

ON THE LS-CATEGORY AND TOPOLOGICAL COMPLEXITY OF PROJECTIVE PRODUCT SPACES

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ABSTRACT. We determine the Lusternik-Schnirelmann category of the projective product spaces introduced by D. Davis. We also obtain an upper bound for the topological complexity of these spaces, which improves the estimate given by J. González, M. Grant, E. Torres-Giese, and M. Xicoténcatl.

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1. INTRODUCTION

We let \bar{n} symbolize an r -tuple (n_1, \dots, n_r) of positive integers with $n_1 \leq \dots \leq n_r$. We consider the product $S_{\bar{n}} := S^{n_1} \times \dots \times S^{n_r}$ and, given $x_i \in S^{n_i}$, we write $\bar{x} = (x_1, \dots, x_r)$ for the corresponding element of $S_{\bar{n}}$. The quotient space

$$P_{\bar{n}} := S_{\bar{n}}/(\bar{x} \sim -\bar{x}) = (S^{n_1} \times \dots \times S^{n_r})/((x_1, \dots, x_r) \sim (-x_1, \dots, -x_r))$$

with respect to the diagonal action of \mathbb{Z}_2 on $S_{\bar{n}}$ has been introduced by D. Davis [5] and is called *projective product space*. This is a manifold of dimension $\dim(P_{\bar{n}}) = \dim(S_{\bar{n}}) = \sum n_i$ and, when $r = 1$, the space $P_{\bar{n}}$ coincides with the usual real projective space P^{n_1} . Recently, the notion of projective product space was used in [2] and generalized in [11].

In this note, we study the (normalized) Lusternik-Schnirelmann category (cat) and Farber's topological complexity (TC) of the space $P_{\bar{n}}$. This study has been initiated by J. González, M. Grant, E. Torres-Giese, M. Xicoténcatl in [10], where the following result is established:

Theorem 1.1. [10, Theorem 3.8] *Let k represent the number of spheres S^{n_q} with n_q even and $q > 1$. Then $\text{TC}(P_{\bar{n}}) < (\text{TC}(P^{n_1}) + 1)(r + k)$.*

In this paper, we first determine the LS-category of $P_{\bar{n}}$:

Theorem 1.2. $\text{cat}(P_{\bar{n}}) = \text{cat}(P^{n_1}) + r - 1 = n_1 + r - 1$.

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This is obtained through the construction of an explicit categorical cover together with the knowledge of the cohomology of $P_{\bar{n}}$ over \mathbb{Z}_2 , which has been determined by Davis [5, Theorem 2.1] (see Theorem 2.2 below).

For the topological complexity, we establish:

Theorem 1.3. $\text{TC}(P_{\bar{n}}) \leq \text{TC}(P^{n_1}) + \sum_{q=2}^r \text{TC}(S^{n_q})$.

In terms of the number k of spheres S^{n_q} with n_q even and $q > 1$, the upper bound of Theorem 1.3 can be written $\text{TC}(P^{n_1}) + r + k - 1$, which permits us to see that Theorem 1.3 improves Theorem 1.1. As mentioned in [10], using Davis' description of $H^*(P_{\bar{n}}; \mathbb{Z}_2)$, the zero-divisor-cup-length of $P_{\bar{n}}$ over \mathbb{Z}_2 can be expressed as $\text{zcl}_{\mathbb{Z}_2}(P^{n_1}) + r - 1$. We then obtain:

Corollary 1.4. *If $\text{zcl}_{\mathbb{Z}_2}(P^{n_1}) = \text{TC}(P^{n_1})$ and n_q is odd for $q > 1$ then $\text{TC}(P_{\bar{n}}) = \text{TC}(P^{n_1}) + \sum_{q=2}^r \text{TC}(S^{n_q}) = \text{TC}(P^{n_1}) + r - 1$.*

The upper bound in Theorem 1.3 is obtained through the construction of an explicit motion planner which uses the characterization of $\text{TC}(P^{n_1})$ in terms of non-singular maps due to M. Farber, S. Yuzvinsky and S. Tabachnikov [8]. We note that, using a strong version of non-singular maps, an explicit motion planner for polyhedral products of real projective spaces has recently been constructed in [1].

2. LUSTERNIK-SCHNIRELMANN CATEGORY OF $P_{\bar{n}}$

As $P_{\bar{n}}$ is a finite, path-connected CW complex, we give all the useful definitions and characterizations for such spaces.

