Smoothness of time series: a new approach to estimation

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Abstract

The assessment of the risk of occurrence of extreme phenomena is inherently linked to the theory of extreme values. In the context of a time series, the analysis of its trajectory towards a greater or lesser smoothness, i.e., presenting a lesser or greater propensity for oscillations, respectively, constitutes another contribution in the assessment of the risk associated with extreme observations. For example, a financial market index with successive oscillations between high and low values shows investors a more unstable and uncertain behavior. This is the idea presented by Ferreira and Ferreira (2021) in the implementation of a smoothness coefficient to ascertain the smoothness of the trajectory of a time series at large values. In stationary time series, the upper tail smoothness coefficient is described by the tail dependence coefficient, a well-known concept first introduced by Sibuya (1960). This work focuses on an inferential analysis of the upper tail smoothness coefficient, based on subsampling techniques for time series. In particular we propose an estimator with reduced bias. We also analyze the estimation of confidence intervals through a block bootstrap methodology and a test procedure to prior detect the presence or absence of smoothness. An application to real data is also presented.

Keywords: extreme value theory; stationary sequences; Jackknife; block bootstrap; tail (in)dependence

1 Introduction

Extreme value theory (EVT) allows to evaluate a stochastic process at the tails (lower or upper). Several risk measures are study within the scope of EVT, in particular, if we are dealing with phenomena with heavier/lighter tails than the usual Gaussian model. The primary result in EVT states the possible limiting laws of the maximum (minimum) of an independent and identically distributed (i.i.d.) sequence of random variables (r.v.) $\{X_i^*\}_{i\geq 1}$ with marginal distribution function (d.f.) F. More precisely, under a convenient linear normalization based on real constants $a_n > 0$ and b_n, if

$$\lim_{n\to\infty} P(\max(X_1^*, ..., X_n^*) \leq a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x)\,,$$

for all continuity points of a non-degenerate d.f.

$$G(x) = \exp\left\{-\left(1+\xi\,\frac{x-\mu_G}{\sigma_G}\right)^{-1/\xi}\right\}\,,\ 1+\xi\,\frac{x-\mu_G}{\sigma_G} > 0\,,$$

where $G(x) = \exp\{-\exp(-(x-\mu_G)/\sigma_G)\}$ if $\xi = 0$, we say that F belongs to the maximum domain of attraction of G. This is usually called Generalized Extreme Value (GEV) d.f., where μ_G and σ_G are, respectively, the location and the scale parameters, and ξ is the shape parameter called tail index. Parameter ξ is of prime importance since it states the type of tail and the respective limit distribution: (reversed) Weibull if $\xi < 0$ with light tails and finite right end-point, Gumbel if $\xi = 0$ characterized by an exponential-type tail and Fréchet if $\xi > 0$ corresponding to a heavy tail with infinite right end-point.

A common assumption within statistics of extreme values is that F belongs to the max-domain of attraction of some GEV. The three domains are quite characterized in literature and reference books are for example, de Haan and Ferreira ([6], 2006) and Beirlant *et al.* ([1], 2004). For instance, Pareto-type models are heavy-tailed and thus belong to the Fréchet max-domain of attraction with positive tail index ξ, They are characterized by a regularly varying tail function at infinity with index $-1/\xi$, i.e., $1-F(x) = x^{-1/\xi}L_F(x)$, where $L_F(x)$ is such that $L(tx)/L(t) \to 1$, as $t \to \infty$, called a slowly varying function at infinity. On the other hand, we also have that $1-F(x^{-1})$ is regularly varying with index $1/\xi$ and $L_F(x^{-1})$ is slowly varying, both at zero.

Now if we move to the stationary context by dropping the independence requisite, we still have a similar result, but another parameter arrives to measure time-dependence. Indeed, consider $\{X_i\}_{i\geq 1}$ a stationary sequence of r.v. with the same marginal d.f. F. Under dependence condition $D(u_n)$ (Leadbetter, [19] 1974), that basically limits the long-range dependence at large values, with $u_n = a_n x + b_n$ for each x such that $G(x) > 0$ and if $P(\max(X_1, ..., X_n) \leq a_n x + b_n)$ converges for some x, then it converges to $H(x) \equiv G^\theta(x)$. Parameter θ is the so-called extremal index. The location and scale parameters of GEV H are affected by θ, but the shape parameter ξ remains equal, that is, H and G have the same type of tail. Clustering of extreme values is a typical phenomenon of risk evaluated through the extremal index related to the mean time permanency of values above a high threshold. However, it lacks information about the tendency for more or less occurrence of oscillations around a high threshold.

Ferreira and Ferreira ([11], 2021) introduced an upper tail smoothness coefficient, which allows distinguishing between more oscillating and smoother trajectories of time series. Smoother time series correspond to more concordant successive r.v., based on the concept of Joe ([18], 1997), i.e., considering r.v. $Y_1, ..., Y_d$ and $W_1, ..., W_d$ where margins Y_i and W_i have the same d.f. F_i, $i = 1, ..., d$, we say that $Y_1, ..., Y_d$ are more concordant than $W_1, ..., W_d$ if $P(Y_1 \leq x_1, ..., Y_d \leq x_d) \geq P(W_1 \leq x_1, ..., W_d \leq x_d)$ and $P(Y_1 > x_1, ..., Y_d > x_d) \geq P(W_1 > x_1, ..., W_d > x_d)$, for all $x_1, ..., x_d \in (-\infty, \infty)$. Thus $Y_1, ..., Y_d$ is more likely to take jointly small values and large values.

Figure 1 illustrates time series with a more oscillating trajectory (left) and with a smoother sample path (right). Thus, on the left plot we see various observations that go up (go down) followed in the next time instant by an observation that goes down (go up) and so on, whilst on the right plot, we observe several successive observations quite close in magnitude.

