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Hyperbolic Polygonal Billiards Close to 1-Dimensional Piecewise Expanding Maps

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Abstract

We consider polygonal billiards with collisions contracting the reflection angle towards the normal to the boundary of the table. In previous work, we proved that such billiards have a finite number of ergodic SRB measures supported on hyperbolic generalized attractors. Here we study the relation of these measures with the ergodic absolutely continuous invariant probabilities (acips) of the slap map, the 1-dimensional map obtained from the billiard map when the angle of reflection is set equal to zero. We prove that if a convex polygon satisfies a generic condition called (*), and the reflection law has a Lipschitz constant sufficiently small, then there exists a one-to-one correspondence between the ergodic SRB measures of the billiard map and the ergodic acips of the corresponding slap map, and moreover that the number of Bernoulli components of each ergodic SRB measure equals the number of the exact components of the corresponding ergodic acip. The case of billiards in regular polygons and triangles is studied in detail.

Keywords Billiards · Hyperbolic systems with singularities · SRB measures · Ergodicity · Piecewise expanding maps

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1 Introduction

The dynamics of billiards has been studied in great detail when the reflection law is the specular one: the angle of reflection equals the angle of incidence. For an account on the subject, we refer the reader to [17]. In a series of recent works, we studied polygonal billiards with a reflection law, i.e., a function f describing the dependence of the angle of reflection from the angle of incidence—both measured with respect to the normal of the billiard table—that is not the identity function as for the specular reflection law, but a strict contraction having the zero angle as its fixed point.

The dynamics of polygonal billiards with contracting reflection laws differ significantly from that of polygonal billiards with specular reflection law: whereas the latter are non-hyperbolic systems, the former generically have uniformly hyperbolic attractors supporting a finite number of ergodic Sinai–Ruelle–Bowen measures (SRB measures for short) [5,6,8]. Some billiards in non-polygonal tables with non-specular reflection laws were studied in [1,2,12].

When the function f is identically equal to 0, i.e., when the angle of reflection θ is identically equal to zero, the billiard map is no longer injective and its image is a 1-dimensional set. The restriction of the billiard map to this subset is a piecewise affine map of the circle called *slap map* [12]. The precise form of the slap map depends only on the polygonal table. If a polygon does not have parallel sides (in fact, a weaker condition introduced later on suffices), then the corresponding slap map is uniformly expanding, and admits a finite number of ergodic absolutely continuous invariant probabilities (acips for short) [7].

Given a polygon P and a contracting reflection law f, we denote by $\Phi_{f,P}$ the map of the billiard in P with reflection law f, and by ψ_P the slap map of P. Precise definitions will be given in Sect. 2. The Lipschitz constant $\lambda(f)$ measures how close $\Phi_{f,P}$ is to $\Phi_{0,P}$ (here $f \equiv 0$), but the image of $\Phi_{0,P}$ is 1-dimensional, and the restriction of $\Phi_{0,P}$ to its image is essentially equal to ψ_P. In this paper, we address the natural question 'what is the relation between the properties of $\Phi_{f,Q}$ and ψ_P when $\lambda(f)$ is small and the polygon Q is close to P?'. In particular, we study the relation between the ergodic SRB measures of $\Phi_{f,Q}$ and the ergodic acips of ψ_P.

The results presented in this paper are formulated for convex polygons only. Analogous results can be obtained for some classes of non-convex polygons, but their proofs are much more involved than the proofs for convex polygons.

Two polygons are *similar* if one polygon can be transformed into the other one by a similarity transformation of the Euclidean plane preserving the orientation (see [6, Sect. 5]). The dynamics of billiards in similar polygons is the same. Similarity is an equivalence relation on the space of polygons with n sides. We denote by $\mathcal{P}_n$ the quotient of such a space by the relation of similarity. In [6, Proposition 5.1], we proved that $\mathcal{P}_n$ is diffeomorphic to an open semialgebraic subset of $\mathbb{P}^1 \times (\mathbb{P}^2)^{n-3} \times \mathbb{P}^1$ with $\mathbb{P}^1$ and $\mathbb{P}^2$ being the real projective line and real projective plane, respectively, and that $\mathcal{P}_n$ is a manifold of dimension $2n-4$. Hence, the metric and the measure of $\mathbb{P}^1 \times (\mathbb{P}^2)^{n-3} \times \mathbb{P}^1$ induce a metric d and a measure m on the set $\mathcal{P}_n$.

Let q be a non-acute vertex of P. Denote by $q_+ \neq q$ and $q_- \neq q$ the intersection points of ∂P with the two lines passing through q each orthogonal to one of the sides of P meeting at q. The sequence $O_+(q) := \{q_0 = q_+, q_1, \ldots\}$ is defined recursively as follows: for each $i \geq 0$ if q_i is a vertex of P, then $q_{i+1} = q_i$, otherwise set q_{i+1} to be the intersection point not equal to q_i of ∂P with the line passing through q_i and orthogonal to ∂P at q_i. Define the

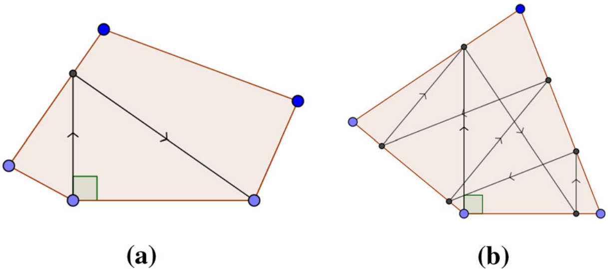

Fig. 1 Polygons that do not satisfy Condition (*): **a** an orthogonal orbit ending at a vertex. **b** An orthogonal orbit that is eventually periodic

sequence $O_-(q)$ similarly by setting $q_0 = q_-$. We call $O_-(q)$ and $O_+(q)$ the *orthogonal orbits* of q.

Definition 1.1 A convex polygon P satisfies Condition (*) if for every non-acute vertex q of P, each set $O_-(q)$ and $O_+(q)$ does not contain any vertex of P and is not eventually periodic.

Figure 1 shows examples of polygons that do not satisfy Condition (*).

If a convex polygon P satisfies Condition (*), then all polygons similar to P satisfy the condition as well. We denote by $\mathcal{P}_n^*$ the subset of $\mathcal{P}_n$ formed by the equivalence classes satisfying Condition (*).

Let $\mathcal{E}(\psi_P)$ be the set of the ergodic acips of ψ_P, and let $\mathcal{E}(\Phi_{f,Q})$ be the set of the ergodic SRB measures of $\Phi_{f,Q}$. The main result of the paper is the following theorem.

Theorem 1.2 *Given $P \in \mathcal{P}_n^*$, there exists $\delta > 0$ such that if f is a contracting reflection law f that is a C^2 embedding with $\lambda(f) < \delta$ and $Q \in \mathcal{P}_n$ with $d(Q, P) < \delta$, then there exists a bijection $\Theta_{f,Q} \colon \mathcal{E}(\psi_P) \to \mathcal{E}(\Phi_{f,Q})$. Moreover,*

1. *the supports of the measures in $\mathcal{E}(\Phi_{f,Q})$ are pairwise disjoint,*
2. *the cardinality of $\mathcal{E}(\Phi_{f,Q})$ is less than or equal to n,*
3. *for every $\nu \in \mathcal{E}(\psi_P)$, the number of Bernoulli components of $\Theta_{f,Q}(\nu)$ equals the number of exact components of ν,*
4. *the union of the basins of the measures in $\mathcal{E}(\Phi_{f,Q})$ is a set of full volume in the domain of $\Phi_{f,Q}$.*

Condition (*) plays a major role in our analysis, and so it is important to know whether the set of polygons satisfying Condition (*) is large in the topological and measure theoretical sense. In Proposition 2.3, we prove that $\mathcal{P}_n^*$ is a full measure residual subset of $\mathcal{P}_n$. Theorem 1.2 and Proposition 2.3 yield immediately that the set of polygons for which the conclusion of Theorem 1.2 holds is generic and has full measure in $\mathcal{P}_n$.

The strategy of the proof of Theorem 1.2 is as follows. First, we prove that for every $P \in \mathcal{P}_n^*$, there exist pairwise disjoint sets $W_1, \ldots, W_k$ with k equal to the cardinality of $\mathcal{E}(\psi_P)$ that are trapping sets for all maps $\Phi_{f,Q}$ sufficiently close to ψ_P. Then, given one of such maps $\Phi_{f,Q}$, we construct the bijection $\Theta_{f,Q}$ by establishing two facts: (1) the support of each measure $\mu \in \mathcal{E}(\Phi_{f,Q})$ is contained in some trapping set W_i, and (2) each W_i contains exactly the support of some measure $\mu \in \mathcal{E}(\Phi_{f,Q})$. The proof of Theorem 1.2 exploits properties of the periodic points of ψ_P that carry over to periodic points of maps $\Phi_{f,Q}$ sufficiently close to ψ_P. Another interesting ingredient of the proof is a novel criterion for the ergodicity of an SRB measure of $\Phi_{f,Q}$. This criterion can be easily generalized to general hyperbolic map with singularities. In fact, we believe that our proof of Theorem 1.2 can be adapted to cover a rather general class of two-dimensional hyperbolic maps with singularities close in a proper sense to piecewise expanding 1-dimensional maps.

Finally, when P is a convex regular polygon or a triangle, using Theorem 1.2, Proposition 2.3 and results from [7], we are able to compute the exact number of ergodic SRB measures and the Bernoulli components of maps $\Phi_{f,Q}$ sufficiently close to ψ_P (see Sect. 6).

The paper is organized as follows. In Sect. 2, we define the objects studied in this paper: the billiard map for a polygonal billiard with a contracting reflection law and the related slap map. Moreover, we prove that Condition (*) is a generic property in the space of polygons. In Sect. 3, we give a sufficient condition for the existence of hyperbolic attractors of a billiard map, and recall the basic notions of Pesin's Theory specialized to our billiards. We also recall a general result on the existence and the spectral decomposition of absolute invariant probabilities of piecewise expanding maps, which applies to the slap maps considered in this paper. The preliminary results necessary to prove Theorem 1.2 are presented in Sect. 4, whereas the final part of its proof is contained in Sect. 5. In Sect. 6, we apply Theorem 1.2 and Proposition 2.3 to billiards in convex regular polygons and triangles.

2 Billiards and Slap Maps

A billiard in a polygon P is a mechanical system formed by a point-particle moving with uniform motion inside P and bouncing off the boundary ∂P according to a given rule, which is a function called *reflection law* whose argument and value are, respectively, the angle of incidence and the angle of reflection of the particle at the collision point. In the usual definition of a billiard, the reflection law is the specular one prescribing the equality between the angle of reflection and the angle of incidence. In this paper, we consider reflection laws that are strict contractions with small (in a sense that will be explained later) Lipschitz constant.

2.1 Polygonal Billiards

Let P be a convex polygon with n sides and perimeter equal to 1. We choose a positively oriented parametrization of ∂P by arc length so that $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ are the values of the arc length parameter corresponding to the vertices of P. The values $s = 0$ and 1 correspond to the same vertex of P. In the following, we identify the points of ∂P with their arc length parameter s (with the additional proviso that $s = 0$ and $s = 1$ denote the same point). In other words, we identify ∂P with the circle S^1 of perimeter 1. Denote by $V_P = \{s_0, \ldots, s_{n-1}\}$ the set of the vertices of P.

Let $M = S^1 \times (-\pi/2, \pi/2)$. We denote by d_{S^1} the standard distance on S^1, and by d_M the Euclidean distance of the cylinder M. Also, we denote by Vol the volume generated by d_M on M, and by $\|\cdot\|$ the Euclidean norm of $\mathbb{R}^2$. Finally, we denote by π_s and π_θ the projections defined by $\pi_s(s,\theta) = s$ and $\pi_\theta(s,\theta) = \theta$ for $(s,\theta) \in M$. A curve $\Gamma \subset M$ is called a *horizontal segment* if $\pi_\theta(\Gamma) = const$.

Let $M_P = \bigcup_{i=0}^{n-1}(s_i, s_{i+1}) \times (-\pi/2, \pi/2) \subset M$. We associate to each element $(s,\theta) \in M_P$ the unit vector v of $\mathbb{R}^2$ with base point s making an angle θ with the inner normal to ∂P at s. Such a normal is not defined at the vertices of P, which is the reason for not including the set $\bigcup_{i=0}^{n-1}\{s_i\} \times (-\pi/2, \pi/2)$ in M_P. Each pair $(s,\theta) \in M_P$ specifies the state of the particle immediately after a collision with ∂P: the collision point is given by s and the velocity is given by the unit vector v.

Given a particle in the state $(s, \theta) \in M_P$, we define $g_P(s, \theta)$ to be the point of collision of the particle with ∂P, and $t_P(s, \theta)$ to be Euclidean distance in $\mathbb{R}^2$ between the points of ∂P corresponding to s and $g_P(s, \theta)$. Let $Y_P = \{s_0, \dots, s_{n-1}\} \times (-\pi/2, \pi/2)$, and let $N_P = g_P^{-1}(Y_P)$. The *standard billiard map for the polygon P* is the map $\Phi_P : M_P \setminus N_P \to M_P$ whose image $\Phi_P(s, \theta) \in M_P$ corresponds to the state of the particle right after the collision at $g_P(s, \theta)$ when the reflection law is the specular one. We denote by $h_P(s, \theta)$ the angle of the particle after the collision. Hence, $\Phi_P(s, \theta) = (g_P(s, \theta), h_P(s, \theta))$. It is not difficult to see that Φ_P is piecewise analytic. For a detailed definition of the map Φ_P, we refer the reader to [4].

2.2 Contracting Reflection Laws

A reflection law is a function

$$f : (-\pi/2, \pi/2) \to (-\pi/2, \pi/2).$$

For instance, the specular reflection law corresponds to the identity function $f(\theta) = \theta$. Given a reflection law f, denote by $R_f : M_P \to M_P$ the map $R_f(s, \theta) = (s, f(\theta))$. The *billiard map for the polygon P with reflection law f* is the transformation $\Phi_{f,P} : M_P \setminus N_P \to M_P$ defined by

$$\Phi_{f,P} = R_f \circ \Phi_P = (g_P(s, \theta), f \circ h_P(s, \theta)) .$$

Note that $\Phi_{f,P}$ is injective if and only if f is, and that $\Phi_{f,P}$ is a C^k, $k > 0$ diffeomorphism onto its image of if and only if f is.

The differential $d_x\Phi_{f,P}$ is given by [6, Sect. 2.5]

$$d_x\Phi_{f,P} = -\begin{pmatrix} \dfrac{\cos\theta}{\cos(h_P(s,\theta))} & \dfrac{t_P(s,\theta)}{\cos(h_P(s,\theta))} \\ 0 & f'(h_P(s,\theta)) \end{pmatrix} . \tag{2.1}$$

Definition 2.1 A reflection law f is called *contracting* if f is of class C^1, $f(0) = 0$ and $\lambda(f) := \sup\{|f'(\theta)| : \theta \in (-\pi/2, \pi/2)\} < 1$.