Definition 2.1. Let X be a finite, path-connected CW complex. A subset $A \subset X$ is called categorical if the inclusion $i : A \hookrightarrow X$ is nullhomotopic. The Lusternik-Schnirelman category $\text{cat}(X)$ of X is defined as the least integer k such that there exists a cover of X by $k + 1$ open categorical subsets $U_0, \dots, U_k \subset X$.

Let R be a commutative unitary ring. Recall that the cup-length over R of a path-connected space X , $\text{cuplength}_R(X)$, is the longest length k of a nonzero product $c_1 \smile \dots \smile c_k \neq 0$ of cohomology classes $c_1, \dots, c_k \in H^+(X; R)$ and provides a lower bound for the Lusternik-Schnirelman category of X , $\text{cuplength}_R(X) \leq \text{cat}(X)$.

As is well-known, the cup-length over \mathbb{Z}_2 of the m -dimensional projective space P^m is equal to m . The \mathbb{Z}_2 -cohomology of $P_{\bar{n}}$ has been determined by Davis:

Theorem 2.2. ([5, Theorem 2.1]) *Let $\bar{n} = (n_1, \dots, n_r)$ such that $n_1 \leq \dots \leq n_r$. If $n_1 < n_2$ or n_1 is odd, the mod 2 cohomology ring of $P_{\bar{n}}$ is given by*

$$H^*(P_{\bar{n}}; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha_1]/(\alpha_1^{n_1+1}) \otimes \Lambda[\alpha_2, \dots, \alpha_r]$$

where $\dim(\alpha_1) = 1$, $\dim(\alpha_i) = n_i$ for $i > 1$, and Λ denotes an exterior algebra. If n_1 is even and $n_1 = \dots = n_k < n_{k+1}$ for some $k > 1$, $H^*(P_{\bar{n}}; \mathbb{Z}_2)$ is the same as above with the extra relation given by $\alpha_i^2 = \alpha_1^{n_1} \alpha_i$ for $2 \leq i \leq k$.

By considering the longest nonzero product $\alpha_1^{n_1} \smile \alpha_2 \cdots \smile \alpha_r \neq 0$ in $H^*(P_{\bar{n}}; \mathbb{Z}_2)$ we obtain:

Proposition 2.3. $\text{cuplength}_{\mathbb{Z}_2}(P_{\bar{n}}) = n_1 + r - 1$.

In order to prove Theorem 1.2, we will construct an explicit cover of $P_{\bar{n}}$ with contractible subsets using the following characterization of the category:

Proposition 2.4. [4, Lemma 1.35] *Suppose that X is a path-connected finite CW-complex. We have $\text{cat}(X) \leq k$ if and only if there exists an increasing sequence of open sets*

$$\emptyset = U_{-1} \subset U_0 \subset U_1 \subset \dots \subset U_k = X$$

such that, for any $i \in \{0, \dots, k\}$, $U_i - U_{i-1}$ is contractible in X .

Remark 2.5. The sets $F_i = U_i - U_{i-1}$ provide a cover of X by $k + 1$ disjoint subsets which are contractible in X . We note that, in [4, Lemma 1.35], X is not supposed to be a finite CW-complex but it is required that each $F_i = U_i - U_{i-1}$ is contained in an open set of X which is contractible in X . Assuming that X is a finite CW-complex (and therefore an ENR space) and adapting the proof of [7, Prop. 4.12], we can just ask that F_i is contractible in X . Actually, when X is a metric ANR space, we know by [12] (see also [9]), that $\text{cat}(X) \leq k$ if and only there exists a cover by $k + 1$ subsets contractible in X without further condition on the subsets.

Proof of Theorem 1.2. We first fix some general notation. Writing

$$S^m = \{(u_0, \dots, u_m) \in \mathbb{R}^{m+1} \mid \sum u_k^2 = 1\},$$

we denote by $p_j : S^m \rightarrow \mathbb{R}$ the obvious projection ($j = 0, \dots, m$) and by a_j the unique point of S^m such that $p_j(a_j) = 1$. Note that, for any $x \in S^m$, $p_j(-x) = -p_j(x)$. When $j = m$, we will use the special notation $A_m := a_m = (0, \dots, 0, 1)$ and we fix a meridian path $\mu_0(A_m, -A_m) : I \rightarrow S^m$ such that $\mu_0(0) = A_m$ and $\mu_0(1) = -A_m$. We will denote by $\mu_0(-A_m, A_m)$ the path given by $\mu_0(-A_m, A_m)(t) = -\mu_0(A_m, -A_m)(t)$, that is $\mu_0(-A_m, A_m) = -\mu_0(A_m, -A_m)$. For non-antipodal points A, B ($A \neq -B$) of S^m , let $\lambda(A, B) : I \rightarrow S^m$ be the geodesic path from A to B . Note that $\lambda(-A, -B) = -\lambda(A, B)$ and that $\lambda(A, A)$ is the constant path.

