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Time-consistent investment-proportional reinsurance strategy under a jump-diffusion model

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Abstract. In this paper, we formulate a mean-variance portfolio selection problem of an insurer who manages her underlying risk by purchasing proportional reinsurance and investing in a financial market consisting of a bank account and a risky asset following jump-diffusion dynamics with random parameters. We then obtain a time-consistent equilibrium strategy via a flow of backward stochastic differential equations. Finally, we apply our results to a mean-reverting Lévy-Ornstein-Uhlenbeck process and obtain closed form solutions.

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1. Introduction

The classical results on the mean-variance formulation for the portfolio allocation problem proposed by [9] in 1952 has inspired several extensions. Due to the presence of a nonlinear expectation, the dynamic mean variance problem is time-inconsistent. Therefore, it does not satisfy the Bellman principle, that is, optimal control today may not be optimal tomorrow. Recently, researches on time-consistent problems of time-inconsistent ones have attracted the attention of many scholars. In [2], it is studied the dynamic mean-variance portfolio problem and its time-consistent solution is derived using dynamic programming. In [3], the authors consider a time-inconsistent problem in a general continuous time framework. They established an extended Hamilton-Jacobi-Bellman (HJB) equation and the associated verification theorem (for more details see [15, 4], and references therein).

Otherwise, managing risks is always an important topic for investors and companies, and reinsurance has proved to be an effective way to control risks for insurers. Many researchers pay much attention to the investment-reinsurance problems under the mean-variance criterion. In [22], the authors studied the optimal time-consistent policies of an investment-reinsurance problem and an investment-only problem under the mean-variance criterion for an insurer. They proved that the two problems have the same investment policies. This problem was then extended to a jump-diffusion case by [23]. An optimal investment-reinsurance problem with delay under

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the diffusion framework is considered in [12]. The authors solved the problem via a maximum principle approach. See e.g. [1, 7].

Most of the works mentioned above considered problems with deterministic parameters under a Markovian setting. However, for long-term investment problems, it is important to consider the randomness of parameters, jump fluctuations on the risky asset price and surplus process, path dependence and memory. This motivates our approach to consider a mean-variance investment-reinsurance problem with jumps under a non-Markovian framework.

Recently, [6] considered a general time-inconsistent linear quadratic problem in a non-Markovian system with random parameters. Through a system of forward backward stochastic differential equations (FBSDEs), the authors derived a necessary and sufficient condition for equilibrium controls and then presented a mean-variance portfolio selection problem as a special case. In [16], their mean-variance problem is extended to incorporate regime-switching. Further, a similar linear quadratic problem with jumps is considered in [13]. The authors applied their results to solve a mean-variance portfolio selection problem in a jump-diffusion financial market with deterministic coefficients. In [18], the authors considered an optimal time-consistent reinsurance-investment strategy selection problem in a financial market with a jump-diffusion risky asset solved by the dynamic programming approach. A similar problem with constant coefficients as in [18] is also considered in [24], but in this case, the insurer is allowed to purchase combining quota-share and excess of loss reinsurance for claims. Other references include [20, 19, 21].

In this paper, we model a claim process using a pure jump process and the stock price using a geometric jump-diffusion process. This model is an extension of [14] to a jump-diffusion case. Although the presence of jumps and random coefficients reflects the reality more, they have shown to bring more mathematical difficulties to the corresponding problem. Following a similar method in [14] and [17], we derive the equilibrium control strategies for investment and reinsurance via a flow of FBSDEs. Then we apply our results to a mean-reverting Lévy-Ornstein-Uhlenbeck stochastic interest rate, where we obtain the corresponding equilibrium strategies by solving partial integro-differential equations (PIDEs). The results obtained in this paper can also be viewed as an extension of a mean-variance selection problem considered in [13] to a more general market model with random parameters.

The rest of the paper is organized as follows: in Section 2, we give a formulation of the optimal investment-reinsurance problem with jumps and define the time-consistent equilibrium strategy. In Section 3, we derive an open-loop equilibrium strategy via a system of FBSDEs and prove the existence and the uniqueness result of the corresponding BSDEs with jumps. We also state the uniqueness result of the equilibrium control strategy. In Section 4, we discuss an example with a stochastic interest rate defined by a mean-reverting Lévy-Ornstein-Uhlenbeck process, which concludes the paper.

2. Model formulation

Let $T > 0$ be a finite time horizon representing the investment period and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ a complete filtered probability space. We define on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ a one dimensional Brownian motion $\{W(t),\, 0 \le t \le T\}$ and two independent Poisson random measures $N(t,\cdot)$ and $N^0(t,\cdot)$ with the corresponding Lévy measures defined by $\nu(\cdot)$ and $\nu^0(\cdot)$, respectively. The compensated Poisson random measures are given by

$$\tilde{N}(dt, d\zeta) := N(dt, d\zeta) - \nu(d\zeta)dt$$

and

$$\tilde{N}^0(dt, d\zeta) := N^0(dt, d\zeta) - \nu^0(d\zeta)dt.$$

Furthermore, we consider the following spaces:

- $\mathbb{L}^p_{\mathcal{F}_t}(\Omega;\mathbb{R})$ - the space of $\mathcal{F}_t$-measurable functions $\xi : \Omega \mapsto \mathbb{R}$, such that $\mathbb{E}[\,|\xi|^p] < \infty$, where $p \ge 1$.
- $\mathbb{H}^p_{\mathcal{F}}(s,t;\mathbb{R})$ - the space of $\mathcal{F}$-adapted functions $Z : [s,t] \times \Omega \mapsto \mathbb{R}$ such that
$$\mathbb{E}\left[\int_s^t |Z(u)|^p du\right] < \infty.$$
- $\mathbb{S}^2_{\mathcal{F}}(\Omega; C([s,t]);\mathbb{R})$ - the space of $\mathcal{F}$-adapted càdlàg processes $Y : \Omega \times [s,t] \mapsto \mathbb{R}$ such that
$$\mathbb{E}[\sup_{t\in[0,T]} |Y(t)|^p] < \infty.$$
- $\mathbb{H}^p_{\nu,\mathcal{F}}$ - the space of predictable processes $\Upsilon : \Omega \times [s,t] \times \mathbb{R} \mapsto \mathbb{R}$, such that
$$\mathbb{E}\left[\int_s^t \int_{\mathbb{R}} |\Upsilon(u,z)|^p \nu(dz)du\right] < \infty.$$

Suppose that the insurance risk process is given by

$$dR(t) = p(t)dt - \int_0^\infty \zeta N^0(dt, d\zeta), \tag{1}$$

where the premium rate $p(t)$ is $\mathcal{F}_t$-progressively measurable and uniformly bounded with values in $\mathbb{H}^2_{\mathcal{F}}(s,t;\mathbb{R})$.

Assume that the insurer receives the premium continuously at the rate $c(t) = (1+k(t))p(t)$, where $k(t)$ is a relative security loading of the insurer. We assume that it is an $\mathcal{F}_t$-adapted process. With no investment and reinsurance strategy, the surplus process $X(t)$ of the insurer is given by

$$\begin{aligned} dX(t) &= c(t)dt - dR(t) \\ &= k(t)p(t)dt + \int_0^\infty \zeta N^0(dt, d\zeta). \end{aligned} \tag{2}$$

In order to control the claim risks, we assume that the insurer acquires a new business by purchasing proportional reinsurance as follows: Let $u_0(t) \ge 0$ be the retention level. The insurer pays $100u_0(t)\%$ of the claim, while the re-insurer pays the rest, i.e., $100(1-u_0(t))\%$ of the claim. Therefore, the insurer should pay $(1+u(t))(1-u_0(t))p(t)$ to the re-insurer, where $u(t) \geq u_0(t)$ is the reinsurance security loading.

Then, the surplus process of the insurer after purchasing the proportional reinsurance is given by

$$\begin{aligned} dX(t) &= (1+k(t))p(t)dt - (1+u(t))(1-u_0(t))p(t)dt - u_0(t)dR(t) \\ &= (k(t)-u(t)+u(t)u_0(t))p(t)dt + \int_0^\infty \zeta u_0(t) N^0(dt,d\zeta). \end{aligned}$$

Furthermore, we assume that the insurer can invest its wealth in the financial market composed by a risk-free asset and a risky share. Suppose that the risk-free asset has price $B(t)$ defined by

$$dB(t) = r(t)B(t)dt,$$

where $r(t)$ is the risk-free interest rate. We assume that the interest rate is continuously bounded and $\mathcal{F}_t$-adapted with values in $\mathbb{H}^2_{\mathcal{F}}(s,t;\mathbb{R})$. The risky asset $S(t)$ is defined by the following geometric jump-diffusion process:

$$dS(t) = S(t)\Big[\mu(t)dt + \sigma(t)dW(t) + \int_{\mathbb{R}} \gamma_S(t,\zeta)\tilde{N}(dt,d\zeta)\Big], \tag{3}$$

where $\mu(t)$, $\sigma(t) \in \mathbb{H}^2_{\mathcal{F}}(s,t;\mathbb{R})$, $\gamma_S(t,\cdot) \in \mathbb{H}^2_{\nu,\mathcal{F}}(s,t;\mathbb{R})$ are $\mathcal{F}_t$-predictable bounded processes on the interval $t \in [0,T]$, representing the appreciation rate, volatility and jump rate, respectively. We also assume that $\gamma_S(t,\cdot)$ is bounded below by -1.

