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On the automorphism group of a toral variety*

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Abstract. Let $\mathbb{K}$ be an uncountable algebraically closed field of characteristic zero. An affine algebraic variety X over $\mathbb{K}$ is toral if it is isomorphic to a closed subvariety of a torus $(\mathbb{K}^*)^d$. We study the group $\mathrm{Aut}(X)$ of regular automorphisms of a toral variety X. We prove that if T is a maximal torus in $\mathrm{Aut}(X)$, then X is a direct product $Y \times T$, where Y is a toral variety with a trivial maximal torus in the automorphism group. We show that knowing $\mathrm{Aut}(Y)$, one can compute $\mathrm{Aut}(X)$. In the case when the rank of the group $\mathbb{K}[Y]^*/\mathbb{K}^*$ is $\dim Y + 1$, the group $\mathrm{Aut}(Y)$ is described explicitly.

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1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. The set of solutions of a system of polynomial equations in an affine space has been studied for a very long time. But some interesting properties may appear when we consider the set of solutions inside a torus $(\mathbb{K}^*)^d$. In other words, we consider only solutions with non-zero coordinates. One of the examples of this approach is the Bernstein-Kushnirenko Theorem; see [2, 6].

In [9], Popov proposed the following definition.

Definition 1. *An irreducible affine algebraic variety X is called* toral *if it is isomorphic to a closed subvariety of a torus* $(\mathbb{K}^*)^d$.

Some authors also use the term a "very affine variety"; see [11, 3]. It can be seen that X is toral if and only if the algebra of regular functions on X is generated by invertible functions; see [9, Lemma 1.14]. One of the reasons why toral varieties are interesting is that they are rigid varieties; see [9, Lemma 1.14].

Definition 2. *An affine algebraic variety X is called* rigid *if there is no non-trivial action of the additive group* $(\mathbb{K}, +)$ *on* X.

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Despite the fact that the automorphism group of an affine algebraic variety has a complicated structure, sometimes it is possible to describe it for rigid varieties. It was proven in [1] that the group of regular automorphisms $\mathrm{Aut}(X)$ of a rigid variety X contains a unique maximal torus T. One can find examples of computation of $\mathrm{Aut}(X)$ for rigid varieties in [1, 7, 8].

In this paper, we study the automorphism group $\mathrm{Aut}(X)$ of a toral variety X. We denote by $\mathbb{K}[X]$ the algebra of regular functions on X and by $\mathbb{K}[X]^*$ the multiplicative group of invertible regular functions on X. Let $E(X)$ be the quotient group $\mathbb{K}[X]^*/\mathbb{K}^*$. By [10], the group $E(X)$ is a free finitely generated abelian group. For a toral variety X the rank of $E(X)$ is not less than $\dim X$.

Any automorphism of X induces an automorphism of $E(X)$. So we obtain a homomorphism from $\mathrm{Aut}(X)$ to $\mathrm{Aut}(E(X))$. We denote by $H(X)$ the kernel of this homomorphism. Note that $H(X)$ consists of automorphisms that multiply invertible functions by constants.

Suppose that X is a closed subvariety of a torus $T_d = (\mathbb{K}^*)^d$. In Proposition 1, we show that the group $H(X)$ is naturally isomorphic to a subgroup in T_d which consists of elements that preserve X under the action by multiplication. In Proposition 2, we propose a way to compute the subgroup $H(X)$.

In Theorem 1, we show that if T is a maximal torus in $\mathrm{Aut}(X)$, then X is isomorphic to a direct product $T \times Y$, where Y is a toral variety with a discrete automorphism group. Here and below we assume that the field $\mathbb{K}$ is uncountable. Theorem 3 gives a way to find $\mathrm{Aut}(X)$ knowing $\mathrm{Aut}(Y)$. If the rank of $E(Y)$ is $\dim Y + 1$, it is possible to describe $\mathrm{Aut}(Y)$ (Theorem 3).

We also consider the case when the rank of $E(X)$ is equal to $\dim X$. By Proposition 3 in this case X is a torus. Moreover, it is the only case when $\mathrm{Aut}(X)$ acts on X with an open orbit.

We use the following notation. If φ is a regular automorphism of an affine variety X, then by φ^* we mean an automorphism of $\mathbb{K}[X]$ dual to φ. If A is a group and B is a normal subgroup in A, then by $[a]$ we denote the image of an element $a \in A$ in the quotient group A/B. If X is a closed subvariety of an affine variety Z, then by $I(X)$ we mean the ideal of regular functions on Z which are equal to zero on X.

2. General facts about toral varieties

Here we prove some initial properties of toral varieties and propose a way to compute the group $H(X)$ for a toral variety X

Let T_r be a torus of dimension r. We recall that the group $\mathrm{Aut}(T_r)$ is isomorphic to $T_r \rtimes \mathrm{GL}_r(\mathbb{Z})$; see [1, Example 2.3]. Here the left factor T_r acts on itself by multiplications and a matrix $(a_{ij}) \in \mathrm{GL}_r(\mathbb{Z})$ defines an automorphism of T_r which is given by the formula

$$t_i \to t_1^{a_{i1}} \dots t_r^{a_{ir}},$$

where $t_1, \dots, t_r$ are coordinate functions on T_r.

Now let X be a toral variety and r is the rank of $E(X)$. One can choose invertible functions $f_1, \dots, f_r \in \mathbb{K}[X]^*$ such that $[f_1], \dots, [f_r]$ form a basis of the group $E(X)$. Then $f_1, \dots, f_r$ generate the algebra $\mathbb{K}[X]$ and define a closed embedding of $\rho : X \hookrightarrow$

T_r. Note that if we choose another $g_1, \ldots, g_r \in \mathbb{K}[X]^*$ such that $[g_1], \ldots, [g_r]$ form a basis of $E(X)$, then the respective embedding $\rho_g : X \hookrightarrow T_r$ differs from ρ by an automorphism of T_r. Indeed, we have

$$g_i = \lambda_i f_1^{a_{i1}} \ldots f_r^{a_{ir}},\ i = 1, \ldots, r$$

for some $\lambda_i \in \mathbb{K}^*$ and $(a_{ij}) \in \mathrm{GL}_r(\mathbb{Z})$. If we consider an automorphism $\tau : T_r \to T_r$ which is given by the formulas

$$\tau(t_i) = \lambda_i t_1^{a_{i1}} \ldots t_r^{a_{ir}},$$

then $\rho_g = \tau \circ \rho$.