As is well-known, $\text{cat}(P^{n_1}) = n_1$. We now define a cover of S^{n_1} which induces a cover of P^{n_1} by $n_1 + 1$ categorical subsets. In our constructions we use a formalism which was inspired by [3].

For a subset $L \subset \{0, 1, \dots, n_1\}$, let $|L|$ denote the cardinality of L and consider

$$S_L = \{x \in S^{n_1} : p_l(x) \neq 0 \text{ if } l \in L\}.$$

By setting

$$U_i = \bigcup_{|L|=(n_1+1)-i} S_L$$

for $0 \leq i \leq n_1$ and $U_{-1} = \emptyset$, we have an increasing sequence of open subsets of S^{n_1} :

$$\emptyset = U_{-1} \subset U_0 \subset \dots \subset U_{n_1} = S^{n_1}.$$

Note that $U_{n_1} = S^{n_1}$ since the projections p_0, \dots, p_{n_1} cannot vanish all at the same time.

Considering, for $L \subset \{0, 1, \dots, n_1\}$,

$$Q_L = \{x \in S^{n_1} : p_l(x) \neq 0 \text{ if } l \in L \text{ and } p_l(x) = 0 \text{ if } l \notin L\}$$

we have, for $0 \leq i \leq n_1$,

$$F_i := U_i - U_{i-1} = \bigcup_{|L|=(n_1+1)-i} Q_L.$$

Note that all the sets S_L , Q_L , U_i , F_i are saturated with respect to the antipodal relation $x \sim -x$ on S^{n_1} . Note also that, for $L_1 \neq L_2$ with same cardinality $|L_1| = |L_2|$, the sets Q_{L_1} and Q_{L_2} are topologically disjoint ($\overline{Q_{L_1}} \cap Q_{L_2} = \emptyset = Q_{L_1} \cap \overline{Q_{L_2}}$) and that $\bigcup_{i=0}^{n_1} F_i$ is a cover of S^{n_1} .

For $L \subset \{0, 1, \dots, n_1\}$, we set $l_0 = \min L$ and consider the map $\psi_L : Q_L \rightarrow (P^{n_1})^I$

$$\psi_L(x) = \begin{cases} [\lambda(x, a_{l_0})], & \text{if } p_{l_0}(x) > 0, \\ [\lambda(x, -a_{l_0})], & \text{if } p_{l_0}(x) < 0. \end{cases}$$

This map is continuous on Q_L and satisfies $\psi_L(x)(0) = [x]$, $\psi_L(x)(1) = [a_{l_0}]$, and $\psi_L(x) = \psi_L(-x)$. This shows that the subset Q_L / \sim is contractible in P^{n_1} . As the distinct sets Q_L with $|L|$ constant are topologically disjoint, we obtain that, for $0 \leq i \leq n_1$, F_i / \sim is the union of topologically disjoint subsets which are contractible in P^{n_1} . Since P^{n_1} is path-connected, we can conclude that F_i / \sim is itself contractible in P^{n_1} .

For $2 \leq q \leq r$, we consider the increasing sequence of open sets

$$V_{-1} = \emptyset \quad V_0 = \{x \in S^{n_q} \mid x \neq \pm A_{n_q}\} \quad V_1 = S^{n_q}$$

as well as the sets $G_0 := V_0 - V_{-1} = V_0$ and $G_1 := V_1 - V_0 = \{\pm A_{n_q}\}$. These sets are saturated with respect to the antipodal relation.

