big[-d\eta^2 + dx^2 + dy^2 + dz^2\Big], \tag{2.11}$$

where

$$\bar{g}_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \qquad \Rightarrow \qquad \bar{g}^{\mu\nu} = a^{-2}(\eta)\eta^{\mu\nu}. \tag{2.12}$$

Then equations (2.6) and (2.9) will be in the forms

$$\mathcal{H}^2 = \left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho}a^2, \tag{2.13}$$

$$\mathcal{H}' = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p})a^2, \tag{2.14}$$

where a prime denotes d/d$\eta$, and

$$\mathcal{H} \equiv \frac{a'}{a} = aH = \dot{a}. \tag{2.15}$$

We note that [6]

$$\mathcal{H}' = \left(\frac{a'}{a}\right)' = \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = (a\dot{a})^{\cdot} - \dot{a}^2 = a^2(\dot{H} + H^2). \tag{2.16}$$



The continuity equation becomes

$$\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{p}) \quad \Rightarrow \qquad \bar{\rho}' = -3\mathcal{H}(1 + w)\bar{\rho}. \tag{2.17}$$

We can also define the adiabatic speed of sound ($c_s$) as

$$c_s^2 \equiv \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} = \frac{\bar{p}'}{\bar{\rho}'}\,. \tag{2.18}$$

This is a background quantity. We note that for an adiabatic fluid, if w = const. then

$$c_s^2 = w. \tag{2.19}$$

In general we have

$$w' = \frac{\rho'}{\rho}(c_s^2 - w) = 3\mathcal{H}(1 + w)(w - c_s^2). \tag{2.20}$$

## 2.3 The Perturbed Universe

In cosmological perturbation theory, the metric of the perturbed FLRW Universe can be written in the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}), \tag{2.21}$$

and its inverse is

$$g^{\mu\nu} = a^{-2}(\eta^{\mu\nu} - h^{\mu\nu}), \tag{2.22}$$

where $h_{\mu\nu}$ is the perturbation. We treat the perturbations (which are scaled by $a^2$) as tensors on Minkowski spacetime [7].
Then we can define

$$h^{\mu}_{\nu} \equiv \eta^{\mu\rho} h_{\rho\nu} \qquad \text{and} \qquad h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}. \tag{2.23}$$

$h_{\mu\nu}$ can be expressed as

$$[h_{\mu\nu}] = \begin{bmatrix} -2\phi & +B_i \\ +B_i & -2\psi\delta_{ij} + 2E_{ij} \end{bmatrix}, \tag{2.24}$$

where $\psi$ carries the trace of the perturbation $h_{ij}$. $E_{ij}$ is traceless so that



$$\psi = -\frac{1}{6} h^i_i, \tag{2.25}$$

$$\delta^{ij} E_{ij} = 0. \tag{2.26}$$

Consequently, the inverse of the perturbation $h_{\mu\nu}$ will be in the form

$$[h^{\mu\nu}] = \begin{bmatrix} 2\phi & +B_i \\ f + B_i & 2\psi\delta_{ij} - 2E_{ij} \end{bmatrix},$$

where lowering or raising the indices of $B_i$ and $E_{ij}$ give the same value. Since for scalar perturbations every 3-vector is a gradient [2], then

$$B_i = B_{,i}. \tag{2.27}$$

Also since every 3-tensor is a double gradient

$$E_{ij} = E_{,ij}. \tag{2.28}$$

Then the line element becomes [7]

$$ds^2 = a^2(\eta)\{-(1+2\phi)d\eta^2 + 2B_{,i}d\eta dx^i + [(1-2\psi)\delta_{ij} + 2E_{,ij}]dx^i dx^j\}. \quad (2.29)$$

See appendix A for a discussion of vector and tensor perturbations.

We will often use the Newtonian (or longitudinal) gauge, defined by

$$E = 0 = B. \quad (2.30)$$

## 2.4 Gauge Transformations

We can link-up between points in our FLRW background and the perturbed spacetimes using the coordinate system $\{x^\mu\}$, so that in any background coordinate system there are other equivalent perturbed coordinate systems.

Taking that to GR perturbations, then gauge transformations represent a transform of coordinate system (say $\hat{x}^\alpha$ and $\tilde{x}^\alpha$) in the perturbed spacetime. Then these coordinates will be connected in the form



$$\tilde{x}^\alpha = \hat{x}^\alpha + \xi^\alpha, \quad (2.31)$$

with the assumption that $\xi^\alpha$ lives in the background spacetime. For more convenience, we illustrate the above explanation in Figure (2.2).
Mathematically, we can express the transformation coordinates of similar points in the perturbed spacetime in the form

$$\tilde{x}^\alpha(\tilde{P}) = \hat{x}^\alpha(\tilde{P}) + \xi^\alpha, \quad (2.32)$$

$$\tilde{x}^\alpha(\hat{P}) = \hat{x}^\alpha(\hat{P}) + \xi^\alpha, \quad (2.33)$$

with a link up in the form

$$\tilde{x}^\alpha(\tilde{P}) = \hat{x}^\alpha(\hat{P}) = x^\alpha(\bar{P}). \quad (2.34)$$

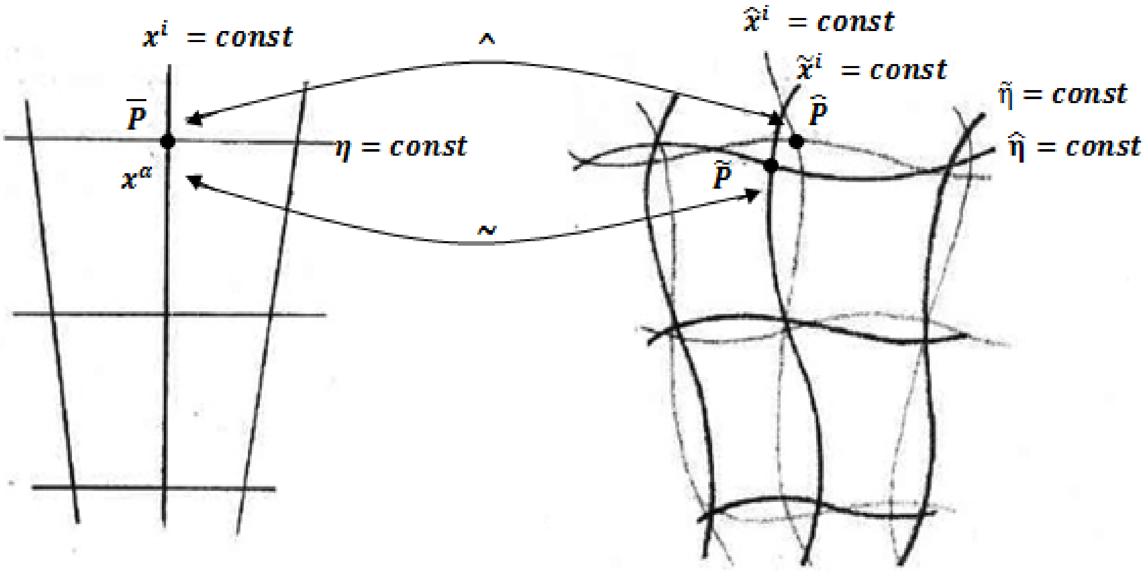


Figure 2.2: Gauge transformation between background (left) and perturbed (right) spacetimes (adapted from [2] and [8]).

Then from equations (2.32),(2.33) and (2.34) we conclude that in a given coordinate system the coordinates of any two different points are in the form

$$\hat{x}^\alpha(\tilde{P}) = \hat{x}^\alpha(\hat{P}) - \xi^\alpha, \tag{2.35}$$

$$\tilde{x}^\alpha(\tilde{P}) = \tilde{x}^\alpha(\hat{P}) - \xi^\alpha. \tag{2.36}$$

In the FLRW background, we have various 4-scalar fields $\bar{s}$, 4-vector fields $\bar{\omega}^\alpha$ and tensor fields $\bar{A}^\alpha_\beta$, $\bar{B}_{\alpha\beta}$. So in the perturbed spacetime we should have correspondence with quantities in the form

$$s = \bar{s} + \delta s, \tag{2.37}$$

$$\omega^\alpha = \bar{\omega}^\alpha + \delta\omega^\alpha, \tag{2.38}$$

$$A^\alpha_\beta = \bar{A}^\alpha_\beta + \delta A^\alpha_\beta, \tag{2.39}$$

$$B_{\alpha\beta} = \bar{B}_{\alpha\beta} + \delta B_{\alpha\beta}. \tag{2.40}$$

We are interested here in the case of 4-scalars ($s$).

Logically, for a point in the perturbed spacetime we can't indicate to a unique value of the background quantity $\bar{s}$ since in different gauges this point will be linked up in the background with other points with different $\bar{s}$ values. Also there is no unique perturbation $\delta s$; it is gauge-dependent. In different gauges the perturbations will be in the form

$$\widehat{\delta s}(x^\alpha) = s(\hat{P}) - \bar{s}(\bar{P}), \tag{2.41}$$

$$\widetilde{\delta s}(x^\alpha) = s(\tilde{P}) - \bar{s}(\bar{P}). \tag{2.42}$$

We can relate $s(\tilde{p})$ and $s(\hat{p})$ in the form [2]

$$s(\tilde{P}) = s(\hat{P}) + \frac{\partial s}{\partial \hat{x}^\alpha}(\hat{P})\Big[\hat{x}^\alpha(\tilde{P}) - \hat{x}^\alpha(\hat{P})\Big] = s(\hat{P}) - \frac{\partial s}{\partial \hat{x}^\alpha}(\hat{P})\xi^\alpha = s(\hat{P}) - \frac{\partial \bar{s}}{\partial x^\alpha}(\bar{P})\xi^\alpha, \tag{2.43}$$

where we used $\frac{\partial s}{\partial \hat{x}^\alpha}(\bar{P}) \approx \frac{\partial \bar{s}}{\partial x^\alpha}(\bar{P})$, since they are second order small given the multiplication by $\xi^\alpha$.

In the FLRW background, $\bar{s} = \bar{s}(\eta, x^i) = \bar{s}(\eta)$, so that

$$\frac{\partial \bar{s}}{\partial x^\alpha}(\bar{P})\xi^\alpha = \frac{\partial \bar{s}}{\partial \eta}(\bar{P})\xi^0 = \bar{s}'\xi^0, \tag{2.44}$$

and then

$$s(\tilde{P}) = s(\hat{P}) - \bar{s}'\xi^0. \tag{2.45}$$



Finally we define the gauge transform rule for $\delta s$ in the form

$$\widetilde{\delta s}(x^\alpha) = s(\hat{P}) - \bar{s}'\xi^0 - \bar{s}(\bar{P}) = \widehat{\delta s}(x^\alpha) - \bar{s}'\xi^0. \tag{2.46}$$

## 2.5 Perturbations in the Energy-Momentum Tensor

We start with the background energy tensor, which is necessarily of the perfect fluid form

$$\bar{T}^{\mu\nu} = (\bar{\rho} + \bar{p})\bar{u}^\mu\bar{u}^\nu + \bar{p}\bar{g}^{\mu\nu}, \tag{2.47}$$

$$\bar{T}^\mu_\nu = (\bar{\rho} + \bar{p})\bar{u}^\mu\bar{u}_\nu + \bar{p}\delta^\mu_\nu. \tag{2.48}$$

Since the FLRW is our background Universe then the homogeneity will put the constraint

$$\bar{\rho} = \bar{\rho}(\eta) \qquad \text{and} \qquad \bar{p} = \bar{p}(\eta). \tag{2.49}$$

The isotropy also will put the constraint: the fluid is at rest so that

$$\bar{u}^i = 0 \Rightarrow \bar{u}^\mu = (\bar{u}^0, 0, 0, 0). \tag{2.50}$$

Since

$$\bar{u}_\mu \bar{u}^\mu = \bar{g}_{\mu\nu} \bar{u}^\mu \bar{u}^\nu = a^2 \eta_{\mu\nu} \bar{u}^\mu \bar{u}^\nu = -a^2(\bar{u}^0) = -1, \tag{2.51}$$

then we have [9]

$$\bar{u}^\mu = \frac{1}{a}(1, \vec{0}) \qquad \text{and} \qquad \bar{u}_\mu = a(-1, \vec{0}). \tag{2.52}$$

For the perturbed Universe the energy tensor is

$$T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu. \tag{2.53}$$

The tensor perturbation has 10 degrees of freedom, 6 of them are physical and 4 are gauge. It can also be divided into scalar + vector + tensor with 4 + 4 + 2 degrees of freedom of which 2 + 2 + 2 are physical.

To distinguish betweeen these degrees of freedom, we keep using the assumption that the perfect fluid degrees of freedom in $\delta T^\mu_\nu$ are those keeping $T_{\mu\nu}$ in the perfect fluid form

$$T^\mu_\nu = (\rho + p) u^\mu u_\nu + p \delta^\mu_\nu. \tag{2.54}$$



Based on that, they can be taken as the density perturbation, pressure perturbation and velocity perturbation in the form

$$\rho = \bar{\rho} + \delta\rho\,, \qquad p = \bar{p} + \delta p\,, \quad \text{and} \quad u^i = \bar{u}^i + \delta u^i. \tag{2.55}$$

In contrast $\delta u^0$ is not an independent degree of freedom because of the constraint

$$u_\mu u^\mu = -1. \tag{2.56}$$

The velocity perturbation is defined by

$$v^i = \frac{dx^i}{d\eta} = \frac{u^i}{u^0} = \frac{u^i}{\bar{u}^0} = a u^i. \tag{2.57}$$

Since the ratio of change in comoving coordinate $dx^i$ to change in conformal time $d\eta$ equals the ratio of the corresponding physical distance $adx^i$ to the change in cosmic time $dt = ad\eta$ then the velocity perturbation $v_i$ should also be equal to the fluid velocity observed by a comoving (which means $x^i = \text{const.}$) observer.

Now we need to express $u^\mu$ and $u_\nu$ in terms of $v_i$ so that

$$
\begin{aligned}
u^\mu &= \bar{u}^\mu + \delta u^\mu \equiv (a^{-1} + \delta u^0, a^{-1}v^1, a^{-1}v^2, a^{-1}v^3), && (2.58)\\
u_\nu &= \bar{u}_\nu + \delta u_\nu \equiv (-a + \delta u_0, \delta u_1, \delta u_2, \delta u_3), && (2.59)
\end{aligned}
$$

where they are related by $u_\nu = g_{\mu\nu}u^\nu$ and $u_\mu u^\mu = -1$. Then with the metric (2.29) we get after dropping any quantities that are higher than first order,

$$
\begin{aligned}
u_0 &= g_{0\mu}u^\mu = a^2(-1-2\phi)(a^{-1} + \delta u^0) + \delta_{ij}a^2 B^i a^{-1} v^j \,, \\
&= -a - a^2\delta u^0 - 2a\phi. && (2.60)
\end{aligned}
$$

From that follows

$$
\delta u_0 = -a^2\delta u^0 - 2a\phi, \tag{2.61}
$$

and likewise

$$
\delta u_i = u_i = g_{i\mu}u^\mu = -aB_i + av_i, \tag{2.62}
$$

and

$$
u_\mu u^\mu = -1 - 2a\delta u^0 - 2\phi = -1 \Rightarrow \delta u^0 = -\frac{1}{a}\phi. \tag{2.63}
$$



So for the 4-veclocity we have

$$
u^\mu = \frac{1}{a}(1-\phi, v_i) \qquad \text{and} \qquad u_\mu = a(1-\phi, v_i + B_i). \tag{2.64}
$$

Inserting this in equation (2.54), we get

$$
\begin{aligned}
T^\mu_\nu &= \bar{T}^\mu_\nu + \delta T^\mu_\nu, \\
&= \begin{bmatrix} -\bar{\rho} & 0 \\ 0 & \bar{p}\delta^i_j \end{bmatrix} + \begin{bmatrix} -\delta\rho & (\bar{\rho}+\bar{p})(v_i + B_i) \\ -(\bar{\rho}+\bar{p})v_i & \delta p \delta^i_j \end{bmatrix}. && (2.65)
\end{aligned}
$$

## 2.6 Perturbed Conservation Equations

The field equation

$$
G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.66}
$$

where $G_{\mu\nu}$ is the Einstein tensor, $G$ is the Newton's constant and $T_{\mu\nu}$ is the total energy-momentum tensor.

With Bianchi identities the above equation implies a local conservation of the total energy and momentum so that [10]

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.67}$$

We can extract from the first-order part of the time component of the continuity equation (2.67) a perturbed energy conservation equation in general gauge (by neglecting the anisotropic stress and non-adiabatic pressure)

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) - 3(\rho + p)\psi' + \nabla^2(\rho + p)(v + B) = 0. \tag{2.68}$$

The space component of equation (2.67) gives an evolution equation for the momentum

$$[(\rho + p)(v + B)]' + (\rho + p)\phi + \delta p + 4\mathcal{H}(\rho + p)(v + B) = 0. \tag{2.69}$$

In terms of Newtonian gauge ($E = 0 = B$) we get

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) = (\rho + p)[3\psi' - \nabla^2 v], \tag{2.70}$$

$$[(\rho + p)v]' + \delta p = -(\rho + p)[\phi + 4\mathcal{H}v]. \tag{2.71}$$



## 2.7 Perturbations of the Curvature Tensor

Starting from Christoffel symbols

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}[\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}], \tag{2.72}$$

we apply Newtonian gauge ($E = 0 = B$) on the general metric equation (2.29)

$$ds^2 = a^2(\eta)[-(1 + 2\phi)d\eta^2 + (1 - 2\psi)\delta_{ij} dx^i dx^j]. \tag{2.73}$$

Then the Christoffel symbols are [1]

$$\Gamma^0_{00} = \frac{a'}{a} + \phi' \ , \qquad \Gamma^0_{0k} = \phi_{,k} \ , \qquad \Gamma^0_{ij} = \frac{a'}{a}\delta_{ij} - \left[2\frac{a'}{a}(\phi + \psi) + \psi'\right]\delta_{ij} \ ,$$

$$\Gamma^i_{00} = \phi_{,i} \ , \qquad \Gamma^i_{0j} = \frac{a'}{a}\delta^j_j - \psi^i\delta^i_j \ , \qquad \Gamma^i_{kl} = -(\psi_{,l}\delta^i_k + \psi_{,k}\delta^i_l) + \psi_{,i}\delta_{kl}. \tag{2.74}$$

Therefore

$$\bar{\Gamma}^0_{00} = \mathcal{H}\,, \qquad \bar{\Gamma}^0_{0k} = 0\,, \qquad \bar{\Gamma}^0_{ij} = \mathcal{H}\delta_{ij}\,, \qquad \bar{\Gamma}^i_{00} = 0\,, \qquad \bar{\Gamma}^i_{0j} = \mathcal{H}\delta^i_j\,, \qquad \bar{\Gamma}^i_{kl} = 0\,,$$

and

$$\delta\Gamma^0_{00} = \phi^{'}\,, \qquad \delta\Gamma^0_{0k} = \phi_{,k}\,, \qquad \delta\Gamma^0_{ij} = -[2\mathcal{H}(\phi+\psi)+\psi^{'}]\delta_{ij}\,,$$

$$\delta\Gamma^i_{00} = \phi_{,i}\,, \qquad \delta\Gamma^i_{0j} = -\psi^{'}\delta^i_j\,, \quad \delta\Gamma^i_{kl} = -(\psi_{,l}\delta^i_k + \psi_{,k}\delta^i_l) + \psi_{,i}\delta_{kl}\,. \tag{2.75}$$

The Ricci tensor components are

$$\begin{aligned} R_{\mu\nu} &= \Gamma^\alpha_{\nu\mu,\alpha} - \Gamma^\alpha_{\alpha\mu,\nu} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\nu\mu} - \Gamma^\alpha_{\nu\beta}\Gamma^\beta_{\alpha\mu}, \\ &= \bar{R}_{\mu\nu} + \delta\Gamma^\alpha_{\nu\mu,\alpha} - \delta\Gamma^\alpha_{\alpha\mu,\nu} + \bar{\Gamma}^\alpha_{\alpha\beta}\delta\Gamma^\beta_{\nu\mu} + \bar{\Gamma}^\beta_{\nu\mu}\delta\Gamma^\alpha_{\alpha\beta} - \bar{\Gamma}^\alpha_{\nu\beta}\delta\Gamma^\beta_{\alpha\mu} \\ &\quad -\bar{\Gamma}^\beta_{\alpha\mu}\delta\Gamma^\alpha_{\nu\beta}. \end{aligned} \tag{2.76}$$

These calculations give the components below

$$R_{00} = -3\mathcal{H}^{'} + 3\psi^{''} + \nabla^2\phi + 3\mathcal{H}(\phi^{'} + \psi^{'}). \tag{2.77}$$

$$R_{0i} = 2(\psi^{'} + \mathcal{H}\phi)_{,i}. \tag{2.78}$$

$$\begin{aligned} R_{ij} &= (\mathcal{H}^{'} + 2\mathcal{H}^2)\delta_{ij} \\ &\quad +[-\psi^{''} + \nabla^2\psi - \mathcal{H}(\phi^{'} + 5\psi^{'}) - (2\mathcal{H}^{'} + 4\mathcal{H}^2)(\phi+\psi)]\delta_{ij} \\ &\quad +(\psi-\phi)_{,ij}. \end{aligned} \tag{2.79}$$

Now we want to get the contravarient components, so we need to raise an index to get $R^\mu_\nu$, but also we should remember that we cannot separately raise the index of the background and perturbation parts, where

$$R^\mu_\nu = g^{\mu\alpha}R_{\alpha\nu} = (\bar{g}^{\mu\alpha} + \delta g^{\mu\alpha})(\bar{R}_{\alpha\nu} + \delta R_{\alpha\nu}) = \bar{R}^\mu_\nu + \delta g^{\mu\alpha}\bar{R}_{\alpha\nu} + \bar{g}^{\mu\alpha}\delta R_{\alpha\nu}, \tag{2.80}$$

then

$$R^0_0 = 3a^{-2}\mathcal{H}^{'} + a^{-2}[-3\psi^{''} - \nabla^2\phi - 3\bar{H}(\phi^{'} + \psi^{'}) - 6\mathcal{H}^{'}\phi]. \tag{2.81}$$

$$R^0_i = -2a^{-2}(\psi^{'} + \mathcal{H}\phi)_{,i}. \tag{2.82}$$

$$R^i_0 = -R^0_i = 2a^{-2}(\psi^{'} + \mathcal{H}\phi)_{,i}. \tag{2.83}$$

$$\begin{aligned} R^i_j &= a^{-2}(\mathcal{H}^{'} + 2\mathcal{H}^2)\delta^i_j + a^{-2}[-\psi^{''} + \nabla^2\psi - \mathcal{H}(\phi^{'} + 5\psi^{'}) - (2\mathcal{H}^{'} + 4\mathcal{H}^2\phi)]\delta^i_j \\ &\quad +a^{-2}(\psi-\phi)^{,i}_{,j}\,. \end{aligned} \tag{2.84}$$

Moreover, we can sum for the curvature scalar

$$\begin{aligned} R &= R^0_0 + R^i_{0i}, \\ &= 6a^{-2}(\mathcal{H}^{'} + \mathcal{H}^2) + a^{-2}[-6\psi^{''} + 2\nabla^2(2\psi - \phi) - 6\mathcal{H}(\phi^{'} + 3\psi^{'}) \\ &\quad -12(\mathcal{H}^{'} + \mathcal{H}^2)\phi] \,. \end{aligned} \tag{2.85}$$

The Einstein tensor will be then [1]

$$\begin{aligned} G^0_0 &= R^0_0 - \frac{1}{2}R, \\ &= -3a^{-2}\mathcal{H}^2 + a^{-2}[-2\nabla^2\psi + 6\mathcal{H}\psi^{'} + 6\mathcal{H}^2\phi]. \end{aligned} \tag{2.86}$$

$$G^0_i = R^0_i = G^0_i. \tag{2.87}$$

$$\begin{aligned} G^i_j &= R^i_j - \frac{1}{2}\delta^i_j R, \\ &= a^{-2}(-2\mathcal{H}^{'} - \mathcal{H}^2)\delta^i_j + a^{-2}[2\psi^{''} + \nabla^2(\phi - \psi) + \mathcal{H}(2\phi^{'} + 4\psi^{'}) + (4\mathcal{H}^{'} + 2\mathcal{H}^2\phi)]\delta^i_j \\ &\quad + a^{-2}(\psi - \phi)^{;i}_{,j} \,. \end{aligned} \tag{2.88}$$

Since the background $\bar{R}^\mu_\nu$ and $\bar{G}^\mu_\nu$ are diagonal then the off-diagonal contain just the perturbation so then we have:

$$R^0_i = G^0_i = \delta R^0_i = \delta G^0_i \,. \tag{2.89}$$

## 2.8 Perturbed Field Equations



In general gauge we extract from the first order perturbed Einstein equations (2.4) two constraint and two evolution equations from the (00),(0i) components and (i-j) trace and tracefree components, respectively, in the form [7]

$$\nabla^2\psi - 3\mathcal{H}(\psi' + \mathcal{H}\phi) = 4\pi G a^2 \delta\rho, \tag{2.90}$$

$$\psi' + \mathcal{H}\phi = -4\pi G a^2(\rho + p)(v + B), \tag{2.91}$$

$$\psi'' + 2\mathcal{H}\psi' + \mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = 4\pi G a^2 \delta p, \tag{2.92}$$

$$\psi - \phi = 0. \tag{2.93}$$

In terms of Newtonian gauge ($E = 0 = B$) these become

$$\nabla^2\phi - 3\mathcal{H}(\phi' + \mathcal{H}\phi) = 4\pi G a^2 \delta\rho, \tag{2.94}$$

$$\phi' + \mathcal{H}\phi = -4\pi G a^2(\rho + p)v, \tag{2.95}$$

$$\phi'' + 3\mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = 4\pi G a^2 \delta p. \tag{2.96}$$

## 2.9 Scalar Perturbations in Fourier Space

We can Fourier expand the perturbations in our flat background. Fourier expansion for an arbitrary perturbation $f = f(\eta, x^i) = f(\eta, \vec{x})$ is usually expressed in the form

$$\phi(x) = \int d^3\vec{k} e^{-\vec{k}\cdot\vec{x}} \phi_k. \tag{2.97}$$

Since perturbations of the first order have independent Fourier component evolution, it is enough to study the progress of one Fourier component $\tilde{f}$. We drop the tilde on $f$ for convenience. Then for any comoving coordinate $\vec{x}$ there is an equivalent comoving wave vector $\vec{k}$ so that with a comoving wave number $k = |\vec{k}|$ and wavelength $\lambda = 2\pi/k$ we can express a physical wavelength in Fourier space in the form

$$k_{ph} = \frac{\pi}{\lambda_{ph}} = \frac{\pi}{a\lambda} = \frac{k}{a}, \qquad \lambda_{ph} = a\lambda. \tag{2.98}$$

The physical meaning of that is as the Universe expands, for a Fourier mode $k$, the wavelength $\lambda_{ph}$ grows in time. In Fourier space the special partial derivative becomes a multiplication by the wave vector

$$\frac{\partial}{\partial x^i} \longrightarrow ik_i, \qquad \nabla^2 \longrightarrow -k^2. \tag{2.99}$$

The perturbation equations in real space are partial differential equations. In the Fourier space they became ordinary differential equations in time, for each independent Fourier mode.

# Chapter 3

# Dynamics of Interacting Fluids

In this part, we work on the general case of dark energy fluids, without any anisotropic stress ($\sigma = 0$), which interact with dark matter. We present a gauge-invariant formalism for the study of scalar perturbations.



## 3.1 Background

For a mixture of fluids, we have $T_A^{\mu\nu}$ for each fluid, where $A = 1, 2...$, and hence

$$T_A^{\mu\nu} = (\rho_A + p_A)u_A^\mu u_A^\nu + p_A g^{\mu\nu}. \tag{3.1}$$

Then Einstein field equations are

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = T^{\mu\nu} = \sum_A T_A^{\mu\nu}, \tag{3.2}$$

$$\Rightarrow \nabla_\nu (G^{\mu\nu} + \Lambda g^{\mu\nu}) \equiv 0 \Rightarrow \nabla_\nu T^{\mu\nu} = 0, \tag{3.3}$$

which represents the conservation of the energy-momentum for the total fluid.
We apply the conservation above in two cases, interacting and non-interacting fluids as follows:
1. Non-interacting fluids.
This case is represented by equation (2.67). The non-interacting fluids may interact only gravitationally, e.g., matter and radiation in the standard cosmology.

2. Interacting fluids, for which

$$\nabla_\nu T_A^{\mu\nu} = Q_A^\mu, \tag{3.4}$$

where $Q^\mu_A$ is the rate of energy-momentum transfer to fluid $A$.