Definition 3. *We will call the embedding ρ described above* canonical.

Note that if $\rho : X \hookrightarrow T_r$ is a canonical embedding, then $\mathbb{K}[X]^* \simeq \mathbb{K}[T_r]^*$ and $E(X) \simeq E(T_r)$. We denote by $\mathrm{Aut}_X(T_r)$ the subgroup of $\mathrm{Aut}(T_r)$ which consists of automorphisms of T_r that preserve X. There is a natural homomorphism $\mathrm{Aut}_X(T_r) \to \mathrm{Aut}(X)$ which sends an automorphism $\varphi \in \mathrm{Aut}_X(T_r)$ to its restriction $\varphi|_X$.

Proposition 1. *Let X be a toral variety and $\rho : X \hookrightarrow T_r$ a canonical embedding. Then*

1. *the homomorphism*

$$\mathrm{Aut}_X(T_r) \to \mathrm{Aut}(X),\ \varphi \to \varphi|_X$$

is an isomorphism;

2. *the subgroup $H(X)$ is the image of the subgroup* $\mathrm{Aut}_X(T_r) \cap T_r$ *with respect to this isomorphism.*

Proof. We denote by $t_1, \ldots, t_r$ coordinate functions on T_r and by $f_1, \ldots, f_r$ the respective invertible regular functions on X. Then $[f_1], \ldots, [f_r]$ is a basis of $E(X)$.

Firstly, we will prove that the homomorphism

$$\mathrm{Aut}_X(T_r) \to \mathrm{Aut}(X),\ \varphi \to \varphi|_X$$

is surjective. Let $\overline{\varphi}$ be an automorphism of X. Then $\overline{\varphi}$ defines an automorphism of the lattice $E(X)$. Therefore,

$$\overline{\varphi}(f_i) = \lambda_i f_1^{a_{i1}} \ldots f_r^{a_{ir}},\ i = 1, \ldots, r,$$

where $\lambda_i \in \mathbb{K}^*$ and $(a_{ij}) \in \mathrm{GL}_r(\mathbb{Z})$. We define an automorphism φ of T_r by the formulas

$$\varphi(t_i) = \lambda_i t_1^{a_{i1}} \ldots t_r^{a_{ir}},\ i = 1, \ldots, r.$$

Then φ preserves X and $\varphi|_X = \overline{\varphi}$.

Now suppose that the image of an automorphism $\psi \in \mathrm{Aut}_X(T_r)$ is a trivial automorphism of X. Then $\psi|_X$ defines a trivial automorphism of the lattice $E(X)$. Hence, ψ defines a trivial automorphism of the lattice $E(T_r)$. So ψ has the form

$$\psi(t_i) = \beta_i t_i$$

for some $\beta \in \mathbb{K}^*$. It means that $\psi \in T_r$. But T_r acts on itself freely. Since ψ preserves all points of X, then ψ is a trivial automorphism of T_r. So the map

$$\mathrm{Aut}_X(T_r) \to \mathrm{Aut}(X)$$

is injective and therefore it is an isomorphism.

It remains to prove the last property. If $\delta \in \mathrm{Aut}_X(T_r) \cap T_r$, then $\delta|_X$ defines a trivial automorphism of $E(X)$. Hence $\delta|_X \in H(X)$.

Conversely, suppose that $\delta|_X \in H(X)$. Then δ is given by the formulas

$$\delta(t_i) = \gamma_i t_i,\ i = 1, \dots, r,$$

for some $\gamma_i \in \mathbb{K}^*$. Therefore, $\delta \in \mathrm{Aut}_X(T_r) \cap T_r$. □

Corollary 1. *Let X be a toral variety and $r = \mathrm{rank}\ E(X)$. Then the group $\mathrm{Aut}(X)$ is isomorphic to a subgroup in $T_r \rtimes \mathrm{GL_r}(\mathbb{Z})$.*

Remark 1. *It follows from Proposition 1 that a toral variety X can be embedded in a torus T_r in such a way that any automorphism X can be uniquely extended to an automorphism of T_r. If X is a subvariety of Z, it is always natural to ask whether an automorphism of X can be extended to an automorphism of Z. Some results concerning this problem can be found in [4, 5].*

Example 1. *Let X be a toral variety and* $\mathrm{rank}\ E(X) = r$. *Then there is a canonical embedding $\rho : X \hookrightarrow T_r$ of X into a torus T_r of dimension r. But in some cases it is also possible to embed X into a torus of lower dimension.*

Consider

$$Y = \{(x, y) \in (\mathbb{K}^*)^2 | yx(x-1)(x-2)\dots(x-k) = 1\}.$$

It is a closed subvariety of a torus $T_2 = (\mathbb{K}^*)^2$, so Y is a toral variety. We see that $x, (x-1), \dots, (x-k)$ are invertible functions on Y. We will show that $[x], [x-1], \dots, [x-k]$ are linearly independent in $E(Y)$. It implies that $\mathrm{rk}\ E(Y) \geq k+1$.

Indeed, otherwise there are $b_0, \dots, b_k \in \mathbb{Z}$ and $\lambda \in \mathbb{K}^*$ such that

$$x^{b_0}(x-1)^{b_1}\dots(x-k)^{b_k} = \lambda. \tag{1}$$

But the polynomial $x^{b_0}(x-1)^{b_1}\dots(x-k)^{b_k} - \lambda$ is not divisible by $yx(x-1)(x-2)\dots(x-k) - 1$ in $\mathbb{K}[x^{\pm 1}, y^{\pm 1}]$. So Equation (1) cannot hold for Y.