Let $L \subset \{0, 1, \dots, n_1\}$ with $l_0 = \min L$ and, for $2 \leq q \leq r$, let $j_q \in \{0, 1\}$. We define

$$\psi_{(L, j_2, \dots, j_r)} : Q_L \times \prod_{q=2}^r G_{j_q} \rightarrow P_{\bar{n}}^I$$

by

$$\psi_{(L, j_2, \dots, j_r)}(x_1, x_2, \dots, x_r) = \begin{cases} [\lambda(x_1, a_{l_0}), \nu_2(x_2), \dots, \nu_r(x_r)], & \text{if } p_{l_0}(x_1) > 0 \\ [\lambda(x_1, -a_{l_0}), \eta_2(x_2), \dots, \eta_r(x_r)], & \text{if } p_{l_0}(x_1) < 0 \end{cases}$$

where

$$\nu_q(x_q) = \begin{cases} \mu_0(-A_{n_q}, A_{n_q}), & \text{if } x_q = -A_{n_q} \\ \lambda(x_q, A_{n_q}), & \text{if } x_q \neq -A_{n_q} \end{cases}$$

and

$$\eta_q(x_q) = \begin{cases} \mu_0(A_{n_q}, -A_{n_q}), & \text{if } x_q = A_{n_q} \\ \lambda(x_q, -A_{n_q}), & \text{if } x_q \neq A_{n_q}. \end{cases}$$

The map $\psi_{(L, j_2, \dots, j_r)}$ is continuous, well-defined on $Q_L \times \prod_{q=2}^r G_{j_q}$ and satisfies, for $\bar{x} = (x_1, \dots, x_r)$,

$$\psi_{(L, j_2, \dots, j_r)}(\bar{x})(0) = [\bar{x}] \quad \text{and} \quad \psi_{(L, j_2, \dots, j_r)}(\bar{x})(1) = [a_{l_0}, A_{n_2}, \dots, A_{n_r}].$$

Moreover, we can check that the map is compatible with the diagonal antipodal relation, that is $\psi_{(L, j_2, \dots, j_r)}(-\bar{x}) = \psi_{(L, j_2, \dots, j_r)}(\bar{x})$. Indeed, we first observe that, for $2 \leq q \leq r$ and $x_q \in S^{n_q}$, we have $\nu_q(x_q) = -\eta_q(-x_q)$. Suppose now that, for $\bar{x} = (x_1, \dots, x_r)$, we have $p_{l_0}(x_1) > 0$. Then $p_{l_0}(-x_1) < 0$ and

$$\begin{aligned} \psi_{(L, j_2, \dots, j_r)}(-\bar{x}) &= [\lambda(-x_1, -a_{l_0}), \eta_2(-x_2), \dots, \eta_r(-x_r)] \\ &= [-\lambda(x_1, a_{l_0}), -\nu_2(x_2), \dots, -\nu_r(x_r)] \\ &= [\lambda(x_1, a_{l_0}), \nu_2(x_2), \dots, \nu_r(x_r)] \\ &= \psi_{(L, j_2, \dots, j_r)}(\bar{x}). \end{aligned}$$

The argument is similar if $p_{l_0}(x_1) < 0$. We thus get a continuous map on the quotient $(Q_L \times \prod_{q=2}^r G_{j_q}) / \sim$ which permits us to see that this quotient is contractible in $P_{\bar{n}}$.

For $s = 0, \dots, n_1 + r - 1$, we now build

$$W_s = \bigcup_{i + \sum_{q=2}^r j_q = s} U_i \times \prod_{q=2}^r V_{j_q} \subset S^{n_1} \times S^{n_2} \dots \times S^{n_r}$$

where $i \in \{0, 1, \dots, n_1\}$ and $j_q \in \{0, 1\}$ for each $2 \leq q \leq r$. Therefore, there exists a tower of open subsets

$$\emptyset = W_{-1} \subset W_0 \subset \dots \subset W_{n_1+r-1} = S^{n_1} \times S^{n_2} \dots \times S^{n_r}.$$

We have

$$W_s - W_{s-1} = \bigcup_{i + \sum_{q=2}^r j_q = s} (U_i - U_{i-1}) \times \prod_{q=2}^r (V_{j_q} - V_{j_q-1}) = \bigcup_{i + \sum_{q=2}^r j_q = s} F_i \times \prod_{q=2}^r G_{j_q}$$

which is a topologically disjoint union and each $F_i \times \prod_{q=2}^r G_{j_q}$ is itself a topologically disjoint union of sets $Q_L \times \prod_{q=2}^r G_{j_q}$ with $|L|$ constant. As already mentioned, all the subsets are saturated with respect to the diagonal antipodal relation. Passing to the quotient we get a tower of open subsets of $P_{\bar{n}}$

$$\emptyset = \tilde{W}_{-1} \subset \tilde{W}_0 \subset \dots \subset \tilde{W}_{n_1+r-1} = P_{\bar{n}}$$

and each set

$$\tilde{W}_s - \tilde{W}_{s-1} = \bigcup_{i+\sum_{q=2}^r j_q=s} \frac{F_i \times \prod_{q=2}^r G_{j_q}}{\sim}$$

