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Construction of new chaotic dynamical systems on a 2D Cantor set

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Abstract. In this paper, we construct new discrete chaotic dynamical systems on a 2D Cantor set by using the shift map, a 0-preadded map and a 2-preadded map. We also obtain a collection of chaotic dynamical systems using the elements of the symmetry group of the square.

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1. Introduction

The notion of dynamical system has several applications in some important fields of mathematics, physics, chemistry and biology (for more details, see [1, 6, 10]). A dynamical system is called a chaotic dynamical system if it satisfies the following conditions: "sensitive dependence on initial conditions", "topological transitivity" and "density of periodic points" (for details, see [6]).

In this paper, we define a chaotic dynamical system on one of the classical self-similar set two-dimensional (2D) Cantor set $C \times C$ by inspiring the shift map on the classical Cantor set C. We also construct new chaotic dynamical systems with the help of the elements of the Dihedral group D_4 (symmetries of the square in the plane). We express these dynamical systems by using code representation of the points of the 2D Cantor set.

The shift map is one of the classical examples of chaotic dynamical systems expressed by code representations of the points. There exist several examples of chaotic dynamical systems defined with the help of this famous map on classical self-similar sets such as the Tent map and Smale's horseshoe map on the Cantor set ([3]). In [2] and [11], the authors define chaotic dynamical systems on the self-similar sets the Sierpinski Triangle and the Box fractal.

For the purpose of this manuscript, we first give a short overview of the notion of iterated function systems.

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1.1. Iterated function systems

Let (X, d) be a complete metric space and $\{w_i : X \to X \mid i = 1, 2, \ldots, n\}$ a finite family of contractions with contractivity ratios $0 < r_i < 1$. The system $\{X; w_i, i = 1, 2, \ldots, n\}$ is called an iterated function system (IFS) (see [5]). In [8], the author proves that there exists a unique nonempty compact set $A \subset X$ such that

$$A = \bigcup_{i=1}^{n} w_i(A),$$

which is called the attractor of the IFS.

The classical Cantor set C is the attractor of the iterated function system $\{\mathbb{R}; \varphi_0, \varphi_2\}$, where

$$\varphi_0(x) = \frac{x}{3} \quad \text{and} \quad \varphi_2(x) = \frac{x}{3} + \frac{2}{3} \tag{1}$$

are the similarities with similarity ratio $\frac{1}{3}$.

1.2. Code representations of the points of $C \times C$

Let $\omega = \omega_1\omega_2\cdots\omega_k$ be a word with length k, where $\omega_i \in \{0, 2\}$ for $i = 1, 2, \ldots, k$. Let $C_\omega := \varphi_\omega(C) = \varphi_{\omega_1\omega_2\cdots\omega_k}(C)$ where $\varphi_{\omega_1\omega_2\cdots\omega_k} = \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_k}$ and φ_0, φ_2 are the functions defined in (1). We call C_ω as a Cantor set of level k (C is the unique Cantor set of level 0). Note that there exist 2^k Cantor sets of level k for a nonnegative integer k. By the Cantor completeness criteria, for any word (with infinite length) $\omega_1\omega_2\cdots\omega_k\cdots$ ($\omega_i \in \{0, 2\}$), the intersection of the nested sets

$$\varphi_{\omega_1}(C) \supset \varphi_{\omega_1\omega_2}(C) \supset \cdots \supset \varphi_{\omega_1\omega_2\cdots\omega_k}(C) \supset \cdots$$

is a singleton $\bigcap\limits_{k=1}^{\infty} \varphi_{\omega_1\omega_2\cdots\omega_k}(C)$ which contains a unique point, say a. Then we call $\omega_1\omega_2\omega_3\cdots$ as a code representation of the point $a \in C$ (for more details, see [9]) and we write $a = \omega_1\omega_2\omega_3\cdots$ throughout the paper. We note that every point of C has a unique code representation since the Cantor set is totally disconnected (for more details, see [5]).