Suppose that the value amount that the insurer invests in the risky asset at time t is denoted by $\pi(t)$. The wealth process is then given by

$$\begin{aligned} dY(t) &= (Y(t)-\pi(t))\frac{dB(t)}{B(t)} + \pi(t)\frac{dS(t)}{S(t)} + dX(t) \\ &= \Big[r(t)Y(t) + u(t)u_0(t)p(t) + (\mu(t)-r(t))\pi(t) + (k(t)-u(t))p(t) \\ &\quad + \int_0^\infty \zeta u_0(t)\nu^0(d\zeta)\Big]dt + \pi(t)\sigma(t)dW(t) + \int_{\mathbb{R}} \pi(t)\gamma_S(t,\zeta)\tilde{N}(dt,d\zeta) \\ &\quad + \int_0^\infty \zeta u_0(t)\tilde{N}^0(dt,d\zeta). \end{aligned} \tag{4}$$

Definition 1. *A strategy* $\{(\pi(t),u_0(t))\}_{t\in[0,T]}$ *is said to be admissible if it satisfies the following conditions:*

1. $(\pi(t),u_0(t)) \in \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R})$, *and*
2. *The SDE* (4) *has a unique strong solution* $Y(t) \in \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R})$.

We denote by $\mathcal{A}$ *a set of all admissible strategies.*

We will now formulate the investment-reinsurance mean variance investment problem without pre-commitment. In order to understand such kind of problems, we first define the pre-commitment mean variance optimization problem. This problem can be described as the maximization over all the admissible strategies $(\pi(\cdot), u_0(\cdot))$ of the following functional:

$$\mathcal{J}(0, Y(0), \pi(\cdot), u_0(\cdot)) := \frac{1}{2}\mathsf{Var}_0[Y(T)] - \lambda\mathbb{E}_0[Y(T)],$$

where λ is the risk aversion parameter. Here, $\mathbb{E}_0[\cdot]$, $\mathsf{Var}_0[\cdot]$ are the expectation and variance conditioned on the event $[Y(0) = y]$, respectively.

For any given $(t, Y(t))$, we define the mean-variance cost functional of the insurer by

$$\mathcal{J}(t, Y(t), \pi(\cdot), u_0(\cdot)) := \frac{1}{2}\mathsf{Var}_t[Y(T)] - \lambda\mathbb{E}_t[Y(T)],$$

where $\lambda > 0$.

Our aim is to solve the following minimization problem of an insurer:

$$\mathcal{J}(t, Y(t), \pi^*(\cdot), u_0^*(\cdot)) = \inf_{(\pi(\cdot),u_0(\cdot))\in\mathcal{A}} \mathcal{J}(t, Y(t), \pi(\cdot), u_0(\cdot)). \tag{5}$$

This problem is a continuous time version of a standard mean-variance investment problem where we want to minimize the risk by the conditional variance $\frac{1}{2}\mathsf{Var}_t[Y(T)]$, while controlling the utility of final wealth $\lambda\mathbb{E}_t[Y(T)]$. Studying the mean-variance portfolio selection problem of an insurer helps investors to make informed investment decisions, manage risks, enhance financial performance, comply with regulations, and gain a competitive advantage in the insurance market. These factors are crucial to insurers to thrive in a complex and competitive financial landscape.

Note that from the expectation function in the variance cost function, our problem (5) is a nonlinear function acting on the conditional expectation, which leads to a time-inconsistent optimization problems pointed out in [3]. Therefore, an optimal strategy at time t does not guarantee the optimality of $\mathcal{J}$ at subsequent moments $s > t$. However, since the time horizon T is very long, the investment-reinsurance preference may change over time; then it becomes very important to formulate the time-consistent optimal investment-reinsurance problem. Following [3, 6, 14], we define an equilibrium strategy which is consistent with time change, i.e., the optimal strategy derived at time t should agree with the optimal strategy at time $t+\epsilon$, $\epsilon > 0$.

Definition 2. *A pair of strategies* $(\pi^*(\cdot), u_0^*(\cdot)) \in \mathbb{H}^2_{\mathcal{F}}(0, T; \mathbb{R}) \times \mathbb{H}^2_{\mathcal{F}}(0, T; \mathbb{R})$ *is an equilibrium control strategy if for any* $t \in [0, T)$ *and* $(\nu_1, \nu_2) \in \mathbb{H}^2_{\mathcal{F}}(0, T; \mathbb{R}) \times \mathbb{H}^2_{\mathcal{F}}(0, T; \mathbb{R})$,

$$\begin{cases} \pi^\epsilon(s) := \pi^*(s) + \nu_1\mathbf{1}_{[t,t+\epsilon]}(s), \text{ for } t \le s < T \\ u_0^\epsilon(s) := u_0^*(s) + \nu_2\mathbf{1}_{[t,t+\epsilon]}(s), \text{ for } t \le s \le T \end{cases}$$

satisfies the property

$$\liminf_{\epsilon\to 0} \frac{\mathcal{J}(t, X^*(t), \pi^\epsilon(\cdot), u_0^\epsilon(\cdot)) - \mathcal{J}(t, X^*(t), \pi^*(\cdot), u_0^*(\cdot))}{\epsilon} \ge 0.$$

The equilibrium value function is defined by

$$\Phi(t, Y^*(t)) = \mathcal{J}(t, Y(t), \pi^*(\cdot), u_0^*(\cdot)). \tag{6}$$

Since the equilibrium strategy above is defined in the class of open-loop controls, $(\pi^*(\cdot), u_0^*(\cdot))$ and $Y^*(\cdot)$ are called an open-loop equilibrium strategy and an open-loop equilibrium state process, respectively.

3. An open-loop equilibrium strategy

We first give a sufficient condition for the equilibrium strategy, which generalizes [17, 14] to a jump-diffusion case.

Theorem 1. *Suppose that the following conditions hold:*

1. *There exists* $Y^*(\cdot) \in \mathbb{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$ *and* $(P(\cdot,t), Z_1(\cdot,t), Z_2(\cdot,t), Z_3(\cdot,t)) \in \mathbb{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\nu,\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\nu^0,\mathcal{F}}(0,T;\mathbb{R})$ *that solve the following forward-backward system of equations:*

$$
\begin{aligned}
dY^*(s) &= \Big[r(s)Y^*(s)+u(s)u_0^*(s)p(s)+(\mu(s)-r(s))\pi^*(s)+(k(s)-u(s))p(s) \\
&\quad + \int_0^\infty \zeta u_0^*(s)\nu^0(d\zeta)\Big]ds+\pi^*(s)\sigma(s)dW(s)+\int_{\mathbb{R}}\pi^*(s)\gamma_S(s,\zeta)\tilde{N}(ds,d\zeta) \\
&\quad + \int_0^\infty \zeta u_0^*(s)\tilde{N}^0(ds,d\zeta), \qquad (7)\\
dP(s,t) &= -r(s)P(s,t)ds + Z_1(s,t)dW(s) + \int_{\mathbb{R}} Z_2(s,t,\zeta)\tilde{N}(ds,d\zeta) \\
&\quad + \int_0^\infty Z_3(s,t,\zeta)N^0(ds,d\zeta) \qquad (8)\\
Y^*(0) &= y_0, \quad P(T,t) = Y^*(T) - \mathbb{E}_t[Y^*(T)] - \lambda.
\end{aligned}
$$

2. *Suppose that for*

$$
\begin{aligned}
\Lambda_1(s,t) &= (\mu(s)-r(s))P(s,t) + \sigma(s)Z_1(s,t) + \int_{\mathbb{R}} \gamma_S(s,\zeta)Z_2(s,t,\zeta)\nu(\zeta) \\
\Lambda_2(s,t) &= \Big(u(s)p(s) + \int_0^\infty \zeta\nu^0(d\zeta)\Big)P(s,t) + \int_0^\infty \zeta Z_3(s,t,\zeta)\nu^0(d\zeta),
\end{aligned}
$$

it holds that

$$
\liminf_{\epsilon\to 0} \frac{1}{\epsilon}\mathbb{E}_t\Big[\int_t^{t+\epsilon} \Lambda_i(s,t)ds\Big] = 0, \quad \text{a.s.,} \quad \text{for} \quad s\in[t,T], \quad \text{where} \quad i=1,2. \qquad (9)
$$

Then, the strategy $(\pi^*(\cdot), u_0^*(\cdot)) \in \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R})$ *is an equilibrium control strategy for any* $t\in[0,T]$.