We are interested in a Universe with interactions in the dark sector:

$$\nabla_\nu T^{\mu\nu}_c = Q^\mu_c, \tag{3.5}$$

$$\nabla_\nu T^\mu_x = Q^\mu_x, \tag{3.6}$$

$$\Rightarrow Q^\mu_c + Q^\mu_x = 0. \tag{3.7}$$

In the background

$$\bar{u}^\mu_c = \bar{u}^\mu_x = \bar{u}^\mu = \delta^\mu_0. \tag{3.8}$$

Then equations (3.5) and (3.6), respectively, give

$$\dot{\bar{\rho}}_c + 3H\bar{\rho}_c = \bar{Q}_c, \tag{3.9}$$

$$\dot{\bar{\rho}}_x + 3H(1+w)\bar{\rho}_x = \bar{Q}_x, \qquad \bar{Q}_x = -\bar{Q}_c. \tag{3.10}$$

In the background there is only energy exchange, given by

$$\bar{Q}^\mu_c = (Q, \vec{0}) = -\bar{Q}^\mu_x. \tag{3.11}$$

The background equations of our FLRW Unvierse are



$$H^2 = \frac{8\pi G}{3}\bar{\rho}, \tag{3.12}$$

$$\dot{H} = -4\pi G(\bar{\rho} + \bar{p}), \tag{3.13}$$

where $\rho$ and $p$ represent here the total energy density and pessure

$$\bar{\rho} = \sum_A \bar{\rho}_A, \qquad \bar{p} = \sum_A \bar{p}_A. \tag{3.14}$$

The energy balance equation for each fluid follows from equation (3.4) as

$$\dot{\bar{\rho}}_A = -3H(\bar{\rho}_A + \bar{p}_A) + \bar{Q}_A. \tag{3.15}$$

## 3.2 Perturbations

In this part we follow references [11,12,13,14].

From section (2.5), the fluid 4-velocities are

$$u_A^\mu = a^{-1}(1-\phi, \partial^i v_A), \tag{3.16}$$

$$u^A_\mu = a(-1-\phi, \partial_i[v_A + B]), \tag{3.17}$$

where $v_A$ represents the peculiar velocity potential. To generalise the Newtonian relation $\theta = \vec{\nabla}.\vec{v}$, we use the volume expansion

$$\theta_A = -k^2(v_A + B). \tag{3.18}$$

The total 4-velocity is

$$u^\mu = a^{-1}(1-\phi, \partial_i v). \tag{3.19}$$

Relative to the total 4-velocity $u^\mu$ we can divide $Q_A^\mu$ into the following parts :

$$Q_A^\mu = Q_A u^\mu + F_A^\mu, \tag{3.20}$$

$$Q_A = \bar{Q}_A + \delta Q_A, \tag{3.21}$$

$$u_\mu F_A^\mu = 0. \tag{3.22}$$



$Q_A$ is the energy density transfer rate and $F_A^\mu$ is the momentum density transfer rate, both relative to $u_\mu$, with

$$F_A^\mu = a^{-1}(0, \partial^i f_A), \tag{3.23}$$

where $f_A$ represents the gauge-invariant momentum transfer potential.

We choose zero momentum flux of the A-fluid relative to $u_A^\mu$ by taking $u_A^\mu$ as the energy-frame 4-velocity, and we make the same choice for the total 4-velocity. Thus

$$T^\mu_{A\nu} u_A^\nu = -\rho_A u_A^\mu, \tag{3.24}$$

$$T^\mu_\nu u^\nu = -\rho u^\mu. \tag{3.25}$$

The energy-momentum tensor for fluid $A$ will be

$$T^\mu_{A\nu} = (\rho_A + p_A) u_A^\mu u^A_\nu + p_A \delta^\mu_\nu, \tag{3.26}$$

where

$$\rho_A = \bar{\rho}_A + \delta\rho_A, \tag{3.27}$$

$$p_A = \bar{p}_A + \delta p_A. \tag{3.28}$$

We can see from the total conserved energy-momentum tensor that

$$(\rho + p)u^\mu u_\nu + p\delta^\mu_\nu + q^\mu u_\nu + q_\nu u^\mu = \sum_A (\rho_A + p_A)u^\mu_A u^A_\nu + \sum_A p_A \delta^\mu_\nu, \tag{3.29}$$

where $q^\mu$ represents the total momentum flux relative to the total 4-velocity $u^\mu$. It follows that

$$\rho = \sum_A \rho_A, \qquad p = \sum_A p_A, \tag{3.30}$$

$$q^0 = 0, \qquad q^i = \sum_A (\rho_A + p_A)\partial^i v_A - (\rho + p)\partial^i v. \tag{3.31}$$

Since ($q^i = 0$) we find

$$(\sum_A \bar{\rho}_A + \sum_A \bar{p}_A)v = \sum_A (\bar{\rho}_A + \bar{p}_A)\, v_A, \tag{3.32}$$

which defines $v$ in equation (3.19).



Equations (3.16),(3.17),(3.20),(3.21),(3.22) and (3.23) lead to

$$Q^A_0 = -a[\bar{Q}_A(1 + \phi) + \delta Q_A], \tag{3.33}$$

$$Q^A_i = a\partial_i[\bar{Q}_A(v + B) + f_A]. \tag{3.34}$$

The first equation defines a perturbed energy transfer consisting of the metric perturbation term $\bar{Q}_A\phi$ and perturbation $\delta Q_A$. The second one defines a perturbed momentum transfer consisting of the intrinsic momentum transfer potential ($f_A$) and the momentum transfer potential $\bar{Q}_A(v + B)$ caused by transport of energy in the direction of the total velocity $v$. The total energy-momentum transfer is zero :

$$0 = \sum \bar{Q}_A = \sum \delta Q_A = \sum f_A. \tag{3.35}$$

Now we are ready to provide our perturbed interaction.

The perturbed energy and momentum equations (2.68) and (2.69) for the non-interacting case now acquire extra terms for the perturbed interaction [13]. In Newtonian gauge:

$$\delta\rho'_A + 3\mathcal{H}(\delta\rho_A + \delta p_A) - 3(\rho_A + p_A)\phi' - k^2(\rho_A + p_A)v_A = aQ_A\phi + a\delta Q_A, \tag{3.36}$$

$$[(\rho_A + p_A)v_A]' + 4\mathcal{H}(\rho_A + p_A)v_A + (\rho_A + p_A)\phi + \delta p_A = aQ_A v + af_A. \tag{3.37}$$

In terms of the dimensionless density perturbaion

$$\delta_A = \frac{\delta \rho_A}{\bar{\rho}_A}, \tag{3.38}$$

equations (3.36) and (3.37) become

$$\begin{aligned} \delta_A' &+ 3\mathcal{H}(c_{eff,A}^2 - w_A)\delta_A - (1+w_A)k^2 v_A - 3\mathcal{H}[3\mathcal{H}(1+w_A)(c_{eff,A}^2 - w_A) + w_A']v_A \\ &- 3(1+w_A)\phi' = \frac{aQ_A}{\rho_A}\Big[\phi - \delta_A - 3\mathcal{H}(c_{eff,A}^2 - w_A)v_A\Big] + \frac{a}{\rho_A}\delta Q_A. \end{aligned} \tag{3.39}$$

$$v_A' + \mathcal{H}(1 - 3c_{eff,A}^2)v_A + \frac{c_{eff,A}^2}{(1+w_A)}\delta_A + \phi = \frac{a}{(1+w_A)\rho_A}\Big[Q_A[v - (1 + c_{eff,A}^2)v_A] + f_A\Big] \tag{3.40}$$

where $c_{eff,A}$ is the physical sound speed. For an adiabatic fluid

$$c_{eff,A} = c_{sA}, \tag{3.41}$$

where $c_{eff,A}$ is the physical sound speed, defined as $\delta p_A / \delta \rho_A$ in the rest frame [13]. For a non-adiabatic fluid, $c_{eff,A} \neq c_{sA}$.



In the comparison between theory and observational parts of dark energy models, it is useful to use the equation of state parametrization and model DE as a fluid. The simplest case then is to take $w_x \equiv w = \text{const}$, called $w$CDM, as a generalization of $\Lambda$CDM.

Any $w$CDM fluid model at the first glance is an adiabatic model. But with dark energy as adiabatic fluid model, we get an imaginary sound speed $c_{sx}$ since $c_{sx}^2 = w < 0$. To avoid these problems [13] we need to take $c_{eff,x}^2 > 0$, i.e. the physical speed is not the adiabatic sound speed.

Here we will take

$$c_{eff,x} = 1, \tag{3.42}$$

i.e. the same value as scalar field dark matter [13].

Finally, the perturbed fluid field equations (2.94),(2.95) and (2.96) are

$$\nabla^2\phi - 3\mathcal{H}(\phi' + \mathcal{H}\phi) = 4\pi G a^2 \rho \delta, \tag{3.43}$$

$$\phi' + \mathcal{H}\phi = -4\pi G a^2(\rho + p)v, \tag{3.44}$$

$$\phi'' + 2\mathcal{H}\phi' + \mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = 4\pi G a^2 \delta p. \tag{3.45}$$

Here $\delta, v$ and $\delta p$ refer to the total fluid. In particular

$$\rho\delta = \rho_b\delta_b + \rho_c\delta_c + \rho_x\delta_x. \tag{3.46}$$

# Chapter 4

# A Simple Model of Interacting Dark Energy: Background Dynamics



The recent cosmological obsevations emphasize the requirement for nonbaryonic cold dark matter (CDM) for galaxies to grow, and for dark energy (DE) to produce a late-time acceleration of the Universe.

The standered models of DE do not include an interaction with CDM. Here we introduce such an interaction, which will affect the dynamics of the Universe.

For our flat FLRW background Universe, after the recombination, the energy balance equations (3.9) and (3.10) can be formulated for cases of radiation (r), baryonic matter (b), cold dark matter (c) and dark energy (x) as

$$\dot{\bar{\rho}}_r = -4H\bar{\rho}_r, \tag{4.1}$$

$$\dot{\bar{\rho}}_b = -3H\bar{\rho}_b, \tag{4.2}$$

$$\dot{\bar{\rho}}_c = -3H\bar{\rho}_c + \bar{Q}_c, \tag{4.3}$$

$$\dot{\bar{\rho}}_x = -3H(1+w)\bar{\rho}_x + \bar{Q}_x, \quad \bar{Q}_x = -\bar{Q}_c, \tag{4.4}$$

where $\bar{Q}_A$ is the rate of energy density transfer, with $\bar{Q}_x > 0$ indicating the direction of energy transfer from dark matter to dark energy, whereas $\bar{Q}_x < 0$ indicates energy transfer from dark energy to dark matter. The dark sector then will interact only gravitationally with baryons and radiation.

The Friedmann equations are

$$H^2 = \frac{8\pi G}{3}(\bar{\rho}_b + \bar{\rho}_c + \bar{\rho}_r + \bar{\rho}_x) \quad \Leftrightarrow \quad \Omega_c + \Omega_b + \Omega_r + \Omega_x = 1, \tag{4.5}$$

$$\dot{H} = -4\pi G[\frac{4}{3}\bar{\rho}_r + \bar{\rho}_b + \bar{\rho}_c + (1+w)\bar{\rho}_x]. \tag{4.6}$$

Now we need to specify a model for the dark sector interaction in the background. There is no fundamental theory yet to predict the interaction [12]. We need to introduce a phenomenological model of transfer. An example is the phenomenolgical model

$$\bar{Q}_x = H(\alpha_c\bar{\rho}_c + \alpha_x\bar{\rho}_x), \tag{4.7}$$

where $\alpha_c$ and $\alpha_x$ are dimensionless constants. A more satisfactory phenomenological model of energy transfer is the one introduced in [12]

$$\bar{Q}_x = \Gamma_c\bar{\rho}_c + \Gamma_x\bar{\rho}_x, \tag{4.8}$$

where $\Gamma_{c,x}$ are the CDM and DE interaction rate constants.



In this thesis we focus on the special case $\Gamma_c = 0$ which is also studied in [15]. The other case $\Gamma_x = 0$ has been studied in detail (e.g.[16]). So from now on we write $\Gamma_x \equiv \Gamma$.

In this case

$$\bar{Q}_x = -\bar{Q}_c = \Gamma\bar{\rho}_x, \tag{4.9}$$

$$\text{where} \begin{cases} \Gamma > 0 & \Rightarrow \quad \text{the interaction is energy transfer from DM to DE,} \\ \Gamma < 0 & \Rightarrow \quad \text{the interaction is decay of DE to DM.} \end{cases}$$

The strength of the interaction is determined by value of $|\Gamma|$. We expect $|\Gamma| < H_0$ in order to avoid too strong an effect from the interaction.

We neglect the radiation component since our model investigates the late Universe. So our background interacting system will be

$$\begin{aligned}
\dot{\bar{\rho}}_b &= -3H\bar{\rho}_b, &(4.10)\\
\dot{\bar{\rho}}_c &= -3H\bar{\rho}_c + \bar{Q}_c, &(4.11)\\
\dot{\bar{\rho}}_x &= -3H(1+w)\bar{\rho}_x + \bar{Q}_x, &(4.12)\\
\dot{H} &= -4\pi G(\bar{\rho}_b + \bar{\rho}_c + \bar{\rho}_x(1+w)). &(4.13)
\end{aligned}$$

Using the dimensionless density $\Omega_A$ (see equation (4.5)) we re-write the above equations in terms of $\Omega_A$ as a function of the scale factor

$$\begin{aligned}
\frac{d\Omega_b}{da} &= \frac{3}{a}w\Omega_b\Omega_x, &(4.14)\\
\frac{d\Omega_c}{da} &= \frac{3}{a}w\Omega_c\Omega_x - \left(\frac{\Gamma}{H_0}\right)\left(\frac{\Omega_x}{ah}\right), &(4.15)\\
\frac{d\Omega_x}{da} &= -\frac{3}{a}w\Omega_x(\Omega_b + \Omega_c) + \left(\frac{\Gamma}{H_0}\right)\left(\frac{\Omega_x}{ah}\right), &(4.16)\\
\frac{dh}{da} &= -\frac{3h}{2a}(1 + w\Omega_x), &(4.17)
\end{aligned}$$

where

$$h = \frac{H}{H_0}, \tag{4.18}$$

is the dimensionless Hubble rate.



We numerically integrate the background equations above, with

$$\begin{aligned}
\Gamma &= 0.3H_0, &(4.19)\\
\Omega_{b0} &= 0.05,\ \Omega_{c0} = 0.23,\ \Omega_{x0} = 0.72. &(4.20)
\end{aligned}$$

We integrate backward from $a_0 = 1$ at present until $a = 10^{-2}$, i.e. until redshift of $z = \frac{1}{a} - 1 = 99$. This will then allow us to integrate the perturbation equations forward in time, with the correct background values at $a = 10^{-2}$. Then with $w = \text{constant} = -0.9$ we find the results in Figure (4.1).

Since $\Gamma > 0$, energy is transferred from DM to DE - thus there must be more DM in the past in order to achieve $\Omega_{c0} = 0.23$. This can be seen in Figure (4.1).

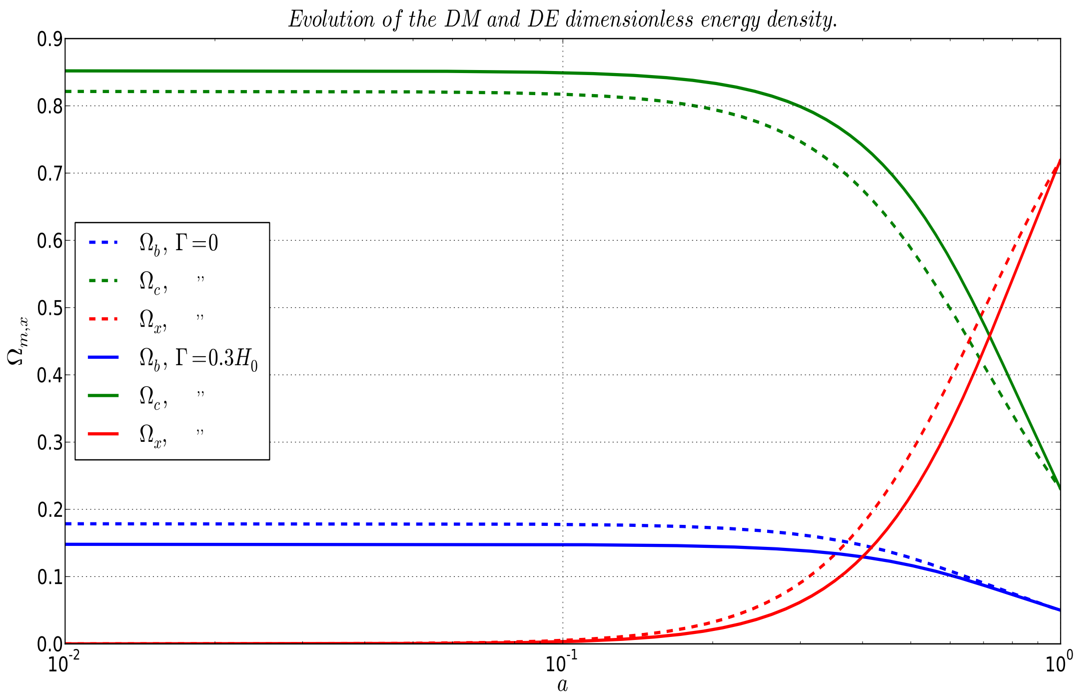


Figure 4.1: Background evolution.

The solution of (4.12) for $\rho_x$ can be found as follows:
multiplying by $\exp\int[3H(1+w)-\Gamma]dt$ one obtains

$$\left[a^{3(1+w)}e^{-\Gamma t}\right]\bar{\rho}_x = C, \tag{4.21}$$

where $C$ is the integration constant. With the initial conditions $a_0 = 1$ at $t = t_0$, we find

$$\bar{\rho}_x = \bar{\rho}_{x0}a^{-3(1+w)}e^{\Gamma(t-t_0)}. \tag{4.22}$$

Clearly $\Gamma > 0$ leads to exponential growth in the DE density. This means that eventually the DM density will became negative. Thus the model is only valid while $\bar{\rho}_c > 0$, and then it breaks down. We require $\bar{\rho}_c > 0$ for $a \leq a_0 = 1$, which holds provided $\Gamma/H_0$ is not too large. With large $\Gamma > 0$ eventually $\bar{\rho}_c$ becomes negative. So if $\bar{\rho}_c < 0$ at $a < 1$ then our model breaks down.
The constraint $\Gamma/H_0 \leq 1$ ensures that $\bar{\rho}_c$ only becomes negative in the future ($a > 1$). This is illustrated in Figure [4.2].
The evolution of the dimensionless Hubble parameter $h$ shows that $H$ is close to $10^3$ at $a = 10^{-2}$. See Figure (4.3).

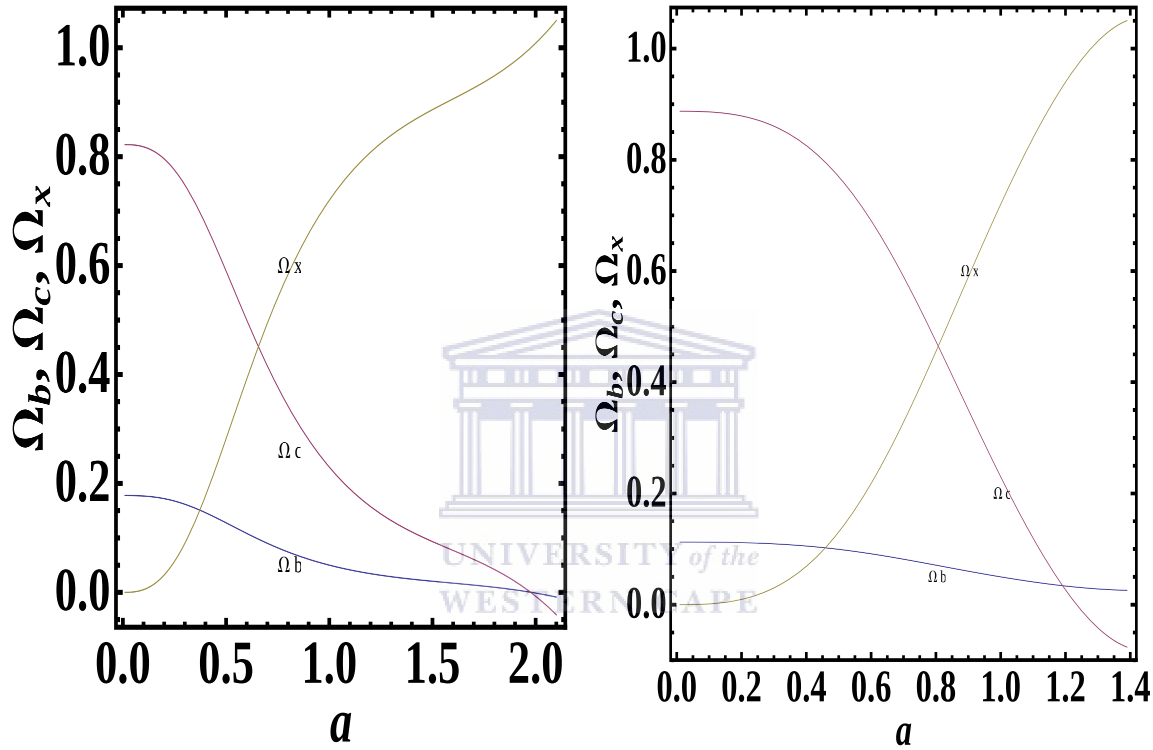


Figure 4.2: $\Gamma = 0.3H_0$ (left) and $1.0H_0$ (right), showing how $\Omega_c$ becames negative at $a > 1$.

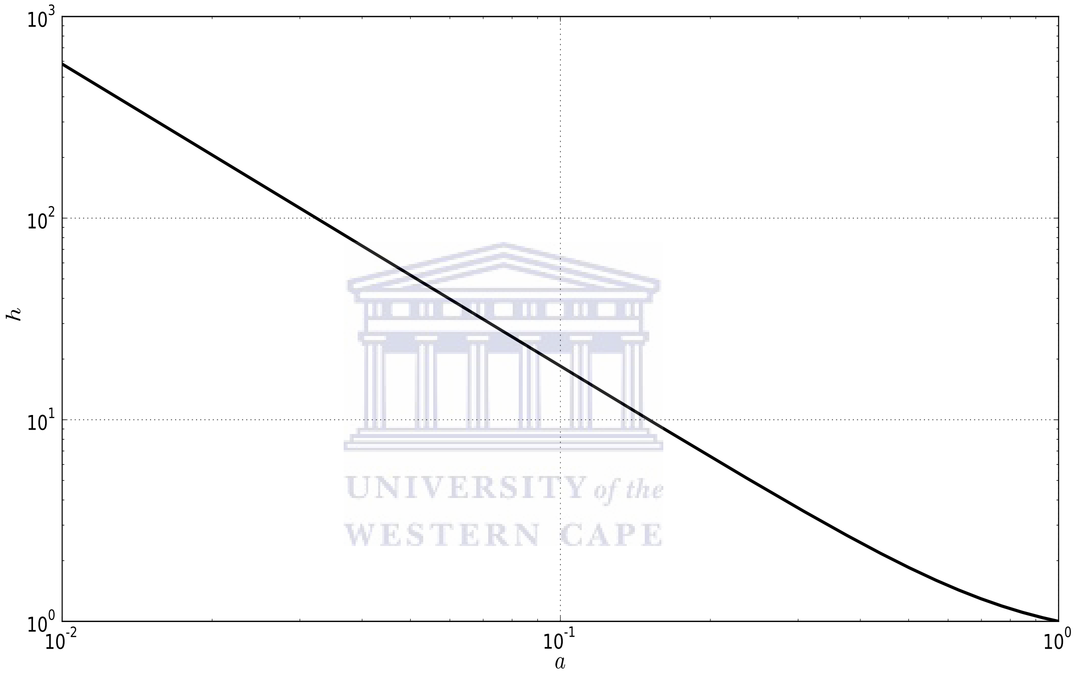


Figure 4.3: Evolution of the dimensionless Hubble parameter $h = H/H_0$ in the background.

# Chapter 5

# A Simple Model of Interacting Dark Energy: Perturbations

In our model of interacting DE the negative sign of the rate of energy density transfer to dark matter ($\bar{Q}_c$) implies that the direction of the energy transfer is from DM to DE.

The perturbed energy transfer is determined by perturbations of $\rho_x$, i.e.

$$Q_x = \Gamma \rho_x = \Gamma \bar{\rho}_x (1 + \delta_x) = -Q_c. \tag{5.1}$$



This means that we treat $\Gamma$ as a constant, with no perturbations.

Momentum transfer can only vanish in one frame. There are two natural choices leading to two models. In the first model, momentum transfer vanishes in the CDM frame. Therefore the energy-momentum transfer 4-vector is parallel to the CDM 4-velocity

$$Q_c^\mu = -\Gamma \bar{\rho}_x (1 + \delta_x) u_c^\mu = -Q_x^\mu. \tag{5.2}$$

In the second model, momentum transfer vanishes in the dark energy frame, i.e. the energy momentum transfer 4-vector is parallel to the dark energy 4-velocity:

$$Q_c^\mu = -\Gamma \bar{\rho}_x (1 + \delta_x) u_x^\mu = -Q_x^\mu. \tag{5.3}$$

We investigate the growth of structure in each case separately. The background evolution for $\Gamma = 0.3H_0, w = -0.9$ is shown in Figure (5.1) for convenience.

In [16], it is shown that an instability in the DE velocity leads to an instability in the

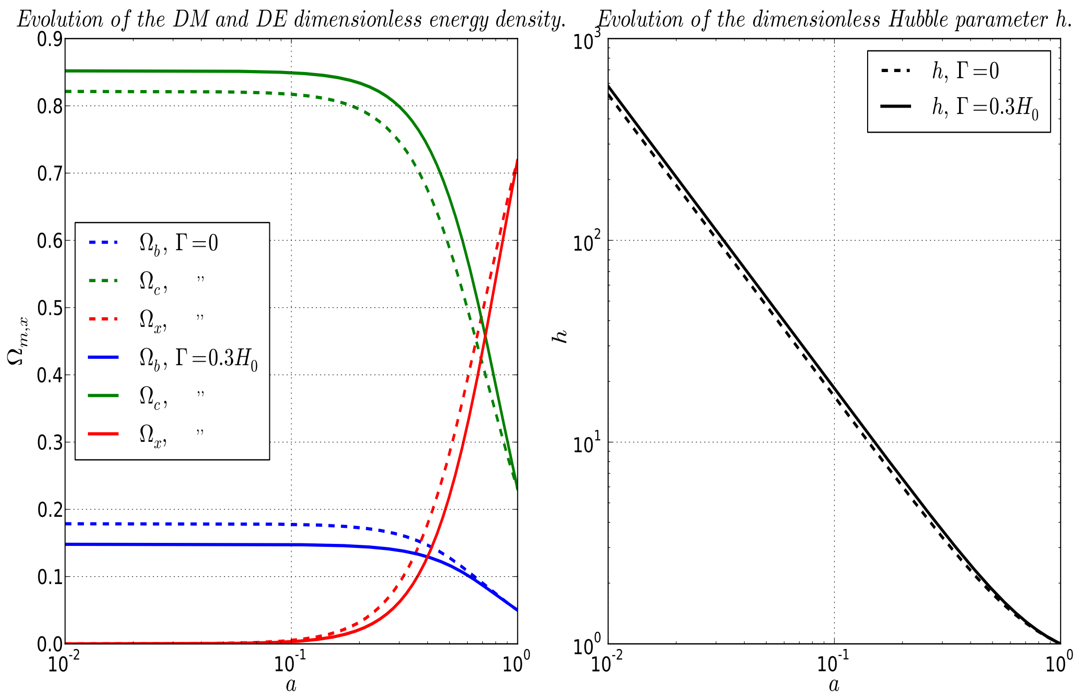


Figure 5.1: Background evolution for $w =$ constant $= -0.9$.

