Remarks on the Vietoris Sequence and Corresponding Convolution Formulas

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Abstract

In this paper we consider the so-called Vietoris sequence, a sequence of rational numbers of the form $c_k = \frac{1}{2^k}\binom{k}{\lfloor \frac{k}{2} \rfloor}$, $k = 0, 1, \dots$. This sequence plays an important role in many applications and has received a lot of attention over the years. In this work we present the main properties of the Vietoris sequence, having in mind its role in the context of hypercomplex function theory. Properties and patterns of the convolution triangles associated with $(c_k)_k$ are also presented.

1 The Vietoris Sequence

For our purpose here, we define the Vietoris sequence $(c_k)_k$ in terms of the "complete central binomial coefficient" as

$$c_k := \frac{1}{2^k}\binom{k}{\lfloor \frac{k}{2} \rfloor}, \quad k = 0, 1, \dots, \tag{1}$$

where $\lfloor \cdot \rfloor$ is the floor function. The first terms of this sequence are

$$1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{35}{128}, \frac{35}{128}, \dots .$$

Some years ago, authors of this paper noticed that the sequence (1) appeared in the construction of sequences of multivariate generalized Appell polynomials [11, 22]. Since then, several studies on this sequence have been carried out (see e.g. [6, 8, 9, 10] and the references therein) and the importance of this sequence in hypercomplex context is unquestionable nowadays. For this reason, we thought it would be interesting to collect the properties that have been obtained over the years, presenting them in a unifying way.

It is worth mentioning that a similar sequence appears in the work [25] of Vietoris in the context of positive trigonometric sums. In fact, in his pioneer work, Vietoris considered the sequence $(a_k)_k$ defined as

$$a_{2k} = a_{2k+1} = \frac{1}{4^k}\binom{2k}{k}, \quad k = 0, 1, \dots . \tag{2}$$

It follows at once that (1) is a subsequence of (2), since $c_k = a_{k+1}$; $k = 0, 1, \ldots$. Despite this small difference, we have coined (1) as the Vietoris number sequence.

This paper is organized as follows: in Sect. 2 we present equivalent definitions of the Vietoris sequence, while in Sect. 3 we list other important properties of $(c_k)_k$. Finally, the paper ends with some new results on the convolution triangles associated with $(c_k)_k$ and $(c_{2k})_k$.

2 Alternative Definitions

One can find in the literature several ways of writing the Vietoris sequence. Some of these representations were obtained independently and using context-dependent arguments in the framework of hypercomplex analysis. In other cases, the alternative definitions come from very well-known identities. In this section we list several ways of defining the Vietoris sequence, being in most of the cases, trivial to prove the equivalence of the definitions. For each case, we also include in the Appendix A, the Wolfram Mathematica code for defining the sequence.

2.1 Representation in Terms of the Generators of $\mathbb{H}$

Let $\{e_1, e_2\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^2$, with a product according to the multiplication rules

$$e_1 e_2 = -e_2 e_1 \qquad \text{and} \qquad e_1^2 = e_2^2 = -1.$$

This non-commutative product generates the well-known algebra of real quaternions $\mathbb{H}$ (with the identification $\mathbf{i} := e_1$, $\mathbf{j} := e_2$ and $\mathbf{k} := e_1 e_2$). The Vietoris sequence has the following representation in terms of e_1 and e_2:

$$c_k = \left[\sum_{s=0}^{k} (-1)^s \binom{k}{s} \left(e_1^{k-s} \times e_2^s \right)^2 \right]^{-1} \tag{3}$$

where the so-called symmetric powers with respect to $\times$ are defined recursively as (see e.g. [20]):

$$\begin{aligned} e_1^m \times e_2^n &:= \underbrace{e_1 \times \cdots \times e_1}_{m} \times \underbrace{e_2 \times \cdots \times e_2}_{n} \\ &= \frac{1}{m+n} [m e_1 ({e_1}^{m-1} \times {e_2}^{n}) + n e_2 ({e_1}^{m} \times {e_2}^{n-1})], \ m, n \in \mathbb{N}, \end{aligned}$$

and for $m = 0$ or $n = 0$, the powers are understood in the ordinary way.

For an algebraic proof of the equivalence of the representations (1) and (3), we refer to [15] (see also [14]). For more details on the properties of the symmetric powers, we mention [20].