is a topologically disjoint union. As $(F_i \times \prod_{q=2}^r G_{j_q})/\sim$ is itself a topologically disjoint union of $(Q_L \times \prod_{q=2}^r G_{j_q})/\sim$ and as these spaces are contractible in the path-connected space $P_{\bar{n}}$, we can conclude that each $\tilde{W}_s - \tilde{W}_{s-1}$ is contractible in $P_{\bar{n}}$. Therefore by Proposition 2.4, we obtain that $\text{cat}(P_{\bar{n}}) \leq n_1 + r - 1$. By Proposition 2.3, we conclude that $\text{cat}(P_{\bar{n}}) = n_1 + r - 1$. \square

3. TOPOLOGICAL COMPLEXITY OF $P_{\bar{n}}$

Definition 3.1. Let X be a finite, path-connected CW complex. The (normalized) topological complexity of X , $\text{TC}(X)$, is the least integer k such that there exists a cover of $X \times X$ by $k + 1$ open subsets $U_0, U_1, \dots, U_k \subset X \times X$ on each of which the fibration

$$ev_{0,1} : X^I \rightarrow X \times X, \gamma \mapsto (\gamma(0), \gamma(1)),$$

admits a continuous section.

In order to establish Theorem 1.3, we will use the following characterization:

Proposition 3.2. [7, Proposition 4.12] *Let X be a finite, path-connected CW complex. We have $\text{TC}(X) \leq k$ if and only if there exists $s : X \times X \rightarrow X^I$ and an increasing sequence of open sets*

$$\emptyset = U_{-1} \subset U_0 \subset U_1 \subset \dots \subset U_k = X \times X$$

such that $ev_{0,1} \circ s = \text{id}$ and, for any $i \in \{0, \dots, k\}$, s is continuous on $U_i - U_{i-1}$.

Remark 3.3. The sets $F_i = U_i - U_{i-1}$ provide a cover of $X \times X$ by $k + 1$ disjoint subsets on each of which there is a continuous section of $ev_{0,1}$. This defines a motion planner for X .

Proof of Theorem 1.3. Through the obvious homeomorphism we think of $P_{\bar{n}} \times P_{\bar{n}} = S_{\bar{n}} \times S_{\bar{n}}/\sim$ as the quotient of $(S^{n_1} \times S^{n_1}) \times \dots \times (S^{n_r} \times S^{n_r})$ with respect to the relation

$$(3.1) \quad (x_1, y_1, \dots, x_r, y_r) \sim (x'_1, y'_1, \dots, x'_r, y'_r) \Leftrightarrow \begin{cases} \forall i & x_i = x'_i \text{ and } y_i = y'_i \\ \text{or } \forall i & x_i = -x'_i \text{ and } y_i = y'_i \\ \text{or } \forall i & x_i = x'_i \text{ and } y_i = -y'_i \\ \text{or } \forall i & x_i = -x'_i \text{ and } y_i = -y'_i \end{cases}$$

We first recall from [8] and [6] the construction of motion planners for the real projective space P^{n_1} and for a sphere S^{n_q} . We will next see how to assemble them in order to obtain a motion planner for $P_{\bar{n}}$.

Suppose that $\text{TC}(P^{n_1}) = k$. As proven in [8], there exists a non-singular map $f = (f_0, \dots, f_k) : \mathbb{R}^{n_1+1} \times \mathbb{R}^{n_1+1} \rightarrow \mathbb{R}^{k+1}$. The $k+1$ scalar maps $f_0, f_1, \dots, f_k : \mathbb{R}^{n_1+1} \times \mathbb{R}^{n_1+1} \rightarrow \mathbb{R}$ satisfy $f_i(ax_1, by_1) = abf_i(x_1, y_1)$ for $(x_1, y_1) \in S^{n_1} \times S^{n_1}$ and $a, b \in \mathbb{R}$ and do not vanish all at the same time (except in $(0, 0)$). Moreover, according to [8, Lemma 11], we can assume that $f_0(x_1, x_1) > 0$ for any $x_1 \in S^{n_1}$.

