STRONG SIMPLICITY OF GROUPS AND VERTEX-TRANSITIVE GRAPHS

by

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Declaration

I declare that “Strong Simplicity of Groups and Vertex-transitive Graphs” is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Emad Alden Sir Al khatim Abraham Fadhal December 2010

Signed:..

Abstract

In the course of exploring various symmetries of vertex-transitive graphs, we introduce the concept of quasi-normal subgroups in groups. This is done since the symmetries of vertex-transitive graphs are intimately linked to those, fait accompli, of groups. With this, we ask if the concept of strongly simple groups has a place for consideration.

We have shown that for $n \geqslant 5$, A_n, the alternating group on n odd elements, is not strongly simple.

December 2010.

Dedication

This thesis is dedicated to my father, who taught me how to use one's time wisely and carefully. He taught me the virtue of perseverance in spite of the seemingly impossibility of the task at hand. It is also dedicated to my mother, who taught me that the truth comes from within. It is also dedicated to my supervisor, who taught me the value of completely immersing and engaging oneself in the task at hand.

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I need also thank my family members: my wonderful mother, Zainab, my great father, Alsir, my sister, my brothers and their children. I owe so much to my family. They are mentioned last to emphasise the special nature of their tremendous encouragement, support and patience all throughout my study.

Last but not the least, this work would have never been completed without the blessings of Allah.

Publication arising from this thesis

Emad Alden Fadhal, Eric Mwambene. **On generalising McKay-Praeger $T(2m^2)$ graphs by representing them as loop graphs**.
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Chapter 1

Introduction

1.1 Introduction and background

A Cayley graph is a representation of the structure of group with respect to a generating subset. The vertex set of the graph is the elements of the group and the arc set is defined by a subset of the group, typically a generating set.

Cayley graphs have many generalisations. Gauyacq [9] generalised the concept of Cayley graph and introduced quasi-Cayley graphs. Once one defines quasi-Cayley graphs, it is easy to see that every Cayley graph is a quasi-Cayley graph.

A graph is said to be vertex-transitive if for any given pair of vertices x, y there is an automorphism which maps x to y. Sabidussi [22] has characterised all vertex-transitive graphs as quotient graphs of Cayley graphs.

Quotient graphs of Cayley graphs on groups exhibit various levels of symmetry depending on the stabiliser subgroup that is used to define the vertices. In that respect, Cayley graphs on groups, quasi-Cayley graphs and the general vertex-transitive graphs have been identified. Quasi-Cayley graphs are intimately linked to quasi-normality of subgroups. It is conceivable that quasi-normality may render the natural concept of strong simplicity in groups worthy investigating. This study has explored this phenomenon.

By looking at groupoids that represent various classes of vertex-transitive graphs, specifically, we study a more generalised concept of normality of subgroups; namely that of quasi-normality. Furthermore, we consider its attendant ramifications in relation to symmetry of graphs. In that respect, we investigate strongly simplicity of groups. In particular, we look at A_n, the alternating group on n elements.

Mwambene [15] has characterised groupoids which represent vertex-transitive graphs. We use this characterisation to identify various layers of symmetry of vertex-transitive graphs. In the course, it becomes natural to define quasi-normality, a concept intimately linked to quasi-Cayley graphs.

In turn, quasi-normality brings to bear the issue of strong simplicity of groups. This thesis explores this phenomenon.

We show that for $n \geqslant 5$, n odd, A_n, the alternating group on n elements is not a strongly simple group. In fact, we have not been able to find such groups save for the obvious cyclic groups of prime order. However, the question is as relevant as ever in the context of symmetry of vertex-transitive graphs.

1.2 Overview of the thesis

In Chapter 2, we introduce basic concepts and definitions of graph theory that we will use in our discussion. In Chapter 3, we describe groupoid graphs as a general of Cayley graphs and their properties. We narrow the concept to introduce quasi-Cayley graph later in Chapter 5. In Chapter 4, we characterise vertex-transitive graphs in terms of groupoids representing them. In Chapter 5, we give a characterisation of quasi-Cayley graphs. Further we represent McKay-Praeger $T(2m^2)$ graphs on loops and generalise the construction into higher dimension. In Chapter 6, we show that for $n \geqslant 5$, n odd, A_n, the alternating group on n elements is not a strongly simple group.

Chapter 2

Definitions and notation

In this chapter we present basic definitions and introduce notation which we use throughout. We also recall fundamental results that we will use.

2.1 Preliminaries

Let V be a finite set and R a relation defined on V. $D = (V, R)$ is called a *digraph* if R is irreflexive, i.e., $(v, v) \notin R$ for all $v \in V$. The elements of V are called *vertices* and the elements of R are called *arcs*. The out-degree of a vertex x is the size of the set $\{y \in V : (x, y) \in R\}$. The in-degree of x is the size of the set $\{y \in V : (y, x) \in R\}$

A *graph* Γ is a digraph with the additional property that R is symmetric. In other words, a graph $\Gamma = (V, E)$ consists of a set of vertices V and a relation E which is

(i) irreflexive;

(ii) symmetric.

The arcs (x, y) and (y, x) are identified into a single *edge* and denoted by $[x, y]$. If it is not clear from the context we will denote V by $V(\Gamma)$ and E by $E(\Gamma)$.

In this sequel, we consider finite graphs only, i.e., a graph Γ for which both $V(\Gamma)$ and $E(\Gamma)$ are finite sets.

2.2 Adjacency, neighbourhood and regularity in graphs

Two vertices x and y of a graph Γ are *adjacent* if there is an edge $e = [x, y]$ joining them. The vertices x and y are said to be *incident* with e. If x and y are adjacent, it is said that they are *neighbours*. Similarly, two distinct edges e and e' are *adjacent* if they have a vertex in common.

Let Γ be a graph. The *neighbourhood* of a vertex $x \in V(\Gamma)$ is the set of vertices that are adjacent to x, and it is denoted by $N_\Gamma(x)$. The *degree* of a vertex x is the size of its neighbourhood, $|N(x)|$, and is denoted $d_\Gamma(x)$.

A graph in which each vertex has the same degree is said to be *regular*. If each vertex has degree r, the graph is *regular of degree* r or r*-regular*. A *complete graph* is a graph in which every two distinct vertices are adjacent. The complete graph on n vertices is denoted by K_n.

Much of graph theory involves 'walks' of various kinds. A *walk* consists of a sequence of vertices, one following after another, in which consecutive vertices are adjacent. A *path* is a special walk $p(x_0x_n)$ of the form

$$V(p(x_0x_n)) = \{x_0, x_1, \ldots, x_n\}, \quad E(p(x_0x_n)) = \{[x_0x_1], [x_1x_2], \ldots, [x_{n-1}x_n]\},$$

and denoted by $x_0x_1 \ldots x_n$. Vertices $x_i, i = 1, \cdots, n-1$ are called interior vertices of the path. For a path $p(x_0x_n)$, if x_0 coincides with x_n, then it is said that $p(x_0x_n)$ is a *cycle*. If in addition, $V(p(x_0x_n)) = V(\Gamma)$, then it is said to be a *Hamiltonian cycle*. For terms not defined, we follow [26].

In order to model resilience in communication networks, sets of paths called routing have been considered in graphs. Routings are defined as follows.

2.1 Definition [18] Let Γ be a graph. A *routing* is a set

$$R = \{p(x, y) : (x, y) \in V(\Gamma) \times V(\Gamma)\}$$

such that

(i) $p(x, y)$ is an xy-path for any $(x, y) \in V(\Gamma) \times V(\Gamma)$;

(ii) $p(x, x) = x$ for any $x \in V(\Gamma)$.

If every $p(x, y) \in R$ is a shortest path joining x and y, then R is said to be a *geodesic routing.*

2.2 Definition The *load* of a vertex u in a routing R is the number of paths $p(x, y) \in R$ such that u is an interior vertex of $p(x, y)$.

A routing R of a graph Γ is uniform if each vertex $u \in V(\Gamma)$ has the same load. Geodesic uniforms routings so far, have revealed three levels of symmetry in vertex-transitive graphs. See [9, 11, 25]. We will be implicitly discussing this symmetry.

2.3 Vertex-transitive graphs

Throughout we study vertex-transitive graphs. In order to introduce vertex-transitive graphs, we consider the following concepts. We closely follow [10].

2.3 Definition (a) Let Γ_1, Γ_2 be graphs. A *homomorphism* from Γ_1 to Γ_2 is a map $\alpha : V(\Gamma_1) \to V(\Gamma_2)$ that preserves adjacency, i.e.,

$$[\alpha(x), \alpha(y)] \in V(\Gamma_2) \text{ for any } [x, y] \in V(\Gamma_1).$$

(b) A homomorphism $\alpha : V(\Gamma_1) \to V(\Gamma_2)$ is an *isomorphism* if

(i) α is a bijection;

(ii) α^{-1} is also a homomorphism.

In this case, it is said that Γ_1 is isomorphic to Γ_2 and written $\Gamma_1 \cong \Gamma_2$.

(c) A homomorphism $\alpha : V(\Gamma_1) \to V(\Gamma_2)$ for which Γ_1 and Γ_2 coincide is called an *endomorphism*. If in addition, α is a permutation, then it is an *automorphism*. In other words, an *automorphism* α of a graph Γ is a permutation of the vertex set with the property that $\alpha(x)$ and $\alpha(y)$ are adjacent if and only if x and y are. The set of automorphisms of Γ form a group under composition and is denoted by $\mathrm{Aut}(\Gamma)$.

Now, we introduce vertex-transitive graphs.

2.4 Definition A graph Γ is *vertex-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, i.e., for any $x, y \in V(\Gamma)$ there is $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha(x) = y$.

Since a homomorphism will map a vertex x of degree r to a vertex y of the same degree, vertex-transitive graphs are necessarily r-regular.

As alluded to in Chapter 1, there are three known levels of symmetry of vertex-transitive graphs. Cayley graphs, which are defined in Chapter 3, are defined on groups. Quasi-Cayley graphs, which are defined in Chapter 5, are defined on loops and the general vertex-transitive graphs, which are defined in Chapter 4, are defined on left loops.

For Cayley graphs, the representation of graphs is closely tied to the action of a group acting on the set of vertices, of which we define below.

2.5 Definition Let G be a group acting on the a set S. The action is said to be *regular* if it satisfies the following conditions:

(i) G is transitive on S;

(ii) given any pair $x, y \in S$, there exists a unique element $\alpha \in G$ such that $\alpha(x) = y$.

The concept of regularity is generalised to that of a loop L acting on a set, instead of a group. We use this to classify quasi-Cayley graphs in Chapter 5.

2.4 Quotient graphs

In Chapter 4 we use quotient graphs to discuss coset graphs.

2.6 Definition Let Γ be a graph, $\mathcal{P} = \{V_i : i \in I\}$ a partition of $V(\Gamma)$. The *quotient graph* $\Gamma/\mathcal{P}$ is defined by

$$V(\Gamma/\mathcal{P}) = \mathcal{P};$$

i.e., the vertices of $\Gamma/\mathcal{P}$ are the cells of $\mathcal{P}$, and

$$[V_i, V_j] \in E(\Gamma/\mathcal{P}) \Longleftrightarrow i \neq j \text{ and } [x, y] \in E(\Gamma) \quad \text{for some } x \in V_i, y \in V_j.$$

Vertex-transitive graphs are represented as quotient graphs of Cayley graphs throughout this work. As will be seen in the characterisation, Sabidussi has shown that every vertex-transitive is a quotient of a Cayley graph. Instead of having cells as vertices of quotient graphs, it is convenient to identify a cell with a representative. We will use this later representation to define left loops.

Chapter 3

Groupoid graphs

3.1 Introduction

In this chapter we introduce Cayley graphs as a protype of other groupoid graphs. Cayley graphs have a long history, appearing in various forms. Cayley introduced them as tools to study groups.

We then generalise them to groupoid graphs. In that vein, we describe some classes of groupoid graphs; namely loop and left loop graphs. As will be noted, the classification of these graphs depends on the algebraic structure defining them.

Now, let us define Cayley graphs as alluded to.

3.2 Cayley graphs

Cayley graphs are defined on groups. The adjacency depends on a well-chosen subset of group which satisfies pertinent conditions elaborated below.

3.1 Definition Let G be a group. A *Cayley set* S is subset of G that satisfies the following:

(i) the identity element is not in S;

(ii) if $s \in S$ then so is s^{-1}.

The first condition of the definition ensures that a graph described by the given Cayley set does not contain *edge-loops*. The second condition ensures that the relation is *symmetric*; so that we obtain edges as opposed to arcs.