Proof. Suppose that $(\pi^*(\cdot), u_0^*(\cdot))$ satisfy (1) and (2) above. For a given strategy $(\pi^\epsilon(\cdot), u_0^\epsilon(\cdot))$, we define the process $Y_1^{t,\epsilon,\nu_1,\nu_2} := Y^{t,\epsilon,\nu_1,\nu_2}(\cdot) - Y^*(\cdot)$, where $Y^*(\cdot)$ and $Y^{t,\epsilon,\nu_1,\nu_2}(\cdot)$ are the state processes associated to the strategies $(\pi^*(\cdot), u_0^*(\cdot))$ and $(\pi^\epsilon(\cdot), u_0^\epsilon(\cdot))$, respectively. Then, from (7), we can easily check that

$$
\begin{aligned}
dY_1^{t,\epsilon,\nu_1,\nu_2}(s) &= \Big[r(s)Y_1^{t,\epsilon,\nu_1,\nu_2}(s) + (\mu(s)-r(s))\nu_1\mathbf{1}_{[t,t+\epsilon]}(s) \\
&\quad + \Big(u(s)p(s) + \int_0^\infty \zeta\nu^0(d\zeta)\Big)\nu_2\mathbf{1}_{[t,t+\epsilon]}(s)\Big]ds \\
&\quad + \sigma(s)\nu_1\mathbf{1}_{[t,t+\epsilon]}(s)dW(s) + \int_{\mathbb{R}} \nu_1\mathbf{1}_{[t,t+\epsilon]}(s)\gamma_S(s,\zeta)\tilde{N}(ds,d\zeta) \\
&\quad + \int_0^\infty \nu_2\zeta\mathbf{1}_{[t,t+\epsilon]}(s)\tilde{N}^0(ds,d\zeta)
\end{aligned}
$$

and $Y_1^{t,\epsilon,\nu_1,\nu_2}(\cdot) \in \mathbb{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$. Note that this is a linear SDE, so its solutions exist and are unique. Moreover, one can check that

$$\mathcal{J}(t, Y^*(t), \pi^\epsilon(\cdot), u_0^\epsilon(\cdot)) - \mathcal{J}(t, Y^*(t), \pi^*(\cdot), u_0^*(\cdot)) = \frac{1}{2}\mathsf{Var}_t[Y_1^{t,\epsilon,\nu_1,\nu_2}(T)] + \mathcal{J}_1(t),$$

where

$$\mathcal{J}_1(t) = \mathbb{E}_t[(Y^*(T) - \mathbb{E}_t[Y^*(T)] - \lambda)Y_1^{t,\epsilon,\nu_1,\nu_2}(T)].$$

Applying Itô's formula for SDEs with jumps (Theorem 1.16, [11]) to $Y_1^{t,\epsilon,\nu_1,\nu_2}(s)P(s,t)$, we have:

$$\begin{aligned}
&dY_1^{t,\epsilon,\nu_1,\nu_2}(s)P(s,t)\\
&\quad= (\Lambda_1(s,t)\nu_1\mathbf{1}_{[t,t+\epsilon]}(s) + \Lambda_2(s,t)\nu_2\mathbf{1}_{[t,t+\epsilon]}(s))ds\\
&\quad\quad+[\sigma(s)\nu_1 P(s,t)\mathbf{1}_{[t,t+\epsilon]}(s) + Y_1^{t,\epsilon,\nu_1,\nu_2}(s)Z_1(s,t)]dW(s)\\
&\quad\quad+\int_{\mathbb{R}}[\nu_1\gamma_S(s,\zeta)(P(s,t)+Z_2(s,t,\zeta))\mathbf{1}_{[t,t+\epsilon]}(s)+Y_1^{t,\epsilon,\nu_1,\nu_2}(s)Z_2(s,t,\zeta)]\tilde{N}(ds,d\zeta)\\
&\quad\quad+\int_0^\infty[\nu_2\zeta(P(s,t)+Z_3(s,t,\zeta))\mathbf{1}_{[t,t+\epsilon]}(s)+Y_1^{t,\epsilon,\nu_1,\nu_2}(s)Z_3(s,t,\zeta)]\tilde{N}^0(ds,d\zeta).
\end{aligned}$$

Taking the conditional expectation, we see that

$$\mathcal{J}_1(t) = \mathbb{E}_t\Big[\int_t^T (\Lambda_1(s,t)\nu_1\mathbf{1}_{[t,t+\epsilon]}(s) + \Lambda_2(s,t)\nu_2\mathbf{1}_{[t,t+\epsilon]}(s))ds\Big].$$

Then, by condition (2) of the theorem, we have

$$\liminf_{\epsilon\to 0} \frac{1}{\epsilon}\mathcal{J}_1(t) = \liminf_{\epsilon\to 0} \frac{1}{\epsilon}\mathbb{E}_t\Big[\int_t^T (\Lambda_1(s,t)\nu_1\mathbf{1}_{[t,t+\epsilon]}(s)+\Lambda_2(s,t)\nu_2\mathbf{1}_{[t,t+\epsilon]}(s))ds\Big] = 0,$$

which completes the proof. □

Next, we are going to derive the equilibrium control strategy such that the first condition in Theorem 1 is satisfied. We construct the solution to (8) by the following ansatz:

$$P(s,t) = P_0(s)\{P_1(s)Y^*(s) - \mathbb{E}_t[P_1(s)Y^*(s)] + \mathbb{E}[\lambda P_2(s)] - \lambda P_3(s)\}, \tag{10}$$

where $(P_0(s), Q_0(s), K_0(s,\cdot))$ and $(P_i(s), Q_i(s), K_i(s,\cdot), M_i(s,\cdot))$, $i = 1,2,3$ solve the following BSDEs:

$$\begin{cases} dP_0(s) = -r(s)P_0(s)ds + Q_0(s)dW(s) + \int_{\mathbb{R}} K_0(s,\zeta)\tilde{N}(ds,d\zeta), \text{ for } 0 \le s < T\\ P_0(T) = 1, \end{cases} \tag{11}$$

and

$$\begin{cases} dP_i(s) = -f_i(s,\cdot)ds + Q_i(s)dW(s) + \int_{\mathbb{R}} K_i(s,\zeta)\tilde{N}(ds,d\zeta)\\ \qquad\qquad + \int_0^\infty M_i(s,\zeta)\tilde{N}^0(ds,d\zeta),\\ P_1(T) = P_3(T) = 1, \quad \text{and} \quad P_2(T) = 0, \end{cases} \tag{12}$$

for $f_i(s,\cdot)$ to be defined later.

For each fixed t, by applying Itô's formula to (10) with respect to s, we have:

$$\begin{aligned}
dP(s,t) = &\Big\{P_0(s)\Big\{P_1(s)\Big[r(s)Y^*(s)+u(s)p(s)u_0^*(s)+(\mu(s)-r(s))\pi^*(s)\\
&+(k(s)-u(s))p(s)+\int_0^\infty \zeta u_0^*(s)\nu^0(d\zeta)\Big]-f_1(s)Y^*(s)+\sigma(s)\pi^*(s)Q_1(s)\\
&+\int_{\mathbb{R}}\pi^*(s)\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)+\int_0^\infty \zeta u_0^*(s)M_1(s,\zeta)\nu^0(d\zeta)\\
&-\mathbb{E}_t\Big[P_1(s)\Big[r(s)Y^*(s)+u(s)p(s)u_0^*(s)+(\mu(s)-r(s))\pi^*(s)\\
&+(k(s)-u(s))p(s)+\int_0^\infty \zeta u_0^*(s)\nu^0(d\zeta)\Big]-f_1(s)Y^*(s)+\sigma(s)\pi^*(s)Q_1(s)\\
&+\int_{\mathbb{R}}\pi^*(s)\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)+\int_0^\infty \zeta u_0^*(s)M_1(s,\zeta)\nu^0(d\zeta)-\lambda f_2(s)\Big]\Big\}\\
&-rP(s,t)+\lambda P_0(s)f_3(s)+Q_0(s)(\sigma(s)\pi^*(s)P_1(s)+Q_1(s)Y^*(s)-\lambda Q_3(s))\\
&+\int_{\mathbb{R}}K_0(s,\zeta)[\pi^*(s)\gamma_S(s,\zeta)(P_1(s)+K_1(s,\zeta))+K_1(s,\zeta)Y^*(s)\\
&-\lambda K_3(s,\zeta)]\nu(d\zeta)\Big\}ds+\Big[P_0(s)\Big(\sigma(s)\pi^*(s)P_1(s)+Q_1(s)Y^*(s)\\
&-\lambda Q_3(s)\Big)+Q_0(s)\Big(P_1(s)Y^*(s)-\mathbb{E}_t[P_1(s)Y^*(s)]+\mathbb{E}[\lambda P_2(s)]\\
&-\lambda P_3(s)\Big)\Big]dW(s)+\int_{\mathbb{R}}\Big[(P_0(s)+K_0(s,\zeta))\,(\pi^*(s)\gamma_S(s,\zeta)(P_1(s)+K_1(s,\zeta))\\
&+K_1(s,\zeta)Y^*(s)-\lambda K_3(s,\zeta))+K_0(s,\zeta)\Big(P_1(s)Y^*(s)-\mathbb{E}_t[P_1(s)Y^*(s)]\\
&+\mathbb{E}[\lambda P_2(s)]-\lambda P_3(s)\Big)\Big]\tilde{N}(ds,d\zeta)+P_0(s)\int_0^\infty\Big(u_0^*(s)\zeta(P_1(s)+M_1(s,\zeta))\\
&+M_1(s,\zeta)Y^*(s)-\lambda M_3(s,\zeta)\Big)\tilde{N}^0(ds,d\zeta). \qquad (13)
\end{aligned}$$