For a subset $L \subset \{0, 1, \dots, k\}$, let

$$S_L = \{(x_1, y_1) \in S^{n_1} \times S^{n_1} : f_l(x_1, y_1) \neq 0 \text{ if } l \in L\}.$$

By setting

$$U_i = \bigcup_{|L|=(k+1)-i} S_L$$

for $0 \leq i \leq k$ and $U_{-1} = \emptyset$, we have an increasing sequence of open subsets of $S^{n_1} \times S^{n_1}$:

$$\emptyset = U_{-1} \subset U_0 \subset \dots \subset U_k = S^{n_1} \times S^{n_1}.$$

Note that $U_k = S^{n_1} \times S^{n_1}$ since the $k+1$ scalar functions f_0, \dots, f_k cannot vanish all at the same time. Considering, for $L \subset \{0, 1, \dots, k\}$,

$$Q_L = \{(x_1, y_1) \in S^{n_1} \times S^{n_1} : f_l(x_1, y_1) \neq 0 \text{ if } l \in L \text{ and } f_l(x_1, y_1) = 0 \text{ if } l \notin L\}$$

we have, for $0 \leq i \leq k$,

$$F_i := U_i - U_{i-1} = \bigcup_{|L|=(k+1)-i} Q_L.$$

Note that all the sets S_L, Q_L, U_i, F_i are saturated with respect to the equivalence relation on $S^{n_1} \times S^{n_1}$ induced by the antipodal relation $x_1 \sim -x_1$ on S^{n_1} . Note also that, for $L_1 \neq L_2$ with same cardinality $|L_1| = |L_2|$, the sets S_{L_1} and S_{L_2} are topologically disjoint and that $\bigcup_{i=0}^k F_i$ is a cover of $S^{n_1} \times S^{n_1}$.

As before, for non-antipodal points A, B ($A \neq -B$) of a sphere S^m , we denote by $\lambda(A, B) : I \rightarrow S^m$ the geodesic path from A, B . Recall that $\lambda(-A, -B) = -\lambda(A, B)$ and that $\lambda(A, A)$ is the constant path.

For $L \subset \{0, 1, \dots, k\}$, we consider the map $\psi_L : Q_L \rightarrow (P^{n_1})^I$

$$\psi_L(x_1, y_1) = \begin{cases} [\lambda(x_1, y_1)], & \text{if } f_{l_0}(x_1, y_1) > 0 \\ [\lambda(-x_1, y_1)], & \text{if } f_{l_0}(x_1, y_1) < 0 \end{cases}$$

where $l_0 = \min L$. Recall that, for any $x_1 \in S^{n_1}$, $f_0(x_1, x_1) = f_0(-x_1, -x_1) > 0$ and, consequently $f_0(x_1, -x_1) = f_0(-x_1, x_1) < 0$. Therefore, if $(\pm x_1, \pm x_1) \in Q_L$, then $l_0 = \min L = 0$. This ensures that ψ_L is well-defined on pairs of antipodal points. The map is continuous on Q_L and satisfies $\psi_L(x_1, y_1) = \psi_L(\pm x_1, \pm y_1)$. As the distinct sets Q_L with $|L|$ constant are topologically disjoint, we obtain, for $0 \leq i \leq k$, a continuous map $\psi_i : F_i \rightarrow (P^{n_1})^I$ by setting $\psi_i|_{Q_L} = \psi_L$. This map satisfies $\psi_i(x_1, y_1) = \psi_i(\pm x_1, \pm y_1)$ and the induced map $\bar{\psi}_i : F_i / \sim \rightarrow (P^{n_1})^I$ gives us an

explicit motion planner on F_i / \sim . Taken all together, the maps $\bar{\psi}_i$ provide a complete motion planner on P^{n_1} which is essentially the one described in [8].

For $2 \leq q \leq r$, we will use the following increasing sequences of open subsets of $S^{n_q} \times S^{n_q}$ together with the associated complements.

When n_q is odd, we consider:

$$\begin{aligned} V_{-1} &= \emptyset \\ V_0 &= \{(x_q, y_q) \in S^{n_q} \times S^{n_q} : y_q \neq \pm x_q\} \\ V_1 &= S^{n_q} \times S^{n_q} \end{aligned}$$

and

$$\begin{aligned} G_0 &= V_0 - V_{-1} = \{(x_q, y_q) \in S^{n_q} \times S^{n_q} : y_q \neq \pm x_q\} \\ G_1 &= V_1 - V_0 = \{(x_q, y_q) \in S^{n_q} \times S^{n_q} : y_q = \pm x_q\}. \end{aligned}$$