We can represent the points of $C \times C$ by using the code representations of the points of C as follows: Let $(a, b) \in C \times C$, $\alpha = \alpha_1\alpha_2\cdots\alpha_k\cdots$ and $\beta = \beta_1\beta_2\cdots\beta_k\cdots$ be the code representations of the points a and b, respectively. Then we set a certain pair (α, β) as the code representation of the point $(a, b) \in C \times C$. We also set $C_{\omega,\theta} := C_\omega \times C_\theta = \varphi_\omega(C) \times \varphi_\theta(C)$ as a 2D Cantor set of level k for the words ω, θ with length k.

Let $(a, b), (c, d) \in C \times C$ and $(\alpha_1\alpha_2\cdots, \beta_1\beta_2\cdots), (\gamma_1\gamma_2\cdots, \delta_1\delta_2\cdots)$ be the code representations of these points, respectively. In this paper, we use the following well-known metric d on $C \times C$:

$$d((a, b), (c, d)) = \sqrt{\left(\sum_{i=1}^{\infty} \frac{\alpha_i - \gamma_i}{3^i}\right)^2 + \left(\sum_{i=1}^{\infty} \frac{\beta_i - \delta_i}{3^i}\right)^2} \tag{2}$$

Remark 1. *Let $\alpha = \alpha_1\alpha_2\cdots\alpha_k\cdots$ and $\beta = \beta_1\beta_2\cdots\beta_k\cdots$ be the code representations of the points $a, b \in C$ respectively. For the sake of clarity, throughout the paper we write $a = \alpha$, $b = \beta$ and $(a, b) = (\alpha, \beta) \in C \times C$.*

2. A new chaotic dynamical system on $C \times C$

In this subsection, we first construct a chaotic dynamical system using the shift map and the mappings called a $0-$preadded map and a $2-$preadded map.

We first remind of the well-known shift map on the Cantor set. Let $a = \alpha_1\alpha_2\alpha_3\cdots \in C$, the mapping $\sigma : C \to C$,

$$\sigma(\alpha_1\alpha_2\alpha_3\cdots) = \alpha_2\alpha_3\alpha_4\cdots$$

is called the shift map. Geometrically, σ maps a Cantor set of level k to a Cantor set of level $(k-1)$, that is, $\sigma(C_{\alpha_1\alpha_2\cdots\alpha_k}) = C_{\alpha_2\cdots\alpha_k}$. It is a similarity map (an extension) with similarity ratio 3.

We now define $0-$preadded and $2-$preadded maps which will be used for the construction of new dynamical systems. Let $a = \alpha_1\alpha_2\alpha_3\cdots \in C$. Then the mappings $\tau_0, \tau_2 : C \to C$ defined as

$$\tau_0(\alpha_1\alpha_2\alpha_3\cdots) := 0\alpha_1\alpha_2\alpha_3\cdots$$
$$\tau_2(\alpha_1\alpha_2\alpha_3\cdots) := 2\alpha_1\alpha_2\alpha_3\cdots$$

are called a $0-$preadded map and a $2-$preadded map respectively. In contrast to the shift map, the $0-$preadded map or the $2-$preadded map, respectively, add 0 or 2 to the beginning of the code of the point. In fact, the image of a Cantor set of level k under τ_0 and τ_2 is a Cantor set of level $(k+1)$. In general, $\tau_0(C_{\alpha_1\cdots\alpha_k}) = C_{0\alpha_1\cdots\alpha_k}$ and $\tau_2(C_{\alpha_1\cdots\alpha_k}) = C_{2\alpha_1\cdots\alpha_k}$. Unlike the shift map, τ_0 and τ_2 are contractions (also similitudes) with similarity ratio $1/3$.

While $\{C; \sigma\}$ is one of the most well-known chaotic dynamical systems in the sense of Devaney ([6]), $\{C; \tau_0\}$ and $\{C; \tau_2\}$ are not chaotic dynamical systems in the sense of Devaney since sensitive dependence on initial conditions does not hold.

However, one can easily show that the dynamical system $\{C\times C; f\}$ is also chaotic where $f(x, y) = (\sigma(x), \sigma(y))$ (for an image example of the extension f on $C \times C$, see Figure 1).

