



Local conjugacy classes for analytic torus flows

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Abstract

If a real-analytic flow on the multidimensional torus close enough to linear has a unique rotation vector which satisfies an arithmetical condition \mathcal{Y} , then it is analytically conjugate to linear. We show this by proving that the orbit under renormalization of a constant \mathcal{Y} -vector field attracts all nearby orbits with the same rotation vector.

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1. Introduction

We are interested in the study of real-analytic flows ϕ^t on the torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, $d \geq 2$, that are topologically conjugate to a linear flow, i.e. $h^{-1} \circ \phi^t \circ h(x) = x + \omega t \pmod{1}$, $t \geq 0$, for some homeomorphism h of \mathbb{T}^d and $\omega \in \mathbb{R}^d$. Our aim, as part of a program, is to find sufficient conditions for which the conjugacy h is guaranteed to be real-analytic.

The general case $d \geq 2$ differs significantly from the lower dimension situation $d = 2$ where there is an invariant of motion, the asymptotic direction of the flow, whose slope is called rotation number. The classical Denjoy's theorem [7] asserts that for sufficiently smooth flows, irrational rotation numbers describe completely topological conjugacy classes. This cannot be generalized to higher dimensions due to the large variety of possible dynamical behaviours; often there is more than one asymptotic direction (rotation vectors). In this paper we look at analytic flows with a unique rotation vector for all orbits, showing that these have similar properties to the ones of $d = 2$, at least for 'typical' rotation vectors.

The two-dimensional theory was further developed by Arnol'd [1], Herman [9] and Yoccoz [23] in the 1960s, 1970s and 1980s, respectively. Mainly working on the discrete-time

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counterpart, circle diffeomorphisms as return maps to flow transversals, they showed that the conjugacy classes correspond to higher regularity (see also [10,11]). However, this will depend on the arithmetical properties of the rotation number due to their crucial role in solving small divisor problems. In particular, in the 1990s Yoccoz proved that for analytic diffeomorphisms close to a rotation, the conjugacy is analytic if the rotation number is of Brjuno type [24]; and this arithmetical condition is optimal. Moreover, the closeness to rotation hypothesis can be dropped if restricting to Yoccoz’s \mathcal{H} set of rotation numbers [25]. Again, this condition is optimal.

The above cited results can be seen as proofs of differential rigidity within topological conjugacy classes of smooth systems. This has been observed to be a property common to different problems as in e.g. [4–6,14], while renormalization being the tool often used. However, in those works the renormalizations are based on the continued fraction expansion of irrational numbers. Here we are able to work on higher dimensions by making use of the multidimensional continued fractions algorithm introduced in [13] (see also [12,17,18]). This roughly corresponds to a flow on the homogeneous space $SL(d, \mathbb{Z}) \backslash SL(d, \mathbb{R})$ that, following Lagarias ideas [19], provides a strongly convergent continued fractions expansion for all vectors (see Section 2 below).

Given $\omega \in \mathbb{R}^d - \{0\}$, define the linear torus flow

$$R_\omega^t : \mathbb{T}^d \rightarrow \mathbb{T}^d, \quad x \mapsto x + \omega t \pmod{1}, \tag{1.1}$$

with $t \geq 0$. The main result in this paper is the following.

Theorem 1.1. *Let $\omega \in \mathcal{Y} \subset \mathbb{R}^d$. If a C^ω flow on \mathbb{T}^d has a unique rotation vector ω and it is C^ω -close enough to linear, then it is C^ω -conjugate to R_ω^t .*

Since the rotation vector is invariant under homeomorphisms (see Section 3), this result yields an immediate consequence:

Corollary 1.2. *Let $\omega \in \mathcal{Y} \subset \mathbb{R}^d$. If a C^ω flow on \mathbb{T}^d is C^0 -conjugate to R_ω^t and it is C^ω -close enough to linear, then the conjugacy is in fact C^ω .*

Therefore, the topological and analytic conjugacy classes of R_ω^t are locally the same for $\omega \in \mathcal{Y}$ and coincide with the set of close-to-linear flows with a rotation vector ω . Note that by allowing time-reparametrizations and recalling that $R_{\lambda\omega}^t = R_\omega^{\lambda t}$, $\lambda \neq 0$, we obtain larger conjugacy classes.

In Proposition 2.6 we will see that the set \mathcal{Y} contains all Diophantine vectors, having full Lebesgue measure. In dimension two \mathcal{Y} corresponds to the set of vectors whose slope is a Brjuno number.

We highlight the fact that no condition such as volume-preservation is required for the main theorem to hold. It is known [8,9,22] that volume-preserving C^ω -flows with a Diophantine rotation vector ω and close to R_ω^t are C^ω -conjugated to R_ω^t . That is a consequence of the existence of a parameter λ which makes a vector field $X + \lambda$ (not necessarily divergence free) conjugated to a translation, where X is close to ω (see also [13]). With R_ω^t ergodic, if the volume is preserved and the rotation vector is ω , then λ has to vanish (Proposition 2.6.1 [9, p. 180]).

We believe that further extensions of Theorem 1.1, namely for C^r -vector fields, should be accessible by the present method. Two open and outstanding problems are the corresponding global result and the determination of an optimal condition on ω for which the theorem holds.

The proof of Theorem 1.1 is a consequence of the convergence under renormalization of vector fields in some small ball around ω . The renormalization operator is basically a coordinate

change and time rescaling related to the continued fractions of ω (cf. [13,16]). Convergence is guaranteed if the rotation vector $\omega \in \mathcal{Y}$. The fact that we have a unique rotation vector permits us to control the distance between ω and the constant Fourier mode of the perturbed vector fields. Our scheme therefore contracts a ball in its domain towards the orbit under renormalization of the constant vector field ω . The differentiable conjugacy then follows.

In Section 2 we review the multidimensional continued fractions scheme introduced in [13], and define the class of \mathcal{Y} -vectors. In Sections 3 and 4 we present the renormalization building blocks, which will be put together in Section 5 in order to determine sufficient conditions for the existence of infinitely renormalizable vector fields. In Section 7 we construct the analytic conjugacy for vector fields which are attracted under renormalization to the orbit of the constant system. The proof of Theorem 1.1 is concluded in Section 8.

For the following we set the notations $\mathbb{N} = \{1, 2, \dots\}$ for the positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for the non-negative integers. Moreover, $A \ll B$ stands for the existence of a constant $C > 0$ such that $A \leq CB$.

2. Multidimensional continued fractions

In this section we present the multidimensional continued fractions algorithm introduced in [13] following ideas of Dani [3], Lagarias [19] and Kleinbock and Margulis [15]. In addition, we define the class of vectors \mathcal{Y} from the properties of the continued fractions expansion in an appropriate way to be used later by the renormalization scheme.

2.1. Flow on homogeneous space

Denote by $G = \text{SL}(d, \mathbb{R})$, $\Gamma = \text{SL}(d, \mathbb{Z})$ and take a fundamental domain $\mathcal{F} \subset G$ of the homogeneous space $\Gamma \backslash G$ (the space of d -dimensional non-degenerate unimodular lattices). On \mathcal{F} consider the flow:

$$\Phi^t : \mathcal{F} \rightarrow \mathcal{F}, \quad M \mapsto P(t)ME^t, \tag{2.1}$$

where

$$E^t = \text{diag}(e^{-t}, \dots, e^{-t}, e^{(d-1)t}) \in G$$

and $P(t)$ is the unique family in Γ that keeps $\Phi^t M$ in \mathcal{F} for every $t \geq 0$.

For the remaining of this paper we fix $\omega = (\alpha, 1) \in \mathbb{R}^d$. We are then interested in the orbit under Φ^t of the matrix

$$M_\omega = \begin{pmatrix} I & \alpha \\ \mathbf{0} & 1 \end{pmatrix}. \tag{2.2}$$

2.2. Growth of the flow

Let the function $\delta : \Gamma \backslash G \rightarrow \mathbb{R}^+$ measuring the shortest vector in the lattice M be

$$\delta(M) = \inf_{k \in \mathbb{Z}^d - \{0\}} \|\top k M\|, \tag{2.3}$$

where $\|\cdot\|$ stands for the ℓ_1 -norm (in the following we will make use of the corresponding matrix norm taken as the usual operator norm). Notice that $\delta(\Phi^t M_\omega) = \delta(M_\omega E^t)$.