The simplest example of a contracting reflection law is $f(\theta) = \sigma\theta$ with $0 < \sigma < 1$. This law was considered in several papers [1,2,5,12].

We denote by $\mathcal{R}$ the set of all contracting reflection laws. It is easy to verify that $\mathcal{R}$ is a Banach space with the norm $\lambda(f)$. We denote by $\mathcal{R}^k$, $k \geq 1$ the set all contracting reflection laws that are C^k diffeomorphisms onto their images. The reflection law $f \equiv 0$ is denoted by 0.

In order to apply Pesin's theory to $\Phi_{f,P} : M_P \setminus N_P \to M_P$ and to establish the existence of stable and unstable local manifolds, $\Phi_{f,P}$ has to be a C^2 diffeomorphism onto its image $\Phi_{f,P}(M_P \setminus N_P)$, which is the case if $f \in \mathcal{R}^2$.

2.3 Slap Maps and Condition (*)

When $f = 0$, the billiard trajectories in P are all orthogonal to ∂P after every collision. Thus, the image of the map $\Phi_{0,P}$ is a subset of the segment $S^1 \times \{0\}$. If $f \in \mathcal{R}$ and $\lambda(f)$ is sufficiently small, then $\Phi_{f,P}$ can be considered as a small perturbation of $\Phi_{0,P}$. Indeed, the two maps have the same domain $M_P \setminus N_P$, and from the definition of $\Phi_{f,P}$ and the expression of $d_x\Phi_{f,P}$, it follows that $\Phi_{f,P}$ and $\Phi_{0,P}$ are $\lambda(f)$-close in the C^1 topology.

We now introduce a 1-dimensional map related to $\Phi_{0,P}$. First, let $I_P = \bigcup_{i=0}^{n-1}(s_i, s_{i+1}) \subset S^1$, and define $F_P\colon I_P \to S^1$ by $F_P(s) = g_P(s, 0)$ for all $s \in I_P$. The map F_P is related to $\Phi_{0,P}$, since $\Phi_{0,P}^n(s, 0) = (F_P^n(s), 0)$ for all $(s, 0) \in M_P \setminus N_P$ and all $n \in \mathbb{N}$. Moreover, F_P is affine and strictly decreasing on each connected component of I_P. For this reason, F_P admits a unique extension to the whole S^1 that is left[1] continuous at each point $s_0, s_1, \ldots, s_{n-1}$. We denote such an extension by $\psi_P\colon S^1 \to S^1$, and call it the *slap map* of P.

The *singular set* of a piecewise expanding map is the set of point where the map does not have continuous second derivatives. It is not difficult to see that ψ_P is analytic at s_i if and only if s_i is a vertex with an acute internal angle. In fact, in that case, $\psi_P(s_i) = s_i$. It follows that the singular set S_P of ψ_P is the set of all non-acute vertices of P.

Condition (*) introduced in Definition 1.1 can be equivalently formulated in terms of the slap map ψ_P as follows: a polygon $P \in \mathcal{P}_n$ satisfies Condition (*) if for every $s \in S_P$, the forward orbits of

$$\psi_P(s^+) := \lim_{t \to s^+} \psi_P(t) \quad \text{and} \quad \psi_P(s^-) := \lim_{t \to s^-} \psi_P(t)$$

do not contain elements of S_P or periodic points of ψ_P.

Remark 2.2 In [9, Sect. 3], we introduced a condition for general piecewise expanding maps of the interval called Condition (*) as well. When specialized to slap maps, that condition becomes Condition (*) as written above.

Proposition 2.3 *The set $\mathcal{P}_n^*$ is a full measure residual subset of $\mathcal{P}_n$.*

Proof We give only the main ideas of the proof. The reader can find the details in the proof of [6, Proposition 5.3] which is very similar to this proof.

Let $P \in \mathcal{P}_n$, and define ℓ_j to be the line supporting the jth side of P. The k-itinerary of an orbit of the slap map ψ_P is a k-tuple $\underline{i} := (i_1, \ldots, i_k)$ with $i_1, \ldots, i_k$ being the labels of the sides of P visited by the first k elements of the orbit. All orbits of ψ_P with a given k-itinerary $\underline{i}$ are solutions of the equation $y = F_{\underline{i}}(u, x)$, where $x \mapsto F_{\underline{i}}(u, x)$ is an affine map, x is a linear coordinate on ℓ_{i_0}, y is the coordinate of the corresponding endpoint in ℓ_{i_k}, and $u \in \mathbb{P}^1 \times (\mathbb{P}^2)^{n-3} \times \mathbb{P}^1$ is the coordinate of the polygon P. Systems of two equations of the form $F_{\underline{i}}(u, 0) = c$ and $F_{\underline{j}}(u, F_{\underline{i}}(u, 0)) = F_{\underline{i}}(u, 0)$ in the unknown u determine the sets of all polygons for which the trajectory of a vertex with coordinate 0 ends up at another vertex with coordinate c or at a pre-periodic point of the slap map of P (see [6, Proposition 5.3] for more details). These sets are closed algebraic sets of codimension 1. Therefore, the complement of $\mathcal{P}_n^*$ is a countable union of algebraic sets of codimension 2 determined by the two equations above. ◻

3 Hyperbolic Polygonal Billiards

3.1 Hyperbolic Attractors

Let P a polygon. For every $f \in \mathcal{R}$ with $f \neq 0$, define

$$K_{f,P} = \left\{ (s, \theta) \in M_P\colon |\theta| < \lambda(f)\frac{\pi}{2} \right\}.$$

[1] We might have as well chosen the extension to be right continuous.

For the special case $f = 0$, define $K_{0,P} = \bigcup_{i=0}^{d-1}(s_i, s_{i+1}) \times \{0\}$, and conventionally choose the boundary of $K_{0,P}$ to be the empty set, that is, $\partial K_{0,P} = \emptyset$. Since f is a contraction, the set $K_{f,P}$ is forward invariant

$$\Phi_{f,P}(K_{f,P} \setminus N_P) \subset K_{f,P}.$$

From now on, we will focus our attention to the restriction of $\Phi_{f,P}$ to $K_{f,P} \setminus N_P$, which by abuse of notation we still denote by $\Phi_{f,P}$.

Not every every element of $K_{f,P} \setminus N_P$ can be iterated indefinitely due to the set N_P. The set of all elements of $K_{f,P} \setminus N_P$ with positive semi-orbit is

$$K^+_{f,P} := \left\{(s,\theta) \in K_{f,P} \colon \Phi_{f,P}(s,\theta) \notin N_P \; \forall n \geq 0\right\}.$$

Then the maximal forward invariant set of $\Phi_{f,P}$ is

$$D_{f,P} := \bigcap_{n \geq 0} \Phi^n_{f,P}(K^+_{f,P}).$$

Note that if $f \in \mathcal{R}^1$, then $D_{f,P}$ is also the maximal invariant set of $\Phi_{f,P}$, meaning that $\Phi^{-1}_{f,P}(D_{f,P}) = D_{f,P}$. Following [14], we call

$$\Lambda_{f,P} := \overline{D}_{f,P}$$

the *attractor* of $\Phi_{f,P}$, and

$$N^+_{f,P} := (N_P \cap K_{f,P}) \cup \partial K_{f,P}$$

the *singular set* of $\Phi_{f,P}$.

In [6,12], it was proved that for every $f \in \mathcal{R}^1$ and every polygon P, the set $D_{f,P}$ has a weak form of hyperbolicity called *dominated splitting*. In this paper, we are interested in the case when $D_{f,P}$ is a *hyperbolic set*, that is, when the tangent space of $K_{f,P}$ at each point $x \in D_{f,P}$ splits into complementary invariant subspaces $E^s(x)$ and $E^u(x)$ that are uniformly contracted and expanded by the differential of $\Phi_{f,P}$.

Definition 3.1 The attractor $\Lambda_{f,P}$ is called *hyperbolic* if $D_{f,P}$ is a hyperbolic set.

Definition 3.2 A polygon P has *parallel sides facing each other*[2] if there exist parallel sides L_1 and L_2 of P and points q_1 and q_2 contained in the interior of L_1 and L_2, respectively, such that the segment joining q_1 and q_2 is contained in P, intersects only the sides L_1 and L_2 of P, and is perpendicular to both L_1 and L_2.

The following proposition was proved in [6, Proposition 3.2 and Corollary 3.4].

Proposition 3.3 *Suppose that $f \in \mathcal{R}^1$. Then $\Lambda_{f,P}$ is hyperbolic if and only if P does not have parallel sides facing each other. Moreover, if $\Lambda_{f,P}$ is hyperbolic, then the unstable direction E^u coincides with the horizontal direction $\theta = const.$ at every point of $D_{f,P}$.*

Remark 3.4 Note that the horizontal direction is always invariant even if $D_{f,P}$ is not hyperbolic. This peculiar property is a consequence of the fact that the angle formed by two trajectories bouncing off the same side of the polygon does not change after the collision no matter how the reflection law $f \in \mathcal{R}$ is chosen.

[2] This notion is not exactly equal to the one given in [6]. According to this definition arbitrarily small perturbations of a polygon without parallel sides facing each other may have parallel sides facing each other.

The next lemma concerns Condition (*) and the property of a polygon of having parallel sides facing each other.

Lemma 3.5 *If $P \in \mathcal{P}_n^*$, then there exists $\delta > 0$ such that if $Q \in \mathcal{P}_n$ and $d(Q, P) < \delta$, then Q does not have vertices with internal right angle and parallel sides facing each other.*

Proof It is not difficult to see that Condition (*) implies that no vertex of P can have right internal angle, and that no segment contained in P with endpoints on two sides of P can be orthogonal to both of them. This last fact implies that P cannot have parallel sides facing each others. The same conclusions hold for every polygon $Q \in \mathcal{P}_n$ sufficiently close to P in the metric d. □

3.2 Pesin Theory and SRB Measures

In this subsection, we recall basic results on the existence of local stable and unstable manifolds for the billiard map $\Phi_{f,P}$ and the definition of SRB measure. We assume $f \in \mathcal{R}^2$.

Let

$$N_{f,P}^- = \Big\{x \in K : \exists y \in N_{f,P}^+ \text{ and } y_n \in K_{f,P} \setminus N_{f,P}^+ \\ \text{such that } y_n \to y \text{ and } \Phi_{f,P}(y_n) \to x\Big\}.$$

The set $N_{f,P}^-$ can be thought of as 'singular set' for the inverse map $\Phi_{f,P}^{-1}$. Next, for every $\epsilon > 0$ and every $l \in \mathbb{N}$, define

$$D_{f,P,\epsilon,l}^+ = \left\{x \in \Lambda_{f,P} \cap K_{f,P}^+ : d_M\left(\Phi_{f,P}^n(x), N_{f,P}^+\right) \geq \frac{e^{-\epsilon n}}{l} \ \forall n \geq 0\right\},$$

$$D_{f,P,\epsilon,l}^- = \left\{x \in D_{f,P} : d_M\left(\Phi_{f,P}^{-n}(x), N_{f,P}^-\right) \geq \frac{e^{-\epsilon n}}{l} \ \forall n \geq 0\right\},$$

$$D_{f,P,\epsilon,l}^0 = D_{f,P,\epsilon,l}^- \cap D_{f,P,\epsilon,l}^+,$$

and

$$D_{f,P,\epsilon}^\pm = \bigcup_{l \geq 1} D_{f,P,\epsilon,l}^\pm, \quad D_{f,P,\epsilon}^0 = D_{f,P,\epsilon}^- \cap D_{f,P,\epsilon}^+.$$

The sets $D_{f,P,\epsilon,l}^0$ play the role of the regular sets in the Pesin theory for smooth maps [13].

Definition 3.6 The attractor $\Lambda_{f,P}$ is called *regular* if there exists $\epsilon_0 > 0$ such that $D_{f,P,\epsilon}^0 \neq \emptyset$ for every $0 < \epsilon < \epsilon_0$.

If $\Lambda_{f,P}$ is hyperbolic and regular, then the Pesin theory for maps with singularities [10] guarantees the existence of an $\epsilon > 0$ such that a local stable manifold $W_{loc}^s(x)$ exists for all $x \in D_{f,P,\epsilon,l}^+$, and a local unstable manifold $W_{loc}^u(x)$ exists for all $x \in D_{f,P,\epsilon,l}^-$ (see [14, Proposition 4]). The local manifolds $W_{loc}^s(x)$ and $W_{loc}^u(x)$ are C^2 (C^k if $f \in \mathcal{R}^k$, $k \geq 2$) embedded submanifolds whose tangent subspaces at x are equal to the stable subspace $E^s(x)$ and the unstable subspace $E^u(x)$, respectively. The size of these manifolds depends on the constants ϵ and l, and they form two transversal invariant laminations. Finally, we observe that for the billiard map $\Phi_{f,P}$, the local unstable manifolds are horizontal segments.

Definition 3.7 Suppose that $\Lambda_{f,P}$ is hyperbolic and regular. Let $\epsilon > 0$ be such that $W_{loc}^s(x)$ exists for all $x \in D_{f,P,\epsilon,l}^+$, and $W_{loc}^u(x)$ exists for all $x \in D_{f,P,\epsilon,l}^-$. An invariant Borel probability measure μ on $\Lambda_{f,P}$ is called *SRB* if $\mu(D^0_{f,P,\epsilon}) = 1$, and the conditional measures of μ on the local unstable manifolds of $\Phi_{f,P}$ are absolutely continuous with respect to the Riemannian volume on the local unstable manifolds.

The precise meaning of 'the conditional measures of μ on the local unstable manifolds of $\Phi_{f,P}$ are absolutely continuous with respect to the Riemannian volume on the local unstable manifolds' in the previous definition requires a technical explanation that can be found in [14]. We remark that what we call here an SRB measure is essentially what is called an *invariant Gibbs u-measure* in [14].

3.3 Expanding Slap Maps

If P does not have parallel sides facing each other, then ψ_P is a piecewise expanding map. This means that there exists $\sigma > 1$ such that $|\psi'_P| > \sigma$. By a well-known result of Lasota and Yorke [11], piecewise expanding maps have absolutely continuous invariant probability measures (for short acips). The theory of piecewise expanding maps applied to slap maps gives the following.

Theorem 3.8 *If P is polygon without parallel sides facing each other, then there exist subsets $A_1, \ldots, A_k$ of S^1 and ergodic acips $\nu_1, \ldots, \nu_k$ of ψ_P with bounded variation densities such that*

(1) $S^1 = A_1 \cup \cdots \cup A_k$ *and* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$;
(2) $\psi_P^{-1}(A_i) = A_i$, $\nu_i(A_i) = 1$ *and* $\psi_P|_{A_i}$ *is ergodic with respect to* ν_i *for every* $i = 1, \ldots, k$;
(3) *for each* $i = 1, \ldots, k$, *there exist disjoint subsets* $A_i^1, \ldots, A_i^{n_i}$ *such that for all* i, j,
   (a) $A_i = A_i^1 \cup \cdots \cup A_i^{n_i}$;
   (b) *each* A_i^j *is* $\psi_P^{n_i}$*-invariant*;
   (c) $\psi_P^{n_i}|_{A_i^j}$ *with the normalized restriction of* ν_i *to* A_i^j *is exact*;
   (d) supp ν_i *consists of finitely many pairwise disjoint intervals*;
   (e) *every open subset of* supp ν_i *contains two periodic points of* ψ_P *whose periods have great common divisor equal to* n_i. *In particular, the periodic points of* ψ_P *are dense in* supp ν_i;
(4) *the union of the basins of* $\nu_1, \ldots, \nu_k$ *has full Lebesgue measure in* S^1.