3.2 Definition Let G be a group and S be a Cayley set of G. The *Cayley graph* $\mathrm{Cay}(G,S)$ is defined as follows:

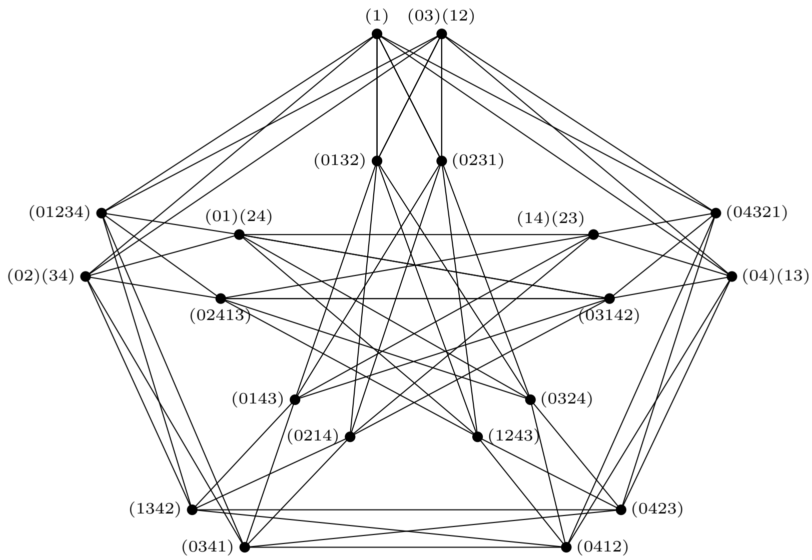
(i) the vertices of $\mathrm{Cay}(G,S)$ are elements of G;

(ii) $[x,y] \in E(\mathrm{Cay}(G,S))$ if there exists $s \in S$ such that $y = xs$.

3.2.1 An example of a Cayley graph

Let G be the Frobenius group of order 20. In our context, we use the representation $G = \{$ (1), (01234), (04321), (02413), (03142), (1342), (1243), (0214), (0412), (0132), (0231), (04)(13), (02)(34), (03)(12), (14)(23), (01)(24), (0423), (0324), (0341), (0143)$\}$ acting on the set $\{0,1,2,3,4\}$. Consider the set $S = \{(01234),(04321),(0132),(0231),(04)(13),(02)(34)\}$. S is clearly a Cayley set; it does not contain the identity and is closed under inverses. $\mathrm{Cay}(G,S)$ is illustrated below.

Figure 3.1: Cayley graph defined on the Frobenius group of order 20



This graph is used to illustrate coset graphs in Section 4.2.1.

3.3 Groupoid graphs

Once one sees the way Cayley graphs are defined, there is a natural generalisation on general algebraic structures. Instead of defining graphs on groups, in [13] graphs are defined on the most general algebraic structure: the groupoid, whilst taking into consideration the nuances that are applicable to Cayley sets in groupoids. We present this here.

3.3 Definition A *groupoid* $(G, *)$ is a set G together with a binary operation $* : G \times G \to G$. As the most general of algebraic structures, besides the binary operation, no another axiom is expected on groupoids. We will write G for $(G, *)$ and ab for $a * b$, when it is clear from the context.

An element u_l is a left unit of a groupoid G, if for all $x \in G$

$$u_l * x = x.$$

A right unit u_r satisfies $x * u_r = x$ for $x \in G$.

An element $u \in G$ is a *unit* of groupoid G if it is both a right and left unit of G.

If we have that (i) $ax = b$; [(ii) $ya = b$] has a unique solution then $(G, *)$ is a left [right] *quasi-group*; in this case we speak of left [right] cancellation of the corresponding equations.

3.4 Definition A *left loop* is a left quasi-group with a unit.

3.5 Definition A *loop* is a groupoid with a unit in which one cancels from both the left and the right.

Now, in order to generalise pertinent properties of Cayley sets in groupoids, we need that the sets do not define edge-loops and defines a symmetric relation. This is done so in the following way.

3.6 Definition Let G be groupoid and S a subset of G. Then S is a *Cayley-set* of G if it satisfies the following.

(i) $as \neq a$, for all $a \in G, s \in S$;

(ii) for any $a \in G, s \in S$ there is an $s' \in S$ such that $(as)s' = a$.

Groupoid graphs are analogously defined as Cayley graphs. In fact, it may have been more apt to define them as Cayley graphs on groupoids.

3.7 Definition Let G be a groupoid and S a Cayley set. The *groupoid graph* $\mathrm{GG}(G, S)$ is a graph with vertices the elements of G and $[x, y] \in E\ \mathrm{GG}(G, S)$ if $y = xs,\ s \in S$.

In the definition above, if G is a left quasi-group[right quasi-group, loop] then we speak of a left quasi-group [right quasi-group, loop] graph respectively. In particular, we focus on the following.

3.8 Definition Let Q be a quasi-group and $S \subseteq Q$, a Cayley set.

(a) The *quasi-group graph* $\Gamma = \mathrm{GG}(Q, S)$ is defined by

$$V(\Gamma) = Q,\ E(\Gamma) = \{(u, us) | u \in Q,\ s \in S\}.$$

(b) If S is quasi-associative, Γ is said to be a *quasi-Cayley graph.*

3.3.1 Example of groupoid graphs

Consider the set $Q = \mathbb{Z}_2 \times \mathbb{Z}_5$ with the binary operation $*$ defined by

$$(a, b) * (c, d) = (a \oplus c, 2^a(abcd + d) + b^4(b + 8(abcd))),$$

where $\oplus$ is addition modulo 2, $+$ is addition modulo 5 respectively.

$*$	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(0,4)$	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$
$(0,0)$	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(0,4)$	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$
$(0,1)$	$(0,1)$	$(0,2)$	$(0,3)$	$(0,4)$	$(0,0)$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,0)$
$(0,2)$	$(0,2)$	$(0,3)$	$(0,4)$	$(0,0)$	$(0,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,0)$	$(1,1)$
$(0,3)$	$(0,3)$	$(0,4)$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,3)$	$(1,4)$	$(1,0)$	$(1,1)$	$(1,2)$
$(0,4)$	$(0,4)$	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(1,4)$	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$
$(1,0)$	$(1,0)$	$(1,2)$	$(1,4)$	$(1,1)$	$(1,3)$	$(0,0)$	$(0,2)$	$(0,4)$	$(0,1)$	$(0,3)$
$(1,1)$	$(1,1)$	$(1,3)$	$(1,0)$	$(1,2)$	$(1,4)$	$(0,1)$	$(0,3)$	$(0,0)$	$(0,2)$	$(0,4)$
$(1,2)$	$(1,2)$	$(1,4)$	$(1,1)$	$(1,3)$	$(1,0)$	$(0,2)$	$(0,4)$	$(0,1)$	$(0,3)$	$(0,0)$
$(1,3)$	$(1,3)$	$(1,0)$	$(1,2)$	$(1,4)$	$(1,1)$	$(0,3)$	$(0,0)$	$(0,2)$	$(0,4)$	$(0,1)$
$(1,4)$	$(1,4)$	$(1,1)$	$(1,3)$	$(1,0)$	$(1,2)$	$(0,4)$	$(0,1)$	$(0,3)$	$(0,0)$	$(0,2)$

Table 3.1: Cayley table of the groupoid on $\mathbb{Z}_2 \times \mathbb{Z}_5$

Note that the Cayley table of this groupoid is a Latin square. Moreover, $(0,0)$ is a unit. Hence $Q = \mathbb{Z}_2 \times \mathbb{Z}_5$ is a loop.

Now, consider the subset $S = \{(1,0), (0,2), (0,3)\}$. By looking at the 3rd, 4th and 6th columns, we have that $as \neq a$ for all $a \in Q$. So, the first condition of Definition 3.6 is satisfied. For the second condition we have that for $i \in \mathbb{Z}_5$,

$((0,i)(0,2))(0,3) = (0,2+i)(0,3) = (0,i)$,
$((1,i)(0,2))(0,3) = (1,4+i)(0,3) = (1,i)$;

$((0,i)(0,3))(0,2) = (0,3+i)(0,2) = (0,i)$,
$((1,i)(0,3))(0,2) = (1,6+i)(0,2) = (1,i)$;

$((0,i)(1,0))(1,0) = (1,0+i)(1,0) = (0,i)$,
$((1,i)(1,0))(1,0) = (0,0+i)(1,0) = (1,i)$.

So the condition is equally satisfied. The graph we obtain is the Petersen graph below. We note that the quasi-group defined by Gauyacq in [9] is different from the above; yet we get the same graph.

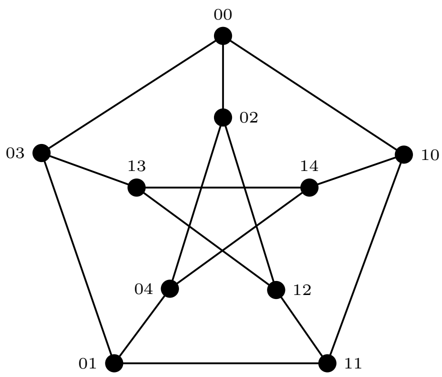


Figure 3.2: A loop graph defined on the groupoid $\mathbb{Z}_2 \times \mathbb{Z}_5$

3.4 Translations in groupoid graphs

Left translations play a distinguishing role in Cayley graphs. They constitute a subgroup of the automorphism group of the graph which acts regularly on the vertices. In fact, Sabidussi [23] has shown that the existence of a subgroup acting regularly is a necessary and sufficient condition for a graph to be Cayley.

Once one generalises Cayley graphs to groupoid graphs, one may ask under what conditions are translations endomorphisms of a given groupoid graph. In this section, we present the conditions under which this is so, closely following [15].

Let $\mathrm{GG}(G,S)$ be a groupoid graph and $a \in G$. Define a left translation $\lambda_a : \mathrm{GG}(G,S) \longrightarrow \mathrm{GG}(G,S)$ by

$$\lambda_a x = ax.$$

λ_a preserves edges of $\mathrm{GG}(G,S)$ provided $[ax, ay]$ is an edge whenever $[x,y]$ is an edge of the graph. In other words, $[ax, a(xs)]$ is an edge whenever $[x, xs]$ is, for any $s \in S$, the Cayley set. This can only happen if for any $s \in S$

$$a(xs) = (ax)s' \quad \text{for some } s' \in S. \tag{3.1}$$

This motivates the following classification of Cayley sets.

3.9 Definition A Cayley set S in a groupoid G is called *quasi-associative* if it satisfies Equation (3.1), i.e., $(xy)S = x(yS)$ for all $x, y \in G$.

With quasi-associativity in context, we have the following result.

3.10 Proposition *Let* $\mathrm{GG}(G, S)$ *be a groupoid graph with the property that* S *is quasi-associative. Then for any* $a \in G$*, the translation* λ_a *is an endomorphism of the graph.*

If in addition we have that we can cancel from the left, then we have more.

3.11 Proposition *Suppose that* $\mathrm{GG}(G, S)$ *is a left quasi-group graph with the property that* S *is quasi-associative. Then for any* $a \in G$*, the translation* λ_a *is an automorphism of the graph.*

3.5 Quasi-associativity and Cayley sets in loops

It is important to note the following results regarding combinations of quasi-associative sets.

3.12 Lemma [19] *In a groupoid in which every element is left regular* ($ab = ac$ *implies* $b = c$) *the set of quasi-associative subsets forms a complete boolean algebra of sets.*

Another feature is that in the presence of quasi-associativity, Cayley sets of left quasi-groups with units take a similar form to the one defined in groups.

3.13 Lemma *Let* G *be a left quasi-group with a unit* u*, and* S *a quasi-associative subset of* G*. Then* S *is Cayley if and only if* $u \notin S^{-1} \subset S$ ($u \notin S^{-1} = S$).

Proof
We only prove sufficiency. Let $S \subset G \setminus \{u\}$ be quasi-associative, and suppose that $S^{-1} \subset S$. Given $a \in G, s \in S$, we have that $a = au = a(ss^{-1}) = (as)s'$, where $s' \in S$. ∎

Chapter 4

Characterisations of vertex-transitive graphs

4.1 Vertex-transitive graphs

In this chapter we present vertex-transitive graphs, a class of graphs which exhibit a high degree of symmetry in terms of their automorphism groups. We further present two of their characterisations: one given by Sabidussi [22] and the recent one given by Mwambene [15]. That is, we show that vertex-transitive graphs are precisely coset graphs of Cayley graphs and that they are also precisely graphs represented by left loops with quasi-associative Cayley sets.

Cast in that light, we show the kind of coset graphs that are Cayley.

In view of Proposition 3.11, left translations on a given group G are automorphism of a Cayley graph $\mathrm{Cay}(G, S)$, for any Cayley set S. Moreover, given any pair $x, y \in V(\mathrm{Cay}(G, S))$, the translation $\lambda_{yx^{-1}}$ maps x to y. Therefore Cayley graphs are vertex-transitive graphs.

4.2 Coset graphs

In this section we show that every vertex-transitive graph is a coset graph. We first define coset graphs.

4.1 Definition Let G be a group, H a subgroup of G and S a Cayley set. A *coset graph* $\Gamma := \text{Cay}(G, S)/H$ is defined by

$$V(\Gamma) := \; G/H;$$

the left cosets of H in G, and

$$[aH, bH] \in E(\Gamma),$$

if and only if there exist $x \in aH$, $y \in bH$ such that $y = xs, s \in S$.