2.2 Double Factorial Representation

The Vietoris sequence can also be written in terms of the double factorial [12, 23]. In fact, using the well-known relations

$$(2k)!! = 2^k k! \quad \text{and} \quad (2k+1)!! = \frac{(2k+1)!}{2^k k!},$$

the relation (1) can be written as

$$c_k = \frac{(2\lfloor \frac{k-1}{2} \rfloor + 1)!!}{(2\lfloor \frac{k-1}{2} \rfloor + 2)!!}. \tag{4}$$

2.3 Recursive Definition

It follows immediately from (4) that c_k can be defined recursively as

$$c_k = \begin{cases} \frac{k}{k+1} c_{k-1}, & \text{if } k \text{ odd} \\ c_{k-1}, & \text{if } k > 1 \text{ even} \\ 1 & \text{if } k = 0 \end{cases}.$$

2.4 Pochhammer Symbol Representation

In the works [5, 11], the elements of the sequence $(c_k)_k$ were obtained through

$$c_k = \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}{(1)_{\lfloor \frac{k+1}{2} \rfloor}},$$

where $(x)_n$ denotes the Pochhammer symbol,

$$(x)_n := x(x+1)\ldots(x+n-1),\ n = 1, 2, \ldots; (x)_0 := 1.$$

This is an immediate consequence of the fact that

$$\left(\tfrac{1}{2}\right)_k = \frac{(2k-1)!!}{2^k} \quad \text{and} \quad (1)_k = k!.$$

2.5 Alternating Sum of a Non-Symmetric Triangle

In the work [12], several arithmetic properties of the triangle

$$T_s^k := \frac{1}{k+1} \frac{\left(\frac{3}{2}\right)_{k-s} \left(\frac{1}{2}\right)_s}{(k-s)!s!}, \quad k = 0, 1, \ldots, s = 0, 1, \ldots, k. \tag{5}$$

were derived. One of these properties allows to express the elements c_k of the Vietoris sequence in terms of the alternating row sum of T_s^k:

$$c_k = \sum_{s=0}^{k} (-1)^s T_s^k.$$

2.6 Gamma Function Representation

Recalling the well known identities

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \quad \Gamma(n+1) = n!, \quad \text{and} \quad \Gamma(n + \tfrac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{2^{2n} n!},$$

we can write

$$c_k = \frac{\Gamma(\frac{1}{2} + \lfloor \frac{k+1}{2} \rfloor)}{\Gamma(\frac{1}{2})\Gamma(1 + \lfloor \frac{k+1}{2} \rfloor)} = \frac{\Gamma(\frac{1}{2} + \lfloor \frac{k+1}{2} \rfloor)}{\sqrt{\pi} \lfloor \frac{k+1}{2} \rfloor !}. \tag{6}$$

2.7 Integral Representation

The Wallis integrals are the terms of the sequence $(\mathcal{I}_k)_k$ defined by

$$\mathcal{I}_k = \int_0^{\frac{\pi}{2}} \cos^k x \, dx,$$

which can be evaluated by using the well-known identity

$$\mathcal{I}_k = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{k}{2} + 1)}.$$

Having in mind the Gamma-representation (6) of c_k and the properties of the Gamma function, we obtain easily

$$c_k = \frac{2}{\pi} \, \mathcal{I}_{2\lfloor \frac{k+1}{2} \rfloor}.$$

2.8 Representation in Terms of Catalan Numbers

The popular Catalan numbers,

$$\mathcal{C}_k = \frac{1}{k+1}\binom{2k}{k};\ k = 0, 1, \ldots,$$

appear in a number of binomial identities. For our purpose here we highlight the following one [1],

$$\mathcal{C}_k = \sum_{s=0}^{k}(-1)^s\binom{k}{s}2^{k-s}\binom{s}{\lfloor\frac{s}{2}\rfloor} = (-2)^k\sum_{s=0}^{k}(-1)^{k-s}\binom{k}{s}c_s,$$

from where it is trivial to obtain the relation

$$c_k = \sum_{s=0}^{k}\binom{k}{s}(-2)^{-s}\mathcal{C}_s,$$

since $\left(\frac{\mathcal{C}_k}{(-2)^k}\right)_k$ is the binomial transform of $(c_k)_k$.