Example 2. *It is also not true that every embedding of a toral variety X with* $\mathrm{rank}\ E(X) = r$ *into a torus T_r is canonical.*

The embedding $X \hookrightarrow T_r$ is canonical if $[t_1|_X], \dots, [t_r|_X]$ is a basis of $E(X)$. If we choose $Y \subseteq T_2$ as in Example 1 above, then the embedding $Y \hookrightarrow T_2 \times T_{r-2} = T_r$, where $z \to (z, p)$ for some fixed point $p \in T_{r-2}$, is not a canonical embedding. Here the restrictions $t_3|_Y, \dots, t_r|_Y$ are constants so $[t_3|_Y] = \dots = [t_r|_Y]$ is a neutral element in $E(Y)$.

Now let X be a closed irreducible subvariety in T_r and let the embedding $X \hookrightarrow T_r$ be canonical. By Proposition 1 we can identify the group $H(X)$ with the subgroup in T_r which preserves X. We will describe the subgroup $H(X)$ as a subgroup in T_r. Let $M \simeq \mathbb{Z}^r$ be the lattice of characters of T_r. For $m = (m_1, \ldots, m_r) \in M$ by χ^m we mean the character $t \to t_1^{m_1} \ldots t_r^{m_r}$. Then each function in $\mathbb{K}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ is a linear combination of characters. For a function $f = \sum_i \alpha_{m_i} \chi^{m_i} \in \mathbb{K}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ by *support of* f we mean the subset

$$\operatorname{Supp} f = \{m_i \in M |\ \alpha_{m_i} \neq 0\} \subseteq M.$$

Let $I(X)$ be the ideal of functions in $\mathbb{K}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ which are equal to zero on X. We say that $f \in I(X)$ is *minimal* if there is no non-zero $g \in I(X)$ such that $\operatorname{Supp} g \subsetneq \operatorname{Supp} f$.

Lemma 1. *Minimal polynomials generate* $I(X)$ *as a vector space.*

Proof. If $f \in I(X)$ is not minimal, then there is a $g \in I(X)$ with $\operatorname{Supp} g \subsetneq \operatorname{Supp} f$. One can choose a constant α such that $\operatorname{Supp}(f - \alpha g) \subsetneq \operatorname{Supp} f$. Applying induction by cardinality of $\operatorname{Supp} f$ we see that g and $f - \alpha g$ can be represented as a sum of minimal polynomials. Then f is also a sum of minimal polynomials. $\square$

Definition 4. *We denote by* $M(X)$ *a subgroup of* M *which is generated by Minkowski sums* $\operatorname{Supp} f + (-\operatorname{Supp} f)$ *for all minimal* $f \in I(X)$.

Proposition 2. *The subgroup* $H(X) \subseteq T_r$ *is given by equations* $\chi^m(t) = 1$ *for all* $m \in M(X)$.

Proof. Let $h \in H(X)$ and $f = \sum_i \alpha_{m_i} \chi^{m_i}$ be a minimal polynomial in $I(X)$. Then $h \circ f = \sum_i \alpha_{m_i} \chi^{m_i}(h) \chi^{m_i}$. The ideal $I(X)$ is invariant under the action of $H(X)$. So $h \circ f \in I(X)$. Suppose that there are $a, b \in M(X)$ such that $\alpha_a, \alpha_b \neq 0$ and $\chi^{m_a}(h) \neq \chi^{m_b}(h)$. Then $g = \chi^{m_a}(h) f - h \circ f$ is a non-zero function in $I(X)$ and $\operatorname{Supp} g \subsetneq \operatorname{Supp} f$. But f is minimal. So $\chi^{m_a}(h) = \chi^{m_b}(h)$. Therefore, $\chi^{m_a - m_b}(h) = 1$ and this implies that $\chi^m(h) = 1$ for all $m \in M(X)$.

Now consider an element $t \in T_r$ such that $\chi^m(t) = 1$, $\forall\ m \in M(X)$. Then every minimal polynomial in $I(X)$ is a semi-invariant with respect to t. But $I(X)$ is a linear span of minimal polynomials. So $I(X)$ is invariant under the action of t. Therefore, $t \in H(X)$.

$\square$

At the end of this section, we note that toral varieties over uncountable fields satisfy the following conjecture formulated by Perepechko and Zaidenberg.

Conjecture 1 (Conjecture 1.0.1 in [8])**.** *If* Y *is a rigid affine algebraic variety over* $\mathbb{K}$*, then the connected component* $\operatorname{Aut}^0(Y)$ *is an algebraic torus of the rank not greater than* $\dim Y$.

Corollary 2. *Suppose that the field* $\mathbb{K}$ *is uncountable. Let* X *be a toral variety over* $\mathbb{K}$*. Then* $\operatorname{Aut}(X)$ *is a discrete extension of an algebraic torus.*

Proof. Indeed, if X is a toral variety, then the group $\mathrm{Aut}(X)/H(X)$ is isomorphic to a subgroup in $\mathrm{Aut}(E(X)) \simeq \mathrm{GL}_r(\mathbb{Z})$, where r is the rank of $E(X)$. If $\mathbb{K}$ is uncountable, then $\mathrm{Aut}(X)/H(X)$ is a discrete group. So $\mathrm{Aut}^0(X)$ is contained in $H(X)$. But $H(X)$ is a quasitorus. Therefore, $\mathrm{Aut}^0(X)$ is a torus and the quotient group $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$ is a discrete group.

□

From this point onwards, we always assume that the field $\mathbb{K}$ is uncountable.

3. The structure of the automorphism group

It follows from Corollary 1 that toral varieties are rigid. By [1, Theorem 2.1], there is a unique maximal torus in the automorphism group of an irreducible rigid variety.

Theorem 1. *Let X be a toral variety over $\mathbb{K}$ and T the maximal torus in* $\mathrm{Aut}(X)$. *Then $X \simeq Y \times T$, where Y is a toral variety with a discrete automorphism group.*

Proof. Let r be the rank of the group $E(X)$ and $\rho : X \hookrightarrow T_r$ a canonical embedding. We denote by M the lattice of characters of T_r and by $M(X)$ the sublattice in M which corresponds to X. One can choose a basis $e_1, \ldots, e_r \in M$ such that $b_1e_1, \ldots, b_le_l$ is a basis of $M(X)$ for some $b_1, \ldots, b_l \in \mathbb{N}$ and $l \leq r$. Denote by $t_1, \ldots, t_r$ coordinates on T_r corresponding to $e_1, \ldots, e_r$.