Now consider the dynamical system $\{C\times C; f = (f_1, f_2)\}$, where $f_1, f_2 \in \{\sigma, \tau_0, \tau_2\}$. The only case where the dynamical system $\{C\times C; f\}$ is chaotic is where $f_1 = f_2 = \sigma$. It can easily be seen that using τ_0 or τ_2 in one component of f (i.e. $f_i = \tau_j$ for some $i \in \{1, 2\}$ and some $j \in \{0, 2\}$), $\{C \times C; f\}$ does not yield a chaotic dynamical system (for a better understanding of the function f see e.g. Figure 2).

But surprisingly, if we define the dynamical system as in (3), we obtain a new chaotic dynamical system on $\{C \times C\}$ as stated in Theorem 1. By defining Φ in (3), we add the term deleted from the first component with σ (indeed, the first term of the code of a point, 0 or 2) to the second component with the corresponding preadded map (τ_0 or τ_2).

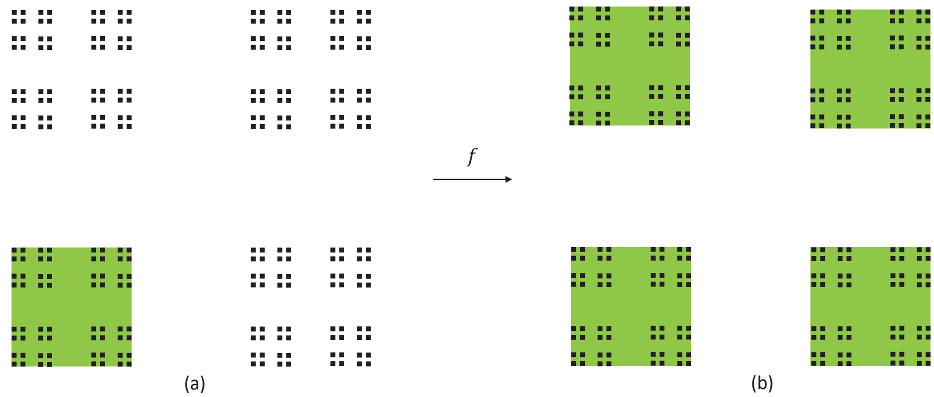

Figure 1: *(a) $C_{0,0}$ (b) $f(C_{0,0})$, where $f(x,y) = (\sigma(x), \sigma(y))$*

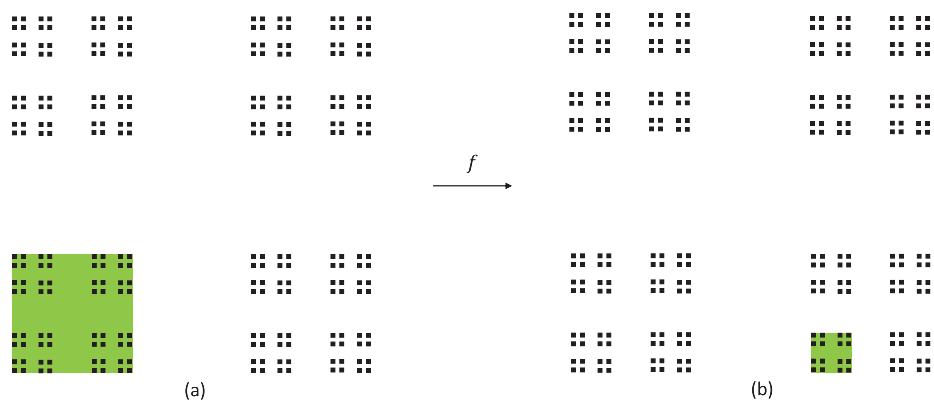

Figure 2: *(a) $C_{0,0}$ (b) $f(C_{0,0})$, where $f(x,y) = (\tau_2(x), \tau_0(y))$ (f contracts the set $C_{0,0}$ horizontally and vertically with ratio $1/3$)*

Theorem 1. *Consider the map $\Phi : C \times C \longrightarrow C \times C$,*

$$\Phi(a,b) = (\sigma(a), \tau_\alpha(b)), \tag{3}$$

where α is the first term of the code representation of the point $a \in C$. Then $\{C \times C; \Phi\}$ is a chaotic dynamical system in the sense of Devaney.