2.9 Representation in Terms of Values of Legendre Polynomials

In the hypercomplex context, the following special holomorphic polynomial of degree k involving the Vietoris coefficients (or their generalizations) have been constructed and its properties have been studied in a series of papers by authors of this work ([4, 5, 6, 11, 14, 22]),

$$\mathbf{P}_k(x_0 + x_1e_1 + x_2e_2) = \sum_{s=0}^{k} c_s\binom{k}{s}x_0^{k-s}(x_1e_1 + x_2e_2)^s, \tag{7}$$

where e_1 and e_2 are the generators of $\mathbb{H}$. We adopt the following notation in what follows: if $x = x_0 + x_1e_1 + x_2e_2 \in \mathbb{H}$, then $|x|^2 = x_0^2 + x_1^2 + x_2^2$, $\mathrm{Re}(x)$ is the real part x_0 of x, $\underline{x}$ designates the vector part of x, i.e., $\underline{x} = x_1e_1 + x_2e_2$ and $\boldsymbol{\omega} := \omega(\underline{x}) = \frac{\underline{x}}{|\underline{x}|}$. Since $\boldsymbol{\omega}^2 = -1$, the right-hand side of (7) can be written as

$$\sum_{s=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^s\binom{k}{2s}c_{2s}x_0^{k-2s}|\underline{x}|^{2s} + \boldsymbol{\omega}\sum_{s=0}^{\lfloor\frac{k-1}{2}\rfloor}(-1)^s\binom{k}{2s+1}c_{2s+1}x_0^{k-2s-1}|\underline{x}|^{2s+1}. \tag{8}$$

This means that the polynomials (7) can easily be rewritten in terms of the real variable $t = \frac{x_0}{|x|}$ as

$$\mathbf{P}_k(t, |x|) = |x|^k\left(f_k(t) + \omega(\underline{x})g_k(t)\right), \tag{9}$$

where $t \in [-1, 1]$ and $f_k(t)$ and $g_k(t)$ are the real functions

$$f_k(t) = \sum_{s=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2s}c_{2s}t^{k-2s}(t^2-1)^s$$

and

$$g_k(t) = \sqrt{1-t^2}\sum_{s=1}^{\lfloor\frac{k+1}{2}\rfloor}\binom{k}{2s-1}c_{2s-1}t^{k-2s+1}(t^2-1)^{s-1}.$$

For more details on the representation (9), we refer to [4]. If we recall the Legendre polynomials of degree k written in the form (see e.g. [24])

$$\mathcal{P}_k(t) = t^k\sum_{s=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2s}\binom{2s}{s}\left(\frac{t^2-1}{4t^2}\right)^s, \tag{10}$$

it is easy to establish a relation between (10) and the real polynomial $f_k(t)$ in (9), concluding in this way that

$$\mathrm{Re}(\mathbf{P}_k(t,|x|)) = |x|^k \mathcal{P}_k(t).$$

For the special case of $x_0 = 0$ and $|x| = 1$, this last relation provides the following form of representing c_k:

$$c_k = (-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor} \mathcal{P}_{2\left\lfloor \frac{k+1}{2} \right\rfloor}(0). \tag{11}$$

We point out that relation (11) can also be obtained directly by using the following well-known property of the Legendre polynomials:

$$\mathcal{P}_n(0) = \begin{cases} \frac{(-1)^m}{4^m}\binom{2m}{m}, & \text{for } n = 2m \\ 0, & \text{for } n = 2m+1 \end{cases}.$$

2.10 Representation in Terms of Values of the Derivatives of the Bessel functions

Consider the Bessel functions of the first kind

$$J_0(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k}(k!)^2} \quad \text{and} \quad J_1(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{2^{2k+1}k!(k+1)!}.$$

Differentiating m-times both Bessel functions we get

$$J_0^{(m)}(z) = \sum_{k=m}^{\infty} (-1)^k \frac{(2k)(2k-1)\cdots(2k-(m-1))z^{2k-m}}{2^{2k}(k!)^2}$$

and

$$J_1^{(m)}(z) = \sum_{k=m}^{\infty} (-1)^k \frac{(2k+1)(2k)\cdots(2k+1-(m-1))z^{2k+1-m}}{2^{2k+1}k!(k+1)!}.$$

Therefore

$$J_0^{(m)}(0) = \begin{cases} (-1)^{\frac{m}{2}} c_m, & \text{if } m \text{ even} \\ 0, & \text{if } m \text{ odd} \end{cases} \quad \text{and} \quad J_1^{(m)}(0) = \begin{cases} 0, & \text{if } m \text{ even} \\ (-1)^{\frac{m-1}{2}} c_m, & \text{if } m \text{ odd} \end{cases}.$$

These two last equalities, allows to write

$$c_k = (-1)^{\lfloor \frac{k}{2} \rfloor} J^{(k)}_{\frac{1-(-1)^k}{2}}(0),$$

which, in turn, leads together with (8) to the following representation of the hypercomplex polynomial (7)

$$\mathbf{P}_k(x_0 + \boldsymbol{\omega}|\underline{x}|) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} |\underline{x}|^s \left(J_0^{(s)}(0) + \omega(\underline{x}) J_1^{(s)}(0) \right).$$

For more details on this subject we mention the works [11, 23].