Then the equations $\chi^m(t) = 1$ for all $m \in M(X)$ define the subgroup $H(X)$ in T_r which consists of elements of the form

$$(\epsilon_1, \ldots, \epsilon_l, t_{l+1}, \ldots, t_r),$$

where $\epsilon_1, \ldots, \epsilon_l$ are the roots of unity of degrees $b_1, \ldots, b_l$, respectively, and $t_{l+1}, \ldots, t_r \in \mathbb{K}^*$. Then the maximal torus in $H(X)$ is the torus

$$T_{r-l} = \{(1, \ldots, 1, t_{l+1}, \ldots, t_r) \in T_r | t_i \in \mathbb{K}^*\}.$$

The group $\mathrm{Aut}(X)/H(X)$ is a discrete group. So the maximal torus of $\mathrm{Aut}(X)$ coincides with the maximal torus of the quasitorus $H(X)$, which is T_{r-l}.

All minimal polynomials in $I(X)$ are semi-invariant with respect to $H(X)$. This means that minimal polynomials in $I(X)$ are homogeneous with respect to each variable $t_{l+1}, \ldots, t_r$. Since functions t_i are invertible, one can choose a set of minimal generators of $I(X)$ which do not depend on $t_{l+1}, \ldots, t_r$. It implies that $X \simeq Y \times T_{r-l}$, where Y is a subvariety of $T_l = \{(t_1, \ldots, t_l, 1, \ldots, 1) \in T_r | t_i \in \mathbb{K}^*\}$.

The variety Y is also a toral variety given by the ideal $I(X) \cap \mathbb{K}[t_1^{\pm 1}, \ldots t_l^{\pm 1}]$. Since the unique maximal torus in $\mathrm{Aut}(X)$ is T_{r-l}, the maximal torus in $\mathrm{Aut}(Y)$ is trivial.

□

Let X be a toral variety and suppose that $X \simeq T_s \times Y$, where Y is a toral variety with a discrete automorphism group and T_s is the torus $(\mathbb{K}^*)^s$. One can see that $\mathrm{Aut}(X)$ contains the following subgroups.

There is a subgroup which is isomorphic to $\mathrm{Aut}(Y)$. This subgroup acts naturally on Y and trivially on T_s. The subgroup $\mathrm{GL_s}(\mathbb{Z})$ acts naturally on T_s and trivially on

Y. Moreover, there is a subgroup which is isomorphic to $(\mathbb{K}[Y]^*)^s \simeq (E(Y) \times \mathbb{K}^*)^s$. This subgroup acts in the following way. If $f_1, \ldots, f_s \in \mathbb{K}[Y]^*$, then we can define an automorphism of $T_s \times Y$ as follows:

$$(t_1, \ldots, t_s, y) \to (f_1(y)t_1, \ldots, f_s(y)t_s, y).$$

The following theorem was proposed to the authors by Gaifullin.

Theorem 2. *Let $X \simeq T_s \times Y$ be a toral variety, where Y is a toral variety with a discrete automorphism group. Then*

$$\mathrm{Aut}(X) \simeq \mathrm{Aut}(Y) \ltimes (\mathrm{GL}_s(\mathbb{Z}) \ltimes (E(Y) \times \mathbb{K}^*)^s).$$

Proof. There is a natural action of T_s on X. We see that $\mathbb{K}[Y]$ is the algebra of invariants of this action. Since T_s is a unique maximal torus in $\mathrm{Aut}(X)$, each automorphism of $T_s \times Y$ preserves $\mathbb{K}[Y]$. So we obtain a homomorphism

$$\Phi : \mathrm{Aut}(X) \to \mathrm{Aut}(Y).$$

Let B be the kernel of Φ. The group $\mathrm{Aut}(Y)$ is naturally embedded into $\mathrm{Aut}(T_s \times Y)$ and it intersects trivially with B. At the same time, $\mathrm{Aut}(Y)$ maps isomorphically to the image of Φ. It implies that

$$\mathrm{Aut}(T_s \times Y) = \mathrm{Aut}(Y) \ltimes B.$$

We denote by $t_1, \ldots, t_s$ coordinate functions on T_s. Then

$$\mathbb{K}[T_s \times Y] \simeq \mathbb{K}[T_s] \otimes \mathbb{K}[Y] = \mathbb{K}[Y][t_1^{\pm 1}, \ldots, t_s^{\pm 1}].$$

Let $\phi \in B$. The algebra $\mathbb{K}[Y]$ is invariant with respect to ϕ^*. So for all $t \in T_s$ and $y \in Y$ we have

$$\phi((t, y)) = (t', y),$$

for some $t' \in T_s$. Therefore, for each $y \in Y$ the automorphism ϕ defines an automorphism $\phi_y : T_s \to T_s$. Hence, for each $y \in Y$ we have

$$\phi^*(t_i)(t, y) = t_i(\phi(t, y)) = t_i((\phi_y(t), y)) = f_i(y)t_1^{a_{i1}(y)} \ldots t_s^{a_{is}(y)},$$

for some non-zero constant $f_i(y)$ and a matrix $A(y) = (a_{ij}(y)) \in \mathrm{GL}_s(\mathbb{Z})$. For reasons of continuity, the matrix $A(y)$ is the same for all $y \in Y$ and $f_i : Y \to \mathbb{K}$ are regular functions on Y. Since $f_i(y) \neq 0$ for all $y \in Y$, the functions f_i are invertible. So we have

$$\phi^*(t_i) = f_i t_1^{a_{i1}} \ldots t_s^{a_{is}}$$

for some $f_i \in \mathbb{K}[Y]^*$ and $A \in \mathrm{GL}_s(\mathbb{Z})$.