Proposition 2.1. (See [13].) *There exist $C_1, C_2 > 0$ such that for all $t \geq 0$*

$$\|\Phi^t M_\omega\| \leq \frac{C_1}{\delta(\Phi^t M_\omega)^{d-1}} \quad \text{and} \quad \|(\Phi^t M_\omega)^{-1}\| \leq \frac{C_2}{\delta(\Phi^t M_\omega)}. \tag{2.4}$$

2.3. Stopping times

Consider a sequence of times, called *stopping times*,

$$t_0 = 0 < t_1 < t_2 < \dots \rightarrow +\infty \tag{2.5}$$

such that the matrices $P(t)$ in (2.1) satisfy

$$P_n := P(t_n) \neq P(t_{n-1}) \tag{2.6}$$

with $n \in \mathbb{N}$. We also set $P_0 = P(t_0) = I$. The sequence of matrices $P_n \in \text{SL}(d, \mathbb{Z})$ are the rational approximates of ω , called the *multidimensional continued fractions expansion*. In addition we define the transfer matrices

$$T_n = P_n P_{n-1}^{-1}, \quad n \in \mathbb{N}, \quad \text{and} \quad T_0 = I. \tag{2.7}$$

The flow of M_ω taken at the time sequence is thus the sequence of matrices

$$M_n = \Phi^{t_n} M_\omega = P_n M_\omega E^{t_n}. \tag{2.8}$$

Using some properties of the flow, the above can be decomposed (see [13]) into

$$M_n = \begin{pmatrix} I & \alpha_n \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \Delta_n & \mathbf{0} \\ \beta_n^\top & \gamma_n \end{pmatrix} \tag{2.9}$$

with γ_n being the d th component of the vector $e^{(d-1)t_n} P_n \omega$.

Define $\omega_n = (\alpha_n, 1)$, $\omega_0 = \omega$ and, for $n \in \mathbb{N}$,

$$\omega_n = \gamma_n^{-1} M_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \lambda_n P_n \omega = \eta_n T_n \omega_{n-1}, \tag{2.10}$$

where

$$\lambda_n = \frac{1}{\gamma_n} e^{(d-1)t_n} \quad \text{and} \quad \eta_n = \frac{\lambda_n}{\lambda_{n-1}}. \tag{2.11}$$

We remark that when $d = 2$, there exists a sequence of stopping times (called Hermitte critical times) that gives an accelerated version of the standard continued fractions of a number α [19].

2.4. *Resonance widths*

We call *resonance widths* to the terms of any decreasing sequence $\sigma : \mathbb{N}_0 \rightarrow \mathbb{R}^+$.

2.5. *Resonance cone*

Given resonance widths $\sigma : \mathbb{N}_0 \rightarrow \mathbb{R}^+$, define the resonant cones to be

$$I_n^+ = \{ \mathbf{k} \in \mathbb{Z}^d : |\mathbf{k} \cdot \boldsymbol{\omega}_n| \leq \sigma_n \|\mathbf{k}\| \}. \tag{2.12}$$

In addition, let

$$A_n = \sup_{\mathbf{k} \in I_n^+ - \{0\}} \frac{\|{}^\top T_{n+1}^{-1} \mathbf{k}\|}{\|\mathbf{k}\|}. \tag{2.13}$$

Proposition 2.2. *There is $c > 0$ such that for any $n \in \mathbb{N}_0$*

$$A_n \leq c e^{-\delta t_{n+1}} \frac{\sigma_n e^{d\delta t_{n+1}} + 1}{\delta(M_n)^{d-1} \delta(M_{n+1})}, \tag{2.14}$$

where $\delta t_{n+1} = t_{n+1} - t_n$.

Proof. Take S_n^\perp to be the hyperspace orthogonal to $\boldsymbol{\omega}_n$. By (2.7) and (2.8), $T_n = M_n E^{-\delta t_n} M_{n-1}^{-1}$ and ${}^\top T_{n+1}^{-1} = {}^\top M_{n+1}^{-1} E^{\delta t_{n+1}} {}^\top M_n$. So, for $\boldsymbol{\xi} \in S_n^\perp$, (2.9) gives

$${}^\top T_{n+1}^{-1} \boldsymbol{\xi} = e^{-\delta t_{n+1}} {}^\top M_{n+1}^{-1} \begin{pmatrix} {}^\top \Delta_n \boldsymbol{\xi}' \\ 0 \end{pmatrix} = e^{-\delta t_{n+1}} {}^\top M_{n+1}^{-1} {}^\top M_n \boldsymbol{\xi},$$

where $\boldsymbol{\xi}' \in \mathbb{R}^{d-1}$ comprises the first $d - 1$ components of $\boldsymbol{\xi}$.

Now, write $\mathbf{k} \in I_n^+ - \{0\}$ as $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ where

$$\mathbf{k}_1 = \frac{\mathbf{k} \cdot \boldsymbol{\omega}_n}{\boldsymbol{\omega}_n \cdot \boldsymbol{\omega}_n} \boldsymbol{\omega}_n \quad \text{and} \quad \mathbf{k}_2 \in S_n^\perp.$$

Hence,

$$\begin{aligned} \|{}^\top T_{n+1}^{-1} \mathbf{k}\| &\leq \|{}^\top T_{n+1}^{-1} \mathbf{k}_1\| + \|{}^\top T_{n+1}^{-1} \mathbf{k}_2\| \\ &\leq \sigma_n \|{}^\top T_{n+1}^{-1}\| \|\mathbf{k}\| + e^{-\delta t_{n+1}} \|{}^\top M_{n+1}^{-1}\| \|{}^\top M_n\| \|\mathbf{k}\| \\ &\leq (\sigma_n \|E^{\delta t_{n+1}}\| + e^{-\delta t_{n+1}}) \|{}^\top M_{n+1}^{-1}\| \|{}^\top M_n\| \|\mathbf{k}\| \end{aligned} \tag{2.15}$$

which can be estimated using Proposition 2.1. \square

2.6. *Class of Diophantine vectors*

A vector $\omega \in \mathbb{R}^d$ is Diophantine with exponent $\beta \geq 0$ if there is a constant $C > 0$ such that

$$|\omega \cdot k| > \frac{C}{\|k\|^{d-1+\beta}}.$$

It is a well-known fact that the sets $DC(\beta)$ of Diophantine vectors with exponent $\beta > 0$ are of full Lebesgue measure [2]. On the other hand, the set $DC(0)$ has zero Lebesgue measure. A vector is said to be Diophantine if it belongs to $DC = \bigcup_{\beta \geq 0} DC(\beta)$. The next proposition gives us a complete characterization of Diophantine vectors in terms of the behaviour of the flow Φ^t of M_ω (the sufficient part appeared already in [13]).

Proposition 2.3. *Let $\beta \geq 0$. Then, $\omega \in DC(\beta)$ iff there is $C' > 0$ such that*

$$\delta(\Phi^t M_\omega) > C' e^{-\theta t}, \quad t \geq 0,$$

with $\theta = \beta / (d + \beta)$.

Proof. Notice first that ${}^T k M_\omega E^t = (e^{-t} \hat{k}, e^{(d-1)t} (k \cdot \omega))$, where $k = (\hat{k}, k_d) \in \mathbb{Z}^d - \{0\}$. Now, we have that

$$C' < e^{\theta t} \inf_{k \neq 0} \|{}^T k M_\omega E^t\| \leq \max\{e^{-(1-\theta)t} \|k\|, e^{(d-1+\theta)t} |k \cdot \omega|\}. \tag{2.16}$$

Let

$$t = t(k) = \begin{cases} \frac{1}{d} \log \frac{\|k\|}{|k \cdot \omega|}, & \|k\| \geq |k \cdot \omega|, \\ 0, & \|k\| \leq |k \cdot \omega|. \end{cases}$$

Using this t in (2.16), the second case trivially means that $|k \cdot \omega| > C' > C \|k\|^{-(d-1+\beta)}$ for some constant $C > 0$. For the first case,

$$C' < \|k\|^{(d-1+\theta)/d} |k \cdot \omega|^{1-(d-1+\theta)/d}.$$

So, $|k \cdot \omega| > C' \|k\|^{d-1+\beta}$ and ω is Diophantine with exponent β .