Proof Even if the results cited in the references below are proved for maps of the interval [0, 1], they continue to hold for maps of the unit circle. The existence of a finite number of ergodic acips of ψ_P and Parts (1), (2) and (3a)–(3d) follow from the general theory of piecewise expanding maps [3, Theorems 7.2.3 and 8.2.2]. Part (3e) is proved in [9, Theorem 3.14 and Proposition 3.15]. For a proof of Part (4), see [18, Corollary 3.14]. □

We call the sets $A_1, \ldots, A_k$ the *ergodic components* of ψ_P, and we call the sets $A_i^1, \ldots, A_i^{n_i}$ the *exact components* of A_i.

4 Preliminary Results

This section contains preliminaries results needed to prove Theorem 1.2.

4.1 Trapping Regions

Let $P \in \mathcal{P}_n$. By Theorem 3.8, the support of an ergodic acip ν of ψ_P consists of finitely many pairwise disjoint closed intervals. In [9, Sect. 3.1], we obtained a characterization of the boundary points of $\operatorname{supp}\nu$. When P satisfies Condition (*), such a characterization can be formulated as follows.

Proposition 4.1 *Suppose that $P \in \mathcal{P}_n^*$, and let ν be an ergodic acip of ψ_P. If $s \in \partial \operatorname{supp}\nu$, then there exist an orbit segment $\{s_0, \dots, s_k\}$, $k \geq 2$ of ψ_P and $0 < j < k$ such that*

(1) $s_0 \in S_P \cap \operatorname{int}(\operatorname{supp}\nu)$,
(2) *either* $s_i = \psi^i(s_0^+)$ *for every* $1 \leq i \leq k$, *or* $s_i = \psi^i(s_0^-)$ *for every* $1 \leq i \leq k$,
(3) $s_i \in \partial \operatorname{supp}\nu$ *for every* $0 < i < k$,
(4) $s_k \in \operatorname{int}(\operatorname{supp}\nu)$,
(5) $s = s_j$.

We call $\{s_0, \dots, s_k\}$ a *boundary segment of* $\operatorname{supp}\nu$.

Remark 4.2 It is not difficult to see that Proposition 4.1 and Condition (*) imply that $\operatorname{supp}\nu_1$ and $\operatorname{supp}\nu_2$ are disjoint for any pair ν_1, ν_2 of distinct ergodic acips of ψ_P.

In the next proposition, given $P \in \mathcal{P}_n^*$ and an ergodic acip ν of ψ_P, we construct a trapping region arbitrarily close to $\operatorname{supp}\nu$ common to all slap maps ψ_Q with Q sufficiently close to P. A similar conclusion was obtained for more general piecewise expanding maps in [9, Lemma 4.3].

Let $\zeta > 0$, and denote by $(\operatorname{supp}\nu)_\zeta$ the ζ-neighborhood of $\operatorname{supp}\nu$.

Proposition 4.3 *Suppose that $P \in \mathcal{P}_n^*$, and let $\nu_1, \dots, \nu_m$ be the ergodic acips of the slap map ψ_P. For every $\zeta > 0$, there exist $\delta > 0$, $\tau > 0$ and pairwise disjoint closed sets $U_1, \dots, U_m$ of S^1 such that if $d(Q, P) < \delta$, then for every $1 \leq i \leq m$,*

(1) $\operatorname{supp}\nu_i \subset \operatorname{int}(U_i) \subset (\operatorname{supp}\nu_i)_\zeta$,
(2) *U_i is a union of finitely many pairwise disjoint closed intervals whose endpoints are not vertices of P,*
(3) $\psi_Q(U_i) \subset \operatorname{int}(U_i)$ *and* $d_{S^1}(\psi_Q(U_i), \partial U_i) > \tau$.

Proof We construct the sets $U_1, \dots, U_m$ inductively. We start with the set U_1. Its construction is also inductive, and requires l steps, i.e., as many steps as the number of distinct boundary segments $\gamma_1, \dots, \gamma_l$ of $\operatorname{supp}\nu_1$. At the kth step, we enlarge $\operatorname{supp}\nu_1$ by enlarging the intervals forming $\operatorname{supp}\nu_1$ whose endpoints lie on the kth boundary segment.

Set $A_0 = \operatorname{supp}\nu_1$. Suppose that A_{k-1} with $1 \leq k \leq l$ is given, and let $\gamma_k = \{s_0, \dots, s_{N_k}\}$ be the kth boundary segment of ν_1. Once again, we construct A_k inductively. Set $B_0 = A_{k-1}$. Suppose that B_{i-1} with $1 \leq i < N_k$ is given. If $s_{N_k-i} \in \operatorname{int}(B_{i-1})$—which happens when there exists $k' < k$ such that $\gamma_{k'}$ and γ_k shares the same point s_{N_k-i+1}—then we set $B_i = B_{i-1}$. Otherwise, if s_{N_k-i} is the right endpoint of an interval of B_{i-1}, then choose $t_{N_k-i} \in (s_{N_k-i}, s_{N_k-i} + \zeta)$ satisfying

(i) $[s_{N_k-i}, t_{N_k-i}] \cap S_P = \emptyset$,
(ii) $[s_{N_k-i}, t_{N_k-i}] \cap B_{i-1} = \emptyset$,
(iii) $[s_{N_k-i}, t_{N_k-i}] \cap (\operatorname{supp}\nu_2 \cup \dots \cup \operatorname{supp}\nu_m) = \emptyset$,
(iv) $\psi_P([s_{N_k-i}, t_{N_k-i}]) \subset \operatorname{int}(B_{i-1})$.

Instead, if s_{N_k-i} is the left endpoint of an interval of B_{i-1}, then choose $t_{N_k-i} \in (s_{N_k-i} - \zeta, s_{N_k-i})$ satisfying conditions analogous to (i)–(iv). Such a t_{N_k-i} exists, because $s_{N_k-i} \notin S_P$, $\psi_P(s_{N_k-i}) \in \operatorname{int}(B_{i-1})$, and the supports of the acips of ψ_P are pairwise disjoint. Next, define $B_i = B_{i-1} \cup [s_{N_k-i}, t_{N_k-i}]$. The previous procedure gives the sets $B_0, \ldots, B_{N_k-1}$. finally, define $A_k = B_{N_k-1}$. Once all the sets $A_0, \ldots, A_l$ have been computed, define $U_1 = A_l$.

An almost identical construction produces the sets $U_2, \ldots, U_m$. Assume that U_{j-1} with $2 \leq j \leq m$ is given, and construct U_j by following the same procedure used for U_1 with the obvious modifications and with Condition (iii) above replaced by

$$[s_i, t_i] \cap \left(U_1 \cup \cdots \cup U_{j-1} \cup \operatorname{supp} \nu_{j+1} \cup \cdots \cup \operatorname{supp} \nu_m\right) = \emptyset.$$

It is easy to see that the sets $U_1, \ldots, U_m$ obtained this way have the wanted properties for the map ψ_P. In fact, besides $\psi_P(U_i) \subset \operatorname{int}(U_i)$ for every $1 \leq i \leq m$, a bit more can be derived from the construction above. Namely, we obtain that there is $\tau > 0$ such that

$$d_{S^1}(\psi_P(U_i), \partial U_i) > 2\tau \quad \text{for every } 1 \leq i \leq m. \tag{4.1}$$

Next, we want to extend the previous conclusion to every map ψ_Q with $d(Q, P)$ sufficiently small. To this end, note that if $d(Q, P)$ is sufficiently small, then there is a natural bijective correspondence between the vertices of Q and P. So for $d(Q, P)$ sufficiently small case, denote by s_Q the vertex of Q corresponding to the vertex s of P. Then, for every vertex s of P, we have $s_Q \to s$ as $d(Q, P) \to 0$. Moreover by Condition (*) and the fact that no vertex of P can have an internal angle equal to $\pi/2$ (see Lemma 3.5), it follows that for every vertex s of P,

$$\psi_Q(s_Q^{\pm}) \longrightarrow \psi_P(s^{\pm}) \quad \text{as } d(Q, P) \to 0. \tag{4.2}$$

By construction of U_i and properties (4.1) and (4.2), it is not difficult to see that $\psi_Q(U_i) \subset \operatorname{int}(U_i)$ and $d_{S^1}(\psi_Q(U_i), \partial U_i) > \tau$ for every $1 \leq i \leq m$ provided that $d(Q, P)$ is sufficiently small. □

Recall that $\Phi_{f,Q} \colon K_{f,Q} \setminus N_Q \to K_{f,Q}$. Next, we show that if a polygon Q is sufficiently close to $P \in \mathcal{P}_n^*$, and f is a reflection law sufficiently close to 0, then the map $\Phi_{f,Q}$ has a trapping region close to $\bigcup_{i=1}^m U_i \times \{0\}$ with $U_1, \ldots, U_m$ being as in Proposition4.3.

Proposition 4.4 *Suppose that* $P \in \mathcal{P}_n^*$. *Given* $\zeta > 0$, *let* $\delta > 0$, $\tau > 0$ *and the sets* $U_1, \ldots, U_m$ *be as in Proposition* 4.3. *There exist* $0 < \delta' < \delta$ *and pairwise disjoint sets* $W_1, \ldots, W_m$ *of* M *defined by*

$$W_i = U_i \times \left(-\frac{\pi}{2}\lambda(f), \frac{\pi}{2}\lambda(f)\right), \quad i = 1, \ldots, m$$

such that if $\lambda(f) < \delta'$ *and* $d(Q, P) < \delta'$, *then* $\Phi_{f,Q}(W_i \setminus N_Q) \subset \operatorname{int}(W_i)$ *for every* $1 \leq i \leq m$.

Proof Given $\zeta > 0$, let $\delta > 0$, $\tau > 0$ and the sets $U_1, \ldots, U_m$ be as in Proposition 4.3. Define W_i as in the statement of the proposition. Clearly, the sets $W_1, \ldots, W_m$ are disjoint because so are $U_1, \ldots, U_m$, and satisfy Condition (1). Note that as U_i, the set W_i depends on the measure ν_i.

Since $|f \circ h_Q| < \pi\lambda(f)/2$, Condition (2) is a consequence of the following property: there exists $0 < \delta' < \delta$ such that if $\lambda(f) < \delta'$ and $d(Q, P) < \delta'$, then $g_Q(W_i \setminus N_Q) \subset \operatorname{int}(U_i)$. This property, by Part (3) of Proposition 4.3, follows immediately from

$$|g_Q(s, \theta) - \psi_Q(s)| \leq \tau, \quad (s, \theta) \in W_i \setminus N_Q.$$

Let $(s, \theta) \in W_i \setminus N_Q$. Using the Mean Value Theorem, we obtain

$$|g_Q(s,\theta) - \psi_Q(s)| = |g_Q(s,\theta) - g_Q(s,0)| \\ \leq \sup_{-\pi\lambda(f)/2<\theta<\pi\lambda(f)/2} \left|\partial_\theta g_Q(s,\theta)\right| |\theta|.$$

Now, $\partial_\theta g_Q(s,\theta) = -t_Q(s,\theta)/\cos(h_Q(s,\theta))$ by (2.1), $t_Q \leq 1$ because the perimeter of Q is equal to 1, and by Lemma 3.5, if $\lambda(f)$ and $d(Q, P)$ are sufficiently small, then $|h_Q(\cdot, 0)|$ is uniformly bounded away from $\pi/2$ on $K_{f,Q} \setminus N_Q$, i.e., there is a constant $A > 0$ depending only on P such that $\cos \circ h_Q > A$ on $K_{f,Q} \setminus N_Q$. Therefore,

$$\left|g_Q(s,\theta) - \psi_Q(s)\right| < \frac{|\theta|}{A} < \frac{\pi}{2A}\lambda(f), \quad (s,\theta) \in W_i \setminus N_Q,$$

provided that $\lambda(f)$ and $d(Q, P)$ are sufficiently small. By taking a smaller $\lambda(f)$ if necessary, we obtain $\pi/(2A)\lambda(f) \leq \tau$. Hence, there exists $0 < \delta' < \delta$ such that if $\lambda(f) < \delta'$ and $d(Q, P) < \delta'$, then

$$\pi_s \circ \Phi_{f,Q}(W_i \setminus N_Q) = g_Q(W_i \setminus N_Q) \subset \text{int}(U_i).$$

□

4.2 Hyperbolicity

Definition 4.5 Given $Q \in \mathcal{P}_n$, $f \in \mathcal{R}$ and $m \in \mathbb{N}$, denote by $\alpha(\Phi^m_{f,Q})$ and $\beta(\Phi^m_{f,Q})$ the infimum and the supremum of $\|d_x\Phi^m_{f,Q}(1,0)^T\|$, respectively, over the subset of $K_{f,Q}$ where $\Phi^m_{f,Q}$ is differentiable.

The next lemma says that the horizontal direction is uniformly expanding for $\Phi_{f,Q}$ whenever $P \in \mathcal{P}^*_n$, $f \in \mathcal{R}$ and $Q \in \mathcal{P}_n$ with $\lambda(f)$ and $d(Q, P)$ sufficiently small. We emphasize that f and $\Phi_{f,P}$ may not be invertible for $f \in \mathcal{R}$.

Lemma 4.6 *Let $P \in \mathcal{P}^*_n$. Then there exist $\delta > 0$ and $1 < \alpha_0 < \beta_0$ such that*

(1) *if $f \in \mathcal{R}^1$ and $d(Q, P) < \delta$, then $\Lambda_{f,Q}$ is hyperbolic*;
(2) *if $f \in \mathcal{R}$ with $\lambda(f) < \delta$ and $d(Q, P) < \delta$, then*

$$\alpha_0 \leq \alpha(\Phi_{f,Q}) \leq \beta(\Phi_{f,Q}) \leq \beta_0.$$

Proof Part (1). By Lemma 3.5, every Q sufficiently close to P has no parallel sides facing each other. For such a Q, Proposition 3.3 guarantees that the attractor $\Lambda_{f,Q}$ is hyperbolic for every $f \in \mathcal{R}^1$.