3 Other Properties of $(c_k)_k$

Without being exhaustive, we list now several properties of the Vietoris sequence that have been proved in the last years.

1. **Alternating series** [8] $\quad \displaystyle\sum_{k=0}^{+\infty} (-1)^k c_k = 1.$

2. **Combinatorial Identity** [21] $\sum_{s=0}^{k}(-1)^{k-s}\binom{-\frac{3}{2}}{k-s}\binom{-\frac{1}{2}}{s} = (k+1)c_k.$

3. **Trigonometric Identities** [6] For $0 < \alpha < \pi$,

$$\sum_{k=0}^{\infty} c_k \cos^k \alpha = \frac{2}{1-\cos\alpha+\sin\alpha} \quad \text{and} \quad \sum_{k=0}^{\infty} c_k \sin^k \alpha = \frac{2}{1-\sin\alpha+\cos\alpha}.$$

4. **Recurrence Relations** [7, 8]

$(k+2)c_{k+1} = c_k + kc_{k-1}, k \geq 1, \quad c_0 = 1, c_1 = \frac{1}{2}$

and

$\Delta c_k = -\frac{1}{2}\sum_{s=0}^{k} c_{k-s}\Delta c_{s-1},$

where $\Delta c_k := c_{k+1} - c_k$ denotes the forward difference with $\Delta c_{-1} := 1$.

5. **Generating Functions**

In [6], an elementary procedure, based on the expansion of the binomial function $(1-t^2)^l$ was used to derive the following generating function of the sequence $(c_k)_k$:

$$F(t) = \frac{\sqrt{1+t}-\sqrt{1-t}}{t\sqrt{1-t}}. \tag{12}$$

Similarly, it can be proved that the even order-terms sequence $(c_{2k})_k$ is generated by the function

$$f(t) = \frac{1}{\sqrt{1-t}}.$$

More recently [7], an exponential generating function of the sequence $(c_k)_k$ was obtained by using methods of the calculus of holonomic differential equations, namely

$$\boldsymbol{F}(t) = I_0(t) + I_1(t),$$

where I_0 and I_1 are the modified Bessel functions of the first kind. This function is closely related to the hypercomplex exponential function that has been studied in the past,

$$\text{Exp}(x_0 + \boldsymbol{\omega}|\underline{x}|) = e^{x_0}\left(J_0(|\underline{x}|) + \omega(\underline{x})J_1(|\underline{x}|)\right).$$

Finally, we mention that the even order-terms sequence $(c_{2k})_k$ has as exponential generating function

$$\boldsymbol{f}(t) = e^{\frac{t}{2}} I_0\left(\tfrac{t}{2}\right).$$

6. **Relation to the central binomial coefficients sequence**

The well-known properties of the central binomial coefficients (CBC) and their relation to the even-order terms of the Vietoris sequence c_{2k}, allows to write down immediately a number of useful relations. In the literature there are a lot of combinatorial identities involving the CBC (see e.g. [19] and the unpublished manuscripts of Gould [16, 17]).

In https://w3.math.uminho.pt/VietorisSequence a Mathematica notebook containing the proofs of several well-known properties written in terms of the sequence $(c_{2k})_k$ was made available.

4 Vietoris Convolution Triangles

We recall that the k^{th} convolution of the sequence $(a_n)_n$ is the sequence $(a_n^{(k)})_n$, defined recursively as

$$\begin{aligned} a_n^{(k)} &= \sum_{s=0}^{n} a_s a_{n-s}^{(k-1)}, \\ a_n^{(0)} &= a_n, \end{aligned}$$

(see e.g. [3, 18]). As it is well known, the convolution of sequences corresponds to the multiplication of their generating functions, i.e., if $F(t)$ is the generating function of the sequence $(a_n)_n$, then the generating function $F_k(t)$ of the k^{th} convolution of the sequence $(a_n)_n$ is $(F(t))^{k+1}$.