In evaluating extremal dependence between r.v. it is usual to take a convenient marginal transformation, like $F_j(X_j)$, where F_j is the d.f. of X_j, and thus $F_j(X_j)$ has standard uniform distribution provided F_j is continuous. Considering oscillations around time-instant i relative to a high threshold u,

$$O_{i,j} = \{F_i(X_i) \leq u < F_j(X_j)\},\ j = i-1, i+1$$

and exceedances of u around instant i,

$$E_i = \{F_j(X_j) > u\},\ j = i-1, i+1$$

the upper tail smoothness of $\{X_i\}_{i\geq 1}$ is evaluated by comparing the number of oscillations with the number of exceedances of threshold u. More precisely,

$$S_{n,m} = 1 - \lim_{u\uparrow 1} \frac{E\left(\sum_{i=n}^{m} \sum_{j\in\{i-1,i+1\}} \mathbf{1}_{O_{i,j}} | \sum_{i=n}^{m} \mathbf{1}_{\{F_i(X_i)>u\}} > 0\right)}{2E\left(\sum_{i=n}^{m} \mathbf{1}_{E_i} | \sum_{i=n}^{m} \mathbf{1}_{\{F_i(X_i)>u\}} > 0\right)}$$

$S_{n,m}$ is the upper tail smoothness coefficient and provides the proportion of exceedances that are oscillations, around each instant $i \in [n, m]$, given that there is at least one exceedance. In more oscillating trajectories, existing, at least, one exceedance of u, $\{F_j(X_j) > u\}$ for some j between instants n and m ($n, m \in \mathbb{N}$), it is expected that the total number of oscillations will be closer to the total number of exceedances. Coefficient $S_{n,m}$ ranges in $[0, 1]$, where bounds 0 and 1 are achieved for, respectively, independent and positive quadrant totally dependent r.v.. Moreover, if $Y_1, ..., Y_d$ are more concordant than r.v. $W_1, ..., W_d$ then $S_{n,m}^{(Y)} \geq S_{n,m}^{(W)}$. Another interesting result is to express $S_{n,m}$ as a function of the tail dependence coefficients (TDC),

$$\lambda(j|i) = \lim_{u\uparrow 1} P(F_j(X_j) > u | F_i(X_i) > u).$$

We have that

$$S_{n,m} = \sum_{i=n}^{m} \frac{\lambda(i+1|i) + \lambda(i|i-1)}{2(m-n+1)},$$

provided $\lambda(j|i)$ exists for all $n \leq i \leq m$ and $j = i-1, i+1$. The TDC is a well-known bivariate tail dependence measure of literature, first introduced in Sibuya ([27], 1960), with many applications in different areas. See, e.g., Ferreira ([12]), Rupa R and Mujumdar ([26], 2018) and references therein.

If $\{X_i\}_{i\geq 1}$ is stationary with common d.f. F, then $\lambda(i+1|i) = \lambda(i|i-1) = \lambda$ and thus

$$S_{n,m} \equiv S = \lambda = \lim_{u\uparrow 1} P(F(X_{i+1}) > u | F(X_i) > u) = 1 - \lim_{u\uparrow 1} \frac{P(F(X_i) \leq u < F(X_{i+1}))}{P(F(X_i) > u)}. \quad (1)$$

Therefore, under stationarity, the smoothness of the trajectory of a series in time $[n, m]$, at large values, only depends on lag-1 bivariate tail dependence.

It is easily seen that independent sequences have $\lambda = 0$. However, $\lambda = 0$ does not imply independence.

In this paper we address the estimation of the upper tail smoothness coefficient under stationarity, i.e., of S in (1) given by the TDC applied to lag-1 random pairs $(F(X_i), F(X_{i+1}))$ of a time series. We consider the empirical counterpart of the last equality in (1) as in Ferreira and Ferreira ([11], 2021). Based on a simulation study it is shown a non-negligible bias. We thus propose a reduced bias Jackknife estimator following the approach in Gomes *et al.* ([14], 2008). The TDC estimators developed in literature and respective asymptotic properties such as normality, are analyzed by considering independence between random pairs. This is not the case here since we must consider successive random pairs $(F(X_1), F(X_2)), (F(X_2), F(X_3)), \ldots$. Therefore, we are going to apply a block bootstrap method in order to estimate confidence intervals, while taking into account the dependency structure (see, e.g., Politis and Romano [25], 1994 and references therein). We also propose an hypotheses test to evaluate the presence of upper tail smoothness in a time series, i.e., if coefficient $S > 0$ or if, on the contrary, S is null which indicates an oscillating behavior similar to an i.i.d. sequence.

The paper is organized as follows: Section 2 is devoted to the estimation methods of the upper tail smoothness coefficient. In particular we use Jackknife technique and present a reduced bias estimator. In Section 3 we propose a test procedure to prior analyze the presence of upper tail smoothness against no smooth at all, which will be fundamental on inference. The methods's performance will be evaluated in Section 4 through simulation, where we include bootstrap confidence intervals (CI) based on block

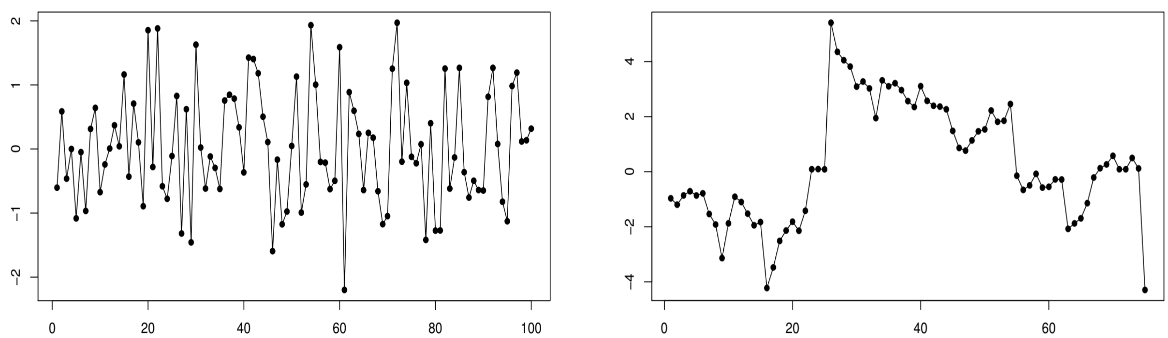

Figure 1: Illustration of a time series with a more oscillating sample path (left) generated from independent standard Gaussian and a time series with a smoother sample path (right) generated from an AR(1) with Cauchy standard marginals and auto-correlation 0.9.

subsampling techniques. An application on financial data illustrates our proposal in Section 5 and after we conclude (Section 6).