In the context of quotient graphs, as given by Definition 2.6, $\text{Cay}(G, S)/H$ is the quotient graph of the Cayley graph $\text{Cay}(G, S)$ modulo the partition of G with respect to the left cosets of H in G.

4.2.1 Example of a coset graph

We present an example showing the coset graph $\text{Cay}(G, S)/H$ from a Cayley graph given by the example in Section 3.2.1 as the Petersen graph. Let $G = \{(1), (01234), (04321), (02413), (03142), (0341), (1243), (0214), (0423), (0132), (01)(24), (04)(13), (02)(34), (03)(12), (14)(23), (0231), (0412), (1342), (0143), (0324)\}$. Let $H = \{(1), (03)(12)\}$ be a subgroup of G. Now the left cosets of H in G are:

$$\begin{aligned}
(1)H &= \{(1), (03)(12)\} \\
(1243)H &= \{(1243), (0324)\} \\
(0132)H &= \{(0132), (0231)\} \\
(0341)H &= \{(0341), (1342)\} \\
(0214)H &= \{(0214), (0143)\} \\
(0423)H &= \{(0423), (0412)\} \\
(01234)H &= \{(01234), (02)(34)\} \\
(02413)H &= \{(02413), (01)(24)\} \\
(03142)H &= \{(03142), (14)(23)\} \\
(04321)H &= \{(04321), (04)(13)\}.
\end{aligned}$$

We have $V(\text{Cay}(G, S)/H) = \{(1)H, (01234)H, (02413)H, (03142)H, (1243)H, (0132)H, (0341)H, (0214)H, (0423)H, (04321)H\}$.

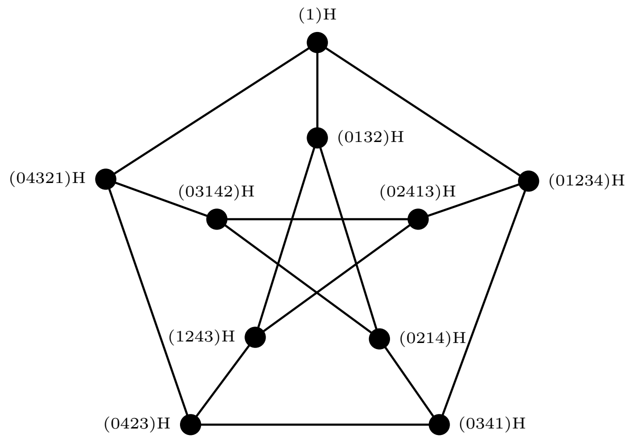


Figure 4.1: The Petersen graph as the coset graph $\mathrm{Cay}(G, S)/H$

4.3 Induced left translations in coset graphs and Sabidussi's theorem

Let $\Gamma = \mathrm{Cay}(G, S)/H$ be a coset graph defined on the group G. Then for each element $a \in G$, there is an induced translation $\overline{\lambda}_a : V(\Gamma) \to V(\Gamma)$ defined by

$$\overline{\lambda}_a(xH) = axH. \tag{4.1}$$

4.2 Lemma *Let G be a group, S a Cayley set and H a subgroup of G. Then for a fixed element $a \in G$, the induced translation $\overline{\lambda}_a$, as defined above* (4.1) *is an automorphism of the coset graph* $\mathrm{Cay}(G, S)/H$.

Proof

$$\begin{array}{rrcl} & \overline{\lambda}_a(xH) & = & \overline{\lambda}_a(yH) \\ \Longleftrightarrow & axH & = & ayH \\ \Longleftrightarrow & a^{-1}(axH) & = & a^{-1}(ayH) \\ \Longleftrightarrow & xH & = & yH. \end{array}$$

Hence $\overline{\lambda}_a$ is one to one.

Suppose $[xH, yH]$ is an edge of $\text{Cay}(G, S)/H$. Then there exist $u \in xH, v \in yH$ such that $v = us, s \in S$. $u = xh,\ v = xhs$. Hence $\overline{\lambda}_a[xH, yH] = [axH, ayH]$ is an edge since we have that axH contains $u' = axh$ and ayH contains $v' = axhs$. ∎

With the abundance of available automorphisms of coset graphs, it is not surprising that they are vertex-transitive. Indeed we have that for any pair of vertices xH, yH, the induced translation $\overline{\lambda}_{yx^{-1}}$ maps xH to yH.

4.3 Proposition *Every coset graph* $\text{Cay}(G, S)/H$ *is vertex-transitive.*

What is interesting is that the converse of the Proposition 4.3 is also true. This is the celebrated result of Sabidussi which we present hereunder.

Note that these coset graphs are defined modulo stabilisers of a fixed vertex in a graph Γ. That is, they are defined modulo subgroups of G of the form

$$G_u = \{\alpha \in G : \alpha(u) = u\},$$

the stabiliser of a fixed vertex u in $V(\Gamma)$.

4.4 Theorem (Sabidussi [22]) *Let* Γ *be a graph and* G *a transitive subgroup of* $\text{Aut}(\Gamma)$. *Fix a vertex* $u \in V(\Gamma)$. *Denote* $G_u := \{\alpha \in G : \alpha(u) = u\}$ *and* $S := \{\beta \in G : [u, \beta(u)] \in E(\Gamma)\}$.

Then Γ *is isomorphic to the coset graph* $\text{Cay}(G, S)/G_u$.

Proof

We first need to show that S is a Cayley set before we establish the isomorphism. Since we are considering loop-less graphs, we have that the identity element is not contained in S. Now, if $\beta \in S$, then $[u, \beta(u)] \in E(\Gamma)$. Since $\beta^{-1} \in G$, we have that $[\beta^{-1}(u), \beta^{-1}(\beta(u))] = [\beta^{-1}(u), u] \in E(\Gamma)$. Hence $\beta^{-1} \in S$, so that S is closed under inverses.

As for the isomorphism, consider the map $f : \text{Cay}(G, S)/G_u \to \Gamma$ given by

$$f(\sigma G_u) = \sigma(u).$$

Now,

$$\begin{array}{rrcl} & f(\sigma G_u) & = & f(\sigma' G_u) \\ \Longleftrightarrow & \sigma u & = & \sigma'(u) \\ \Longleftrightarrow & u & = & \sigma^{-1}\sigma'(u) \\ \Longleftrightarrow & & = & \sigma^{-1}\sigma' \in G_u \\ \Longleftrightarrow & \sigma G_u & = & \sigma' G_u. \end{array}$$

Hence f is one to one.
By orbit-stabiliser formula, the number of vertices of Γ coincides with that of $\mathrm{Cay}(G,S)/G_u$. By the pigeonhole principle, we have that f is also onto.

For the preservation of edges: suppose $[\alpha G_u, \tau G_u] \in \mathrm{Cay}(G,S)/G_u$. Then there are elements $\alpha' \in \alpha G_u, \tau' \in \tau G_u$ such that $\tau' = \alpha'\sigma, \sigma \in S$. Therefore

$$\begin{array}{lcc} [\alpha G_u, \tau G_u] & = & [\alpha' G_u, \alpha'\sigma G_u], \text{ so that} \\ f[\alpha G_u, \tau G_u] & = & f[\alpha' G_u, \alpha'\sigma G_u] \\ & = & [\alpha'(u), \alpha'\sigma(u)] \\ & = & \alpha'[u, \sigma(u)]. \end{array}$$

Since $\sigma \in S, [u, \sigma(u)] \in E(\Gamma)$ and since α' is an automorphism, we have that $\alpha'[u, \sigma(u)]$ is an edge in Γ, so f preserves edges.

Now, suppose $[\alpha(u), \alpha'(u)] \in E(\Gamma)$. We need to show that $[\alpha G_u, \alpha' G_u] \in \mathrm{Cay}(G,S)/G_u$. By definition, it is enough to show that $[\alpha, \alpha'] \in \mathrm{Cay}(G,S)/G_u$. Now, $[\alpha(u), \alpha'(u)] \in E(\Gamma)$ implies that $[u, \alpha^{-1}\alpha'(u)] \in E(\Gamma)$. Therefore $\alpha^{-1}\alpha' \in S$, i.e.,

$$\begin{array}{lrcll} & \alpha^{-1}\alpha' & = & \beta \in S & \\ \Rightarrow & \alpha' & = & \alpha\beta \Rightarrow & [\alpha', \alpha] \in \mathrm{Cay}(G,S)/G_u, \end{array}$$

So f^{-1} preserves edges.

∎

This completes the characterisation of vertex-transitive graphs as coset graphs.

As for a characterisation of vertex-transitive graphs in respect with representation of graphs on groupoids, recall that, in Chapter 3, we showed that left translations $\lambda_a, a \in G$ are automorphisms of $\mathrm{GG}(G,S)$ as long as G is a left quasi-group and S is quasi-associative. If in addition, G contains an identity 1, then λ_a maps the vertex 1 to a, for any $a \in G$.

Now if we are given a pair of vertices $a, b \in G$, then the inverse automorphism $(\lambda_a)^{-1}$ maps a to 1 and λ_b maps 1 to b. Composing the two, we obtain an automorphism mapping a to b. In view of this, we have:

4.5 Proposition *Let Q be a left loop and S a quasi-associative Cayley set. Then the groupoid graph* $\mathrm{GG}(Q, S)$ *is vertex-transitive.*

A surprising result of Mwambene [15] is to the effect that the converse also holds. The rest of this section is devoted to the presentation of this important result.

Let Γ be a vertex-transitive graph. Fix a vertex $u \in V(\Gamma)$. Let G be a subgroup of $\mathrm{Aut}(\Gamma)$ which acts transitively on $V(\Gamma)$ and consider the stabiliser of u in G:

$$G_u := \{\alpha \in G : \alpha(u) = u\}.$$

Let T be a left transversal of the left cosets of G_u, i.e., a set containing exactly one distinct element α of each of the left coset αG_u. Note that given any $\sigma, \tau \in T$,

$$\sigma = \tau \Longleftrightarrow \sigma(u) = \tau(u). \tag{4.2}$$

Define a binary operation $*$ on T as follows. Given $\sigma, \tau \in T$, let $\sigma * \tau$ be the representative of the coset $\sigma\tau G_u$. Thus

$$(\sigma * \tau)(u) = \sigma\tau(u). \tag{4.3}$$

Denote by ϵ_T the representative of G_u in T. It is clear that $(T, *)$ has an identity and $\sigma * x = \tau$ for all $\sigma, \tau \in T$ has a unique solution. Hence $Q_T := (T, *)$ is a left loop. Now, let $S := \{\alpha \in G : [u, \alpha(u)] \in E(\Gamma)\}$, and put

$$S_T := S \cap T.$$

4.6 Lemma *S_T is a Cayley set.*

Proof

Clearly the right unit ϵ_T is not in S_T. Moreover, let $\tau \in T, \alpha \in S_T$. Then $\tau = \tau * \epsilon_T = \tau * (\alpha * \alpha') = (\tau * \alpha) * \alpha'$ for some $\alpha' \in S_T$. Hence $\tau \in (\tau * \alpha) * S_T$. We therefore have that S_T satisfies the hypothesis of Lemma 3.13, and hence (i) and (ii) conditions of Definition 3.6. ∎

4.7 Lemma *S_T is quasi-associative in Q_T.*

Proof
Let $\tau, \sigma \in T, \alpha \in S_T$ and $S := \{\alpha \in G : \alpha(u) \in N(u)\}$. Then $[u, \alpha(u)] \in E(\Gamma)$ implies

$$\sigma\tau\alpha(u) \in N((\sigma\tau)(u)), \tag{4.4}$$

where $N((\sigma\tau)(u))$ is the set of neighbours of $(\sigma\tau)(u)$. By using (4.3), $((\sigma * \tau)^{-1}(\sigma\tau))(u) = u$, hence applying $(\sigma*\tau)^{-1}$ to (4.4) we get $((\sigma*\tau)^{-1}(\sigma\tau\alpha))(u) \in N(u)$. Therefore $((\sigma * \tau)^{-1}(\sigma\tau\alpha))(u) = \alpha'(u)$ for some unique $\alpha' \in S_T$. Thus we have

$$\begin{aligned} (\sigma\tau\alpha)(u) &= (\sigma * \tau)(\alpha'(u)) \\ &= ((\sigma * \tau) * \alpha')(u) \quad \text{(because } (\sigma * \tau), \alpha' \in \mathrm{T}). \end{aligned}$$

So using (4.3) twice, we have $(\sigma * (\tau * \alpha))(u) = ((\sigma * \tau) * \alpha')(u)$.

∎

4.8 Theorem (Mwambene [15]) *Suppose Γ is vertex-transitive graph. Then there is a left loop Q with a right unit and a quasi-associative Cayley set S and Γ is isomorphic to* $\mathrm{GG}(Q, S)$.

Proof
Let the notation be as in Lemma 4.6 and Lemma 4.7. It is enough to show that $\mathrm{GG}(Q_T, S_T) \cong \Gamma$.