Comparing the coefficients of $dW(s)$, $\tilde{N}(ds,d\zeta)$ and $\tilde{N}^0(ds,d\zeta)$ with (8), we get:

$$\begin{aligned}
Z_1(s,t) &= P_0(s)\Big(\sigma(s)\pi^*(s)P_1(s)+Q_1(s)Y^*(s)-\lambda Q_3(s)\Big)+Q_0(s)\Big(P_1(s)Y^*(s)\\
&\quad -\mathbb{E}_t[P_1(s)Y^*(s)]+\mathbb{E}[\lambda P_2(s)]-\lambda P_3(s)\Big)\,; \qquad (14)\\
Z_2(s,t,\zeta) &= (P_0(s)+K_0(s,\zeta))(\pi^*(s)\gamma_S(s,\zeta)(P_1(s)+K_1(s,\zeta))+K_1(s,\zeta)Y^*(s)\\
&\quad -\lambda K_3(s,\zeta))+K_0(s,\zeta)\Big(P_1(s)Y^*(s)-\mathbb{E}_t[P_1(s)Y^*(s)]+\mathbb{E}[\lambda P_2(s)]\\
&\quad -\lambda P_3(s)\Big); \qquad (15)\\
Z_3(s,t,\zeta) &= P_0(s)\Big(u_0^*(s)\zeta(P_1(s)+M_1(s,\zeta))+M_1(s,\zeta)Y^*(s)-\lambda M_3(s,\zeta)\Big). \qquad (16)
\end{aligned}$$

Note that from the limit function in (9), we can deduce that

$$\Lambda_1(t,t)=0 \ \text{ and } \ \Lambda_2(t,t)=0, \quad \forall\, t\in[0,T].$$

Then,

$$(\mu(s)-r(s))P(s,s)+\sigma(s)Z_1(s,s)+\int_{\mathbb{R}}\gamma_S(s,\zeta)Z_2(s,s,\zeta)\nu(\zeta)=0,$$
$$\Big(u(s)p(s)+\int_0^\infty \zeta\nu^0(d\zeta)\Big)P(s,s)+\int_0^\infty \zeta Z_3(s,s,\zeta)\nu^0(d\zeta)=0,$$

which implies that

$$\pi^*(s)=\phi_1(s)Y^*(s)+\psi_1(s), \tag{17}$$
$$u_0^*(s)=\phi_2(s)Y^*(s)+\psi_2(s), \tag{18}$$

where

$$\phi_1(s)=-\frac{\sigma(s)Q_1(s)+\int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)}{P_1(s)\Big(\sigma^2(s)+\int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)\gamma_S^2(s,\zeta)\nu(d\zeta)\Big)}; \tag{19}$$

$$\begin{aligned}\psi_1(s)=&-\frac{1}{P_1(s)\Big(\sigma^2(s)+\int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)\gamma_S^2(s,\zeta)\nu(d\zeta)\Big)}\times\\&\times\Big[\lambda(P_2(s)-P_3(s))\Big(\mu(s)-r(s)+\sigma(s)\frac{Q_0(s)}{P_0(s)}+\int_{\mathbb{R}}\gamma_S(s,\zeta)\frac{K_0(s,\zeta)}{P_0(s)}\nu(d\zeta)\Big)\\&-\lambda\Big(\sigma(s)Q_3(s)+\int_{\mathbb{R}}\gamma_S(s,\zeta)\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)K_3(s,\zeta)\nu(d\zeta)\Big)\Big];\end{aligned} \tag{20}$$

$$\phi_2(s)=-\frac{1}{\int_0^\infty\zeta^2(P_1(s)+M_1(s,\zeta))\nu^0(d\zeta)}\int_0^\infty\zeta M_1(s,\zeta)\nu^0(d\zeta)\quad\text{and} \tag{21}$$

$$\begin{aligned}\psi_2(s)=&-\frac{1}{\int_0^\infty\zeta^2(P_1(s)+M_1(s,\zeta))\nu^0(d\zeta)}\Big[\lambda\Big(up+\int_0^\infty\zeta\nu^0(d\zeta)\Big)(P_2(s)-P_3(s))\\&-\lambda\int_0^\infty\zeta M_3(s,\zeta)\nu^0(d\zeta)\Big].\end{aligned} \tag{22}$$

Furthermore, we compare the ds term in (8) with that in (13) to obtain the following equation:

$$\begin{aligned}&P_0(s)\Big\{r(s)Y^*(s)P_1(s)+u_0^*(s)\Big(u(s)p(s)P_1(s)+\int_0^\infty\zeta(P_1(s)+M_1(s,\zeta))\nu^0(d\zeta)\Big)\\&+\pi^*(s)\Big((\mu(s)-r(s))P_1(s)+\sigma(s)Q_1(s)+\int_{\mathbb{R}}\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)\Big)\\&+(k(s)-u(s))p(s)P_1(s)-f_1(s)Y^*(s)\Big\}+\lambda P_0(s)f_3(s)+\pi^*(s)P_1(s)\Big(\sigma(s)Q_0(s)\\&+\int_{\mathbb{R}}\gamma_S(s,\zeta)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)K_0(s,\zeta)\nu(d\zeta)\Big)+Y^*(s)\Big(Q_0(s)Q_1(s)\\&+\int_{\mathbb{R}}K_0(s,\zeta)K_1(s,\zeta)\nu(d\zeta)\Big)-\lambda Q_0(s)Q_3(s)-\lambda\int_{\mathbb{R}}K_0(s,\zeta)K_3(s,\zeta)\nu^0(d\zeta)\end{aligned}$$

$$-P_0(s)\mathbb{E}_t\Big[r(s)Y^*(s)P_1(s)+u_0^*(s)\Big(u(s)p(s)P_1(s)+\int_0^\infty \zeta(P_1(s)+M_1(s,\zeta))\nu^0(d\zeta)\Big)$$
$$+\pi^*(s)\Big((\mu(s)-r(s))P_1(s)+\sigma(s)Q_1(s)+\int_{\mathbb{R}}\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)\Big)$$
$$+(k(s)-u(s))p(s)P_1(s)-f_1(s)Y^*(s)+\lambda f_2(s)\Big]=0. \quad (23)$$

Note that the above equation is of the form $A(s)+b(s)\mathbb{E}_t[B(s)]=0$, where

$$A(s):=P_0(s)\Big\{r(s)Y^*(s)P_1(s)+u_0^*(s)\Big(u(s)p(s)P_1(s)+\int_0^\infty\zeta(P_1(s)+M_1(s,\zeta))\nu^0(d\zeta)\Big)$$
$$+\pi^*(s)\Big((\mu(s)-r(s))P_1(s)+\sigma(s)Q_1(s)+\int_{\mathbb{R}}\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)\Big)$$
$$+(k(s)-u(s))p(s)P_1(s)-f_1(s)Y^*(s)\Big\}+\lambda P_0(s)f_3(s)+\pi^*(s)P_1(s)\Big(\sigma(s)Q_0(s)$$
$$+\int_{\mathbb{R}}\gamma_S(s,\zeta)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)K_0(s,\zeta)\nu(d\zeta)\Big)+Y^*(s)\Big(Q_0(s)Q_1(s)$$
$$+\int_{\mathbb{R}}K_0(s,\zeta)K_1(s,\zeta)\nu(d\zeta)\Big)-\lambda Q_0(s)Q_3(s)-\lambda\int_{\mathbb{R}}K_0(s,\zeta)K_3(s,\zeta)\nu^0(d\zeta)$$

and

$$B(S):=r(s)Y^*(s)P_1(s)+u_0^*(s)\Big(u(s)p(s)P_1(s)+\int_0^\infty\zeta(P_1(s)+M_1(s,\zeta))\nu^0(d\zeta)\Big)$$
$$+\pi^*(s)\Big((\mu(s)-r(s))P_1(s)+\sigma(s)Q_1(s)+\int_{\mathbb{R}}\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)\Big)$$
$$+(k(s)-u(s))p(s)P_1(s)-f_1(s)Y^*(s)+\lambda f_2(s).$$

Following these arguments, equation (23) is satisfied if $A(s)=0$, i.e.,

$$P_0(s)\Big\{r(s)Y^*(s)P_1(s)+u_0^*(s)\Big(u(s)p(s)P_1(s)+\int_0^\infty\zeta(P_1(s)+M_1(s,\zeta))\nu^0(d\zeta)\Big)$$
$$+\pi^*(s)\Big((\mu(s)-r(s))P_1(s)+\sigma(s)Q_1(s)+\int_{\mathbb{R}}\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)\Big)$$
$$+(k(s)-u(s))p(s)P_1(s)-f_1(s)Y^*(s)\Big\}+\lambda P_0(s)f_3(s)+\pi^*(s)P_1(s)\Big(\sigma(s)Q_0(s)$$
$$+\int_{\mathbb{R}}\gamma_S(s,\zeta)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)K_0(s,\zeta)\nu(d\zeta)\Big)+Y^*(s)\Big(Q_0(s)Q_1(s$$
$$+\int_{\mathbb{R}}K_0(s,\zeta)K_1(s,\zeta)\nu(d\zeta)\Big)-\lambda Q_0(s)Q_3(s)-\lambda\int_{\mathbb{R}}K_0(s,\zeta)K_3(s,\zeta)\nu^0(d\zeta)=0.$$