2 Estimation of the upper tail smoothness coefficient

In this section we present an inference methodology to estimate the upper tail smoothness coefficient S in (1) for stationary time series $\{X_i\}_{i\geq 1}$. Our approach is based on the empirical counterpart of (1), given by

$$\widehat{S}_u = 1 - \frac{\sum_{i=1}^{n-1} \mathbf{1}_{\{\widehat{F}(X_i)\leq u<\widehat{F}(X_{i+1})\}}}{\sum_{i=1}^{n} \mathbf{1}_{\{\widehat{F}(X_i)>u\}}},$$

where $\widehat{F}$ denotes the empirical d.f. of F (see Ferreira and Ferreira, [11] 2021 and references therein). Following the Generalized Jackknife methodology studied in Gomes *et al.* ([14], 2008), we are going to propose a reduced bias estimator of S.

Consider $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ the order statistics (o.s.) of $\{X_i\}_{i\geq 1}$. Replacing the threshold u by $\widehat{F}(X_{n-k:n})$, we obtain the upper tail smoothness estimator $\widehat{S}_k$ as a function of the number k of o.s. larger than the threshold, i.e.,

$$\widehat{S}_k = 1 - \frac{\sum_{i=1}^{n-1} \mathbf{1}_{\{\widehat{F}(X_i)\leq 1-k/n<\widehat{F}(X_{i+1})\}}}{k}, \tag{2}$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function. Observe that the last term of $\widehat{S}_k$ corresponds to the number of upcrossings of threshold $\widehat{F}(X_{n-k:n})$ within the k exceedances of the same threshold.

Gomes *et al.* (2008) stated that, for several dependent structures, such estimator presents a bias that is a function of k with two dominant components, as long as k satisfies the intermediate sequence condition,

$$k = k_n \to \infty,\ k = o(n), \text{ as } n \to \infty, \tag{3}$$

More precisely, the bias is assumed to be

$$d_1(S)(k/n) + d_2(S)(1/k) + o(1/k) + o(k/n). \tag{4}$$

According to Gray and Schucany ([15], 1972), taking three estimators with the same type of bias in (4), e.g., $\widehat{S}_k$, $\widehat{S}_{\lfloor \delta k \rfloor+1}$ and $\widehat{S}_{\lfloor \delta^2 k \rfloor+1}$, where $\lfloor \delta x \rfloor$ denotes the integer part of x and $\delta \in (0,1)$ is a tuning parameter, as suggested in Gomes *et al.* (2008), the Generalized Jackknife (GJ) class of unbiased estimators corresponds to

$$\widehat{S}_k^{GJ(\delta)} = \left| \begin{array}{ccc} \widehat{S}_{\lfloor \delta^2 k \rfloor+1} & \widehat{S}_{\lfloor \delta k \rfloor+1} & \widehat{S}_k \\ \delta^2 & \delta & 1 \\ 1/\delta^2 & 1/\delta & 1 \end{array} \right| \Bigg/ \left| \begin{array}{ccc} 1 & 1 & 1 \\ \delta^2 & \delta & 1 \\ 1/\delta^2 & 1/\delta & 1 \end{array} \right|,$$

where $|M|$ denotes the determinant of matrix M. The analysis in Gomes *et al.* (2008) led to the heuristic choice $\delta = 1/4$, from which we derive

$$\widehat{S}_k^{GJ} \equiv \widehat{S}_k^{GJ(1/4)} = 5\widehat{S}_{\lfloor k/2 \rfloor+1} - 2\left(\widehat{S}_{\lfloor k/4 \rfloor+1} + \widehat{S}_k\right). \tag{5}$$

3 Asymptotic tail independence

In estimating very small values of S, the question of whether S is zero or not arises. A null smoothness coefficient tell us that the time series sample path is very oscillating, similar to an i.i.d. sequence corresponding to the least concordant case. This is the scenario of asymptotic tail independence, which can be ascertained prior to any inference. By asymptotic tail independence we mean dependence structures for which the TDC λ in (1) is null and thus smoothness $S = 0$. In this section we present an hypotheses test to evaluate this question.

It is straightforward that i.i.d. sequences have a null TDC but the opposite is not always true. The well-known auto-regressive Gaussian time series with auto-correlation dependence coefficient ρ has $\lambda = 0$ for all $\rho \in (-1, 1)$. Such cases present a weak tail dependence that can be captured by the rate of convergence of the bivariate upper tail towards zero, as the threshold gets higher and closer to the right end-point 1.

This feature was noticed by Ledford and Tawn ([20, 21], 1996/1997), who proposed the model:

$$P(F(X_2) > 1-t|F(X_1) > 1-t) = t^{1/\eta-1}L(t), \text{as}, \ t \downarrow 0, \tag{6}$$

where $\eta \in (0,1]$ and $L(t)$ is a slowly varying function at 0. In short, we call (6) the L&T model. If $\eta = 1$ and $L(t)$ converges to some positive constant c, then probability in (6) also converges to positive c leading to asymptotic tail dependence, i.e., TDC $\lambda = c > 0$. The case $0 < \eta < 1$ or $L(t) \to 0$ as $t \downarrow 0$, corresponds to asymptotic tail independence and TDC $\lambda = 0$. Exact independence takes place under $\eta = 1/2$ and $L(t) = 1$, positive association whenever $1/2 < \eta < 1$ and negative association if $0 < \eta < 1/2$. Observe that we can generalize (6) to lag-m, for any positive integer m, i.e.,

$$P(F(X_{1+m}) > 1-t|F(X_1) > 1-t) = t^{1/\eta_m-1}L_m(t), \text{as}, \ t \downarrow 0,$$

where $\eta_m \in (0,1]$ and $L_m(t)$ is a slowly varying function at 0, with $\eta \equiv \eta_1$ and $L(t) \equiv L_1(t)$.