When n_q is even, we consider:

$$\begin{aligned} V_{-1} &= \emptyset \\ V_0 &= \{(x_q, y_q) \in S^{n_q} \times S^{n_q} : y_q \neq \pm x_q\} \\ V_1 &= S^{n_q} \times S^{n_q} \setminus \{(x_q, y_q) \in S^{n_q} \times S^{n_q} : y_q = \pm x_q \text{ and } x_q = \pm A_{n_q}\} \\ V_2 &= S^{n_q} \times S^{n_q} \end{aligned}$$

and

$$\begin{aligned} G_0 &= V_0 - V_{-1} = \{(x_q, y_q) \in S^{n_q} \times S^{n_q} : y_q \neq \pm x_q\} \\ G_1 &= V_1 - V_0 = \{(x_q, y_q) \in S^{n_q} \times S^{n_q} : y_q = \pm x_q, x_q \neq \pm A_{n_q}\} \\ G_2 &= V_2 - V_1 = \{(x_q, y_q) \in S^{n_q} \times S^{n_q} : y_q = \pm x_q, x_q = \pm A_{n_q}\}. \end{aligned}$$

Here, as before, $A_{n_q} = (0, \dots, 0, 1) \in S^{n_q}$.

Recall that the classical motion planner for a sphere can be given as follows:

- For $(x_q, y_q) \in G_0$ and for $(x_q, x_q) \in G_1 \cup G_2$, we consider the geodesic path $\lambda(x_q, y_q)$.
- For $(x_q, -x_q) \in G_1$, we consider the geodesic meridian $\mu(x_q, -x_q)$ from x_q to $-x_q$ in the direction of $\chi(x_q)$. Here χ is the symmetric tangent vector field on $S^{n_q} \subset \mathbb{R}^{n_q+1}$ given by $\chi(u_1, v_1, \dots, u_m, v_m) = (-v_1, u_1, \dots, -v_m, u_m)$ if $n_q = 2m - 1$ is odd and by $\chi(u_1, v_1, \dots, u_m, v_m, u_{m+1}) = (-v_1, u_1, \dots, -v_m, u_m, 0)$ if $n_q = 2m$ is even. Note that $\mu(-x_q, x_q) = -\mu(x_q, -x_q)$.
- For $(A_{n_q}, -A_{n_q})$, we fix a meridian $\mu_0(A_{n_q}, -A_{n_q})$ from A_{n_q} to $-A_{n_q}$ and we set $\mu_0(-A_{n_q}, A_{n_q}) = -\mu_0(A_{n_q}, -A_{n_q})$.

We first assemble these motion planners in the following way.

Let $L \subset \{0, 1, \dots, k\}$ with $l_0 = \min L$ and, for $2 \leq q \leq r$, let $j_q \in \{0, 1\}$ when n_q is odd or $j_q \in \{0, 1, 2\}$ when n_q is even. We define

$$\psi_{(L, j_2, \dots, j_r)} : Q_L \times \prod_{q=2}^r G_{j_q} \rightarrow P_{\bar{n}}^I$$

by

$$\psi_{(L, j_2, \dots, j_r)}(x_1, y_1, \dots, x_r, y_r) = \begin{cases} [\lambda(x_1, y_1), \nu_2(x_2, y_2), \dots, \nu_r(x_r, y_r)], & f_{l_0}(x_1, y_1) > 0 \\ [\lambda(-x_1, y_1), \eta_2(x_2, y_2), \dots, \eta_r(x_r, y_r)], & f_{l_0}(x_1, y_1) < 0 \end{cases}$$

where, for n_q odd,

$$\nu_q(x_q, y_q) = \begin{cases} \mu(x_q, y_q), & \text{if } y_q = -x_q \\ \lambda(x_q, y_q), & \text{otherwise,} \end{cases}$$

$$\eta_q(x_q, y_q) = \begin{cases} \mu(-x_q, y_q), & \text{if } y_q = x_q \\ \lambda(-x_q, y_q), & \text{otherwise,} \end{cases}$$

and, for n_q even,

$$\nu_q(x_q, y_q) = \begin{cases} \mu(x_q, y_q), & \text{if } y_q = -x_q, x_q \neq \pm A_{n_q} \\ \mu_0(x_q, y_q), & \text{if } y_q = -x_q, x_q = \pm A_{n_q} \\ \lambda(x_q, y_q), & \text{otherwise,} \end{cases}$$

$$\eta_q(x_q, y_q) = \begin{cases} \mu(-x_q, y_q), & \text{if } y_q = x_q, x_q \neq \pm A_{n_q} \\ \mu_0(-x_q, y_q), & \text{if } y_q = x_q, x_q = \pm A_{n_q} \\ \lambda(-x_q, y_q), & \text{otherwise.} \end{cases}$$