Part (2). Let $f \in \mathcal{R}$, and let $Q \in \mathcal{P}_n$. By (2.1), we have

$$\alpha_{f,Q}(x) := \left\| d_x\Phi_{f,Q}\begin{pmatrix}1\\0\end{pmatrix}\right\| = \frac{\cos\theta}{\cos(h_Q(s,\theta))}$$

for every $x = (s, \theta) \in K_{f,Q} \setminus N^+_{f,Q}$. Denote by $u(x)$ the unit vector of $\mathbb{R}^2$ parallel to the side $L(x)$ of Q containing s and having the same orientation of $L(x)$ (induced by the parametrization of ∂P). Also, denote by $0 < \omega_Q(x) < 2\pi$ the smallest angle of the counterclockwise rotation of $\mathbb{R}^2$ mapping $u(x)$ to $u(\Phi_{f,Q}(x))$. A simple computation shows that $h_Q(s,\theta) = \pi - \omega_Q(x) - \theta$. Hence

$$\alpha_{f,Q}(x) = \frac{1}{-\cos\omega_Q(x) + \tan\theta \sin\omega_Q(x)}. \tag{4.3}$$

Now, let $Q = P \in \mathcal{P}_n^*$. Since P satisfies Condition (*), P does not have sides parallel facing each other and adjacent sides that are perpendicular (see Lemma 3.5). It is still possible for P to have parallel sides, but two consecutive collisions x and $\Phi_{f,P}(x)$ at parallel sides never occur. It follows that if $\lambda(f)$ is sufficiently small, then the angle $\omega_P(x)$ must satisfy the property: there exist $\tau > 0$ depending only on P and $\delta_0 > 0$ such that if $\lambda(f) < \delta_0$, then for all $x \in K_{f,P} \setminus N^+_{f,P}$,

$$\omega_P(x) \in \left(\frac{\pi}{2} + \tau, \pi - \tau\right) \cup \left(\pi + \tau, \frac{3}{2}\pi - \tau\right). \tag{4.4}$$

Recall that if $x \in K_{f,P}$, then $|\theta| < \pi\lambda(f)/2$ for $f \neq 0$, and $\theta = 0$ for $f = 0$. This together (4.4) and (4.3) implies that there exists $0 < \delta_1 \leq \delta_0$ and $1 < \alpha_0 < \beta_0$ such that if $\lambda(f) < \delta_1$, then

$$\alpha_0 < \alpha_{f,P}(x) < \beta_0 \quad \text{for all } x \in K_{f,P} \setminus N^+_{f,P}. \tag{4.5}$$

By Lemma 3.5, every Q sufficiently close to P does not have parallel sides facing each other and adjacent sides that are perpendicular. From this, it is not difficult to see that there must exist $0 < \delta \leq \delta_1$ such that (4.4) with the same τ, and therefore (4.5) with the same α_0 and β_0 continue to hold for every $Q \in \mathcal{P}_n$ with $d(Q, P) < \delta$ and every $f \in \mathcal{R}$ with $\lambda(f) < \delta$. This implies the wanted conclusion. □

For every $s \in V_Q$ and every $r > 0$, define

$$I(s,r) = (s - r, s) \cup (s, s + r) \subset S^1,$$

and

$$H_Q(r) = \bigcup_{s \in V_Q} I(s,r) \times (-r, r) \subset M.$$

The first conclusion of the next proposition is an obvious consequence of Lemma 4.6. The second conclusion says, roughly speaking, that for every $m \in \mathbb{N}$, the map $\Phi^m_{f,Q}$ is differentiable on a sufficiently small neighborhood of the 'vertices' of the polygon provided that P satisfies (*) and $\Phi_{f,Q}$ is sufficiently close to $\Phi_{0,P}$. From this, it follows that when a sufficiently short curve γ in $K_{f,Q}$ is iterated forward m times, it cannot be cut more than once by the singular set of $\Phi_{f,Q}$.

Proposition 4.7 *Let $P \in \mathcal{P}_n^*$. For every $\bar{\alpha} > 1$, there exist $\delta > 0$, $m \in \mathbb{N}$ and $r > 0$ such that if $f \in \mathcal{R}$ with $\lambda(f) < \delta$ and $d(Q, P) < \delta$, then*

(1) $\bar{\alpha} \leq \alpha(\Phi^m_{f,Q}) \leq \beta(\Phi^m_{f,Q}) \leq \beta_0^m$ *with β_0 be as in Lemma 4.6,*
(2) $\Phi^m_{f,Q}|_{H_Q(r)}$ *is differentiable.*

Proof Let $P \in \mathcal{P}_n^*$, and let $\bar{\alpha} > 1$. By Lemma 4.6, there exist $m \in \mathbb{N}$ and $\delta_0 > 0$ such that

$$\bar{\alpha} \leq \alpha_0^m \leq \alpha(\Phi^m_{f,Q}) \leq \beta(\Phi^m_{f,Q}) \leq \beta_0^m$$

for all $f \in \mathcal{R}$ with $\lambda(f) < \delta_0$ and all $Q \in \mathcal{P}_n$ with $d(Q, P) < \delta_0$. The value of m will be kept fixed throughout the rest of the proof.

Recall that $Y_P = V_P \times (-\pi/2, \pi/2)$. Since P satisfies Condition (*), if $s \in S_P$ (i.e., s is a singular point of ψ_P), then the forward orbits of $\psi_P(s^+)$ and $\psi_P(s^-)$ do not visit any vertex of P. Also, recall that each vertex of P in $V_P \setminus S_P$ is a fixed point of ψ_P. Hence, there exists $r_0 > 0$ such that $d_{S^1}(V_P, \psi_P^i(I(s, r_0))) > 0$ for all $0 \le i \le m$ and all $s \in V_P$. Equivalently, in terms of the map $\Phi_{0,P}$,

$$d_M(Y_P, \Phi^i_{0,P}(I(s, r_0) \times \{0\})) > 0$$

for all $0 \le i \le m$ and all $s \in V_P$. It is not difficult to see that the conclusion remains valid for every Q sufficiently close to P. More precisely, there is $0 < \delta_1 \le \delta_0$ such that if $d(Q, P) < \delta_1$, then

$$d_M(Y_Q, \Phi^i_{0,Q}(I(s, r_0) \times \{0\})) > 0 \quad \forall 0 \le i \le m \quad \forall s \in V_Q. \tag{4.6}$$

Now, arguing as in the proof of Proposition 4.4, one can show that there is $0 < r \le r_0$ such that (4.6) holds even when $I(s, r_0) \times \{0\}$ is replaced by $I(s, r) \times \{-r, r\}$, and that there is $0 < \delta \le \delta_1$ such that if $f \in \mathcal{R}$ with $\lambda(f) < \delta$ and $d(Q, P) < \delta$, then

$$d_M(Y_Q, \Phi^i_{f,Q}(I(s, r) \times (-r, r))) > 0$$

for all $0 \le i \le m$ and all $s \in V_Q$. By taking a sufficiently small δ, one can even guarantee that

$$\Phi^i_{f,Q}(I(s, r) \times (-r, r)) \subset S^1 \times (-r, r)$$

for all $0 \le i \le m$ and all $s \in V_Q$. What we have just proved can be reformulated in terms of the set $H_Q(r)$ as follows. There exist $r > 0$ and $\delta > 0$ such that if $f \in \mathcal{R}$ with $\lambda(f) < \delta$ and $d(Q, P) < \delta$, then

$$d_M(Y_Q, \Phi^i_{f,Q}(H_Q(r))) > 0 \quad \text{and} \quad \Phi^i_{f,Q}(H_Q(r)) \subset S^1 \times (-r, r)$$

for every $0 \le i \le m$. The first inequality implies that $\Phi^m_{f,Q}$ is differentiable on $H_Q(r)$. Indeed, suppose that the claim was not true. Then there would exist $0 \le i < m$ such that $\Phi^i_{f,Q}(H_Q(r)) \cap N^+_{f,Q} \neq \emptyset$ implying $d_M(Y_Q, \Phi^i_{f,Q}(H_Q(r))) = 0$, which is impossible. □

Given a C^1-curve $\Gamma \subset K_{f,Q}$, denote by $|\Gamma|$ the length of Γ induced by the metric d_M.

Lemma 4.8 *Let $P \in \mathcal{P}_n^*$. Then there exist $\delta > 0$ and $\eta > 0$ such that if $f \in \mathcal{R}$, $Q \in \mathcal{P}_n$ with $\lambda(f) < \delta$ and $d(Q, P) < \delta$, and Γ is a horizontal segment, then there exist a segment $\gamma \subset \Gamma$ and $k \in \mathbb{N}$ with the property that $\Phi^k_{f,Q}(\gamma)$ is a horizontal segment with $|\Phi^k_{f,Q}(\gamma)| > \eta$.*

Proof Choose[3] $\bar{\alpha} = 3$, and let $m > 0$, $r > 0$ and $\delta > 0$ be as in Proposition 4.7. Let $f \in \mathcal{R}$ with $\lambda(f) < \delta$, and let $Q \in \mathcal{P}_n$ with $d(Q, P) < \delta$.

Let Γ be a horizontal segment in $K_{f,Q}$. Define recursively a sequence of horizontal segments $\{\Gamma_j\}$ as follows: let $\Gamma_0 = \Gamma$, and let Γ_{j+1} be any interval of maximal length of $\Phi^m_{f,Q}(\Gamma_j)$ for every $j \ge 0$. We claim that $|\Gamma_j| > r$ for some $j \ge 0$. It is easily seen that the conclusion of the lemma with $\eta = r$ and $k = mj$ is a direct consequence of our claim.

To prove the claim, we study separately the two alternatives: (1) $\Phi^m_{f,Q}(\Gamma_j)$ consists of more than two horizontal segments for some $j \ge 0$, and (2) $\Phi^m_{f,Q}(\Gamma_j)$ consists of one or two horizontal segments for every j. If alternative 1 occurs, then there exist $1 \le i \le m$ and a segment $\Gamma' \subset \Phi^i_{f,Q}(\Gamma_j)$ with both endpoints on Y_Q. By Proposition 4.7,

$$\Gamma' \subset \Phi^i_{f,Q}(H_Q(r)) \subset S^1 \times (-r, r),$$

[3] This choice is arbitrary, every $\bar{\alpha} > 2$ will do.

and so the intersection $\Gamma' \cap H_Q(r)$ contains a segment Γ'' of length r. Since $\Phi^{m-i}_{f,Q}$ is continuous on $H_Q(r)$ and $\alpha(\Phi_{f,Q}) > 1$,

$$|\Gamma_{j+1}| \geq |\Phi^{m-i}_{f,Q}(\Gamma'')| > r.$$

If alternative 2 occurs, then we clearly have $|\Gamma_{j+1}| \geq |\Gamma_j|\bar{\alpha}/2$, and so

$$|\Gamma_j| \geq |\Gamma_0| \left(\frac{\bar{\alpha}}{2}\right)^j > r$$

for some $j \geq 0$. Hence, in both cases, there exists $j \geq 0$ such that $|\Gamma_j| > r$. This completes the proof. □

4.3 Existence of SRB Measures

Recall that Vol denotes the volume generated by the Riemannian metric d_M on M. Given a C^1-curve $\Gamma \subset K_{f,Q}$, denote by Vol_Γ the normalized volume on Γ induced by the metric d_M. Finally, denote by $N^+_{f,Q}(r)$ the neighborhood of $N^+_{f,Q}$ in $K_{f,Q}$ of radius $r > 0$.

Proposition 4.9 *If $P \in \mathcal{P}^*_n$, then there exists $\delta > 0$ such that if $f \in \mathcal{R}^2$ with $\lambda(f) < \delta$ and $d(Q, P) < \delta$, then the following properties hold*:

(A) *there are positive constants $C = C(f, Q)$ and $r_0 = r_0(f, Q)$ such that*

$$\mathrm{Vol}(\Phi^{-n}_{f,Q}(N^+_{f,Q}(r))) < Cr \quad \forall n \geq 1 \quad \forall 0 < r < r_0,$$

(B) *there is $r_1 = r_1(f, Q) > 0$ such that for every horizontal segment Γ, there exists $B = B(f, Q, \Gamma) > 0$ for which*

$$\mathrm{Vol}_\Gamma(\Gamma \cap \Phi^{-n}_{f,Q}(N^+_{f,Q}(r))) < Br \qquad \forall n \geq 1 \quad \forall 0 < r < r_1.$$

Proof In [8, Lemma 4.9 and Theorem 4.15], we demonstrated that properties (A) and (B) are consequences of the *m-step expansion condition* (c.f. [4, Inequality (5.38)]): there exists $m \in \mathbb{N}$ such that

$$\liminf_{\tau \to 0^+} \sup_{\Gamma \in \mathcal{H}(\tau)} \sum_{\gamma \in S_m(\Gamma)} \frac{1}{a_m(\gamma)} < 1, \tag{4.7}$$

where $\mathcal{H}(\tau)$ is the set of horizontal segments $\Gamma \subset K_{f,Q}$ with $|\Gamma| < \tau$, $S_m(\Gamma)$ is the set of maximal subsegments γ of Γ such that $\Phi^m_{f,Q}|_\gamma$ is differentiable, and $a_m(\gamma) = \inf_{x\in\gamma} \|d_x\Phi^m_{f,Q}(1,0)^T\|$. Accordingly, to prove the proposition, it suffices to show that the m-step expansion condition holds for a proper $m \in \mathbb{N}$.

Let $P \in \mathcal{P}^*_n$, and denote by L the length of the shortest side (or sides) of P. Choose $\bar{\alpha} > 2$, and let δ, m, r and β_0 be the positive constants as in Proposition 4.7. Consider $f \in \mathcal{R}^2$ with $\lambda(f) < \delta$ and $Q \in \mathcal{P}_n$ with $d(Q, P) < \delta$. If necessary, take a smaller δ so that the length of the shortest side (or sides) of Q is less than $3L/2$. Choose $\delta = \beta_0^{-m} \cdot \min\{r, l\}$, and let $\Gamma \in \mathcal{H}(\tau)$.

If $N^+_{f,Q} \cap \Phi^i_{f,Q}(\Gamma) = \emptyset$ for every $0 \leq i < m$, then $\Phi^i_{f,Q}|_\gamma$ is differentiable, and $\Phi^i_{f,Q}(\Gamma)$ consists of a single segment. Thus $S_m(\Gamma) = \{\Gamma\}$, $a_m(\Gamma) \geq \bar{\alpha}$ and

$$\sup_{\Gamma \in \mathcal{H}(\tau)} \sum_{\gamma \in S_m(\Gamma)} \frac{1}{a_m(\gamma)} = \frac{1}{\bar{\alpha}} < \frac{1}{2}.$$

Now, suppose that $N^+_{f,Q} \cap \Phi^i_{f,Q}(\Gamma) \neq \emptyset$ for some $0 \leq i < m$, and let $0 \leq j < m$ be the smallest i with such a property. It follows that $\Phi^j_{f,Q}(\Gamma)$ consists of several disjoint horizontals segments whose total length is less than $L\beta_0^{-m+j} < L$, because β_0^m is the supremum of the expansion along the horizontal direction. Since Q is convex, and the length of its shortest side is less than $3L/2$, $\Phi^j_{f,Q}(\Gamma)$ must consist exactly of two horizontal segments, both having one endpoint in Y_Q. The length of each segment is less than $r\beta_0^{-m+j} < r$. Hence, both segments are contained in $H_Q(r)$. By Proposition 4.7, $\Phi^m_{f,Q}|_{H_Q(r)}$ is differentiable and $\alpha(\Phi^m_{f,Q}) \geq \bar{\alpha}$. So $\pi_m(\Gamma)$ consists of two segments, and $a_m(\gamma) \geq \bar{\alpha} > 2$ for every $\gamma \in \pi_m(\Gamma)$. Therefore,

$$\sup_{\Gamma \in \mathcal{H}(\tau)} \sum_{\gamma \in S_m(\Gamma)} \frac{1}{a_m(\gamma)} = \frac{2}{\bar{\alpha}} < 1.$$

The previous estimates imply the desired property,

$$\liminf_{\tau \to 0^+} \sup_{\Gamma \in \mathcal{H}(\tau)} \sum_{\gamma \in S_m(\Gamma)} \frac{1}{a_m(\gamma)} < \frac{2}{\bar{\alpha}} < 1.$$

□

Remark 4.10 Property (A) is a version adapted to billiards of a condition introduced by Sataev [15] for general maps with singularities, which in turn is a stronger version of Condition H4 in [14]. Condition H4 is key in proving the existence of SRB measures for hyperbolic maps with singularities.