Inference on η has been studied in EVT's literature, e.g., Draisma *et al.* ([8], 2004), Chiapino *et al.* ([3], 2019), and references therein, in the context of i.i.d. sequences of random vectors.

In the L&T model, considering $T = \min\left((1-F(X_1))^{-1}, (1-F(X_2))^{-1}\right)$, we have that

$$P\left(T > t^{-1}\right) = P(F(X_2) > 1-t, F(X_1) > 1-t) = P(F(X_2) > 1-t|F(X_1) > 1-t)P(F(X_1) > 1-t) = t^{1/\eta}L(t).$$

Therefore, the tail of T is regularly varying with index $1/\eta$ and so T belongs to the Fréchet max-domain of attraction with tail index η. There are several estimators developed under this framework for the tail index, like Hill ([16], 1975), Moments ([10], 1989), maximum likelihood (ML) of Smith ([28], 1987), Pickands ([17], 1975), among others.

Considering $T_{1:n} \leq ... \leq T_{n:n}$ are the o.s. of $T_i = \min\left((1-F(X_i))^{-1}, (1-F(X_{i+1}))^{-1}\right)$, $i = 1, ..., n$, the classic Hill estimator is defined by,

$$\widehat{\eta}_{k,n} := \frac{1}{k}\sum_{i=1}^{k} \log T_{n-i+1:n} - \log T_{n-k:n}, \tag{7}$$

Consistency is achieved whenever $k \equiv k_n$ is an intermediate sequence as in (3) and each choice of k generates an estimate. If k is too large a high variance takes place but if it is too small the bias increases. In practice, plotting the sample path of estimates help us in finding a stable region for a plausible choice of k.

Since our context implies dependence between r.v. $T_1 = \min\left((1-F(X_1))^{-1}, (1-F(X_2))^{-1}\right)$, $T_2 = \min\left((1-F(X_2))^{-1}, (1-F(X_3))^{-1}\right)$,$T_3 = \min\left((1-F(X_3))^{-1}, (1-F(X_4))^{-1}\right)$, ... we will resort to asymptotic results in Drees ([9], 2003) valid for a class of tail index estimators under a dependence framework, which includes all the above mentioned ones. The package *ExtremeRisks* of software R ([22], 2020) allows estimating the asymptotic variance $\widehat{\sigma^2}$ of the referred estimators under dependency, according to the methods in Drees ([9], 2003). These will be here applied to test the hypotheses $H_0 : \eta = 1$ vs. $H_1 : \eta < 1$. If $\hat{\eta} < 1 + z_{0.05}\widehat{\sigma}$, where $z_{0.05}$ is the 5% quantile of the standard Gaussian d.f., we reject H_0 in favor of H_1 and conclude that $\lambda = 0$, i.e., $S = 0$ and that the series is asymptotic tail independent with large values resembling an i.i.d. behavior.

4 Simulation study

Based on simulation, we analyze the performance of the test procedure in Section 3 and of the upper tail smoothness estimator (2) and the GJ estimator (5) in Section 2. We recall that the upper tail smoothness coefficient S is given by the TDC λ under stationarity, as stated in (1). We also estimate 95% CI through block resampling bootstrap, in order to preserve serial dependence. We use the R package *blocklength* ([29], 2022) where the block length choice is based on an automatic selection method proposed by Politis and White ([24], 2004) and corrected in Patton *et al.* ([23], 2009). We compute Normal, basic and percentile bootstrap CI of Davison and Hinkley ([5], 1997; Chapter 5), available in R package *boot*.

In the following we present some models and respective TDC, in which we base the simulation study:

- 1st order max auto-regressive (MAR), $X_i = \max(\phi X_{i-1}, \epsilon_i)$, $i \geq 1$, $X_0 = \epsilon_1/(1-\phi)$, $\{\epsilon_i\}$ i.i.d. with standard Fréchet marginal F (Davis and Resnick [4], 1989) and quantile function F^{-1}, for which

$$\lambda = 2 - \lim_{u \uparrow 1} \frac{1 - P(F(X_1) \leq u, F(X_2) \leq u)}{1 - P(F(X_1) \leq u)} = 2 - \lim_{u \uparrow 1} \frac{1 - uP(\epsilon_2 \leq F^{-1}(u))}{1-u} = 2 - \lim_{u \uparrow 1} \frac{1 - u^{2-\phi}}{1-u} = \phi$$

  We take $\phi = 0.1$ leading to $S = 0.1$.

- moving maxima $X_i = \max_{j=1,\ldots,d} a_j Z_{i-j}$ with $\{Z_i\}$ i.i.d. standard Fréchet (Frec) (Deheuvels [7], 1983), where $d = 2$ and parameters $\alpha_0 = 1/6$, $\alpha_1 = 1/2$, and $\alpha_2 = 1/3$, for which $\lambda = 0.5$, since

$$\begin{aligned}\lambda = &\ 2 - \lim_{u \uparrow 1} \frac{1 - P(F(X_1) \leq u, F(X_2) \leq u)}{1 - P(F(X_1) \leq u)} = 2 - \lim_{u \uparrow 1} \frac{1 - P(\max(a_0 Z_1, a_1 Z_0, a_2 Z_{-1}, a_0 Z_2, a_1 Z_1, a_2 Z_0) \leq F^{-1}(u))}{1-u} \\ = &\ 2 - \lim_{u \uparrow 1} \frac{1 - P(a_2 Z_{-1} \leq F^{-1}(u)) P(a_0 Z_2 \leq F^{-1}(u)) P(Z_1 \leq F^{-1}(u)/(\max(a_0, a_1))) P(Z_0 \leq F^{-1}(u)/(\max(a_1, a_2)))}{1-u}\end{aligned}$$

  In our case, $\max(a_0, a_1) = \max(a_1 0, a_2) = a_1$ and since $a_0 + a_1 + a_2 = 1$, we have

$$\lambda = 2 - \lim_{u \uparrow 1} \frac{1 - u^{1+a_1}}{1-u} = 1 - a_1$$

  and thus $S = 1/2$.