Remark 2. *In the book [6], the author defines a chaotic dynamical system so-called two-sided shift map on the two-dimensional Cantor set. One can easily prove that this dynamical system (the two-sided shift map) is homeomorphic to Φ given in Theorem 1, which says Φ is also chaotic.*

Although Φ is homeomorphic to the two-sided shift map in [6], Φ is expressed more clearly by using directly the codes of the points of $C \times C$, which gives us some advantages (computation of periodic points, etc.) to understand the chaotic behaviour of the dynamical system Φ on the two-dimensional Cantor set.

Although we know that the dynamical system Φ is chaotic since the two-sided shift map is, we give the proof of Theorem 1 by using our notations for a better *understanding of the proofs of the theorems in the next section.*

Proof. We first note that the expression of the function Φ in terms of the code of the point $(a,b) = (\alpha_1\alpha_2\alpha_3\cdots, \beta_1\beta_2\beta_3\cdots) \in C \times C$ is

$$\Phi(a,b) = (\alpha_2\alpha_3\cdots, \alpha_1\beta_1\beta_2\beta_3\cdots).$$

Since Φ is a continuous map, according to [4] it is enough to show that the conditions of topological transitivity and density of periodic points are satisfied. On the other hand, here the conditions topological transitivity and existence of a dense orbit are equivalent (due to the Proposition 1 in [7]), thus we show that Φ has a dense orbit instead of topological transitivity.

We first prove that the set of periodic points is dense in $C \times C$. By a simple calculation we get

$$\Phi^n(a,b) = (\alpha_{n+1}\alpha_{n+2}\alpha_{n+3}\cdots, \alpha_n\alpha_{n-1}\cdots\alpha_2\alpha_1\beta_1\beta_2\beta_3\cdots), \tag{4}$$

where $\Phi^n = \underbrace{\Phi \circ \Phi \circ \cdots \circ \Phi}_{n\ times}$ for $n \geq 1$. Thus clearly, the $2n$-periodic points of Φ can be expressed as

$$(\overline{\alpha_1\alpha_2\cdots\alpha_n\beta_n\beta_{n-1}\cdots\beta_2\beta_1}, \overline{\beta_1\beta_2\cdots\beta_n\alpha_n\alpha_{n-1}\cdots\alpha_2\alpha_1}).$$

To show that the periodic points are dense, we must find a periodic point close enough to each point of $C \times C$. Let $(p,q) = (p_1p_2\cdots, q_1q_2\cdots) \in C \times C$. Choose $r \in \mathbb{N}$ such that $\frac{\sqrt{2}}{3^r} < \varepsilon$ for a given $\varepsilon > 0$. It is easy to verify that the point

$$(x,y) = (\overline{p_1p_2\cdots p_rq_rq_{r-1}\cdots q_2q_1}, \overline{q_1q_2\cdots q_rp_rp_{r-1}\cdots p_2p_1})$$

is a $2r$-periodic point. Furthermore, the point (x,y) is close enough to the given (p,q) since

$$d((x,y),(p,q)) \leq \sqrt{\left(\frac{2}{3^{r+1}} + \frac{2}{3^{r+2}} + \cdots\right)^2 + \left(\frac{2}{3^{r+1}} + \frac{2}{3^{r+2}} + \cdots\right)^2} = \frac{\sqrt{2}}{3^r} < \varepsilon,$$

which says that the periodic points of Φ are dense in $C \times C$.

Let B_n (n block) be all combinations of length n consisting of 0 and 2 for a positive even integer n. For example, B_2 and B_4 are as follows:

$B_2 = 00\mathbf{02}20\mathbf{22}$
$B_4 = 0000\mathbf{0002}0020\mathbf{0022}0200\mathbf{0202}0220\mathbf{0222}2000\mathbf{2002}2020\mathbf{2022}2200\mathbf{2202}2220\mathbf{2222}$.

