UNIVERSITY

of the

WESTERN CAPE

DEPARTMENT OF MATHEMATICS

AND

APPLIED MATHEMATICS

A model of pension portfolios with salary and surplus process

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A project report submitted in partial fulfilment of the requirements for the degree of Master of Science

February 2010

Abstract

Essentially this project report is a discussion of mathematical modelling in pension funds, presenting sections from Cairns, A.J.D., Blake, D., Dowd, K., *Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans*, Journal of Economic Dynamics and Control, Volume 30, Issue 2006, Pages 843-877, with added details and background material in order to demonstrate the mathematical methods. In the investigation of the management of the investment portfolio, we only use one risky asset together with a bond and cash as other assets in a continuous time framework. The particular model is very much designed according to the members' preference and then the funds are invested by the fund manager in the financial market. At the end, we are going to show various simulations of these models. Our methods include stochastic control for utility maximisation among others. The optimisation problem entails the optimal investment portfolio to maximise a certain power utility function. We use MATLAB and MAPLE programming languages to generate results in the form of graphs and tables.

JEL Classification: G11; G23; C61
2000 AMS Subject Classification: 91B28

Keywords: Stochastic control theory, optimal investment strategy, model of pension fund, utility function surplus process.

Declaration

I declare that *A model of pension portfolios with salary and surplus process* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Nyika Mtemeri February 2010

Signed:..

Acknowledgement

I would like to express my heartfelt gratitude to Prof. Peter Witbooi for his unwaivering commitment and supervision towards my project. The journey was not easy but would have been worse without moral support and encouragement from all of my friends which are too numerous to mention but I could not do without mentioning Tongai Foto, Walter Mudzimbabwe, Vuyokazi Manona and Yamkela Madikazi, who lifted me to a little higher level so that I could see a little bit further. I thank you all.

I wish to thank the University of the Western Cape, in particular, the Departments of Mathematics and Applied Mathematics, Statistics and Finance for providing facilities and a pleasant atmosphere to conduct this work.

I am also grateful to Dr Lorna Holtman and the Wright family for all kind of support that they have provided me throughout my studies and what is to come.

Last but not least I would like to thank my parents, brothers, sisters and all my friends in Christ for their prayers, inspiring motivation and support.

Dedication...

To my twin brother, Felix.

List of Symbols

$S(t)$: Price of a riskless asset at time t;

r : Constant risk free nominal rate of interest;

$r(t)$: Short-rate interest process at time t;

$R(t)$: Price of a risky asset at time t;

$W(t)$: Wealth of the pension fund at time t;

$\tilde{W}(t)$: Augmented pension wealth at time t;

$\mathcal{A}$: Set of admissible controls;

$Y(t)$: Plan member salary at time t;

$X(t)$: Ratio of the final pension wealth to final salary at time t;

π : Proportion of premium paid from plan member salary at time t;

$Z(t),$: Standard Brownian motion;

$Z_r(t)$: One dimensional Brownian motion;

$\tilde{Z}(t)$: Standard Q-Brownian motion t;

$\tilde{\zeta}$: Market price of the risk;

$B(t,T)$: Price of a bond with maturity T at time t;

$A(t,T)$: Price of a bond with maturity T at time t;

$p(t)$: proportion of the wealth invested at time t;

$q(t)$: proportion of the augmented pension wealth invested at time t;

$Y(t)$: Plan member salary at time t;

σ : Volatility;

μ : Drift coefficient;

$\mathcal{F}$: Information available for pension funds;

Ω : State of the economy;

P : Probability space;

Q : Unique risk-neutral measure;

$U(t)$: Utility function at time t;

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Chapter 1

Introduction

In many countries, pension funds have recently become a popular and more important role player in the financial sector. There are two main types of radically different approaches to pension funds which are; the Defined Contribution (DC) Scheme and Defined Benefits (DB) Scheme which are clearly explained by Cairns [5].

The former is in effect a savings account which the employer establishes for his employees. The employer plays a role only in contributing additional funds to the plan but whatever happens at maturity and afterwards is for the employee to decide for himself or herself. The employee bears all the risk of the funds investment performance. Investment earnings in these retirement plans are not subject to tax until maturity, i.e, the day funds are withdrawn, see Cairns et al [6].

In Defined Benefit, the fund manager holds an obligation to provide a specific annual retirement benefit. The payments are an obligation of the employer, and the assets in the pension fund provide collateral security for the promised benefits [5]. If the investments perform poorly, the pension manager is obliged to make up for the shortfall by contributing additional assets to the fund, and this can be found in the paper by Deelstra et al [10]. Investment earnings in these retirement plans are also not subject to tax until maturity.

Strategic asset allocation for managing equity risk during the accumulation phase, is the most important task of the fund manager. Usually at the beginning, all the individual plan member contributions are invested in equities. Before the retirement age, all the assets are converted to less risky assets at a rate known to be equal to the inverse of the length of convertable period. The reason for doing this is because the fund manager will be risk averse, so he wants to waive the risk of stock market crashes and hedge against such crashes (see the paper [10] of Deelstra et al). On the date of retirement, all the assets will be held in bonds which are then sold to purchase a life annuity. This life annuity should then provide the necessary pension. One of the drawbacks of this strategy is that it has interest rate risk from annuity purchase decision. When it comes to stochastic lifestyling, it intends to provide to the contributor an income which is closely related to his salary just before retirement. The main objective is to aid the contributor in maintaining the same standard of living that he was in immediately prior to retirement, as explained by Cairns et al [5], and Boulier et al [4].

The presentation in this project report is as follows. Chapter 2 covers the mathematical preliminaries that will be used throughout the project. In chapter 3, we present a general overview of pension funds and the benefits associated with each type. We also include a survey of relevant literature. An analysis of the simple stochastic model of Cairns et al. is discussed in chapter 4. In this chapter we outline the concept of optimal expected utility of the pension funds. A more general stochastic model of Cairns et al. is discussed in chapter 5 where the interest rate is a function of time. We simulate the optimal path of risky assets in chapter 6 and show how the wealth process evolves over time. In chapter 7 we formulate without proving the three fund theorem of Cairns-Blake-Dowd. We also show in the case of this theorem, how the wealth process evolves stochastically over time. The main observations are discussed in the concluding chapter 8.

Chapter 2

Preliminaries

2.1 Stochastic processes

In this section we introduce same basic concepts and notation, together with some standard results that will be used in the project report.

The concept of *stochastic process* is fundamental in financial modelling. A particularly basic tool in this regard is the *Brownian motion/Wiener Process*. We use the symbol $W(t)$ to denote a Wiener process. We shall avoid a long discussion and refer to the popular reference book of Øksendal [20]. Also, we refer to the same book for the definition and notation of a stochastic integral. In particular, we shall be using the Itô integral. In this project report, the triple $(\Omega, \mathcal{F}, \mathbb{P})$ will commonly be used to denote a suitable probability space. We shall normally consider a filtration $\mathcal{F}(t)$ of $\mathcal{F}$.

2.2 Itô formula

An Itô formula is basically the sum of an initial value, a time integral and a stochastic integral. A time integral has instantaneous increments whose mean vary over time stochastically. Stochastic integrals also have instantaneous increments whose variance and covariances may vary stochastically over time, as explained by Etheridge [11].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration, satisfying the usual conditions, and let $W(t)$ be a Wiener process on Ω. Consider a C^2 map, $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$. Then the Itô formula states that

$$\begin{aligned} f(t, W(t)) &= f(0, W(0)) + \int_0^t \frac{\partial f}{\partial x}(s, W(s))dW(s) \\ &\quad + \int_0^t \frac{\partial f}{\partial s}(s, W(s))ds + \int_0^t \frac{\partial^2 f}{\partial x^2}(s, W(s))ds. \end{aligned} \tag{2.1}$$

In differential form equation (2.1) is written as

$$df(t, W(t)) = f_x(t, W(t))dW(t) + f_t(t, W(t))dt + \frac{1}{2}f^{''}(t, W(t))dt.$$

We assume that the dynamics of the stock prices movement is described by the following SDE:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t). \tag{2.2}$$

Here μ and σ are assumed to be constants. By applying the Itô formula the explicit solution is

$$S(t) = S(0) \exp\{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)\}.$$

2.3 Ornstein-Uhlenbeck process

This is a conditional mean reverting one-factor process. The strength of this effect is governed by the positive number a which is the speed of adjustment. Assuming that α and σ are constants, then the following process r is an Ornstein-Uhlenbeck process:

$$dr = (\alpha - ar)dt + \sigma dW.$$

If $a = 0$, then $dr = \alpha dt + \sigma dW$ and if $a \neq 0$ then $\alpha = ar - a(b - r)$ for $b = \frac{\alpha}{a}$.