- AR(1) with Uniform marginals (ARUnif), $X_i = (1/r)X_{i-1} + \epsilon_i$, $r \geq 2$, $X_0 \sim U(0,1)$ independent of $\{\epsilon_i\}$ i.i.d. having Uniform d.f. with support $\{0, 1/r..., (r-1)/r\}$. This model was developed in Chernick *et al.* ([2], 1991), from which it can be stated

$$\lim_{u \uparrow 1} P(F(X_1) \leq u | F(X_2) > u) = 1 - 1/r$$

  Thus, by (1) we derive $\lambda = 1/r$. We take $r = 3$ which leads to $S = 1/3$.

- AR(1) with Cauchy standard marginals (ARCau), $X_i = \rho X_{i-1} + \epsilon_i$, $\rho > 0$, $\{\epsilon_i\}$ i.i.d. having Cauchy d.f. with mean 0 and scale $1 - \rho$ and X_0 is standard Cauchy. This model was studied in Chernick *et al.* ([2], 1991) and based on Corollary 1.3 of this latter reference, we obtain

$$\lim_{u \uparrow 1} P(F(X_2) \leq u | F(X_1) > u) = 1 - \rho$$

  and thus $\lambda = \rho$. We take $\rho = 0.8$ and so $S = 0.8$.

- AR(1) process, $X_i = \phi X_{i-1} + \epsilon_i$, $i \geq 1$, $\{\epsilon_i\}$ i.i.d. $N(0,1)$, $X_0 \frown N(0, 1/(1-\phi^2))$, with parameter $\phi = 0.5$, where $\lambda = 0$, i.e., $S = 0$ (see, e.g., Frahm *et al.* [13] 2005).

- an i.i.d. sequence of Fréchet r.v. and thus $\lambda = 0$, i.e., $S = 0$.

We simulated 1000 replicas of each given model with size $n = 1000$.

We start by testing the null hypothesis of $\eta = 1$, i.e., $\lambda > 0$ against the alternative $\eta < 1$, i.e., $\lambda = 0$. In order to estimate η by the Hill estimator in (7), we take the $k = 100^{th}$ upper o.s. of each ordered sample and compute the acceptance proportion of $\lambda = 0$. The results in Table 1 are quite promising. At the significance level 5%, models AR and FrecInd with null TDC λ present an acceptance proportion of this condition above 95% and the remaining models with $\lambda > 0$ present an acceptance proportion of a null TDC below 5%, except the ARUnif with a slightly larger value (6.2%).

We computed the absolute bias (abias) and the root mean squared error (rmse) of the upper tail smoothness estimators $\widehat{S}_k$ in (2) and reduced-bias GJ $\widehat{S}_k^{GJ}$ in (5, plotted in Figures 2 and 3, respectively, as functions of $1 < k < n$ on the x-axis. The full line corresponds to estimator in (2) and the dashed line to the GJ estimator in (5).

It is evident that $\widehat{S}_k$ presents lower biases for small values of k, i.e., for higher thresholds and an increasingly sharp bias as the threshold decreases. As expected, the GJ estimator $\widehat{S}_k^{GJ}$ reduces the bias showing a sample path with a quite long stable region, even for large values of k. Exception is made for the ARUnif model were both estimators perform well and similar for $k \lesssim 350$ but have an irregular pattern in the remaining trajectory. Another exception is the AR model, in which, despite being smaller,

$\widehat{S}_k^{GJ}$ still has a considerable bias, perhaps due to correlation $\rho = 0.5$ and therefore a non-negligible dependence, despite a quasi-independence in the tail. Looking at the rmse, we still observe a decrease within the GJ estimator $\widehat{S}_k^{GJ}$ for larger values of k. In the AR and ARUnif models, we continue to see the same types of drawbacks already observed in their biases.

Table 1: Acceptance (%) of the hypothesis of $\lambda = 0$.

ARCau	Frec	ARUnif	MAR	AR	FrecInd
0	4.5	6.2	0	96.4	1

Figure 4 presents block bootstrap 95% CI results for both estimators, $\widehat{S}_k$ and $\widehat{S}_k^{GJ}$, with $k = 100$, for Normal, basic and percentile methods. Namely, the bars height correspond to the proportion of CI that included the true value of S (coverage, on the left) and the coverage divided by the mean range width (coverage/range, on the right).

We can see that again the AR model does not benefit from the method. The percentile CI seems to be the best choice.

5 Application

In this section we analyze the upper tail smoothness of daily EUR/USD exchange rate, $\{R_t\}$, obtained from `https://finance.yahoo.com/` in two time periods. The first period corresponds to years 2007-2009 and the second period to years 2020-2022. More precisely, we look at volatility by taking the absolute value of successive log-returns, i.e., $\{|\log(R_t/R_{t-1})|\}$, with length 784. The data plots are represented, respectively, in the left and right panels of Figure 5. At a first glance it is not so easy to infer the degree of upper tail smoothness in each data set. We can see that in the earliest period the magnitude of observations is higher than in the more recent data. However, the upper tail smoothness coefficient does not measure the magnitude of data, only the propensity for more or less oscillations in successive observations. We shall see that the least oscillating dataset is in the first period.

Before estimating the upper tail smoothness coefficient, we first apply the asymptotic tail independence test described in Section 3. The test procedure is based on the L&T coefficient η estimated by the Hill method in (7). In Figure 6 we can see the sample path of estimates for each k^{th} upper o.s. with $1 < k < n$ and respective 95% confidence bands (dotted lines). In the first plot of period 2007-2009, it is plausible to have $\eta = 1$, since it is included in the confidence bands for several values of k, contrarily to the second plot of period 2020-2022, where the confidence bands exclude the value 1 for η. Thus we conclude that the upper tail smoothness coefficient S is positive in the earliest period and $S = 0$ in the recent years 2020-2022.