Let $x := B_2B_4B_6\cdots$. Let $(p,q) = (p_1p_2p_3\cdots, q_1q_2q_3\cdots)$ be an arbitrary point in $C \times C$ and $\varepsilon > 0$. There exists a block

$$q_kq_{k-1}\cdots q_2q_1p_1p_2\cdots p_k$$

in x, where q_1 is the n^{th} term of x. From equation (4)

$$\Phi^n(x,y) = (p_1p_2\cdots p_k\cdots, q_1q_2\cdots q_k\cdots)$$

for any $y \in C$ and thus we obtain $d(\Phi^n(x,y),(p,q)) \leq \frac{\sqrt{2}}{3^k} < \varepsilon$, which says that the point $(x,y) \in C \times C$ has a dense orbit. $\square$

3. The construction of a family of chaotic dynamical systems on $C \times C$ via the symmetries of the unit square

It is possible to obtain new dynamical systems by combining the function Φ defined in (3) and the elements of the Dihedral group D_4. Indeed, we use the restriction of the functions of the well-known Dihedral group $D_4 = \{\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \delta_1, \delta_2\}$ to the 2D Cantor set to define a new dynamical system. In Figure 3, we give a pictorial description of the maps $\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \delta_1$ and δ_2 on the unit square $[0,1] \times [0,1]$.

Let $(a,b) \in C \times C$. The symmetries in the group D_4 restricted to $C \times C$ can be expressed explicitly as

$$\begin{array}{llll} \rho_0(a,b) = (a,b) & \rho_1(a,b) = (\tilde{b},a) & \rho_2(a,b) = (\tilde{a},\tilde{b}) & \rho_3(a,b) = (b,\tilde{a}), \\ \mu_1(a,b) = (\tilde{a},b) & \mu_2(a,b) = (a,\tilde{b}) & \delta_1(a,b) = (\tilde{b},\tilde{a}) & \delta_2(a,b) = (b,a), \end{array}$$

which can be easily verified (see also Figure 4(b) for verification).

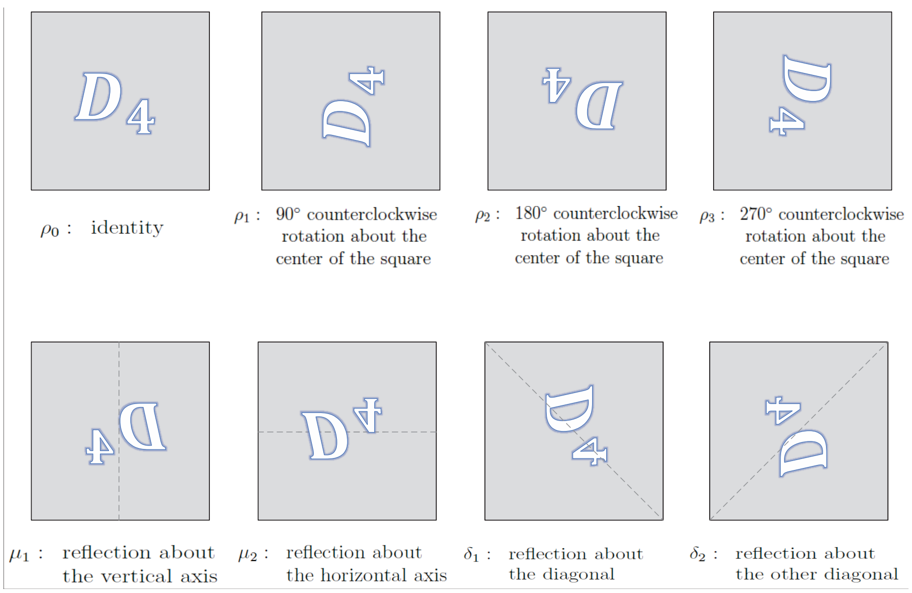

Figure 3: *Pictorial description of the the symmetries of the square*

For a point $a = \alpha_1\alpha_2\alpha_3 \cdots \in C$, we define the point $\tilde{a} := \tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3 \cdots$, where $\tilde{x} = 2 - x$ for $x \in \{0,2\}$, i.e. $\tilde{x} = 2$ if $x = 0$ and $\tilde{x} = 0$ if $x = 2$. By definition, it is obvious that the point $\tilde{a}$ is the reflection of the point a about the point $x = 1/2$ (see Figure 4(a)). Similarly, let $(a,b) \in C \times C$. Then the points $(\tilde{a}, b)$ and $(a, \tilde{b})$ are the reflections of the point (a,b) about the lines $x = 1/2$ and $y = 1/2$, respectively, and $(\tilde{a}, \tilde{b})$ is the point obtained by rotating the point (a,b), 180° (counterclockwise) about the origin O of the unit square (see Figure 4(b)).