For example, in the case of the Vietoris sequence and taking into account (12) we obtain

$$\begin{aligned} F_0(t) =& \frac{\sqrt{t+1}-\sqrt{1-t}}{t\sqrt{1-t}} \\ =& 1+\frac{t}{2}+\frac{t^2}{2}+\frac{3t^3}{8}+\frac{3t^4}{8}+\frac{5t^5}{16}+\frac{5t^6}{16}+\frac{35t^7}{128}+\frac{35t^8}{128}+\dots \\ F_1(t) =& \frac{2\left(\sqrt{1-t^2}-1\right)}{(t-1)t^2} \\ =& 1+t+\frac{5t^2}{4}+\frac{5t^3}{4}+\frac{11t^4}{8}+\frac{11t^5}{8}+\frac{93t^6}{64}+\frac{93t^7}{64}+\frac{193t^8}{128}+\dots \\ F_2(t) =& \frac{\left(\sqrt{t+1}-\sqrt{1-t}\right)^3}{(1-t)^{3/2}t^3} \\ =& 1+\frac{3t}{2}+\frac{9t^2}{4}+\frac{11t^3}{4}+\frac{27t^4}{8}+\frac{123t^5}{32}+\frac{281t^6}{64}+\frac{309t^7}{64}+\frac{681t^8}{128}+\dots \end{aligned}$$

The convolution triangle, written in rectangular form, of the sequence $(a_n)_n$ is an array whose k^{th} column is the sequence $(a_n^{(k-1)})_n$, $k=1,2,\dots$ (see e.g. [3, 18]).

Tables 1-2 show the convolution triangles $\mathcal{T}_V$ and $\mathcal{T}_{EV}$ corresponding to the Vietoris sequence $(c_n)_n$ and the even order-term sequence $(c_{2n})_n$. Tables 3-4 present the triangles $\mathcal{T}_{CC}$ and $\mathcal{T}_C$ associated to the related sequences

$$u_n = \binom{n}{\lfloor \frac{n}{2} \rfloor} \quad \text{and} \quad v_n = \binom{2n}{n} = u_{2n} \tag{13}$$

of the complete central binomial coefficients and of the central binomial coefficients.

For the sake of better visibility, we also write on the right side of each table, the corresponding left justified triangle. The Mathematica code to produce the tables is presented in the Appendix B.

Denote by $\mathcal{M}_V=(m_{ij})$, $\mathcal{M}_{EV}=(n_{ij})$, $\mathcal{M}_C=(\tilde{n}_{ij})$ and $\mathcal{M}_{CC}=(\tilde{m}_{ij})$ the $r\times r$ matrices formed by using as elements the first r rows of the triangles $\mathcal{T}_V$, $\mathcal{T}_{EV}$, $\mathcal{T}_C$ and $\mathcal{T}_{CC}$ in rectangular form, respectively. It is easy to see that

$$\tilde{m}_{ij} = 2^{i-1} m_{ij} \quad \text{and} \quad \tilde{n}_{ij} = 4^{i-1} n_{ij};\ i=1,2,\dots,r, \tag{14}$$

since $c_n^{(k)} = \frac{1}{2^n} u_n^{(k)}$ and $c_{2n}^{(k)} = \frac{1}{4^n} v_n^{(k)}$, $k=0,1,\dots$. In other words,

$$\mathcal{M}_{CC} = \begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 2^2 & & \\ & & & \ddots & \\ & & & & 2^{r-1} \end{pmatrix} \mathcal{M}_V \quad \text{and} \quad \mathcal{M}_C = \begin{pmatrix} 1 & & & & \\ & 4 & & & \\ & & 4^2 & & \\ & & & \ddots & \\ & & & & 4^{r-1} \end{pmatrix} \mathcal{M}_{EV}.$$

Therefore

$$\det \mathcal{M}_{CC} = \prod_{i=1}^{r} 2^{i-1} \det \mathcal{M}_V = 2^{\frac{r(r-1)}{2}} \det \mathcal{M}_V$$