From this, we substitute $\pi^*(s)$ and $u_0^*(s)$ from (17)-(18) to get the following generator functions (suppressing the variable s):

$$\begin{aligned}
&f_1(s,P_1,Q_1,K_1,M_1) \qquad (24)\\
&\quad = rP_1 + \frac{1}{P_0}\Big(Q_0Q_1 + \int_{\mathbb{R}} K_0K_1\nu(d\zeta)\Big)\\
&\quad - \frac{\sigma(s)Q_1(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)}{P_1(s)\Big(\sigma^2(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)\gamma_S^2(s,\zeta)\nu(d\zeta)\Big)} \times\\
&\quad \times\Big[(\mu-r)P_1 + \sigma Q_1 + \sigma\frac{Q_0}{P_0}P_1 + \int_{\mathbb{R}}\Big(K_1 + \frac{K_0}{P_0}\left(P_1+K_1\right)\Big)\gamma_S\nu(d\zeta)\Big]\\
&\quad - \frac{\int_0^\infty \zeta M_1\nu^0(d\zeta)}{\int_0^\infty \zeta^2(P_1(s)+M_1(s,\zeta))\nu^0(d\zeta)} \times \left[upP_1 + \int_0^\infty \zeta(P_1+M_1)\nu^0(d\zeta)\right];
\end{aligned}$$

$$\begin{aligned}
&f_3(s,P_3,Q_3,K_3,M_3) \qquad (25)\\
&\quad = \frac{1}{P_0}\Big(Q_0Q_3 + \int_{\mathbb{R}} K_0K_3\nu(d\zeta)\Big) - \frac{k-u}{\lambda}pP_1\\
&\quad + \frac{1}{P_1(s)\Big(\sigma^2(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)\gamma_S^2(s,\zeta)\nu(d\zeta)\Big)}\\
&\quad \times\Big[(\mu-r)P_1 + \sigma Q_1 + \sigma\frac{Q_0}{P_0}P_1 + \int_{\mathbb{R}}\Big(K_1 + \frac{K_0}{P_0}\left(P_1+K_1\right)\Big)\gamma_S\nu(d\zeta)\Big]\\
&\quad \times\Big[(P_2-P_3)\Big(\mu - r + \sigma\frac{Q_0}{P_0} + \int_{\mathbb{R}}\gamma_S\frac{K_0}{P_0}\nu(d\zeta)\Big) - \sigma Q_3\\
&\quad - \int_{\mathbb{R}}\gamma_S(s,\zeta)\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)K_3(s,\zeta)\nu(d\zeta)\Big]\\
&\quad + \frac{1}{\int_{\mathbb{R}}\gamma_S(s,\zeta)\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)K_3(s,\zeta)\nu(d\zeta)}\\
&\quad \times\Big[upP_1 + \int_0^\infty \zeta(P_1+M_1)\nu^0(d\zeta)\Big]\\
&\quad \times\Big[(P_2-P_3)\Big(up + \int_0^\infty \zeta\nu^0(d\zeta)\Big) - \int_0^\infty \zeta M_3\nu^0(d\zeta)\Big].
\end{aligned}$$

Moreover, the term under the expected value in (23) should also be zero, i.e., $B(s) = 0$. Hence

$$\begin{aligned}
&r(s)Y^*(s)P_1(s) + u_0^*(s)\Big(u(s)p(s)P_1(s) + \int_0^\infty \zeta(P_1(s)+M_1(s,\zeta))\nu^0(d\zeta)\Big)\\
&+\pi^*(s)\Big((\mu(s)-r(s))P_1(s) + \sigma(s)Q_1(s) + \int_{\mathbb{R}}\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)\Big)\\
&+(k(s)-u(s))p(s)P_1(s) - f_1(s)Y^*(s) + \lambda f_2(s) = 0,
\end{aligned}$$

which similarly implies that (suppressing the variable s)

$$
\begin{aligned}
&f_2(s,P_2,Q_2,K_2,M_2) \qquad (26)\\
&\quad = f_3(s,P_3,Q_3,K_3,M_3) - \frac{1}{P_0}\Big(Q_0Q_3 + \int_{\mathbb{R}} K_0K_3\nu(d\zeta)\Big)\\
&\quad + \frac{Y^*}{\lambda P_0}\Big[Q_0Q_1 + \int_{\mathbb{R}} K_0K_1\nu(d\zeta)\\
&\quad - \frac{\sigma(s)Q_1(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)}{P_1(s)\Big(\sigma^2(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)\gamma_S^2(s,\zeta)\nu(d\zeta)\Big)}\\
&\quad \times\Big(\sigma Q_1 + \int_{\mathbb{R}} \gamma_S K_1\nu(d\zeta)\Big)\Big]\\
&\quad - \frac{\sigma(s)Q_1(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\gamma_S(s,\zeta)K_1(s,\zeta)\nu(d\zeta)}{P_1(s)\Big(\sigma^2(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)\gamma_S^2(s,\zeta)\nu(d\zeta)\Big)}\\
&\quad \times\Big[(P_2-P_3)\Big(\mu - r + \sigma\frac{Q_0}{P_0} + \int_{\mathbb{R}}\gamma_S\frac{K_0}{P_0}\nu(d\zeta)\Big) - \sigma Q_3\\
&\quad - \int_{\mathbb{R}}\gamma_S(s,\zeta)\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)K_3(s,\zeta)\nu(d\zeta)\Big].
\end{aligned}
$$

Remark 1. *It is easy to see that* (11) *is a linear BSDE with jumps. Then, following Proposition 3.4.1 in [5], there exists a unique solution* $(P_0,\, Q_0,\, K_0) \in \mathbb{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\nu,\mathcal{F}}(0,T;\mathbb{R})$ *such that*

$$P_0 = \mathbb{E}_t\Big[e^{\int_t^T r(s)ds}\Big],$$

for any $t \in [0,T]$*. Moreover,*

$$\int_t^T Q_0(s)dW(s) \quad \text{and} \quad \int_t^T\int_{\mathbb{R}} K_0(s,\zeta)\tilde{N}(ds,d\zeta)$$

are bounded mean oscillation martingales, with Q_0 *and* K_0 *derived by the martingale representation theorem.*

Furthermore, we state the following proposition for the existence and uniqueness of the solution to (12), with the generators given by (24)-(26), respectively.

Proposition 1. *The system of BSDEs in* (12) *admits a unique solution* $(P_i, Q_i, K_i, M_i) \in \mathbb{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\nu,\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\nu^0,\mathcal{F}}(0,T;\mathbb{R})$*, for any* $p > 1$*.*

Proof. Note that the generator function f_1 is quadratic w.r.t diffusion control Q_1 and the jump controls K_1 and M_1. Then, one can check if f_i satisfies the assumptions (H_1) and (H_2) in [10]. The result then follows from ([8], Proposition 5.1) and ([10], theorems 1 and 2). To prove (12), for $i = 2, 3$, we first define the following BSDE

$(P_4, Q_4, K_4, M_4) := (P_2 - P_3, Q_2 - Q_3, K_2 - K_3, M_2 - M_3)$. Then

$$\begin{cases} dP_4(s) = \Big\{ \frac{1}{P_0}\Big(Q_0 Q_3 + \int_{\mathbb{R}} K_0 K_3 \nu(d\zeta)\Big) - \frac{Y^*}{\lambda P_0}\Big[Q_0 Q_1 + \int_{\mathbb{R}} K_0 K_1 \nu(d\zeta) \\ \quad - \frac{\sigma(s) Q_1(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\gamma_S(s,\zeta) K_1(s,\zeta)\nu(d\zeta)}{P_1(s)\left(\sigma^2(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)\gamma_S^2(s,\zeta)\nu(d\zeta)\right)} \\ \quad \times\Big(\sigma Q_1 + \int_{\mathbb{R}} \gamma_S K_1 \nu(d\zeta)\Big)\Big] \\ \quad + \frac{\sigma(s) Q_1(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\gamma_S(s,\zeta) K_1(s,\zeta)\nu(d\zeta)}{P_1(s)\left(\sigma^2(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)\gamma_S^2(s,\zeta)\nu(d\zeta)\right)} \\ \quad \times\Big[P_4\Big(\mu - r + \sigma\frac{Q_0}{P_0} + \int_{\mathbb{R}} \gamma_S \frac{K_0}{P_0}\nu(d\zeta)\Big) \\ \quad -\sigma Q_3 - \int_{\mathbb{R}} \gamma_S(s,\zeta)\left(1 + \frac{K_0(s,\zeta)}{P_0(s)}\right) K_3(s,\zeta)\nu(d\zeta)\Big]\Big\} ds \\ \quad + Q_4(s) dW(s) + \int_{\mathbb{R}} K_4(s,\zeta)\tilde{N}(ds,d\zeta) + \int_0^\infty M_4(s,\zeta)\tilde{N}^0(ds,d\zeta), \\ P_4(T) = -1. \end{cases} \tag{27}$$

This is a linear BSDE with jumps. Then, by ([5], propositions 3.3.1 and 3.4.1), there exists a unique solution $(P_4, Q_4, K_4, M_4) \in \mathbb{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\nu,\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\nu^0,\mathcal{F}}(0,T;\mathbb{R})$.