Then we have a homomorphism $\overline{\Phi} : B \to \mathrm{GL}_s(\mathbb{Z})$, $\phi \to A$. Again, the group $\mathrm{GL}_s(\mathbb{Z})$ is naturally embedded into B in the following way. The matrix $(d_{ij}) \in \mathrm{GL}_s(\mathbb{Z})$ corresponds to an automorphism

$$(t_1, \ldots, t_s, y) \to (t_1^{d_{11}} \ldots t_s^{d_{1s}}, \ldots, t_1^{d_{s1}} \ldots t_s^{d_{ss}}, y).$$

The group $\mathrm{GL}_s(\mathbb{Z})$ maps isomorphically to $\mathrm{GL}_s(\mathbb{Z})$ under $\overline{\Phi}$. So

$$B = \mathrm{GL}_s(\mathbb{Z}) \ltimes \mathrm{Ker}\ \overline{\Phi}.$$

The kernel of $\overline{\Phi}$ consists of automorphisms $\varphi \in \mathrm{Aut}(T_s \times Y)$ which have the following form:

$$\varphi(t_1, \ldots, t_s, y) = (f_1(y)t_1, \ldots, f_s(y)t_s, y),$$

for some $f_1, \ldots, f_s \in \mathbb{K}[Y]^*$. We see that for all $f_1, \ldots, f_s \in \mathbb{K}[Y]^*$ this formula defines an automorphism of $T_s \times Y$, so Ker $\overline{\Phi} \simeq (\mathbb{K}[Y]^*)^s \simeq (E(Y) \times \mathbb{K}^*)^s$. □

4. The case rk $E(X) = \dim X$

Let X be a toral variety. Then rk $E(X) \geq \dim X$. Indeed, suppose that $f_1, \ldots, f_r$ are invertible functions and $[f_1], \ldots, [f_r]$ is a basis in $E(X)$. Then $f_1, \ldots, f_r$ generate $\mathbb{K}[X]$. So $r \geq \mathrm{tr.deg}\ \mathbb{K}[X] = \dim X$.

The following result shows that if rk $E(X) = \dim X$, then X is a torus. Moreover, this is the only case when $\mathrm{Aut}(X)$ acts with an open orbit on X.

Proposition 3. *Let X be a toral variety. Then the following conditions are equivalent:*

1. *X is a torus;*
2. rk $E(X) = \dim X$;
3. $\mathrm{Aut}(X)$ *acts on X with an open orbit.*

Proof. Implication 1) $\Rightarrow$ 2) is trivial.

Suppose that rk $E(X) = \dim X$. Then one can choose invertible functions $f_1, \ldots, f_n$ such that $[f_1], \ldots, [f_n]$ is a basis of $E(X)$. Then $\mathbb{K}[X]$ is generated by

$$f_1, f_1^{-1}, \ldots, f_n, f_n^{-1}.$$

But $f_1, \ldots, f_n$ are algebraically independent, otherwise $\dim X <$ rk $E(X)$. So $\mathbb{K}[X]$ is isomorphic to the algebra of Laurent polynomials. So we obtain implication 2) $\Rightarrow$ 1).

Implication 1) $\Rightarrow$ 3) is trivial. Suppose X is a toral variety and $\mathrm{Aut}(X)$ acts on X with an open orbit U.

Let T be the maximal torus in $\mathrm{Aut}(X)$. Since the quotient group $\mathrm{Aut}(X)/T$ is a discrete group, the set U is a countable union of orbits of T. Since $\mathbb{K}$ is uncountable, it implies that one of the orbits of T is open in X. Then $\dim X = \dim T$. By Theorem 1, we have $X \simeq T \times Y$ for some toral variety Y. But since $\dim T = \dim T \times Y$, we obtain that Y is a point and $X \simeq T$.

□

5. The case rk $E(X) = \dim X + 1$

By Theorem 1, any toral variety over an algebraically closed uncountable field of characteristic zero is a direct product $T \times Y$, where T is a torus and Y is a toral variety with a discrete automorphism group. By Theorem 2, one can find $\mathrm{Aut}(X)$ knowing $\mathrm{Aut}(Y)$. In this section, we provide a way to find $\mathrm{Aut}(Y)$ when $\mathrm{rk}\ E(Y) = \dim\ Y+1$.

Let Y be a toral variety with a trivial maximal torus in $\mathrm{Aut}(Y)$. Let r be the rank of $E(Y)$. We suppose that $r = \dim\ Y + 1$.

There is a canonical embedding of Y into the torus T_r as a hypersurface. The variety T_r is factorial so there is an irreducible polynomial $h \in \mathbb{K}[t_1^{\pm1}, \ldots, t_r^{\pm1}]$ such that $I(Y) = (h)$. The polynomial h has a form

$$h = \sum_{m\ \in\ \mathrm{Supp}\ h} \alpha_m \chi^m.$$

Let M be the lattice of characters of T_r and $M(Y)$ a sublattice in M which corresponds to Y; see Definition 4. Since the maximal torus in $\mathrm{Aut}(Y)$ is trivial, the rank of the lattice $M(Y)$ is equal to r. It means that the elements $m_a - m_b$ with $m_a, m_b \in \mathrm{Supp}\ h$ generate a sublattice of full rank in M.

We denote by $\mathrm{GAff}(M,h)$ the group of all invertible integer affine transformations φ of M, which preserve $\mathrm{Supp}\ h$ and for any linear combination

$$\sum_{m \in \mathrm{Supp}\ h} a_m m = 0,$$

where $a_m \in \mathbb{Z}$ and $\sum_m a_m = 0$, the affine transformation φ satisfies

$$\prod_{m \in \mathrm{Supp}\ h} (\alpha_m)^{a_m} = \prod_{m \in \mathrm{Supp}\, h} \left(\alpha_{\varphi(m)}\right)^{a_m}. \tag{2}$$

Theorem 3. *Let Y be a toral variety with a trivial maximal torus in* $\mathrm{Aut}(Y)$. *Suppose that* $\mathrm{rk}\ E(Y) = \dim Y + 1$. *Then*

$$\mathrm{Aut}(Y)/H(Y) \simeq \mathrm{GAff}(M,h).$$

Proof. Let ψ be an automorphism of Y. By Proposition 1, the automorphism ψ can be extended to an automorphism of T_r. We denote by ψ^* the respective automorphism of $\mathbb{K}[t_1^{\pm1}, \ldots, t_r^{\pm1}]$. Then ψ^* has the form

$$\psi^*(t_i) = \lambda_i t_1^{a_{i1}} \ldots t_r^{a_{ir}},$$

where $\lambda_i \in \mathbb{K}^*$ and $(a_{ij}) \in \mathrm{GL}_r(\mathbb{Z})$. We denote by λ the element $(\lambda_1, \ldots, \lambda_r) \in T_r$ and by $\overline{\psi}$ the automorphism of M that corresponds to the matrix (a_{ij}). Then

$$\psi^*(\chi^m) = \chi^m(\lambda)\chi^{\overline{\psi}(m)}$$

for all $m \in M$.