It is enough to show that the map $f : T \to V(\Gamma)$ defined by

$$f(\tau) = \tau(u),$$

is an isomorphism $\mathrm{GG}(Q_T, S_T) \to \Gamma$.

Since T is a left transversal of left cosets of G_u, f is clearly a bijection.

(i) f preserves adjacency: for $\alpha \in S_T$, the edge $[\tau, \tau * \alpha]$ is mapped to $[\tau(u), (\tau * \alpha)(u)] = [\tau(u), \tau\alpha(u)] = \tau[u, \alpha(u)] \in E(\Gamma)$.

(ii) f^{-1} preserves adjacency: let $[x, y] \in E(\Gamma)$. There is unique $\tau \in T$ such that $\tau(u) = x$. Let $\alpha \in T$ such that $\alpha(u) = \tau^{-1}(y)$. Since $\tau^{-1}(y) \in N(u)$ we have that $\alpha \in S_T$, and hence $[x, y] = [\tau(u), \tau\alpha(u)] = [\tau(u), \tau * \alpha(u)] = f[\tau, \tau * \alpha]$.

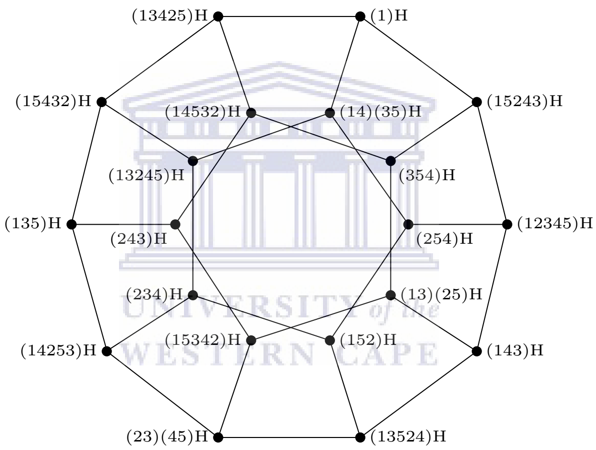
∎

4.3.1 Example of a left loop graph

Consider the alternating group A_5, and $H = \{(1), (123), (132)\}$ a subgroup of A_5. Consider the Cayley set $S = \{(14235), (15324), (14352), (12534), (15243), (13425), (14)(35), (15)(24), (25)(34)\}$.
We have the coset graph $\Gamma = \mathrm{Cay}(G, S)/H$, the dodecahedron.

Since the dodecahedron does not admit geodesic uniform routings [25], it can not be represented on a loop. What we have here is the best there is in terms of the complexity of the groupoid that can represent the dodecahedron graph.

Figure 4.2: Coset graph $\mathrm{Cay}(G, S)/H$ defined on A_5, the dodecahedron



For a fixed $\alpha \in A_5$, by Equation (4.1) $\overline{\lambda}_a$ is an automorphism. Fix $u = (1)H$. $G_u = \{\alpha \in A_5 : \lambda_\alpha(1)(H) = (1)(H)\}$ coincides with H. Now, the left cosets in A_5 modulo H are:

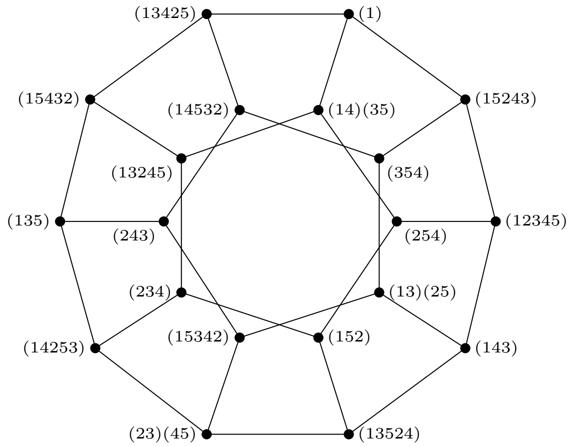
$$
\begin{aligned}
(1)H &= \{(1),(123),(132)\}\\
(243)H &= \{(243),(124),(13)(24)\}\\
(152)H &= \{(152),(153),(15)(23)\}\\
(234)H &= \{(234),(12)(34),(134)\}\\
(135)H &= \{(135),(235),(12)(35)\}\\
(143)H &= \{(143),(14)(23),(142)\}\\
(13)(25)H &= \{(13)(25),(253),(125)\}\\
(254)H &= \{(254),(12543),(13254)\}\\
(354)H &= \{(354),(12354),(13542)\}\\
(12345)H &= \{(12345),(13452),(345)\}\\
(15432)H &= \{(15432),(154),(15423)\}\\
(14532)H &= \{(14532),(145),(14523)\}\\
(13245)H &= \{(13245),(245),(12453)\}\\
(13524)H &= \{(13524),(24)(35),(12435)\}\\
(14253)H &= \{(14253),(14325),(14)(25)\}\\
(15243)H &= \{(15243),(15324),(15)(24)\}\\
(15342)H &= \{(15342),(15)(34),(15234)\}\\
(13425)H &= \{(13425),(25)(34),(12534)\}\\
(14)(35)H &= \{(14)(35),(14235),(14352)\}\\
(23)(45)H &= \{(23)(45),(12)(45),(13)(45)\}.
\end{aligned}
$$

Choose the left transversal $T = \{(1),(243),(152),(234),(135),(143),(13)(25),$ $(254),(354),(12345),(15432),(14532),(13245),(13524),(14253),(15243),$ $(15342),(13425),(14)(35),(23)(45)\}$.

Now from the multiplication defined on the left transversal below, we show that $a * x = b$ for any $a, b \in Q_T$ has a unique solution. We therefore obtain $Q_T = (T, *)$, a left loop.

We have $S_T := S \cap T = \{(15243),(13425),(14)(35)\}$. Clearly the identity element is not in S_T and is closed under the inverses. So S_T is a Cayley set.

Figure 4.3: The $\mathrm{GG}(Q_T, S_T)$ graph isomorphic to $\mathrm{Cay}(G, S)/H$ graph



4.4 Characterisations of Cayley graphs

As alluded to in Chapter 3 Sabidussi has shown that a graph Γ is a Cayley graph if the automorphism of Γ, $\mathrm{Aut}(\Gamma)$ contains a subgroup which acts regularly on the vertices [23, Lemma 4]. The result reads as follows:

4.9 Proposition *Let Γ be a vertex-transitive graph. Suppose there is a subgroup G of $\mathrm{Aut}(\Gamma)$ which acts regularly on $V(\Gamma)$. Then Γ is a Cayley graph.*

The converse of this proposition is immediate. On the other hand, the proof of Proposition 4.9 is exactly that of Theorem 4.4 in the preceding section. The enabling facet is that the subgroup used is the regular group G and G_u

is 1 so that $\mathrm{Cay}(G, S)/G_u$ is precisely $\mathrm{Cay}(G, S)$.

In this section, we introduce another characterisation that depends on the normality of the stabilisers of vertices. We do this in order to motivate why we introduce the concept of quasi-normal subgroups later in the thesis. In other words, we do so in order to put into context the classification of vertex-transitive graphs depending on the strength of the normality of stabilisers of vertices; the running theme of this thesis.

4.10 Theorem *Let Γ be a graph and $u \in V(\Gamma)$. Γ is Cayley graph if and only if there is a subgroup G of $\mathrm{Aut}(\Gamma)$ acting transitively on $V(\Gamma)$ and G_u, the stabiliser of u, is normal in G.*

Proof
Suppose $\Gamma :=\mathrm{Cay}(G, S)$. Clearly the set of translations $\Lambda_G := \{\lambda_a : a \in G\}$ is such a subgroup with the stabiliser, the trivial subgroup 1 which is normal in Λ_G.

On the other hand, suppose there is a subgroup G acting regularly on $V(\Gamma)$ and G_u is normal in G. Consider the set S/G_u, where

$$S := \{\sigma \in G, [u, \sigma(u)] \in E(\Gamma)\}$$

as defined in Theorem 4.4.
Claim: S/G_u is Cayley set in G/G_u.
Since for any $\sigma \in S, \sigma(u)$ is a neighbour of u, $S \cap G_u = \phi$. Therefore $G_u \notin S/G_u$. Again, since S is Cayley set, we easily have that $\alpha^{-1}G_u \in S/G_u$, for any $\alpha G_u \in S/G_u$.

Now, consider the Cayley graph $\mathrm{Cay}(G/G_u, S/G_u)$ and define a map $f : \mathrm{Cay}(G/G_u, S/G_u) \to \Gamma$ by

$$f(\alpha G_u) = \alpha(u).$$

By an identical argument as in the proof of Theorem 4.4, f is a bijection.

For the preservation of edges: suppose $[\alpha G_u, \sigma G_u] \in \mathrm{Cay}(G/G_u, S/G_u)$. Then $\sigma G_u = \alpha G_u.\sigma G_u = \alpha\sigma G_u$, $\sigma G_u \in S/G_u$. Therefore

$$\begin{array}{rcl} [\alpha G_u, \sigma G_u] & = & [\alpha G_u, \alpha\sigma G_u], \text{ so that} \\ f[\alpha G_u, \sigma G_u] & = & f[\alpha G_u, \alpha\sigma G_u] \\ & = & [\alpha(u), \alpha\sigma(u)] \\ & = & \alpha[u, \sigma(u)]. \end{array}$$

Since $\sigma \in S, [u, \sigma(u)] \in E(\Gamma)$ and since α is an automorphism, we have that $\alpha[u, \sigma(u)]$ is an edge in Γ. So, f preserves edges. ∎

*	(1)	(243)	(152)	(234)	(135)	(143)	(254)	(354)	(13)(25)	(12345)
(1)	(1)	(243)	(152)	(234)	(135)	(143)	(254)	(354)	(13)(25)	(12345)
(243)	(243)	(234)	(15243)	(1)	(13245)	(143)	(354)	(23)(45)	(13524)	(13)(25)
(152)	(152)	(15432)	(13)(25)	(15342)	(135)	(15243)	(143)	(14)(35)	(1)	(12345)
(234)	(234)	(1)	(15342)	(243)	(13425)	(143)	(23)(45)	(254)	(12345)	(13524)
(135)	(135)	(13524)	(1)	(14)(35)	(152)	(354)	(13425)	(15342)	(13)(25)	(14532)
(143)	(143)	(243)	(14)(35)	(1)	(14532)	(234)	(254)	(23)(45)	(14253)	(152)
(254)	(254)	(13)(25)	(15432)	(13425)	(354)	(14253)	(13245)	(13524)	(23)(45)	(234)
(354)	(354)	(13524)	(15432)	(135)	(23)(45)	(14)(35)	(13425)	(12345)	(254)	(1)
(13)(25)	(13)(25)	(254)	(135)	(14253)	(152)	(13425)	(243)	(15243)	(1)	(14532)
(12345)	(12345)	(14532)	(234)	(13245)	(13425)	(23)(45)	(152)	(13)(25)	(15342)	(13524)
(15432)	(15432)	(15342)	(254)	(152)	(23)(45)	(15243)	(14)(35)	(14532)	(354)	(1)
(14532)	(14532)	(12345)	(14253)	(23)(45)	(143)	(13245)	(135)	(1)	(14)(35)	(15432)
(13245)	(13245)	(12345)	(243)	(14532)	(15243)	(23)(45)	(135)	(152)	(13524)	(14253)
(13524)	(13524)	(354)	(13245)	(14)(35)	(15243)	(135)	(234)	(15342)	(243)	(14253)
(14253)	(14253)	(13)(25)	(14)(35)	(254)	(143)	(13425)	(13245)	(243)	(14532)	(15432)
(15243)	(15243)	(15432)	(13524)	(152)	(13245)	(15342)	(143)	(14532)	(243)	(13)(25)
(15342)	(15342)	(152)	(13425)	(15432)	(234)	(15243)	(14532)	(143)	(12345)	(354)
(13425)	(13425)	(13)(25)	(234)	(14253)	(15342)	(254)	(13245)	(15243)	(12345)	(14)(35)
(14)(35)	(14)(35)	(13524)	(14532)	(354)	(143)	(135)	(13425)	(234)	(14253)	(15432)
(23)(45)	(23)(45)	(12345)	(15432)	(13245)	(254)	(14532)	(135)	(13)(25)	(354)	(243)