DM and DE density perturbations:

$$-\frac{\Gamma}{1+w} > 0 \quad \Rightarrow \quad \text{instability.} \tag{5.4}$$

We avoid this instability:

$$w > -1 \;\text{ and }\; \Gamma > 0 \quad \Rightarrow \quad \text{no instability.} \tag{5.5}$$

## 5.1 Model with No Momentum Transfer in the CDM Frame

In this model with equations (3.11),(3.16),(3.17) and (4.8) we extract the expressions

$$\delta Q_x^{\mu} = -Q_c^{\mu} = a\Gamma\bar{\rho}_x\Big[1+\phi+\delta_x,\ \partial_i(v_c+B)\Big], \tag{5.6}$$

$$\delta Q_x = \Gamma\bar{\rho}_x\delta_x = -\delta Q_c. \tag{5.7}$$

Also from equations (3.16),(3.17),(3.19),(3.23),(3.33) and (3.34) we provide in the following steps an important result

$$Q_c^\mu = Q_c u_c^\mu \quad \Rightarrow \quad \begin{cases} Q_c^0 = Q_c u_c^0 & \Leftrightarrow \quad \bar{Q}_c(1-\phi) + \delta Q_c = (\bar{Q}_c + \delta Q_c)(1-\phi), \\ Q_c^\mu = Q_c u_c^\mu & \Leftrightarrow \quad a^{-1}\partial^i(f_c + \bar{Q}_c v) = a^{-1}\bar{Q}_c[\partial^i v_c]. \end{cases} \tag{5.8}$$

These equations imply

$$a^{-1} f_c + a^{-1}\bar{Q}_c v = a^{-1}\bar{Q}_c v_c, \tag{5.9}$$

$$\bar{Q}_c(1-\phi) + \delta Q_c = Q_c(1-\phi) + \delta Q_c. \tag{5.10}$$

Thus

$$f_c = \bar{Q}_c(v_c - v) = -\bar{Q}_x(v_c - v) = -f_x. \tag{5.11}$$

Now we introduce

$$w_b = 0 = w_c, \quad w_x = w = \text{constant}, \quad c_{sb} = 0 = c_{sc}, \quad c_{eff,x} = 1. \tag{5.12}$$

Then equations (3.39) and (3.40) give

$$\delta_b' - k^2 v_b - 3\phi' = 0, \tag{5.13}$$

$$\delta_c' - k^2 v_c - 3\phi' = a\Gamma\left(\frac{\bar{\rho}_x}{\bar{\rho}_c}\right)(\delta_c - \delta_x - \phi), \tag{5.14}$$

$$\begin{aligned}\delta_x' &+ 3\mathcal{H}(1-w)\delta_x - (1+w)k^2 v_x - 9\mathcal{H}^2(1-w^2)v_x \\ &- 3(1+w)\phi' = a\Gamma\Big[\phi - 3\mathcal{H}(1-w)v_x\Big],\end{aligned} \tag{5.15}$$

$$v_c' + \mathcal{H}v_c + \phi = 0, \tag{5.16}$$

$$v_x' - 2\mathcal{H}v_x + \frac{\delta_x}{(1+w)} + \phi = \frac{a\Gamma}{(1+w)}(v_c - 2v_x), \tag{5.17}$$

$$v_b' + \mathcal{H}v_b + \phi = 0. \tag{5.18}$$

We notice that baryons and DM have the same velocity evolution equations, therefore they have the same velocity: $v_b = v_c \equiv v$.

We now define the dimensionless velocity perturbation and Fourier mode as

$$u_A = v_A H_0, \tag{5.19}$$

$$l = \frac{k}{H_0}. \tag{5.20}$$

Then equations (3.43) and (5.13) to (5.18) can be written in dimensionless variables as:

$$\begin{aligned}
\frac{du}{da} &= -\frac{u}{a}-\frac{\phi}{a^2h}, && (5.21)\\
\frac{du_x}{da} &= \frac{2}{a}u_x-\frac{\delta_x}{a^2h(1+w)}-\frac{\phi}{a^2h}+\frac{\Gamma}{H_0}\left(\frac{u-2u_x}{ah(1+w)}\right), && (5.22)\\
\frac{d\delta_b}{da} &= \frac{l^2}{a^2}\left(\frac{u}{h}\right)-\frac{3}{a}\phi-\frac{9}{2}h\Big[(\Omega_b+\Omega_c)u+u_x(1+w)\Omega_x\Big], && (5.23)\\
\frac{d\delta_c}{da} &= \frac{l^2}{a^2}\left(\frac{u}{h}\right)-\frac{3}{a}\phi-\frac{9}{2}h\Big[(\Omega_b+\Omega_c)u+u_x(1+w)\Omega_x\Big]\\
&+ \frac{\Gamma}{H_0}\left(\frac{\Omega_x}{\Omega_c}\frac{\delta_c}{a\,h}\right)\\
&- \frac{\Gamma}{H_0}\left(\frac{\Omega_x}{\Omega_c}\frac{\phi}{ah}\right)-\frac{\Gamma}{H_0}\left(\frac{\delta_x}{a\,h}\frac{\Omega_x}{\Omega_c}\right), && (5.24)\\
\frac{d\,\delta_x}{da} &= -\frac{3}{a}\,(1-w)\,\delta_x+(1+w)\,\Big[\frac{l^2}{a^2}\frac{u_x}{h}-\frac{3}{a}\phi\\
&- \frac{9}{2}h\,((\Omega_b+\Omega_c)\,u+\Omega_x(1+w)\,u_x)\Big]\\
&+ 9\,h\,(1-w^2)\,u_x+\frac{\Gamma}{H_0}\left(\frac{\phi}{a\,h}\right)-3\frac{\Gamma}{H_0}(1-w)\,u_x, && (5.25)\\
\frac{d\phi}{da} &= -\frac{\phi}{a}-\frac{3}{2}\,h\Big[(\Omega_b+\Omega_c)\,u+\Omega_x\,(1+w)\,u_x\Big]. && (5.26)
\end{aligned}$$

The numerical solution for the total interaction system [background equations (4.14) to (4.17) and perturbation equations (5.21) to (5.26)], both normalised to current values of

$$\Omega_{b0}=0.05,\ \Omega_{c0}=0.23,\ \Omega_{x0}=0.72, \tag{5.27}$$

with initial conditions at $a=10^{-2}$:

$$\begin{aligned}
\delta_{ci} &= \delta_{bi}=10^{-4},\ \delta_{xi}=10^{-6},\ u_{ci}=u_{bi}=5.9\times10^{-8},\ u_{xi}=4.2\times10^{-8},\\
\phi_i &= -1.5\times10^{-5}, && (5.28)
\end{aligned}$$

and

$$\Gamma=0.3H_0,\ w=-0.9. \tag{5.29}$$

The results are shown in Figures (5.2) to (5.6) on sub-Hubble scales:

$$\frac{k}{a_iH_i}=10, \tag{5.30}$$

and in Figures (5.7) to (5.15) on super-Hubble scales:

$$\frac{k}{a_iH_i}=0.1. \tag{5.31}$$

The perturbed velocity of DM grows until DE dominates the background expansion whereas the DE velocity decays away. The interaction suppresses the matter velocity but has very little impact on the DE velocity. The suppression of velocity follows from the suppression of $\phi$.

The density perturbations of DM grows more than the baryon perturbations because there was more CDM in the past relative to the non-interaction case. DE density perturbations started with oscillations at early times and then grow as DE starts to dominate the background. The gravitational potential does not change initially but then starts to grow as DE starts to dominate the background. The potential is constant while matter dominates, but changes with DE starts to domiate.

In matter domination

$$\delta p = 0, \text{ and } 2\mathcal{H}' + \mathcal{H}^2 = 0. \tag{5.32}$$

Then equation (2.96) shows that the potential satisfies the equation

$$\phi'' + 3\mathcal{H}\phi' = 0, \tag{5.33}$$

with solution

$$\phi = \phi_i + \frac{C}{a^3}, \tag{5.34}$$



where $\phi_i$ and $C$ are constants. The $C$ mode is decaying and can be neglected. Thus

$$\phi = \phi_i = \text{constant}. \tag{5.35}$$

In Figure (5.6) we see that $\phi$ would be flat if $\rho_x = 0$. $\Gamma > 0$ increases growth of $\delta_c$ and so the effect of DE is weaker than in the case of $\Gamma = 0$. Therefore the potential is suppressed.

## Negative $\Gamma$

For the case of negative $\Gamma$ there is an instability: see equation (5.4). As $|\Gamma|$ inecreases, the blow-up starts latter. See Figures (5.12) and (5.13).

This follows because a stronger interaction, i.e. a larger $|\Gamma|$ causes a stronger suppression of $\bar{\rho_x}$ (see equation (4.22)). However the strength of the instability is greater as $|\Gamma|$ increases.

An interesting point to note is the positive values of the gravitational potential $\Phi$ during the blow-up. See Figures (5.14) and (5.15). It starts negative and then becomes positive since the perturbations of the DM becomes negative. That clearly can be understood with equation (3.43).

A more interesting point is that we cannot find blow-up in the perturbations on sub-Hubble scales.

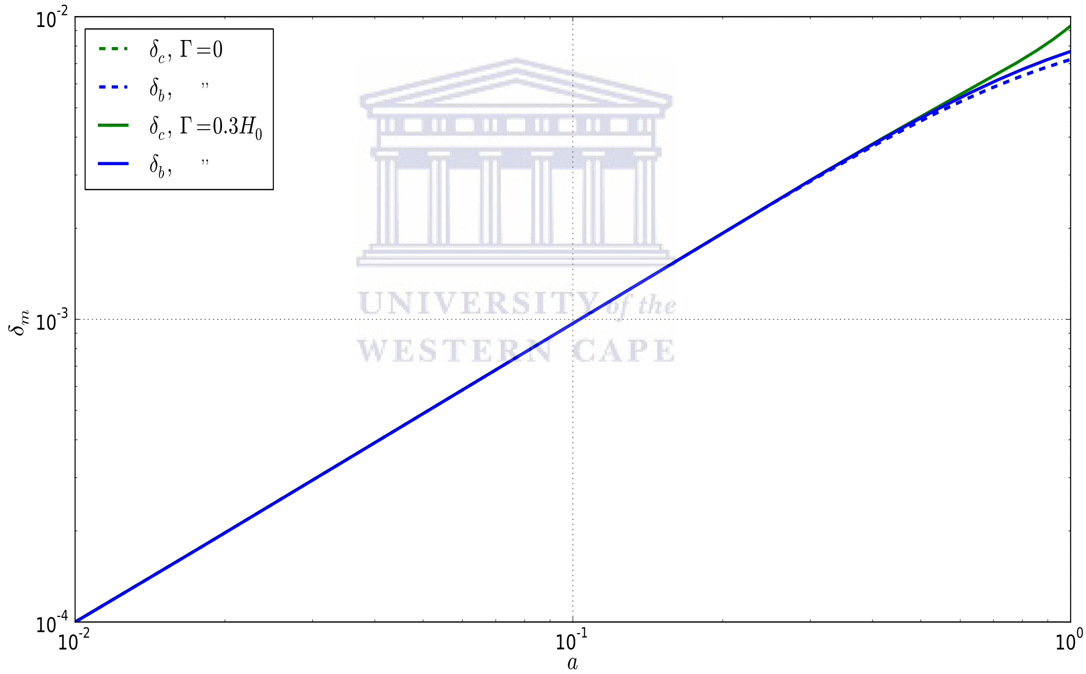


Figure 5.2: Baryonic Matter and CDM density perturbations (sub-Hubble).

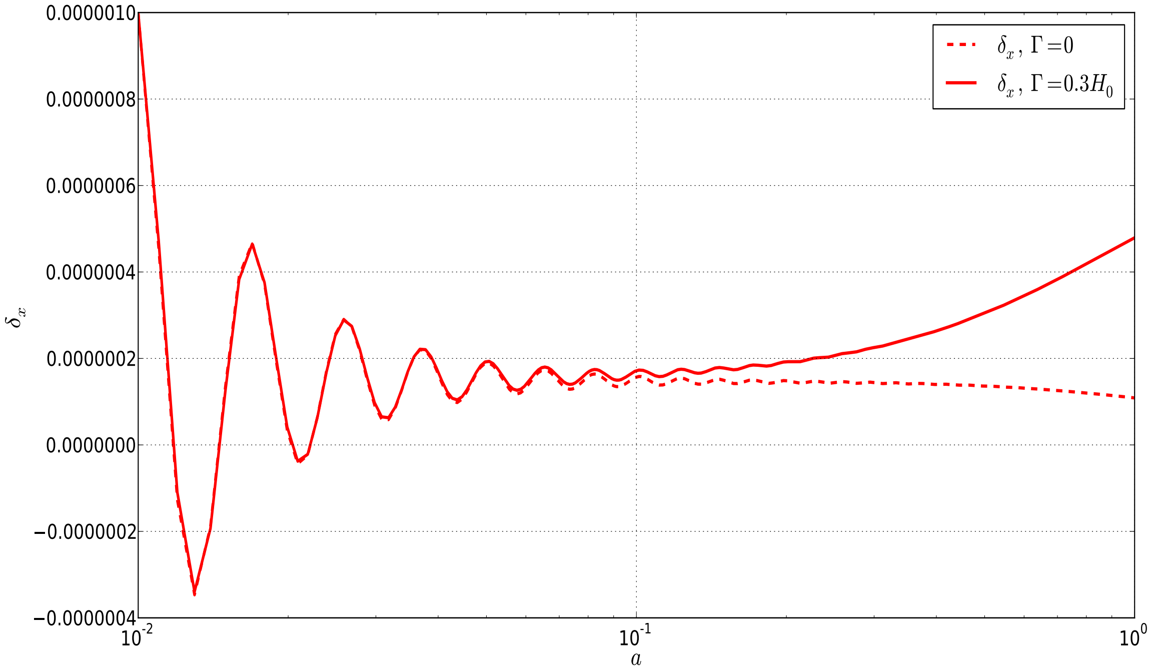


Figure 5.3: DE density perturbations (sub-Hubble).

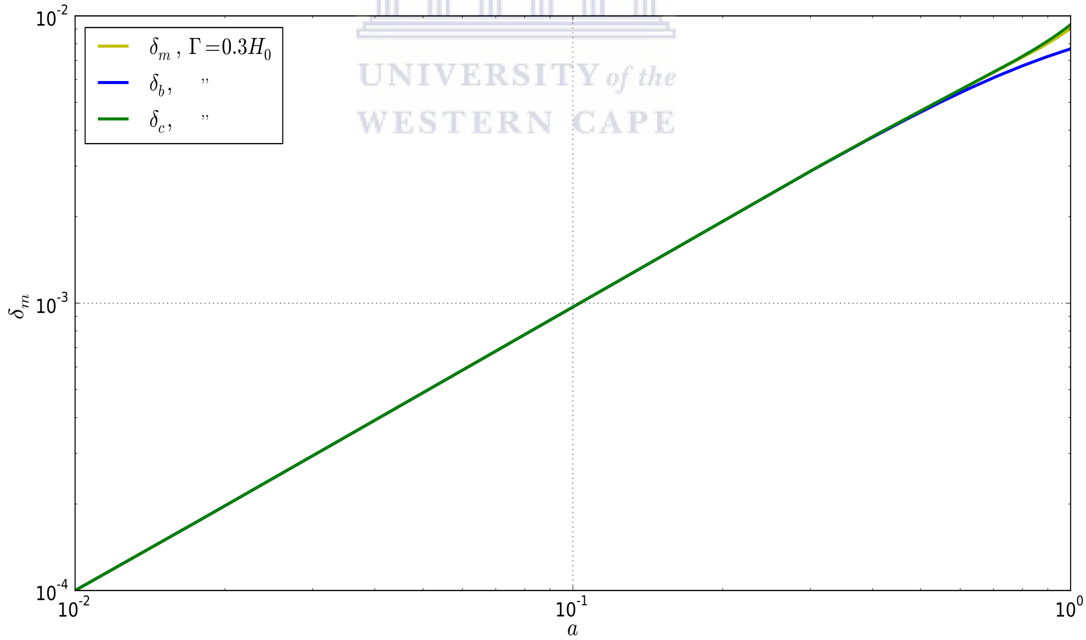


Figure 5.4: Total Matter, CDM and Baryonic Matter density perturbations (sub-Hubble).

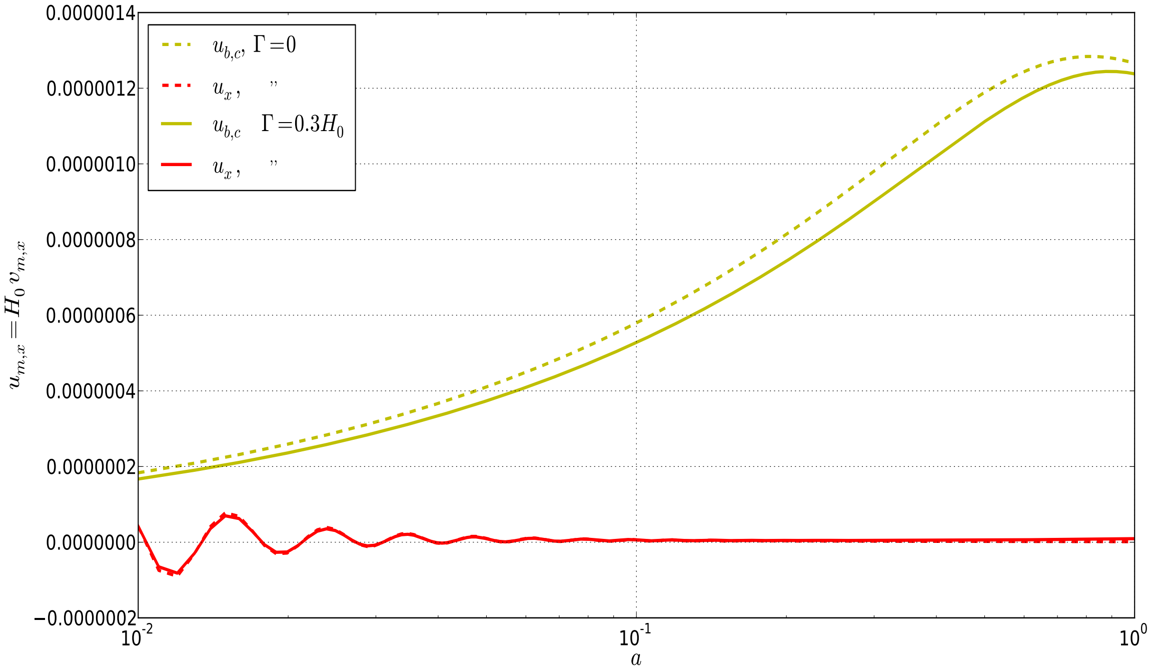


Figure 5.5: Baryonic Matter, CDM and DE velocities (sub-Hubble).



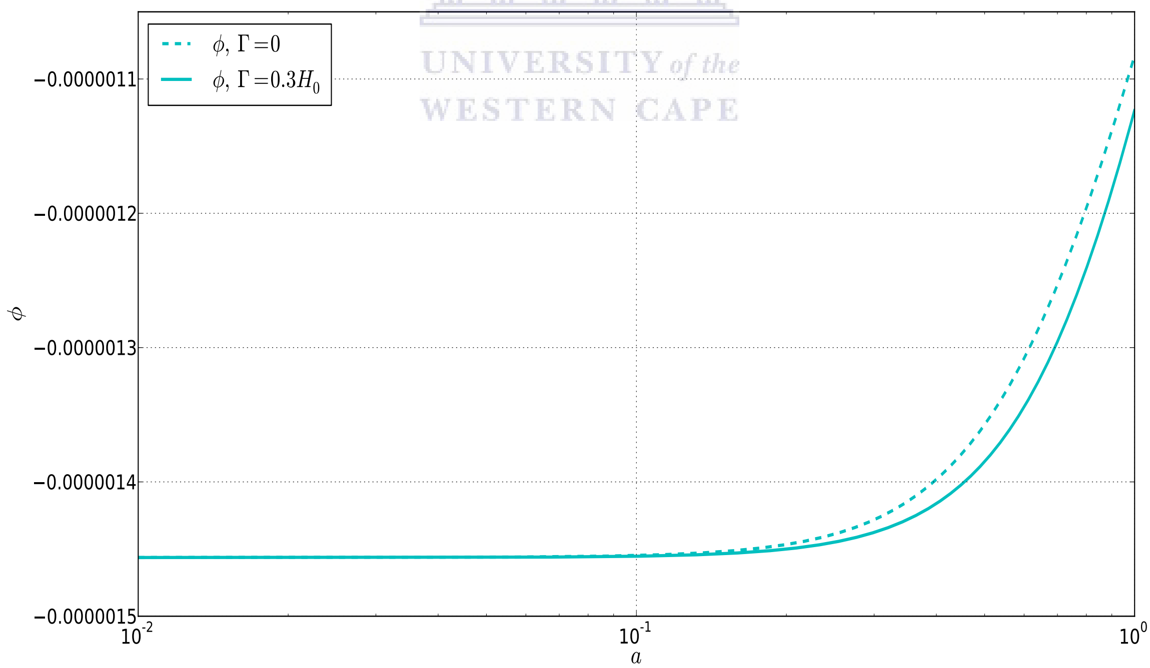


Figure 5.6: The gravitationl potential (sub-Hubble).

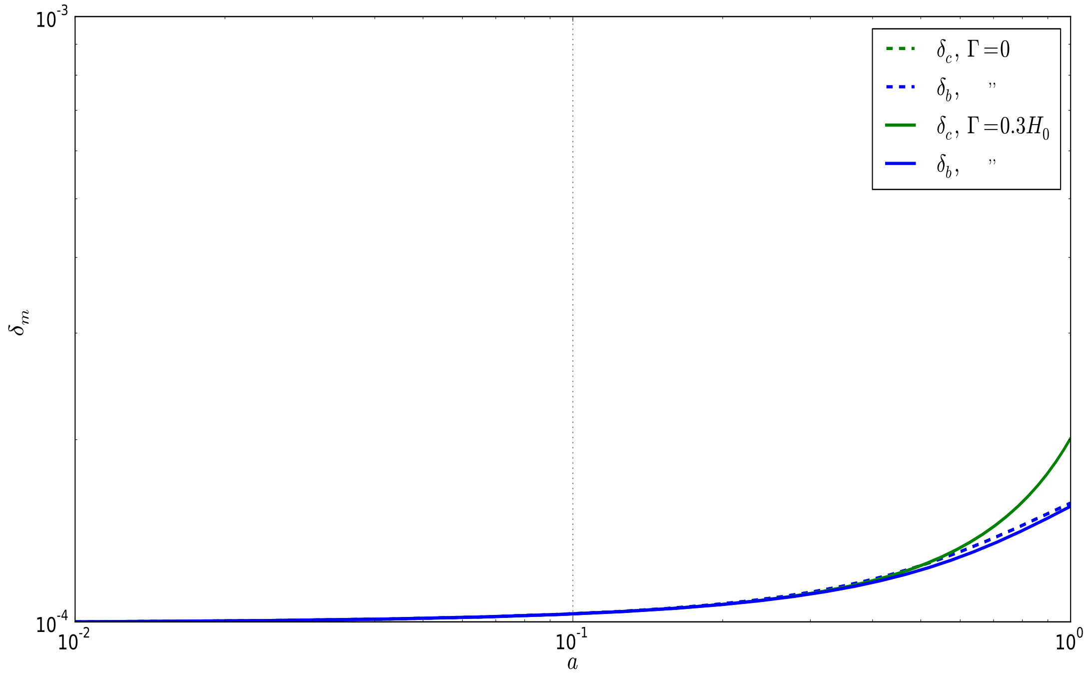


Figure 5.7: Baryonic Matter and CDM density perturbations (super-Hubble).



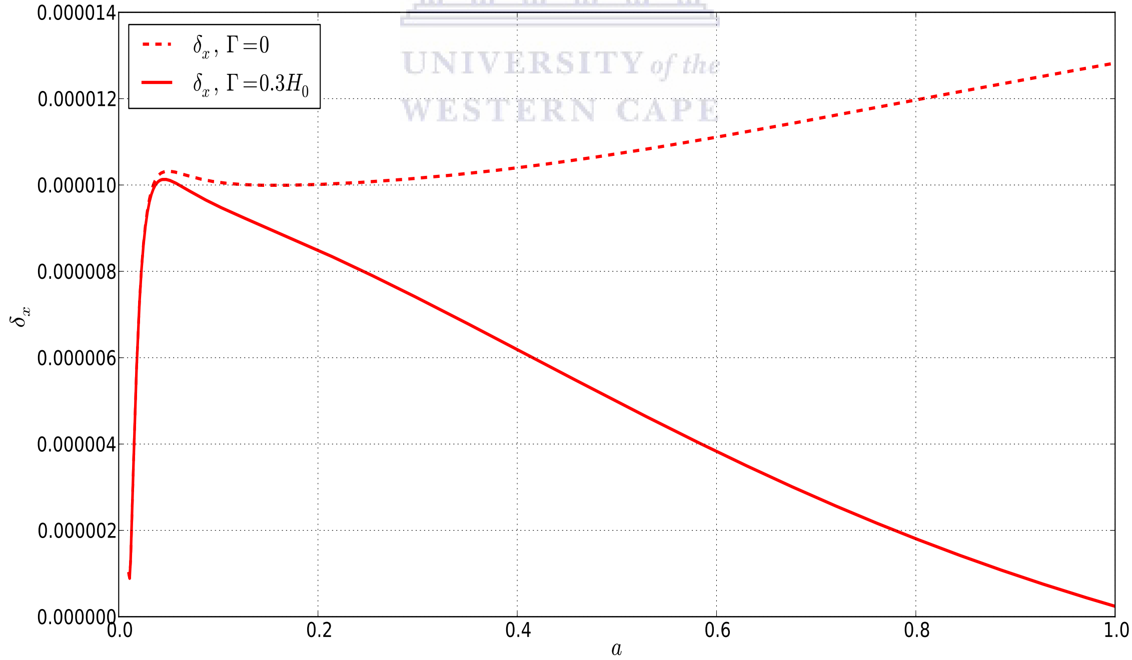


Figure 5.8: DE density perturbaions (super-Hubble).

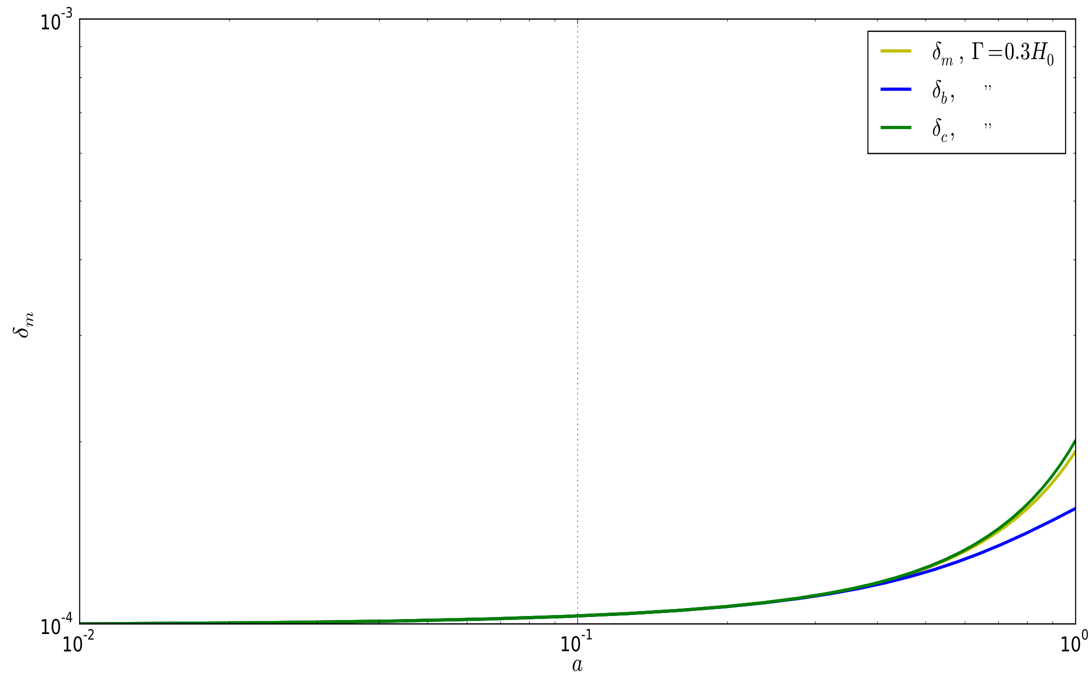


Figure 5.9: Total Matter, CDM and Baryonic Matter density perturbations (super-Hubble).



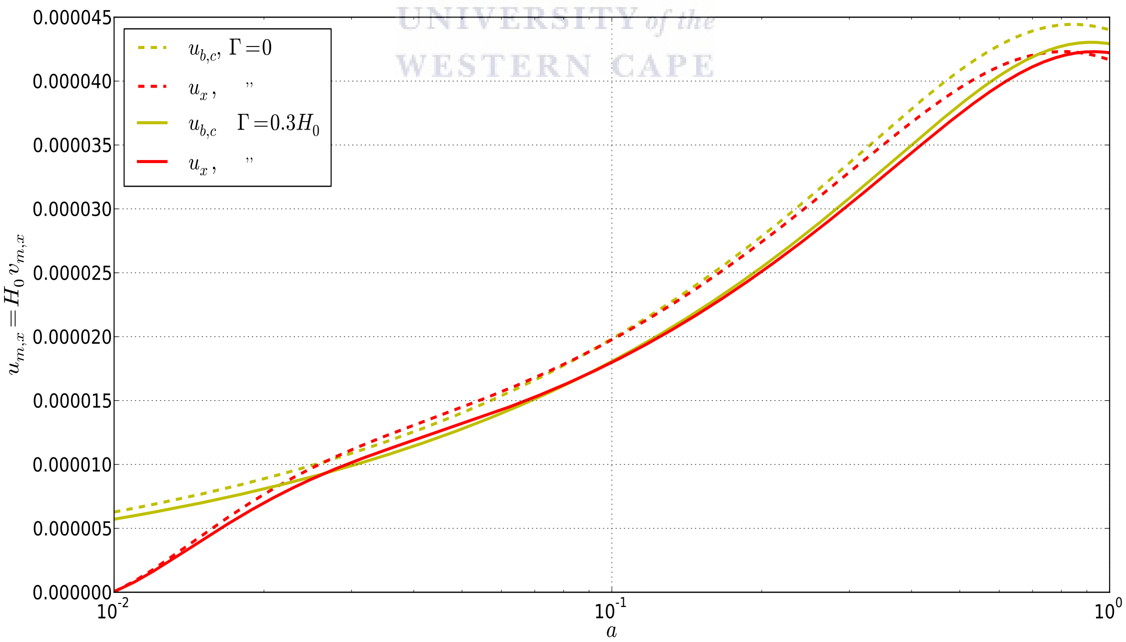


Figure 5.10: Baryonic Matter, CDM and DE velocities (super-Hubble) .