The polynomial $\psi^*(h)$ also generates $I(Y)$. So it differs from h by an invertible element of $\mathbb{K}[t_1^{\pm1}, \ldots, t_r^{\pm1}]$. Then

$$\psi^*(h) = \alpha\chi^v h$$

for some $\alpha \in \mathbb{K}^*$ and $v \in M$. Therefore, we have the equation

$$\psi^*(h) = \sum_{m \in \mathrm{Supp}\ h} \alpha_m \chi^m(\lambda) \chi^{\overline{\psi}(m)} = \alpha \sum_{m \in \mathrm{Supp}\ h} \alpha_m \chi^{m+v}. \tag{3}$$

It implies that $\overline{\psi}(m) - v$ belonge to Supp h for all $m \in$ Supp h. We define the map $\varphi : M \to M$ by the following formula:

$$\varphi(m) = \overline{\psi}(m) - v.$$

Then φ is an affine transformation of M which preserves Supp h.

We will prove that φ belonge to $\mathrm{GAff}(M, h)$. So we consider a linear combination as in (2):

$$\sum_{m \in \mathrm{Supp}\ h} a_m m = 0,$$

where $a_m \in \mathbb{Z}$ and $\sum_m a_m = 0$.

Equation (3) can be written as

$$\sum_{m \in \mathrm{Supp}\ h} \alpha_m \chi^m(\lambda) \chi^{\varphi(m)} = \alpha \sum_{m \in \mathrm{Supp}\ h} \alpha_m \chi^m = \alpha \sum_{m \in \mathrm{Supp}\ h} \alpha_{\varphi(m)} \chi^{\varphi(m)},$$

and it implies

$$\frac{\alpha_{m_1} \chi^{m_1}(\lambda)}{\alpha_{m_2} \chi^{m_2}(\lambda)} = \frac{\alpha_{m_1}}{\alpha_{m_2}} \chi^{m_1 - m_2}(\lambda) = \frac{\alpha_{\varphi(m_1)}}{\alpha_{\varphi(m_2)}}$$

for all $m_1, m_2 \in$ Supp h.

We fix some $m_0 \in$ Supp h. Then we have

$$\begin{aligned}
\prod_{m \in \mathrm{Supp}\ h} \left(\alpha_{\varphi(m)}\right)^{a_m} &= \prod_{m \in \mathrm{Supp}\ h} \left(\frac{\alpha_{\varphi(m)}}{\alpha_{\varphi(m_0)}}\right)^{a_m} = \prod_{m \in \mathrm{Supp}\ h} \left(\frac{\alpha_m}{\alpha_{m_0}} \chi^{m - m_0}(\lambda)\right)^{a_m} \\
&= \prod_{m \in \mathrm{Supp}\ h} \left(\frac{\alpha_m}{\alpha_{m_0}}\right)^{a_m} (\chi^{\sum_m a_m (m - m_0)}(\lambda)) \\
&= \prod_{m \in \mathrm{Supp}\ h} \left(\frac{\alpha_m}{\alpha_{m_0}}\right)^{a_m} = \prod_{m \in \mathrm{Supp}\ h} (\alpha_m)^{a_m}
\end{aligned}$$

So $\varphi \in \mathrm{GAff}(M, h)$. Then we obtain a homomorphism

$$\eta : \mathrm{Aut}(Y) \to \mathrm{GAff}(M, h), \ \psi \to \varphi.$$

Moreover, we see that the kernel of η is $H(Y)$. Now we will show that η is surjective.

Let $\varphi \in \mathrm{GAff}(M, h)$ and $f_1, \ldots, f_r$ be a basis in $M(Y)$. Again, we fix some $m_0 \in \mathrm{Supp}\, h$. Then there are $a_{m,j} \in \mathbb{Z}$ for $m \in \mathrm{Supp}\, h$ such that

$$f_j = \sum_{m \in \mathrm{Supp}\ h} a_{m,j} (m - m_0).$$

There is a $\lambda \in T_r$ such that

$$\chi^{f_j}(\lambda) = \prod_{m \in \text{Supp } h} \left(\frac{\alpha_m}{\alpha_{m_0}}\right)^{-a_{m,j}} \prod_{m \in \text{Supp } h} \left(\frac{\alpha_{\varphi(m)}}{\alpha_{\varphi(m_0)}}\right)^{a_{m,j}}$$

for all $j = 1, \ldots, r$.

Let $d\varphi$ be the linear part of φ, i.e., $d\varphi(m) = \varphi(m) - \varphi(0)$. We define an automorphism ψ^* of $\mathbb{K}[t_1^{\pm 1} \ldots t_r^{\pm 1}]$ by the following rule:

$$\psi^*(\chi^m) = \chi^m(\lambda)\chi^{d\varphi(m)}.$$

Let us check if ψ^* preserves $I(Y)$. We have

$$\psi^*(h) = \sum_{m \in \text{Supp } h} \alpha_m \chi^m(\lambda)\chi^{d\varphi(m)}.$$

We denote $\varphi(0)$ by v. Then

$$\varphi(m) = d\varphi(m) + v$$

and

$$\chi^v \psi^*(h) = \sum_{m \in \text{Supp } h} \alpha_m \chi^m(\lambda)\chi^{\varphi(m)}.$$