We can now rewrite the BSDE for (P_3, Q_3, K_3, M_3) with

$$\begin{aligned} & f_3'(s, P_3, Q_3, K_3, M_3) \\ & \quad = \frac{1}{P_0}\Big(Q_0 Q_3 + \int_{\mathbb{R}} K_0 K_3 \nu(d\zeta)\Big) - \frac{k-u}{\lambda} p P_1 \\ & \qquad + \frac{1}{P_1(s)\left(\sigma^2(s) + \int_{\mathbb{R}}\left(1+\frac{K_0(s,\zeta)}{P_0(s)}\right)\left(1+\frac{K_1(s,\zeta)}{P_1(s)}\right)\gamma_S^2(s,\zeta)\nu(d\zeta)\right)} \\ & \qquad \times\Big[(\mu - r)P_1 + \sigma Q_1 + \sigma\frac{Q_0}{P_0}P_1 + \int_{\mathbb{R}}\Big(K_1 + \frac{K_0}{P_0}(P_1 + K_1)\Big)\gamma_S \nu(d\zeta)\Big] \\ & \qquad \times\Big[P_4\Big(\mu - r + \sigma\frac{Q_0}{P_0} + \int_{\mathbb{R}} \gamma_S \frac{K_0}{P_0}\nu(d\zeta)\Big) - \sigma Q_3 \\ & \qquad - \int_{\mathbb{R}} \gamma_S(s,\zeta)\left(1 + \frac{K_0(s,\zeta)}{P_0(s)}\right) K_3(s,\zeta)\nu(d\zeta)\Big] \\ & \qquad + \frac{1}{+\int_{\mathbb{R}} \gamma_S(s,\zeta)\left(1 + \frac{K_0(s,\zeta)}{P_0(s)}\right) K_3(s,\zeta)\nu(d\zeta)} \\ & \qquad \times\Big[upP_1 + \int_0^\infty \zeta(P_1 + M_1)\nu^0(d\zeta)\Big] \\ & \qquad \times\Big[P_4\Big(up + \int_0^\infty \zeta\nu^0(d\zeta)\Big) - \int_0^\infty \zeta M_3 \nu^0(d\zeta)\Big]. \end{aligned}$$

According to ([5], propositions 3.3.1 and 3.4.1), there exists a unique solution $(P_3, Q_3, K_3, M_3) \in \mathbb{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\nu,\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\nu^0,\mathcal{F}}(0,T;\mathbb{R})$.

Then, the unique solution for (P_2, Q_2, K_2, M_2) is given by $(P_2, Q_2, K_2, M_2) = (P_4 + P_3, Q_4 + Q_3, K_4 + K_3, M_4 + M_3)$. This completes the proof. □

The following theorem states the main result of this paper.

Theorem 2. *Let* $(\pi^*(\cdot),\, u_0^*(\cdot))$ *be given by* (17)-(18) *and* $Y^*(\cdot)$ *the corresponding solution to* (4). *If*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Q_0^2(t)|^2\Big] < \infty, \tag{28}$$

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\int_{\mathbb{R}}|K_0^2(t,\zeta)|^2\nu(d\zeta)\Big] < \infty. \tag{29}$$

Then, $(\pi^*(\cdot),\, u_0^*(\cdot))$ *is an open-loop equilibrium strategy for our problem* (5).

Proof. Applying the Itô formula in the ansatz $P(\cdot,\cdot)$ in (10), the controls $Z_1(\cdot,\cdot)$, $Z_2(\cdot,\cdot,\cdot)$, $Z_3(\cdot,\cdot,\cdot)$ in (14)-(16) were derived by comparing the coefficients of (8) and (13). Therefore, $(P(\cdot,\cdot), Z_1(\cdot,\cdot), Z_2(\cdot,\cdot,\cdot), Z_3(\cdot,\cdot,\cdot))$ is the solution to BSDE (8). Moreover, Equation (7) is a linear SDE. Therefore, from (28), (29) and Proposition 1, both $\phi_j(\cdot)$ and $\psi_j(\cdot)$, $j = 1, 2$, in (19)-(22) are uniformly bounded, which implies that $(\pi^*(\cdot),\, u_0^*(\cdot))$ is bounded. Hence, $Y^*(\cdot) \in \mathbb{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$ and $(\pi^*(\cdot),\, u_0^*(\cdot)) \in \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathbb{H}^2_{\mathcal{F}}(0,T;\mathbb{R})$. Then, by Theorem 1, $(\pi^*(\cdot),\, u_0^*(\cdot))$ is an open-loop equilibrium strategy. □

Following arguments similar to [13], Theorem 4.2, one can prove the following uniqueness result.

Theorem 3. *Let* (P_0, Q_0, K_0) *and* (P_i, Q_i, K_i, M_i), $i = 1, 2, 3$ *be the solutions to BSDEs* (11)-(12). *Then,* $(\pi^*(\cdot),\, u_0^*(\cdot))$ *given by* (17)-(18) *is the unique equilibrium investment strategy for our optimization problem stated in* (5).

Next, we show the equilibrium value function and the efficient frontier.

Proposition 2. *The equilibrium value function at time* t *is given by*

$$\begin{aligned}\Phi(t, Y^*(t)) = {} & \frac{1}{2}\mathbb{E}_t\Big\{\int_t^T \Big[(Y^*(s)Q_1(s) + \sigma(s)\pi^*(s)P_1(s) - \lambda Q_2(s))^2 \\ & + \int_{\mathbb{R}}(Y^*(s)K_1(s,\zeta) + \gamma_S(s,\zeta)\pi^*(s)P_1(s) - \lambda K_2(s,\zeta))^2\nu(d\zeta) \\ & + \int_0^\infty (Y^*(s)M_1(s,\zeta) + \zeta u_0^*(s)P_1(s) - \lambda Q_2(s))^2\nu^0(d\zeta)\Big] ds\Big\} \\ & -\lambda(Y^*(t)P_1(t) - \lambda P_2(t)).\end{aligned} \tag{30}$$

Furthermore, the corresponding efficient frontier is given by

$$\begin{aligned}Var_t[Y^*(T)] = {} & \mathbb{E}_t\Big\{\int_t^T \Big[(Y^*(s)Q_1(s) + \sigma(s)\pi^*(s)P_1(s) - \lambda Q_2(s))^2 \\ & + \int_{\mathbb{R}}(Y^*(s)K_1(s,\zeta) + \gamma_S(s,\zeta)\pi^*(s)P_1(s) - \lambda K_2(s,\zeta))^2\nu(d\zeta) \\ & + \int_0^\infty (Y^*(s)M_1(s,\zeta) + \zeta u_0^*(s)P_1(s) - \lambda Q_2(s))^2\nu^0(d\zeta)\Big] ds\Big\}.\end{aligned} \tag{31}$$

Proof. By Itô's formula, it is easy to check that

$$\begin{aligned} d(Y^*P_1 - \lambda P_2) &= (Y^*Q_1 + \sigma\pi^*P_1 - \lambda Q_2)dW(s) \\ &\quad + \int_{\mathbb{R}} (Y^*K_1 + \gamma_S\pi^*(P_1 + K_1) - \lambda K_2)\tilde{N}(ds, d\zeta) \\ &\quad + \int_0^{\infty} (Y^*M_1 + \zeta u_0^*(P_1 + M_1) - \lambda Q_2)\tilde{N}^0(ds, d\zeta). \end{aligned}$$

Then

$$\begin{aligned} Y^*(T) &= Y^*(T)P_1(T) - \lambda P_2(T) \\ &= Y^*(t)P_1(t) - \lambda P_2(t) + \int_t^T (Y^*Q_1 + \sigma\pi^*P_1 - \lambda Q_2)dW(s) \\ &\quad + \int_t^T \int_{\mathbb{R}} (Y^*K_1 + \gamma_S\pi^*(P_1 + K_1) - \lambda K_2)\tilde{N}(ds, d\zeta) \\ &\quad + \int_t^T \int_0^{\infty} (Y^*M_1 + \zeta u_0^*(P_1 + M_1) - \lambda Q_2)\tilde{N}^0(ds, d\zeta) \end{aligned}$$

and

$$\mathbb{E}_t[Y^*(T)] = Y^*(t)P_1(t) - \lambda P_2(t).$$

Following similar arguments as above in $(Y^*(t)P_1(t) - \lambda P_2(t))^2$, we have

$$\begin{aligned} \mathbb{E}_t[(Y^*(T))^2] &= (\mathbb{E}_t[(Y^*(T))])^2 + \mathbb{E}_t\Big\{\int_t^T \Big[(Y^*(s)Q_1(s) + \sigma(s)\pi^*(s)P_1(s) - \lambda Q_2(s))^2 \\ &\quad + \int_{\mathbb{R}} (Y^*(s)K_1(s,\zeta) + \gamma_S(s,\zeta)\pi^*(s)P_1(s) - \lambda K_2(s,\zeta))^2\nu(d\zeta) \\ &\quad + \int_0^{\infty} (Y^*(s)M_1(s,\zeta) + \zeta u_0^*(s)P_1(s) - \lambda Q_2(s))^2\nu^0(d\zeta)\Big]ds\Big\}. \end{aligned}$$

Then (30) and (31) follow from (6) and the definition of the efficient frontier, respectively. □

4. Application to the stochastic interest rate

We consider the interest rate $r(\cdot)$ defined by the mean-reverting Lévy-Ornstein-Uhlenbeck as follows:

$$\begin{cases} dr(t) = \alpha(a - r(t))dt + \beta dW(t) + \int_{\mathbb{R}} \gamma\zeta\tilde{N}(dt, d\zeta), \text{ for } 0 \le t \le T \\ r(0) = r_0 > 0, \end{cases} \tag{32}$$

where α, a, β, $\gamma > 0, \zeta \in \mathbb{R}$.

Moreover, we assume that for the risky asset (3), $\sigma > 0$ and $\gamma_S > 0$ are deterministic bounded functions and $\mu(\cdot) = r(\cdot) + \theta(\cdot)$, where $\theta(\cdot) > 0$ is also a deterministic bounded function.