2.4 The finite time horizon stochastic control problem

We follow the approach of Fleming and Soner [12], chapter IV. We consider a time horizon $[s, T]$ for fixed s, T. The notion of Markov process is important in this regard. *A process $x(t)$ is said to be a Markov process if for any sequence of time ticks $s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n$ with $s < s_1$ and $s_n < T$ and an $s_n < t < T$ and any $B \subseteq \mathbb{R}$, we have $\mathbb{P}\left[x(t) \in B | x(s_1), x(s_2), \cdots, x(s_n)\right] = \mathbb{P}\left[x(t) \in B | x(s_n)\right]$.*

We consider an n-dimensional stochastic process of the form:

$$dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \tag{2.3}$$

where $u(t)$ is a control and $W(t)$ is an n dimensional Brownian motion. The values of $u(t)$ are restricted to a given closed set U, and the functions u are restricted to belong to a set $\mathcal{A}$. Let Ψ be a function of two real variables. Now let us define the quantity J as a function of three variables for some functions L and Ψ.

$$J(s, x, u) = \mathbb{E}_{sx}\left[\int_s^T L(t, x(t), u(t))dt + \Psi(T, x(T))\right]. \tag{2.4}$$

Here $\mathbb{E}_{sx}$ means expectation conditional on the event $x(s) = x$. Our problem is to find the maximum of J :

$$V = \max_{\{u\}} J(s, x, u). \tag{2.5}$$

Now let us define $\mathcal{H}$ as follows;

$$\mathcal{H} = \sup_{\{u\}} \mathcal{G} \tag{2.6}$$

where

$$\mathcal{G} = -f(t, x, u) \cdot \text{grad}V - \frac{1}{2}\sum_{i,j} \sigma_{ij} \frac{(\partial)^2}{\partial x_i \partial x_j} V + L(t, x, v),$$

the summation being over all the pairs (i, j), a total of n^2, and with σ_{ij} being the entries of the $n \times n$ matrix σ. More precisely, σ_{ij} is the coefficient of $dW_j(t)$ in the expression for $dx_i(t)$ in the equation (2.3). The solution of the problem (2.5) can be shown (see the book by Fleming and Soner [12] Chapter III.7) to satisfy the so-called Hamilton -Jacobi-Bellman (HJB) equation:

$$-\frac{\partial V}{\partial t} + \mathcal{H} = 0. \tag{2.7}$$

Remark 2.2 In particular, we note that solving the problem (2.5) implies finding the maximum of $\mathcal{G}$.

Chapter 3

General survey of pension funds

In a nutshell, a pension is an arrangement that is meant to provide an income when a person is no longer earning a salary or wage from regular employment. Pension schemes may be set up by employers, governments, insurance companies or other institutions and organisations. Some pension plans will provide for members in the event that they suffer a disability. This may take the form of early entry into a retirement plan for an affected member below the normal retirement age. A lot can be covered in this area but our goal is to just give a brief description of how pension funds work.

3.1 Stochastic versus deterministic lifestyling

According to Cairns et al [5], stochastic lifestyling is when the pension plan manager aims to achieve a retirement pension plan that is closely related to the salary. By so doing, the pension manager enables the plan members to maintain their standards of living after retirement.

Many asset fund managers adopt an asset allocation strategy which invests the entire wealth of the fund in the risky asset over the first period of the pension plan. Then, in the other half towards the lapsing of the contract, the risk portion is gradually reallocated to the risk free asset. The reason for considering this strategy is to prevent the losses in the pension fund's wealth due to stock market crashes toward the end of the pension term. The strategy has its own benefits and drawbacks. One of the disadvantages is that it does not take into account the risk aversion of the individual plan member according to Cairns et al [5]. They outlined that when we look at its expected utility, it appears to be outperformed by a suboptimal strategy. In this suboptimal strategy, the pension plan manager invests over the lifetime at a constant equity fraction. After a period of time, he will additionally consider hedging demand which will be caused by the unfavourable changes in the plan member's salary, as explained by Mark and Davis [23].

The main difference between stochastic lifestyling and deterministic lifestyling is that the projected standard of living and the projected salary are deterministic whilst in stochastic lifestyling, the two are stochastic.

3.2 Attainable benefits

There are various different designs of the benefits and contribution structure but in this project report we consider three specific ones and for other ways of managing a pension funds, we refer to other schools of thought such as Wahal [27].

3.2.1 Defined benefits

A traditional form of a defined benefit pension plan is the final salary plan, under which the pension paid is equal to the number of years worked, multiplied by the member's salary at retirement, multiplied by a factor known as the accrual rate. The final accrued amount is available as a monthly pension or a lump sum. The retirement and possibly other benefits are calculated according to length of service or years of membership in the fund, and average salary over the last few years before retirement. Members of a defined benefit pension fund do not suffer if the fund's performance deteriorates. If return on investment declines, the employer has to make up the difference so that payments to members are maintained at the predetermined level, and for more see the paper by Black and Perold [3].

In an unfunded defined benefit pension, no assets are set aside and the benefits are paid for by the employer or other pension sponsor as and when funds are available. Pension arrangements provided by the state in most countries in the world are unfunded, with benefits paid directly from current workers' contributions. In a funded plan, contributions from the employer, and sometimes also from plan members, are invested in a fund towards meeting the benefits. The future returns on the investments, and the future benefits to be paid, are not known in advance, so there is no guarantee that a given level of contributions will be enough to meet the benefits. Typically, the contributions to be paid are regularly reviewed in a valuation of the plan's assets and liabilities carried out by an actuary, to ensure that the pension fund will meet future payment obligations. This means that in a defined benefit pension, investment risk and investment rewards are typically assumed by the sponsor or employer and not by the individual, as well explained by Cairns et al [5] and Deelstra et al [10].

3.2.2 Defined contribution

A retirement plan in which a certain amount or percentage of money is set aside each year by a company for the benefit of the employee. The benefits in defined contribution plans are tied directly to financial market returns. The contributions are invested, for example in the stock market, and the returns on the investment (which may be positive or negative) are credited to the individual's account. On retirement, the member's account is used to provide retirement benefits, sometimes through the purchase of an annuity which then provides a regular income, as explained by Cairns et al [6].

3.2.3 Targeted money purchase

These are money purchase schemes targeted on the individual needs of scheme members. They aim to target as closely as possible the pension that a member of a final salary scheme would get on his or her chosen retirement date. This objective is achieved by a planned strategy of increasing contribution rates and changing the asset allocation of the fund away from equities towards fixed income securities throughout the life of the scheme.

A targeted money purchase scheme might have an initial contribution rate of, say 7% of salary and be invested entirely in equities. Over time the contribution rate increases reaching, say 19% of salary in the year before retirement. In addition, the accumulating assets in the scheme are gradually transferred into fixed income securities. The aim is to benefit from higher expected return on shares in the early life of the scheme but to reduce the potential volatility in the value of the fund as retirement approaches by re-allocating into less risky bonds.

This targeting strategy however remains approximate and might have difficulty in dealing with sudden and unexpected falls in asset values just prior to retirement or with unexpected early retirement, see Sundaresan et al [26].

Chapter 4

The simple stochastic model of Cairns et al.

Our main goal in this section is to determine the optimal asset allocation method under the simple pension fund model of Cairns et al [5], with a deterministic nominal interest rate. Assets are exposed to different types of risk. In order to avoid losses, hedging comes into play hence some assets will be hedged while others being non-hedgeable. We assume that the financial market is fully hedgeable and complete. Furthermore we also consider one risky asset instead of many. We present the structure of the model and the utility function to the optimised, and then solve the optimisation problem.

4.1 The risky assets

Pension fund managers face the difficulty on how to diversify a portfolio of assets in the financial market in order to maximise the expected returns. One can choose to invest in a best portfolio of assets chosen in such a way that the risk must be at minimum but with high returns. In this section, we consider only two underlying assets, the risk-free asset $R_0(t)$ and one risky asset $R(t)$. The risk-free asset represents the cash fund and the risky asset represents an equity fund. At any given time t, the cash fund is

$$R_0(t) = R_0 \exp\{rt\}.$$

Clearly, the riskless asset increases at an exponential rate $\exp\{rt\}$, where r is the constant interest rate.

The risky asset has a price $R_1(t)$ at time t and the remainder in $R_2(t)$.

Proposition 4.1 *The price $R_1(t)$ of the risky asset at a given time t, is assumed to satisfy the SDE*

$$dR_1(t) = R_1(t)[(r + \tilde{\zeta}_1\sigma_1)dt + \sigma_1 dZ_1(t)] \tag{4.1}$$

where $Z_1(t)$ is a Standard Brownian Motion and $\tilde{\zeta}_1$ and σ_1 are constants. □

Proposition 4.2 *The explicit solution of the equation* (4.1) *is*

$$R_1(t) = R_1(0)\exp\left\{(r + \tilde{\zeta}_1\sigma_1 - \frac{1}{2}\sigma_1^2)t + \sigma_1 Z_1(t)\right\}.$$

Proof. If we let $f(t, R_1(t)) = \log R_1(t)$, then

$$d\log(R_1(t)) = \frac{1}{R_1(t)}dR_1(t) + \frac{1}{2}\left[-\frac{1}{(R_1(t))^2}(dR_1(t))^2\right].$$

From the equation (4.1) above, $(dR_1(t))^2 = (R_1(t))^2(\sigma_1^2 dt)$. Therefore

$$d(\log(R_1(t))) = \left[(r + \tilde{\zeta}_1\sigma_1)dt + \sigma_1 dZ_1(t)\right] - \frac{1}{2}\sigma_1^2 dt.$$

The proof is concluded as follows:

$$\begin{aligned}
\int_0^t d(\log(R_1(s))) &= \int_0^t \left[(r + \tilde{\zeta}_1\sigma_1) - \frac{1}{2}\sigma_1^2\right] ds + \int_0^t \sigma_1 dZ_1(s) \\
\log\left(\frac{R_1(t)}{R_1(0)}\right) &= \left[(r + \tilde{\zeta}_1\sigma_1) - \frac{1}{2}\sigma_1^2\right] t + \sigma_1 Z_1(t) \\
R_1(t) &= R_1(0)\exp\left\{(r + \tilde{\zeta}_1\sigma_1 - \frac{1}{2}\sigma_1^2)t + \sigma_1 Z_1(t)\right\}
\end{aligned}$$

. (4.2)

□

4.2 The plan member's salary

At any given time t, we denote the pension plan member's salary by $Y(t)$. A certain portion of his/her salary is invested in the pension fund. The contributions will then be made continuously up to the date of retirement. In defined contribution pension funds, the idea of considering a fixed interest rate is difficult to accept because the contribution period is very long, generally from 20 to 40 years. It is crucial to allow stochastic term structure for the plan member salary, which is based on the argument by Deelstra et al [10]. The pension fund manager will invest on the basis of the contribution of the plan member. At any given time t, the dynamics of $Y(t)$ is governed by the following SDE,

$$dY(t) = Y(t)\left[(r + \mu_Y)dt + \sigma_{Y_1} dZ_1(t)\right], \tag{4.3}$$

where μ_Y is a constant and $Z_1(t)$ is the same standard Brownian motion, as in section 4.1.