The converse is proved in [13, Lemma 2.3]. \square

Proposition 2.4. *(See [13].) If $\omega \in DC(\beta)$, $\beta \geq 0$, there are constants $c_1, c_2, c_3, c_4, c_5, c_6, c_7 > 0$ such that, for any stopping-time sequence $t : \mathbb{N}_0 \rightarrow \mathbb{R}$,*

$$\|M_n\| \leq c_1 \exp[(d - 1)\theta t_n], \tag{2.17}$$

$$\|M_n^{-1}\| \leq c_2 \exp(\theta t_n), \tag{2.18}$$

$$\|P_n\| \leq c_3 \exp[(d\theta + 1 - \theta)t_n], \tag{2.19}$$

$$\|P_n^{-1}\| \leq c_4 \exp[(d - 1 + \theta)t_n], \tag{2.20}$$

$$\|T_n\| \leq c_5 \exp[(1 - \theta)\delta t_n + d\theta t_n], \tag{2.21}$$

$$\|T_n^{-1}\| \leq c_6 \exp[(d - 1)(1 - \theta)\delta t_n + d\theta t_n], \tag{2.22}$$

and

$$c_7 \exp\left[-\theta\left(\frac{d^2}{1-\theta} - (d-1)\right)t_n\right] \leq |\gamma_n| \leq c_1 \exp[(d-1)\theta t_n], \tag{2.23}$$

where $\delta t_n = t_n - t_{n-1}$ and $\theta = \beta/(d + \beta)$.

The above estimates were already proved in [13], and are a direct consequence of Proposition 2.3 applied to Proposition 2.1.

Proposition 2.5. *If $\omega \in DC(\beta)$, $\beta \geq 0$, then there is $c > 0$ such that for any $n \in \mathbb{N}_0$,*

$$A_n \leq c e^{-(1-\theta)\delta t_{n+1} + d\theta t_n} (\sigma_n e^{d\delta t_{n+1}} + 1). \tag{2.24}$$

Proof. It follows immediately by applying Proposition 2.3 to Proposition 2.2. \square

2.7. Class of \mathcal{Y} -vectors

In this paper we will be dealing with a class of vectors which satisfies some arithmetical properties more general than the Diophantine ones (cf. Section 2.6).

A vector $\omega \in \mathbb{R}^d$ belongs to \mathcal{Y} if there exist sequences of resonance widths $\sigma : \mathbb{N}_0 \rightarrow \mathbb{R}^+$ and of stopping times $t : \mathbb{N}_0 \rightarrow \mathbb{R}$ such that

$$\sum_{n \geq 0} A_0 \cdots A_n \log\left(|\eta_{n+1}| \|T_{n+1}\| \frac{\sigma_n \|\omega_{n+1}\|}{\sigma_{n+1}}\right) < \infty \tag{2.25}$$

and

$$\lim_{n \rightarrow +\infty} 2^{2n} \sigma_n \|P_n^{-1}\| \|T_0\|^2 \cdots \|T_n\|^2 = 0. \tag{2.26}$$

The conditions above should be seen as lower and upper bounds on σ_n . It follows immediately that $\omega_N \in \mathcal{Y}$ for any fixed $N \in \mathbb{N}$, using shifted sequences σ_{N+n} and t_{N+n} .

The class of \mathcal{Y} -vectors contains the full probability set of Diophantine vectors as it is proved below.

Proposition 2.6. $DC \subset \mathcal{Y}$.

Proof. Let $0 < \xi < 1 - \theta$, the stopping-time sequence given recursively by $t_{n+1} = e^{(1-\theta-\xi)t_n}$, $n \in \mathbb{N}$, $t_0 = 0$, and the sequences of resonance widths given by $\sigma_n = e^{-d\delta t_{n+1}}$ for n large enough.

For $\omega \in DC(\beta)$, $\beta \geq 0$, using Propositions 2.4 and 2.5, the terms in the series in (2.25) can be bounded from above by

$$C^n t_{n+2} e^{-(1-\theta)t_{n+1} + d\theta \sum_{i=0}^n t_i},$$

for some constant $C > 0$. By our choice of the stopping times, the above can be estimated from above by $C' e^{(1-\theta-\xi)t_{n+1} - (1-\theta-\xi')t_{n+1}}$ with $C' > 0$ and $0 < \xi' < \xi$. Therefore, the series in (2.25) converges.

On the other hand, the expression in the limit of (2.26) can be bounded by

$$2^{2n} e^{-d\delta t_{n+1} + (d+1-\theta)t_n + 2\theta d \sum_{i=0}^n t_i}$$

which goes to zero as $n \rightarrow +\infty$ by our present choice of stopping times.

This completes the proof that $\omega \in \mathcal{Y}$. \square

Remark 2.7. If we restrict to $d = 2$ it is natural to use the standard continued fractions expansion. In this case it can be shown that \mathcal{Y} corresponds to the Brjuno vectors, i.e. vectors whose slope is a Brjuno number (see [21]).

3. Rotation vectors

We define the rotation vector of a flow ϕ^t at each $\mathbf{x} \in \mathbb{T}^d$ to be the asymptotic direction of the corresponding orbit of the lift $\Phi^t(\mathbf{x})$ to the universal cover:

$$\text{Rot}(\phi)(\mathbf{x}) = \lim_{t \rightarrow \infty} \frac{\Phi^t(\mathbf{x}) - \mathbf{x}}{t}, \tag{3.1}$$

if the limit exists.

Remark 3.1. If the rotation vector exists at \mathbf{x} for a flow ϕ^t generated by a vector field X on \mathbb{T}^d (i.e. $\frac{d}{dt}\phi^t = X \circ \phi^t$), it is the time average of the vector field along the orbit:

$$\text{Rot}(\phi)(\mathbf{x}) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X \circ \phi^s(\mathbf{x}) ds. \tag{3.2}$$

When the rotation vector exists for all $\mathbf{x} \in \mathbb{T}^d$, the rotation set of ϕ is

$$\text{Rot}(\phi) = \{ \text{Rot}(\phi)(\mathbf{x}) : \mathbf{x} \in \mathbb{T}^d \}. \tag{3.3}$$

Throughout this text we denote by $\text{Homeo}(M)$ and $\text{Diff}^r(M)$, $r \in \mathbb{N} \cup \{\infty, \omega\}$, the set of homeomorphisms and C^r -diffeomorphisms on M . Moreover, we add a subscript 0 to distinguish the case of isotopic to the identity maps. Finally, $\text{Vect}^r(M)$ stands for the set of C^r -vector fields on M .

Lemma 3.2. *Let $h \in \text{Homeo}_0(\mathbb{T}^d)$, $\lambda \neq 0$ and $T \in \text{GL}(d, \mathbb{Z})$. If $\text{Rot}(\phi) \neq \emptyset$, then*

$$\text{Rot}(h^{-1} \circ \phi \circ h) = \text{Rot}(\phi) \quad \text{and} \quad \text{Rot}(T^{-1} \circ \phi^\lambda \circ T) = \lambda T^{-1} \text{Rot}(\phi). \tag{3.4}$$

Proof. Writing $h^{-1} = \text{Id} + \varphi$ with φ a \mathbb{Z}^d -periodic function, we get

$$\text{Rot}(h^{-1} \circ \phi \circ h)(\mathbf{x}) = \lim_{t \rightarrow \infty} \frac{1}{t} [\Phi^t \circ h(\mathbf{x}) + \varphi \circ \phi^t \circ h(\mathbf{x}) - h(\mathbf{x}) + h(\mathbf{x}) - \mathbf{x}]. \tag{3.5}$$

The fact that φ is bounded and that there is a rotation vector for all points in \mathbb{T}^d , yields $\text{Rot}(h^{-1} \circ \phi \circ h)(\mathbf{x}) = \text{Rot}(\phi)(h(\mathbf{x}))$. Thus the first assertion.

The second claim follows from

$$\begin{aligned} \text{Rot}(T^{-1} \circ \phi^{\lambda \cdot} \circ T)(\mathbf{x}) &= \lim_{t \rightarrow \infty} \frac{1}{t} (T^{-1} \circ \Phi^{\lambda t} \circ T\mathbf{x} - \mathbf{x}) \\ &= \lim_{t \rightarrow \infty} \frac{1}{\lambda t} \lambda T^{-1} (\Phi^{\lambda t} \circ T\mathbf{x} - T\mathbf{x}) \\ &= \lambda T^{-1} \text{Rot}(\phi)(T\mathbf{x}). \quad \square \end{aligned} \tag{3.6}$$

Proposition 3.3. *Let ϕ^t be the flow generated by $X \in \text{Vect}^0(\mathbb{T}^d)$ and $\omega \in \mathbb{R}^d$. If $\text{Rot} \phi = \{\omega\}$, then*

$$\|\mathbb{E}X - \omega\| \leq d \|X - \mathbb{E}X\|_{C^0}, \tag{3.7}$$

where $\mathbb{E}X = \int_{\mathbb{T}^d} X \, dm$ and m denotes the Lebesgue measure on \mathbb{T}^d .