As in [14, 17], we point out that $r(\cdot)$ in (32) is not bounded as previously assumed in this paper. However, the boundedness of the coefficients is only used to prove the existence and uniqueness of the solutions of the BSDEs and the integrability of the wealth process.

We consider the following important lemma, which extends lemma 4.1 in [17] to the jump-diffusion case.

Lemma 1. *For any constant $\kappa \geq 0$, we have $\mathbb{E}\Big[\sup_{t\in[0,T]} e^{\kappa|r(t)|}\Big] < \infty$.*

Proof. Note that (32) is a linear SDE with jumps. Its solution is then given by

$$r(t) = e^{-\alpha t}\Big[r_0 + a\Big(e^{\alpha t} - 1\Big) + \int_0^t \beta e^{\alpha s} dW(s) + \int_0^t \int_{\mathbb{R}} \gamma\zeta e^{\alpha s}\tilde{N}(ds, d\zeta)\Big].$$

Then for any $\kappa \geq 0$,

$$\begin{aligned} e^{\kappa r(t)} &= \exp\Big\{\kappa e^{-\alpha t}\Big[r_0 + a\Big(e^{\alpha t} - 1\Big) + \int_0^t \beta e^{\alpha s} dW(s) + \int_0^t \int_{\mathbb{R}} \gamma\zeta e^{\alpha s}\tilde{N}(ds, d\zeta)\Big]\Big\} \\ &\leq D\exp\Big\{e^{-\alpha t}\Big[\frac{1}{2}\int_0^t |\kappa\sigma e^{\alpha s}|^2 ds + \int_0^t \int_{\mathbb{R}}\Big(e^{\gamma\kappa\zeta e^{\alpha s}} - 1 - \gamma\kappa\zeta e^{\alpha s}\Big)\nu(d\zeta)ds\Big]\Big\} \\ &\quad \times(\mathcal{M}(t))^{e^{-\alpha t}}, \end{aligned}$$

where $D > 0$ is a constant and

$$\begin{aligned} \mathcal{M}(t) = \exp\Big\{&-\int_0^t \Big[\frac{1}{2}|\kappa\sigma e^{\alpha s}|^2 + \int_{\mathbb{R}}\Big(e^{\gamma\kappa\zeta e^{\alpha s}} - 1 - \gamma\kappa\zeta e^{\alpha s}\Big)\nu(d\zeta)\Big]ds \\ &+ \int_0^t \kappa\sigma e^{\alpha s} dW(s) + \int_0^t \int_{\mathbb{R}}\Big(e^{\gamma\kappa\zeta e^{\alpha s}} - 1\Big)\tilde{N}(ds, d\zeta)\Big\}. \end{aligned}$$

One can easily check that

$$d\mathcal{M}(t) = \mathcal{M}(t-)\Big[\kappa\sigma e^{\alpha t} dW(t) + \int_{\mathbb{R}}\Big(e^{\gamma\kappa\zeta e^{\alpha t}} - 1\Big)\tilde{N}(dt, d\zeta)\Big]$$

and

$$\sup_{t\in[0,T]} \mathcal{M}(t) < \infty.$$

Hence

$$\mathbb{E}\Big[\sup_{t\in[0,T]} (\mathcal{M}(t))^{e^{-\alpha t}}\Big] < \infty, \quad \text{which implies that} \quad \mathbb{E}\Big[\sup_{t\in[0,T]} e^{\kappa r(t)}\Big] < \infty.$$

Then, the result follows. □

In the following proposition, we derive the solutions to BSDEs (11) and (12).

Proposition 3. *Suppose that the interest rate $r(\cdot)$ is given by (32) and let $\phi(t) = \frac{1}{\alpha}\Big(1 - e^{-\alpha(T-t)}\Big)$. Then, the solutions to BSDEs (11) and (12) are given by:*

$$\begin{aligned} (P_0(t), Q_0(t), K_0(t,\zeta)) &= \Big(G_0(t)e^{\phi(t)r(t)}, \beta\phi(t)G_0(t)e^{\phi(t)r(t)}, G_0(t)e^{\phi(t)r(t)}\Big(e^{\gamma\zeta} - 1\Big)\Big), \\ (P_1(t), Q_1(t), K_1(t,\zeta)) &= \Big(G_1(t)e^{\phi(t)r(t)}, \beta\phi(t)G_1(t)e^{\phi(t)r(t)}, G_1(t)e^{\phi(t)r(t)}\Big(e^{\gamma\zeta} - 1\Big)\Big), \\ (P_2(t), Q_2(t), K_2(t,\zeta)) &= (G_2(t), 0, 0), \\ (P_3(t), Q_3(t), K_3(t,\zeta)) &= (G_2(t) - G_3(t), 0, 0), \end{aligned}$$

where

$$G_0(t) = \exp\Big\{\int_t^T\Big[a\alpha\phi(s)+\frac{1}{2}\beta^2\phi^2(s)+\int_{\mathbb{R}}\Big(e^{\gamma\zeta\phi(s)}-1-\gamma\zeta\phi(s)\Big)\nu(d\zeta)\Big]ds\Big\}, \quad (33)$$

$$G_1(t) = \exp\Big\{\int_t^T\Big[\phi(s)\Big(a\alpha+\frac{3}{2}\beta^2\phi(s)\Big)+\int_{\mathbb{R}}\Big(\Big(e^{\gamma\zeta}-1\Big)^2+e^{\gamma\zeta\phi(s)}-1-\gamma\zeta\phi(s)\Big)\nu(d\zeta)$$
$$-\frac{\sigma\beta\phi(s)+\int_{\mathbb{R}}\gamma_S e^{\gamma\zeta}\Big(e^{\gamma\zeta}-1\Big)\nu(d\zeta)}{\sigma^2+\int_{\mathbb{R}}\gamma_S^2 e^{2\gamma\zeta}\nu(d\zeta)}\times\Big[\theta(s)+2\sigma\beta\phi(s)$$
$$+\int_{\mathbb{R}}\gamma_S\Big(e^{2\gamma\zeta}-1\Big)\nu(d\zeta)\Big]\Big]ds\Big\}, \quad (34)$$

$$G_3(t) = e^{\int_t^T\Gamma_1(s)ds}\Big(\int_t^T\Gamma_2(s)e^{-\int_s^T\Gamma_1(z)dz}ds-1\Big),$$

$$G_2(t) = \int_t^T\Big((1-\Gamma_1(s))G_3(s)+\Gamma_2(s)\Big)ds. \textit{ nonumber}$$

The functions $\Gamma_1(\cdot)$ *and* $\Gamma_2(\cdot)$ *are given by equations* (36)-(37) *below.*

Proof. From the definition of the interest rate in (32), we can conclude that $M_1 = M_2 = M_3 = 0$. We consider the following partial integro-differential equation (PIDE) using the generalized Feynman-Kac formula:

$$F_{i,t}(t,r)+\alpha(a-r)F_{i,r}(t,r)+\frac{1}{2}\beta^2F_{i,rr}(t,r)+\int_{\mathbb{R}}\Big(F_i(t,r+\gamma\zeta)-F_i(t,r)$$
$$-\gamma\zeta F_{i,r}(t,r)\Big)\nu(d\zeta)+f_i\Big(t,F_i(t,r),\beta F_{i,r}(t,r),F_i(t,r+\gamma\zeta)-F_i(t,r)\Big)=0. \quad (35)$$

If (35) admits a unique classical solution, then by Itô's formula, we know that

$$(P_i,Q_i,K_i)=(F_i(t,r),\beta F_{i,r}(t,r),F_i(t,r+\gamma\zeta)-F_i(t,r)), \quad i=0,1,2,3$$

satisfy BSDEs (11) and (12).

Moreover, we observe that $\phi(t)=\frac{1}{\alpha}\Big(1-e^{-\alpha(T-t)}\Big)$ satisfies the following ordinary differential equation (ODE):

$$\begin{cases}\phi'(t)-\alpha\phi(t)+1=0, \text{ for } 0\le t\le T\\ \phi(T)=0.\end{cases}$$

We consider the following ansatz:

$$F_0(t,r)=G_0(t)e^{\phi(t)r(t)}, \quad F_1(t,r)=G_1(t)e^{\phi(t)r(t)},$$
$$F_2(t,r)=G_2(t) \text{ and } F_3(t,r)=G_2(t)-G_3(t).$$