Proposition 4.3 *The explicit solution of equation* (4.3) *is*

$$Y(t) = Y(0)\exp\{(r + \mu_Y - \frac{1}{2}\sigma_{Y_1})t + \sigma_{Y1} Z_1(t)\}.$$

Proof. The proof is similar as for Proposition 4.2.

□

4.3 Asset allocation

Let $p(t)$ be the proportion of the assets invested in the risky asset. At any given time, the pension manager will be concerned about maximising the expected terminal utility of the investment. It becomes a question of optimising the dynamic asset allocation strategy, $p(t)$. We need to find a process which after retirement will give the plan member an equivalent income in order to meet the standard of living, and that will be the main objective of a pension scheme. This was clearly explained in the papers of Cairns et al [6], and Deelstra et al [9], [10].

Stating without proving, the SDE of the wealth process as explained by Cairns et al [5] is

$$dW(t) = W(t)[(r(t) + p(t)'C\tilde{\zeta})dt + p(t)'CdZ(t)] + \pi Y(t)dt.$$

Usually utility is given by a certain real-valued function. Let us define a new state of variable, $X(t) = \frac{W(t)}{Y(t)}$, which is the ratio of the wealth to salary. In this case the utility associated with a given value of $X(t)$ will be calculated as $u(X(t)) = \gamma^{-1}X(t)^{\gamma}$. The final salary $Y(T)$ at retirement age is directly related to the utility or the level of consumption, given exogenously as $(1-\pi)Y(T)$, where π is the proportion of the salary contributed towards the pension fund.

Proposition 4.4 *The dynamics of $X(t)$ is given by the SDE:*

$$\begin{aligned} dX(t) &= \left[\pi + X(t)(-\mu_Y + p(t)\sigma_1(\tilde{\zeta}_1 - \sigma_{Y_1}) + \sigma^2_{Y_1}\right] dt) \\ &+ X(t)(p(t)\sigma_1 - \sigma_{Y_1})dZ(t). \end{aligned} \tag{4.4}$$

Proof. On the defined state variable, $X(t) = \frac{W(t)}{Y(t)}$ we apply the stochastic product rule:

$$dX(t) = d\left(\frac{W(t)}{Y(t)}\right) = \frac{1}{Y(t)}dW(t) + W(t)d\frac{1}{Y(t)} + d\left\langle W(t), \frac{1}{Y(t)}\right\rangle.$$

Now using (4.3)

$$d\left(\frac{1}{Y(t)}\right) = -\frac{1}{Y^2(t)}dY(t) + \frac{1}{Y^3(t)}(dY(t))^2$$

and then computing $(dY(t))^2$ we obtain $Y^2(t)\sigma_Y^2 dt$. This will yield

$$\begin{aligned} d\left(\frac{1}{Y(t)}\right) &= -\frac{1}{Y(t)}\left[(r+\mu_Y)dt + \sigma_{Y_1}dZ(t)\right] - \frac{\sigma_Y^2}{Y(t)}dt \\ &= \frac{1}{Y(t)}\left[-(r+\mu_Y+\sigma_Y^2)dt - \sigma_{Y_1}dZ(t)\right]. \end{aligned} \tag{4.5}$$

Using the above equation,

$$\begin{aligned}
dX(t) &= \frac{W(t)}{X(t)}\Big\{(r+p(t)\tilde{\zeta}_1\sigma_1)dt + p(t)\sigma_1 dZ_1(t)\Big\} + \frac{\pi Y(t)}{Y(t)}dt \\
&+ \frac{W(t)}{Y(t)}\left\{(\sigma_Y^2 - (r+\mu_Y)dt - \sigma_{Y_1} dZ_1(t)\right\} - \sigma_1\sigma_{Y_1}p(t)dt \\
&= X(t)\Big[(r+p(t)\tilde{\zeta}_1\sigma_1)dt + p(t)\sigma_1 dZ_1(t)\Big] + \pi \\
&+ X(t)\left[\sigma_{Y_1}^2 - (r+\mu)\right]dt - X(t)\sigma_{Y_1}dZ_1 - X(t)\sigma_1\sigma_{Y1}p(t)dt \\
&= \Big[\pi + X(t)(-\mu_Y + p(t)\sigma_1(\tilde{\zeta}_1 - \sigma_{Y_1}) + \sigma_{Y_1}^2\Big]dt \\
&+ X(t)(p(t)\sigma_1 - \sigma_{Y_1})dZ(t).
\end{aligned}$$

□

4.4 Market price premiums

Considering that we have our contribution rate, $\pi > 0$, this shows that the stream of payments of premiums is regular. Due to the properties of completeness of a market, we are able to completely hedge the future premiums.

Remark 4.1 The market price at time t for the premiums payable between time $[t, T]$, (i.e. their discounted value) can be written as explained by these two papers of Cairns et al [5] and Deelstra et al [10]

$$\mathcal{R} = \mathbb{E}_{\mathbb{Q}}\left[\int_t^T \exp\left(-r(s-t)\right)\pi Y(s)ds|\mathcal{F}_t\right]. \tag{4.6}$$

Proposition 4.5 *For $\mathcal{R}$ as given in equation (4.6), the value is*

$$\mathcal{R} = \pi Y(t) f(t) \tag{4.7}$$

where

$$f(t) = \frac{\exp((\mu_Y - \tilde{\zeta}_1\sigma_{Y_1})(s-t)) - 1}{\mu - \tilde{\zeta}_1\sigma_{Y_1}}.$$

Proof. Consider the quantity $\widetilde{E}_{\mathbb{Q}}$ to be the expectation under risk-neutral probability measure $\mathbb{Q}$ which happens to be equivalent to the real world probability measure.

Under $\mathbb{Q}$,

$$\begin{aligned} dR_1(t) &= R_1(t)\left[(r+\tilde{\zeta}_1\sigma_1)dt+\sigma_1 dZ_1(t)\right] \\ &= R_1(t)\left[rdt+\sigma_1(\tilde{\zeta}_1 dt+dZ_1(t))\right] \\ &= R_1(t)\left[rdt+\sigma_1 d\tilde{Z}_1(t)\right]. \end{aligned} \quad (4.8)$$

This is obtained by using the Girsanov theorem found in the book by Etheridge [11], and hence $\tilde{Z}_1 = \tilde{\zeta}_1 + Z_1(t)$, eventually $\frac{d\mathbb{Q}}{d\mathbb{P}} = \tilde{Z}_1(t) = \exp\{-1/2(\tilde{\zeta}_1(t)) - \tilde{\zeta}_1 Z_1(t)\}$. Under the same probability measure,

$$\begin{aligned} dY(t) &= Y(t)\left[(r+\mu_Y-\tilde{\zeta}_1\sigma_{Y_1})dt+\sigma_{Y_1}d\tilde{Z}_1(t)\right] \\ &= Y(0)\exp\{(r+\mu_Y-\tilde{\zeta}_1\sigma_{Y_1}-\frac{1}{2}\sigma_{Y_1}^2)t-\tilde{\zeta}_1 Z_1(t)\}, \end{aligned} \quad (4.9)$$

where $\tilde{Z}_1(t)$ is a Standard $\mathbb{Q}$ -Brownian Motion.

$$\begin{aligned}
\mathcal{R} &= \widetilde{\mathbb{E}}_{\mathbb{Q}}\left[\int_t^T \exp\{-r(s-t)\}\pi Y(s)\,ds\middle|\mathcal{F}_t\right] \\
&= \widetilde{\mathbb{E}}_{\mathbb{Q}}\left[\int_t^T \exp\{-r(s-t)\}\pi Y(t)\exp\{(r+\mu_Y-\tilde{\zeta}_1\sigma_{Y_1}-\frac{1}{2}\sigma_Y^2)s+\sigma_{Y_1}\tilde{Z}_1(s)\}ds\middle|\mathcal{F}_t\right] \\
&= \pi Y(t)\int_t^T \exp\{-r(s-t)\}\exp\{(r+\mu_Y-\tilde{\zeta}_1\sigma_{Y_1}-\frac{1}{2}\sigma_Y^2)(s-t)\} \\
&\cdot \widetilde{E}_{\mathbb{Q}}(\exp\{\sigma_{Y_1}\tilde{Z}_1(s-t)\}\Big|\mathcal{F}_t)ds \\
&= \pi Y(t)\int_t^T \exp\{-r(s-t)\}\exp\{(r+\mu_Y-\tilde{\zeta}_1\sigma_{Y_1}-\frac{1}{2}\sigma_Y^2)(s-t)\} \\
&\cdot \widetilde{E}_{\mathbb{Q}}(\exp\{\sigma_{Y_1}\tilde{Z}_1(s-t)\})ds \\
&= \pi Y(t)\int_t^T \exp\{-r(s-t)\}\exp\{(r+\mu_Y-\tilde{\zeta}_1\sigma_{Y_1}-\frac{1}{2}\sigma_Y^2)(s-t)\} \\
&\cdot \exp\{\frac{1}{2}\sigma_{Y_1}^2(s-t)\}ds \\
&= \pi Y(t)\int_t^T \exp\{(\mu_Y-\tilde{\zeta}_1\sigma_{Y_1})(s-t)\}ds \\
&= \pi Y(t)\frac{1}{(\mu_Y-\tilde{\zeta}_1\sigma_{Y_1})}\exp\{(\mu_Y-\tilde{\zeta}_1\sigma_{Y_1})(s-t)\}\Big|_t^T \\
&= \pi Y(t)\left[\frac{\exp((\mu_Y-\tilde{\zeta}_1\sigma_{Y_1})(s-t))-1}{\mu-\tilde{\zeta}_1\sigma_{Y_1}}\right] \\
&= \pi Y(t)f(t).
\end{aligned}$$

$\square$

This market price will give us an access to work with the future premiums as if they were part of the current assets of the pension plan. With this, it then leads us to what we call the augmented pension wealth, which is well explained in the paper by Boulier et al [4].