Now, consider the dynamical system Φ_η on $C \times C$ depending on $\eta \in D_4$ as

$$\Phi_\eta(a,b) = (\Phi \circ \eta)(a,b), \tag{5}$$

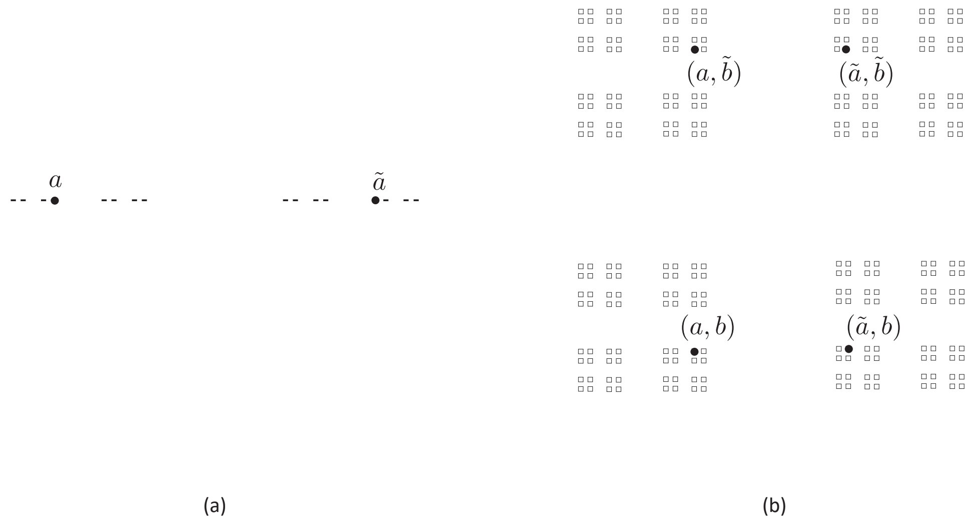

Figure 4: *(a) A point in C and its symmetry about the point $\frac{1}{2}$, (b) A point in $C \times C$ and its image under the symmetries $\rho_0, \mu_2, \rho_2, \mu_1$.*

where Φ is the function defined in (3).

Theorem 2. *Let Φ_η be the function given in (5).*

i. If $\eta \in \{\rho_0, \rho_2, \mu_1, \mu_2\}$, then $\{C \times C; \Phi_\eta\}$ is a chaotic dynamical system,

ii. If $\eta \in \{\rho_1, \rho_3, \delta_1, \delta_2\}$, then $\{C \times C; \Phi_\eta\}$ is not a chaotic dynamical system

in the sense of Devaney.

Proof. **i)** Since Φ_η is a continuous map, it is enough to show that the density of periodic points and the existence of a dense orbit are satisfied. First, consider the case $\eta = \mu_1$. We can express the function Φ_{μ_1} as

$$\Phi_{\mu_1}(a, b) = (\tilde{\alpha}_2\tilde{\alpha}_3\tilde{\alpha}_4 \cdots, \tilde{\alpha}_1\beta_1\beta_2\beta_3 \cdots),$$

where $(a, b) = (\alpha_1\alpha_2\alpha_3 \cdots, \beta_1\beta_2\beta_3 \cdots) \in C \times C$.
Density of periodic points: To show that the set of periodic points is dense in $C \times C$, we need to find a periodic point close enough to the arbitrary $(a, b) = (\alpha_1\alpha_2\alpha_3 \cdots, \beta_1\beta_2\beta_3 \cdots) \in C \times C$. We obtain

$$\Phi^n_{\mu_1}(a, b) = (\alpha_{n+1}\alpha_{n+2}\alpha_{n+3} \cdots, \alpha_n\tilde{\alpha}_{n-1}\alpha_{n-2} \cdots \alpha_2\tilde{\alpha}_1\beta_1\beta_2\beta_3 \cdots), \tag{6}$$

for all positive even integers n. In this case, the $2n-$periodic points are in the following form:

$$(\overline{\alpha_1\alpha_2 \cdots \alpha_n\tilde{\beta}_n\beta_{n-1} \cdots \tilde{\beta}_2\beta_1}, \overline{\beta_1\beta_2 \cdots \beta_n\alpha_n\tilde{\alpha}_{n-1} \cdots \alpha_2\tilde{\alpha}_1}). \tag{7}$$