Table 1: The Vietoris convolution triangle $\mathcal{T}_V$

1	1	1	1	1	1	1	$\dots$	1							
$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	$\dots$	$\frac{1}{2}$	1						
$\frac{1}{2}$	$\frac{5}{4}$	$\frac{9}{4}$	$\frac{7}{2}$	5	$\frac{27}{4}$	$\frac{35}{4}$	$\dots$	$\frac{1}{2}$	1	1					
$\frac{3}{8}$	$\frac{5}{4}$	$\frac{11}{4}$	5	$\frac{65}{8}$	$\frac{49}{4}$	$\frac{35}{2}$	$\dots$	$\frac{3}{8}$	$\frac{5}{4}$	$\frac{3}{2}$	1				
$\frac{3}{8}$	$\frac{11}{8}$	$\frac{27}{8}$	$\frac{109}{16}$	$\frac{195}{16}$	$\frac{321}{16}$	$\frac{497}{16}$	$\dots$	$\frac{3}{8}$	$\frac{5}{4}$	$\frac{9}{4}$	2	1			
$\frac{5}{16}$	$\frac{11}{8}$	$\frac{123}{32}$	$\frac{69}{8}$	$\frac{541}{32}$	$\frac{483}{16}$	$\frac{805}{16}$	$\dots$	$\frac{5}{16}$	$\frac{11}{8}$	$\frac{11}{4}$	$\frac{7}{2}$	$\frac{5}{2}$	1		
$\frac{5}{16}$	$\frac{93}{64}$	$\frac{281}{64}$	$\frac{341}{32}$	$\frac{45}{2}$	$\frac{689}{16}$	$\frac{1225}{16}$	$\dots$	$\frac{5}{16}$	$\frac{11}{8}$	$\frac{27}{8}$	5	5	3	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\ddots$

Table 2: The even order-terms Vietoris convolution triangle $\mathcal{T}_{EV}$

1	$\mathbf{1}$	1	$\mathbf{1}$	1	$\mathbf{1}$	1	$\dots$	1						
$\frac{1}{2}$	$\mathbf{1}$	$\frac{3}{2}$	$\mathbf{2}$	$\frac{5}{2}$	$\mathbf{3}$	$\frac{7}{2}$	$\dots$	$\frac{1}{2}$	1					
$\frac{3}{8}$	$\mathbf{1}$	$\frac{15}{8}$	$\mathbf{3}$	$\frac{35}{8}$	$\mathbf{6}$	$\frac{63}{8}$	$\dots$	$\frac{3}{8}$	1	1				
$\frac{5}{16}$	$\mathbf{1}$	$\frac{35}{16}$	$\mathbf{4}$	$\frac{105}{8}$	$\mathbf{10}$	$\frac{231}{16}$	$\dots$	$\frac{5}{16}$	1	$\frac{3}{2}$	1			
$\frac{35}{128}$	$\mathbf{1}$	$\frac{315}{128}$	$\mathbf{5}$	$\frac{1155}{128}$	$\mathbf{15}$	$\frac{3003}{128}$	$\dots$	$\frac{35}{128}$	1	$\frac{15}{8}$	2	1		
$\frac{63}{256}$	$\mathbf{1}$	$\frac{693}{256}$	$\mathbf{6}$	$\frac{3003}{256}$	$\mathbf{21}$	$\frac{9009}{256}$	$\dots$	$\frac{63}{256}$	1	$\frac{35}{16}$	3	$\frac{5}{2}$	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 3: The complete central binomial coefficients convolution triangle $\mathcal{T}_{CC}$

1	1	1	1	1	1	...		1						
1	2	3	4	5	6	...		1	1					
2	5	9	14	20	27	...		2	2	1				
3	10	22	40	65	98	...		3	5	3	1			
6	22	54	109	195	321	...		6	10	9	4	1		
10	44	123	276	541	966	...		10	22	22	14	5	1	
⋮	⋮	⋮	⋮	⋮	⋮			⋮	⋮	⋮	⋮	⋮	⋮	⋱

and

$$\det \mathcal{M}_C = 4^{\frac{r(r-1)}{2}} \det \mathcal{M}_{EV}.$$

The determinant of the matrices $\mathcal{M}_C$ and $\mathcal{M}_{CC}$ can be easily computed (see e.g. [2]),

$$\det \mathcal{M}_C = 2^{\frac{r(r-1)}{2}} \quad \text{and} \quad \det \mathcal{M}_{CC} = 1,$$

and this leads to the interesting equality

$$\det \mathcal{M}_V = \det \mathcal{M}_{EV} = 2^{-\frac{r(r-1)}{2}}.$$

For example, for $r = 4$ we obtain

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ \frac{3}{8} & 1 & \frac{15}{8} & 3 \\ \frac{5}{16} & 1 & \frac{35}{16} & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ \frac{1}{2} & \frac{5}{4} & \frac{9}{4} & \frac{7}{2} \\ \frac{3}{8} & \frac{5}{4} & \frac{11}{4} & 5 \end{vmatrix} = \frac{1}{64}.$$

It is worthwhile to recall that the sequences $(u_n)_n$ and $(v_n)_n$ in (13) have as generating functions,

$$G(t) = \frac{-1 + 2t + \sqrt{1-4t^2}}{2t - 4t^2} \quad \text{and} \quad g(t) = \frac{1}{\sqrt{1-4t}},$$

respectively, being, as expected, related to the generating functions F and f of $(c_n)_n$ and $(c_{2n})_n$, respectively, through

$$G(t) = F(2t) \quad \text{and} \quad g(t) = f(4t).$$

This last relation could also have been used to obtain (14).