We see that Supp $\chi^v\psi^*(h) =$ Supp h. We will show that there is an $\alpha \in \mathbb{K}$ such that

$$\chi^v\psi^*(h) = \alpha h.$$

For any $b, c \in$ Supp h there are numbers $d_j \in \mathbb{Z}$ such that

$$b - c = \sum_{j=1}^{r} d_j f_j.$$

So

$$\begin{aligned}\frac{\alpha_b \chi^b(\lambda)}{\alpha_c \chi^c(\lambda)} =& \frac{\alpha_c}{\alpha_b}\chi^{b-c}(\lambda) = \frac{\alpha_b}{\alpha_c}\chi^{\sum_j d_j f_j}(\lambda) \\ =& \frac{\alpha_b}{\alpha_c}(\prod_{j=1}^{r} \chi^{f_j}(\lambda))^{d_j} = \frac{\alpha_b}{\alpha_c}\prod_{m,j=1}\left(\frac{\alpha_m}{\alpha_{m_0}}\right)^{-d_j a_{m,j}} \prod_{m,j}\left(\frac{\alpha_{\varphi(m)}}{\alpha_{\varphi(m_0)}}\right)^{d_j a_{m,j}}.\end{aligned} \tag{4}$$

We have a combination

$$0 = b-c-\sum_j d_j f_j = b-c-\sum_{m,j} d_j a_{m,j}(m-m_0) = b-c-\sum_{m,j} d_j a_{m,j} m + (\sum_{m,j} d_j a_{m,j}) m_0.$$

The sum of all coefficients in the last sum is equal to 0. Since $\varphi \in \text{Gaff}(M, h)$ we obtain

$$\frac{\alpha_b}{\alpha_c}\prod_{m,j}\left(\frac{\alpha_m}{\alpha_{m_0}}\right)^{-d_j a_{m,j}} = \frac{\alpha_{\varphi(b)}}{\alpha_{\varphi(c)}}\prod_{m,j}\left(\frac{\alpha_{\varphi(m)}}{\alpha_{\varphi(m_0)}}\right)^{-d_j a_{m,j}}. \tag{5}$$

It follows from equations 4 and 5 that

$$\frac{\alpha_b \chi^b(\lambda)}{\alpha_c \chi^c(\lambda)} = \frac{\alpha_{\varphi(b)}}{\alpha_{\varphi(c)}}.$$

So the coefficients of the polynomials $\chi^v \psi^*(h)$ and h are proportional. Then there is an $\alpha \in \mathbb{K}$ such that $\chi^v \psi^*(h) = \alpha h$. Hence $\psi^*(h) = \alpha \chi^{-v} h \in I(Y)$. Therefore, ψ^* preserves $I(Y)$ and defines an automorphism ψ. It is a direct check that $\eta(\psi) = \varphi$. So η is surjective.

□

Corollary 3. *Let Y be a toral variety with a trivial maximal torus in* $\mathrm{Aut}(Y)$. *Suppose that* $\mathrm{rk}\ E(Y) = \dim Y + 1$. *Then* $\mathrm{Aut}(Y)$ *is a finite group.*

Proof. Indeed, the group $H(Y)$ is finite in this case. As mentioned before, the sublattice $M(Y)$ is of full rank and generated by the finite set $\mathrm{Supp}\ h + (-\mathrm{Supp}\ h)$. Then any affine transformation of M is uniquely defined by the image of the set $\mathrm{Supp}\ h + (-\mathrm{Supp}\ h)$. Therefore, the group $\mathrm{GAff}(M, h)$ is finite. Then the group $\mathrm{Aut}(Y)$ is also finite.

□

It is natural to formulate the following question.

Conjecture 2. *Let Y be a toral variety with a trivial maximal torus in* $\mathrm{Aut}(Y)$. *Is* $\mathrm{Aut}(Y)$ *a finite group?*

Note that this is not true for a general rigid variety. One can find a counterexample in [7].

At the end, we give three examples illustrating Theorem 3.

Example 3. *Let Y be the affine line $\mathbb{A}^1$ without two points. Then Y is isomorphic to an open set of the torus $\mathbb{K}^*$:*

$$Y = \{t \in \mathbb{K}^* | t \neq 1\} \subseteq \mathbb{K}^*.$$

Hence, Y can be given in $(\mathbb{K}^*)^2$ as the set of solutions of the equation

$$h = t_1(t_2 - 1) - 1 = 0,\ (t_1, t_2) \in (\mathbb{K}^*)^2.$$

So Y is a toral variety. We have

$$\mathbb{K}[Y] = \mathbb{K}[t_1^{\pm 1}, t_2^{\pm 1}]/(t_1(t_2 - 1) - 1) \simeq \mathbb{K}[t_2^{\pm 1}]_{t_2 - 1},$$

where $\mathbb{K}[t_2^{\pm 1}]_{t_2-1}$ denotes the localization of $\mathbb{K}[t_2^{\pm 1}]$ at $t_2 - 1$. Hence, all invertible elements of $\mathbb{K}[Y]$ have the form $\lambda(t_2 - 1)^a t_2^b = \lambda t_1^a t_2^b$, where $\lambda \in \mathbb{K}^*$. Therefore, $[t_1], [t_2]$ is a basis of $E(Y)$. So the rank of $E(Y)$ is equal to $\dim\ Y + 1$ *and the embedding $Y \hookrightarrow (\mathbb{K}^*)^2$ as a set of zeros*

$$h = t_1(t_2 - 1) - 1 = t_1 t_2 - t_1 - 1 = 0,\ (t_1, t_2) \in (\mathbb{K}^*)^2$$

is a canonical embedding. We can apply Theorem 3 to find $\mathrm{Aut}(Y)$.

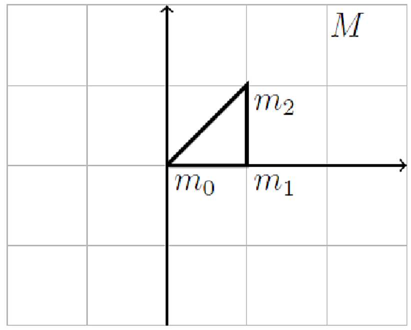

Figure 1: Supp $(t_1 t_2 - t_1 - 1)$

Let $M \simeq \mathbb{Z}^2$ be the lattice of characters of $(\mathbb{K}^*)^2$. The set Supp h *consists of points $m_0 = (0,0)$, $m_1 = (1,0), m_2 = (1,1)$; see Figure 1.*

We see that the lattice $M(Y)$ contains elements $(1,0),(0,1)$, so $M(Y) = M$. Therefore, $H(Y)$ is a trivial group.