Table 4.1: Cayley table defined on the left transversal T of H on A_5

(15432)	(14532)	(13245)	(13524)	(14253)	(15243)	(15342)	(13425)	(14)(35)	(23)(45)
(15432)	(14532)	(13245)	(13524)	(14253)	(15243)	(15342)	(13425)	(14)(35)	(23)(45)
(15432)	(14253)	(13425)	(12345)	(14)(35)	(15342)	(152)	(135)	(14532)	(254)
(14253)	(13245)	(23)(45)	(354)	(243)	(254)	(13425)	(234)	(13524)	(14532)
(15432)	(14)(35)	(135)	(13)(25)	(14532)	(152)	(15243)	(13245)	(14253)	(354)
(234)	(23)(45)	(13245)	(15243)	(254)	(243)	(143)	(14253)	(15432)	(12345)
(354)	(15342)	(15243)	(13245)	(13425)	(13524)	(135)	(13)(25)	(12345)	(15432)
(15243)	(143)	(1)	(135)	(14532)	(152)	(15342)	(12345)	(14)(35)	(243)
(15342)	(143)	(243)	(13245)	(14253)	(15243)	(152)	(13)(25)	(14532)	(234)
(13524)	(12345)	(23)(45)	(15432)	(234)	(354)	(14)(35)	(143)	(15342)	(13245)
(1)	(354)	(14)(35)	(14253)	(15432)	(143)	(243)	(15243)	(254)	(135)
(14253)	(243)	(234)	(12345)	(13524)	(13425)	(13)(25)	(135)	(13245)	(143)
(13)(25)	(15243)	(15342)	(234)	(13524)	(13425)	(254)	(354)	(243)	(152)
(1)	(254)	(13425)	(15342)	(354)	(234)	(143)	(14)(35)	(15432)	(13)(25)
(12345)	(13)(25)	(254)	(15432)	(1)	(23)(45)	(14532)	(143)	(152)	(13425)
(13524)	(15342)	(152)	(1)	(12345)	(135)	(354)	(23)(45)	(234)	(15243)
(14)(35)	(13425)	(254)	(23)(45)	(234)	(354)	(135)	(1)	(12345)	(14253)
(14253)	(13524)	(135)	(1)	(13245)	(13)(25)	(254)	(23)(45)	(243)	(14)(35)
(243)	(354)	(135)	(152)	(23)(45)	(1)	(143)	(14532)	(15432)	(13524)
(12345)	(152)	(15243)	(243)	(13)(25)	(13245)	(23)(45)	(254)	(1)	(15342)
(152)	(143)	(234)	(13425)	(14)(35)	(15342)	(15243)	(13524)	(14253)	(1)

Chapter 5

Quasi-Cayley graphs

Having described vertex-transitive graphs as left loop graphs with quasi-associative sets in Chapter 4, our attention now turns to groupoid graphs $\mathrm{GG}(G, S)$ for which G is a loop and S is quasi-associative.

5.1 Definition Let Q be a quasi-group with a right unit element. Let S be a quasi-associative Cayley set. The *quasi-Cayley graph* defined on Q by the set S is the groupoid graph $\mathrm{GG}(Q, S)$. As a further qualification, we write $\mathrm{QCay}(Q, S)$ for quasi-Cayley graphs $\mathrm{GG}(Q, S)$.

Since for quasi-groups left cancellation holds and the Cayley sets are quasi-associative, in view of Proposition 3.11 quasi-Cayley graphs are vertex-transitive. However, in addition, because right cancellation also holds, quasi-Cayley graphs are distinguishable from the general vertex-transitive graphs.

While right cancellability may, at first glance, not seem important in the context of automorphisms, this feature has ramifications in distinguishing vertex-transitive graphs.

Gauyacq [9] considered quasi-Cayley graphs in the context of uniform routings. She showed that (a) quasi-Cayley graphs as a class of graphs is bigger than Cayley graphs; (b) a graph in this class admits geodesic uniform routings; and (c) that there are some classes of vertex-transitive graphs which are not quasi-Cayley graphs.

It has been shown that the dodecahedron does not admit uniform geodesic routings. In view of the fact that not all vertex-transitive graphs do not accept uniform routing, this class is worthy consideration in its own right. We have shown that the dodecahedron is representable on a left loop and it is known that it does not admit geodesic uniform routings [25]. The hierarchy of symmetry is therefore worth considering.

In this chapter we first present a characterisation of quasi-Cayley graphs in terms of a set of automorphisms acting regularly on the set of vertices before we consider their characterisations in the context of the proof of Theorem 4.8. The second characterisation is the main thrust of our thesis.

5.1 Characterisation of quasi-Cayley graphs

Gauyacq [9] showed that a graph is quasi-Cayley if and only if the automorphism group of the graph contains a set of automorphisms that acts regularly on the set of vertices. Her proof is based on defining a binary operation on the set of vertices of a given graph. We depart from that approach. Instead, we define a binary operation on the set of automorphisms acting regularly on the vertices to describe a new graph and link the graph we define to the given graph by an isomorphism. The proof is in the same spirit with that of Sabidussi's characterisation of Cayley graphs [23, Lemma 4], and those that exploit similar phenomenon. See, for instance, [22, Theorem 2] and [15, Theorem 9].

Let Γ be a graph and F a set of automorphism of Γ acting regularly on $V(\Gamma)$. Fix $u \in V(\Gamma)$, a base point. Denote $f_x \in F$ an automorphism that maps u to x, i.e.,

$$f_x(u) = x.$$

Because F is acting regularly on $V(\Gamma)$, f_x is uniquely determined.

Define a binary operation $*$ on F by

$$f_x * f_y = f_z, \tag{5.1}$$

where $f_x(f_y(u)) = z$, i.e., $f_x(y) = z$.

That the binary operation is well-defined follows from the fact that F acts regularly on $V(\Gamma)$.

5.2 Lemma *F together with the binary operation defined by* (5.1) *is a loop, i.e., both right and left cancellation hold and there is a right unit.*

Proof
Suppose

$$f_x * f_y = f_{x'} * f_y = f_z.$$

Then $f_x(y) = f_{x'}(y)$. Hence $x = x'$. (by regularity of F).
Therefore $f_x = f_{x'}$, so we have right cancellation.

Suppose

$$f_x * f_y = f_x * f_{y'} = f_{z'}.$$

Then $f_x(y) = f_x(y')$. Hence $y = y'$. (by regularity of F).
Therefore $f_y = f_{y'}$, so we also have left cancellation.

As for a right-unit of $(F, *)$, consider f_u.

$$\begin{aligned} f_x * f_u(u) &= f_x(u), \\ &= x. \end{aligned}$$

Therefore $f_x * f_u = f_x$. Hence f_u is the right-unit. ∎

Let S be the set of neighbours of u. Denote F_S the automorphisms in F which assign u to its neighbours, i.e.,

$$F_S := \{f \in F : [u, f(u)] \in E(\Gamma)\}. \tag{5.2}$$

1 Claim *Let $(F, *)$ be the loop defined in Lemma 5.2. Then F_S as defined in (5.2) is Cayley set.*

Proof
We have $[u, f_u(u)] = [u, u] \notin E(\Gamma)$, hence $f_u \notin F_S$.

Suppose $f_x \in F_S$. Then $[u, x] \in E(\Gamma)$. We have $(f_x)^{-1}(x) = u$. Hence

$$\begin{aligned} f_x^{-1}[u, x] &= [f_x^{-1}(u), f_x^{-1}(x)] \in E(F), \\ &= [f_x^{-1}(u), u] \in E(\Gamma). \end{aligned}$$

Therefore $f_x^{-1} \in F_S$. ∎

2 Claim *Let $(F, *)$ be the loop defined in Lemma 5.2. Then F_S is quasi-associative.*

Proof
It is enough to show that

$$f_x * (f_y * (F_S)) = (f_x * f_y) * F_S. \tag{5.3}$$

Note that $f_x[S]$ is the neighbourhood of x since for all $s \in S$,

$$\begin{aligned} f_x[u, s] &= [f_x(u), f_x(s)], \\ &= [x, f_x(s)] \in E(\Gamma), \end{aligned}$$

in view of the fact that f_x is an automorphism of Γ.

Now, $f_x * (f_y * F_S) = f_x * F_W = F_{W_x}$, where

$$\begin{aligned} F_W : &= \{f_w \in F; [y, w] \in E(\Gamma)\}; \quad w \in W \\ F_{W_x} : &= \{f_{w_x} \in F; [f_x(y), w_x] \in E(\Gamma)\}; \quad w_x \in W_{w_x}. \end{aligned}$$

$(f_x * f_y) * F_S = f_z * F_S = F_{W'}$, where $z = f_x(y)$ and $W' = N(z) = N(f_x(y))$. Therefore $F_{W'} = F_{W_x}$. Hence Equation (5.1) holds. ∎

5.3 Lemma *Let* $(F, *)$ *be the loop defined in Lemma* 5.2 *and* F_S *the Cayley set defined by* (5.2). *Then* $\mathrm{QCay}(F, F_S)$ *is isomorphic to* Γ.

Proof
Consider the map $\sigma : F \to V(\Gamma)$ defined by

$$\sigma(f) = f(u).$$

(a) σ is one to one:

$$\begin{aligned} \sigma(f) &= \sigma(f') \\ \iff f(u) &= f'(u) \\ \iff f &= f' \quad \text{(by regularity of } F) \end{aligned}$$

(b) σ is onto:

For any $x \in V(\Gamma)$, consider $f \in F$ such that $f(u) = x$. Indeed such an f exists, because of regularity of F. Then $\sigma(f) = f(u) = x$. Therefore, σ is onto.

(c) σ preserves edges:

Suppose $[f_x, f_y] \in E(\mathrm{QCay}(F, F_S))$

$$\begin{aligned} \Rightarrow f_y &= f_x * f_s, f_s \in F_S; \text{ where } y = f_x(s) \\ \Rightarrow \sigma[f_x, f_y] &= [\sigma(f_x), \sigma(f_y)] \\ &= [f_x(u), f_y(u)] \\ &= [x, f_x(s)] \\ &= f_x[u, s] \in E(\Gamma), \end{aligned}$$

since $[u, s] \in E(\Gamma)$ and f_x is an automorphism. ∎

5.4 Theorem (Gauyacq [9]) *Let Γ be a graph. The automorphism group of Γ contains a set acting regularly on $V(\Gamma)$ if and only if Γ is a quasi-Cayley graph.*

Proof

Suppose $\Gamma = \mathrm{QCay}(Q, S)$ is a quasi-Cayley graph. Then, for any fixed $a \in Q$, define a map $\lambda_a : V(\Gamma) \to V(\Gamma)$ by

$$\lambda_a(x) = ax.$$

That λ_a is an automorphism is the content of Proposition 3.11.

Consider the set $\Lambda_A := \{\lambda_a : a \in Q\}$. We need to show that Λ_A is regular.

For any $(x, y) \in V(\Gamma)$ there is unique automorphism $\lambda_a \in \Lambda_A$ such that

$$\lambda_a(x) \quad = y \Rightarrow ax = y; \tag{5.4}$$

because in a loop, Equation (5.2) has a unique solution.
Also, suppose

$$\begin{aligned} \lambda_a(x) &= \lambda_b(x) \\ ax &= bx \\ \Rightarrow \quad a &= b. \end{aligned}$$

Therefore $\lambda_a = \lambda_b$. Hence, there is exactly one automorphism that maps x to y.

Now, suppose Γ is a graph and $\mathrm{Aut}(\Gamma)$ contains a set F of automorphism acting regularly on $V(\Gamma)$. Then by Lemma 5.3, Γ is isomorphic to a quasi-Cayley graph, completing the proof of the theorem. ∎

Whilst Theorem 5.4 suffices to characterise quasi-Cayley graphs, it does not necessarily reveal the structure and properties of transitive subgroups of graphs which give rise to them.

The characterisation below motivates the concept of strongly simple groups we consider in Chapter 6.

5.5 Theorem *A graph* Γ *is quasi-Cayley if and only if there is a transitive subgroup* H *of* $\mathrm{Aut}(\Gamma)$ *and a left transversal* T *of a stabiliser* K *for some* $u \in V(\Gamma)$*, such that* $(T, *)$ *is a loop, where* $*$ *is the binary operation defined by* (4.3)*.*

Proof

Suppose $\mathrm{Aut}(\Gamma)$ contains a subgroup acting transitively on $V(\Gamma)$ and a left transversal T such that $(T, *)$ is a loop as in the hypothesis. Then $\mathrm{GG}(Q_T, S_T)$ as defined in Theorem 4.8 describes a quasi-Cayley graph.

Suppose Γ is quasi-Cayley. Fix $u \in V(\Gamma)$. By Theorem 5.4, there is a set of automorphisms T which acts regularly on $V(\Gamma)$. It is easy to see that T is a left transversal of a stabiliser K of $u \in V(\Gamma)$.

Let α, β, $\beta' \in T$.
Suppose

$$\begin{aligned} \alpha * \beta &= \alpha * \beta' \\ (\alpha\beta)u &= (\alpha\beta')u \end{aligned}$$

Then

$$\begin{aligned} \alpha^{-1}(\alpha\beta)(u) &= \alpha^{-1}(\alpha\beta')(u) \\ \beta(u) &= \beta'(u) \\ \beta &= \beta'. \text{ (by regularity of } T\text{)}. \end{aligned}$$

Let α, α', $\beta \in T$.
Suppose

$$\begin{aligned} \alpha * \beta &= \alpha' * \beta \\ \alpha\beta(u) &= \alpha'\beta(u) \\ \alpha(\beta(u)) &= \alpha'(\beta(u)) \\ \alpha &= \alpha'. \text{ (by regularity of } T\text{)}. \end{aligned}$$

Hence $(T, *)$ is a loop. ∎

5.1.1 An example of a quasi-Cayley graph

While admitting that K_5 can be represented as a Cayley graph on $\mathbb{Z}_5$, a much more sophisticated groupoid, we represent it here on a loop defined in the context of (4.3). We do so in part to illustrate that A_5, the alternating group on 5 points, whilst admittedly a simple group, has a subgroup that defines a loop, hence not strongly simple.