We now establish the existence of SRB measures for the map $\Phi_{f,Q}$.

Theorem 4.11 *Suppose that $P \in \mathcal{P}_n^*$. Then there exists $\delta > 0$ such that for every $f \in \mathcal{R}^2$ with $\lambda(f) < \delta$ and every $Q \in \mathcal{P}_n$ with $d(Q, P) < \delta$, the attractor $\Lambda = \Lambda_{f,Q}$ is hyperbolic and regular, and there exist countably many ergodic SRB measures $\mu_1, \mu_2, \dots$ of $\Phi = \Phi_{f,Q}$ and countably many Borel subsets $E_0, E_1, E_2, \dots$ of Λ such that*

(1) $\Lambda = \bigcup_{i=0} E_i$ *and* $E_i \cap E_j = \emptyset$ *for all* $i \neq j$;
(2) $E_i \subset D$, $\Phi(E_i) = E_i$, $\mu_i(E_i) = 1$ *and* $\Phi|_{E_i}$ *is ergodic with respect to* μ_i *for every* $i \geq 1$;
(3) *for every* $i \geq 1$, *there exist* $k_i \in \mathbb{N}$ *disjoint subsets* $B_i^1, \dots, B_i^{k_i}$ *such that*
   (a) $E_i = \bigcup_{j=1}^{k_i} B_i^j$;
   (b) $\Phi(B_i^j) = B_i^{j+1}$ *for* $j = 1, \dots, k_i - 1$, *and* $\Phi(B_i^{k_i}) = B_i^1$;
   (c) $\Phi^{k_i}|_{B_i^j}$ *with the normalized restriction of* μ_i *to* B_i^j *is a Bernoulli automorphism*;
(4) *If* μ *is an SRB measure of* Φ, *then there exist* $\alpha_1, \alpha_2, \dots$ *with* $\sum_{i=1} \alpha_i = 1$ *such that* $\mu = \sum_i \alpha_i \mu_i$;
(5) *if* $x \in D_\epsilon^-$ *and* ν *is a probability measure on* M *supported on* $W^u_{loc}(x)$ *absolutely continuous with respect to the Riemannian volume on* $W^u_{loc}(x)$ *and with density* $\kappa(x, \cdot)$ (*see* [14, Proposition 6]), *then every weak-* limit point of* $\mu_n = n^{-1} \sum_{k=0}^{n-1} \Phi_*^k \nu$ *is an SRB measure of* Φ;
(6) *the set of periodic points of* Φ *is dense in* Λ;
(7) *for every* $i \geq 1$, *there exist* $C > 0$, $\alpha > 0$ *and* $r_0 > 0$ *such that* $\mu_i(N^+_{f,Q}(r)) \leq Cr^\alpha$ *for every* $0 < r < r_0$.

Proof The theorem follows from results by Pesin on the existence and properties of SRB measures for general hyperbolic piecewise smooth maps. More precisely, conclusions (1)–(4) follow from [14, Theorem 4], conclusion (5) follows from [14, Theorem 1], conclusion (6) follows from [14, Theorem 11], and conclusion (7) follows from [14, Proposition 12]. See also [15], where Sataev obtained results that are stronger than those of Pesin (but under stronger hypotheses). To justify our claim, we show that the map $\Phi_{f,Q}$ satisfies the hypothesis of Pesin's paper, i.e., the conditions called H1–H4 and the condition that $\Lambda_{f,Q}$ is hyperbolic.

The map $\Phi_{f,Q}$ satisfies conditions H1 and H2, because so does the billiard map Φ_P with the specular reflection law (see [10, Theorem 7.2]), and f and f^{-1} have bounded second derivates since $f \in \mathcal{R}^2$. Since $P \in \mathcal{P}_n^*$, Lemma 4.6 and Proposition 4.9 implies, respectively, the hyperbolicity of $\Lambda_{f,Q}$ and Properties (A) and (B) for every $f \in \mathcal{R}^1$ and every $Q \in \mathcal{P}_n$ sufficiently close to P. Finally, Property (A) implies H3 (the regularity of $\Lambda_{f,Q}$) by [14, Proposition 3], and (B) implies H4. □

We call the sets $E_1, \ldots, E_m$ the *ergodic components* of Φ, and we call the sets $B_i^1, \ldots, B_i^{k_i}$ the *Bernoulli components* of E_i.

Remark 4.12 Under the extra hypothesis that $f'(\theta) > 0$ for every $\theta \in (-\pi/2, \pi/2)$, the previous theorem follows from a general result on polygonal billiards with contracting reflection laws [8, Theorem 4.12]. Theorem 4.11 shows that the condition $f'(\theta) > 0$ can be dropped when $\lambda(f)$ is sufficiently small.

4.4 Continuation of Periodic Points

Throughout this section, P and Q are assumed to be polygons in $\mathcal{P}_n$ without parallel sides facing each other, and f is assumed to be a reflection law in $\mathcal{R}$. Note that for such an f, the map $\Phi_{f,Q}$ is not necessarily invertible. Since P does not have parallel sides facing each other, ψ_P is piecewise expanding.

Denote by $B(x, r)$ the open ball of $S^1 \times (-\pi/2, \pi/2)$ centered at x of radius $r > 0$. Given $x \in D^+_{f,Q,\epsilon}$, we call the two curves contained in $W^s_{loc}(x)$ having as endpoints x and a point of $\partial W^s_{loc}(x)$ the *components* of $W^s_{loc}(x)$. In the next theorem, we prove that each periodic point of ψ_P admits a continuation to a hyperbolic periodic point of the billiard map $\Phi_{f,Q}$ provided that $\lambda(f)$ and $d(Q, P)$ are sufficiently small.

Theorem 4.13 *Let s be a periodic point of ψ_P of period $m \in \mathbb{N}$ whose orbit does not visit V_P. Then there exist positive constants δ, r and ℓ such that if $\lambda(f) < \delta$ and $d(Q, P) < \delta$, then*

(1) *$\Phi_{f,Q}$ has exactly one hyperbolic periodic point $x_{f,Q}$ of period m in $B((s, 0), r)$ converging to $(s, 0)$ as $\lambda(f) + d(Q, P) \to 0$,*
(2) *the slope of $E^s(x_{f,Q})$ is smaller than $-1/(2t_P(s, 0))$,*
(3) *if γ is a component of $W^s_{loc}(x_{f,Q})$), then $|\gamma| \geq \ell$.*

Note that $x := (s, 0)$ is a hyperbolic periodic point of $\Phi_{0,P}$ of period m.

Definition 4.14 We call $x_{f,Q}$ the *continuation* of s [or of $x = (s, 0)$].

To prove Theorem 4.13, we need Lemma 4.15. Let d_1 be the C^1-distance between maps.

Lemma 4.15 *Let $s \in S^1$, and suppose that there exists $m \in \mathbb{N}$ such that $\psi_P^i(s) \notin V_P$ for every $0 \leq i \leq m - 1$. There are $\delta_0 > 0$ and $r_0 > 0$ such that if $\lambda(f) < \delta_0$, $d(P, Q) < \delta_0$ and $x = (s, 0)$, then the restrictions $\Phi^m_{0,P}|_{B(x,r_0)}$ and $\Phi^m_{f,Q}|_{B(x,r_0)}$ are both differentiable.*

Proof By the hypothesis on s, there exists $r_0 > 0$ such that the restriction $\Phi^m_{0,P}|_{B(x,r_0)}$ is differentiable. This implies that the components of the map $\Phi^m_{f,Q}$ and the entries of the matrix $d_x\Phi^m_{f,Q}$ are continuous in the variables f and Q at $f = 0$ and $Q = P$. Hence, there exists $\delta_0 > 0$ such that if $\lambda(f) < \delta_0$ and $d(Q, P) < \delta_0$, then $\Phi^n_{f,Q}|_{B(x,r_0)}$ is differentiable as well. □

Remark 4.16 By Lemma 4.15, $\Phi^n_{f,Q}|_{B(x,r_0)}$ can be thought as a perturbation of $\Phi^m_{0,P}|_{B(x,r_0)}$, and Theorem 4.13 is a corollary of a general theorem on the persistence of hyperbolic periodic points of smooth maps (without singularities) under small perturbations [19, Theorem 2.6]. However, to be able to apply this theorem to our maps with singularities, we need the following observation. The map $\Phi^m_{0,P}|_{B(x,r_0)}$ is an endomorphism, whereas [19, Theorem 2.6] assumes that the unperturbed map is a C^1 diffeomorphism. Nevertheless, it continues to hold when the unperturbed map is just a C^1 endomorphism, because even if [19, Lemma 2.5]—the key step in the proof of Theorem 2.6—is formulated for hyperbolic linear isomorphisms, it is actually valid for hyperbolic linear endomorphisms (see also [20, Sect. 2.1]).

Proof of Theorem 4.13 Since the orbit of s does not visit V_P, the $\Phi_{0,P}$-orbit of $x = (s, 0)$ is defined, and $x = (s, 0)$ is a fixed point of $\Phi^m_{0,P}$. Moreover, since P does not have parallel sides facing each other, x is a hyperbolic periodic point. We can then apply Lemma 4.15 to $\Phi^m_{0,P}$ and x. Let $r_0 > 0$ and $\delta_0 > 0$ be as in the lemma. Next, we apply [19, Theorem 2.6] to $\Phi^m_{0,P}|_{B(x,r_0)}$ and its perturbation $\Phi^m_{f,Q}|_{B(x,r_0)}$ with $\lambda(f) < \delta_0$ and $d(Q, P) < \delta_0$. So there exist $0 < \delta < \delta_0$ and $0 < r < r_0$ such that if $\lambda(f) < \delta$ and $d(Q, P) < \delta$, then $\Phi^m_{f,Q}|_{B(x,r)}$ has a unique fixed point $x_{f,Q}$ with the property that $x_{f,Q} \to x$ as $\lambda(f)+d(Q, P) \to 0$. Since Q does not have sides facing each other, the map $\Phi_{f,Q}$ is uniformly hyperbolic, implying that $x_{f,Q}$ is hyperbolic. This proves conclusion (1) of the theorem.

Now, consider the fixed point x of $\Phi^m_{0,P}$. Note that the $(2, 2)$-entry of $d_x\Phi^m_{0,P}$ is equal to 0, since $f = 0$ in this case. Then, using (2.1), one can easily show that the slope of the stable direction of x is equal to $-1/t_P(x)$. Since the entries of the matrix $d_{x_{f,Q}}\Phi^m_{f,Q}$ are continuous functions of (f, Q) at $f = 0$ and $Q = P$, so is the slope of the stable direction of $x_{f,Q}$. Thus, by further shrinking δ, we obtain conclusion (2) of the theorem with the lower bound for the slope equal to $-1/(2t_P(x))$.

Now, we assume that $\lambda(f) < \delta$ and $d(Q, P) < \delta$. Since $x_{f,Q}$ is a periodic point, the distance of its orbit from the singular set $N^+_{f,Q}$ is bounded away from zero uniformly in f and Q (chosen as above). This implies that $x_{f,Q} \in D^+_{f,Q,\epsilon,l}$ for some $l \in \mathbb{N}$ uniformly in (f, Q). By Pesin's theory, there exists $\ell > 0$ such that the length of each component of $W^s_{loc}(x)$ is greater than or equal to ℓ for every $x \in D^+_{f,Q,\epsilon,l}$. This implies conclusion (3) of the theorem. □

4.5 A Criterion for Ergodicity

Throughout of this section, we will assume implicitly that P is a polygon without parallel sides facing each other and that $f \in \mathcal{R}^2$. Hence, $\Lambda_{f,P}$ is hyperbolic by Proposition 3.3. In particular, every periodic point of $\Phi_{f,P}$ is hyperbolic.

The next lemma plays a crucial role in the proof of Theorem 4.22. It tells us about the points contained in a given horizontal segment where local stable manifolds exist (c.f. [15, Proposition 3.4] and [14, Lemma 1]). The lemma is a consequence of Property (B) in Proposition 4.9.

Lemma 4.17 *Let Γ be a horizontal segment. For every $0 < \tau < 1$, there exists $l_+ \in \mathbb{N}$ such that* $\mathrm{Vol}_\Gamma(\Gamma \cap D^+_{f,P,\epsilon,l}) \geq 1 - \tau$ *for all $l \geq l_+$.*

Proof Note that if $x \in \Gamma \setminus D^+_{f,P,\epsilon,l}$, then $d_M(\Phi^n_{f,P}(x), N^+_{f,P}) < l^{-1}e^{-\epsilon n}$ for some $n \geq 0$. Thus,

$$\Gamma \setminus D^+_{f,P,\epsilon,l} \subset \bigcup_{n=0}^{\infty} \Gamma \cap \Phi^{-n}_{f,P}\left(N^+_{f,P}(l^{-1}e^{-\epsilon n})\right).$$

By Property (B) of Proposition 4.9, we obtain

$$\begin{aligned}\mathrm{Vol}_\Gamma(\Gamma \setminus D^+_{f,P,\epsilon,l}) &\leq \sum_{n=0}^{\infty} \mathrm{Vol}_\Gamma\left(\Gamma \cap \Phi^{-n}_{f,P}\left(N^+_{f,P}(l^{-1}e^{-\epsilon n})\right)\right)\\ &\leq \frac{B}{l}\sum_{n=0}^{\infty} e^{-\epsilon n} = \frac{B}{l(1-e^{-\epsilon})}.\end{aligned}$$

Hence,

$$\mathrm{Vol}_\Gamma(\Gamma \cap D^+_{f,P,\epsilon,l}) = 1 - \mathrm{Vol}_\Gamma(\Gamma \setminus D^+_{f,P,\epsilon,l}) \geq 1 - \frac{B}{l(1-e^{-\epsilon})},$$

which yields the wanted conclusion. □

Denote by $C(\Lambda_{f,P})$ be set of all continuous functions on the attractor $\Lambda_{f,P}$. For every $\varphi \in C(\Lambda_{f,P})$, let

$$\varphi^+(x) = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi\left(\Phi^k_{f,P}(x)\right)$$

be the forward Birkhoff average of φ. Also, let $\mu_1, \mu_2, \ldots$ be the ergodic SRB measures of $\Phi_{f,P}$, and let $E_1, E_2, \ldots$ be the corresponding sets as in Theorem 4.11.