Proof. We first show that for each $1 \leq i \leq d$ there is $\mathbf{y}^{(i)} \in \mathbb{T}^d$ such that $X_i(\mathbf{y}^{(i)}) = \omega_i$. (We represent the i th coordinate of vectors by the subscript i .) This follows from the fact that $\text{Rot} \phi(\mathbf{x}) - \omega = 0$. I.e. for any $\mathbf{x} \in \mathbb{T}^d$ and $1 \leq i \leq d$,

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t [X_i \circ \phi^s(\mathbf{x}) - \omega_i] \, ds = 0.$$

The continuous function $\psi(\mathbf{x}, t) = X_i \circ \phi^t(\mathbf{x}) - \omega_i$ on $\mathbb{T}^d \times \mathbb{R}^+$ has $m = \min \psi$ and $M = \max \psi$ because X is continuous on a compact set. So, for any (\mathbf{x}, t) we have $m \leq \frac{1}{t} \int_0^t \psi(\mathbf{x}, s) \, ds \leq M$. Taking the limit, $m \leq 0 \leq M$ and we can find a zero of ψ , hence of $X_i - \omega_i$.

Now, using the above points $\mathbf{y}^{(i)}$,

$$\begin{aligned} \|\mathbb{E}X - \omega\| &= \sum_{i=1}^d |\mathbb{E}X_i - \omega_i| = \sum_{i=1}^d |X_i(\mathbf{y}^{(i)}) - \mathbb{E}X_i| \\ &\leq \sum_{i=1}^d \max_{\mathbf{x}} |X_i(\mathbf{x}) - \mathbb{E}X_i| \leq d \|X - \mathbb{E}X\|_{C^0}. \quad \square \end{aligned}$$

Remark 3.4. We will be interested in vector fields generating flows that possess the same rotation vector for all orbits. Hence, for a vector field X we will write $\text{Rot} X$ to mean the unique rotation vector associated to the flow generated by X .

4. Preliminaries

4.1. Definitions

The transformation of $X \in \text{Vect}(M)$ on a manifold M by $\psi \in \text{Diff}(M)$ is given by the pull-back of X under ψ :

$$\psi^* X = (D\psi)^{-1} X \circ \psi.$$

As $T\mathbb{T}^d \simeq \mathbb{T}^d \times \mathbb{R}^d$, we identify the set of vector fields on $\text{Vect}(\mathbb{T}^d)$ with the set of functions $C(\mathbb{T}^d, \mathbb{R}^d)$, that can be regarded as \mathbb{Z}^d -periodic maps of \mathbb{R}^d by lifting to the universal cover. We will make use of the analyticity to extend to the complex domain, so we will deal with complex analytic functions.

4.2. Space of vector fields

Let $\rho > 0$ and the domain

$$D_\rho = \left\{ \mathbf{x} \in \mathbb{C}^d : \|\text{Im } \mathbf{x}\| < \frac{\rho}{2\pi} \right\}, \tag{4.1}$$

for the norm $\|\mathbf{u}\| = \sum_i |u_i|$ on \mathbb{C}^d .

Take complex analytic functions $f : D_\rho \rightarrow \mathbb{C}^d$ that are \mathbb{Z}^d -periodic and on the form of the Fourier series

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} f_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \tag{4.2}$$

with $f_{\mathbf{k}} \in \mathbb{C}^d$. The Banach spaces \mathcal{A}_ρ and \mathcal{A}'_ρ are the subspaces of such functions with the respective finite norms

$$\|f\|_\rho = \sum_{\mathbf{k} \in \mathbb{Z}^d} \|f_{\mathbf{k}}\| e^{\rho \|\mathbf{k}\|} \quad \text{and} \quad \|f\|'_\rho = \sum_{\mathbf{k} \in \mathbb{Z}^d} (1 + 2\pi \|\mathbf{k}\|) \|f_{\mathbf{k}}\| e^{\rho \|\mathbf{k}\|}. \tag{4.3}$$

Consider also the norm $\|f\|_{C^0} = \max_{\mathbf{x} \in \mathbb{T}^d} \|f(\mathbf{x})\|$.

Some of the properties of the above spaces are of easy verification. For instance, given any $f, g \in \mathcal{A}'_\rho$ we have:

- $\|f(\mathbf{x})\| \leq \|f\|_{C^0} \leq \|f\|_\rho \leq \|f\|'_\rho$ where $\mathbf{x} \in D_\rho$,
- $\|f\|_{\rho-\delta} \leq \|f\|_\rho$ with $0 \leq \delta \leq \rho$,
- $\|Df\|_\rho \leq \delta^{-1} \|f\|_{\rho+\delta}$ with $\rho, \delta > 0$ (Cauchy’s estimate).

Write the constant Fourier mode of $f \in \mathcal{A}_\rho$ through the projection

$$\mathbb{E}f = \int_{\mathbb{T}^d} f \, dm = f_0 \in \mathbb{C}^d. \tag{4.4}$$

We will only be interested in the above vector fields which are fixed-point-free, i.e. $f(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in D_\rho$.

4.3. Far from resonance modes

Given $\sigma > 0$, we call *far from resonance* modes with respect to $\mathbf{v} \in \mathbb{R}^d$ to the Fourier modes with indices in

$$I_\sigma^- = \{ \mathbf{k} \in \mathbb{Z}^d : |\mathbf{v} \cdot \mathbf{k}| > \sigma \|\mathbf{k}\| \}. \tag{4.5}$$

The *resonant modes* are the ones in $I_\sigma^+ = \mathbb{Z}^d - I_\sigma^-$ which correspond to the resonant cones defined in (2.12). We also have the projections \mathbb{I}_σ^+ and \mathbb{I}_σ^- over the spaces of vector fields by restricting the modes to I_σ^+ and I_σ^- , respectively. The identity operator is $\mathbb{I} = \mathbb{I}_\sigma^+ + \mathbb{I}_\sigma^-$.

4.4. *Uniformization*

We call uniformization of a vector field to the action of a diffeomorphism that produces a new vector field with only resonant modes.

Given $\rho, \varepsilon, \nu > 0$, denote by \mathcal{V}_ε the open ball in $\mathcal{A}'_{\rho+\nu}$ centred at $\mathbf{v} \neq 0$ with radius ε .

Theorem 4.1. (See [13,20].) *Let $\sigma < \|\mathbf{v}\|$ and*

$$\varepsilon = \frac{\sigma}{42} \min \left\{ \frac{\nu}{4\pi}, \frac{\sigma}{72\|\mathbf{v}\|} \right\}. \tag{4.6}$$

*There exists a 1-parameter smooth family of maps $\mathcal{U}_t: \mathcal{V}_\varepsilon \rightarrow \mathcal{A}'_\rho$ and $\mathcal{U}_t: \mathcal{V}_\varepsilon \rightarrow \mathbb{I}_\sigma^+ \mathcal{A}_\rho \oplus (1-t)\mathbb{I}_\sigma^- \mathcal{A}'_{\rho+\nu}$ given by $\mathcal{U}_t(X) = \mathcal{U}_t(X) * X$ such that*

$$\mathbb{I}_\sigma^- \mathcal{U}_t(X) = (1-t)\mathbb{I}_\sigma^- X, \quad t \in [0, 1], \tag{4.7}$$

and

$$\begin{aligned} \|\mathcal{U}_t(X) - \text{Id}\|'_\rho &\leq \frac{42t}{\sigma} \|\mathbb{I}_\sigma^- X\|_\rho, \\ \|\mathcal{U}_t(X) - \mathbf{v}\|_\rho &\leq (3-t)\|X - \mathbf{v}\|'_{\rho+\nu}. \end{aligned} \tag{4.8}$$

Moreover, if X is real-analytic, $\mathcal{U}_t(X)(\mathbb{R}^{2d}) \subset \mathbb{R}^{2d}$.