The Table 2 also reveals another pattern related to the k^{th} convolution of sequence $(\tilde{c}_n)_n = (c_{2n})_n$, when $k = 2m - 1$ is odd, as we next point out.

Observe that the columns $2m$ of $\mathcal{T}_{EV}$ contain the Taylor series coefficients of the functions $(1-t)^{-m}$ (cf. (12)), which as it is well known is, for each fixed m, the generating function of the sequence $(b_n)_n$, where

$$b_n = \binom{n+m-1}{m-1}, \; n = 0, 1, \ldots.$$

Table 4: The central binomial coefficients convolution triangle $\mathcal{T}_C$

1	1	1	1	1	1	...	1						
2	4	6	8	10	12	...	2	1					
6	16	30	48	70	96	...	6	4	1				
20	64	140	256	420	649	...	20	16	6	1			
70	256	630	1280	2310	3840	...	70	64	30	8	1		
252	1024	2772	6144	12012	21504	...	252	256	140	48	10	1	
⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋱

In other words, the triangle formed by the even columns of the triangle $\mathcal{T}_{EV}$, is the rectangular Pascal triangle.

For the particular case of $k = 1$ (i.e., $m = 1$) we find the convolution formula

$$\tilde{c}_n^{(1)} = \sum_{s=0}^{n} \tilde{c}_s \tilde{c}_{n-s} = \binom{n}{0} = 1,$$

which is just a rewrite of the well-known identity for convolution of central binomial coefficients:

$$\sum_{s=0}^{n} \binom{2s}{s} \binom{2(n-s)}{n-s} = 4^n.$$

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A Alternative definitions

Vietoris sequence

```
c[k_Integer?NonNegative] := 1/2^k Binomial[k, Floor[k/2]]
```

Definition in terms of the generators of $\mathbb{H}$

The implementation of (3) requires the use of the free Mathematica package `QuaternionAnalysis`[13].

```
e1=Quaternion[0,1,0,0];e2=Quaternion[0,0,1,0];
c1[k_]:=(-1)^k
Sum[Binomial[k,s]QPower[SymmetricPower[e1,k-s,e2,s],2],{s,0,k}]^-1;
```

The function `Quaternion` defines a quaternion object, while `QPower` implements the usual quaternions powers. Both functions are included in the Mathematica package `QuaternionAnalysis`, where the arithmetic operations are also defined. The code of the function `SymmetricPower` is presented below. For more details on the use of the package we refer to the user guide included in the package documentation.

```
SymmetricPower[q1_, k_, q2_, s_] := SymmetricPower[q1, k, q2, s] =
1/(k + s) (k q1 ** SymmetricPower[q1, k - 1, q2, s] +
s q2 ** SymmetricPower[q1, k, q2, s - 1])
SymmetricPower[q1_, 0, q2_, s_] := QPower[q2, s];
SymmetricPower[q1_, k_, q2_, 0] := QPower[q1, k];
```

Double Factorial representation

```
c2[k_Integer?NonNegative] := (2 Floor[(k-1)/2]+1)!!/(2Floor[(k-1)/2]+2)!!
```

Recursive definition

```
c3[k_?((OddQ[#] && Positive[#]) &)] := c3[k] = k/(k + 1) c3[k - 1]
c3[k_?((EvenQ[#] && Positive[#]) &)] := c3[k] = c3[k - 1]
c3[0] = 1;
```

Pochhammer symbol representation

```
c4[k_Integer?NonNegative] :=
Pochhammer[1/2, Floor[(k + 1)/2]]/Floor[(k + 1)/2]!;
```

Alternating sum of a non-symmetric triangle

```
c5[k_Integer?NonNegative] := Sum[(-1)^s (Pochhammer[3/2, k - s]
Pochhammer[1/2, s])/((k - s)! s!), {s, 0, k}]/(k + 1);
```

We point out that the function `Ck[k,n]` included in `QuaternionAnalysis` defines, for the choice $n = 2$, the Vietoris sequence, using the form (5).