Definition 4.18 For every i, define

$$\mu_i(\varphi) = \int_\Lambda \varphi(x)\, d\mu_i(x) \quad \forall\varphi \in C(\Lambda_{f,P}),$$

and

$$R_i = \left\{x \in E_i : \varphi^+(x) = \mu_i(\varphi)\ \forall\varphi \in C(\Lambda_{f,P})\right\}.$$

Since the sets E_i's are $\Phi_{f,P}$-invariant and pairwise disjoint, so are the sets R_i's. Moreover, the separability of $C(\Lambda_{f,P})$ and the Birkhoff Ergodic Theorem imply that $\mu_i(R_i) = \mu_i(E_i) = 1$ for every i.

Definition 4.19 For every i, define Δ_i to be the set of all $x \in D^-_{f,P,\epsilon}$ for which there exists an open disk V_x in $W^u_{loc}(x)$ containing x such that $\mathrm{Vol}_{V_x}(V_x \cap R_i) = 1$.

Remark 4.20 By the property of the conditional measures of an SRB measure, it follows that $\mu_i(\Delta_i) = 1$ (see the definition of a u-measure and the paragraph before [14, Proposition 9]).

The next results play a central role in the proofs of Theorem 1.2, permitting to characterize the sets E_i's and the number of their Bernoulli components using the periodic points of the map $\Phi_{f,P}$. We observe that our Theorem 4.22 is similar to [16, Theorem 5.1].

Proposition 4.21 *Let $x \in \Delta_i$, and suppose that there are a periodic point x_0 of $\Phi_{f,P}$ and an integer $n \geq 0$ such that*

$$W^s_{loc}(x_0) \cap \Phi^n_{f,P}(V_x) \neq \emptyset,$$

where V_x is as in the definition of Δ_i. Then there exists an open disk W of x_0 in $W^s_{loc}(x_0)$ and a set $W' \subset W \cap R_i$ such that $\mathrm{Vol}_W(W') = 1$.

Proof Let p be the period of x_0. By hypothesis, there exists $y \in W^s_{loc}(x_0) \cap \Phi^n_{f,P}(V_x)$. Define $y_k = \Phi^{kp}_{f,P}(y)$ for every $k \geq 0$. Clearly, $y_k \in W^s_{loc}(x_0)$ and $\lim_{k\to+\infty} y_k = x_0$.

Since the attractor $\Lambda_{f,P}$ is hyperbolic, there exists $k_0 \geq 0$ such that

$$W^u_{loc}(y_k) \cap \Phi^{n_1+kp}_{f,P}(V_x) = W^u_{loc}(y_k) \quad \text{for all } k \geq k_0.$$

Let $k \geq k_0$. We have $V_k := \Phi^{-(n_1+kp)}_{f,P}(W^u_{loc}(y_k)) \subset V_x$. Moreover, since the probability measure $(\Phi^{n_1+kp}_{f,P})_*(\mathrm{Vol}_{V_k})$ is equivalent to $\mathrm{Vol}_{W^u_{loc}(y_k)}$, and the set R_i is Φ-invariant, we have

$$\mathrm{Vol}_{W^u_{loc}(y_k)}(W^u_{loc}(y_k) \cap R_i) = 1.$$

The periodic point x_0 is hyperbolic, and so $x_0 \in D^-_{f,P,\epsilon,l_-}$ for some $l_- \in \mathbb{N}$. It follows that there is $k_1 \geq k_0$ such that

$$y_k \in D^-_{f,P,\epsilon,2l_-} \quad \text{for all } k \geq k_1. \tag{4.8}$$

Hence, the size of $W^u_{loc}(y_k)$ is uniformly bounded from below by some positive constant depending on ϵ and l_-. Since $D^-_{f,P,\epsilon,2l_-}$ is closed, by [14, Propositions 1 and 4], the sequence of curves $W^u_{loc}(y_k)$ converges in the C^1-topology to an open disk $W \subset W^u_{loc}(x_0)$ containing x_0.

By Lemma 4.17, for every $0 < \delta < 1$, there exists $l_+ \geq 0$ such that

$$\mathrm{Vol}_W(W \cap D^+_{f,P,\epsilon,l_+}) \geq 1 - \delta.$$

This together with (4.8) implies that there exist an open disk $W_1 \subset W$ containing x_0 and an integer $k_2 \geq k_1$ such that

$$\mathrm{Vol}_W(W_1 \cap D^+_{f,P,\epsilon,l_+}) \geq 1 - \delta/2,$$

and

$$W^s_{loc}(w) \cap W^u_{loc}(y_{k_2}) \neq \emptyset \quad \text{for all } w \in W_1 \cap D^+_{f,P,\epsilon,l_+}.$$

Let

$$W_2 = \left\{ w \in W_1 \cap D^+_{f,P,\epsilon,l_+} : W^s_{loc}(w) \cap W^u_{loc}(y_{k_2}) \cap R_i \neq \emptyset \right\}.$$

It is a well known fact that if $\varphi \in C(\Lambda)$, $w \in D^+_{f,P,\epsilon}$ and $\varphi^+(w)$ exists, then $\varphi^+(z) = \varphi^+(w)$ for every $z \in W^s_{loc}(x)$. Therefore, if $w \in W_2$, then $W^s_{loc}(w) \cap R_i \neq \emptyset$, and so $\varphi^+(w) = \mu_i(\varphi)$ for every $\varphi \in C(\Lambda_{f,P})$. By the absolute continuity of the stable foliation (see [14, Proposition 10]),

$$\mathrm{Vol}_W(W_2) = \mathrm{Vol}_W(W_1 \cap D^+_{f,P,\epsilon,l_+}) \geq 1 - \delta/2.$$

We have proved that for every $0 < \delta < 1$, there is a subset $W_2 = W_2(\delta) \subset W$ such that $\mathrm{Vol}_W(W_2) \geq 1 - \delta/2$, and $\varphi^+|_{W_2} = \mu_i(\varphi)$ for every $\varphi \in C(\Lambda_{f,P})$. We obtain immediately the existence of a set $W' \subset W$ such that $\mathrm{Vol}_W(W') = 1$ and $\varphi^+|_{W'} = \mu_i(\varphi)$ for every $\varphi \in C(\Lambda_{f,P})$. □

Theorem 4.22 *Let $x_1 \in \Delta_i$ and $x_2 \in \Delta_j$, and suppose that there are a periodic point x_0 of $\Phi_{f,P}$ and two integers $n_1 \geq 0$ and $n_2 \geq 0$ such that $W^s_{loc}(x_0) \cap \Phi^{n_1}_{f,P}(V_{x_1}) \neq \emptyset$ and $W^s_{loc}(x_0) \cap \Phi^{n_2}_{f,P}(V_{x_2}) \neq \emptyset$, where V_{x_1} and V_{x_2} are the sets corresponding to x_1 and x_2 as in the definition of R_i. Then $i = j$.*

Proof Both x_1 and x_2 satisfy the hypotheses of Proposition 4.21. Thus, there exist two open disks W_1 and W_2 of x_0, and two sets $W_1' \subset W_1 \cap R_i$ and $W_2' \subset W_2 \cap R_j$ such that $\mathrm{Vol}_{W_1}(W_1') = \mathrm{Vol}_{W_2}(W_2') = 1$. It follows that $R_i \cap R_j \neq \emptyset$. Hence $R_i = R_j$, i.e., $i = j$. □

We now prove a proposition that allows us to estimate the number of Bernoulli components of an ergodic SRB measure.

Let $\mu = \mu_i$ be one of the ergodic SRB measures of $\Phi = \Phi_{f,P}$, and let $E = E_i$ be the corresponding ergodic component. Suppose that μ has n Bernoulli components. Then, it follows from Theorem 4.11 that there exist an integer $n > 0$ and a probability measure μ' such that Φ^n endowed with μ' is a Bernoulli automorphism, and μ is the arithmetic average of $\mu', \Phi_*\mu', \ldots, \Phi^{n-1}_*\mu'$. Define the sets R' and Δ' for the measure μ' and the map Φ^n exactly as the sets R_i and Δ_i for the measure μ_i and the map Φ in Definitions 4.18 and 4.19. Remark 4.20 applies to Δ' and μ' as well, and so $\mu'(\Delta') = 1$. Also, given $x \in \Delta'$, let V_x be the open disk in $W^u_{loc}(x)$ containing x as in Definition 4.19.

Proposition 4.23 *Let $x \in \Delta'$, and suppose that there exist an integer $j \geq 0$ and a periodic point x_0 of Φ of period p such that*

$$W^s_{loc}(x_0) \cap \Phi^j(V_x) \neq \emptyset.$$

Then n is a divisor of p.

Proof By Proposition 4.21, there exists a neighborhood $W \subset W^u_{loc}(x_0)$ of x_0 and a set $W' \subset W \cap R'$ with $\mathrm{Vol}_W(W') = 1$. Since x_0 is periodic and $W' \subset W^u_{loc}(x_0)$, we have $\Phi^{-p}(W') \subset W'$. Moreover, $\mathrm{Vol}_W(\Phi^{-p}(W')) > 0$ because the measure $\Phi^p_* \mathrm{Vol}_W$ is equivalent to $\mathrm{Vol}_{\Phi^{-p}(W)}$. This yields $W' \cap \Phi^{-p}(W') \neq \emptyset$. Since $W' \subset R'$, the set W' is contained in a Bernoulli component of μ. Hence, n must be a divisor of p. □

5 Proof of Theorem 1.2

In this section, we prove Theorems 5.11, 5.14 and 5.15 and Corollary 5.12, which all together form Theorem 1.2. Throughout this section, we assume that $P \in \mathcal{P}^*_n$. Recall that $\mathcal{E}(\psi_P)$ and $\mathcal{E}(\Phi_{f,Q})$ denote the set of ergodic acips of ψ_P and the set of the ergodic SRB measures of $\Phi_{f,Q}$, respectively.

5.1 The Set $\mathcal{F}(\nu)$

Consider $\nu \in \mathcal{E}(\psi_P)$. Let $\eta > 0$ be as in Lemma 4.8. From Part (3) of Theorem 3.8 and Condition (*), it follows that there exists a finite subset $\mathcal{F}(\nu)$ of $\mathrm{int}(\mathrm{supp}\,\nu)$ with the following properties:

(1) $\mathcal{F}(\nu)$ consists of periodic points of ψ_P whose orbits do not visit any vertex of P,
(2) $\mathcal{F}(\nu)$ is $\eta/6$-dense in $\mathrm{supp}\,\nu$,
(3) for every $s \in \mathcal{F}(\nu)$, there exists $z \in (s - \eta/9, s + \eta/9) \cap \mathcal{F}(\nu)$ with $z \neq s$ such that the great common divisor of the periods of s and z equals the number of the exact components of ν.

Given $s \in S^1$ and $r > 0$, let $B(s,r) = (s-r, s+r)$ be the open interval of S^1 centered at s of radius r.

Lemma 5.1 *For any pair $s_1, s_2 \in \mathcal{F}(\nu)$, there exist an open interval $I \subset B(s_1, \eta/3) \cap \operatorname{supp} \nu$, integers $k, m \geq 0$ and $s_0 \in \mathcal{F}(\nu)$ such that*

(1) $\psi_P^{k+m}|_I$ *is differentiable [so $\psi_P^k(I)$ and $\psi_P^{k+m}(I)$ are intervals],*
(2) $\psi_P^k(I) \subset B(s_2, \eta/3) \cap \operatorname{supp} \nu$,
(3) $|\psi_P^{k+m}(I)| > \eta$,
(4) $B(s_0, \eta/3) \subset \psi_P^{k+m}(I) \subset \operatorname{supp} \nu$.

Proof Let $B_i = B(s_i, \eta/3)$ for $i = 1, 2$. From $s_i \in \operatorname{int}(\operatorname{supp} \nu)$, it follows that $B_i \cap \operatorname{supp} \nu$ is an interval, and so $\nu(B_i) > 0$. Since ν is ergodic, there exist $s \in B_1 \cap \operatorname{supp} \nu$ and an integer $k \geq 0$ such that $\psi_P^k(s) \in B_2$. The set of points whose forward orbit meets a vertex of P has zero ν-measure. Then, we can assume without loss of generality that $\psi_P^i(s) \notin V_P$ for every $0 \leq i \leq k$. Thus, there exists a subinterval I_1 of $B_1 \cap \operatorname{supp} \nu$ with $s \in \operatorname{int}(I_1)$ such that $\psi_P^k(I_1)$ is differentiable. In particular, $\psi_P^k(I_1)$ is a subinterval of $B_2 \cap \operatorname{supp} \nu$.

By Lemma 4.8, there are an integer $m \geq 0$ and an open interval $J \subset \psi_P^k(I_1)$ such that $\psi_P^m|_J$ is differentiable, and $\psi_P^m(J)$ is an interval contained in $\operatorname{supp} \nu$ with $|\psi_P^m(J)| > \eta$. We conclude that there exists an open interval $I \subset I_1$ with $\psi_P^k(I) = J$ such that ${\psi_P^{k+m}}_I$ is differentiable, $\psi_P^{k+m}(I) \subset B_2 \cap \operatorname{supp} \nu$, $\psi_P^{k+m}(I) \subset \operatorname{supp} \nu$ and $|\psi_P^{k+m}(I)| > \eta$. Finally, since $\mathcal{F}(\nu)$ is $\eta/6$-dense in $\operatorname{supp} \nu$, there exists $s_0 \in \mathcal{F}(\nu)$ such that $B(s_0, \eta/3) \subset \psi_P^{k+m}(I)$. □

Lemma 5.2 *There exists $\delta_1 > 0$ such that for any $s_1, s_2 \in \mathcal{F}(\nu)$, there are integers $m_1, m_2 \geq 0$ and $s_0 \in \mathcal{F}(\nu)$ for which if $\lambda(f) < \delta_1$, $d(Q, P) < \delta_1$ and Γ_1 and Γ_2 are horizontal segments satisfying $B(s_i, \eta/3) \subset \pi_s(\Gamma_i)$ for $i = 1, 2$, then*

$$B(s_0, \eta/3) \subset \pi_s\left(\Phi_{f,Q}^{m_i}(\Gamma_i)\right), \quad i = 1, 2.$$

Proof For any pair of points $s_1, s_2 \in \mathcal{F}(\nu)$, denote by $I(s_1, s_2)$, $s_0(s_1, s_2)$, $m(s_1, s_2)$ and $k(s_1, s_2)$ the interval, the point of $\mathcal{F}(\nu)$ and the two positive integers as in Lemma 5.1. Also, define

$$m_1(s_1, s_2) = m(s_1, s_2) + k(s_1, s_2) \quad \text{and} \quad m_2(s_1, s_2) = k(s_1, s_2).$$

Since $\mathcal{F}(\nu)$ is finite, m_1 and m_2 are bounded functions on $\mathcal{F}(\nu) \times \mathcal{F}(\nu)$. For this reason, the assumption on Γ_1 and Γ_2 and Lemma 5.1, we can find a $\delta_1 > 0$ such that if $\lambda(f) < \delta_1$ and $d(Q, P) < \delta_1$, then for any pair $s_1, s_2 \in \mathcal{F}(\nu)$, the sets $\Phi_{f,Q}^{m_1(s_1,s_2)}(\Gamma_1)$ and $\Phi_{f,Q}^{m_2(s_1,s_2)}(\Gamma_2)$ will be so close to the intervals $\psi_P^{m_1(s_1,s_2)}(I(s_1, s_2))$ and $\psi_P^{m_2(s_1,s_2)}(I(s_1, s_2))$ that

$$B\left(s_0(s_1, s_2), \eta/3\right)) \subset \pi_s\left(\Phi_{f,Q}^{m_i(s_1,s_2)}(\Gamma_i)\right), \quad i = 1, 2.$$

□

5.2 The Set $\mathcal{F}_{f,Q}(\nu)$

Given $s \in \mathcal{F}(\nu)$, denote by $\delta(s)$, $\kappa(s) = 2t_P(s, 0)$ and $\ell(s)$ the constants in Theorem 4.13. Define $\bar{\delta} = \min_{s \in \mathcal{F}(\nu)} \delta(s)$, $\bar{\kappa} = \min_{s \in \mathcal{F}(\nu)} \kappa(s)$ and $\bar{\ell} = \min_{s \in \mathcal{F}(\nu)} \ell(s)$. If $\lambda(f) < \min\{\delta_1, \bar{\delta}\}$ and $d(Q, P) < \min\{\delta_1, \bar{\delta}\}$, then by Theorem 4.13 there exists a continuation $x_{f,Q} = (s_{f,Q}, \theta_{f,Q})$ for every $(s, 0)$ with $s \in \mathcal{F}(\nu)$.