We will discuss the concept of strong simplicity and the question of groups which contain subgroups that give rise to loops in Chapter 6.

Consider the alternating group, $G = A_5$ and $H = \{(1), (145), (154), (15)(34), (13)(45), (134), (143), (135), (153), (345), (354), (14)(35)\}$ a subgroup of G. We have the left cosets in G/H as follows.

$(1)H$=$\{(1), (145),\ (154),\ (15)(34), (13)(45), (134), (143), (135), (153), (345),\ (354), (14)(35)\}$
$(12345)H$=$\{(12345), (12354), (123), (124), (12)(35), (12453), (12)(45), (12534), (12)(34), (12435), (125), (12543)\}$
$(15432)H$=$\{(15432), (243), (14325), (253), (14)(23), (15)(23), (15324), (23)(45), (13254), (132), (14532), (13245)\}$
$(13524)H = \{(13524), (13)(25), (13452), (14523), (25)(34), (14352), (235), (15243), (245), (14)(25), (15234), (152)\}$
$(14253)H = \{(14253), (15342), (24)(35), (13542), (15)(24), (254), (13425), (142), (14235), (15423), (13)(24), (234)\}$.

Choose the left transversal $T = \{(1), (12345), (15432), (13524), (14253)\}$, and define a binary operation as in (4.3).

*	(1)	(12345)	(15432)	(13524)	(14253)
(1)	(1)	(12345)	(15432)	(13524)	(14253)
(12345)	(12435)	(13524)	(1)	(14253)	(15432)
(15432)	(15432)	(1)	(14253)	(12345)	(13524)
(13524)	(13524)	(14253)	(12345)	(15432)	(1)
(14253)	(14253)	(15432)	(13524)	(1)	(12345)

Table 5.1: Cayley table of the quasi-Cayley graph on A_5

Note that since the Cayley table is a Latin square $(T, *)$ is loop. Consider $S_T = T \setminus \{(1)\}$. Clearly S_T is both a Cayley set and quasi-associative.

The graph it defines as expected is K_5 below.

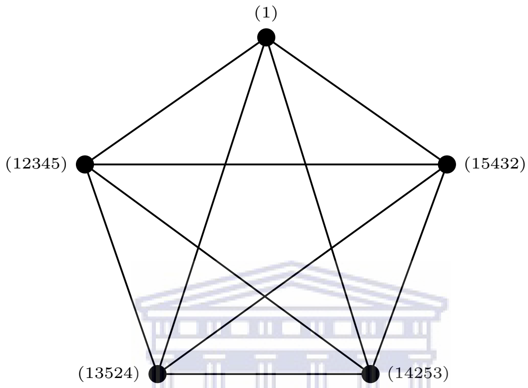


Figure 5.1: Quasi-Cayley graph defined on A_5

5.2 Extensions of McKay-Praeger graphs

B. McKay and C. Praeger [12] introduced, amongst others, a class of non-Cayley vertex-transitive graphs, which for lack of better name have been called the McKay-Praeger $T(2m^2)$ graphs by Guayacq [9]. They are defined on the direct product of three cyclic groups, one of which is $\mathbb{Z}_2$ and the definition of their adjacency relation involves some interaction between the groups.

In order to give an indication in the richness of quasi-Cayley graphs, in this section, we present these graphs as groupoid graphs defined on loops with quasi-associative Cayley sets, and in the course, provide a natural extension of the class in two ways. First, by observing the pertinent characteristics of the Cayley sets describing these graphs, we give a general description of the sets. Second, loops that represent these graphs are given in a more general setting. The twisting maps that define the loops are generalised in higher dimension as was done in [16] in defining generalised Petersen graphs in higher dimension. For a positive integer r, we therefore as a general case define graphs on $\prod_i^r(\mathbb{Z}_{m_i^2}) \times \mathbb{Z}_2$. The content of this section is under review in [8]. That McKay-Praeger $T(2m^2)$ graphs are quasi-Cayley graphs was first shown by Gauyacq [9] by presenting a set of automorphisms that acts regularly on the vertices.

The McKay-Praeger $T(2m^2)$ graphs we are alluding to are defined as follows:

For $m \geqslant 3$, an integer, define the graph $T = T(2m^2)$ of order $(2m^2)$ as follows:

$V(T) = \mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_2$ and $E(T) = E_1 \cup E_2 \cup E_3$,
where

$$\begin{array}{rcl} E_1 &=& \{(x,y,0)(x+1,y,0),(x,y,1)(x,y+1,1) \mid x,y \in \mathbb{Z}_m\} \\ E_2 &=& \{(x,y,0)(x+1,y-1,0),(x,y,1)(x+1,y+1,1) \mid x,y \in \mathbb{Z}_m\} \\ E_3 &=& \{(x,y,0)(x-1,y-1,1),(x,y,0)(x-1,y+1,1),(x,y,0)(x+1,y-1,1), \\ && (x,y,0)(x+1,y+1,1) \mid x,y \in \mathbb{Z}_m\}. \end{array}$$

5.3 Loop graphs representing McKay-Praeger $T(2m^2)$ graphs

Let $\mathcal{L} = \mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_2$. Define a binary operation $\oplus$ on $\mathcal{L}$ by

$$(a,b,c) \oplus (a',b',c') = (a + (1-c)a' - cb', b + (1-c)b' + ca', c + c').$$

We now show that $\mathcal{L}$ defines a loop under the binary operation $\oplus$.

First, it is clear that $(0,0,0)$ is a two-sided identity.

5.6 Lemma *$\mathcal{L}$ is a loop.*

Proof
It is enough to show that left and right cancellation hold.
Now, suppose that $(a,b,c) \oplus (a',b',c') = (a,b,c) \oplus (a'',b'',c'')$. We have two cases:

(i) $c = 0$: in this case the binary operation coincides with the direct product of groups;

(ii) $c = 1$: the left hand side is $(a - b', b + a', 1 + c')$ and the right hand side is $(a - b'', b + a'', 1 + c'')$ so that $b' = b'', a' = a''$ and $c' = c''$.

On the other hand, if $(a', b', c') + (a, b, c) = (a'', b'', c'') + (a, b, c)$ then the left hand side is $(a' + (1 - c')a - c'b, b' + (1 - c')b + c'a, c' + c)$ and the right hand side is $(a'' + (1 - c'')a - c''b, b'' + (1 - c'')b + c''a, c'' + c)$. Hence $c' = c''$. So, write the right hand side as $(a'' + (1 - c')a - c'b, b'' + (1 - c')b + c'a, c' + c)$ and hence cancellation follows. ∎

In view of Lemma 3.13, let us first discuss quasi-associativity of the subsets we consider.

5.7 Lemma *Let S be a subset of $\mathcal{L}$ such that $s' = (-a, -b, -c) \in S$ whenever $s = (a, b, c) \in S$. Then S is quasi-associative in $\mathcal{L}$.*

Proof

We need to show that for any $x, y \in \mathcal{L}, s \in S$, there exists an $s' \in S$ such that $(x \oplus y) \oplus s = x \oplus (y \oplus s')$.

Write $x = (a, b, c); y = (a', b', c')$. We have four cases to consider: (i) $c = c' = 0$, (ii) $c = 0, c' = 1$, (iii) $c = 1, c' = 0$ and (iv) $c = c' = 1$. In the first three cases, clearly $s' = s$ will do.

As for the last case, we have $s = (a_s, b_s, c_s), x = (a, b, 1), y = (a', b', 1)$ so that $(x \oplus y) \oplus s = (a - b', b + a', 0) + (a_s, b_s, c_s) = (a - b' + a_s, b + a' + b_s, c_s)$.

Consider $s' = (-a_s, -b_s, -c_s)$. Then $x \oplus (y \oplus s') = (a, b, 1) \oplus ((a', b', 1) \oplus (-a_s, -b_s, -c_s)) = (a, b, 1) \oplus (a' + b_s, b' - a_s, 1 - c_s) = (a - b' + a_s, b + a' + b_s, -c_s)$, as required. ∎

For subsets which are closed under inverses, we have that $(a_s, a_s, 1)^{-1} = (-a_s, a_s, 1)$ for any $a_s \in \mathbb{Z}_m$ and $(a_{s'}, b_{s'}, 0)^{-1} = (-a_{s'}, -b_{s'}, 0)$ for any $a_{s'}, b_{s'} \in \mathbb{Z}_m$. The subsets $S := \{(a_s, a_s, 1), (-a_s, a_s, 1) : a_s \in \mathbb{Z}_m\}$ and $S' := \{(a_{s'}, b_{s'}, 0), (-a_{s'}, -b_{s'}, 0) : a_{s'}, b_{s'} \in \mathbb{Z}_m\}$ are therefore closed under inverses.

We thus have that the subsets of the form $S \cup S'$ are quasi-associative Cayley sets in $\mathcal{L}$ so that the Cayley graphs Cay$(\mathcal{L}, S \cup S')$ are quasi-Cayley graphs. Now, as for the representation of McKay-Praeger $T(2m^2)$ graphs as Cayley graphs, we need to identify the Cayley sets that describe them.

5.8 Proposition *The graphs $T(2m^2)$ are quasi-Cayley.*

Proof

In view of the discussion above, we have that the subset

$$S = \{(111), (-1-11), (-111), (1-11), (100), (-100), (1-10), (-110)\}$$

is a Cayley set. It is therefore enough to show that edges in $T(2m^2)$ correspond to edges in $\mathrm{Cay}(\mathcal{L}, S)$; implicitly using the identity map as the considered isomorphism between the two graphs.

Now, edges of the form $[(x, y, 0), (x \pm 1, y \pm 1, 1)]$ are exactly edges of the form $[(x, y, 0), (x, y, 0) \oplus s]$ where $s = (\pm 1, \pm 1, 1)$ in $\mathrm{Cay}(\mathcal{L}, S)$. Those of the form $[(x, y, 0), (x+1, y, 0)], [(x, y, 0), (x+1, y-1, 0)]$ correspond to edges $[(x, y, 0), (x, y, 0) \oplus (1, 0, 0)], [(x, y, 0), (x, y, 0) \oplus (1, -1, 0)]$ respectively. As for edges of the form $[(x, y, 1), (x, y+1, 1)], [(x, y, 1), (x+1, y+1, 1)]$, they correspond to $[(x, y, 1), (x, y, 1) \oplus (1, 0, 0)], [(x, y, 1), (x, y, 1) \oplus (1, -1, 0)]$ respectively.

Note that $(x+1, y, 0), (x+1, y-1, 0)$ coincide with $(0, 0, 0)$ in the edges of the form $[(x, y, 0), (x+1, y, 0)], [(x, y, 0), (x+1, y-1, 0)]$ respectively when $x = m-1, y = 0$ and $x = m-1, y = 1$ respectively which in $\mathrm{Cay}(\mathcal{L}, S)$ corresponds to edges $[(x, y, 0), (x, y, 0) \oplus (-1, 0, 0)], [(x, y, 0), (x, y, 0) \oplus (-1, 1, 0)]$. ∎

Once we see the nature of the Cayley sets, there is no point in restricting to graphs of degree 8; as long as m is big enough one can have a regular graph of degree a multiple of 8.

5.4 Generalising McKay-Praeger graphs

As with the generalisation of the generalised Petersen graphs in higher dimension [16], here too, it is easy to extend this class of graphs into higher dimension. This, we do in the following way.

For $m_i \geq 3, i = 1, \cdots, r$, let $\mathcal{L}(\Pi_{i=1}^{r} m_i)$ be the set $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_2} \times \cdots \mathbb{Z}_{m_r} \times \mathbb{Z}_{m_r} \times \mathbb{Z}_2$. Consider the binary operation $\oplus$ on $\mathcal{L}(\Pi_{i=1}^{r} m_i)$ given by $(a_1, b_1, a_2, b_2, \cdots, a_r, b_r, c) \oplus (a_1', b_1', a_2', b_2', \cdots, a_r', b_r', c') = (a_1 + (1-c)a_1' - cb_1', b_1 + (1-c)b_1' + ca_1', \cdots, a_r + (1-c)a_r - cb_r', b_r + (1-c)b_r' + ca_r', c + c')$.

Again, it is easy to see that $(0, \cdots, 0)$ is two-sided identity.

Now, we also show that left and right cancellation hold.