This map is continuous and well-defined on $Q_L \times \prod_{q=2}^r G_{j_q}$. Moreover, this space is saturated with respect to the relation (3.1) and we can check that the map is compatible with this relation. This follows from the fact that, for $2 \leq q \leq r$ and $(x_q, y_q) \in S^{n_q} \times S^{n_q}$, one has the following equalities:

$$\begin{aligned} \nu_q(-x_q, -y_q) &= -\nu_q(x_q, y_q), & \eta_q(-x_q, -y_q) &= -\eta_q(x_q, y_q), \\ \eta_q(-x_q, y_q) &= \nu_q(x_q, y_q), & \eta_q(x_q, -y_q) &= -\nu_q(x_q, y_q). \end{aligned}$$

For $i \in \{0, \dots, k\}$, we now define

$$\psi_{(i, j_2, \dots, j_r)} : F_i \times \prod_{q=2}^r G_{j_q} \rightarrow P_{\bar{n}}^I$$

by setting $\psi_{(i, j_2, \dots, j_r)}|_{Q_L \times \prod_{q=2}^r G_{j_q}} = \psi_{(L, j_2, \dots, j_r)}$. Passing to the quotient, we get a continuous map

$$\tilde{\psi}_{(i, j_2, \dots, j_r)} : (F_i \times \prod_{q=2}^r G_{j_q}) / \sim \rightarrow P_{\bar{n}}^I$$

which is a motion planner on the set $(F_i \times \prod_{q=2}^r G_{j_q}) / \sim$.

For $s = 0, \dots, k + \sum_{q=2}^r \text{TC}(S^{n_q}) = \text{TC}(P^{n_1}) + \sum_{q=2}^r \text{TC}(S^{n_q})$, we built

$$W_s = \bigcup_{i + \sum_{q=2}^r j_q = s} U_i \times \prod_{q=2}^r V_{j_q} \subset (S^{n_1} \times S^{n_1}) \times (S^{n_2} \times S^{n_2}) \times \dots \times (S^{n_r} \times S^{n_r})$$

where $i \in \{0, 1, \dots, k\}$, $j_q \in \{0, 1\}$ when n_q is odd and $j_q \in \{0, 1, 2\}$ when n_q is even ($2 \leq q \leq r$). Therefore, there exists a tower of open subsets

$$\emptyset = W_{-1} \subset W_0 \subset \dots \subset W_{k+\sum_{q=2}^r \text{TC}(S^{n_q})} = (S^{n_1} \times S^{n_1}) \times (S^{n_2} \times S^{n_2}) \times \dots \times (S^{n_r} \times S^{n_r}).$$

We have

$$W_s - W_{s-1} = \bigcup_{i+\sum_{q=2}^r j_q=s} (U_i - U_{i-1}) \times \prod_{q=2}^r (V_{j_q} - V_{j_q-1}) = \bigcup_{i+\sum_{q=2}^r j_q=s} F_i \times \prod_{q=2}^r G_{j_q}$$

which is a topologically disjoint union. As already mentioned, all the subsets are saturated subsets of $S^{n_1} \times S^{n_1} \times S^{n_2} \times S^{n_2} \times \dots \times S^{n_r} \times S^{n_r} \cong S_{\bar{n}} \times S_{\bar{n}}$ with respect to the relation (3.1). In particular, the tower of W_s provides the following tower of open subsets by passing to the quotient space

$$\emptyset = \tilde{W}_{-1} \subset \tilde{W}_0 \subset \dots \subset \tilde{W}_{k+\sum_{q=2}^r \text{TC}(S^{n_q})} \cong P_{\bar{n}} \times P_{\bar{n}}$$

and, for $s = 0, \dots, k + \sum_{q=2}^r \text{TC}(S^{n_q})$, the set

$$\tilde{W}_s - \tilde{W}_{s-1} = \bigcup_{i+\sum_{q=2}^r j_q=s} \frac{F_i \times \prod_{q=2}^r G_{j_q}}{\sim}$$

is a topologically disjoint union. Assembling the maps $\tilde{\psi}_{(i, j_2, \dots, j_r)}$ we get a continuous motion planner on $\tilde{W}_s - \tilde{W}_{s-1}$ and, by Proposition 3.2, we can conclude that

$$\text{TC}(P_{\bar{n}}) \leq k + \sum_{q=2}^r \text{TC}(S^{n_q}) = \text{TC}(P^{n_1}) + \sum_{q=2}^r \text{TC}(S^{n_q}).$$

□

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