Lemma 5.3 *There is* $0 < \delta_2 < \min\{\delta_1, \bar{\delta}\}$ *such that if* $\lambda(f) < \delta_2$ *and* $d(Q, P) < \delta_2$*, then for every* $s \in \mathcal{F}(\nu)$*, the local stable manifold* $W^s_{loc}(x_{f,Q})$ *of the continuation of* $(s, 0)$ *intersects both lines* $\theta = -\lambda(f)\pi/2$ *and* $\theta = \lambda(f)\pi/2$ *at points with* s*-coordinate contained in the interval* $(s - 2\eta/9, s + 2\eta/9)$*.*

Proof Suppose that $\lambda(f) < \min\{\delta_1, \bar{\delta}\}$ and $d(Q, P) < \min\{\delta_1, \bar{\delta}\}$. A straightforward computation—which we omit—shows that if we also require $\lambda(f) < \min\{\bar{\ell}/(\pi\sqrt{1 + \bar{\kappa}^2}), \eta/(9\pi)\}$, then for every $s \in \mathcal{F}(\nu)$, the local stable manifold of the continuation $x_{f,Q}$ of $(s, 0)$ intersects both lines $\theta = -\lambda(f)\pi/2$ and $\theta = \lambda(f)\pi/2$ at points with s-coordinate contained in $(s_{f,Q} - \eta/9, s_{f,Q} + \eta/9)$. The existence of $0 < \delta_2 < \min\{\delta_1, \bar{\delta}\}$ with the wanted property follows from the fact that $x_{f,Q} \to (s, 0)$ as $\lambda(f) + d(P, Q) \to 0$. □

For $\lambda(f) < \delta_2$ and $d(Q, P) < \delta_2$, define

$$\mathcal{F}_{f,Q}(\nu) = \left\{x_{f,Q} \in M_Q : s \in \mathcal{F}(\nu)\right\}.$$

Fix $0 < \zeta < \eta/6$. Propositions 4.3 and 4.4 imply that there exists $0 < \delta_3 < \delta_2$ such that if $\lambda(f) < \delta_3$ and $d(Q, P) < \delta_3$, then there are trapping regions $U(\nu)$ and $W(\nu)$ for ψ_Q and $\Phi_{f,Q}$, respectively. Finally, note that $\mathcal{F}(\nu)$ is $\eta/3$-dense in $U(\nu)$ because of our choice of ζ.

5.3 The Set $\mathcal{G}$

Let $\mathcal{B} \subset S^1$ be the union of the basins of the ergodic acips of ψ_P. For every $s \in S^1$ and $r > 0$, define

$$\Sigma(s, r) = (s - r, s + r) \times (-\pi r/2, \pi r/2).$$

Lemma 5.4 *There exist a* $\eta/2$*-dense finite set* $\mathcal{G} \subset \mathcal{B}$ *in* S^1 *and* $\delta_4 > 0$ *such that if* $\lambda(f) < \delta_4$ *and* $d(Q, P) < \delta_4$*, then for every* $s \in \mathcal{G}$*, there are an ergodic acip* $\tilde{\nu} \in \mathcal{E}(\psi_P)$ *and an integer* $k \geq 0$ *for which* $\Phi^k_{f,Q}(\Sigma(s, \delta_4)) \subset \mathrm{int}(W(\tilde{\nu}))$*.*

Proof Let $\mathcal{S}$ be the set of all $s \in \mathcal{B}$ such that $\psi^i_P(s)$ is a vertex of P for some $i \geq 0$. $\mathcal{S}$ is at most countable, and $\mathcal{B}$ has full Lebesgue measure by Part (4) of Theorem 3.8. Hence, $\mathcal{B} \setminus \mathcal{S}$ has full Lebesgue measure as well, and so it contains a finite set $\mathcal{G}$ that is $\eta/2$-dense in S^1.

Let $s \in \mathcal{G}$. Then, there exists $\tilde{\nu} \in \mathcal{E}(\psi_P)$ such that $s \in B(\tilde{\nu})$. Since $\mathrm{supp}\,\tilde{\nu} \subset \mathrm{int}(U(\tilde{\nu}))$ (see Proposition 4.4), Urysohn's Lemma guarantees the existence of a continuous function $\phi : I \to [0, 1]$ such that ϕ is identically equal to 1 on $\mathrm{supp}\,\tilde{\nu}$ and is identically equal to 0 on the closure of the complement of $S^1 \setminus U(\tilde{\nu})$. Since $s \in B(\tilde{\nu})$,

$$\lim_{k \to +\infty} \frac{1}{k} \sum_{i=0}^{k-1} \phi(\psi_P(s)) = \int_{S^1} \phi(s) d\tilde{\nu}(s) = 1.$$

Hence, there exists $k \geq 0$ such that $\psi^k_P(s) \in \mathrm{int}(U(\tilde{\nu}))$. Since $s \notin \mathcal{S}$, it follows that $\psi^i_P(s) \notin V_P$ for every i, and so

$$\Phi^k_{0,P}(s, 0) = \left(\psi^k_P(s), 0\right).$$

This and the fact that $U(\tilde{\nu}) \times \{0\} = W(\tilde{\nu})$ imply $\Phi^k_{0,P}(s, 0) \in \mathrm{int}(W(\tilde{\nu}))$.

Finally, note that the map $(f, Q, x) \mapsto \Phi^k_{f,Q}(x)$ is continuous at $(0, P, (s, 0))$. Thus, there exits $\delta_4 > 0$ such that

$$\Phi^k_{f,Q}(\Sigma(s, \delta_4)) \subset \operatorname{int}(W(\tilde{\nu}))$$

provided that $\lambda(f) < \delta_4$ and $d(Q, P) < \delta_4$. □

5.4 Ergodic SRB Measures

The constant δ_3 introduced at the end of Sect. 5.2 depends on the ergodic acip ν. To emphasize such a dependence, we write $\delta_3(\nu)$. Let $\bar{\delta}_3$ be the minimum of $\delta_3(\nu)$ over all $\nu \in \mathcal{E}(\psi_P)$.

Let δ be as in Theorem 4.11, and choose $0 < \delta_5 < \min\{\delta, \bar{\delta}_3, \delta_4\}$. By Theorem 4.11, for every $f \in \mathcal{R}^2$ with $\lambda(f) < \delta_5$ and every $Q \in \mathcal{P}_n$ with $d(Q, P) < \delta_5$, the map $\Phi_{f,Q}$ admits ergodic SRB measures. In the rest of this subsection, we will implicitly assume that $f \in \mathcal{R}^2$ with $\lambda(f) < \delta_5$ and that $Q \in \mathcal{P}_n$ with $d(Q, P) < \delta_5$.

Definition 5.5 For every $\nu \in \mathcal{E}\psi_P$, define

$$H_{f,Q}(\nu) = \left\{\mu \in \mathcal{E}(\Phi_{f,Q}) \colon \mu(\operatorname{supp} \mu \cap W(\nu)) = 1\right\}.$$

Given $\mu \in H_{f,Q}(\nu)$, denote by $R(\mu)$ and $\Delta(\mu)$ the sets corresponding to μ as in Definitions 4.18 and 4.19, respectively. By Remark 4.20, $\mu\,(\Delta(\mu) \cap W(\nu)) = 1$.

Lemma 5.6 *Let $\mu \in H_{f,Q}(\nu)$. If $x \in \Delta(\mu) \cap W(\nu)$, then there exist $j \geq 0$ and $s \in \mathcal{F}(\nu)$ such that $B(s, \eta/3) \subset \pi_s(\Phi^j_{f,Q}(V_x))$, where $V_x \subset W^u_{loc}(x)$ is the set associated to x as in Definition 4.19.*

Proof Since V_x is a horizontal segment, Lemma 4.8 implies that there exists $j \geq 0$ such that $\Phi^j_{f,Q}(V_x)$ contains a horizontal segment Γ with $|\Gamma| > \eta$. Since $\mathcal{F}(\nu)$ is $\eta/6$-dense in $U(\nu)$, and $\pi_s(W(\nu)) = U(\nu)$, there is $s \in \mathcal{F}(\nu)$ such that $B(s, \eta/3) \subset \pi_s(\Gamma) \subset \pi_s(\Phi^j_{f,Q}(V_x))$. □

Lemma 5.7 $\#H_{f,Q}(\nu) \leq 1$.

Proof Let $\mu_{n_1}, \mu_{n_2} \in H_{f,Q}(\nu)$. For $i = 1, 2$, pick $x_i \in \Delta(\mu_{n_i}) \cap W(\nu)$. Such an x_i exists, because $\mu_i(\Delta(\mu_{n_i}) \cap W(\nu)) = 1$. By Lemma 5.6, there exist $j_1, j_2 \geq 0$ and $s_1, s_2 \in \mathcal{F}(\nu)$ such that

$$B(s_i, \eta/3) \subset \pi_s\left(\Phi^{j_i}_{f,Q}(V_{x_i})\right) \quad \text{for } i = 1, 2. \tag{5.1}$$

Hence, there exist two horizontal segments $\Gamma_1 \subset \Phi^{j_i}_{f,Q}(V_{x_1})$ and $\Gamma_2 \subset \Phi^{j_i}_{f,Q}(V_{x_2})$ whose images under π_s contain the intervals $B(s_1, \eta/3)$ and $B(s_2, \eta/3)$, respectively.

By applying Lemma 5.2 to Γ_1 and Γ_2, we can conclude that there exist two integers $m_1, m_2 \geq 0$ and $s_0 \in \mathcal{F}(\nu)$ such that for each $i = 1, 2$,

$$B(s_0, \eta/3) \subset \pi_s\left(\Phi^{j_i+m_i}_{f,Q}(V_{x_i})\right). \tag{5.2}$$

Let $x_0 \in \mathcal{F}_{f,Q}(\nu)$ be the periodic point of $\Phi_{f,Q}$ corresponding to s_0. By Lemma 5.3, $W^s_{loc}(x_0)$ intersects both lines $\theta = \pm\lambda(f)\pi/2$ at points with s-coordinate contained in $B(s_0, \eta/3)$. Since each $\pi_\theta\left(\Phi^{j_i+m_i}_{f,Q}(V_{x_i})\right)$ is contained in the strip $|\theta| < \lambda(f)\pi/2$, it follows that $W^s_{loc}(x_0) \cap \Phi^{j_i+m_i}_{f,Q}(V_{x_i}) \neq \emptyset$. for each $i = 1, 2$. We can now apply Theorem 4.22 to μ_{n_1} and μ_{n_2}, and conclude that $n_1 = n_2$, i.e., $\mu_{n_1} = \mu_{n_2}$. □

Lemma 5.8 $\#H_{f,Q}(\nu) = 1$.

Proof By Lemma 5.7, it is enough to prove that $H_{f,Q}(\nu) \neq \emptyset$. Let $x \in \mathcal{F}_{f,Q}(\nu)$. Since x is a hyperbolic periodic point, we have $x \in D^-_{f,Q,\epsilon}$. Parts (5) of Theorem 4.11 applied to x implies that $\Phi_{f,Q}$ has an SRB measure $\tilde{\mu}$. Since $\mathcal{F}_{f,Q}(\nu) \subset W(\nu)$, it follows from Proposition 4.4 that supp $\tilde{\mu}$ is contained in the closure of $W(\nu)$. By the ergodic decomposition of $\tilde{\mu}$ [see Part (4) of Theorem 4.11], there exists $\mu \in \mathcal{E}(\Phi_{f,Q})$ such that

$$\operatorname{supp} \mu \subset \operatorname{supp} \tilde{\mu} \subset \overline{W(\nu)} = U(\nu) \times \left[-\frac{\pi}{2}\lambda(f), \frac{\pi}{2}\lambda(f)\right].$$

Now, by Part (7) of Theorem 4.11, $\tilde{\mu}(\operatorname{supp} \mu \cap \partial K_{f,Q}) = 0$. Since $\partial K_{f,Q} \subset S^1 \times \{-\pi\lambda(f)/2, \pi\lambda(f)/2\}$, we have $\mu(\operatorname{supp} \mu \cap W(\nu)) = 1$. We conclude that $\mu \in H_{f,Q}(\nu)$. □

Lemmas 5.7 and 5.8 allow us to define $\Theta_{f,Q} \colon \mathcal{E}(\psi_P) \to \mathcal{E}(\Phi_{f,Q})$ by $\Theta_{f,Q}(\nu) = \mu$ with $\mu \in H_{f,Q}(\nu)$. Next, we prove that $\Theta_{f,Q}$ is a bijection.

Lemma 5.9 *$\Theta_{f,Q}$ is one-to-one.*

Proof Suppose that there exist two distinct measures $\nu_1, \nu_2 \in \mathcal{E}(\psi_P)$ such that $\mu = \Theta_{f,Q}(\nu_1) = \Theta_{f,Q}(\nu_2)$. Then $\mu(W(\nu_1) \cap W(\nu_2)) = 1$, contradicting $W(\nu_1) \cap W(\nu_2) = \emptyset$ (see Proposition 4.4). □

Lemma 5.10 *$\Theta_{f,Q}$ is onto.*

Proof We prove that given $\mu \in \mathcal{E}(\Phi_{f,Q})$, there exists $\nu \in \mathcal{E}(\psi_P)$ such that $\mu \in H_{f,Q}(\nu)$, i.e., $\mu(\operatorname{supp} \mu \cap W(\nu)) = 1$.