4.5 Augmented pension wealth

The idea behind this augmented pension wealth rests on the neoclassical theory. From this, we can say that the current value of the asset should be equal to the present value of the salary inflows. The total net of the pension wealth is viewed as the capitalised value of future benefits. The net value is used here in order to be consistent with the neoclassical notion of wealth. For our augmented pension wealth, which we denote by $\tilde{W}(t)$, we have, as found in theory of constant proportion portfolio insurance by Black and Perold [3],

$$\tilde{W}(t) = W(t) + \pi Y(t) f(t),$$

where $\tilde{W}(t)$ is the augmented pension wealth.
Solving $\tilde{W}(t)$, using stochastic product rule we are going to yield

$$\begin{aligned}
d\tilde{W}(t) &= dW(t) + \pi f(t) dY(t) + \pi Y(t) df(t) + \pi < Y(t), f(t) > \\
&= dW(t) + \pi f(t)\left[Y(t)\left((r+\mu_Y)dt + \sigma_{Y_1} dZ_1(t)\right)\right] \\
&+ \pi Y(t)\left[-\exp\left((\mu_Y - \tilde{\zeta}_1 \sigma_{Y_1})(T-t)\right)\right] \\
&= W(t)[(r + p(t)\tilde{\zeta}_1\sigma_1)dt + p(t)\sigma_1 dZ_1(t)] + \pi Y(t)dt + \pi f(t)[Y(t)(r+\mu_Y)dt \\
&+ \sigma_{Y_1} dZ_1(t))] + \pi Y(t)\{-\exp\left[\left(\mu_Y - \tilde{\zeta}_1\sigma_{Y_1}\right)(T-t)\right] dt\} \\
&= [W(t)(r + p(t)\tilde{\zeta}_1\sigma_1) + \pi Y(t) + \pi Y(t) f(t)(r+\mu_Y) \\
&+ \pi Y(t)\{-\exp(\mu_Y - \tilde{\zeta}_1\sigma_{Y_1})(T-t)\}]dt + [W(t)p(t)\sigma_1 \\
&+ \pi Y(t) f(t) \sigma Y_1] dZ_1(t).
\end{aligned}$$

Synthetic asset

We use the concept of complete market to construct a self financing strategy of synthetic asset, $R_2(t)$. Its dynamics is then given by the following SDE

$$\frac{dR_2(t)}{R_2(t)} = (r + \tilde{\zeta}_1 \sigma_{Y_1})dt + \sigma_{Y_1} dZ_1(t),$$

where r is the riskless rate, σ_{Y_1} is the volatility and $Z_1(t)$ is the Brownian motion. There is a perfect correlation between salary risk and the synthetic asset and hence it can be fully used to hedge the stream of contributions.

The augmented wealth $\tilde{W}(t)$ is going to be divided into proportions. Let $q(t)$ be the proportion invested in the asset with the price $R_1(t)$ and the remainder be the proportion invested in the asset with the price $R_2(t)$.

4.6 Optimal expected utility

The utility function that we consider on wealth is $u(\tilde{W}(T)) = \gamma^{-1}\tilde{W}(T)^{\gamma}$. We seek to maximise the expected terminal utility. Let us write

$$J(t, x, r, p) = \mathbb{E}\left[u\left(W(T)\right) \mid X(t) = x\right].$$

We seek to

$$\text{maximize}_p J(t, x, r; p) \tag{4.10}$$

with

$$dr = \mu_r dt + \sigma_{r_1} dZ_1 + \sigma_{r_2} dZ_2 \tag{4.11}$$

$$dx = \beta dt + \alpha_1 dZ_1 + \alpha_1 dZ_2 \tag{4.12}$$

where $\beta = X(-\mu_Y + p^{'}C(\tilde{\zeta} - \sigma_Y) + \sigma_Y^{'}\sigma_Y) + \pi$
and $\alpha = (p^{'}C - \sigma_Y^{'})$.

Here p is the portfolio, the optimal one of which will be denoted by p^*. Let $V(t)$ be the value function corresponding to the problem (4.10). The following proposition describes p^*.

Proposition 4.6 *The optimal path* p^* *solving the problem* (4.10) *satisfies the equation*

$$p^*(t,x,r;V) = C^{'-1}\left(\sigma_Y - (\tilde{\zeta} - \sigma_Y)\frac{V_x}{xV_{xx}} - \sigma_r(r)\frac{V_{rx}}{xV_{xx}}\right). \tag{4.13}$$

Proof. The HJB equation (see (2.7)) for this problem is $-\frac{\partial V}{\partial t} + \mathcal{H} = 0$. $\mathcal{G}(V)$ is given by

$$\mathcal{G}(V) = \mu_r V_r + \mu_x V_x + \frac{1}{2}\upsilon_{rr}V_{rr} + \upsilon_{rx}V_{rx} + \frac{1}{2}\upsilon_{xx}V_{xx}. \tag{4.14}$$

In order to maximise $\mathcal{G}$, we differentiate $\mu_x^p, \mu_r^p, \upsilon_{rr}^p, \upsilon_{rx}^p, \upsilon_{xx}^p$ with respect to p_1 and p_2 and equate to zero to obtain the vector expression.

Thus

$$\begin{aligned} \mu_x^{'p} &= xC(\tilde{\zeta} - \sigma_Y) & (4.15)\\ \upsilon_r x^{'p} &= C\sigma_r(r)x & (4.16)\\ \upsilon_{xx}^{'p} &= 2CC^{'}p - C\sigma_Y, & (4.17) \end{aligned}$$

and we obtain the vector equation

$$\mathcal{G}^{'}(V) = \mu_r V_r + V_x(xC(\tilde{\zeta} - \sigma_Y)) + xC\sigma_r(r)V_{rx} + \frac{1}{2}x^2(2CC^{'}p - 2C\sigma_Y)V_{xx}. \tag{4.18}$$

Equating to zero and making p the subject, it then becomes

$$\begin{aligned} x^2CC^{'}pV_{xx} &= x^2C\sigma_Y V_{xx} - \mu_r V_r + V_x(xC(\tilde{\zeta} - \sigma_Y)) - xC\sigma_r(r)V_{rx} \\ p^*(t,x,r;V) &= C^{'-1}\left(\sigma_Y - (\tilde{\zeta} - \sigma_Y)\frac{V_x}{xV_{xx}} - \sigma_r(r)\frac{V_{rx}}{xV_{xx}}\right). & (4.19) \end{aligned}$$

□

The strategy with respect to augmented wealth

Let $q(t)$ denote the proportion of the augmented wealth invested in the risky asset. We start off with the assumption that the value function associated with maximisation of u, takes the form:

$$V(t, X(t)) = h(t)(\tilde{X(t)})^{\gamma}, \tag{4.20}$$

$$V(T, X(T)) = \left[\frac{1}{\gamma}(\tilde{X(T)})^{\gamma}\right] = \mathbb{E}\left[u(\tilde{W}(T), Y(T)\right].$$

$$V = \max_{q^*} \mathbb{E}\left[\frac{1}{\gamma}(\tilde{X})^{\gamma}\right].$$

Using the HJB-equation,

$$\frac{\partial V}{\partial t} = \max_{q^*}\left\{V'\mu + \frac{1}{2}\sigma^2 V''\right\},$$

thus we need,

$$\frac{\partial}{\partial q^*}\left(V' + \frac{1}{2}\sigma^2 V''\right) = 0.$$

Solving for the above, $V' = (\tilde{X})^{(\gamma-1)}$ and $V'' = (\gamma-1)\tilde{X}^{(\gamma-2)}$ and eventually we will obtain

$$0 = V'\tilde{X}(\tilde{\zeta}_1 - \sigma_{Y_1})(\sigma_1 - \sigma_{Y_1}) + \frac{1}{2}V''(2q^*\tilde{X}^2(\sigma_1 - \sigma_{Y_1})^2).$$

This is done in such a manner that a bigger portion is in R_1 and the remainder of it invested in R_2, see Cairns et al [5]. Now the value of $q^*(t)$ is then given by

$$\begin{aligned} q^*(t, \tilde{X}(t)) &= -\left(\frac{V'\tilde{X}(\tilde{\zeta} - \sigma_{Y_1})(\sigma_1 - \sigma_{Y_1})}{V'\tilde{X}^2(\sigma_1 - \sigma_{Y_1})^2}\right) \\ &= \left[\frac{\tilde{X}}{1-\gamma} \cdot \frac{\tilde{X}(\tilde{\zeta}_1 - \sigma_{Y_1})(\sigma_1 - \sigma_{Y_1})}{\tilde{X}^2(\sigma_1 - \sigma_{Y_1})^2}\right] \\ &= \frac{(\tilde{\zeta}_1 - \sigma_{Y_1})(\sigma_1 - \sigma_{Y_1})}{(1-\gamma)(\sigma_1 - \sigma_{Y_1})^2} \\ &= \frac{(\tilde{\zeta}_1 - \sigma_{Y_1})}{(1-\gamma)(\sigma_1 - \sigma_{Y_1})}. \end{aligned}$$

This is then the partitioning into two different amounts invested in different portions. The function $q(t, \tilde{X}(t))$ is basically dependent on two variables, time t and $\frac{W(t)}{Y(t)}$. Eventually, we can simply determine the amount of pension wealth which is to be invested in proportions but being expressed in the form of $Y(t)$. This is because the plan member's salary appears to be the back bone of the wealth investments.