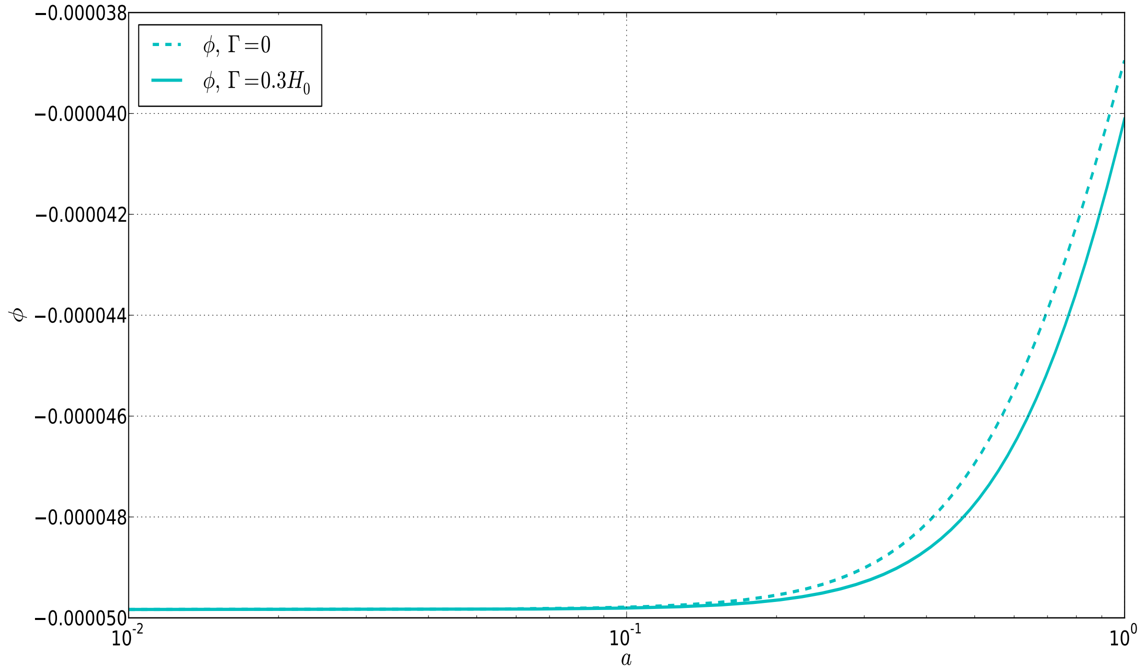


Figure 5.11: The gravitationl potential (super-Hubble).



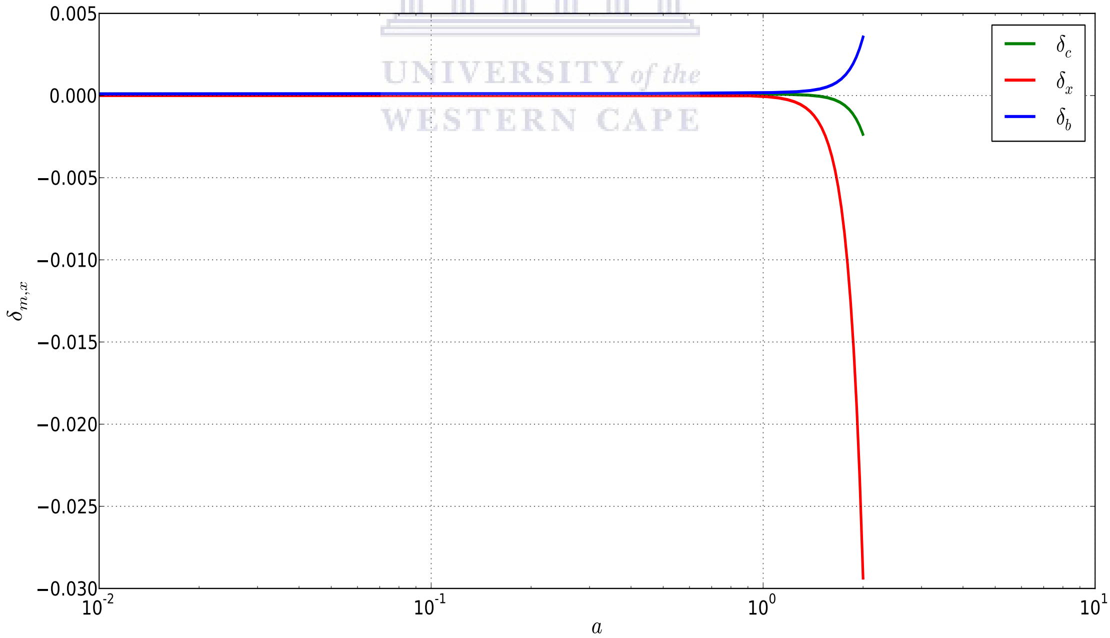


Figure 5.12: Negative CDM and DE density perturbations for $\Gamma = -0.38H_0$ (super-Hubble).

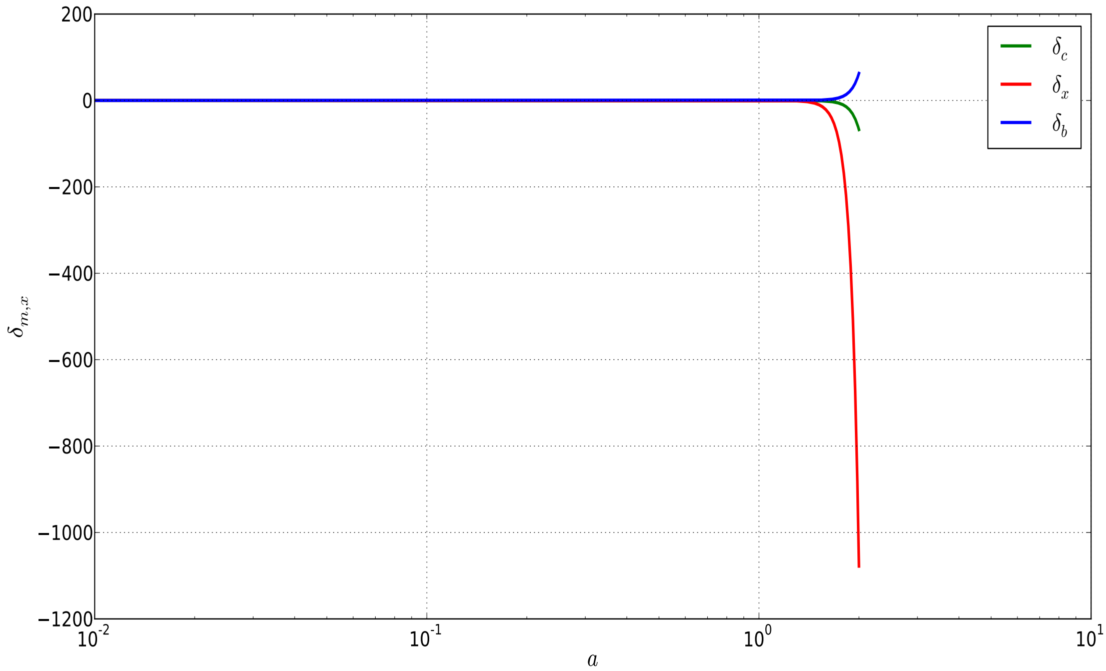


Figure 5.13: Blow-up in CDM and DE density perturbations for $\Gamma = -0.65H_0$ (super-Hubble).



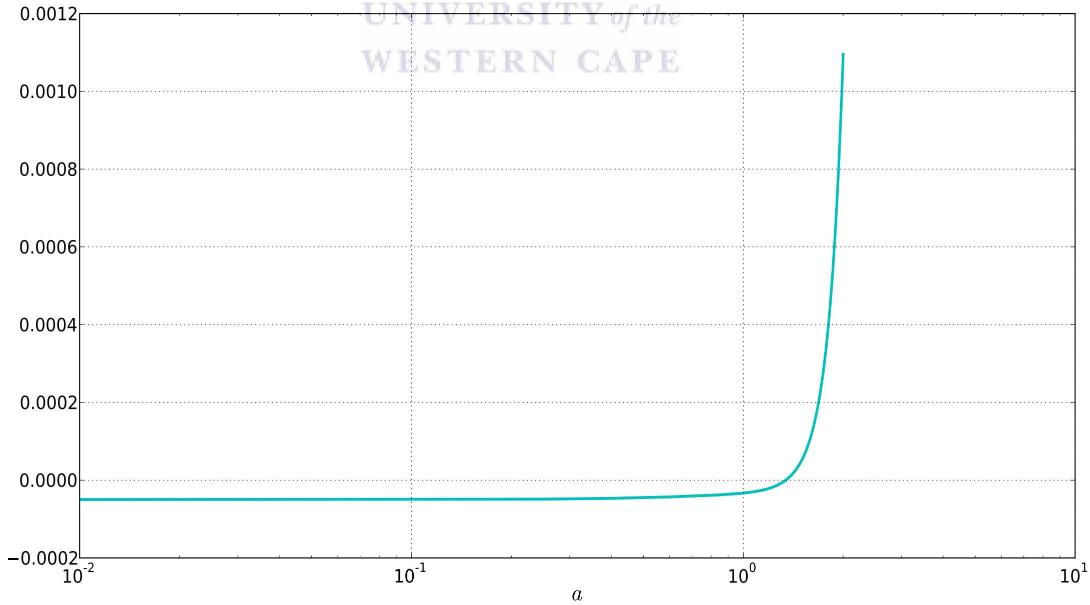


Figure 5.14: The gravitational potential for $\Gamma = -0.38H_0$ (super-Hubble).

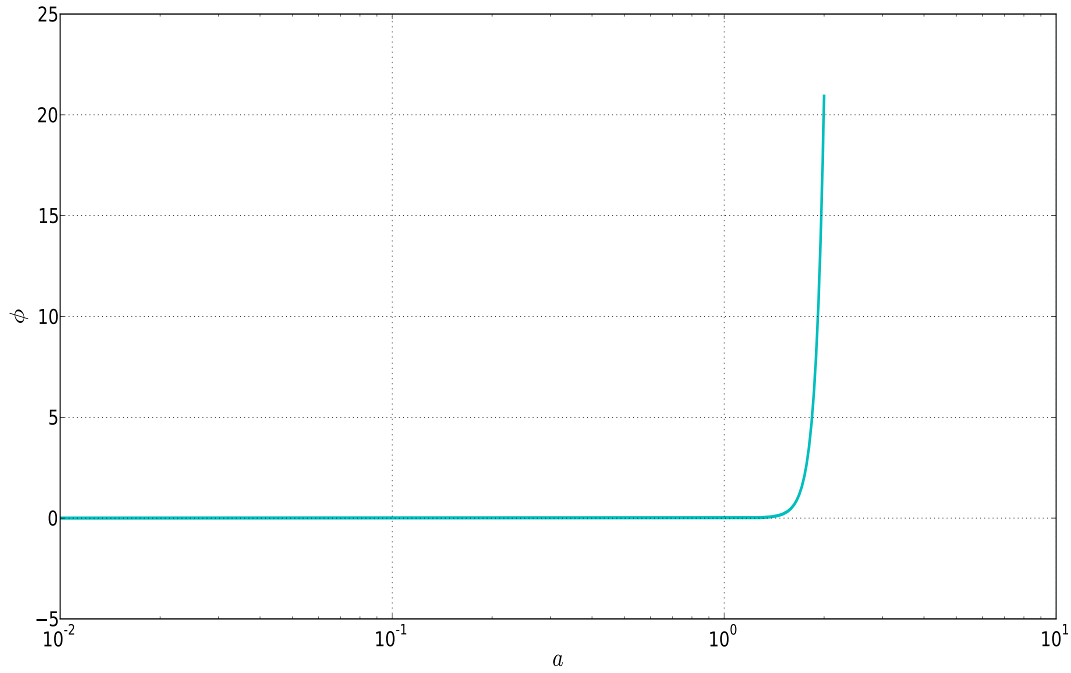


Figure 5.15: The gravitational potential for $\Gamma = -0.65H_0$ (super-Hubble).



## 5.2 Model with No Momentum Transfer in the DE Frame

In this model we have $Q_x^\mu = Q_x u_x^\mu$. Then equation (5.11) will become

$$f_c = \bar{Q}_c(v_x - v) = -\bar{Q}_x(v_x - v) = -f_x. \tag{5.36}$$

Because $\delta Q_x$ is unchanged, the density perturbaion and the gravitational potential equations are the same as for the $Q_c^\mu$ model of section 5.1. Thus the numerical results for the density perturbaions and the gravitational potential in this model are the same as the $Q_c^\mu$ model of section 5.1 (see Figures (5.16),(5.17),(5.21),(5.22),(5.23) and (5.27)). However, the difference in $f_x$ means that the velocity perturbation equations are different. By equation (3.40) we have

$$v_c' + \mathcal{H}v_c + \phi = a\Gamma\left(\frac{\bar{\rho}_x}{\bar{\rho}_c}\right)(v_c - v_x), \tag{5.37}$$

$$v_x' - 2\mathcal{H}v_x + \frac{\delta_x}{(1+w)} + \phi = -\frac{a\Gamma v_x}{(1+w)}. \tag{5.38}$$

In the dimensionless variables of section 5.1, we find

$$\frac{du_b}{da} = - \frac{u_b}{a} - \frac{\phi}{a^2 h}, \tag{5.39}$$

$$\frac{du_c}{da} = - \frac{u_c}{a} - \frac{\phi}{a^2 h} + \frac{\Gamma}{H_0}\left(\frac{\Omega_x}{\Omega_c}\frac{1}{ah}\right)(u_c - u_x), \tag{5.40}$$

$$\frac{du_x}{da} = 2\frac{u_x}{a} - \frac{\delta_x}{a^2 h(1+w)} - \frac{\phi}{a^2 h} - \frac{\Gamma}{H_0}\left(\frac{u_x}{ah(1+w)}\right), \tag{5.41}$$

$$\frac{d\delta_b}{da} = \frac{l^2}{a^2}\left(\frac{u_b}{h}\right) - 3\frac{\phi}{a} - \frac{9}{2}h\Big[\Omega_b u_b + \Omega_c u_c + u_x(1+w)\Omega_x\Big], \tag{5.42}$$

$$\begin{aligned}\frac{d\delta_c}{da} = {} & \frac{l^2}{a^2}\left(\frac{u_c}{h}\right) - 3\frac{\phi}{a} - \frac{9}{2}h\Big[\Omega_b u_b + \Omega_c u_c + u_x(1+w)\Omega_x\Big] \\ & + \frac{\Gamma}{H_0}\left(\frac{\Omega_x}{\Omega_c}\frac{\delta_c}{ah}\right) \\ & - \frac{\Gamma}{H_0}\left(\frac{\Omega_x}{\Omega_c}\frac{\phi}{ah}\right) - \frac{\Gamma}{H_0}\left(\frac{\delta_x}{ah}\frac{\Omega_x}{\Omega_c}\right),\end{aligned} \tag{5.43}$$

$$\begin{aligned}\frac{d\delta_x}{da} = {} & - \frac{3}{a}(1-w)\delta_x + (1+w)\Big[\frac{l^2}{a^2}\left(\frac{u_x}{h}\right) - 3\frac{\phi}{a} \\ & - \frac{9}{2}h\left(\Omega_b u_b + \Omega_c u_c + \Omega_x(1+w)u_x\right)\Big] \\ & + 9h(1-w^2)u_x + \frac{\Gamma}{H_0}\left(\frac{\phi}{ah}\right) - 3\frac{\Gamma}{H_0}(1-w)u_x,\end{aligned} \tag{5.44}$$

$$\frac{d\phi}{da} = - \frac{\phi}{a} - \frac{3}{2}h\Big[\Omega_b u_b + \Omega_c u_c + \Omega_x(1+w)u_x\Big]. \tag{5.45}$$

The $u_c$ equation now has a source term

$$\Gamma\frac{\bar{\Omega}_x}{\bar{\Omega}_c}(u_c - u_x). \tag{5.46}$$

The $u_b$ equation is unchanged and has no source term. Therefore the CDM and baryon velocities will *not* be equal in this model. This is illustrated in Figures (5.18),(5.19),(5.20),(5.24),(5.25) and (5.26).

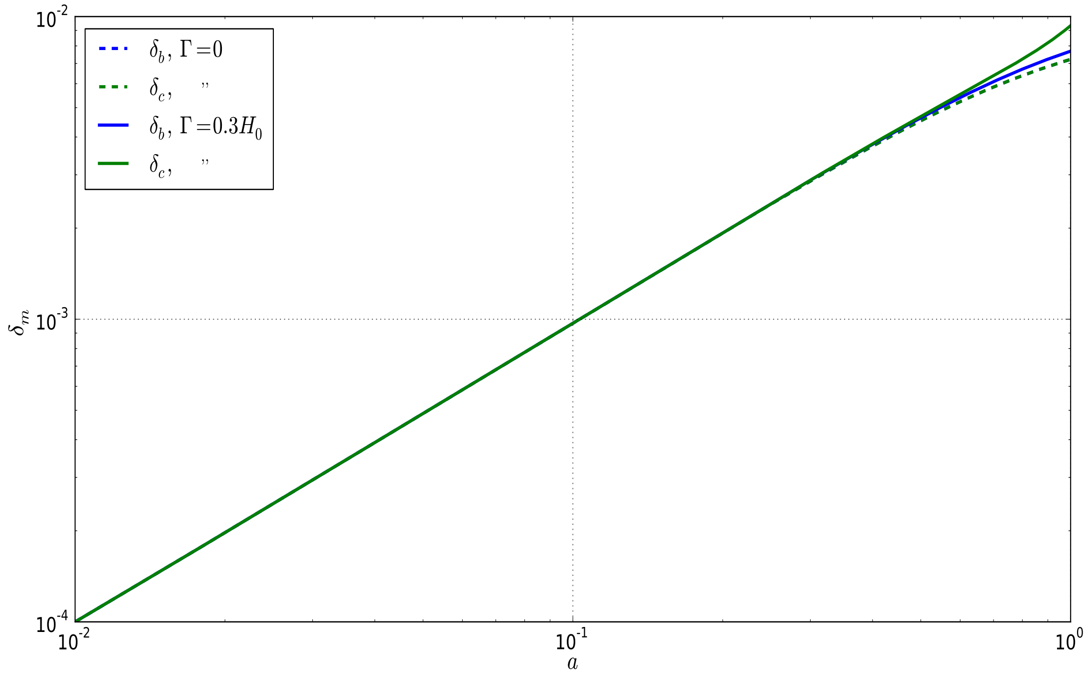


Figure 5.16: Baryonic Matter and CDM density perturbations (sub-Hubble).



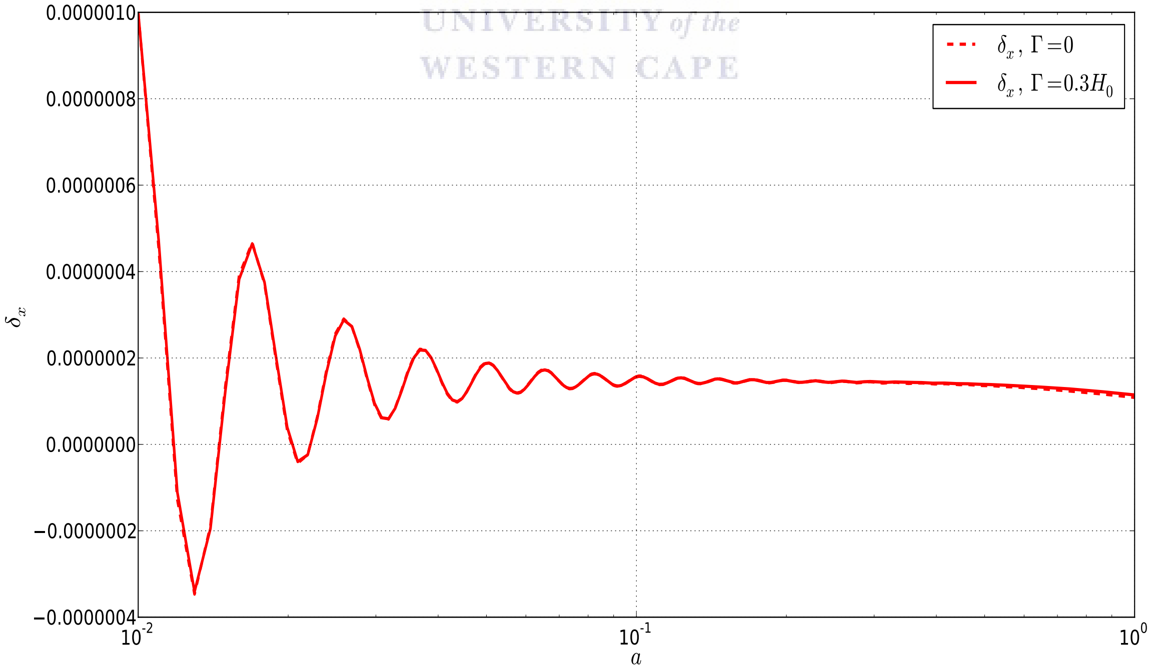


Figure 5.17: DE density perturbations (sub-Hubble).

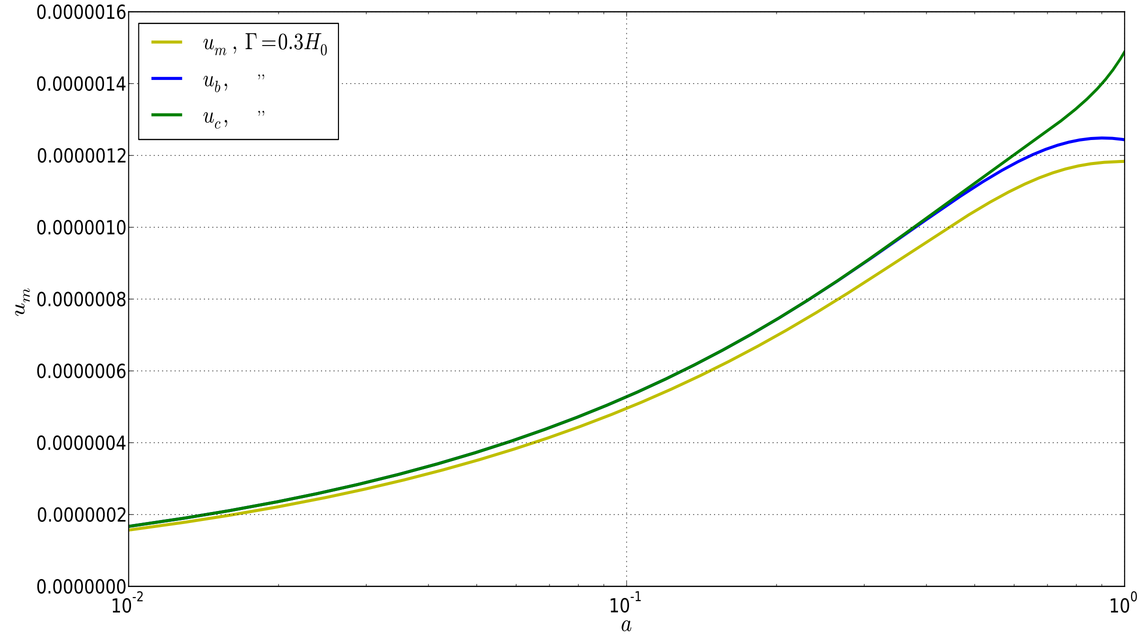


Figure 5.18: Total Matter, CDM and Baryonic Matter velocities (sub-Hubble).



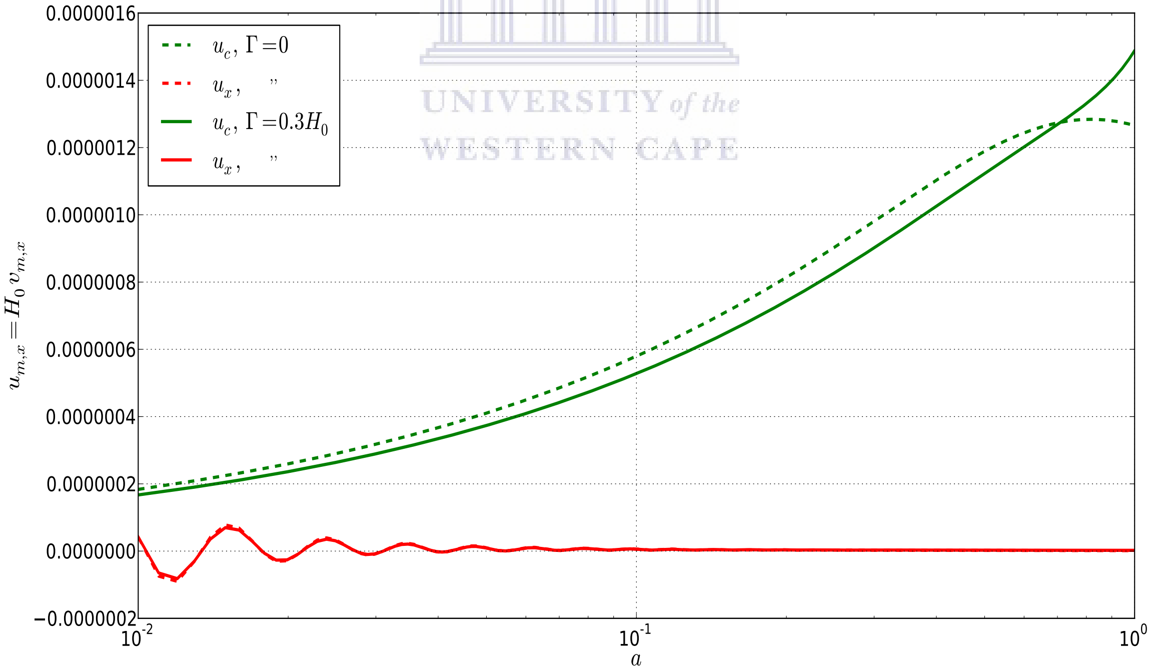


Figure 5.19: CDM and DE velocities (sub-Hubble).

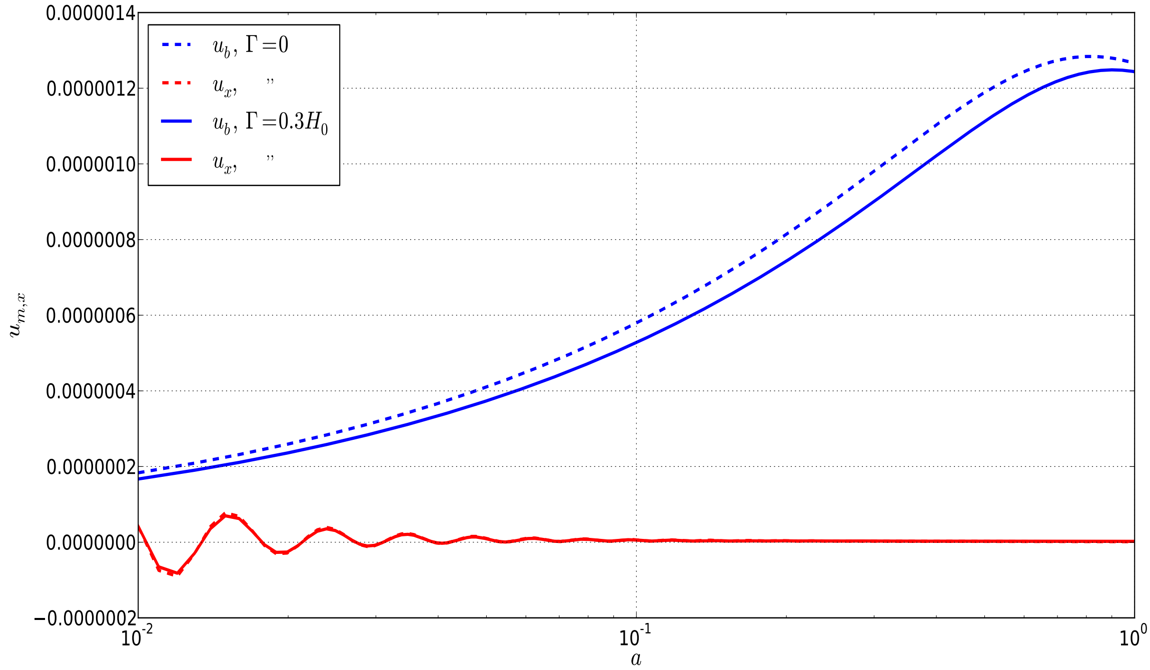


Figure 5.20: Baryonic Matter and DE velocities (sub-Hubble).