In Figure 7 are the estimates of $\widehat{S}_k$ (black full line), $\widehat{S}_k^{GJ}$ (red full line) and respective 95% confidence bands (dashed lines). We used bootstrap percentile CI as it revealed an overall better performance in the simulation study. The left panel corresponds to the first period 2007-2009. Our guess is that the upper tail smoothness coefficient is around 0.3 (horizontal line). As for the second period 2020-2022 in the right panel, the plots seem to corroborate the test result of a null upper tail smoothness coefficient. Recall that in the simulation results, it was found that $\widehat{S}_k$ presented better behavior for smaller values of k. The plotted estimates of the second period data are close to zero for small k. As for the $\widehat{S}_k^{GJ}$ estimator, its better performance in the simulation study was associated to higher values of k. The trajectory of this estimator in the second period lower panel, although with some interruptions in which it moves away from zero, it returns to zero for large values of k. Thus the exchange rate's volatility presents a smoother upper tail behavior in the first period than in the second one where it is more oscillating and resembles an i.i.d. sequence, meaning more instability.

6 Conclusion

Extreme value theory and extreme value statistics assume a very important role in the implementation of adequate tools for risk assessment. The upper tail smoothness coefficient discussed here is another such example. By evaluating the degree of smoothness of a time series at large values, we can see whether its behavior is typically more uncertain or not, which may help analysts' decision-making. It is a simple measure and easy to implement. The proposal introduced here for a reduced bias estimator proved to be promising in the study carried out and constitutes an important step in the inference. Motivated by this work, we intend to deepen subsampling methodologies to proceed improving the smoothness estimation, namely, in reducing the root mean squared error. The study of asymptotic properties of estimator (5) making use of the theory of empirical tail processes for stationary time series is another way to proceed with this work.

Disclosure Statement

The authors report there are no competing interests to declare.

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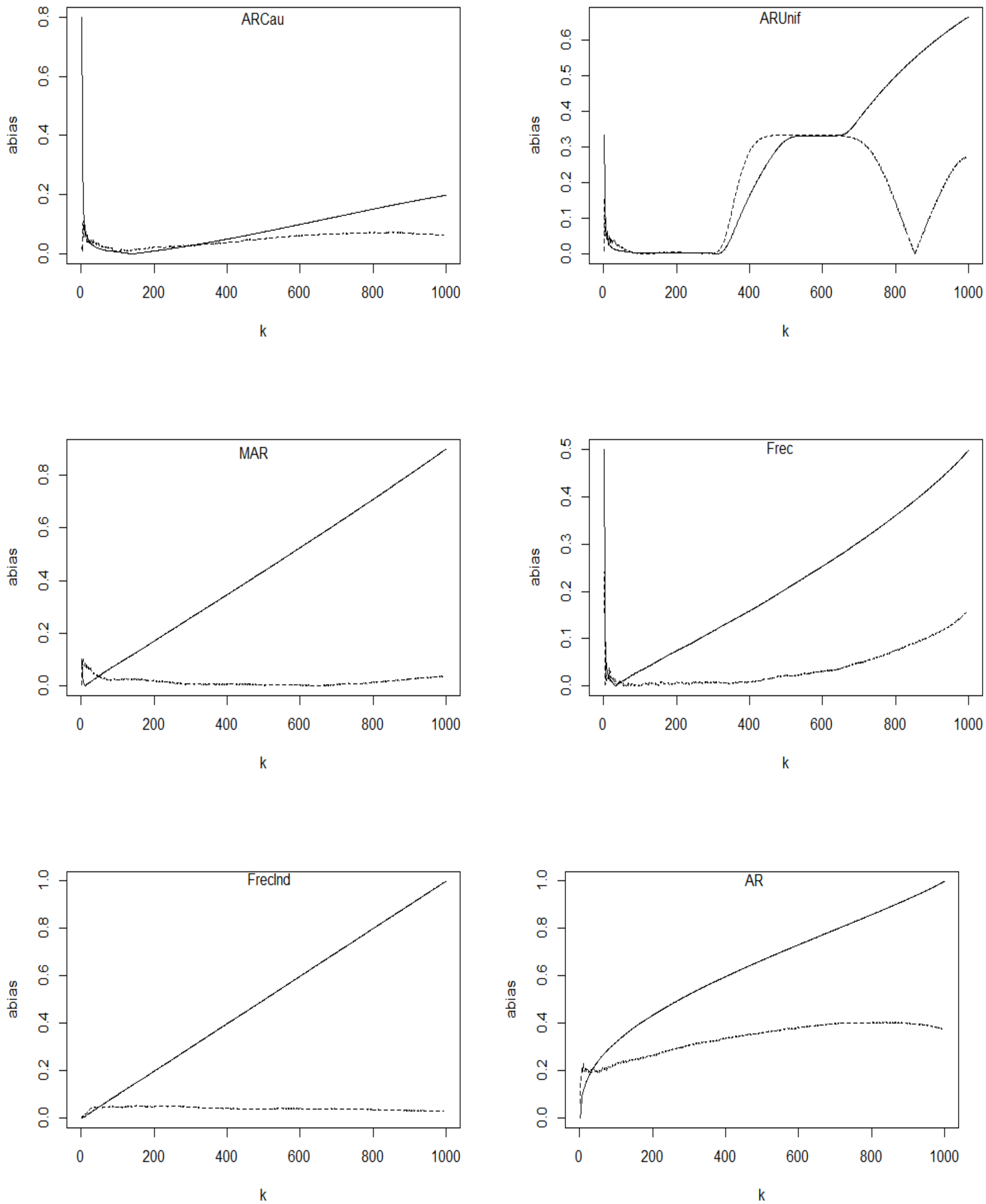

Figure 2: Absolute bias (abias) of the upper tail smoothness estimator $\widehat{S}_k$ in (2) (full line) and the GJ estimator $\widehat{S}_k^{GJ}$ in (5) (dashed line), for $1 < k < n$.