To illustrate this, we consider the following set of parameters,

$\mu_y = 0, \tilde{\zeta}_1 = 0.2, \sigma_1 = 0.2, \sigma_{Y1} = 0.05, \pi = 0.1, T = 20.$

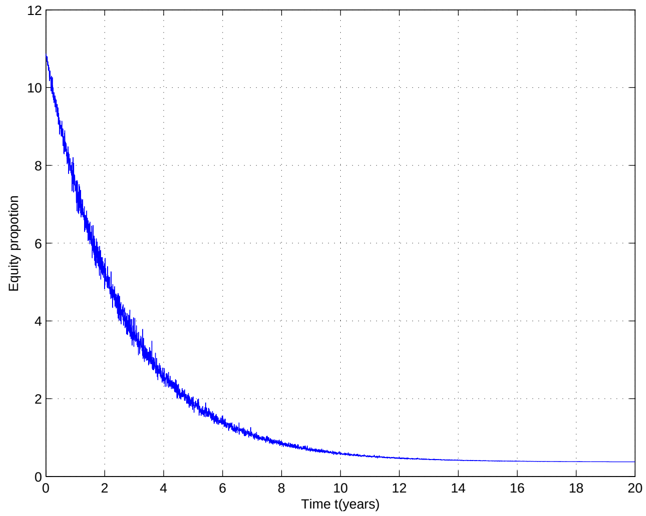

Figure 4.1: Optimal equity proportions

Considering long term-term average salary increase, our value for $\mu_y = 0$, is reasonable good enough. This is beacause the average long-run interest rates is almost similar to long term-term average salary increase. $r = 0.06$ is the nominal interest rate. Since we are assuming power utility, setting $\pi = 0.1$ wont be of any loss of generality.

Using Matlab, the simulated path converges to a very small value 0.375, which shows merely a stochastic lifestyling. The value of the equity proportion will be higher when t is low and this is because $X(t)$ will also be lower, almost 0.375. From the fact that the value is lower at the beginning causes a greater volatility which shows an operation of stochastic lifestyling. On that note, the optimal equity proportion varies stochastically on each time interval.

Chapter 5

The more general stochastic model of Cairns et al.

In this chapter, we focus on the discussion of the more general stochastic model of Cairns et al [5]. As compared to what we had in the previous chapter, in this case introduce a stochastic risk-free nominal rate of interest $r(t)$. Instead of working with n risky assets as proposed in the original paper, we work with only 2 risky assets which are a stock and a bond.

5.1 Risk free interest rates

Decomposing the risk-free rate of interest with only 2 risk-assets, the time homogeneous SDE is

$$\frac{dr(t)}{r(t)} = \mu_r dt + \left[\sigma_{r_1} dZ_1(t) + \sigma_{r_2} dZ_2(t)\right], \tag{5.1}$$

with $Z_1(t)$ and $Z_2(t)$ being two independent Brownian motions so that r is now a function of t.

Proposition 5.1 *The explicit formula for $r(t)$ is the following*

$$r(t) = r(0) \exp\left\{\left[\mu - \frac{1}{2}(\sigma_{r_1}^2 + \sigma_{r_2}^2)\right] t + \sigma_{r_1} Z_1(t) + \sigma_{r_2} Z_2(t)\right\}.$$

Proof. We apply the multifactor Itô formula, see for instance [11], section 7.2, on the function $r(t)$ stated in the proposition.

Then:

$$\begin{aligned} dr(t) &= r(t)\left[\mu - \frac{1}{2}(\sigma_{r_1}^2 + \sigma_{r_2}^2)\right] + \sigma_{r_1} r(t) dZ_1 + \sigma_{r_2} r(t) dZ_2 \\ &\quad + \frac{1}{2}\sigma_{r_1}^2 r(t) dt + \frac{1}{2}\sigma_{r_2}^2 r(t) dt \\ &= r(t)\left[\mu - \frac{1}{2}(\sigma_{r_1}^2 + \sigma_{r_2}^2)\right] + \sigma_{r_1} r(t) dZ_1 + \sigma_{r_2} r(t) dZ_2 \\ &\quad + \frac{1}{2} r(t)\left[\sigma_{r_1}^2 + \sigma_{r_2}^2\right] dt. \end{aligned}$$

Thus equation (5.1) follows from the given $r(t)$. □

Our cash is the risk-free asset and is to be subjected to the risk-free nominal rate of interest defined by $r(t)$. To illustrate this, we are going to make use of the simulations, figure 5.1 below, based on the Euler method as in Cyganowski et al [7]. This

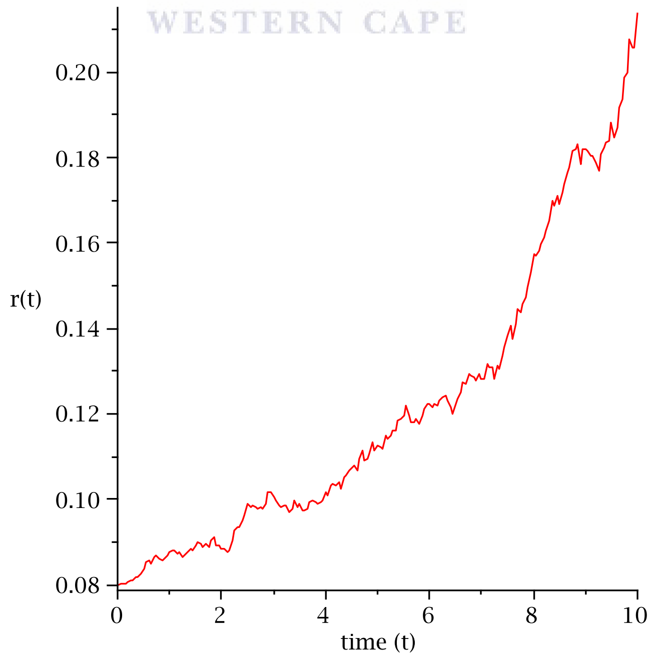

Figure 5.1: Interest rate

shows that the interest rate is affected by the change in time, with time in years, and its movement is stochastic. Figure 5.1 illustrates a simulation of $r(t)$, where the parameters are assigned values as $\sigma_{r_1} = 0.02, \sigma_{r_2} = 0.05, \mu_r = 0.1, r_0 = 0.08$.

Proposition 5.2 *The value of the riskless asset, R_0 satisfies the equation*

$$\frac{R_0(t)}{R_0(0)} = \exp \int_0^t r(s)ds.$$

Proof. This is because $dR_0(t) = r(t)R_0(t)dt$. □

5.2 Risky assets

Different assets are invested in the market and each one of them having a different return. Let R_i be the total investment into the i^{th} asset. With only two risky assets, the two assets evolves according to the SDE given by

$$\frac{dR_i(t)}{R_i(t)} = \left(r(t) + \sigma_{i_1}\tilde{\zeta}_1 + \sigma_{i_2}\tilde{\zeta}_2\right)dt + \sigma_{i_1}dZ_1(t) + \sigma_{i_2}dZ_2(t).$$

Proposition 5.3 *The explicit formula for $R_i(t)$ is the following*

$$R_i(t) = R_i(0)\exp\left\{\left(r(t) + \sigma_{i_1}\tilde{\zeta}_1 + \sigma_{i_2}\tilde{\zeta}_2 - \frac{1}{2}(\sigma_{i_1}^2 + \sigma_{i_2}^2)\right)t + \sigma_{i_1}Z_1(t) + \sigma_{i_2}Z_2(t)\right\}.$$

Proof. Again this is as Proposition 5.1 and we skip the detail. □

5.3 Plan member salary

The plan member's salary is denoted by $Y(t)$ is going to evolve in accordance to a given SDE. Doing the computations, we use the SDE

$$\frac{dY(t)}{Y(t)} = \left[(r(t) + \mu(t)dt + \sigma_{Y_1}dZ_1(t) + \sigma_{Y_2}dZ_2(t)\right]. \tag{5.2}$$

Proposition 5.4 *The explicit formula for $Y(t)$ is the following*

$$Y(t) = Y(0)\exp\left\{(r(t) + \mu(t) - \frac{1}{2}(\sigma_{Y_1}^2 + \sigma_{Y_2}^2)\right\}dt + \sigma_{Y_1}dZ_1(t) + \sigma_{Y_2}dZ_2(t).$$