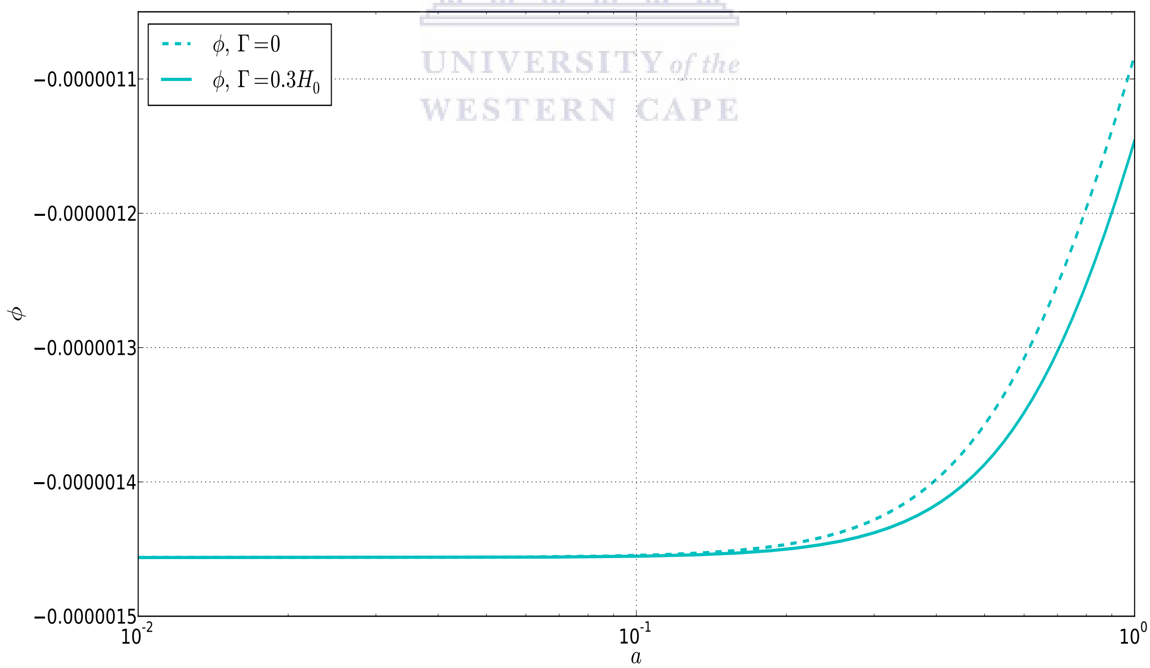


Figure 5.21: The gravitational potential (sub-Hubble).

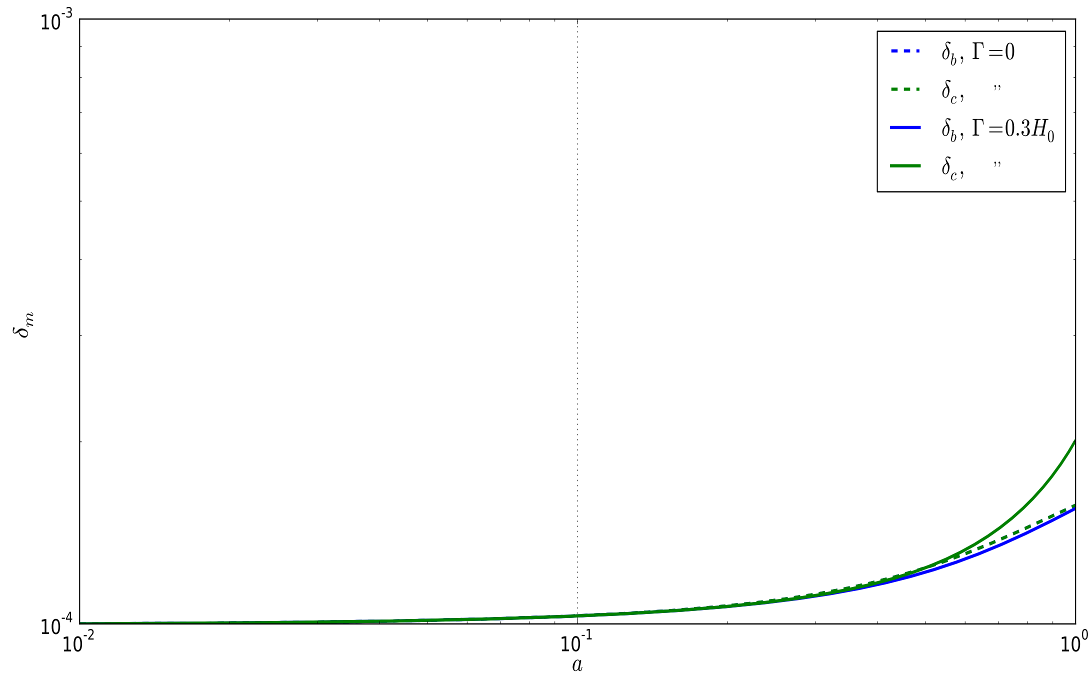


Figure 5.22: Baryonic Matter and CDM density perturbations (super-Hubble).

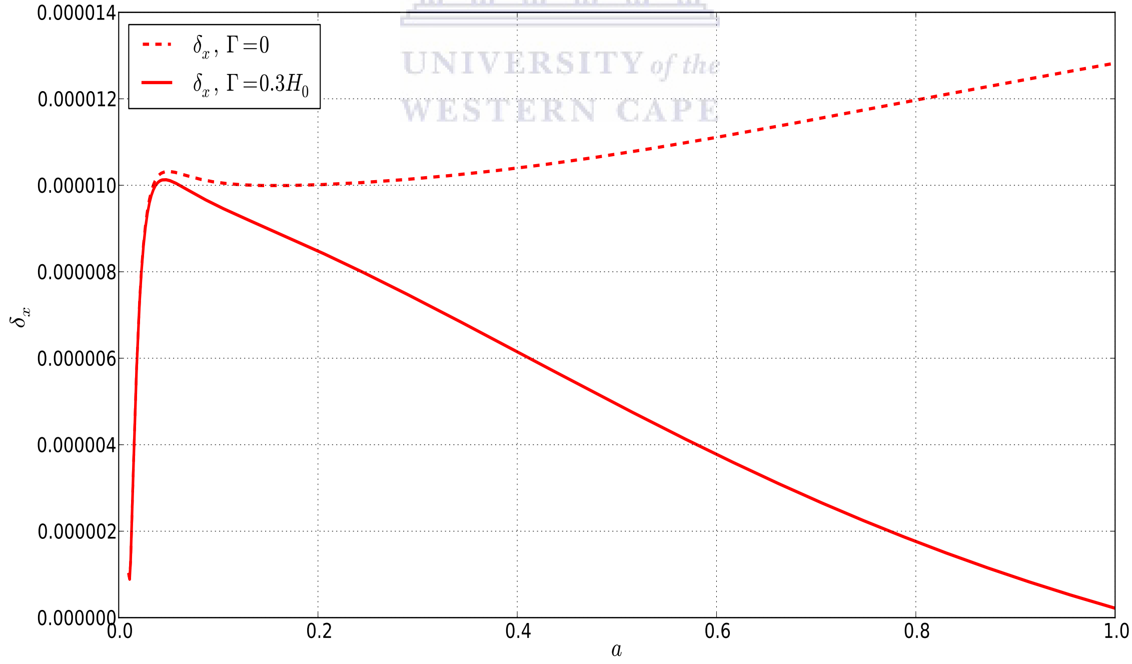


Figure 5.23: DE density perturbations (super-Hubble).

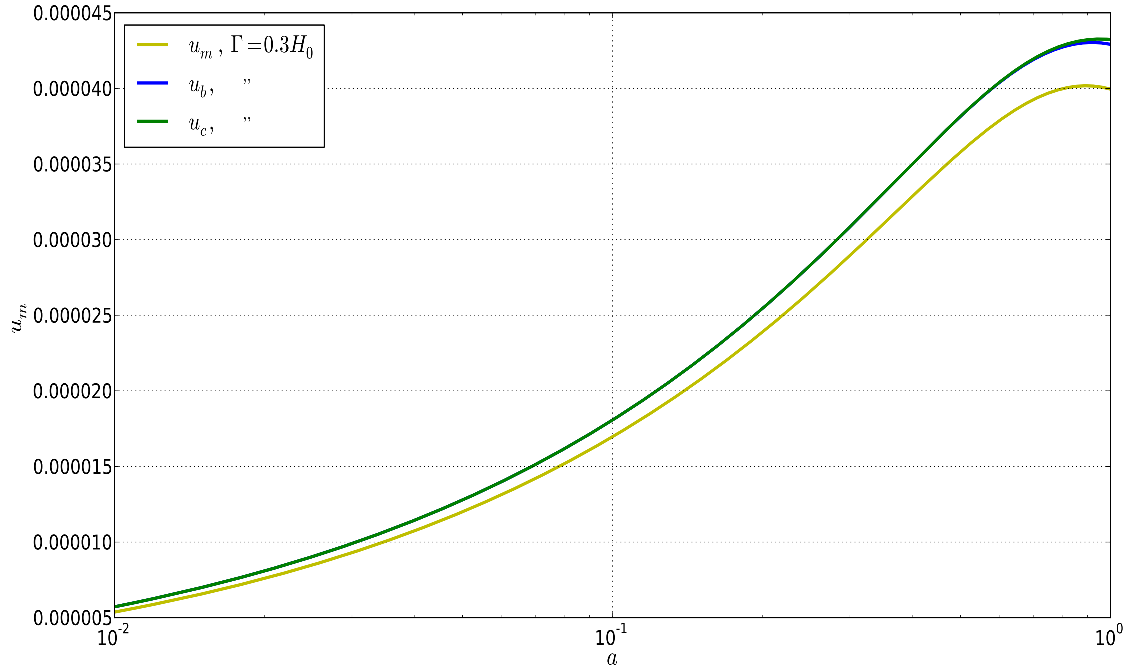


Figure 5.24: Total Matter, CDM and Baryonic Matter velocities (super-Hubble).



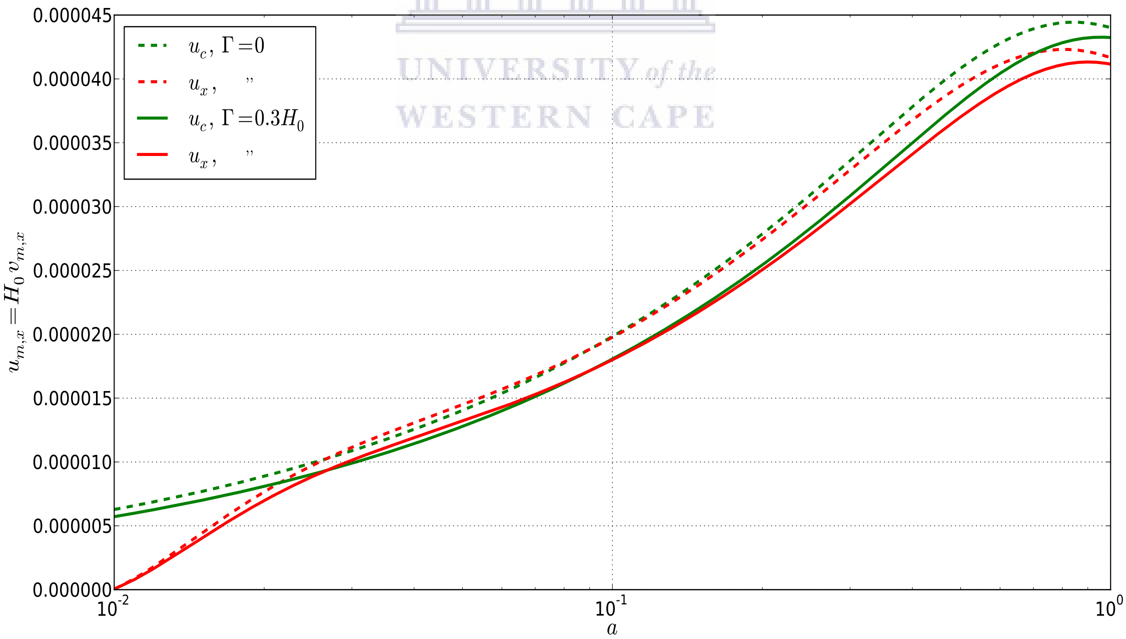


Figure 5.25: CDM and DE velocities (super-Hubble).

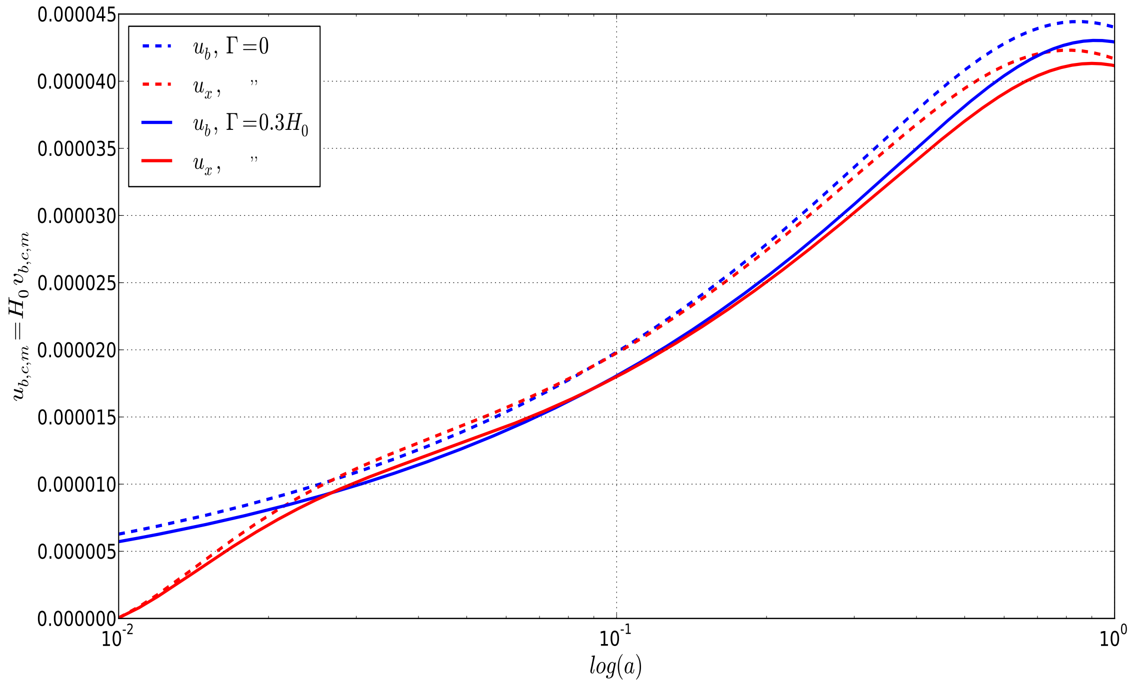


Figure 5.26: Baryonic Matter and DE velocities (super-Hubble).



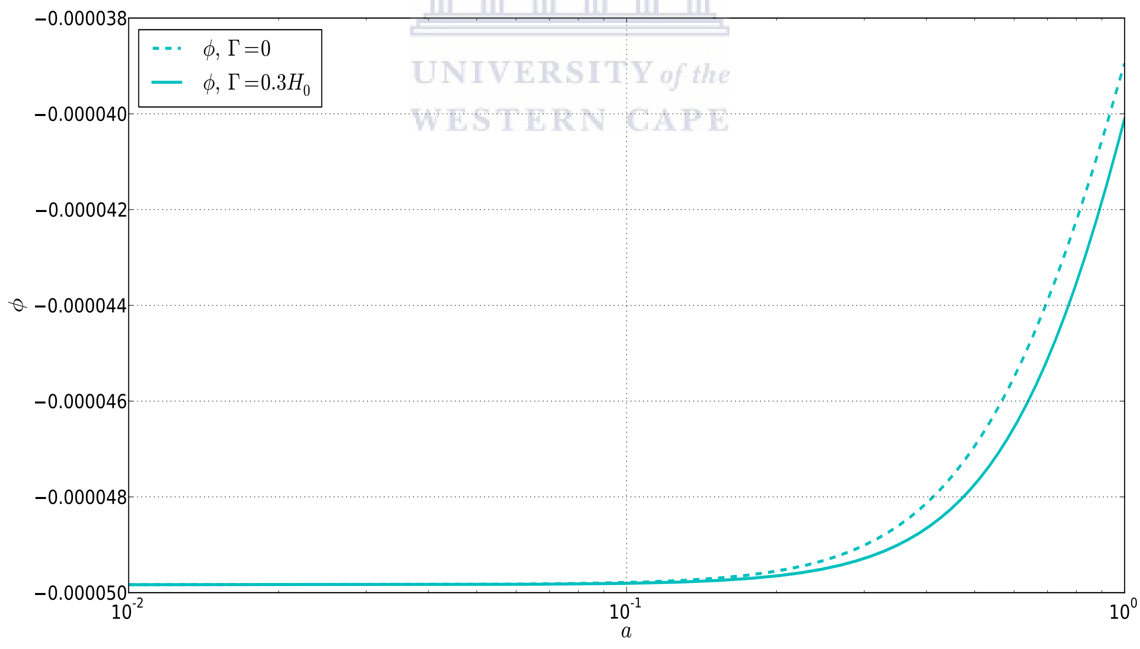


Figure 5.27: The gravitational potential (super-Hubble).

## 5.3 Newtonian Approximation

We can neglect the change in the gravitational potential compared to matter density perturbations on small cosmological scales, i.e. on sub-Hubble scales (Newtonian regime) $a/k \ll H^{-1}$ at which structure formation takes place [12]. Then the effect of the high sound speed of the DE density perturbations means that we can neglect these perturbations. In the model $Q^{\mu}_{x} = Q_x u^{\mu}_{c}$ in the Newtonian regime (discard the $\phi'$ terms) we get from equation (3.43)

$$\left(\frac{k^2}{a^2}\right)\phi = -4\pi G(\bar{\rho}_c\delta_c + \bar{\rho}_b\delta_b), \tag{5.47}$$

$$\dot{v}_c = -Hv_c - \frac{\phi}{a}, \tag{5.48}$$

$$\dot{\delta}_c = \frac{k^2 v_c}{a} + \Gamma\left(\frac{\bar{\rho}_x}{\bar{\rho}_c}\right)\delta_c, \tag{5.49}$$

we are using cosmic proper time. We can drive an equation for $\delta_c$ as follows [12]. First we differentiate equation (5.49) to get

$$\ddot{\delta}_c - \left(\frac{k^2}{a}\right)\dot{v}_c + \left(\frac{H\,k^2}{a}\right)v_c - \Gamma\frac{\bar{\rho}_x}{\bar{\rho}_c}\dot{\delta}_c = \frac{\Gamma}{\bar{\rho}_c}\left\{\bar{\rho}_x - \frac{\bar{\rho}_x}{\bar{\rho}_c}\dot{\bar{\rho}}_c\right\}\delta_c. \tag{5.50}$$

From equations (4.1) we can write

$$\dot{\bar{\rho}}_x - \left(\frac{\bar{\rho}_x}{\bar{\rho}_c}\right)\dot{\bar{\rho}}_c = \Gamma\bar{\rho}_x - 3Hw\bar{\rho}_x + \Gamma\left(\frac{\bar{\rho}_x^2}{\bar{\rho}_c}\right). \tag{5.51}$$



Substituting equation (5.51) and equations (5.49) into (5.50) yields

$$\begin{aligned}\ddot{\delta}_c \;+\; & 2\,H\left\{1 - \left(\frac{\Gamma}{2\,H}\right)\left(\frac{\bar{\rho}_x}{\bar{\rho}_c}\right)\right\}\dot{\delta}_c + \left(\frac{k^2}{a^2}\right)\phi \;-\; H^2\left(\frac{\Gamma}{H}\right)\left(\frac{\bar{\rho}_x}{\bar{\rho}_c}\right)\times \\ & \left\{2 - 3\,w + \left(\frac{\Gamma}{H}\right)\left(1 + \left(\frac{\bar{\rho}_x}{\bar{\rho}_c}\right)\right)\right\}\delta_c = 0.\end{aligned} \tag{5.52}$$

If we substitute equation (3.43) in equation (5.49) and Friedmann equation (1.4) into equation (5.52) we get

$$\begin{aligned}\ddot{\delta}_c \;+\; & 2H\left\{1 - \left(\frac{\Gamma}{\mathbf{2}H}\right)\left(\frac{\bar{\rho}_x}{\bar{\rho}_c}\right)\right\}\dot{\delta}_c - 4\,\pi\,G\Bigg\{\bar{\rho}_c\Big[1 + \left(\frac{2}{3a}\right)\left(\frac{\Gamma}{H}\right)\left(\frac{\bar{\rho}_x}{\bar{\rho}_c}\right) \\ \times\;\; & \{2 - 3w + \left(\frac{\Gamma}{H}\right)\left(1 + \frac{\bar{\rho}_x}{\bar{\rho}_c}\right)\}\Big]\delta_c + \bar{\rho}_b\delta_b\Bigg\} = 0.\end{aligned} \tag{5.53}$$

From equation (5.53) above, we find a disagreement with [12] in their equation (37) regarding the factor of (1/2) written in bold.

The effect of the interaction on the growth of density perturbaions may be summarized in 2 contributions

1. Modify the Hubble term $H \longrightarrow H\,(1 - \Gamma\bar{\rho}_x/2H\bar{\rho}_c)$.
2. Modify the effective Newtonian constant for the DM case to be in the form:

$$\frac{G_{eff}}{G} = 1 + \left(\frac{2}{3\,a}\right)\left(\frac{\Gamma}{H}\right)\left(\frac{\bar{\rho}_x}{\bar{\rho}_c}\right)\left\{2 - 3\,w + \left(\frac{\Gamma}{H}\right)\left(1 + \frac{\bar{\rho}_x}{\bar{\rho}_c}\right)\right\}. \tag{5.54}$$

## 5.4 The Case $w = -1$

In [16] $w = -1$ is excluded because of the singularity in the perturbation equations (5.39) to (5.45). We can avoid this singularity by rearranging the equations. We multiply the DE velocity evolution equation in (5.41) by $(1+w)$ and then set $(1+w)$ to 0 to get

$$u_x = -\frac{\Gamma}{aH_0}\delta_x. \tag{5.55}$$

Then we substitute the value of $w = -1$ in the DE density evolution equation (5.44) to get

$$\frac{d\delta_x}{da} = -\frac{6}{a}\delta_x + \frac{\Gamma}{H_0}\frac{1}{h}\left[\frac{\phi}{a} - 6hu_x\right]. \tag{5.56}$$



Then we solve these numerically with the interaction system of equations (5.39),(5.40), (5.42),(5.43) and (5.45) with the same previous initial conditions at $w = -1$.

For $\Gamma < 0$, we find no instability: see Figures (5.28) to (5.31). However $\Gamma > 0$ leads to an instability: see Figures (5.32) and (5.33). Thus we can extend the instability conditions (5.4) and (5.5) to the $w = -1$ case:

$$w = -1, \quad \Gamma > 0 \quad \Rightarrow \quad \text{instability}, \tag{5.57}$$

$$w = -1, \quad \Gamma < 0 \quad \Rightarrow \quad \text{no instability}. \tag{5.58}$$

We get the results in Figures (5.28) to (5.38).

**For** $\Gamma = -0.3H_0$: from equation (5.55) we see $u_x$ is proportional to $\delta_x$ and as $a$ increases then $\delta_x$ decreases. It follows that $u_x$ decreases rapidly. We can see at late times $\delta_x$ and $u_c$ start to grow. The gravitational potential almost has the same formation as in the original case.

**For** $\Gamma = 0.111709H_0$ blow-up in CDM density perturbation on both sub-Hubble and super-Hubble scales, as in Figures (5.31) and (5.34). We also find a negative CDM density at $a > 1$ as in Figure (5.37).

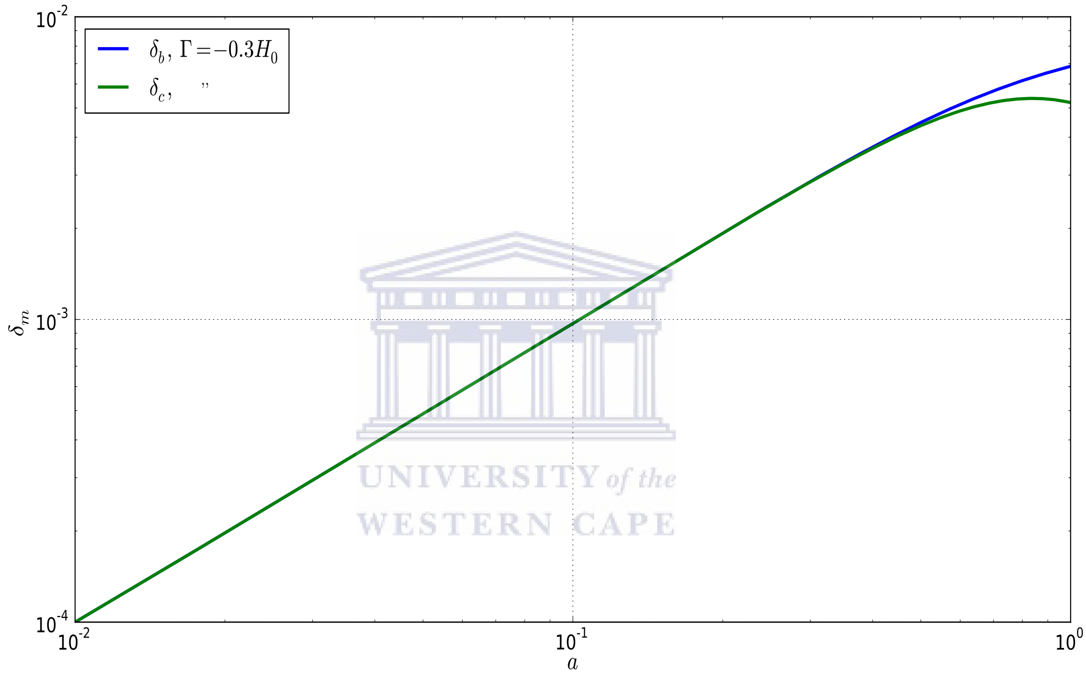


Figure 5.28: Baryonic Matter and CDM density perturbations for $w = -1$ (sub-Hubble).

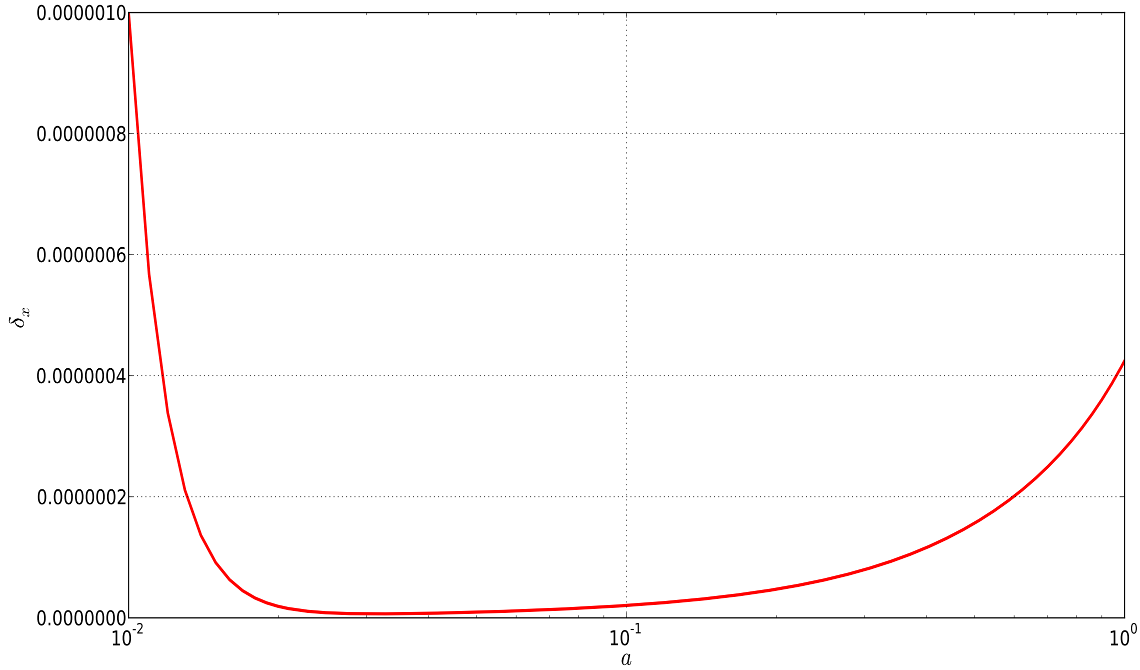


Figure 5.29: DE density perturbations for $w = -1$ (sub-Hubble).