Let the translation R_z on \mathbb{C}^d for each $z \in \mathbb{C}^d$ be

$$R_z: \mathbf{x} \mapsto \mathbf{x} + z. \tag{4.9}$$

Lemma 4.2. *In the conditions of Theorem 4.1, if $\mathbf{x} \in \mathbb{R}^d$ and $X \in \mathcal{V}_\varepsilon$, then*

$$\mathcal{U}_t(X \circ R_x) = R_x^{-1} \circ \mathcal{U}_t(X) \circ R_x \tag{4.10}$$

on D_ρ .

Proof. Notice that $R_x(D_\rho) = D_\rho$. If $U_t = \mathcal{U}_t(X)$ is a solution of the homotopy equation (4.7) on D_ρ , then $\tilde{U}_t = R_x^{-1} \circ \mathcal{U}_t(X) \circ R_x$ solves the same equation for $\tilde{X} = X \circ R_x$, i.e. $\mathbb{I}_\sigma^- \tilde{X} \circ \tilde{U}_t = (1-t)\mathbb{I}_\sigma^- \tilde{X}$, on D_ρ . \square

4.5. Rescaling

The fundamental step of the renormalization is a linear transformation of the domain of definition of our vector fields. This is done by a change of basis using the multidimensional continued fractions matrices $T_n \in \text{SL}(d, \mathbb{Z})$ of a vector $\omega \in \mathbb{R}^d$. The normalizing scalars η_n in (2.11) are used for a linear time rescaling.

Consider $X \in \mathcal{A}_\rho$. We are interested in the following coordinate and time linear changes:

$$L_n : \mathbf{x} \mapsto T_n^{-1} \mathbf{x}, \quad t \mapsto \eta_n t. \tag{4.11}$$

(Negative time rescaling means inverting the direction of time.) These determine a new vector field as the image of the map

$$X \mapsto \mathcal{L}_n(X) = \eta_n L_n^* X = \eta_n T_n X \circ T_n^{-1}.$$

Thus the following relation holds:

$$L_n^* R_z^* = R_{T_n z}^* L_n^*, \quad z \in \mathbb{C}^d. \tag{4.12}$$

Recall the definition of A_n given by (2.13) for some choice of σ_n . Notice that, according to (2.12), $I_n^\pm = I_{\sigma_n}^\pm$ the far from resonance modes with respect to ω_n , and we write $\mathbb{I}_n^\pm = \mathbb{I}_{\sigma_n}^\pm$.

Lemma 4.3. *If $\delta > 0$ and*

$$\rho'_n \leq \frac{\rho_{n-1}}{A_{n-1}} - \delta, \tag{4.13}$$

then $\tilde{\mathcal{L}}_n$ given by \mathcal{L}_n restricted to $(\mathbb{I}_{n-1}^+ - \mathbb{E})\mathcal{A}_{\rho_{n-1}}$ into $(\mathbb{I} - \mathbb{E})\mathcal{A}'_{\rho'_n}$ is bounded with

$$\|\tilde{\mathcal{L}}_n\| \leq |\eta_n| \|T_n\| \left(1 + \frac{2\pi}{\delta}\right). \tag{4.14}$$

Proof. Let $f \in (\mathbb{I}_{n-1}^+ - \mathbb{E})\mathcal{A}_{\rho_{n-1}}$. Then,

$$\|f \circ L_n\|'_{\rho'_n} \leq \sum_{\mathbf{k} \in I_{n-1}^+ - \{0\}} (1 + 2\pi \|\top T_n^{-1} \mathbf{k}\|) \|f_{\mathbf{k}}\| e^{(\rho'_n - \delta + \delta) \|\top T_n^{-1} \mathbf{k}\|}. \tag{4.15}$$

By using the relation $\xi e^{-\delta \xi} \leq \delta^{-1}$ with $\xi \geq 0$, and (2.13), we get

$$\|f \circ L_n\|'_{\rho'_n} \leq \left(1 + \frac{2\pi}{\delta}\right) \sum_{I_{n-1}^+ - \{0\}} \|f_{\mathbf{k}}\| e^{A_{n-1}(\rho'_n + \delta) \|\mathbf{k}\|} \leq \left(1 + \frac{2\pi}{\delta}\right) \|f\|_{\rho_{n-1}}. \tag{4.16}$$

Finally, $\|\tilde{\mathcal{L}}_n f\|'_{\rho'_n} \leq |\eta_n| \|T_n\| \|f \circ L_n\|'_{\rho'_n}$. \square

4.6. Analyticity strip cut-off

Consider the operator $\mathcal{I}: \mathcal{A}_\rho \rightarrow \mathcal{A}_{\rho'}$ obtained by restricting $X \in \mathcal{A}_\rho$ to the domain $D_{\rho'}$. When restricted to non-constant modes, its norm can be estimated as follows.

Lemma 4.4. *If $0 < \phi \leq e^{\rho-\rho'}$, then $\|\mathcal{I}(\mathbb{I} - \mathbb{E})\| \leq \phi^{-1}$.*

The proof is immediate.

5. Renormalization

5.1. Renormalization operator

We are only interested in vectors $\omega \in \mathbb{R}^d$ that satisfy $\omega \cdot k \neq 0$ for any $k \in \mathbb{Z}^d - \{0\}$. Associated to σ_{n-1} (specifying the cones I_{n-1}^\pm) and ϕ_n (the size of the analyticity strip width cut-off in 4.6, related to the domain we are considering), the n th step renormalization operator is defined to be $\mathcal{I}_n \circ \mathcal{L}_n \circ \mathcal{U}_{n-1}$, where \mathcal{U}_{n-1} is the full elimination of the modes in I_{n-1}^- as in Theorem 4.1 with $v = \omega_{n-1}$ (for $t = 1$). Vector fields in the domain of the renormalization operator for some pair (σ_{n-1}, ϕ_n) are said to be renormalizable.

We denote the iteration of the renormalizations up to the step n by

$$\mathcal{R}_n = \mathcal{I}_n \circ \mathcal{L}_n \circ \mathcal{U}_{n-1} \circ \mathcal{R}_{n-1} \quad \text{with } \mathcal{R}_0 = \text{Id},$$

which is defined on \mathcal{A}'_ρ . We highlight the fact that each \mathcal{R}_n is specified by a given set of pairs $\{(\sigma_{k-1}, \phi_k)\}_{1 \leq k \leq n}$. A vector field inside the domain of \mathcal{R}_n , for all n , is called infinitely renormalizable. Notice that $\mathcal{R}_n(\omega_0 + v) = \omega_n$, for every $v \in \mathbb{C}^d$ and any choice of a sequence pair (σ, ϕ) .

Following the previous sections, the map \mathcal{R}_n is analytic on its domain. Also, in case a vector field X is real-analytic, the same is true for $\mathcal{R}_n(X)$.

5.2. Coordinate changes

Assuming that X is in the domain of \mathcal{R}_n , denote by $X_n = \mathcal{R}_n(X)$ so that

$$X_n = \lambda_n (U_0 \circ T_1^{-1} \cdots \circ U_{n-1} \circ T_n^{-1})^*(X) \quad \text{and} \quad X_0 = X, \tag{5.1}$$

where $U_n(X) = \mathcal{U}_n(X_n)$. Thus,

$$P_n^* X_n = \lambda_n W_{n-1}(X)^* \cdots W_0(X)^*(X) \tag{5.2}$$

with the isotopic to the identity diffeomorphisms

$$W_n(X) = P_n^{-1} \circ U_n(X) \circ P_n. \tag{5.3}$$

If X is real-analytic, then $W_n(X)(\mathbb{R}^d) \subset \mathbb{R}^d$, since this property holds for $U_n(X)$. We also have $W_n(\mathbb{I}^+ X) = \text{Id}$.

Finally, by Proposition 3.2, $\text{Rot } \mathcal{R}_n(X) = \lambda_n P_n \text{Rot } X$.

5.3. Infinitely renormalizable vector fields

We want to find sufficient conditions on sequences of pairs (σ, ϕ) in order to obtain infinitely renormalizable vector fields.