Proof. Again this is as for Proposition 5.1 and we skip the detail. □

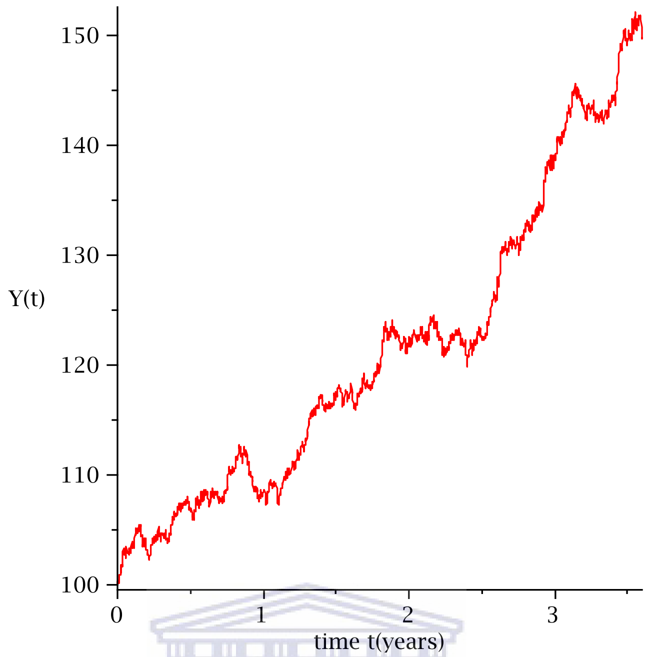

Figure 5.2: Plan member salary

To illustrate this, we are going to make use of the figure 5.2 above.

From figure 5.2 the plan member salary therefore evolve stochastically and in this case our parameters are assigned values as $\sigma_{Y1} = 0.002$, $\sigma_{Y2} = 0.05$, $\sigma_{r_1} = 0.05$, $\sigma_{r_2} = 0.01$, $\mu_Y = 0.005$, $\mu_r = 0.005, r_0 = 0.01$.

Chapter 6

Simulating the optimal path of risky assets

In this chapter, we are going to perform simulations of the paths to show how our risky assets stochastically behave in a market. We will get into more detail on the simulations of stochastic lifestyling process and show the randomness of the assets.

6.1 Optimal equity proportion

This is the proportion amount invested in optimal investment in equities and the value of $X(0)$ will never attain the value of zero. From this, we can say $X(0) = 0$ is the asymptote for the horizontal axis. This implies that since $X(0)$ is indomitable by the salary of the plan member and the wealth at time T, there is no way it can ever be nothing. The value of this proportion will start at a high level, provided that $\frac{\sigma_1 - \sigma_{Y1} q^*}{\sigma_1}$ is above zero. The dynamics will ultimately show that there will be lower drift as $f(t)$ dwindles and $X(t)$ increases. As $X(0)$ tends to infinity, the value of $p^*(0, X(0))$ tend to $\frac{\sigma_1 - \sigma_{Y1} q^*}{\sigma_1}$, which shows is stochastic otherwise it is deterministic. Now if we express the relative risk aversion with parameters $\mu_Y = 0, \tilde{\zeta}_1 = 0.2, \sigma_{Y1} = 0.05, \pi = 0.1$ and $T = 20$. Since for our asymptotic value, as $X(0)$ tends to infinity, the value of $p^*(0, X(0))$ is then 0.375. This will then give all the support to depict the optimal asset-allocation as in stochastic lifestyling.

6.2 Optimal asset allocation

The optimal asset allocation of the more general stochastic model is derived step by step from the HJB equation. It is then going to take the form given by

$$p^*(t,x,r;V) = C'^{-1}\left(\sigma_Y - (\tilde{\zeta} - \sigma_Y)\frac{V_x}{xV_{xx}} - \sigma_r(r)\frac{V_{xr}}{xV_{xx}}\right), \tag{6.1}$$

where $\frac{V_x}{xV_{xx}} = \frac{x+f(t)}{\gamma-1}$ and $\frac{V_{xr}}{xV_{xx}} = B(\gamma,(T-t)) = \gamma d e^{-\alpha_r(T-t)}$.
Simplifying this, we obtain:

$$\begin{aligned}
p^*(t,x,r;V) &= \Delta \cdot \left[\begin{pmatrix} \sigma_{Y1} \\ \sigma_{Y2} \end{pmatrix} - \left(\begin{pmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{pmatrix} - \begin{pmatrix} \sigma_{Y1} \\ \sigma_{Y2} \end{pmatrix}\right)\right. \\
&\quad \left. \cdot \left(\frac{x+f(t)}{\gamma-1}\right) - \begin{pmatrix} \sigma_r r_1 \\ \sigma_r r_2 \end{pmatrix} \cdot \left(\gamma d_1 e^{-\alpha_r(T-t)}\right)\right] \\
&= \Delta \cdot \left[\begin{pmatrix} \sigma_{Y1} - \left(\frac{x+f(t)}{\gamma-1}\right)\left(\tilde{\zeta}_1 - \sigma_{Y1}\right) \\ \sigma_{Y2} - \left(\frac{x+f(t)}{\gamma-1}\right)\left(\tilde{\zeta}_2 + \sigma_{Y2}\right) \end{pmatrix}\right. \\
&\quad \left. - \begin{pmatrix} \left(\gamma d_1 e^{-\alpha_r(T-t)}\right)\sigma_r r_1 \\ \left(\gamma d_1 e^{-\alpha_r(T-t)}\right)\sigma_r r_2 \end{pmatrix}\right] \\
&= \Delta \cdot \begin{pmatrix} \sigma_{Y1} - \left(\frac{x+f(t)}{\gamma-1}\right)\left(\tilde{\zeta}_1 - \sigma_{Y1}\right) - \left(\gamma d_1 e^{-\alpha_r(T-t)}\right)\sigma_r r_1 \\ \sigma_{Y2} - \left(\frac{x+f(t)}{\gamma-1}\right)\left(\tilde{\zeta}_2 + \sigma_{Y2}\right) - \left(\gamma d_1 e^{-\alpha_r(T-t)}\right)\sigma_r r_2 \end{pmatrix} \\
&= \begin{pmatrix} k_1 \\ k_2 \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
k_1 = \frac{1}{c_1c_4 - c_2c_3}&\left[c_4\left(\sigma_{Y1} - \left(\frac{x+f(t)}{\gamma-1}\right)\left(\tilde{\zeta}_1 - \sigma_{Y1}\right) - \left(\gamma d_1 e^{-\alpha_r(T-t)}\right)\sigma_r r_1\right)\right. \\
&\left. -c_3\left(\sigma_{Y2} - \left(\frac{x+f(t)}{\gamma-1}\right)\left(\tilde{\zeta}_2 + \sigma_{Y2}\right) - \left(\gamma d_1 e^{-\alpha_r(T-t)}\right)\sigma_r r_2\right)\right],
\end{aligned}$$

$$k_2 = \frac{1}{c_1c_4 - c_2c_3}\left[-c_2\left(\sigma_{Y1} - \left(\frac{x+f(t)}{\gamma-1}\right)\left(\tilde{\zeta}_1 - \sigma_{Y1}\right) - \left(\gamma d_1 e^{-\alpha_r(T-t)}\right)\sigma_r r_1\right)\right.$$
$$\left. + c_1\left(\sigma_{Y2} - \left(\frac{x+f(t)}{\gamma-1}\right)\left(\tilde{\zeta}_2 + \sigma_{Y2}\right) - \left(\gamma d_1 e^{-\alpha_r(T-t)}\right)\sigma_r r_2\right)\right]$$

and

$$\Delta = \frac{1}{c_1c_4 - c_2c_3}\begin{pmatrix} c_4 & -c_3 \\ -c_2 & c_1 \end{pmatrix}.$$

6.3 Wealth process of Cairns et al.

We consider the wealth process given in the paper of Cairns et al [5] which thereof is given by

$$dW(t) = W(t)[(r(t) + p(t)'C\tilde{\zeta})dt + p(t)'CdZ(t)] + \pi Y(t)dt. \tag{6.2}$$

The optimal process $p(t)$ which is a vector is now being substituted by $p^*(t, x, r, V)$. Substituting this we are going to obtain

$$\begin{aligned} dW(t) &= W(t)[(r(t) + p^*(t,x,r,V)C\tilde{\zeta})dt + p^*(t,x,r,V)CdZ(t)] + \pi Y(t)dt \\ &= [(r(t) + C'^{-1}(\sigma_Y - (\tilde{\zeta} - \sigma_Y)\frac{V_x}{xV_{xx}} - \sigma_r(r)\frac{V_{xr}}{xV_{xx}}C\tilde{\zeta})dt \\ &+ C'^{-1}(\sigma_Y - (\tilde{\zeta} - \sigma_Y)\frac{V_x}{xV_{xx}} - \sigma_r(r)\frac{V_{xr}}{xV_{xx}}CdZ(t)] + \pi Y(t)dt, \end{aligned} \tag{6.3}$$

where $V(t,x,r)$ is in the form $\gamma^{-1}g(t,r)^{1-\gamma}\cdot(x+\pi f(t))^{\gamma}$. In this regards, our $V(t,x,r)$ is going to take the form $\gamma e^{\gamma g(t,x,r)}\cdot(x+\pi f(t))^{\gamma}$.