Suppose $(a_1, b_1, \cdots, a_r, b_r, c) \oplus (a'_1, b'_1, a'_2, b'_2, \cdots, a'_r, b'_r, c') = (a_1, b_1, a_2, b_2, \cdots, a_r, b_r, c) \oplus (a''_1, b''_1, a''_2, b''_2, \cdots, a''_r, b''_r, c'')$. Again, we have two cases to consider:

Case (i) $c = 0$

The left hand side simplifies to $(a_1 + a'_1, b_1 + b'_1, \cdots, a_r + a'_r, b_r + b'_r, c')$ and the right hand side to $(a_1 + a''_1, b_1 + b''_1, \cdots, a_r + a''_r, b_r + b''_r, c'')$. This implies that $a'_1 = a''_1$, $b'_1 = b''_1$, $a'_r = a''_r$, $b'_r = b''_r$, $c' = c''$.

Case (ii) $c = 1$, Here again, the same argument works element-wise as was the case in Section 3.

Again, to show that right cancellation holds, the proof is as in Lemma 5.

Now for quasi-associativity, we have the following.

3 Claim *Let S be a subset of $\mathcal{L}$ such that $s' = (-a_1, -b_1, \cdots, -a_r, -b_r, c) \in S$, whenever $s = (a_1, b_1, \cdots, a_r, b_r, c) \in S$. Then S is quasi-associative.*

We need to show that for any $x, y \in \mathcal{L}$, $s \in S$, there exists $s' \in S$ such that $(x \oplus y) \oplus s = x \oplus (y \oplus s')$. We have four cases; namely, (1) $c = c' = 0$, (2) $c = 0, c' = 1$, (3) $c = 1, c' = 0$ and (4) $c = c' = 1$.

Clearly for the first three cases, $s = s'$ will do. For the last case, we have the following:

$c = c' = 1$: Let $x = (a_1, b_1, a_2, b_2, \cdots, a_r, b_r, c)$, $y = (a'_1, b'_1, a'_2, b'_2, \cdots, a'_r, b'_r, c')$, $s = (a_s, b_s, \cdots, a_{sr}, b_{sr}, c_s)$ then $(x \oplus y) \oplus s = x \oplus (y \oplus s')$, given by the following. The left hand side is $(a_1, b_1, a_2, b_2, \cdots, a_r, b_r, 1) \oplus (a'_1, b'_1, a'_2, b'_2, \cdots, a'_r, b'_r, 1)$. So we have $(a_1 - b'_3 1, b_1 + a'_1, \cdots, a_r - b'_r, b_r + a'_r, 0)$. This implies $(a_1 - b'_1, b_1 + a'_1, \cdots, a_r - b'_r, b_r + a'_r, 0) \oplus (a_s, b_s, \cdots, a_{sr}, b_{sr}, c_s)$
$= (a_1 - b'_1 + a_s, b_1 + a'_1 + b_s, \cdots, a_r - b'_r + a_{sr}, b_r + a'_r + b_{sr}, c_s)$.

Consider $s' = (-a_s, -b_s, \cdots, a_{sr}, -b_{sr}, -c_s)$. Then $x \oplus (y \oplus s')$ is $(a'_1, b'_1, a'_2, b'_2, \cdots, a'_r, b'_r, 1) \oplus (-a_s, -b_s, \cdots, a_{sr}, -b_{sr}, -c_s)$. This evaluates to $(a'_1 + b_s, b'_1 - a, \cdots, a'_r + b_{sr}, b'_r - a_{sr}, 1 - c_s)$. Hence we have $(a_1, b_1, a_2, b_2, \cdots, a_r, b_r, 1) \oplus (a'_1 + b_s, b'_1 - a_s, \cdots, a'_r + b_{sr}, b'_r - a_{sr}, 1 - c_s) = (a_1 - b'_1 + a_s, b_1 + a'_1 + b_s, \cdots, a_r - b'_r + a_{sr}, b_r + a'_r + b_{sr}, -c_s)$. ∎

As for Cayley sets, we have the following.

5.9 Lemma *The sets*

$$S := \{(a_s, b_s, \cdots, a_r, b_r, 1), (-a_s, b_s, \cdots, -a_r, b_r, 1) : a_s, b_s, \cdots, a_r, b_r \in \mathbb{Z}_m\}$$

and

$$S' := \{(a_s, b_s, \cdots, a_r, b_r, 0), (-a_s, -b_s, \cdots, -a_r, -b_r, 0) : a_s, b_s, \cdots, a_r, b_r \in \mathbb{Z}_m\}$$

are Cayley sets.

Proof

In view of Lemma 3.13, it is enough to show that the sets are closed under inverses.

(i) For any $s = (a_s, b_s, \cdots, a_r, b_r, 1)$, $s' = (-a_s, b_s, \cdots, -a_r, b_r, 1)$, then $s + s' = (0, 0, 0, \cdots, 0, 0)$.

(ii) Since the binary operation is that of direct product of groups in this case, this follows immediately.

∎

Note that we need both sets defined in Lemma 5.9 for quasi-associativity. With this set, we define generalised McKay-Praeger graphs as follows.

5.10 Definition The generalised McKay-Praeger graph of dimension r is the Cayley graph $\mathrm{Cay}(\mathcal{L}(\Pi_{i=1}^{r} m_i), S \cup S')$, where S and S' are the Cayley sets defined in Lemma 5.9.

Chapter 6

Strongly simple groups

6.1 Introduction

Motivated by classification of vertex-transitive graphs, in this chapter we introduce the concept of strongly simple groups. Besides $\mathbb{Z}_p$, p is prime, we have not been able to identify a strongly simple group. However, given the classification of vertex-transitive graphs, we would like to motivate that such classification of groups may not be vacuous.

Theorem 4.10 is to the effect that Cayley graphs are intimately linked to normal subgroups. Given a vertex-transitive graph Γ, if all transitive subgroups of $\text{Aut}(\Gamma)$, including $\text{Aut}(\Gamma)$ itself, are simple and do not act regularly on $V(\Gamma)$, then Theorem 4.10 is to the effect that Γ is not Cayley.

In Chapter 5, we presented another class of vertex-transitive graphs; namely, quasi-Cayley graphs. In the context of Proposition 4.5, for a given graph Γ, $u \in V(\Gamma)$ and H a subgroup of $\text{Aut}(\Gamma)$ acting transitively on $V(\Gamma)$, the stabilisers of u render themselves to classification in the following way.

6.1 Definition A subgroup H of a group G is said to be *quasi-normal* in G if there exists a left transversal T of H such that with the binary operation $*$ defined by (4.3) on T, $(T, *)$ is a loop.

For example, let G, H and $*$ be defined as in Section 5.1.1. It was shown that, for $T = \{(1), (12345), (15432), (13524), (14253)\}$, $(T, *)$ is a loop.

Let Γ be a vertex-transitive graph and fix $u \in V(\Gamma)$. If for any transitive subgroup G of $\mathrm{Aut}(\Gamma)$, G_u, the stabiliser of u, is not quasi-normal, then Γ is not quasi-Cayley. This motivates the following definition.

6.2 Definition A group G is strongly simple if it does not contain a quasi-normal subgroup.

Clearly, $\mathbb{Z}_p$, p is prime, is vacuously strongly simple. In view of 5.1.1, A_5 is not strongly simple. In the following section, we will show that A_n, the alternating group on n elements, for n odd, $n \geqslant 5$, is not strongly simple.

6.2 Simple but not strongly simple alternating groups

Here, we show that for $n \geqslant 5$, $n \equiv 1 \bmod 2$, A_n, the alternating group on n elements, for n odd, is not strongly simple. We do this in terms of factorisations of a complete graph K_n.

In order to identify a left transversal that give rise to a loop, we need the following.

For the complete graph K_n, we replace each edge $[x, y]$ by arcs (x, y) and (y, x). The result is a complete bi-directed graph $\vec{K_n}$ with $V(K_n) = \{1, \cdots, n\}$. This digraph can be factored into $(n-1)$ 1-regular directed Hamiltonian subgraphs.

Let $\mathcal{F}$ be such a factorisation. That is, $\mathcal{F}$ satisfies the following conditions:

(i) $V(F) = V(G)$ for any $F \in \mathcal{F}$, i.e., F is a spanning directed cycle;

(ii) $\vec{E}(F) \cap \vec{E}(F') = \phi$ for any two distinct F, $F' \in \mathcal{F}$, that is, F and F' are arc-disjoint;

(iii) $\vec{E}(G) = \bigcup_{F \in \mathcal{F}} \vec{E}(F)$;

(iv) $d_F^+(x) = d_F^-(x) = 1$ for any $x \in V(G)$ and $F \in \mathcal{F}$.

Since each $F \in \mathcal{F}$ is a directed Hamiltonian cycle, $\mathcal{F}$ is said to be a directed *Hamiltonian decomposition.*

We now turn to the problem of existence of such factorisations. We present a directed Hamiltonian decomposition given by Lucas and attributed to Walecki. Our work is wholly that of Alspach [1].

Let us construct for all values of $2n+1$ a directed Hamiltonian factorisation. Label the vertices of K_{2n+1} as $x_0, x_1, x_2, \cdots, x_{2n}$ and let ϕ be the permutation whose disjoint cycle decomposition representation is

$$\phi = (x_1 x_2 \cdots x_{2n}).$$

Consider a directed Hamiltonian cycle C_1 given by

$$C_1 = x_0 x_1 x_2 x_{2n} x_3 x_{2n-1} \cdots x_n x_{n+2} x_{n+1} x_0. \tag{6.1}$$

Then denote

$$C_i = \phi^{i-1}(C_1), i = 1, 2, \cdots, n. \tag{6.2}$$

6.3 Proposition [1, p. 9] *Let C_i be defined as in* (6.2). *K_{2n+1} admits a decomposition $\mathcal{C} = \{\phi^{i-1}(C_1) : i = 1, 2, \cdots, n\}$, i.e.,*

$$K_{2n+1} = C_1 \oplus C_2 \oplus \cdots \oplus C_n. \tag{6.3}$$

Proof
We consider a directed Hamiltonian cycle C_i. Define the length of an arc $[x_i, x_j]$ by $j - i \pmod{2n}$ where $i, j \neq 0$ and $i, j \in \{1, 2, \cdots, n\}$. The arcs of odd length r are $[x_{\frac{(4n-r+1)}{2}}, x_{\frac{(r+1)}{2}}]$ and $[x_{\frac{(2n-r+1)}{2}}, x_{\frac{(2n+r+1)}{2}}]$, $1 \leq r < n$. Every arc of odd length r appears once in $C_1, C_2, \cdots, C_n$, since ϕ^{i-1} preserves length. The arcs $[x_{\frac{(4n-r+2)}{2}}, x_{\frac{(r+2)}{2}}]$ and $[x_{\frac{(2n-r+2)}{2}}, x_{\frac{(2n+r+2)}{2}}]$, $1 < 2 < n$, have even length r, so that every arc of even length r appears once as well. The arcs of length n are $[x_i, x_{i+n}]$, $1 \leq n$, and there is one of them in each directed Hamiltonian cycle. Finally, the arcs incident with x_0 in C_i are $[x_0, x_i]$ and $[x_0, x_{n+i}]$ so that all arcs incident with x_0 are used. ∎

Now, for a given directed Hamiltonian factorisation $\mathcal{F}$ of K_n, define a map $\alpha : \mathcal{F} \to S_n$, the symmetric group on n points, by

$$\alpha(F) = \begin{pmatrix} 1 & 2 & \cdots & n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}, \tag{6.4}$$

if $(i, x_i) \in F$, $i = 1, 2, \cdots, n$. Since each vertex has out-degree 1 and in-degree 1, $\alpha(F)$ is a directed cycle.

Suppose $F \neq F'$. Then for any $(x, y) \in E(F)$, $(x, z) \in E(F')$ we have $y \neq z$. Therefore $\alpha(F) \neq \alpha(F')$. Hence α is one to one.

Now, if $n \geqslant 5$, n is odd, then $\alpha(\mathcal{F}) \subset A_n$. As A_n, the alternating group acts on n points, A_n as a subgroup of $\mathrm{Aut}(K_n)$, we have:

6.4 Lemma *Let $\mathcal{F}$ be a given directed Hamiltonian factorisation, the map defined in Equation* (6.4) *and α be as above. Then $T := \alpha(\mathcal{F}) \cup \{(1)\}$ is a left transversal of $(A_n)_1$ in A_n, where $(A_n)_1$ is the stabiliser of* 1.

Proof
Since $\sigma \in \tau(A_n)_1$ if and only if $\sigma(1) = \tau(1)$, it is enough to show that there is exactly one permutation in T which maps 1 to i, for each $i = 1. \cdots, n$.