For any $\sigma : \mathbb{N}_0 \rightarrow \mathbb{R}^+$ satisfying $\sigma_n < \|\omega_n\|$, we choose $\phi : \mathbb{N} \rightarrow (1, \infty)$ given by

$$\phi_n = \max \left\{ 1, 2(d + 1) \|\tilde{\mathcal{L}}_n\| \frac{\varepsilon_{n-1}}{\varepsilon_n} \right\}. \tag{5.4}$$

Here ε_n is as in (4.6) for the n th step. Then, for any $\rho > 0$, we associate the sequence of analyticity strip widths

$$\rho_n = \frac{\rho - \mathcal{B}_n(\omega, \sigma)}{A_0 \cdots A_{n-1}}, \quad n \in \mathbb{N}, \tag{5.5}$$

where

$$\mathcal{B}_n(\omega, \sigma) = \sum_{i=0}^{n-1} A_0 \cdots A_i \log(e^{\delta + \nu/A_i} \phi_{i+1}) > 0 \tag{5.6}$$

and ν and δ are the constants given in Theorem 4.1 and Lemma 4.3, respectively, taken to be the same for all n . In case $\rho_n \leq 0$ the renormalization procedure stops at step n . Recall that each A_n also depends on σ_n as in (2.13). Finally, define the function

$$\mathcal{B}(\omega, \sigma) = \lim_{n \rightarrow +\infty} \mathcal{B}_n(\omega, \sigma) \tag{5.7}$$

whenever the limit exists (enough being bounded from above).

Theorem 5.1. *If $X \in \mathcal{A}'_\rho$ is real-analytic and*

- $\text{Rot } X = \omega$,
- $\|(\mathbb{I} - \mathbb{E})X\|'_\rho < \varepsilon_0/(d + 1)$,
- $\rho > \mathcal{B}(\omega, \sigma)$,

then X is infinitely renormalizable and

$$\|X_n - \omega_n\|'_\rho < \varepsilon_n, \quad n \in \mathbb{N}. \tag{5.8}$$

Remark 5.2. Our choice (5.4) of ϕ is the “smallest” that we have achieved here so that X is infinitely renormalizable.

Proof of Theorem 5.1. Firstly we remark that by Proposition 3.3,

$$\begin{aligned} \|X - \omega\|'_\rho &\leq \|X - \mathbb{E}X\|'_\rho + \|\mathbb{E}X - \omega\| \\ &\leq (d + 1) \|(\mathbb{I} - \mathbb{E})X\|'_\rho \\ &< \varepsilon_0. \end{aligned} \tag{5.9}$$

So, we can consider the bound on $\|X - \omega\|'_\rho$ instead of $\|(\mathbb{I} - \mathbb{E})X\|'_\rho$.

Notice that if

$$\|X_n - \omega_n\|'_{\rho_n} < \varepsilon_n \tag{5.10}$$

(meaning that at each step X_n is in the domain of \mathcal{U}_n), then X_n is renormalizable and $X_{n+1} = \mathcal{R}_n(X)$. Being true for any $n \in \mathbb{N}$, then X is infinitely renormalizable. The inequality (5.10) can be estimated using Proposition 3.3 and Lemmas 4.4 and 4.3 by

$$\begin{aligned} \|X_n - \omega_n\|'_{\rho_n} &= \|\mathcal{I}_n \mathcal{L}_n \mathcal{U}_{n-1}(X_{n-1}) - \omega_n\|'_{\rho_n} \\ &\leq \|(\mathbb{I} - \mathbb{E})\mathcal{I}_n \mathcal{L}_n \mathcal{U}_{n-1}(X_{n-1})\|'_{\rho_n} + \|\mathbb{E}\mathcal{L}_n(X_{n-1}) - \omega_n\| \\ &\leq (d + 1) \|\mathcal{I}_n(\mathbb{I} - \mathbb{E})\mathcal{L}_n(X_{n-1})\|'_{\rho_n} \\ &\leq (d + 1) \frac{\|\tilde{\mathcal{L}}_n\|}{\phi_n} \|(\mathbb{I} - \mathbb{E})\mathcal{U}_{n-1}(X_{n-1})\|'_\xi, \end{aligned} \tag{5.11}$$

where $\xi = A_{n-1}(\rho_n + \log \phi_n + \delta)$.

We now proceed by induction. Assuming that (5.8) holds for $n - 1$ and substituting the value of ϕ_n ,

$$\|X_n - \omega_n\|'_{\rho_n} \leq (d + 1) \frac{2\|\tilde{\mathcal{L}}_n\|}{\phi_n} \|X_{n-1} - \omega_{n-1}\|'_{\xi'} < \varepsilon_n, \tag{5.12}$$

where $\xi' = \xi + \nu = \rho_{n-1}$. \square

5.4. Width of resonance cones

In the following sections we assume that the conditions of Theorem 5.1 are satisfied. In particular, we will be interested in those vectors ω for which we can find σ as given in the lemma below so that $\mathcal{B}(\omega, \sigma)$ converges and is less than ρ .

Let

$$R_n = \frac{1}{2^n \|T_0\| \cdots \|T_n\|}.$$

Lemma 5.3. *If $\sigma : \mathbb{N}_0 \rightarrow \mathbb{R}^+$ satisfies $\sigma_n \leq \|\omega_n\|$ and*

$$\lim_{n \rightarrow +\infty} \frac{\sigma_n \|P_n^{-1}\|}{R_n (R_{n-1} - R_n) \|\omega_n\|} = 0, \tag{5.13}$$

then there is $N \in \mathbb{N}$ such that, for $n \geq N$,

$$R_n \leq \frac{\rho_n}{\|P_n\|}, \quad \frac{42\|P_n^{-1}\|\varepsilon_n}{\sigma_n} \leq \frac{R_{n-1} - R_n}{2\pi} \tag{5.14}$$

and

$$\lim_{n \rightarrow +\infty} \frac{\|P_n^{-1}\| \varepsilon_n}{R_n(R_{n-1} - R_n)\sigma_n} = 0, \quad \lim_{n \rightarrow +\infty} \frac{\|P_n^{-1}\| \varepsilon_n}{|\lambda_n|} = 0. \tag{5.15}$$

Proof. For the first estimate in (5.14) it is sufficient to check that $\inf_k \rho_k > 0$. Indeed we get the bound for N large enough such that $2^{-N} \leq \inf_k \rho_k$. Now, as $\mathcal{B}_n \leq \mathcal{B}$ (notice the simplification of notation) we have that $\rho - \mathcal{B}_n \geq \rho - \mathcal{B} > 0$. Hence we only need to check that $\sup_n A_0 \cdots A_{n-1} < \infty$. This follows from $\sum_{n \geq 0} A_0 \cdots A_{n-1} < \delta^{-1} \mathcal{B} < \infty$, so that $\lim_{n \rightarrow +\infty} A_0 \cdots A_{n-1} = 0$.

From (5.13) for any choice of $\kappa > 0$ we can find $N \in \mathbb{N}$ so that for $n \geq N$,

$$\sigma_n \leq \kappa \frac{(R_{n-1} - R_n) \|\omega_n\|}{\|P_n^{-1}\|}.$$

Therefore, the second estimate in (5.14) follows using the definition of ε_n , i.e. $\varepsilon_n \leq \sigma_n^2 / \|\omega_n\|$. This also proves the first limit in (5.15).

Finally, notice that (2.10) yields

$$|\lambda_n|^{-1} \leq \|P_n\| \|\omega\| \|\omega_n\|^{-1} \leq \|T_0\| \cdots \|T_n\| \|\omega\| \|\omega_n\|^{-1}.$$

So,

$$\frac{\|P_n^{-1}\| \varepsilon_n}{|\lambda_n|} \ll \frac{\|P_n^{-1}\| \sigma_n^2}{R_n \|\omega_n\|^2}.$$

We then obtain the second limit in (5.15) from (5.13). \square

Remark 5.4. It is simple to check that $\|g \circ P_n\|_{R_n} \leq \|g\|_{\rho_n}$.