A linear combination

$$a_0 m_0 + a_1 m_1 + a_2 m_2 = (a_1 + a_2, a_2)$$

with $a_0 + a_1 + a_2 = 0$ is equal to zero if and only if $a_0 = a_1 = a_2 = 0$. But then equations (2) are trivial. By affine transformations of M we can permute all points in Supp h. *Therefore,*

$$\mathrm{Aut}(Y) \simeq \mathrm{GAff}(M,h) \simeq S_3.$$

The answer looks natural since the affine line without two points is the projective line without three points.

In this case, $\mathrm{Aut}(Y)$ *is generated by the automorphisms ψ_1, ψ_2, where*

$$\psi_1((t_1,t_2)) = (-t_1 t_2, t_2^{-1}),\ \psi((t_1,t_2)) = (-t_2, t_1^{-1} t_2^{-1}).$$

Example 4. *Now let Y be the set of solutions of the equation*

$$Y = \{(t_1,t_2,t_3) \in (\mathbb{K}^*)^3 |\ h = t_3(t_1^2 + t_2^2 - 1) - 1 = 0\} \subseteq (\mathbb{K}^*)^3.$$

Then Y is a toral variety and

$$\mathbb{K}[Y] = \mathbb{K}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}/(t_3(t_1^2 + t_2^2 - 1) - 1) = \mathbb{K}[t_1^{\pm 1}, t_2^{\pm 1}]_{t_1^2 + t_2^2 - 1}.$$

Therefore, $[t_1],[t_2],[t_3]$ is a basis of $E(Y)$ and the embedding of Y in $(\mathbb{K}^*)^3$ is a canonical embedding.

We have $h = t_3(t_1^2 + t_2^2 - 1) - 1 = t_1^2 t_3 + t_2^2 t_3 - t_3 - 1$ and

$$\mathrm{Supp}\, h = \{m_0 = (0,0,0), m_1 = (0,0,1), m_2 = (2,0,1), m_3 = (0,2,1)\} \subseteq M \simeq \mathbb{Z}^3.$$

The vectors $(2,0,0),(0,2,0)$ and $(0,0,1)$ form a basis of $M(Y)$. Then the group $H(Y) \subseteq (\mathbb{K}^*)^3$ consists of elements

$$H(Y) = \{(\pm 1, \pm 1, 1) \in (\mathbb{K}^*)^3\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The group of invertible affine transformations of M preserving $\operatorname{Supp} h$ *is isomorphic to S_3 and permutes points m_1, m_2, m_3 preserving m_0. The sum*

$$a_0m_0 + a_1m_1 + a_2m_2 + a_3m_3 = (2a_2, 2a_3, a_1 + a_2 + a_3)$$

with $a_0 + a_1 + a_2 + a_3 = 0$ is equal to zero if and only if $a_0 = a_1 = a_2 = a_3 = 0$. So equations (2) are trivial and $\operatorname{GAff}(M, h) \simeq \operatorname{Aut}(Y)/H(Y) \simeq S_3$.

The group $\operatorname{Aut}(Y)$ *is generated by $H(Y)$ and the automorphisms ψ_1 and ψ_2 which are defined by the formulas:*

$$\psi_1((t_1, t_2, t_3)) = (t_2, t_1, t_3),\ \ \psi_2((t_1, t_2, t_3)) = (-t_2^{-1}, it_1t_2^{-1}, -t_2^2t_3).$$

One can check that ψ_1 and ψ_2 generate the subgroup in $\operatorname{Aut}(Y)$ *which is isomorphic to S_3 and trivially intersects with $H(Y)$. So*

$$\operatorname{Aut}(Y) \simeq H(Y) \rtimes S_3.$$

The automorphism ψ_2 does not commute with the element $(1, -1, 1) \in H(Y)$*. Therefore,* $\operatorname{Aut}(Y)$ *is not a direct product of $H(Y)$ and S_3.*

Remark 2. *It is natural to ask if it is true that, under the conditions of Theorem 3, we have* $\operatorname{Aut}(Y) \simeq H(Y) \rtimes \operatorname{Gaff}(M, h)$? *The authors do not know the answer to this question.*

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References

[1] I. Arzhantsev, S. Gaifullin, *The automorphism group of a rigid affine variety*, Math. Nachr. **290**(2017), 662–671.

[2] D. Bernstein, *The number of roots of a system of equations*, Funct. Anal. Appl. **9**(1975), 183–185.

[3] J. Huh, *The maximum likelihood degree of a very affine variety*, Compos. Math. **149**(2013), 1245–1266.

[4] S. Kaliman, *Extensions of isomorphisms between affine algebraic subvarieties of k^n to automorphisms of k^n*, Proc. Amer. Math. Soc. **113**(1991), 325–334.

[5] S. Kaliman, D. Udumyan, *On automorphisms of flexible varieties*, Adv. Math. **396**(2022), Article no. 108112.

[6] A. Koushnirenko, *Newton polytopes and the Bezout theorem*, Funct. Anal. Appl. **10**(1976), 82–83.

[7] A. Perepechko, *Automorphisms of surfaces of Markov type*, Math. Notes **110**(2021), 732–737.

[8] A. Perepechko, M. Zaidenberg, *Automorphism group of affine rigid surfaces: the identity component*, arXiv:2208.09738.

[9] V. Popov, *On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties*, CRM Proc. Lect. Notes **54**(2011), 289–311.

[10] M. Rosenlicht, *Some rationality questions on algebraic groups*, Ann. Mat. Pura Appl. **43**(1957), 25–50.
[11] J. Tevelev, *Compactifications of subvarieties of tori*, Amer. J. Math. **129**(2007), 1087–1104.