Now, if $i = 1$, then the identity map (1) will do. If $i \neq 1$, let F be the factor containing the arc $(1, i)$. Then $\alpha(F)$ maps, by Equation (6.4), 1 to i. ∎

6.5 Lemma *Let $\mathcal{F}$ be a factorisation of K_n constructed by algorithm in* (6.4) *and T be the corresponding left transversal defined in Lemma* 6.4. *Then $(T, *)$, where $*$ is the binary operation defined in* (4.3), *is a loop.*

Proof
Denote α_i the map in T such that $\alpha_i(1) = i$. Note that this is well-defined. Otherwise $\mathcal{F}$ would not be a well-defined factorization.
For left cancellation, we have:

$$\begin{array}{crcl} & \alpha_i * \alpha_j & = & \alpha_i * \alpha_{j'} \\ \Longleftrightarrow & \alpha_i\alpha_j(A_n)_1 & = & \alpha_i\alpha_{j'}(A_n)_1 \\ \Longleftrightarrow & \alpha_i\alpha_j(1) & = & \alpha_i\alpha_{j'}(1) \\ \Longleftrightarrow & \alpha_i(j) & = & \alpha_i(j') \\ \Longleftrightarrow & j & = & j' \text{ (since } \alpha_i \text{ is a permutation)} \\ \Longleftrightarrow & \alpha_j & = & \alpha_{j'}. \end{array}$$

For right cancellation, we have:

$$\begin{array}{rrcl}
 & \alpha_i * \alpha_j & = & \alpha_{i'} * \alpha_j \\
\Longleftrightarrow & \alpha_i\alpha_j(A_n)_1 & = & \alpha_{i'}\alpha_j(A_n)_1 \\
\Longleftrightarrow & \alpha_i\alpha_j(1) & = & \alpha_{i'}\alpha_j(1) \\
\Longleftrightarrow & \alpha_i(j) & = & \alpha_{i'}(j) \\
\Longleftrightarrow & (j,\alpha_i(j)) & = & (j,\alpha_{i'}(j)) \text{ (as arcs in } K_n) \\
\Longleftrightarrow & \alpha^{-1}(\alpha_i) & = & \alpha^{-1}(\alpha_{i'}) \\
\Longleftrightarrow & \alpha_i & = & \alpha_{i'}.
\end{array}$$

■

We now present our main result.

6.6 Proposition *For $n \geqslant 5$, A_n, the alternating group on n odd elements is not a strongly simple group.*

Proof
We need to show that A_n, the alternating group on n elements contains a left transversal T of a subgroup H such $(T, *)$ is a loop, where $*$ is the binary operation defined by (4.3).

Now, such a T is defined in Lemma 6.5 ■

6.3 Example of a simple but not strongly simple group

Let $G = A_7$ be the alternating group acting naturally on $\Omega = \{1, \cdots, 7\}$, and $H = (A_7)_1$ the stabiliser of the point 1. Consider K_7 and the cycle as given in (6.1). Now a directed Hamiltonian cycle C_1 is $(x_1\ \ x_2\ \ x_3\ \ x_7\ \ x_4\ \ x_6\ \ x_5)$, and the directed Hamiltonian factorisation of K_7 given by

$$\begin{array}{rcl}
\phi & = & (234567),\ \phi(C_1) = C_2 \text{ is } (x_1\ \ x_3\ \ x_4\ \ x_2\ \ x_5\ \ x_7\ \ x_6); \\
\phi^2 & = & (246)(357),\ \phi^2(C_1) = C_3 \text{ is } (x_1\ \ x_4\ \ x_5\ \ x_3\ \ x_6\ \ x_2\ \ x_7); \\
\phi^3 & = & (25)(36)(47),\ \phi^3(C_1) = C_4 \text{ is } (x_1\ \ x_5\ \ x_6\ \ x_4\ \ x_7\ \ x_3\ \ x_2); \\
\phi^4 & = & (264)(375),\ \phi^4(C_1) = C_5 \text{ is } (x_1\ \ x_6\ \ x_7\ \ x_5\ \ x_2\ \ x_4\ \ x_3); \\
\phi^5 & = & (276543),\ \phi^5(C_1) = C_6 \text{ is } (x_1\ \ x_7\ \ x_2\ \ x_6\ \ x_3\ \ x_5\ \ x_4).
\end{array}$$

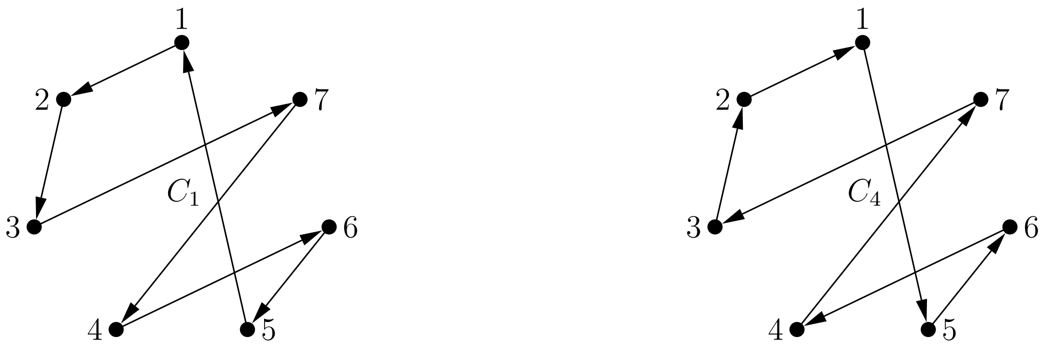


Figure 6.1: Factors C_1 and C_4

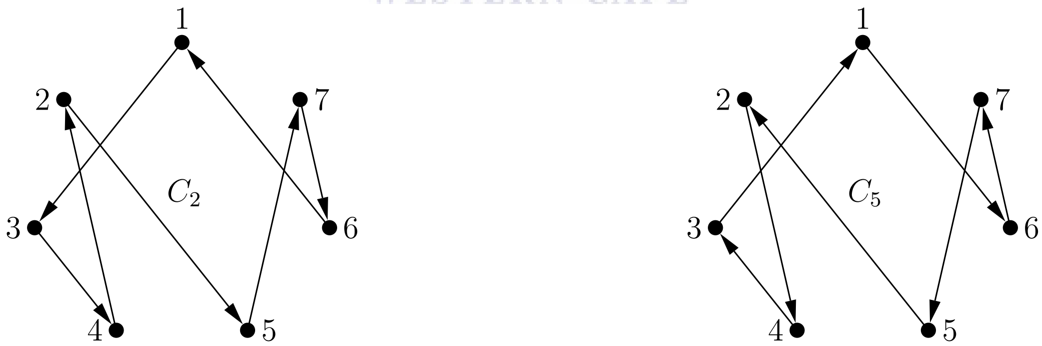


Figure 6.2: Factors C_2 and C_5

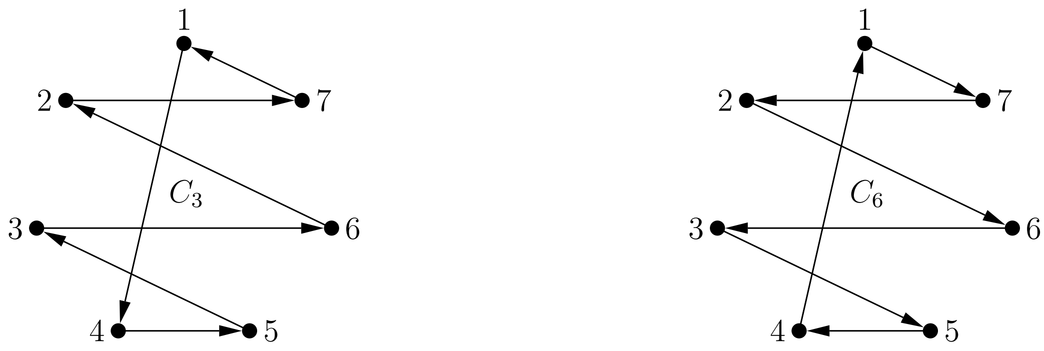


Figure 6.3: Factors C_3 and C_6

$$
\text{Now,} \quad \begin{array}{rcl}
\alpha(C_1) & = & (1237465) \\
\alpha(C_2) & = & (1342576) \\
\alpha(C_3) & = & (1453627) \\
\alpha(C_4) & = & (1564732) \\
\alpha(C_5) & = & (1675243) \\
\alpha(C_6) & = & (1726354).
\end{array}
$$

We obtain the following.

T is $\{(1), (1237465), (1342576), (1453627), (1564732), (1675243), (1726354)\}$, and then we have $(T, *)$ a loop, as seen in Table 6.1.

*	(1)	(1237465)	(1342576)	(1453627)	(1564732)	(1675243)	(1726354)
(1)	(1)	(1237465)	(1342576)	(1453627)	(1564732)	(1675243)	(1726354)
(1237465)	(1237465)	(1342576)	(1564732)	(1726354)	(1)	(1453627)	(1675243)
(1342576)	(1342576)	(1726354)	(1453627)	(1675243)	(1237465)	(1)	(1564732)
(1453627)	(1453627)	(1675243)	(1237465)	(1564732)	(1726354)	(1342576)	(1)
(1564732)	(1564732)	(1)	(1726354)	(1237465)	(1675243)	(1237465)	(1453627)
(1675243)	(1675243)	(1564732)	(1)	(1237465)	(1453627)	(1726354)	(1342576)
(1726354)	(1726354)	(1453627)	(1675243)	(1)	(1342576)	(1564732)	(1237465

Table 6.1: Cayley table of A_7 isomorphic to factorisation of K_7

Bibliography

[1] B. Alspach, The wonderful Walecki construction, *Bull. Inst. Comb. Appl.* **52** (2008), 7-20.

[2] B. Alspach, Isomorphism and Cayley graphs on abelian groups, in *Graph Symmetry: Algebraic Methods and Applications, Dordrecht,* 1997, eds. G. Sabidussi and G. Hahn, NATO ASI Series, Vol. 497, (Kluwer Academic Publishers, The Netherlands).

[3] B. Bollobás, *Graph Theory, An Introductory Course*, Springer, New York, 1979.

[4] C. Berge, *Graphes et hypergraphes*, Dunod, Paris, 1970.

[5] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Springer, New York, 1999.

[6] R.H. Bruck, *A Survey of Binary Systems*, Springer, 1996.

[7] O. Chein, H. O. Pfugfelder, and J. D. H. Smith, *Quasigroups and Loops: Theory and Applications*, Heldermann, 1990.

[8] E. Fadhal and E. Mwambene, On generalising McKay-Praeger $T(2m^2)$ graphs by representing them as loop graphs. submitted 2010.

[9] G. Gauyacq, On quasi-Cayley graphs, *Discrete Appl. Math* **77** (1997), 43-58.

[10] P. Hell and J. Nesetril, *Graphs and Homomorphisms*, Oxford University Press, England, 2004.

[11] M.C. Heydemann, Cayley Graphs and Interconnection Networks, in *Graph Symmetry: Algebraic Methods and Applications, Dordrecht,* 1997, eds. G. Sabidussi and G. Hahn, NATO ASI Series, Vol. 497, (Kluwer Academic Publishers, The Netherlands).

[12] B. McKay and C. Praeger, Vertex-transitive graphs which are not Cayley graphs I, *J. Aust. Math. Soc.Ser A* **56** (1994) 53-63.

[13] E. Mwambene, *On groupoids graph*, Technical Report UWC-TRB, 2003.

[14] E. Mwambene, Characterisation of regular graphs as loop graphs, *Quaes. Math* **28** 2 (2005), 243-250.

[15] E. Mwambene, Representing vertex-transitive graphs on groupoids, *Quaes. Math* **28** 3 (2006), 279-284.

[16] E. Mwambene, Cayley graphs on left quasi-groups and groupoids representing $k-$generalised Petersen graphs, *Discrete Math* **309** (2009), 2544-2547.

[17] E. Mwambene, Multiples of left loops and vertex-transitive graphs, *Centr. Europ. J. Math* **3(2)**(2005), 245-250.

[18] E. Mwambene, *Routings of Vertex-transitive Graphs*, MSc Thesis, University of Zimbabwe, 1998.

[19] E. Mwambene, *Representing Graphs on Groupoids, Symmetry and Form*, PhD thesis, University of Vienna, 2001.

[20] C. Praeger, Finite transitive permutation groups and finite vertex-transitive, in *Graph Symmetry: Algebraic Methods and Applications, Dordrecht,* 1997, eds. G. Sabidussi and G. Hahn, NATO ASI Series, Vol. 497, (Kluwer Academic Publishers, The Netherlands).

[21] J. Rose, *A course on group theory*, Cambrige University Press, Cambridge, 1978.

[22] G. Sabidussi, Vertex-transitive graphs, *Monatsh. Math* **68** (1968), 426-438.

[23] G. Sabidussi, On a class of fixed-point-free graph, *Proc. Amer. Math. Soc.* **5** (1958), 800-804.

[24] R. Scapellato, Vertex-transitive graphs and digraphs, in *Graph Symmetry: Algebraic Methods and Applications, Dordrecht,* 1997, eds. G. Sabidussi and G. Hahn, NATO ASI Series, Vol. 497, (Kluwer Academic Publishers, The Netherlands).

[25] S. Shim, J. Širáň and J. Žerovnik, Counterxamples to uniform shortest path routing conjecture for vertex-transitive graphs, *Discrete Appl. Math* **119** (2002), 281-286.

[26] R.J. Wilson, *Introduction to Graph Theory*, Longman Group, England, 1996.