Pick $x \in \Delta(\mu)$, and let V_x be the open disk of $W^u_{loc}(x)$ as in Definition 4.19. By Lemma 4.8, there exists an integer $i > 0$ such that $\Phi^i_{f,Q}(V_x)$ contains a horizontal segment Γ_1 with $|\Gamma_1| > \eta$.

Let $\mathcal{G}$ be the $\eta/2$-dense set in S^1 as in Lemma 5.4. Since Γ_1 is contained in $S^1 \times (-\pi\lambda(f)/2, \pi\lambda(f)/2)$, we have $\Gamma_1 \cap \Sigma(s, \delta_5) \neq \emptyset$ for some $s \in \mathcal{G}$. By Lemma 5.4, there exist an integer $k \geq 0$, a measure $\nu \in \mathcal{E}(\psi_P)$ and a horizontal segment $\Gamma_2 \subset \Gamma_1 \cap \Sigma(s, \delta_5)$ such that $\Gamma' := \Phi^k_{f,Q}(\Gamma_2)$ is a horizontal segment contained in $W(\nu)$.

Combining the previous conclusions, we obtain that there is a horizontal segment $\Gamma_0 \subset V_x$ such that $\Gamma' = \Phi^{k+i}_{f,Q}(\Gamma_0) \subset W(\nu)$. Now, recall that $\mathrm{Vol}_{V_x}(V_x \cap R(\mu)) = 1$, and that $R(\mu)$ is $\Phi_{f,Q}$-invariant. Since the push-forward of Vol_{Γ_0} by $\Phi^{k+i}_{f,Q}$ is equivalent to $\mathrm{Vol}_{\Gamma'}$, it follows that $\mathrm{Vol}_{\Gamma'}(\Gamma' \cap R(\mu)) = 1$. But $\Gamma' \subset W(\nu)$, and so $R(\mu) \cap W(\nu) \neq \emptyset$.

We claim that $\mu(R(\mu) \cap W(\nu)) = 1$. Indeed, let $A = R(\mu) \setminus W(\nu)$, and suppose that $\mu(A) > 0$. Let $z \in A$ be a μ-point of density of A. Since $K_{f,Q} \setminus W(\nu)$ is open, there are a compact neighborhood C of z and an open neighborhood O of z such that $C \subset O \subset K_{f,Q} \setminus W(\nu)$. By Urysohn's Lemma, there is a continuous $\phi \colon K_{f,Q} \to [0, 1]$ such that $\phi|_C \equiv 1$ and $\phi|_{K_{f,Q} \setminus O} \equiv 0$. Let $x \in R(\mu) \cap W(\nu)$. Using the function ϕ as in the proof of Lemma 5.4, one can show that $\Phi^j_{f,Q}(x) \in O$ for some $j \in \mathbb{N}$. In particular, $\Phi^j_{f,Q}(x) \notin W(\nu)$. However, that is impossible, since $W(\nu)$ is $\Phi_{f,Q}$-forward invariant by Proposition 4.4, and so $\Phi^j_{f,Q}(x) \in W(\nu)$. Therefore, $\mu(A) = 0$, whic is equivalent to $\mu(R(\mu) \cap W(\nu)) = 1$. Since $\mu(R(\mu) \cap \operatorname{supp} \mu) = 1$, we conclude that $\mu(\operatorname{supp} \mu \cap W(\nu)) = 1$. □

The previous propositions prove the following.

Theorem 5.11 *Let $P \in \mathcal{P}^*_n$. There is $\delta_5 > 0$ such that if $\lambda(f) < \delta_5$ and $d(Q, P) < \delta_5$, then there exists a bijection $\Theta_{f,Q} \colon \mathcal{E}(\psi_P) \to \mathcal{E}(\Phi_{f,Q})$. Moreover, for every $\nu \in \mathcal{E}(\psi_P)$, the support of $\Theta_{f,Q}(\nu)$ is contained in the closure of the trapping set $W(\nu)$.*

The next corollary is a direct consequence of Theorem 5.11 and the fact that the number of ergodic acips of ψ_P is bounded from above by the cardinality of the singular set S_P, which is not larger than n, the number of sides of P.

Corollary 5.12 *Under the hypotheses of Theorem 5.11, the ergodic SRB measures of $\Phi_{f,Q}$ enjoy the following properties: their number equals the number of the ergodic acips of ψ_P, which is bounded from above by n, and their supports are pairwise disjoint.*

5.5 Bernoulli Components

We prove that for every $\nu \in \mathcal{E}(\psi_P)$, the number of Bernoulli components of $\Theta_{f,Q}(\nu)$ equals the number of exact components of ν.

Proposition 5.13 *Under the hypotheses of Theorem 5.11, if $\nu \in \mathcal{E}(\psi_P)$, then the number of Bernoulli components of $\Theta_{f,Q}(\nu)$ is a multiple of the number of exact components of ν.*

Proof Let ν be an ergodic acip of $\psi = \psi_P$, and suppose that ν has m exact components. Accordingly, ψ^m has m exact invariant measures $\nu_1, \ldots, \nu_m$ whose arithmetic average is equal to ν and such that $\psi_*(\nu_i) = \nu_{i+1}$ for each $i = 1, \ldots, m-1$, and $\psi_*\nu_m = \nu_1$. Of course, $\nu_1, \ldots, \nu_m$ are ergodic acips of ψ^m. In fact, they are the only ergodic acips of ψ^m with support contained in supp ν.

The map ψ^m is piecewise expanding, and satisfies Condition (*), since so does ψ. Hence, Proposition 4.1 and Remark 4.2 apply to the measures $\nu_1, \ldots, \nu_m$, and so their supports are pairwise disjoint, and consist of finitely many closed intervals. Since $\operatorname{supp} \nu = \bigcup_{i=1}^m \operatorname{supp} \nu_i$, the trapping set $U(\nu)$ of ψ in Proposition 4.3 can be written as $U(\nu) = \bigcup_{i=1}^m U_i$, where U_i is the subset of $U(\nu)$ containing supp ν_i. Since ψ_* permutes cyclically $\nu_1, \ldots, \nu_m$, we have $\psi(U_i) \subset U_{i+1}$ for each $i = 1, \ldots, m-1$, and $\psi(U_m) \subset U_1$.

In view of the last conclusion, the trapping set $W(\nu)$ of $\Phi = \Phi_{f,Q}$ in Proposition 4.4 can be written as $W(\nu) = \bigcup_{i=1}^m W_i$, where $W_i = U_i \times (-\pi\lambda(f)/2, \pi\lambda(f)/2)$. Moreover, from the properties of $U_1, \ldots, U_m$, it follows that $\Phi(W_i) \subset W_{i+1}$ for each $i = 1, \ldots, m$, and $\Phi(W_m) \subset W_1$.

Suppose that $\mu = \Theta_{f,Q}(\nu)$ has n Bernoulli components $B_1, \ldots, B_n$. For every $1 \le i \le n$ and every $1 \le j \le m$, define $B_{i,j} = B_i \cap W_j$. We claim that for each i, there exists k such that $\mu(B_{i,k}) = \mu(B_i)$. The proof of the claim is as follows. Since $\Phi^m(W_j) \subset W_j$ for every j and $\Phi^n(B_i) = B_i$ for every i, each $B_{i,j}$ is Φ^{mn}-forward invariant. Moreover, since $W_1, \ldots, W_m$ are pairwise disjoint, so are $B_{i,1}, \ldots, B_{i,m}$. But the normalization of μ to B_i is mixing for Φ^{mn}, and so $B_{i,1}, \ldots, B_{i,m}$ are Φ^{mn}-forward invariant and pairwise disjoint only if there exists k such that $\mu(B_{i,k}) = \mu(B_i)$.

Now, consider the set $B_{i,k}$ such that $\mu(B_{i,k}) = \mu(B_i)$. By the invariance of μ, we have $\mu(\Phi^n(B_{i,k})) = \mu(B_i)$, and so $\mu(B_{i,k} \cap \Phi^n(B_{i,k})) = \mu(B_i) > 0$. Thus, there exists a nonempty set $B \subset B_{i,k}$ such that $\Phi^n(B) \in B_{i,k} \subset W_k$. Since $W_k \cap \Phi^i(W_k) = \emptyset$ for all $1 \le i \le m-1$, and $\Phi^m(W_k) \subset W_k$, we can conclude that n must be a multiple of m. □

Theorem 5.14 *Under the hypotheses of Theorem 5.11, for every $\nu \in \mathcal{E}(\psi_P)$, the number of Bernoulli components of $\Theta_{f,Q}(\nu)$ equals the number of exact components of ν.*

Proof Let $\nu \in \mathcal{E}(\psi_P)$, and suppose that ν has m exact components. Let E be the ergodic component of $\Phi_{f,Q}$ corresponding to $\mu = \Theta_{f,Q}(\nu)$. The measure μ is the arithmetic average of n Bernoulli invariant measures of $\Phi' = \Phi^n_{f,Q}|_E$. Let μ' be one of these measures. Next, define the sets R' and Δ' for the measure μ' and the map Φ^n exactly as the sets R_i and Δ_i

have been defined for the measure μ_i and the map Φ in Definitions 4.18 and 4.19. Finally, given $x \in \Delta'$, let V_x be the open disk in $W^u_{loc}(x)$ containing x as in Definition 4.19.

Remark 4.20 applies to Δ' and μ', and so $\mu'(\Delta') = 1$. Let $x \in \Delta' \cap W(\nu)$. By Lemma 5.6, there exist an integer $j \geq 0$ and $s_1 \in \mathcal{F}(\nu)$ such that $B(s_1, \eta/3) \subset \pi_s(\Phi'^j(V_x))$, and by property (3) of the definition of $\mathcal{F}(\nu)$, there is $s_2 \in \mathcal{F} \cap B(s_1, \eta/9)$ with $s_2 \neq s_1$ such that the great common divisor of s_1 and s_2 is m.

Let x_1 and x_2 be the points in $\mathcal{F}_{f,Q}(\nu)$ corresponding to s_1 and s_2. Lemma 5.3 guarantees that for each $i = 1, 2$, $W^s_{loc}(x_i)$ intersects both lines $\theta = -\pi\lambda(f)/2$ and $\theta = \pi\lambda(f)/2$ at points x_i^- and x_i^+ with s-coordinate contained in $B(s_i, 2\eta/9)$. This together with $s_2 \in B(s_1, \eta/9)$ implies that $\pi_s(x_i^-)$ and $\pi_s(x_i^+)$ are both contained in $B(s_1, \eta/3)$, and so $W^s_{loc}(x_i)$ intersects $\Phi'^j(V_x)$. By Proposition 4.23, n is a divisor of m. On the other hand, by Proposition 5.13, n is a multiple of n. We conclude that $n = m$. □

5.6 Basins of the Ergodic SRB Measures

In [8, Theorem 5.1], we proved that for every polygon Q without parallel sides facing each other and for every reflection law $f \in \mathcal{R}^2$ satisfying the additional condition $f' > 0$, the ergodic SRB measures of $\Phi_{f,Q}$ have the property that the union of their basins is a subset of M_Q of full Vol-measure. The extra hypothesis $f' > 0$ was required to make sure that the billiard map admits SRB measures, and it can be safely replaced by the weaker condition $f \in \mathcal{R}^2$, once we know that $\Phi_{f,Q}$ does admit ergodic SRB measures. Accordingly, from [8, Theorem 5.1], we obtain the following.

Theorem 5.15 *Under the hypotheses of Theorem 5.11, the union of the basins of the ergodic SRB measures of $\Phi_{f,Q}$ is a subset of M_Q of full* Vol-*measure.*

6 Billiards in Regular Polygons and Triangles

In this section, we apply Theorem 1.2 and Proposition 2.3 to billiards in convex regular polygons with an odd number of sides and to billiards in triangles. The case of the billiard in an equilateral triangle was first studied in [2].

Lemma 6.1 *Each convex regular polygon with an odd number n of sides satisfies Condition (*).*

Proof Let P be a convex regular polygon with an odd number n of sides. Using the symmetry of P, we showed in [6, Sect. 6.2] that the map $\phi_n : [0, 1[\to [0, 1[$ defined by

$$\phi_n(x) = -\frac{1}{\cos\left(\frac{\pi}{n}\right)}\left(x - \frac{1}{2}\right) \pmod 1$$

is a factor of ψ_P, and that ϕ_n enjoys the following property: the forward orbit of $x = 0$ does not contain any periodic points and the discontinuity point $x = 1/2$ of ϕ_n. In view of this, it is not difficult to see that ψ_P satisfies Condition (*). □

Corollary 6.2 *Let P be a convex regular polygon with an odd number $n \geq 3$ of sides. There exists $\delta > 0$ such that if $f \in \mathcal{R}^2$ with $\lambda(f) < \delta$ and $Q \in \mathcal{P}_n$ with $d(Q, P) < \delta$, then the following hold*:

(1) *if $n = 3$, then $\Phi_{f,Q}$ has a unique ergodic SRB measure, and this measure has a single Bernoulli component*;

(2) *if* $n = 5$, *then* $\Phi_{f,Q}$ *has a unique ergodic SRB measure, and this measure has two Bernoulli components*;

(3) *if* $n \geq 7$, *then* $\Phi_{f,Q}$ *has exactly n ergodic SRB measures. All these measures have* $2^{m(n)}$ *Bernoulli components, where* $m(n)$ *is the integer part of* $-\log_2(-\log_2 \cos(\pi/n))$.

Proof By [7, Theorem 1.1], the slap map ψ_P has a unique ergodic acip if $n = 3$ or 5, and exactly n ergodic acips if $n \geq 7$. Moreover, every acip has $2^{m(n)}$ exact components, where $m(n)$ is the integer part of $-\log_2(-\log_2 \cos(\pi/n))$. In particular, $m(3) = 0$ and $m(5) = 1$. The conclusion of the corollary now follows from Theorem 1.2 and Lemma 6.1. □

Next, we consider billiards in triangles.

Corollary 6.3 *There exists a residual and full measure subset* $\mathcal{X}_3$ *of the set of triangles* $\mathcal{P}_3$ *with the following property: for any* $P \in \mathcal{X}_3$, *there is* $\delta > 0$ *such that if* $f \in \mathcal{R}^2$ *with* $\lambda(f) < \delta$ *and* $Q \in \mathcal{P}_3$ *with* $d(Q, P) < \delta$, *then the billiard map* $\Phi_{f,Q}$ *has a unique ergodic SRB measure. This measure is Bernoulli if P is acute, and has an even number of Bernoulli components, otherwise.*

Proof The corollary follows from Theorem 1.2 and Proposition 2.3 and the ergodic properties of slap maps of triangles [7, Theorem 1.2]. □

Corollary 6.4 *Let P be an acute triangle. If* $\lambda \in \mathcal{R}^2$ *and* $\lambda(f)$ *is sufficiently small, then* $\Phi_{f,P}$ *has a unique ergodic SRB measure, and this measure has a single Bernoulli component.*

Proof By [7, Theorem 1.2], the slap map of any acute triangle has a unique ergodic acip, which is also exact. Since acute triangles trivially satisfy Condition (*), the wanted conclusion follows from Theorem 1.2. □

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