$$
\begin{aligned}
dW(t) = W(t) & \left[\left(r(t) + \begin{pmatrix} p_1^* & p_2^* \end{pmatrix} \cdot \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \cdot \begin{pmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{pmatrix}\right) dt + \begin{pmatrix} p_1^* & p_2^* \end{pmatrix} \cdot \right. \\
& \left. \cdot \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} dZ(t_1) \\ dZ(t_2) \end{pmatrix} \right] + \pi Y(t)dt \\
= W(t) & \left[\left(r(t) + \begin{pmatrix} k_1 & k_2 \end{pmatrix} \cdot \begin{pmatrix} c_1\tilde{\zeta}_1 + c_2\tilde{\zeta}_2 \\ c_3\tilde{\zeta}_1 + c_4\tilde{\zeta}_2 \end{pmatrix}\right) dt + \begin{pmatrix} k_1 & k_2 \end{pmatrix} \cdot \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \cdot \right. \\
& \left. \begin{pmatrix} dZ(t_1) \\ dZ(t_2) \end{pmatrix} \right] + \pi Y(t)dt \\
= W(t) & \left[\left(r(t) + \left(k_1\left(c_1\tilde{\zeta}_1 + c_2\tilde{\zeta}_2\right) + k_2\left(c_3\tilde{\zeta}_1 + c_4\tilde{\zeta}_2\right)\right)\right) dt + \right. \\
& \left. \begin{pmatrix} k_1c_1 + k_2c_3 & k_1c_2 + k_2c_4 \end{pmatrix} \cdot \begin{pmatrix} dZ(t_1) \\ dZ(t_2) \end{pmatrix} \right] + \pi Y(t)dt \\
= W(t) & \left[\left(r(t) + \left(k_1\left(c_1\tilde{\zeta}_1 + c_2\tilde{\zeta}_2\right) + k_2\left(c_3\tilde{\zeta}_1 + c_4\tilde{\zeta}_2\right)\right)\right) dt \right. \\
& \left. + \left(k_1c_1 + k_2c_3\right) dZ(t_1) + \left(k_1c_2 + k_2c_3\right) dZ(t_2) \right] + \pi Y(t)dt.
\end{aligned}
\tag{6.4}
$$

At this juncture, we are going to quote the theorem in Cairns et al [5] and use it to evaluate the value of the function (6.3). Evaluating $V(t,x,r)$ which is now given by

$$V(t,x,r) = \frac{1}{\gamma}e^{A(\gamma,T-t)+\gamma\varphi(\gamma)(T-t)} \cdot e^{B(\gamma,T-t)}(x+\pi f(t))^{\gamma}, \tag{6.5}$$

then

$$g(t,x,r)^{1-\gamma} = e^{[A(\gamma,T-t)+B(\gamma,T-t)r(t)+\gamma\varphi(\gamma)(T-t)]}. \tag{6.6}$$

Using (6.5) and (6.6) we are going to yield the value of $V(t,x,r)$ in the form given by Cyganowski et al [7]

$$V(t,x,r) = \gamma e^{\gamma(t,x,r)} \cdot (x+f(t))^{\gamma}. \tag{6.7}$$

The function $f(t)$ is given by

$$\begin{aligned} f(t) &= \int_t^T e^{-\sigma_Y' \tilde{\zeta}(s-t)} ds \\ &= e^{\sigma_Y' \tilde{\zeta} t} \int_t^T e^{-\sigma_Y' \tilde{\zeta} s} ds \\ &= \frac{1}{\sigma_Y' \tilde{\zeta}} [1 - e^{-\sigma_Y' \tilde{\zeta} t(T-t)}]. \end{aligned} \tag{6.8}$$

The detail of γ is for our purpose unimportant, except that,

$$\frac{\partial}{\partial r} V(t, x, r) = B(\gamma, T - \varepsilon). \tag{6.9}$$

In this regards, the value of B is the same as in (6.5), we say it is unimportant because there will be much cancellation in simplifying the expression for the optimal proportion p^*.

Using the equation (6.5), its first derivative with respect to x is given by

$$V_x = \gamma e^{\gamma} (x + f(t))^{\gamma - 1}. \tag{6.10}$$

Furthermore, the second derivative with respect to x is

$$V_{xx} = \gamma(\gamma - 1) e^{\gamma} (x + f(t))^{\gamma - 2}. \tag{6.11}$$

Lastly, the partial derivative with respect to x and then r will be

$$V_{xr} = \gamma e^{\gamma} \frac{\partial}{\partial r} ((x + f(t))^{\gamma - 1}). \tag{6.12}$$

Considering (6.10) and (6.11), we are going to obtain

$$\begin{aligned} \frac{V_x}{V_{xx}} &= \frac{\gamma e^{\gamma} (x + f(t))^{\gamma - 1}}{\gamma(\gamma - 1) e^{\gamma} (x + f(t))^{\gamma - 2}} \\ &= \frac{x + f(t)}{\gamma - 1} \\ &= \frac{1}{\gamma - 1} (x + f(t)), \end{aligned} \tag{6.13}$$

and (6.12) and (6.11), we are going to obtain

$$\begin{aligned}\frac{V_{xr}}{V_{xx}} &= \frac{\gamma e^{\gamma}\frac{\partial\gamma}{\partial r}(x+f(t))^{\gamma-1}}{\gamma(\gamma-1)e^{\gamma}(x+f(t))^{\gamma-2}} \\ &= \frac{\partial}{\partial r}\left(\frac{(x+f(t))}{\gamma-1}\right) \\ &= B(\gamma, T-t).\end{aligned} \tag{6.14}$$

The wealth ratio to salary is given by $X(t) = \frac{W(t)}{X(t)}$. After computing the equation by straight forward application of the product formula, we obtain the SDE:

$$dX(t) = X(t)\left[\left(-\mu_Y(t) + p(t)'C(\tilde{\zeta} - \sigma_Y) + \sigma_Y'\sigma_Y\right)dt + (p(t)'C - \sigma_Y')dZ(t)\right] + \pi dt. \tag{6.15}$$

Furthermore, we are going to substitute $p(t)$ with $p^*(t,x,r;V)$ and obtain:

$$\begin{aligned}dX(t) &= X(t)\left[\left(-\mu_Y(t) + \begin{pmatrix} k_1 & k_2 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}\begin{pmatrix} \tilde{\zeta}_1 - \sigma_{Y1} \\ \tilde{\zeta}_2 - \sigma_{Y2} \end{pmatrix} + \begin{pmatrix} \sigma_{Y1} & \sigma_{Y2} \end{pmatrix}\right.\right. \\ &\left.\left.\begin{pmatrix} \sigma_{Y1} \\ \sigma_{Y2} \end{pmatrix}\right)dt + \left(\begin{pmatrix} k_1 & k_2 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} - \begin{pmatrix} \sigma_{Y1} & \sigma_{Y2} \end{pmatrix}\right)\begin{pmatrix} dZ(t_1) \\ dZ(t_2) \end{pmatrix}\right] \\ &+ \pi dt \\ &= X(t)\left[\left(-\mu_Y(t) + \begin{pmatrix} c_1k_1 + c_3k_2 & c_2k_1 + c_4k_2 \end{pmatrix}\begin{pmatrix} \tilde{\zeta}_1 - \sigma_{Y1} \\ \tilde{\zeta}_2 - \sigma_{Y2} \end{pmatrix}\right.\right. \\ &+(\sigma_{Y1}^2 + \sigma_{Y2}^2)dt + \begin{pmatrix} c_1k_1 + c_3k_2 & c_2k_1 + c_4k_2 \end{pmatrix} - \begin{pmatrix} \sigma_{Y1} & \sigma_{Y2} \end{pmatrix} \\ &\left.\left.\begin{pmatrix} dZ(t_1) \\ dZ(t_2) \end{pmatrix}\right)\right] + \pi dt \\ &= X(t)\left[\left(-\mu_Y(t) + (c_1k_1 + c_3k_2)(\tilde{\zeta}_1 - \sigma_{Y1}) + (c_2k_1 + c_4k_2)(\tilde{\zeta}_2 - \sigma_{Y2})\right.\right. \\ &\left.+(\sigma_{Y1}^2 + \sigma_{Y2}^2)\right)dt + (c_1k_1 + c_3k_2 - \sigma_{Y1})dZ(t_1) + (c_2k_1 + c_4k_2 - \sigma_{Y2})dZ(t_2)\Big] \\ &+ \pi dt.\end{aligned}$$

To illustrate this, we are going to make use of the figure 6.1 above and time is in years. From figure 6.1 the wealth process therefore evolve stochastically and in this case our paremeter are assigned values as $\alpha = 0.0025, \sigma_{Y1} = 0.02, \sigma_{Y2} = 0.02, \sigma r_1 =$

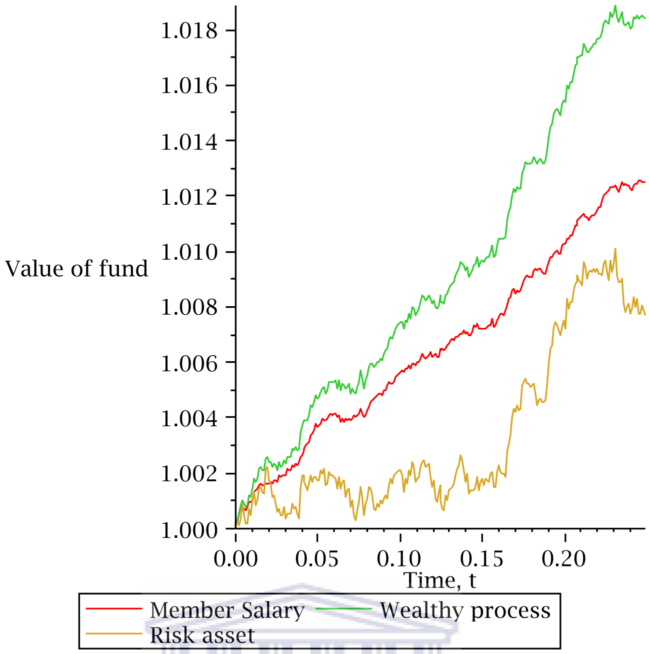

Figure 6.1: Different stochastic processes

$-0.02, \sigma r_2 = 0.5, \mu_Y = 0.5, \mu_r = 0.5, \gamma = -5, \tilde{\zeta}_1 = 0.002, \tilde{\zeta}_2 = 0.003$. This implies that the wealth process is stochastically modelled with change in time variable.