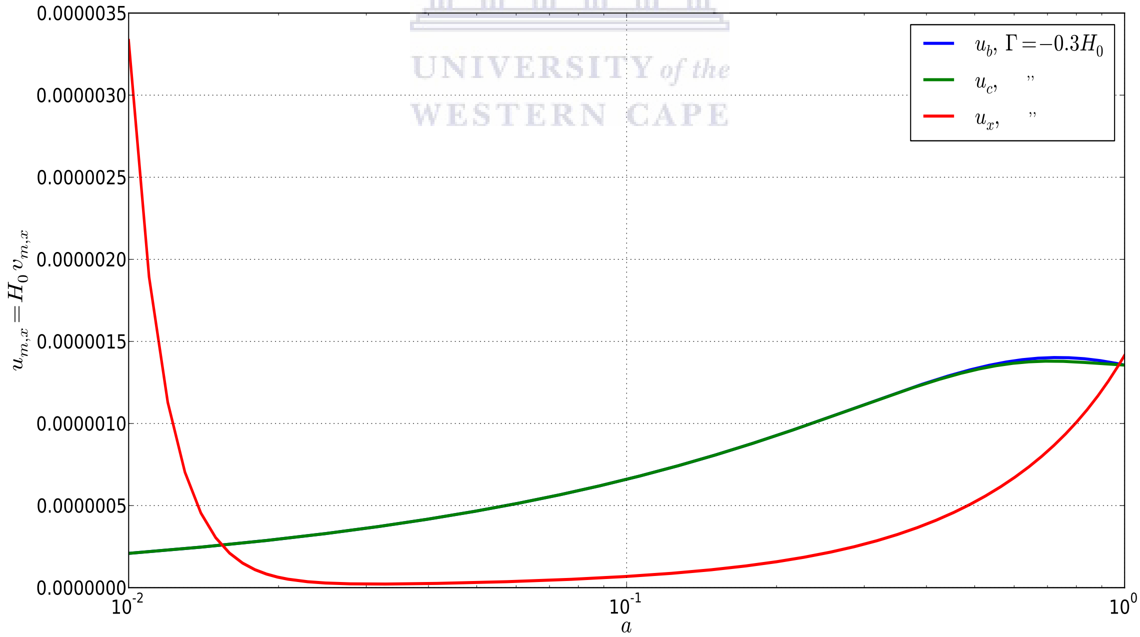




Figure 5.30: Baryonic Matter, CDM and DE velocities for $w = -1$ (sub-Hubble).

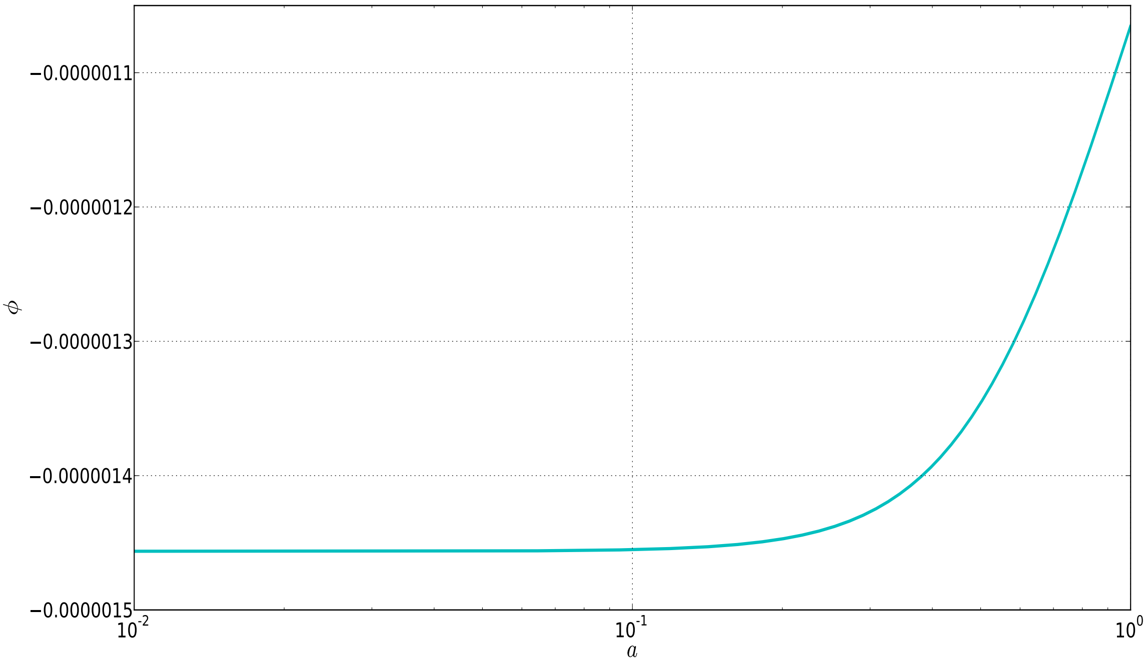


Figure 5.31: The gravitatioanl potential for $\Gamma = -0.3H_0$ and $w = -1$ (sub-Hubble).



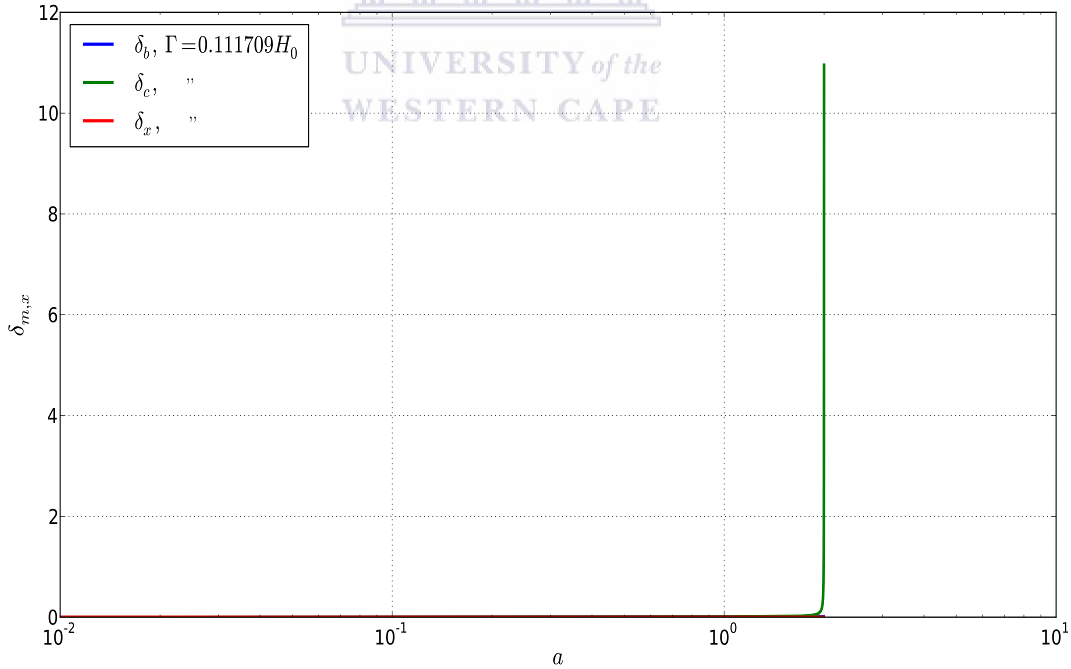


Figure 5.32: Blow-up in CDM density perturbation for $w = -1$ (sub-Hubble) .

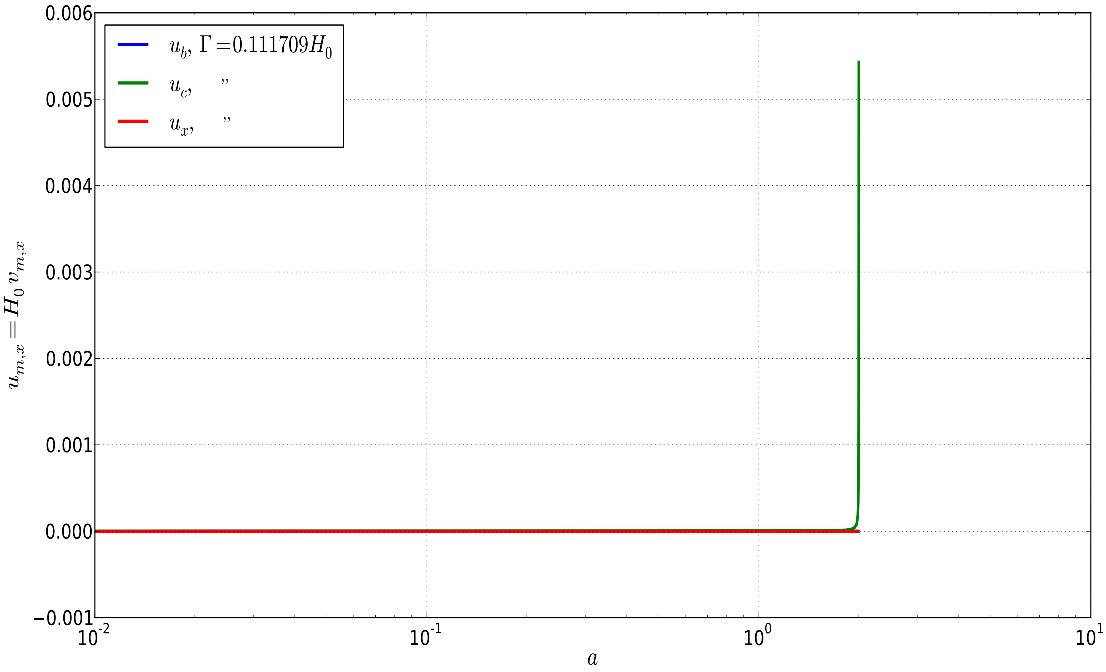


Figure 5.33: DM and DE velocities for $w = -1$ (sub-Hubble).



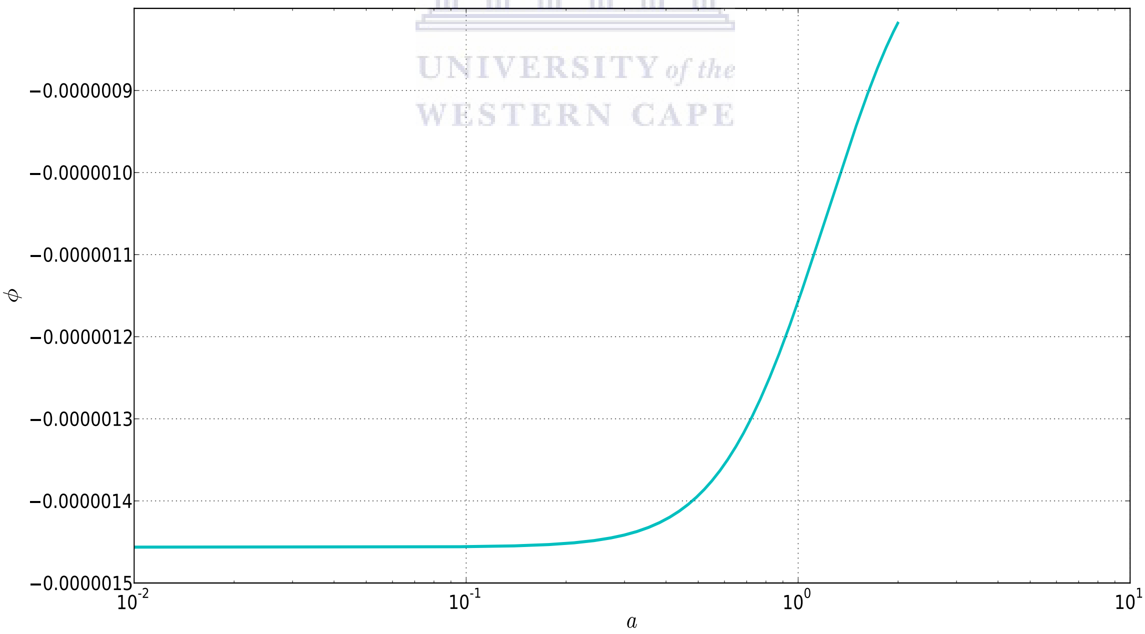


Figure 5.34: The gravitational potential for $w = -1$ and $\Gamma = 0.111709 H_0$ (sub-Hubble).

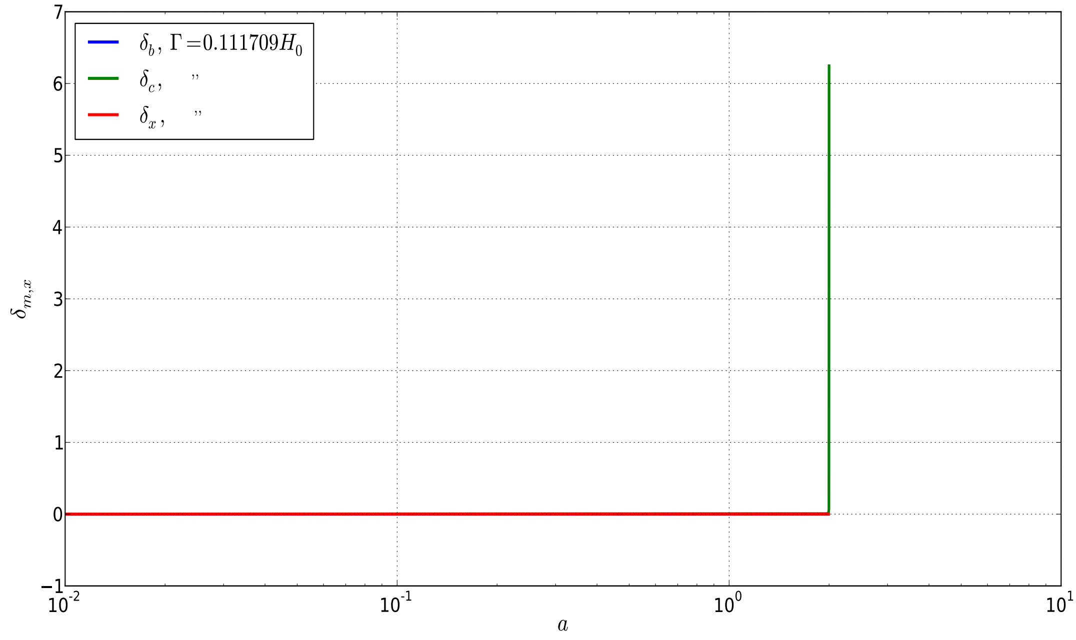


Figure 5.35: Blow-up in CDM density perturbation for $w = -1$ (super-Hubble).



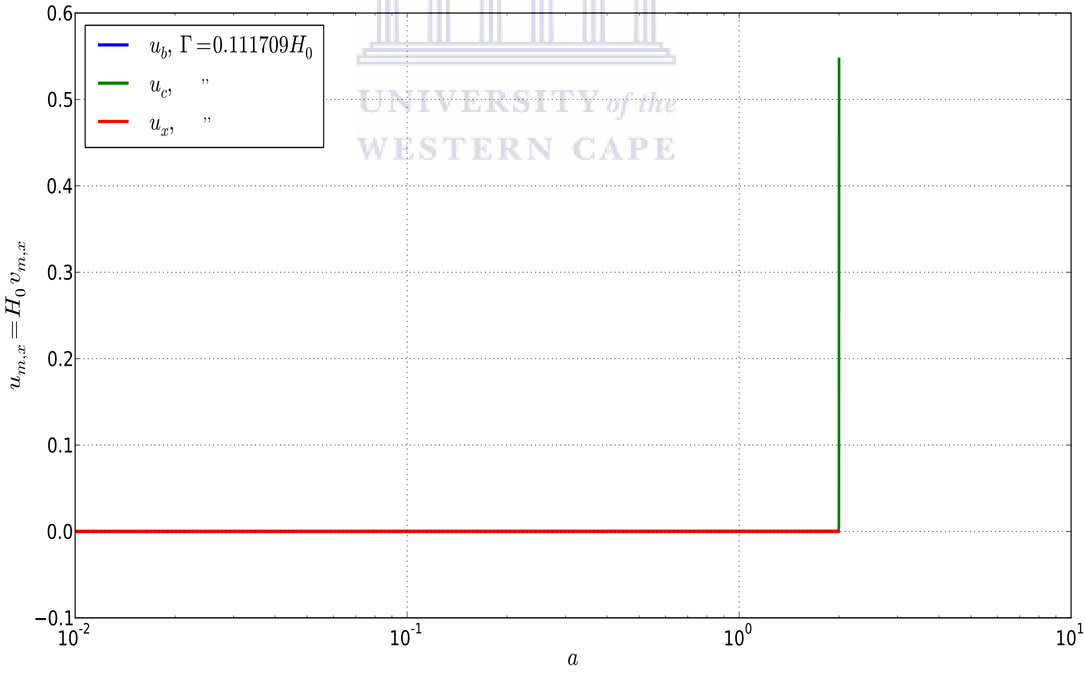


Figure 5.36: DM and DE velocities for $w = -1$ (super-Hubble).

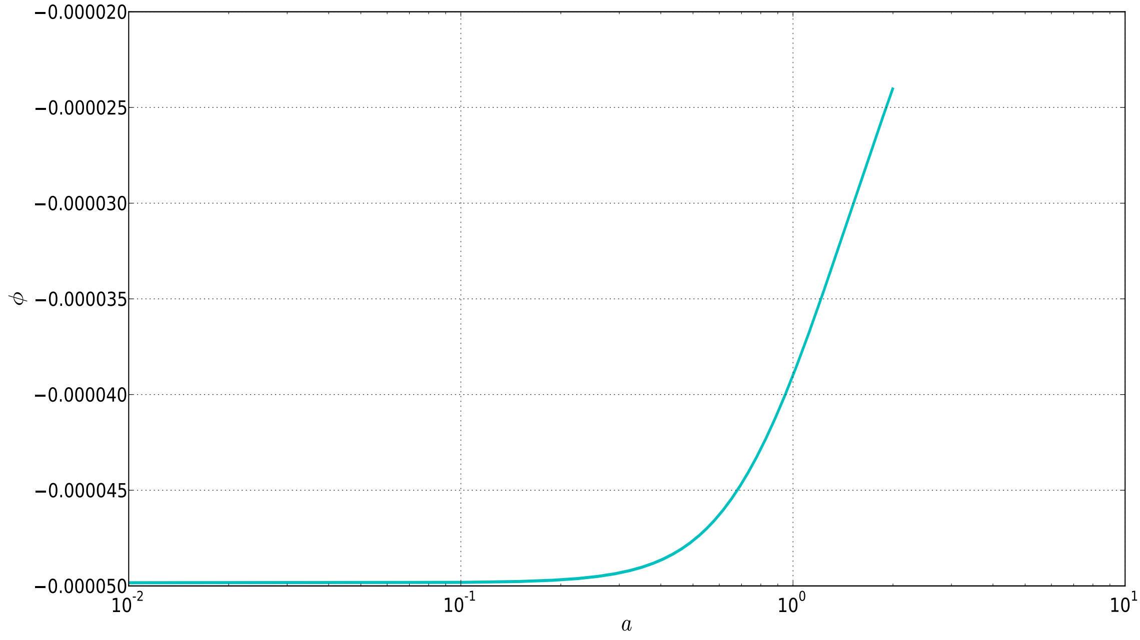


Figure 5.37: The gravitational potential for $\Gamma = 0.111709H_0$ and $w = -1$ (super-Hubble).



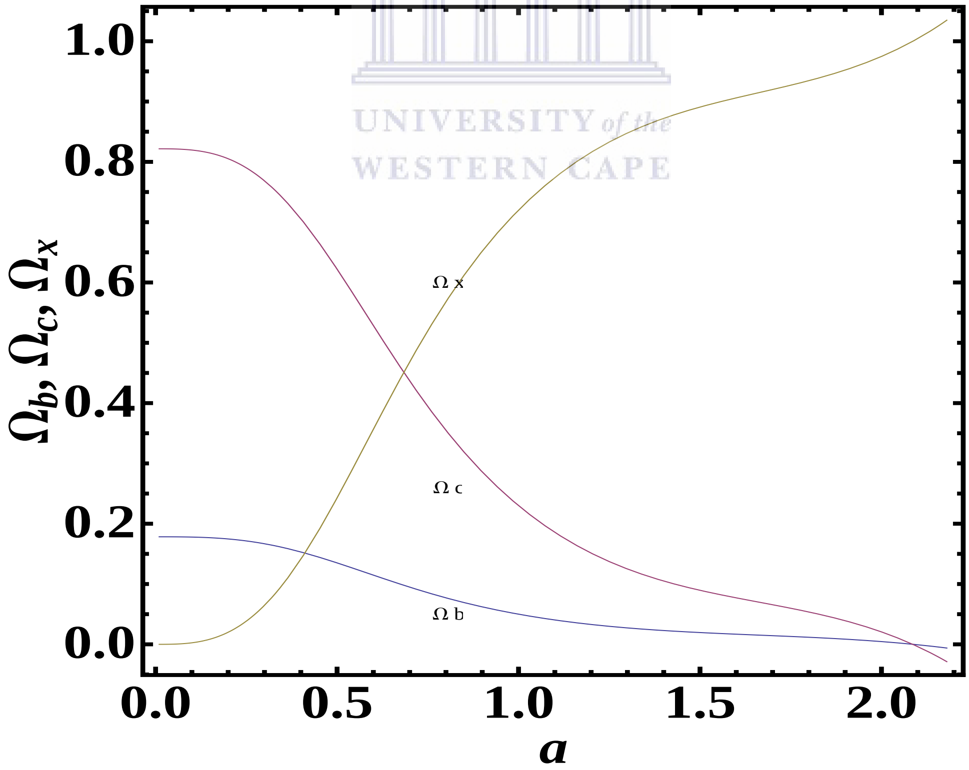


Figure 5.38: $\Gamma = 0.111709H_0$ for $w = -1$ shows how $\Omega_c$ becames negative at $a > 1$.

## 5.5 Variable $w$ Model. I

Now we consider $w$ as a function of the scale factor ($a$). The simplest case is a linear function

$$
\begin{aligned}
w(a) &= w_0 + w_a\,(1 - a), \\
&= -w_a\,a + (w_0 + w_a).
\end{aligned}
\tag{5.59}
$$

See Figure (5.39) for $w(a)$. Since the values of $a$ starts from 0.01 up to 1.0 we can take $w_0 + w_A = 0,\ w_a = 0.9$ so that $w$ is nearly zero initially. Thus we take

$$
w = -0.9\,a. \tag{5.60}
$$

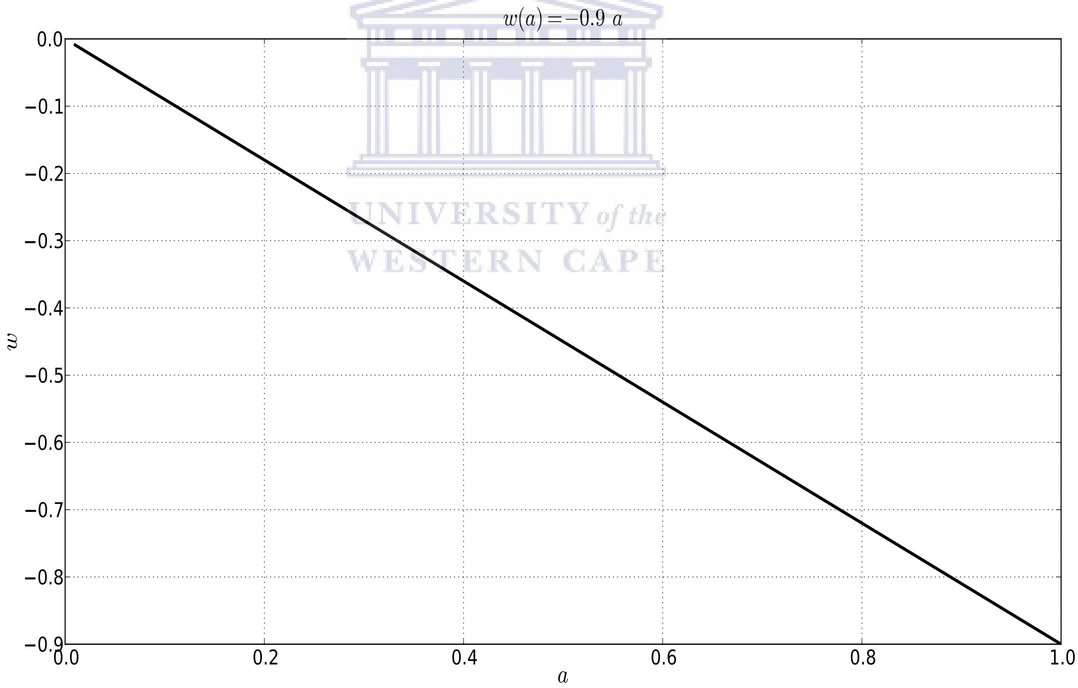


Figure 5.39: The linear function $w(a) = -0.9\,a$ .

The background evolution is shown in Figure (5.40). When we apply this modification to the $Q^\mu_x = Q_x u^\mu_c$ model on sub-Hubble scales, we get the results in Figures (5.41) to (5.44).

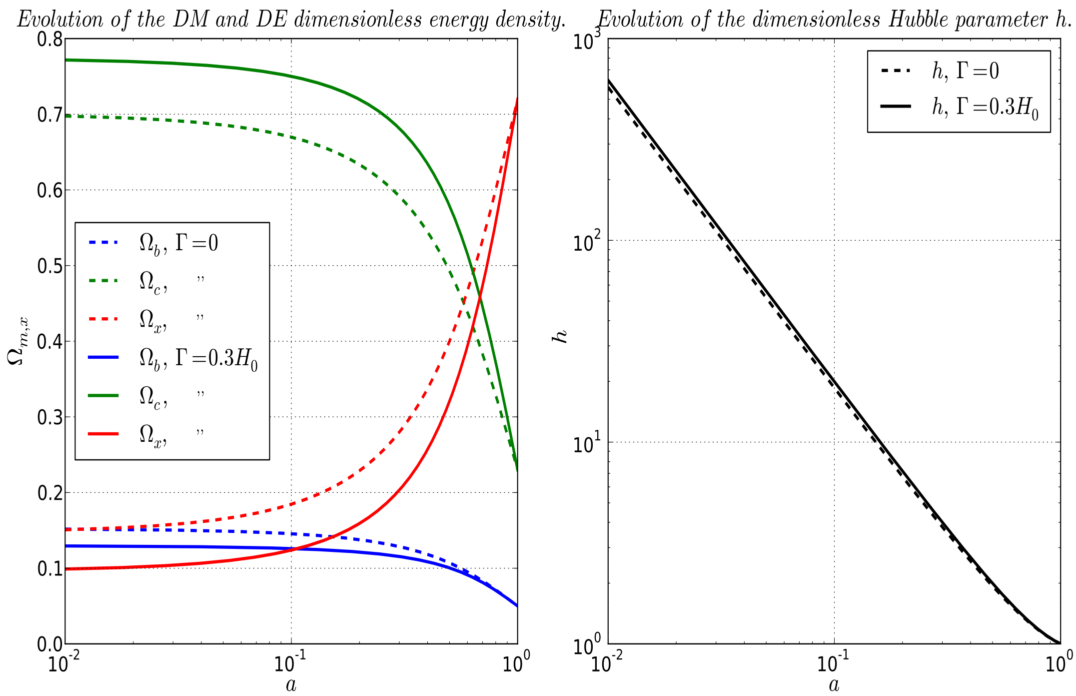


Figure 5.40: The background evolution for $w = -0.9\ a$ .



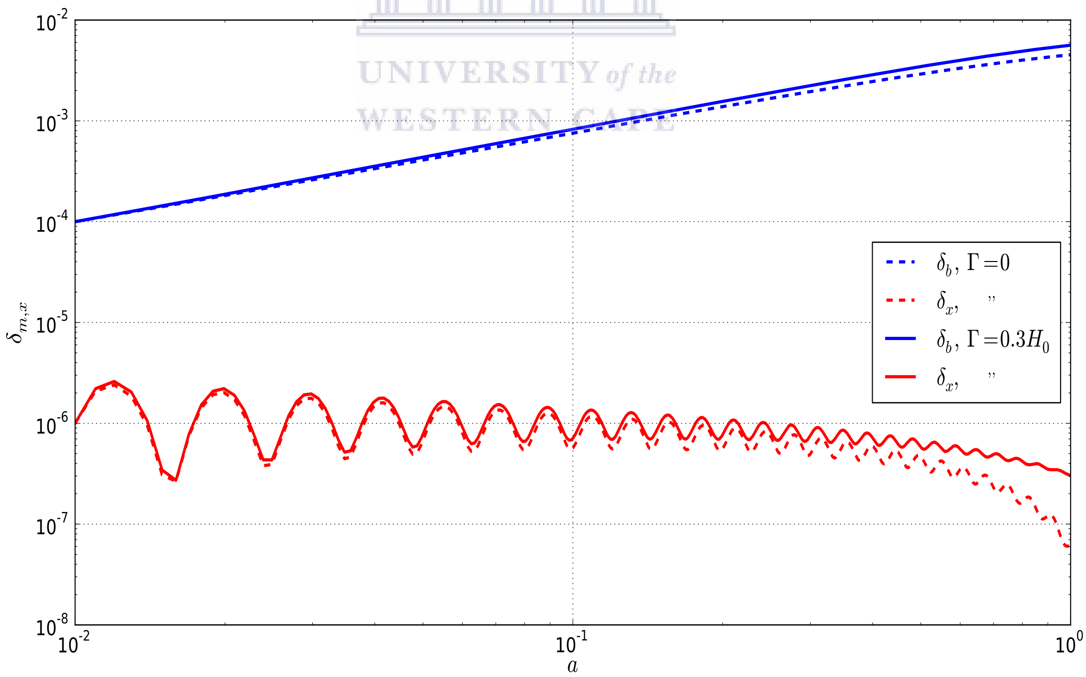


Figure 5.41: Baryonic Matter and DE density perturbations for $w = -0.9a$.