Then, inserting $F_0(t,r)$ and $F_1(t,r)$ into (35), after some algebraic calculations, we obtain the following linear ODEs:

$$\begin{cases}G_{0,t}(t)+G_0(t)\Big[a\alpha\phi(t)+\frac{1}{2}\beta^2\phi^2(t)+\int_{\mathbb{R}}\Big(e^{\gamma\zeta\phi(t)}-1-\gamma\zeta\phi(t)\Big)\nu(d\zeta)\Big]=0, \ t\in[0,T]\\ G_0(T)=1,\end{cases}$$

and

$$\begin{cases} G_{1,t}(t) + G_1(t)\Big\{\phi(s)\Big(a\alpha + \frac{3}{2}\beta^2\phi(s)\Big) \\ + \int_{\mathbb{R}}\Big(\Big(e^{\gamma\zeta} - 1\Big)^2 + e^{\gamma\zeta\phi(s)} - 1 - \gamma\zeta\phi(s)\Big)\nu(d\zeta) \\ -\frac{\sigma\beta\phi(s)+\int_{\mathbb{R}}\gamma_S e^{\gamma\zeta}\left(e^{\gamma\zeta}-1\right)\nu(d\zeta)}{\sigma^2+\int_{\mathbb{R}}\gamma_S^2 e^{2\gamma\zeta}\nu(d\zeta)} \\ \times\Big[\theta(s) + 2\sigma\beta\phi(s) + \int_{\mathbb{R}}\gamma_S\Big(e^{2\gamma\zeta} - 1\Big)\nu(d\zeta)\Big]\Big\} = 0,\ t \in [0, T] \\ G_1(T) = 1, \end{cases}$$

which imply (33) and (34). We now have to solve for $G_2(\cdot)$ and $G_3(\cdot)$. Motivated by (27), we first look for the solution to (P_4, Q_4, K_4), since $M_4 = 0$. We look for the ansatz function $F_4(t, r)$ in the form $F_4(t, r) = G_3(t)$. After some algebraic calculations, we obtain the following linear ODE:

$$\begin{cases} G_{3,t}(t) - \Gamma_1(t)G_3(t) + \Gamma_2(t) = 0,\ t \in [0, T] \\ G_3(T) = -1, \end{cases}$$

where

$$\Gamma_1(t) = \frac{\sigma\beta\phi(t) + \int_{\mathbb{R}}\gamma_S e^{\gamma\zeta}\Big(e^{\gamma\zeta} - 1\Big)\nu(d\zeta)}{\sigma^2 + \int_{\mathbb{R}}\gamma_S^2 e^{2\gamma\zeta}\nu(d\zeta)} \times \Big(\theta(t) + \sigma\beta\phi(t) + \int_{\mathbb{R}}\gamma_S\Big(e^{\gamma\zeta} - 1\Big)\nu(d\zeta)\Big), \tag{36}$$

$$\Gamma_2(t) = \Big[\frac{\Big(\sigma\beta\phi(t) + \int_{\mathbb{R}}\gamma_S e^{\gamma\zeta}\Big(e^{\gamma\zeta} - 1\Big)\nu(d\zeta)\Big)^2}{G_0(t)\Big(\sigma^2 + \int_{\mathbb{R}}\gamma_S^2 e^{2\gamma\zeta}\nu(d\zeta)\Big)} - \Big(\beta^2\phi^2(t) + \int_{\mathbb{R}}\Big(e^{\gamma\zeta} - 1\Big)^2\nu(d\zeta)\Big)e^{\phi(t)r}\Big]\frac{Y^*(t)G_1(t)}{\lambda}. \tag{37}$$

Its solution is then given by

$$G_3(t) = e^{\int_t^T\Gamma_1(s)ds}\Big(\int_t^T\Gamma_2(s)e^{-\int_s^T\Gamma_1(z)dz}ds - 1\Big).$$

Finally, to solve for $i = 2$, we can see that, based on (26) and the ansatz $F_2(t, r) = G_2(t)$, PIDE (35) becomes the following linear ODE:

$$G_{2,t}(t) + (1 - \Gamma_1(t))G_3(t) + \Gamma_2(t) = 0,$$

with the terminal condition $G_2(T) = 0$. We then have that

$$G_2(t) = \int_t^T\Big((1 - \Gamma_1(s))G_3(s) + \Gamma_2(s)\Big)ds,$$

which completes the proof. □

The strategy $(\pi^*(t), u_0^*(t))$ in (17)-(18) becomes

$$\pi^*(s) = \phi_1(s)Y^*(s) + \psi_1(s),$$
$$u_0^*(s) = \psi_2(s),$$

where

$$\phi_1(s) = -\frac{\sigma\beta\phi(s) + \int_{\mathbb{R}} \gamma_S e^{\gamma\zeta}\Big(e^{\gamma\zeta} - 1\Big)\nu(d\zeta)}{\sigma^2 + \int_{\mathbb{R}} \gamma_S^2 e^{2\gamma\zeta}\nu(d\zeta)};$$
$$\psi_1(s) = -\frac{\lambda G_3(s)\Big(\theta(s) + \sigma\beta\phi(s) + \int_{\mathbb{R}} \gamma_S\Big(e^{\gamma\zeta} - 1\Big)\nu(d\zeta)\Big)}{G_1(s)\Big(\sigma^2 + \int_{\mathbb{R}} \gamma_S^2 e^{2\gamma\zeta}\nu(d\zeta)\Big)} e^{-\phi(s)r(s)};$$
$$\psi_2(s) = -\frac{\lambda\Big(up + \int_0^\infty \zeta\nu^0(d\zeta)\Big)G_3(s)}{G_1(s)\int_0^\infty \zeta^2\nu^0(d\zeta)} e^{-\phi(s)r(s)},$$

and Y^* is the corresponding surplus process.

Furthermore, conditions (28) and (29) are also satisfied, i.e.,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |Q_0^2(t)|^2\Big] = \mathbb{E}\Big[\sup_{t\in[0,T]} \beta\phi^2(t)G_0^2(t)e^{2\phi(t)r(t)}\Big] < \infty;$$
$$\mathbb{E}\Big[\sup_{t\in[0,T]} \int_{\mathbb{R}} |K_0^2(t,\zeta)|^2\nu(d\zeta)\Big] = \mathbb{E}\Big[\sup_{t\in[0,T]} \int_{\mathbb{R}} G_0^2(t)e^{2\phi(t)r(t)}\Big(e^{\gamma\zeta} - 1\Big)^2\nu(d\zeta)\Big] < \infty.$$

Then, by Theorem 2, $(\pi^*(t), u_0^*(t))$ describe the equilibrium strategy.

Finally, from (30) and (31), the equilibrium value function and the corresponding efficient frontier become:

$$\begin{aligned}
\Phi(t, Y^*(t)) = {}& \frac{1}{2}\mathbb{E}_t\Big\{\int_t^T \Big[(\beta\phi(s)G_1(s)Y^*(s) + \sigma\pi^*(s)G_0(s))^2 \\
&+ \int_{\mathbb{R}} (\gamma_S\pi^*(s)G_1(s) + Y^*(s)G_1(s)(e^{\gamma\zeta} - 1))^2\nu(d\zeta) \\
&+ \int_0^\infty \zeta^2(u_0^*(s)G_1(s))^2\nu^0(d\zeta)\Big]e^{2\phi(s)r(s)}ds\Big\} \\
&-\lambda(Y^*(t)G_1(t)e^{\phi(t)r(t)} - \lambda G_2(t)), \\
Var_t[Y^*(T)] = {}& \mathbb{E}_t\Big\{\int_t^T \Big[(\beta\phi(s)G_1(s)Y^*(s) + \sigma\pi^*(s)G_0(s))^2 \\
&+ \int_{\mathbb{R}} (\gamma_S\pi^*(s)G_1(s) + Y^*(s)G_1(s)(e^{\gamma\zeta} - 1))^2\nu(d\zeta) \\
&+ \int_0^\infty \zeta^2(u_0^*(s)G_1(s))^2\nu^0(d\zeta)\Big]e^{2\phi(s)r(s)}ds\Big\}.
\end{aligned}$$

Remark 2. *If the interest rate is a constant $r > 0$, then $\alpha = a = \beta = \gamma = 0$. It is easy to check that $\phi(t) = T-t$, $G_0(t) = G_1(t) = 1$, $G_3(t) = -1$ and $G_2(t) = -(T-t)$.*

The equilibrium strategy, the value function and the efficient frontier become:

$$\begin{aligned}
\pi^*(t) &= \frac{\lambda\theta e^{-(T-t)r}}{\sigma^2+\int_{\mathbb{R}}\gamma_S^2\nu(d\zeta)},\\
u_0^*(t) &= \frac{\lambda\Big(up+\int_0^\infty\zeta\nu^0(d\zeta)\Big)}{\int_0^\infty\zeta^2\nu^0(d\zeta)}e^{-(T-t)r},\\
\Phi(t,Y^*(t)) &= \frac{1}{2}\int_t^T\Big[\sigma^2(\pi^*(s))^2+\int_{\mathbb{R}}\gamma_S^2(\pi^*(s))^2\nu(d\zeta)\\
&\quad+\int_0^\infty\zeta^2(u_0^*(s))^2\nu^0(d\zeta)\Big]e^{2(T-s)r}ds\\
&\quad-\lambda(Y^*(t)e^{(T-t)r}+\lambda(T-t))\\
Var_t[Y^*(t)] &= \int_t^T\Big[\sigma^2(\pi^*(s))^2+\int_{\mathbb{R}}\gamma_S^2(\pi^*(s))^2\nu(d\zeta)\\
&\quad+\int_0^\infty\zeta^2(u_0^*(s))^2\nu^0(d\zeta)\Big]e^{2(T-s)r}ds.
\end{aligned}$$

We observe that when the interest rate is constant, the equilibrium strategy is independent of the surplus process. Moreover, the optimal investment strategy is an increasing function of time and expected return θ. This implies that as the time approaches maturity, the insurer will keep more investments in the risky asset. Moreover, it is decreasing with respect to volatility σ and jump rate γ_S. In this case, when σ and γ_S increased, the insurer would reduce his investment in the risky asset. The findings are consistent with the results existing in the literature, see e.g., [18, 19, 21].

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