6. Class of vectors

Theorem 6.1. *If $\omega \in \mathcal{Y}$, then there exists $\sigma : \mathbb{N}_0 \rightarrow \mathbb{R}^+$ in the conditions of Lemma 5.3 such that $\mathcal{B}(\omega, \sigma) < \infty$.*

Proof. We need to estimate $\mathcal{B}(\omega, \sigma)$. We start by noticing that

$$\phi_{n+1} \ll |\eta_{n+1}| \|T_{n+1}\| \frac{\varepsilon_n}{\varepsilon_{n+1}} \ll |\eta_{n+1}| \|T_{n+1}\| \|\omega_n\| \frac{\sigma_n^2}{\sigma_{n+1}^2}.$$

From (2.26) we have σ satisfying (5.13). So,

$$\mathcal{B}(\omega, \sigma) \ll \sum_{n \geq 0} A_0 \cdots A_n \log \left(|\eta_{n+1}| \|T_{n+1}\| \frac{\sigma_n \|\omega_{n+1}\|}{\sigma_{n+1}} \right)$$

converges whenever $\omega \in \mathcal{Y}$ as in (2.25). \square

7. Analytic conjugacy

7.1. C^1 -conjugacy

Starting from a vector field satisfying $\text{Rot } X = \omega$, we first use Theorem 5.1 to show that there is a smooth conjugacy. Let $\Delta \subset \mathcal{A}'_\rho$ be the subset of all infinitely renormalizable real-analytic vector fields that verify the conditions of Theorem 5.1 while (5.13) holds. Moreover, we fix $N \in \mathbb{N}$ as in Lemma 5.3.

We use the notation $\text{Diff}^r_{\text{per}}$ for a set of isotopic to the identity \mathbb{Z}^d -periodic C^r -diffeomorphisms, i.e. identify $\text{Diff}^r_{\text{per}}(\mathbb{R}^d)$ with $\text{Diff}^r_0(\mathbb{T}^d)$.

Lemma 7.1. *For all $n \geq N$, $W_n : \Delta \rightarrow \text{Diff}^\omega_{\text{per}}(D_{R_n}, D_{R_{n-1}})$ is analytic satisfying*

$$\|W_n(X) - \text{Id}\|_{R_n} \leq \frac{42}{\sigma_n} \|P_n^{-1}\| \|X_n - \omega_n\|'_{\rho_n}. \tag{7.1}$$

Proof. For any $X \in \Delta$, by (4.8) and the first inequality in (5.14),

$$\|W_n(X) - \text{Id}\|_{R_n} = \|P_n^{-1}(U_n(X) - \text{Id})P_n\|_{R_n} \leq \frac{42}{\sigma_n} \|P_n^{-1}\| \|X_n - \omega_n\|'_{\rho_n}.$$

Now, for $x \in D_{R_n}$,

$$\begin{aligned} \|\text{Im } W_n(X)(x)\| &\leq \|\text{Im}(W_n(X)(x) - x)\| + \|\text{Im } x\| \\ &< \|W_n(X) - \text{Id}\|_{R_n} + R_n/2\pi \leq R_{n-1}/2\pi, \end{aligned}$$

where we have used the second inequality in (5.14). Hence $W_n(X) : D_{R_n} \rightarrow D_{R_{n-1}}$. From the properties of \mathfrak{U}_n , $W_n : \Delta \rightarrow \text{Diff}^\omega_{\text{per}}(D_{R_n}, D_{R_{n-1}})$ is analytic. \square

For $n \geq m \geq 0$ consider the analytic map $H_{m,n} : \Delta \rightarrow \text{Diff}^\omega_{\text{per}}(D_{R_n}, \mathbb{C}^d)$ defined by

$$H_{m,n}(X) = W_m(X) \circ \dots \circ W_n(X). \tag{7.2}$$

In particular, by the above lemma, $H_{m,n}(X) : D_{R_n} \rightarrow D_{R_{m-1}}$ whenever $m \geq N$.

Lemma 7.2. *For $X \in \Delta$ and $n > m \geq N$,*

$$\begin{aligned} \|H_{m,n}(X) - H_{m,n-1}(X)\|_{R_n} &\leq \frac{R_{m-1}}{\frac{(R_{n-1}-R_n)\sigma_n}{2.42\|P_n^{-1}\|\|X_n-\omega_n\|'_{\rho_n}} - 1}, \\ \|H_{m,n}(X) - \text{Id}\|_{R_n} &\leq 42 \sum_{i=m}^n \frac{\|P_i^{-1}\|}{\sigma_i} \|X_i - \omega_i\|'_{\rho_i}. \end{aligned} \tag{7.3}$$

Proof. For each $k = m, \dots, n - 1$, consider the transformations

$$G_k(z, X) = (W_k(X) - \text{Id}) \circ (\text{Id} + G_{k+1}(z, X)) + G_{k+1}(z, X),$$

$$G_n(z, X) = z(W_n(X) - \text{Id}),$$

with $(z, X) \in \{z \in \mathbb{C}: |z| < 1 + d_n\} \times \Delta$, where

$$d_n = \frac{R_{n-1} - R_n}{2\pi} \frac{\sigma_n}{42 \|P_n^{-1}\| \|X_n - \omega_n\|'_{\rho_n}} - 1$$

is positive by the second inequality in (5.14). If the image of D_{R_n} under $\text{Id} + G_{k+1}(z, X)$ is inside the domain of $W_k(X)$, or simply

$$\|G_{k+1}(z, X)\|_{R_n} \leq (R_k - R_n)/2\pi,$$

then G_k is well defined as an analytic map into $\text{Diff}_{\text{per}}^{\omega}(D_{R_n}, \mathbb{C}^d)$, and

$$\|G_k(z, X)\|_{R_n} \leq \|W_k(X) - \text{Id}\|_{R_k} + \|G_{k+1}(z, X)\|_{R_n}. \tag{7.4}$$

An inductive scheme shows that

$$\begin{aligned} \|G_n(z, X)\|_{R_n} &= |z| \|W_n(X) - \text{Id}\|_{R_n} \\ &\leq (R_{n-1} - R_n)/2\pi, \\ \|G_k(z, X)\|_{R_n} &\leq \sum_{i=k}^{n-1} \|W_i(X) - \text{Id}\|_{R_i} + \|G_n(z, X)\|_{R_n} \\ &\leq (R_{k-1} - R_n)/2\pi, \end{aligned}$$

using Lemmas 7.1 and 5.3.

It is easy to check that $G_m(1, X) = H_{m,n}(X) - \text{Id}$ and $G_m(0, X) = H_{m,n-1}(X) - \text{Id}$. In particular, from (7.4) and Lemma 7.1

$$\begin{aligned} \|H_{m,n}(X) - \text{Id}\|_{R_n} &= \|G_m(1, X)\|_{R_n} \leq \sum_{i=m}^n \|W_i(X) - \text{Id}\|_{R_i} \\ &\leq \sum_{i=m}^n \frac{42 \|P_i^{-1}\|}{\sigma_i} \|X_i - \omega_i\|'_{\rho_i}. \end{aligned} \tag{7.5}$$

Finally, by Cauchy’s formula

$$\begin{aligned} \|H_{m,n}(X) - H_{m,n-1}(X)\|_{R_n} &= \|G_m(1, X) - G_m(0, X)\|_{R_n} \\ &= \left\| \frac{1}{2\pi i} \oint_{|z|=1+d_n/2} \frac{G_m(z, X)}{z(z-1)} dz \right\|_{R_n} \\ &\leq \frac{2}{d_n} \sup_{|z|=1+d_n/2} \|G_m(z, X)\|_{R_n} \leq \frac{R_{m-1}}{\pi d_n}. \quad \square \end{aligned}$$

Lemma 7.3. *There exists an analytic map $H : \Delta \rightarrow \text{Diff}_0^1(\mathbb{T}^d)$ such that*

$$H(X) = \lim_{n \rightarrow +\infty} H_{0,n}(X). \tag{7.6}$$

Proof. Consider $C_{\text{per}}^1(\mathbb{R}^d, \mathbb{C}^d)$ to be the Banach space of the \mathbb{Z}^d -periodic C^1 functions between \mathbb{R}^d and \mathbb{C}^d with norm

$$\|f\|_{C^1} = \max_{k \leq 1} \max_{x \in \mathbb{R}^d} \|D^k f(x)\|. \tag{7.7}$$

Let $m \geq N$. Since the domains D_{R_n} are shrinking, consider the restrictions of $W_n(X)$ and $H_{m,n}(X)$ to \mathbb{R}^d . So, for any $X \in \Delta$, Cauchy’s estimate and Lemma 7.2 holds that

$$\begin{aligned} \|H_{m,n}(X) - H_{m,n-1}(X)\|_{C^1} &\leq \max_{k \leq 1} \sup_{x \in D_{R_n/2}} \|D^k [H_{m,n}(X)(x) - H_{m,n-1}(X)(x)]\| \\ &\leq C_1 \frac{1}{R_n} \|H_{m,n}(X) - H_{m,n-1}(X)\|_{R_n} \end{aligned} \tag{7.8}$$

goes to zero by (7.3) and the first bound in (5.15), where $C_1 > 0$ is a constant. Thus, $H_{m,n}(X)$ converges to $H_m(X)$ in $C_{\text{per}}^1(\mathbb{R}^d, \mathbb{C}^d)$. Similarly, there is a constant $C_2 > 0$ such that $\|H_m(X) - \text{Id}\|_{C^1} \leq C_2 \sigma_m^{-1} \|P_m^{-1}\| \|X_m - \omega_m\|'_{\rho_m}$. For m large enough we can use the first limit in (5.15) to show that $\|H_m(X) - \text{Id}\|_{C^1} < 1$. Hence, $H_m(X)$ is a diffeomorphism isotopic to the identity.