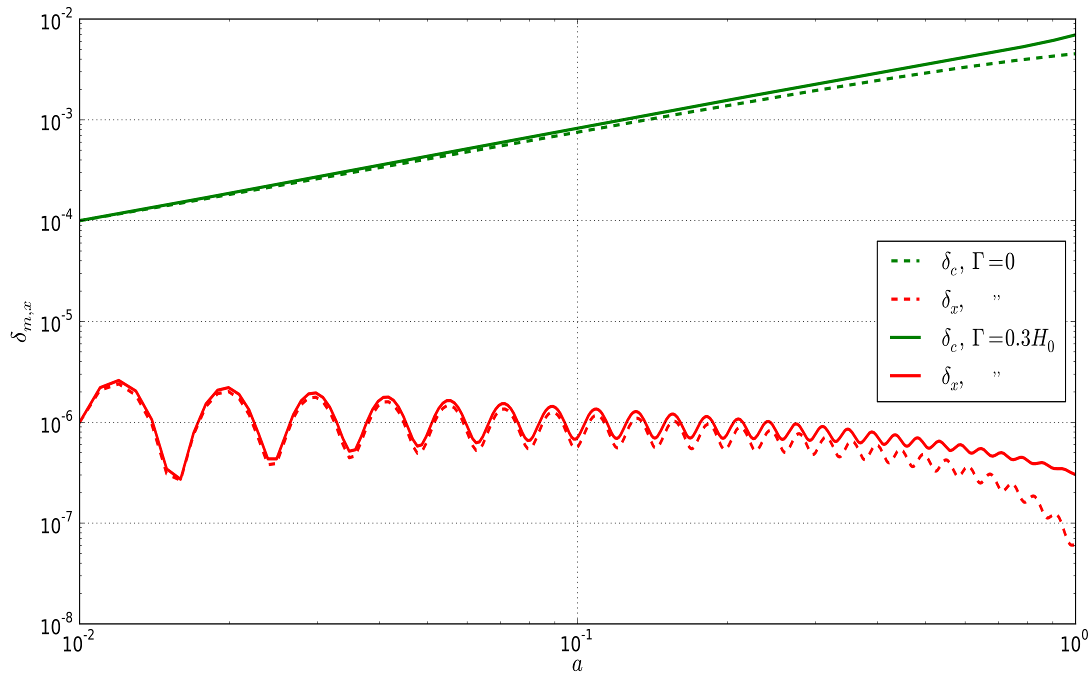


Figure 5.42: CDM and DE density perturbations for $w = -0.9a$.



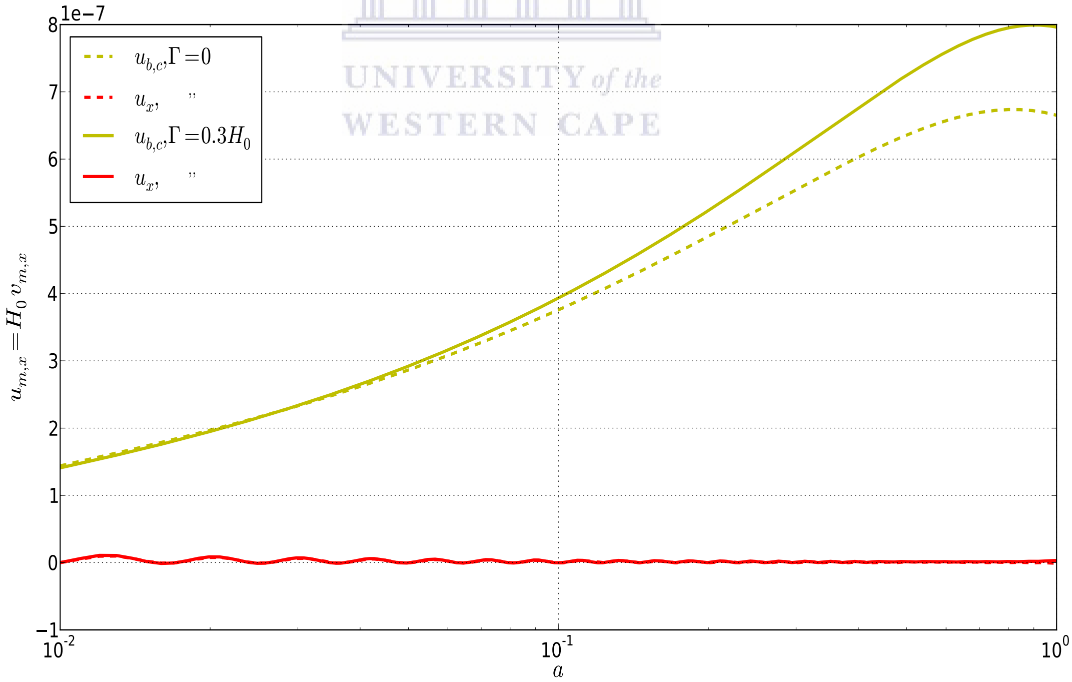


Figure 5.43: Baryonic Matter, CDM and DE velocities for $w = -0.9a$.

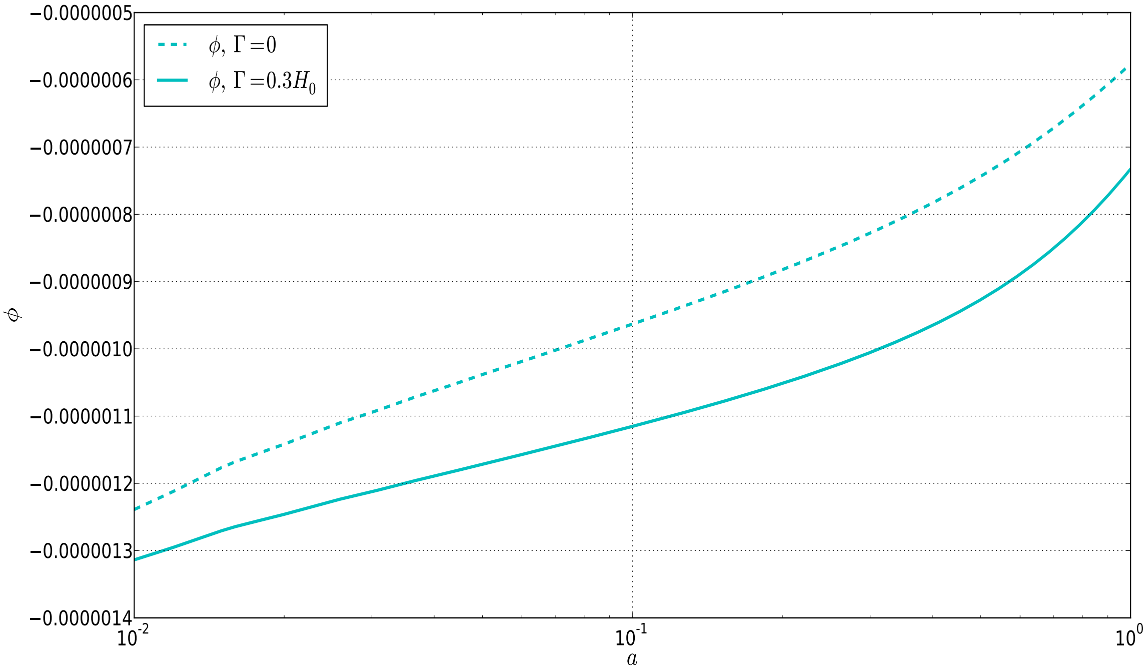


Figure 5.44: The gravitational potential for $w = -0.9a$.



For the $Q_x^\mu = Q_x u_x^\mu$ model on sub-Hubble scales, we find the results shown in Figures (5.45) to (5.49):

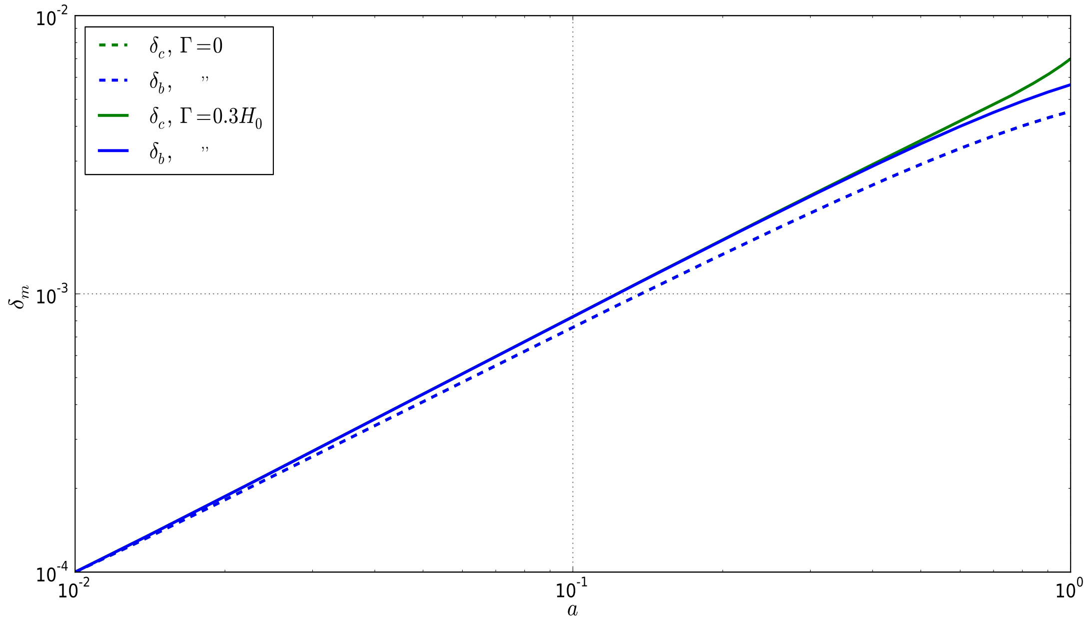


Figure 5.45: Baryonic Matter and CDM density perturbations for $w = -0.9a$.

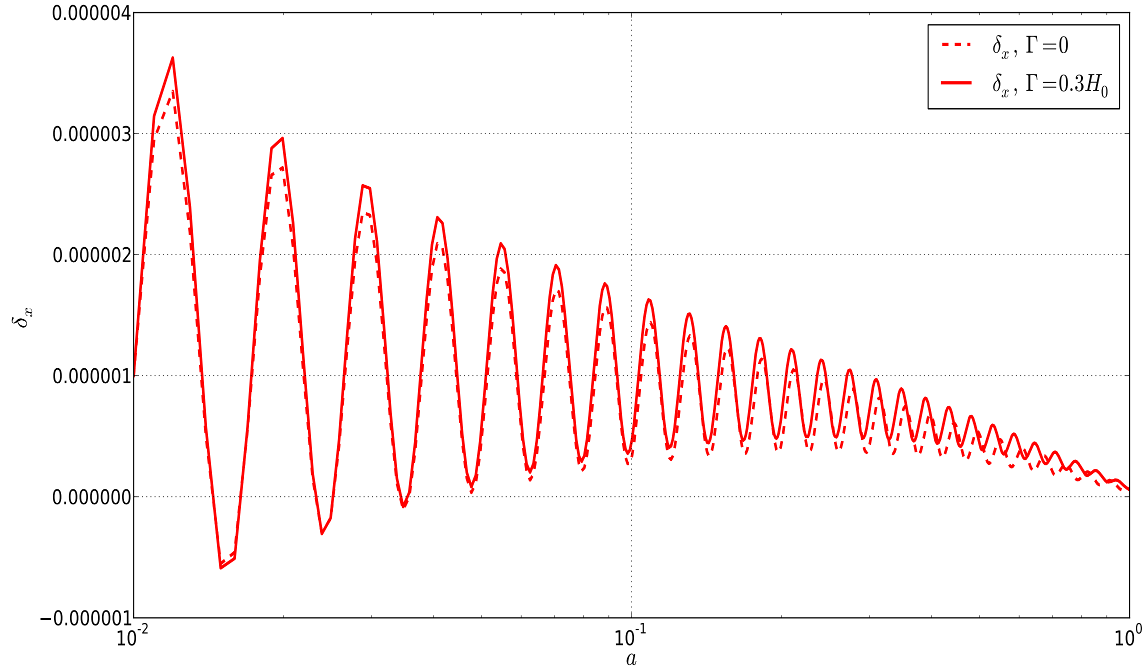


Figure 5.46: DE density perturbations for $w = -0.9a$.



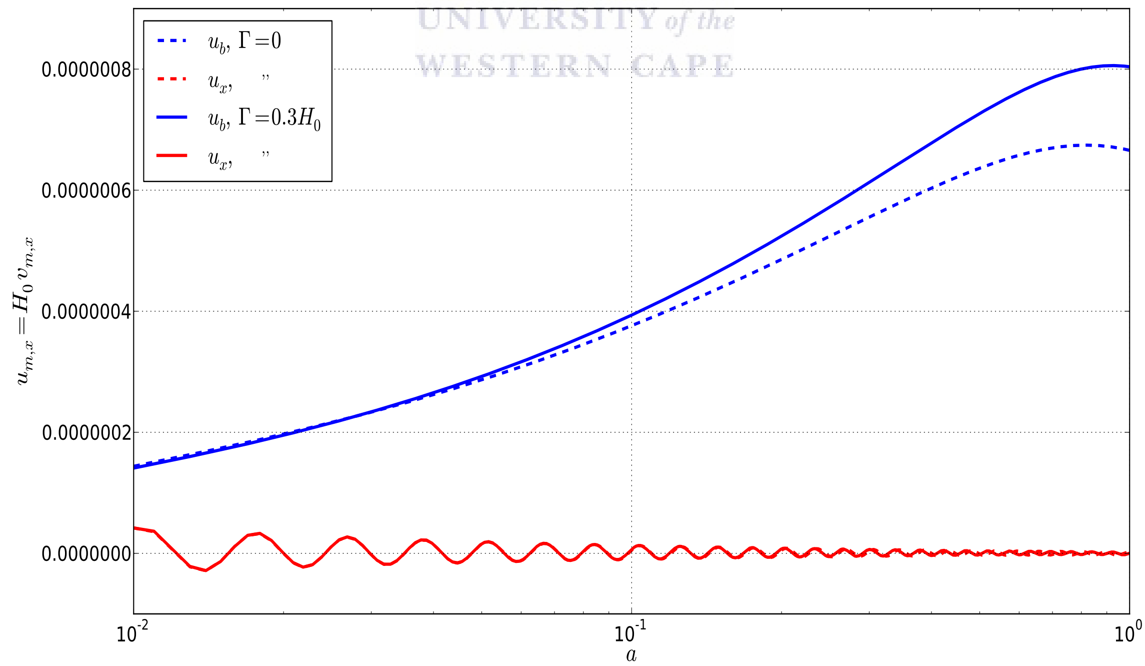


Figure 5.47: Baryonic Matter and DE velocities for $w = -0.9a$.

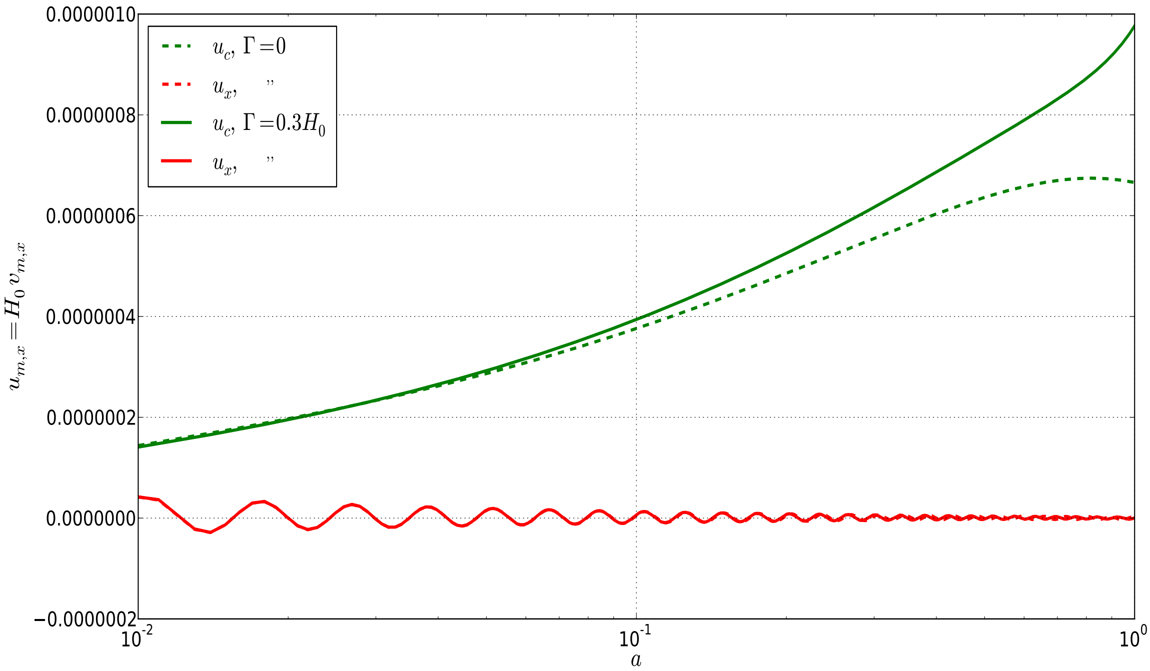


Figure 5.48: CDM and DE velocities for $w = -0.9a$.



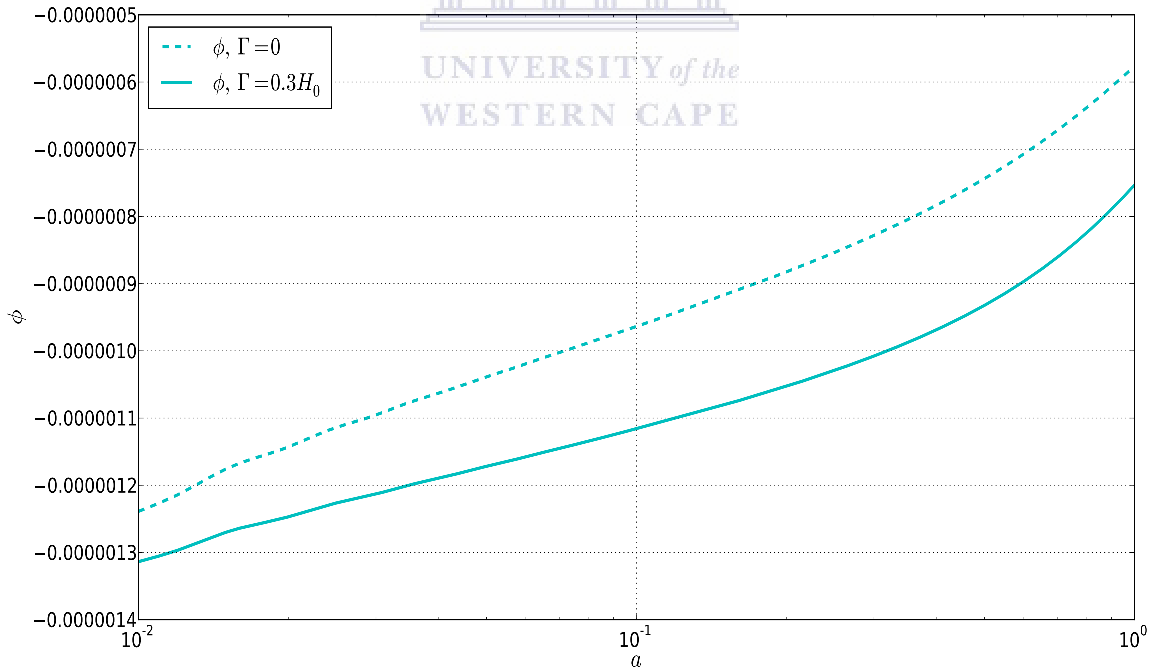


Figure 5.49: The gravitational potential for $w = -0.9a$.

We notice that the gravitational potential evolves much more rapidly than in the $w =$ constant models.

## 5.6 Variable $w$ Model. II

A more realistic model for $w(a)$ is one which evolves more smoothly from nearly 0 at early times to nearly -1 at late times. We choose a hyberbolic tangent shape

$$w = A + B\tanh[\alpha(a - C)], \tag{5.61}$$

where $A, B, C$ and $\alpha$ are parameters. $C$ determines where the transition takes place: we choose $C = \frac{1}{2}$. For $w = -1$ at $a = \infty$, and $w = 0$ at $a = 0$, we choose $A = B = -0.5$. Finally we fix $\alpha$ by requiring $w = -0.9$ at $a = 1$. Then $\alpha = 2.2$. This gives

$$w = -0.5 - 0.5\tanh[2.2(a - 0.5)]. \tag{5.62}$$

See Figure (5.50) for $w(a)$. The background evolution shown in Figure (5.51).



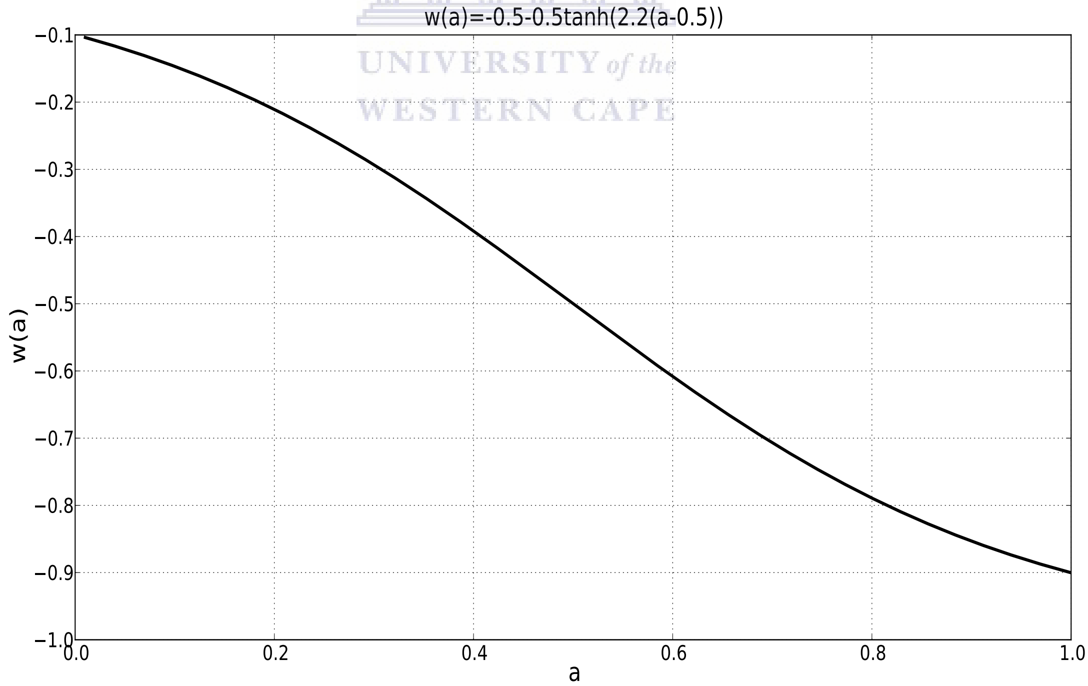


Figure 5.50: The function $w(a) = -0.5 - 0.5\tanh[2.2(a - 0.5)]$ .

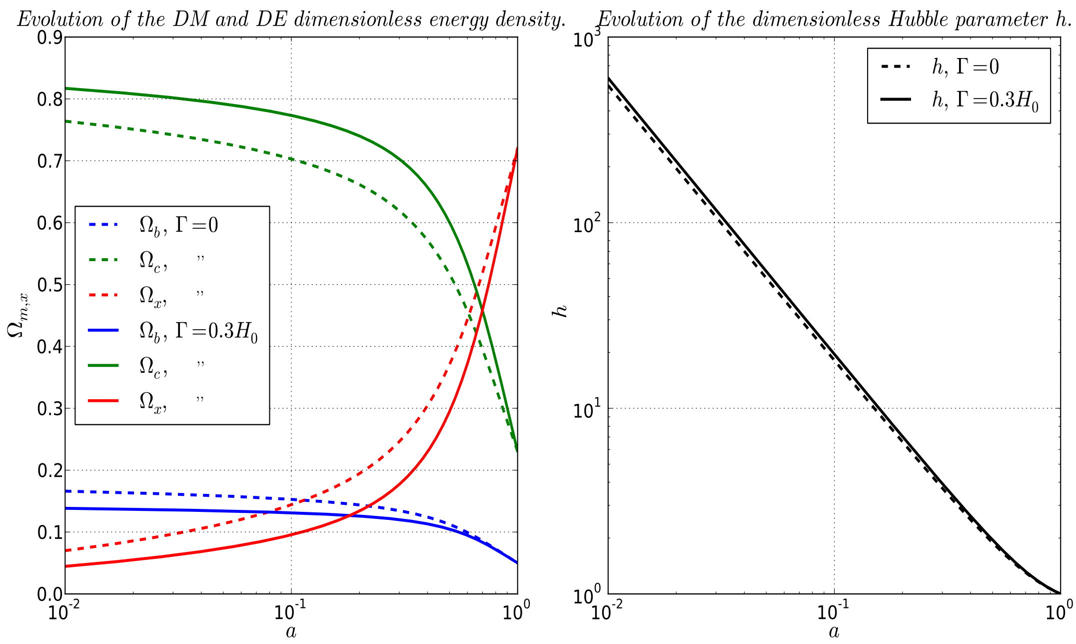


Figure 5.51: The background evolution for $w(a)$ given by equation (5.62).



For the $Q_x^\mu = Q_x u_c^\mu$ model we get the results shown in Figures (5.52) to (5.55) on sub-Hubble scales:

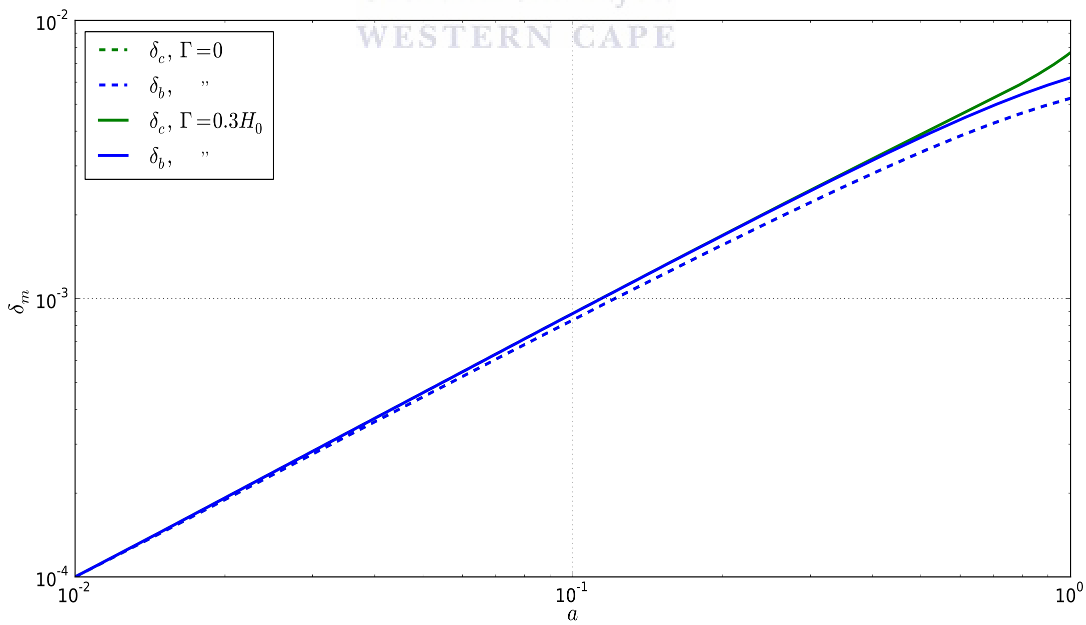


Figure 5.52: Baryonic Matter and CDM density perturbations for $w$ given by equation (5.62).