Gamma function representation

```
c6[k_Integer?NonNegative] :=  Gamma[1/2 + Floor[(k + 1)/2]]/
(Gamma[1/2] Gamma[1 + Floor[(k +  1)/2]]);
```

Integral representation

```
c7[k_Integer?NonNegative] :=
2/Pi Integrate[Cos[x]^(2 Floor[(k + 1)/2]), {x, 0, Pi/2}]
```

Catalan Numbers

```
c8[k_Integer?NonNegative] := Sum[(-2)^-s Binomial[k, s] CatalanNumber[s],
{s, 0, k}]
```

Legendre Polynomials

```
c9[k_Integer?NonNegative] := (-1)^Floor[(k+1)/2]
LegendreP[2 Floor[(k+1)/2], 0]
```

Bessel functions

```
c10[k_Integer?NonNegative] := Limit[(-1)^Floor[k/2]
D[BesselJ[1/2 (1 - (-1)^k), x], {x, k}], x -> 0]
```

Bessel functions of the first kind with integer order are entire functions; here we have use the limit to avoid the indetermine form provided by a direct evaluation of the derivatives in Mathematica.

B Convolution triangles

To produce the convolution triangle of a sequence in its rectangular form, one can use the function `TriangleRect`, in one of the following forms:

1. `TriangleRect[{a0,a1,...,an},k]`
   gives the $(n+1)\times(k+1)$ matrix corresponding to the first k convolutions of the sequence whose first $n+1$ terms are `a0,...,an`;

2. `TriangleRect[exp,n,k]`
   gives the $(n+1)\times(k+1)$ matrix corresponding to the first k convolutions of the sequence whose general term is given by the expression `expr`.

```
SeqConv[list1_List, list2_List] :=
Module[{dim1 = Length[list1], dim2 = Length[list2]},
If[dim1 == dim2,
Table[Sum[list1[[k]] list2[[n - k + 1]], {k, 1, n}], {n, 1, dim1}],
Message[SeqConv::dim]]];
SeqConv::dim = "Lists in the argument must have the same lenght.";

SeqConv[list_List, k_Integer] := NestList[SeqConv[list, #] &, list, k]
SeqConv[a_, b_, n_] :=
SeqConv[Table[a[k], {k, 0, n}], Table[b[k], {k, 0, n}]]
SeqConv[a_, n_Integer, k_Integer] :=
Module[{list = Table[a[j], {j, 0, n}]}, SeqConv[list, k]]

TriangleRect[list_List, k_Integer] := Transpose[SeqConv[list, k]];
TriangleRect[a_, n_Integer, k_Integer] := Transpose[SeqConv[a, n, k]];
```

For example, the code

```
c[k_] := 1/2^k Binomial[k, Floor[k/2]]
TriangleRect[c, 6, 6] // TableForm
```

produces the matrix in the left hand side of Table 1. This result can also be obtained by using

```
TriangleRect[{1, 1/2, 1/2, 3/8, 3/8, 5/16, 5/16}, 6] // TableForm
```

The left justified form of the convolution triangle can be obtained by the use of the function `TriangleLeft` whose syntax is analogous to that of the function `TriangleRect`.

```
Matrix2Triangle[matrix_?MatrixQ] :=
Module[{dim = Dimensions[matrix], m, n}, {m, n} = dim;
If[n >= m, (Cases[#1, Except[Null]] & ) /@
Transpose[MapThread[PadLeft[Drop[#1, -#2], m, Null] & ,
{Take[Transpose[matrix], m], Range[m] - 1}]],
(Cases[#1, Except[Null]] & ) /@
Transpose[MapThread[PadLeft[Drop[#1, -#2],
m, Null] & , {Transpose[matrix], Range[n] - 1}]]]]
```

```
TriangleLeft[list_List, k_Integer] :=
Module[{n = Length[list]},
If[n < k + 1, Message[TriangleLeft::order, n - 1]];
Matrix2Triangle[Transpose[SeqConv[list, k]]]];
TriangleLeft[a_, k_Integer] := Matrix2Triangle[SeqConv[a, k, k]];
TriangleLeft::order = "Showing only the triangle of order ‘1‘.";
```

To obtain the table in the right hand side of Table 1 we just have to use:

```
TriangleLeft[c, 6] // TableForm
```

or

```
TriangleLeft[{1, 1/2, 1/2, 3/8, 3/8, 5/16, 5/16}, 6] // TableForm
```