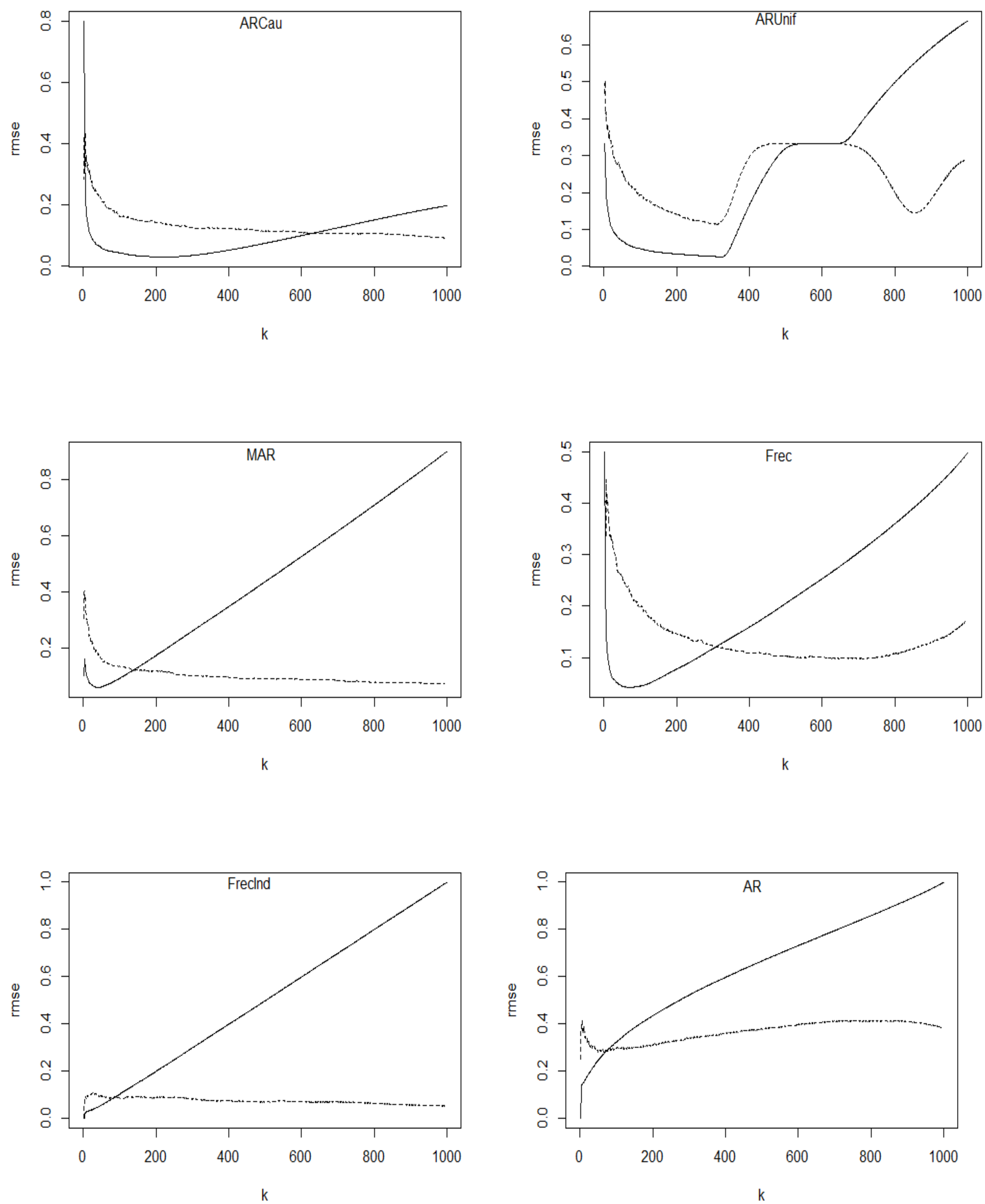

Figure 3: Root mean squared error (rmse) of the upper tail smoothness estimator $\widehat{S}_k$ in (2) (full line) and the GJ upper tail smoothness estimator $\widehat{S}_k^{GJ}$ in (5) (dashed line), for $1 < k < n$.

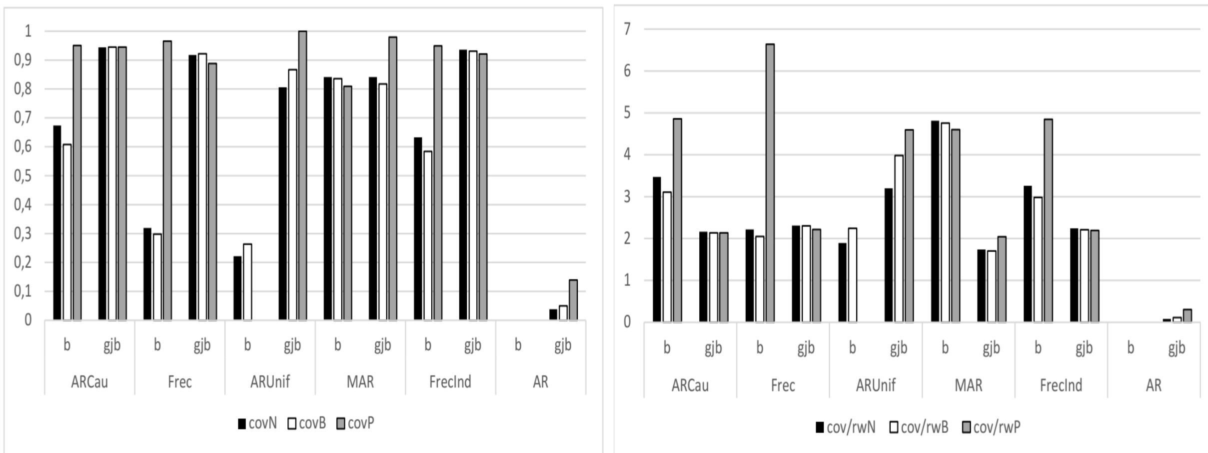

Figure 4: Percentage of the estimated block bootstrap 95% CI that included the true value of S (coverage) on the left and the rate given by the coverage over the mean intervals range width (coverage/range) on the right.