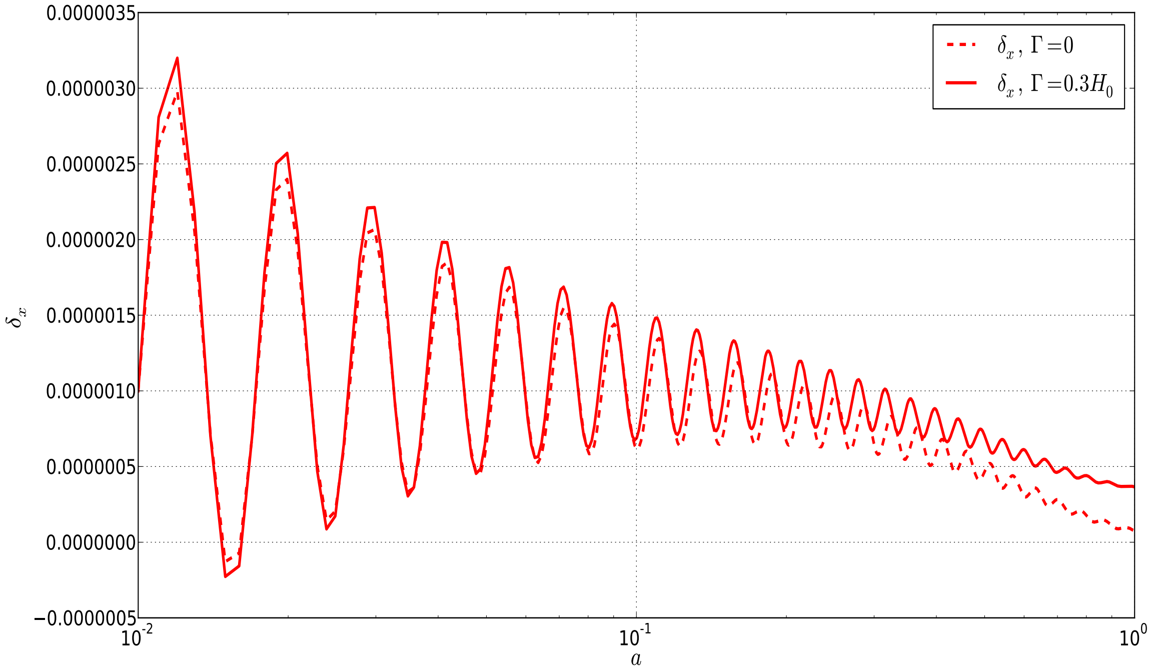


Figure 5.53: DE density perturbations for $w$ given by equation (5.62).



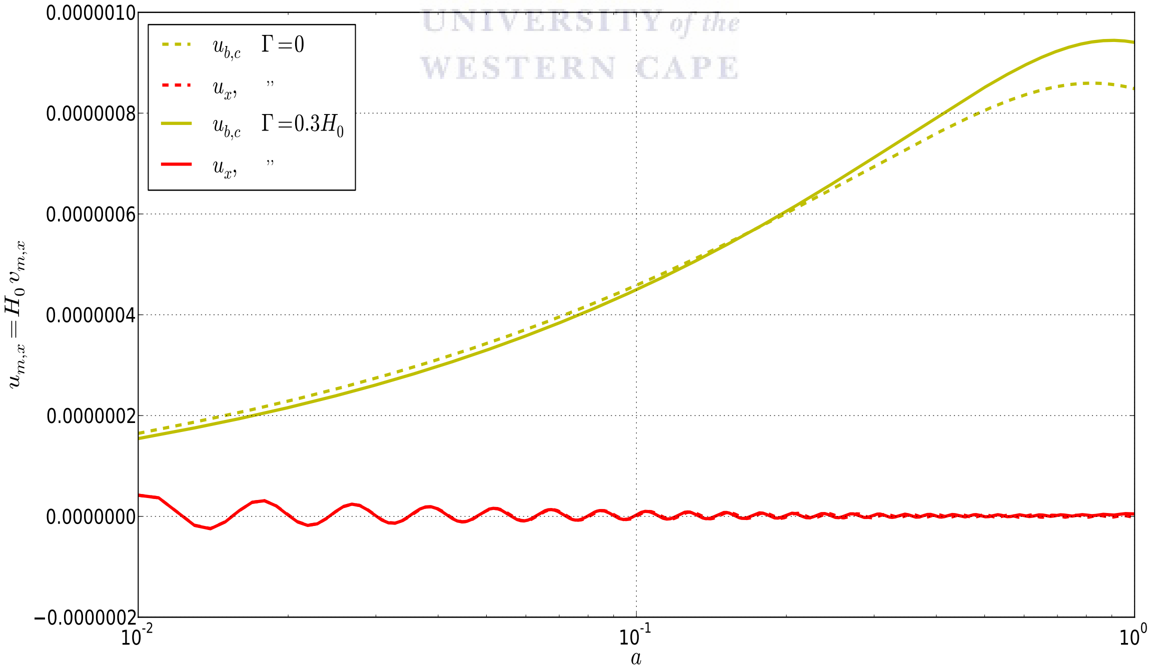


Figure 5.54: Baryonic Matter,CDM and DE velocities for $w$ given by equation (5.62).

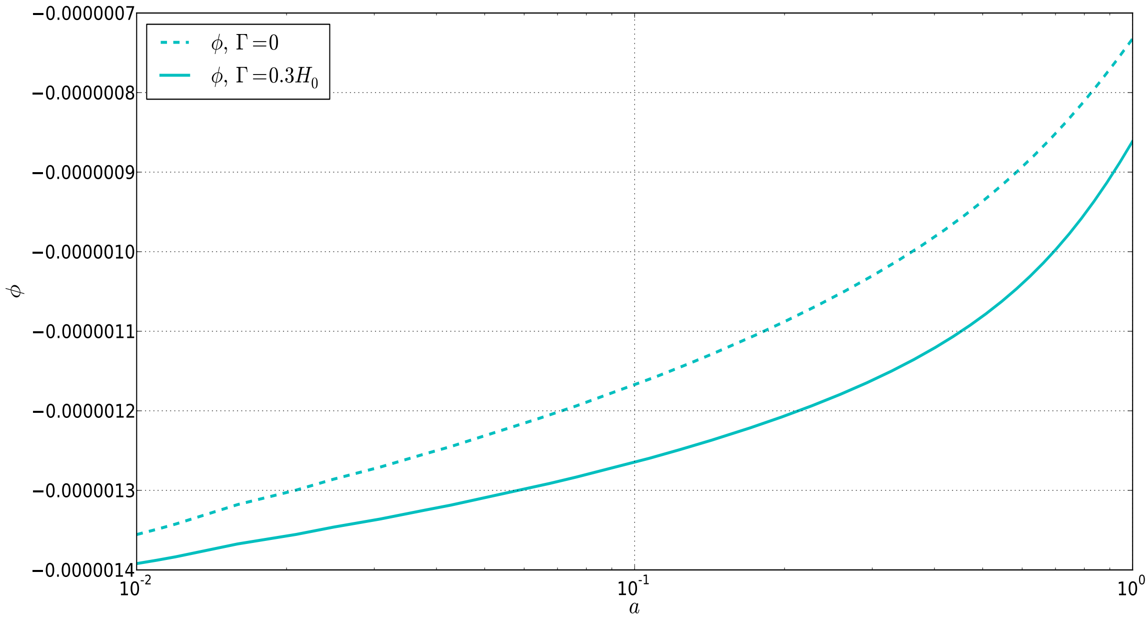


Figure 5.55: The gravitatioanl potential for $w$ given by equation (5.62).



For the $Q_x^\mu = Q_x u_x^\mu$ model on sub-Hubble scales we find the results shown in Figures (5.56) to (5.60):

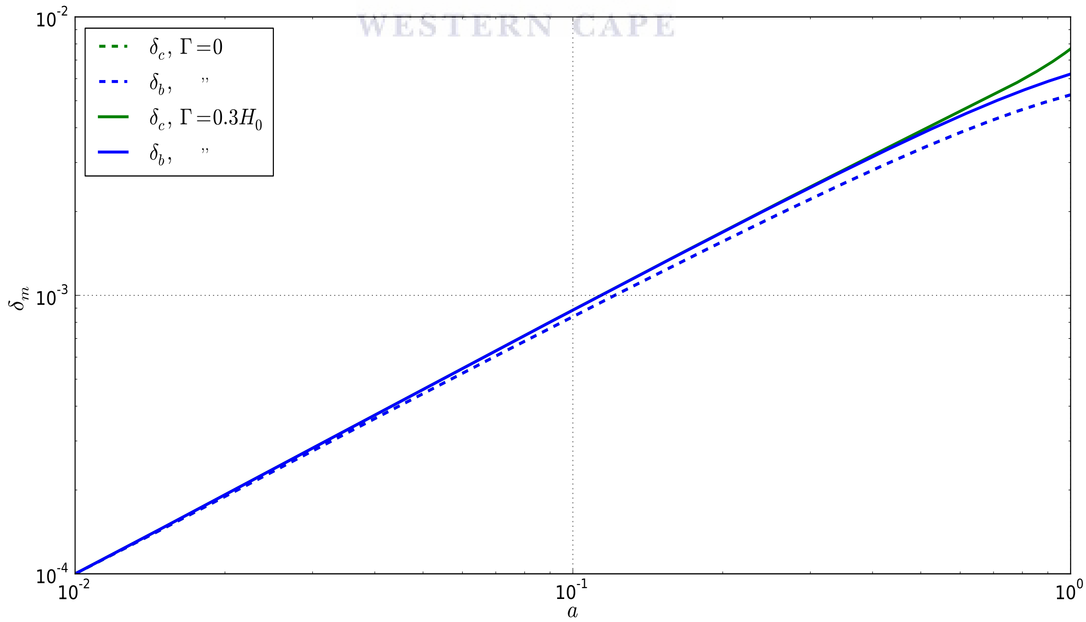


Figure 5.56: Baryonic Matter and CDM density perturbations for $w$ given by equation (5.62).

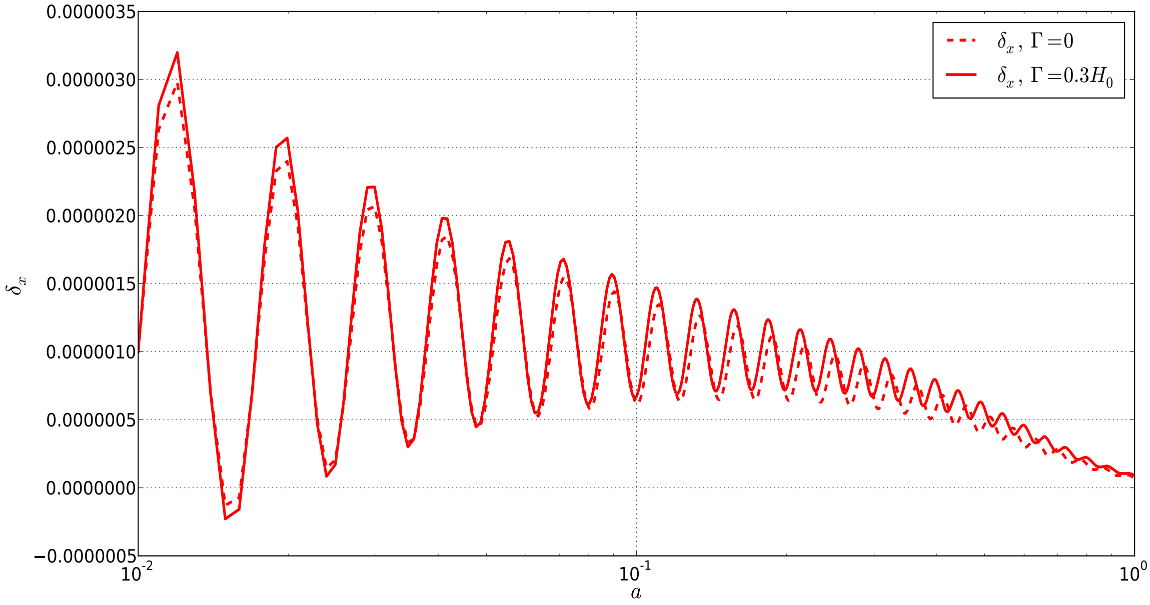


Figure 5.57: DE density perturbations for $w$ given by equation (5.62).



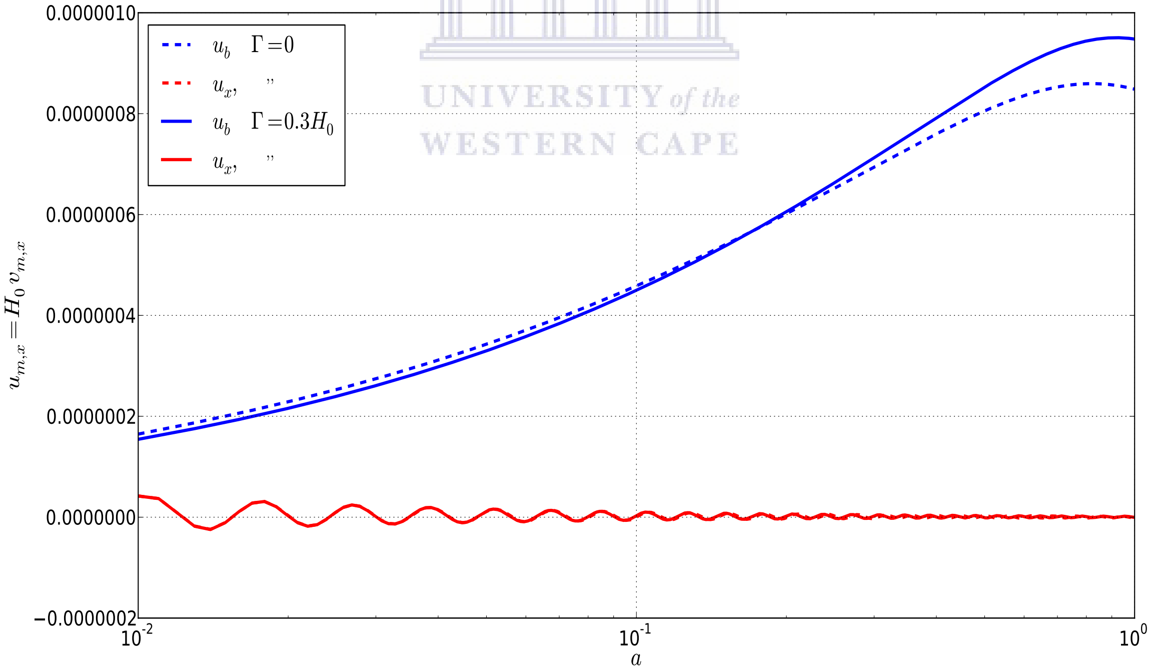


Figure 5.58: Baryonic Matter and DE velocities for $w$ given by equation (5.62).

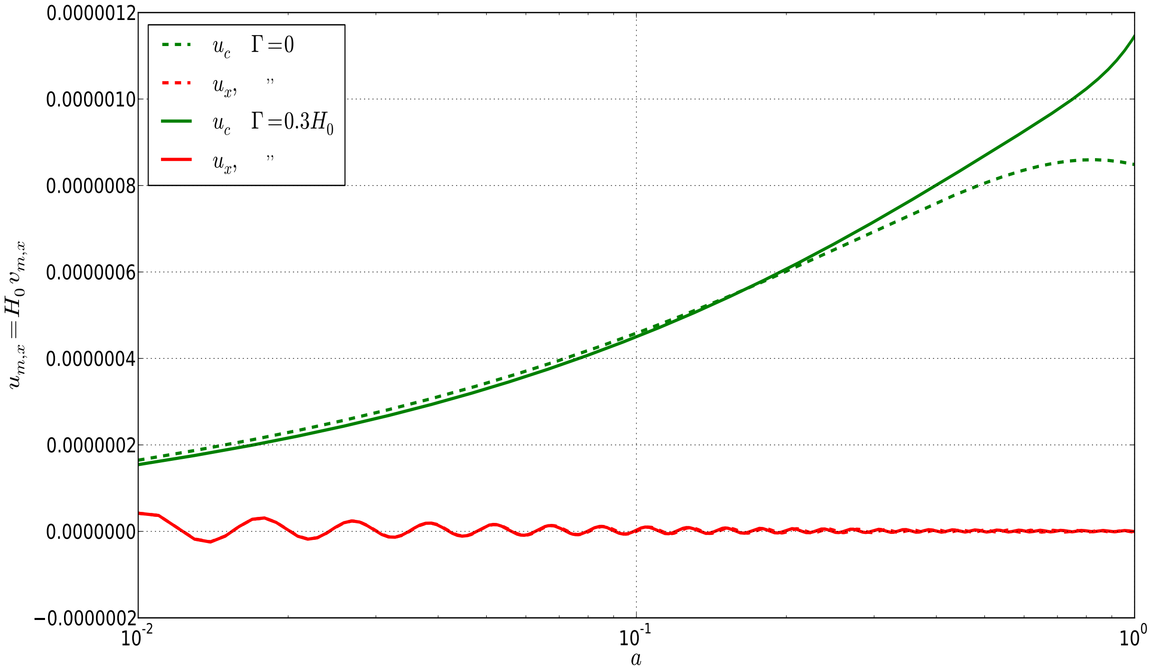


Figure 5.59: CDM and DE velocities for $w$ given by equation (5.62).



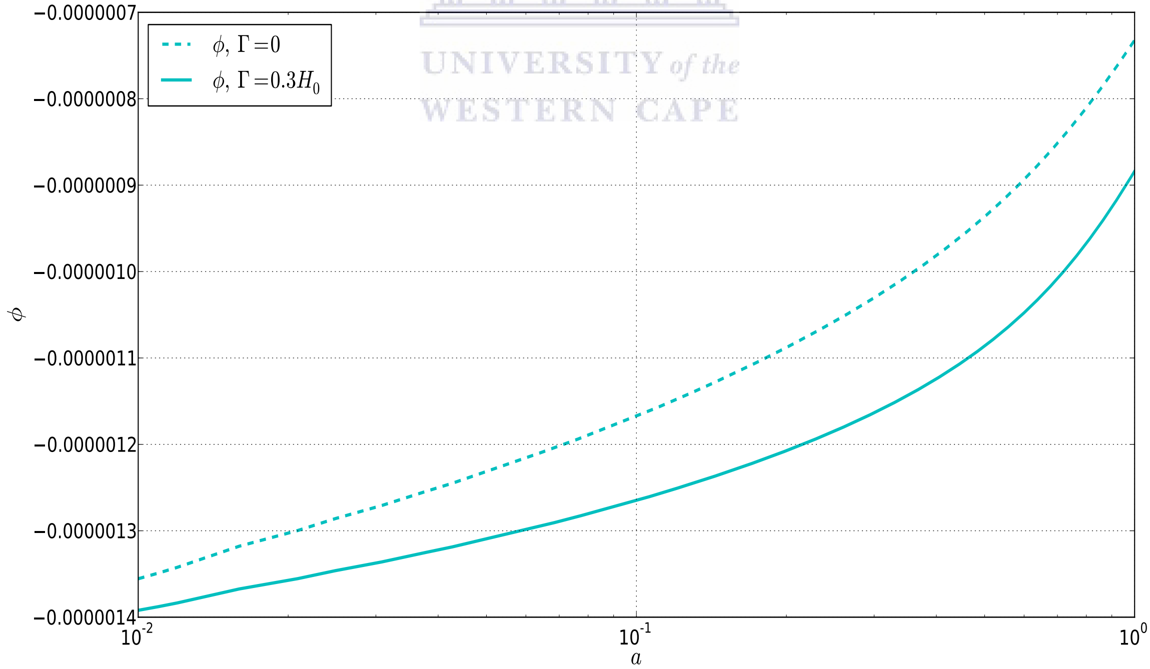


Figure 5.60: The gravitational potential for $w$ given by equation (5.62)

# Chapter 6

# Conclusions

In this thesis we have studied the scalar perturbation theory in cosmology and have employed it to investigate the growth of structure in the current Universe using an elegant model of interacting dark energy.

In Chapter 1 we introduced the idea of cosmological principle and how it is embedded into background spacetime, the components of the Universe and how it dominates during the history of the Universe.

In Chapter 2 we gave the detailed description of the scalar perturbations theory defining their concepts in cosmology in terms of background and perturbed background models.



In Chapter 3 we provided an alternative dynamics to the cosmological constant by introducing a dynamics of fluid dark energy models with constant equation of state parameter $w_x \neq -1$. Interactions of the DE with DM models usually change the evolution of the DM and also the formation of large scale structure such as galaxies and cluster of galaxies. Those interactions could explain why DE comes to dominate only after galaxy formation.

In Chapter 4 we analysed the dynamics of the background for an interacting DE fluid. Then we described an interacting system between DM and DE model with balance equations in general form presented in equations (4.1) to (4.4). Equation (4.8) expresses this model as an interaction term proportional to a linear density combination of the dark sector.

In Chapter 5 we illustrated the dynamics of the perturbations in our IDE model to investigate the growth of structure.

With a covariant 4-vector describing an interacting DE of the form:

$$\begin{aligned} Q^{\mu}_{c} &= Q_c u^{\mu}_{c} \quad \text{and} \quad Q^{\mu}_{c} = Q_c u^{\mu}_{x}, \\ Q_c &= \bar{Q}_c + \delta Q_c = -\Gamma \bar{\rho}_x (1+\delta_x), \end{aligned} \tag{6.1}$$

we investigate the background evolution and structure formation.

In the case of DM transfer to DE ($\Gamma > 0$), we find enhanced growth of matter density perturbations due to more DM density in the past for the interaction case. For large values of $\Gamma > 0$, we have $\rho_c < 0$ at $a < 1$. We avoided that using the observational constraint $\Gamma \leq H_0$, so that $\rho_c$ becomes negative only in the future.

In the case of DE transfer to DM ($\Gamma < 0$), the lower DM density in the past suppresses the growth.

For ($\Gamma < 0$) there is an instability in the perturbations: gradually we see blow-up mode in the perturbations. It is possible for the case of $\Gamma < 0$ to use $w < -1$ to avoid instabilities.



In the case of energy-momentum transfer parallel to DE ($Q^{\mu}_{x} = Q_x u^{\mu}_{x}$), we get higher velocity perturbations of the DM. The reason is at ($Q^{\mu}_{x} = Q_x u^{\mu}_{x}$) the DM receives more momentum as shown in the first part of equation (5.37) compared to the first part in equation (5.16) at ($Q^{\mu}_{x} = Q_x u^{\mu}_{c}$) model, where that is reflected in more DM velocity structure formation.

The mechanism of creating the interaction represents the cornerstone in enhancing or suppressing the growth, where observations can be used to correct the mechanisms of these interactions.

Interacting DE models can be tested against observations, using baryon acoustic oscillations, the growth rate of matter perturbations and weak lensing. Radio telescopes like the Square Kilometre Array (SKA) and its pathfinders (LOFAR, MeerKAT, ASKAP, etc.) can map the distribution of neutral hydrogen and can give us new data to constrain interacting DE, which will supplement the data from optical galaxy surveys.

# Appendix A

## A.1 Classification of the Metric Perturbations

In the perturbed universe we can split the line element into time-time part, time-space part and space-space part in the form

$$ds^2 = a^2(\eta)\{-(1+2A)d\eta^2 - 2B_i d\eta dx^i + [(1-2D)\gamma_{ij} + 2E_{ij}]dx^i dx^j\}. \quad \text{(A.1)}$$

Also we can split it further into three parts: scalar, vector and tensor.



## A.2 Separation into Scalar, Vector, and Tensor Perturbations

In the perturbaions of GR theory there are two modes for transforming the coordinates:

First: gauge transformation (no change in the coordinates of the background, only change in the coordinates of the perturbed spacetime).

Second: the one that we keep the gauge (fixed link up between the background and the perturbed spacetime points, but transform the coordinates of the perturbed spacetime).

Practical point is the coordinate system of the background is chosen in a way that coincides with the symmetries of the background. Moreover in cosmological perturbaion theory, the background coordinates are chosen in order to correspond with its homogeneity. Physically, that leaves us in two status [2]

1. always have homogeneous transformations for the coordinates of the time (like the switching from the case of cosmic time to a conformal time).

2. the space cordinates in a transformations of the form:

$$x^{i'} = X^{i'}_k x^k, \tag{A.2}$$

where $X^{i'}_k$ is time independent. But we want to keep our previous choice about the coordinates of the FLRW background so that the 3-metric remain Euclidian

$$g_{ij} = a^2 \delta^i_j,$$

that leaves us in rotations. With that the rotations of the backgroud space is established. Then, the transformation matrix will be [2]

$$\mathbf{X}^{\mu'}{}_{\rho} = \begin{bmatrix} 1 & 0 \\ 0 & X^{i'}_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R^{i'}_k \end{bmatrix} \quad \text{and} \quad X^{\mu}_{\rho'} = \begin{bmatrix} 1 & 0 \\ 0 & R^{i}_{k'} \end{bmatrix}.$$

The element $R^{i'}_k$ has the property that $R^T R = I$ so that $R^{i'}_k R^{i'}_l = (R^T R)_{kl} = \delta_{kl}$ where $R^{i'}_k$ is a rotation matrix. We can simply deduce: $R^T = R^{-1}$ so that $R^{i'}_k = R^k_{i'}$.



Taking these coordinate transfromations into the background, that will coorespond to the perturbed space in the form

$$x^{\mu'} = X^{\mu'}_\rho x^\rho, \tag{A.3}$$

and that will produce the metric

$$g_{\mu\nu} = a^2 \begin{bmatrix} -1-2A & -B_i \\ -B_i & (1-2D)\delta_{ij} + 2E_{ij} \end{bmatrix} = a^2 \eta_{\mu\nu} + a^2 \begin{bmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix}.$$

Now, if we transform the metric $g_{\rho'\sigma'} = X^{\mu}_{\rho'} X^{\nu}_{\sigma'} g_{\mu\nu}$ we obtain the following components

$$
\begin{aligned}
g_{0'0'} &= X^{\mu}_{\mu'}X^{\nu}_{\nu'}g_{\mu\nu} = X^{0}_{0'}X^{0}_{0'}g_{00} = g_{00} = a^2(-1-2A), && \text{(A.4)}\\
g_{0'l'} &= X^{\mu}_{0'}X^{\nu}_{l'}g_{\mu\nu} = X^{0}_{0'}X^{j}_{l'}g_{0j} = -a^2R^{j}_{l'}B_j, && \text{(A.5)}\\
g_{k'l'} &= X^{i}_{k'}X^{j}_{l'}g_{ij} = a^2(-2D\delta_{ij}R^{i}_{k'}R^{j}_{l}{}' + 2E_{ij}R^{i}_{k'}R^{j}_{l}{}') = \\
& a^2(-2D\delta_{kl} + 2E_{ij}R^{i}_{k'}R^{j}_{l'}), && \text{(A.6)}
\end{aligned}
$$

so that the perturbations in the new coordinates will be

$$
\begin{aligned}
A' &= A, && \text{(A.7)}\\
D' &= D, && \text{(A.8)}\\
B_{l'} &= R^{j}_{l'}B_j, && \text{(A.9)}\\
E_{k'l'} &= R^{i}_{k'}R^{j}_{l'}E_{ij}. && \text{(A.10)}
\end{aligned}
$$

We conclude that under rotation in the background spacetime coordinates, A and B transform as scalars, $B_i$ as 3-vectors and $E_{ij}$ as 3-tensors. So, as we stay in one gauge as we think of them as scalar, vector and tensor fields on the Euclidian 3-D background space. Based on that, in the Euclidian 3-vector calculus a vector field can be divided into two components; one with zero curl and the other with zero divergence:

$$
\vec{B} = \vec{B}^S + \vec{B}^V \quad , \qquad \text{so that} \qquad \nabla \times \vec{B}^S = 0 \qquad \text{and} \qquad \nabla.\vec{B}^V = 0, \qquad \text{(A.11)}
$$

where $\vec{B}^S$ can be expressed as a gradient of scalar field with negative sign

$$
\vec{B}^S = -\nabla B. \qquad \text{(A.12)}
$$

then in component expression it takes the form

$$
B_i = -B_{,i} + B^V_i \qquad \text{since} \qquad \delta^{ij}B^V_{i,j} = 0. \qquad \text{(A.13)}
$$

Also, a symmetric traceless tensor field $E_{ij}$ can be divided into scalar, vector and tensor in the form

$$
E_{ij} = E^S_{ij} + E^V_{ij} + E^T_{ij}. \qquad \text{(A.14)}
$$

Again we express $E_{ij}^S$ and $E_{ij}^V$ in terms of scalar and tensor fields, $E$ and $E_i$, respectively

$$\begin{aligned} E_{ij}^S &= (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E = E_{,ij} - \frac{1}{3}\delta^{ij}\delta^{kl}E_{,kl}, && \text{(A.15)} \\ E_{ij}^V &= -\frac{1}{2}(E_{i,j} + E_{j,i}) \qquad \text{since} \qquad \delta_{i,j} = \nabla.\vec{E} = 0, && \text{(A.16)} \\ \delta^{ik}E_{ij,k}^T &= 0 \qquad \text{and} \qquad \delta^{ij}E_{ij}^T = 0. && \text{(A.17)} \end{aligned}$$

Based on the above calcualtions, the metric perturbation can be divided into scaler part, which consists of A, B, D and E, vector part, which consists of $B_i^V$ and $E_i$, and tensor part, which consists of $E_{ij}^k$.

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