In order to prove that the periodic points are dense in $C \times C$, we must find a periodic point close enough to each arbitrary point in $C \times C$. Let $(p, q) = (p_1p_2 \cdots, q_1q_2 \cdots) \in$

$C \times C$ and $\varepsilon > 0$ is given. Choose $r \in \mathbb{N}$ such that $\frac{\sqrt{2}}{3^r} < \varepsilon$. If we choose

$$x = \overline{p_1 p_2 \cdots p_r \tilde{q}_r q_{r-1} \cdots \tilde{q}_2 q_1} \text{ and } y = \overline{q_1 q_2 \cdots q_r p_r \tilde{p}_{r-1} \cdots p_2 \tilde{p}_1},$$

then the point (x, y) is the $2r-$periodic point of the map. Since the first r terms of x and p are the same, just like y and q, it follows that

$$d((x, y), (p, q)) \leq \frac{\sqrt{2}}{3^r} < \varepsilon.$$

Thus the periodic points of Φ_{μ_1} are dense in $C \times C$.
Existence of a dense orbit: Let B_n (an n block) be all combinations of length n consisting of 0 and 2 for a positive even integer n and let $x := B_2 B_4 B_6 \cdots$. Let $(p, q) = (p_1 p_2 p_3 \cdots, q_1 q_2 q_3 \cdots) \in C \times C$ and $\varepsilon > 0$. There exists a block

$$\tilde{q}_k q_{k-1} \cdots \tilde{q}_2 q_1 p_1 p_2 \cdots p_k$$

in x, where q_1 is the n^{th} term of x and k is an odd integer. From equation (6)

$$\Phi^n_{\mu_1}(x, y) = (p_1 p_2 \cdots p_k \cdots, q_1 q_2 \cdots q_k \cdots)$$

for any $y \in C$, so then

$$d(\Phi^n_{\mu_1}(x, y), (p, q)) \leq \frac{\sqrt{2}}{3^k} < \varepsilon.$$

Thus the point $(x, y) \in C \times C$ has a dense orbit and consequently $\{C \times C; \Phi_{\mu_1}\}$ is a chaotic dynamical system.

Now consider the case $\eta = \mu_2$. Some $2n$-periodic points of Φ_{μ_2} are in the form

$$(\overline{\alpha_1 \alpha_2 \cdots \alpha_n \tilde{\beta}_n \beta_{n-1} \cdots \tilde{\beta}_2 \beta_1}, \overline{\beta_1 \beta_2 \cdots \beta_n \alpha_n \tilde{\alpha}_{n-1} \cdots \alpha_2 \tilde{\alpha}_1})$$

for all positive even integers n. Similarly, we can obtain the $2n$-periodic points of Φ_{ρ_0} and $2m$-periodic points of Φ_{ρ_2} as

$$(\overline{\alpha_1 \alpha_2 \cdots \alpha_n \beta_n \beta_{n-1} \cdots \beta_1}, \overline{\beta_1 \beta_2 \cdots \beta_n \alpha_n \alpha_{n-1} \cdots \alpha_1})$$

and

$$(\overline{\alpha_1 \alpha_2 \cdots \alpha_m \beta_m \beta_{m-1} \cdots \beta_1}, \overline{\beta_1 \beta_2 \cdots \beta_m \alpha_m \alpha_{m-1} \cdots \alpha_1})$$

for all positive integers n and for all positive even integers m, respectively. Using these facts, the statement can be proven similarly as done $\eta = \mu_1$.
ii) First, we consider the case $\eta = \rho_1$. In this case, the function Φ_{ρ_1} can be expressed as

$$\Phi_{\rho_1}(a, b) = (\tilde{\beta}_2 \tilde{\beta}_3 \tilde{\beta}_4 \cdots, \tilde{\beta}_1 \alpha_1 \alpha_2 \alpha_3 \cdots).$$