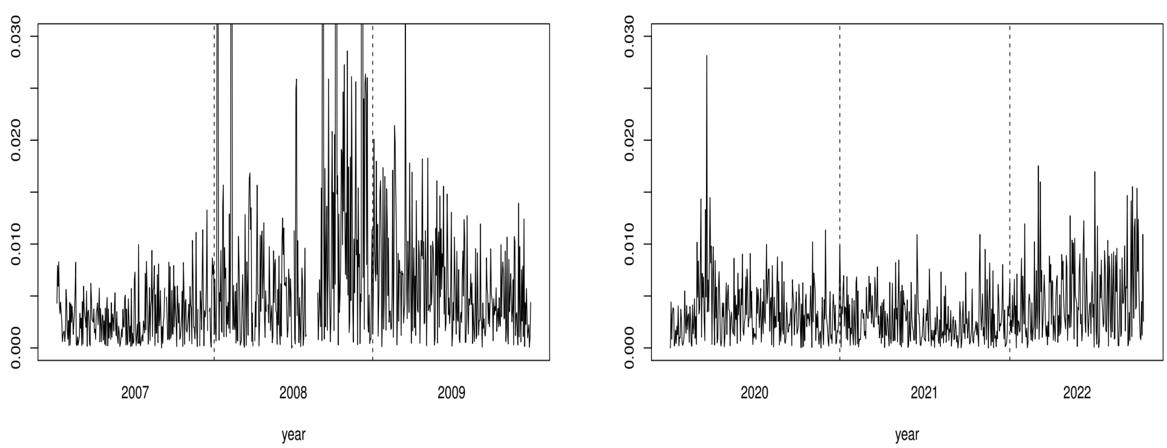

Figure 5: Absolute log-returns of EUR/USD daily exchange rates in years 2007-2009 (left) and years 2020-2022 (right).

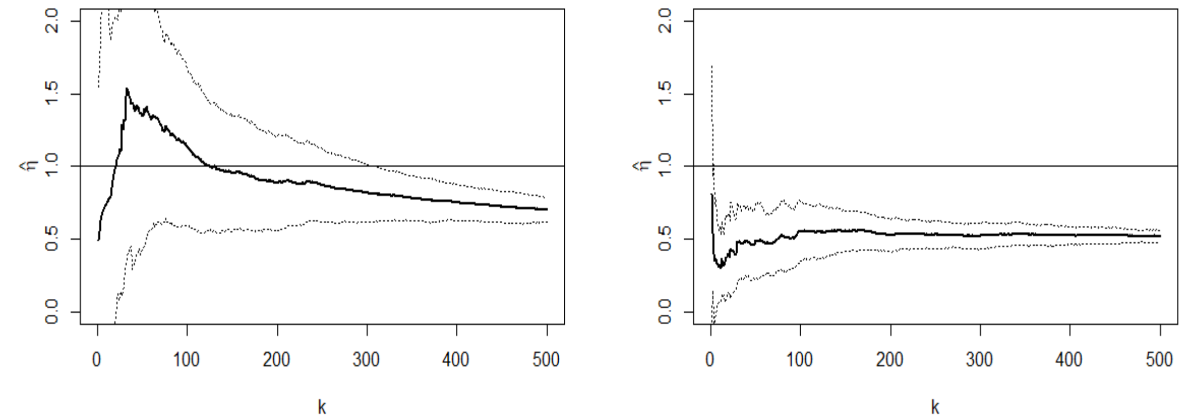

Figure 6: Estimates of η through Hill estimator (7), as function of $1 < k < n$ upper o.s. (full line). The upper and lower dotted lines are, respectively, the upper and lower 95% confidence bands, in periods 2007-2009 (left panel) and 2020-2022 (right panel).

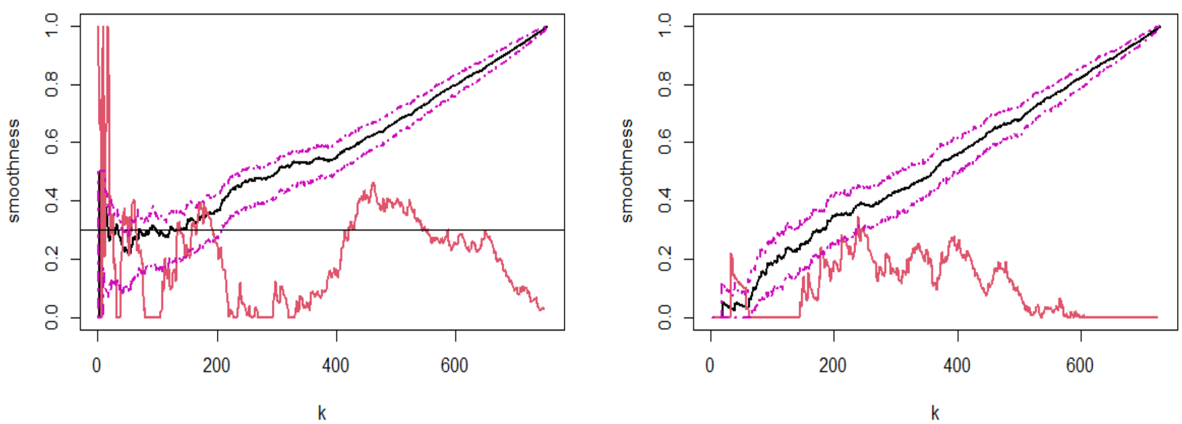

Figure 7: Estimates of $\widehat{S}_k$ (black full line), $\widehat{S}_k^{GJ}$ (red full line) and respective 95% confidence bands (dashed lines) based on bootstrap percentile CI, in periods 2007-2009 (left panel) and 2020-2022 (right panel).