Chapter 7

The three fund theorem of Cairns-Blake-Dowd

The theorem of Cairns et al that we consider in this section, is a special case, because in general there are other influencing variables such as inflation and wage income which are stochastic and the pension fund manager wants to hedge against unfavourable outcomes by purchasing securities correlated to these variables. Nevertheless, the three fund theorem is quite informative. In every financial sector, it is important to follow the optimal asset allocation so as to maximise returns on investments. In the three fund theorem, the final wealth is portioned and invested in three different sectors with unique risk measurements. For this purpose, we consider three funds in which to invest, which are cash, bond and equity.

Investors hold a portfolio comprising of three funds; the risk free asset, the market portfolio and a third portfolio, chosen in such a way that its return is perfectly correlated with the return on the risk free asset.

Before we go into more detail in the three fund assets, we need to explore on how the dynamics are derived. To do this, we are going to make use of the Hamilton-Jacobi-Bellman equation, fully explained in [13]. Eventually we obtain

$$p^*(t,x,r,V) = C'^{-1}\left(\sigma_Y - (\tilde{\zeta}-\sigma_Y)\frac{V_x}{xV_{xx}} - \sigma_r(r)\frac{V_{xr}}{xV_{xx}}\right). \tag{7.1}$$

Simplifying (7.1), we eventually obtain the PDE in the simplest form given by,

$$V_t + \mu_r(r)V_r + (\pi - \tilde{\mu}_Y(t)x + \sigma_Y'(\tilde{\zeta} - \sigma_Y)x)V_x + \frac{1}{2}\sigma_r(r)'\sigma_r(r)V_{rr}$$
$$-\frac{1}{2}(\tilde{\zeta} - \sigma_Y)'(\tilde{\zeta} - \sigma_Y)\frac{V_x^2}{V_{xx}} - (\tilde{\zeta} - \sigma_Y)'\sigma_r(r)\frac{V_x V_{xr}}{V_{xx}} - \frac{1}{2}\sigma_r(r)'\sigma_r(r)\frac{V_{xr}^2}{V_{xx}} = 0. \quad (7.2)$$

7.1 Optimal asset mix

We now formulate informally state without proving it, the so-called Three fund theorem of Cairns-Blake-Dowd, where at any given time the investment consists of three efficient mutual funds as follows:

$$p^*(t, x, r, V) = \theta_A p_A + \theta_B p_B + \theta_C p_C, \quad (7.3)$$

where

Cash fund, $\theta_A(t, x, r) = 1 - \frac{V_{xr} - d_a(r)V_x}{d_a(r)xV_{xx}}$

Bond fund, $\theta_B(t, x, r) = \frac{V_x}{d_a(r)xV_{xx}}$

Stock fund, $\theta_C(t, x, r) = 1 - \theta_A - \theta_B = \frac{V_x}{xV_{xx}}$

with

$P_A = C'^{-1}\sigma_Y$

$P_B = C'^{-1}\,(\sigma_Y d_a(r)\sigma_r(r))$

$P_C = C'^{-1}\zeta.$

7.1.1 Cash fund

At any given time, the fund manager is absolute sure that the plan member will receive a salary $Y(t)$. Cash fund, as compared to other portfolios, it is the minimum risk portfolio being measured relative to the salary numeraire. Since there might be risk in salary, this fund is reserved to hedge the salary risk. If there is not be any other source of cash besides salary, then this fund contains only cash growth. In other words, if there is a correlation between asset returns and cash growth, then the cash fund contains other asset apart from cash only. This mutual fund undergoes steady growth and by the evolution of (7.1), it is dependent on $t, X(t)$ and $r(t)$.

7.1.2 Bond fund

In the case of an annuity, the major risk to the insurance company is that the person may live a very long life requiring more payments than the insurance company expected. Another risk is that the company may not be able to earn as great a return on its investments as planned, and so it may have less money to make payments when they are due. This fund is mainly dominated by bonds and the returns tend to be highly correlated with annuity yields. Bond fund is the minimum risk portfolio measured relatively to $\frac{Y(t)}{a(t,r(t))}$. Since there is a high correlation of annuity yields, it is used to hedge against annuity risks. These mutual funds also contain constant growth but varying over time and it only respond to changes in $r(t)$. It is also dependent on $t, X(t)$ and $r(t)$.

7.1.3 Stock fund

These are mutual funds and the objective is for long-term growth through capital appreciation, although dividends and interest are also sources of revenue. In some cases, specific equity funds may focus on a certain sector of the market or may be geared toward a certain level of risk. This is a risk portfolio and tend to be efficient when measured relative to both $Y(t)$ and $\frac{Y(t)}{a(t,r(t))}$. Both bond and cash are in cooperated here and then the stock fund is there so as to satisfy the risk appetite of the plan member. This mutual fund maintains a constant proportion of assets but then it also depends on $t, X(t)$ and $r(t)$.

To illustrate this, we are going to make use of the figure 7.1 below. From figure 7.1, the wealth process therefore evolve stochastically and in this case our paremeter are

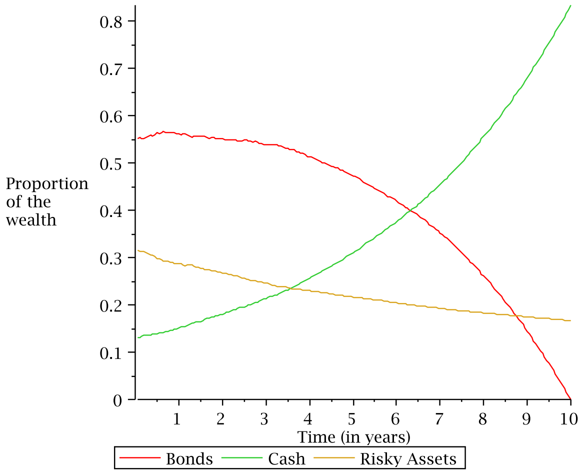

Figure 7.1: The three fund theorem

assigned values as $\alpha = 0.0025, \sigma_{Y1} = 0.02, \sigma_{Y2} = 0.02, \sigma_{r_1} = -0.02, \sigma_{r_2} = 0.5, \mu_Y = 0.5, \mu_r = 0.5, \gamma = -5, \tilde{\zeta}_1 = 0.002, \tilde{\zeta}_2 = 0.003$. This implies that the wealth process is stochastically modelled with change in time variable.

Concerning the global management of the funds, figure 7.1 shows the evolution of the funds from time $t = 0$ to time $T = 10$, assuming that the plan member will retire after 10 years. Since the plan member is going to have a salary, the cash asset will start at a value above zero and will increase more towards the retirement age because all the assets will be converted in cash.Bonds are purchased at time $t = 0$. The maturity of the bonds in this case is taken to be the retirement age hence they must be paid up at time $T = 10$. Less funds are invested in the risky assets at the beginning as compared to other assets and the proportion will continue to reduce as the plan member approaches retirement age. This is also because all the asset will be converted in cash [10].

Chapter 8

Conclusion

The level of volatility when it comes to deterministic is lower and the value of $p^*(t, X(t))$ is even lower. During the interval before assets are converted to bonds, the value of $p^*(t, X(t))$ can be slightly higher than that of stochastic. Towards the retirement age, all assets are converted to bonds, which is more risky as compared to stochastic lifestyling. In general, we can say the equity proportion dwindles to as much as zero level irrespective of the plan member's degree of risk aversion or salary dynamics.

On the contrary, we can now make a simple comparison of our general model of chapter 5 and chapter 6, by mainly looking at how they behave from time $t = 0$ to time $t = T$. The optimal equity proportion for stochastic lifestyling will be having high volatility during first few years and low value of $X(t)$. As $X(t)$ increases, the value of $p^*(t, X(t))$ tend to $\frac{\sigma_1 - \sigma_{Y1} q^*}{\sigma_1}$ but will never be zero. The level of its non-zero nature depends on how risk averse is the plan member and the correlation with the plan member's salary.

Using the three mutual funds, we noticed that there is the high risk one and low risk ones, all serving different but important purposes. The high risk asset, equity fund, have been used to satisfy the risk appetite of the plan member. The low risk assets were cash and bond funds. The bond funds, have been the default low risk investment while the cash fund as a hedger against annuity rate risk. As with deterministic, there is a gradual change from high risk to low risk assets as the retirement date approaches. In the optimal stochastic lifestyle, during the early stages of the plan, cash fund dominate more in low risk component but as retirement date get near it then switches from cash into bonds. Basically, stochastic lifestyle involves switching between different assets of low risk.

A weakness of the model is that the variable risk free interest rate is modeled as a geometric Browian motion, which may potentially grow out of bounds. A revision of the model, with interest rate taken as mean reverting seems a better alternative.

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