$\{C \times C; \Phi_{\rho_1}\}$ is not a chaotic dynamical system in the Devaney sense:
It is enough to show that the condition of topological transitivity does not hold. For that purpose, we choose open sets as $U \subset C_{00,00}$ and $V \subset C_{02,02}$. Let $(a, b) \in C_{00,00}$. The code representation of the point is $(00\alpha_3\alpha_4 \cdots, 00\beta_3\beta_4 \cdots)$. When the orbit of the point is examined, then for all $n \geq 1$ we obtain

$$\Phi^n_{\rho_1}(a, b) = \Phi^{n+4}_{\rho_1}(a, b). \tag{8}$$

In fact, (8) is satisfied for all points in $C\times C$. On the other hand, according to (8), one can easily verify that $\Phi^n_{\rho_1}(a,b)\notin V$, for all $n\in\mathbb{N}$. Obviously, $\Phi^n_{\rho_1}(C_{00,00})\cap V=\varnothing$, for all $n\in\mathbb{N}$, and since $U\subset C_{00,00}$, we conclude that

$$\Phi^n_{\rho_1}(U)\cap V=\varnothing,$$

which means that Φ_{ρ_1} does not satisfy the condition of topological transitivity. Therefore, $\{C\times C;\Phi_{\rho_1}\}$ is not chaotic in the sense of Devaney.

For the case $\eta=\rho_3$ one can verify that $\Phi^n_{\rho_3}(a,b)=\Phi^{n+4}_{\rho_3}(a,b)$ for all $(a,b)\in C\times C$ and $n\geq 1$, i.e. each point in $C\times C$ is a periodic point with period 4 of ρ_3, as in the case $\eta=\rho_1$. Similarly, for the cases $\eta=\delta_1$ and $\eta=\delta_2$ we obtain $\Phi^n_{\eta}(a,b)=\Phi^{n+2}_{\eta}(a,b)$ for all $(a,b)\in C\times C$ and $n\geq 1$, i.e. each point in $C\times C$ is a periodic point with period 2 of δ_1 and δ_2. Thus, similarly to the case $\eta=\rho_1$, one can show that the related dynamical system is not chaotic in these cases since the topological transitivity condition does not hold.

□

In (5), to define Φ_η, we choose τ_0 or τ_2 according to the code representation of the first component of the point $\eta(a,b)$. We can express the function Φ_η in a different form as

$$\Phi_\eta(a,b)=(\sigma(\eta_x),\tau_\alpha(\eta_y)),$$

where $(\eta_x,\eta_y)=\eta(a,b)$ and α is the first term of the code representation of the point $\eta_x\in C$. More clearly, in Φ_η, after applying η, σ is applied to the first component and τ_0 or τ_2 is applied to the second component depending on η_x which is the first component of $\eta(a,b)$. If we choose τ_0 or τ_2 according to $a\in C$ not η_x, we can obtain different dynamical systems on $C\times C$. Although different dynamical systems have been obtained with this choice, the chaoticity characters of these new dynamical systems are the same for each η, as stated in Theorem 3.

Theorem 3. *Let $\eta\in D_4$ and $\Psi_\eta(a,b)=(\sigma(\eta_x),\tau_\alpha(\eta_y))$ be the function on $C\times C$, where $(\eta_x,\eta_y)=\eta(a,b)$ and α is the first term of the code representation of the point $a\in C$. Then*

i. If $\eta\in\{\rho_0,\rho_2,\mu_1,\mu_2\}$, then $\{C\times C;\Psi_\eta\}$ is a chaotic dynamical system,

ii. If $\eta\in\{\rho_1,\rho_3,\delta_1,\delta_2\}$, then $\{C\times C;\Psi_\eta\}$ is not a chaotic dynamical system

in the sense of Devaney.

Proof. We left the easy proof to the readers which can be done in a similar way to the proof of Theorem 2. □

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