The convergence of $H_{m,n}$ is uniform in Δ so H_m is an analytic map. The fact that, for real-analytic X , $H_m(X)$ takes real values for real arguments, follows from the same property for each $W_n(X)$.

Take now the analytic map

$$H = W_0 \cdots W_{m-1} \circ H_m.$$

It follows from the above that for every real-analytic $X \in \Delta$, $H(X) \in \text{Diff}_0^1(\mathbb{T}^d)$. \square

Theorem 7.4. *For every real-analytic $X \in \Delta$, $H(X)^*(X) = \omega$ on \mathbb{R}^d .*

Proof. From (5.2) we have

$$H_{0,n}(X)^*(X) - \omega = \lambda_n^{-1} P_n^*(X_n) - \omega = \lambda_n^{-1} P_n^*(X_n - \omega_n).$$

Since $\|\lambda_n^{-1} P_n^*(X_n - \omega_n)\|_{C^0} \leq |\lambda_n^{-1}| \|P_n^{-1}\| \|X_n - \omega_n\|'_{\rho_n} \rightarrow 0$ by the second limit in (5.15) and $H_{0,n} \rightarrow H$ as $n \rightarrow +\infty$, we complete the proof. \square

7.2. *From C^1 to C^ω -conjugacy*

Because of the analyticity dependence of the conjugacy map H with respect to the vector field X , we will show that the conjugacy can be extended analytically to a complex strip.

Lemma 7.5. *If $X \in \Delta$ and $x \in \mathbb{R}^d$, then*

$$H(X \circ R_x) = R_x^{-1} \circ H(X) \circ R_x. \tag{7.9}$$

Proof. The relations (4.10) and (4.12) yield that $\mathcal{U}_n(X \circ R_x) = \mathcal{U}_n(X) \circ R_x$ and $\mathcal{L}_n(X \circ R_x) = \mathcal{L}_n(X) \circ R_{T_n x}$. This implies immediately that

$$\mathcal{R}_n(X \circ R_x) = \mathcal{R}_n(X) \circ R_{P_n x}. \tag{7.10}$$

Next, from a simple adaptation of (4.10) and the formula $R_{P_n z} = P_n R_z P_n^{-1}$ for $z \in \mathbb{C}^d$, we get

$$\begin{aligned} W_n(X \circ R_x) &= P_n^{-1} \circ \mathcal{U}_{a_n}(\mathcal{L}_n \mathcal{R}_{n-1}(X \circ R_x)) \circ P_n \\ &= R_x^{-1} \circ W_n(X) \circ R_x. \end{aligned} \tag{7.11}$$

Thus, $H_{0,n}(X \circ R_x) = R_x^{-1} \circ H_{0,n}(X) \circ R_x$. The convergence of $H_{0,n}$ implies (7.9). \square

Theorem 7.6. *For every real-analytic $X \in \Delta$, $H(X) \in \text{Diff}_0^{\omega}(\mathbb{T}^d)$.*

Proof. Take r such that $\rho > r > \mathcal{B}(\omega, \sigma)$. For each $X \in \Delta$ and $z \in D_\eta$ with $\eta = \rho - r > 0$, $X \circ R_z \in \mathcal{A}'_r$ and $\|X \circ R_z - \omega\|_r \leq \|X - \omega\|_\rho$. So, we define the set of infinitely renormalizable vector fields $\Delta_r \subset \mathcal{A}'_r$ and can use the previous results to show the existence of the conjugacy $H(X \circ R_z)$.

We now need to analytically extend the conjugacy $H(X)$ to a complex neighbourhood of \mathbb{R}^d . Let $F(z) = z + H(X \circ R_z)(0)$. The maps $z \mapsto X \circ R_z$, $X \mapsto H(X)$ and $C^1_{\text{per}}(\mathbb{R}^d, \mathbb{C}^d) \ni g \mapsto g(0)$ are analytic, thus F is also analytic on D_η and $F - \text{Id}$ is \mathbb{Z}^d -periodic. It remains to show that F is an analytic extension of $H(X)$. From (7.9), for any $x \in \mathbb{R}^d$, we have

$$\begin{aligned} F(x) &= x + R_x^{-1} \circ H(X) \circ R_x(0) \\ &= x + H(X)(x) - x \\ &= H(X)(x). \quad \square \end{aligned} \tag{7.12}$$

8. Small analyticity strip

The above results (viz. Theorems 5.1 and 7.6) can be generalized for a small analyticity radius ρ . Thus, through Theorem 6.1, we conclude the proof of Theorem 1.1.

Theorem 8.1. *Let $\omega \in \mathcal{Y}$. If $X \in \mathcal{A}_r$, $r > 0$, $\text{Rot } X = \omega$ and X is sufficiently close to constant, then there is an analytic diffeomorphism ψ such that $\tilde{X} = \psi^* X$ is in the conditions of Theorem 5.1, thus infinitely renormalizable.*

Proof. First, we observe that $X \in \mathcal{A}'_\rho$ for some $\rho < r$ because $\|X\|'_\rho \leq (1 + 2\pi/(r - \rho))\|X\|_r$. Moreover, by Proposition 3.3,

$$\|X - \omega\|'_\rho \leq (d + 1)\|(\mathbb{I} - \mathbb{E})X\|'_\rho.$$

By considering a sufficiently large N we want to apply Theorem 5.1 to

$$\tilde{X} = \mathcal{L}_N \mathcal{U}_{N-1} \cdots \mathcal{L}_1 \mathcal{U}_0(X)$$

in \mathcal{A}'_{ρ_N} , with $\|(\mathbb{I} - \mathbb{E})X\|'_\rho$ small enough. (Notice that we are not including any operator \mathcal{I} , thus no need for ϕ .) Under a suitable choice of resonance width and stopping-time sequences up to step N , we recover the large strip case since

$$\rho_N = \frac{\rho - \sum_{i=0}^{N-1} A_0 \cdots A_i (\delta + \nu/A_i)}{A_0 \cdots A_{N-1}}. \tag{8.1}$$

The numerator above can be made larger than some positive constant for any choice of a finite N . It remains to check that $\rho_N > \mathcal{B}(\omega_N, \sigma')$, now for $\sigma'_n = \sigma_{N+n}$ given by Theorem 6.1. As

$$\mathcal{B}(\omega_N, \sigma') = \frac{(\mathcal{B} - \mathcal{B}_N)(\omega, \sigma)}{A_0 \cdots A_{N-1}} \tag{8.2}$$

we compare (8.2) with (8.1) by noticing that $\mathcal{B}_N \rightarrow \mathcal{B}$ as $N \rightarrow +\infty$. So, for N sufficiently large (but finite), (8.2) can be made less than ρ_N . \square

Theorem 8.2. *Let $\omega \in \mathcal{Y}$. If $v \in \text{Vect}^\omega(\mathbb{T}^d)$ generates a flow with rotation vector ω and v is sufficiently close to constant, then there exists $h \in \text{Diff}_0^\omega(\mathbb{T}^d)$ such that*

$$h^*(v) = \omega. \tag{8.3}$$

Proof. The lift to \mathbb{R}^d of v is assumed to have an analytic extension in D_r . Theorem 8.1 then gives $\tilde{v} \in \mathcal{A}_\rho$ and, as long as v is close enough to constant, we have $\tilde{v} \in \Delta$. Then, by Theorems 7.6 and 7.4, the analytic diffeomorphism $h = H(\tilde{v}) \bmod 1$ verifies (8.3). \square

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