Bounds on 10th moments of (x, x^3) for ellipsephic sets

Theresa C. Anderson, Bingyang Hu, Yu-Ru Liu, and Alan Talmage

Abstract. Let $\mathcal{A}$ be an ellipsephic set which satisfies digital restrictions in a given base. Using the method developed by Hughes and Wooley, we bound the number of integer solutions to the system of equations

$$\sum_{i=1}^{2}\left(x_i^3 - y_i^3\right) = \sum_{i=3}^{5}\left(x_i^3 - y_i^3\right)$$
$$\sum_{i=1}^{2}(x_i - y_i) = \sum_{i=3}^{5}(x_i - y_i),$$

with $\mathbf{x}, \mathbf{y} \in \mathcal{A}^5$. The fact that ellipsephic sets with small digit sumsets have fewer solutions of linear equations allows us to improve the general bounds obtained by Hughes and Wooley and also the corresponding efficient congruencing estimates. We also generalize our result from the curve (x, x^3) to $(x, \phi(x))$, where ϕ is a polynomial with integer coefficients and $\deg(\phi) \geq 3$.

1. introduction

The discrete restriction conjecture has recently been of wide interest for researchers in both harmonic analysis and number theory (for example, see [**8**], [**9**]

2020 *Mathematics Subject Classification.* 11L07, 42B05, 35Q53.

Key words and phrases. mean value estimates, ellipsephic sets, KdV-like equations.
T. C. Anderson was supported by an NSF DMS Grant in analysis and number theory.

Y.-R. Liu was supported by an NSERC Discovery Grant.

The authors would like to thank K. Biggs and T. Wooley for helpful discussions related to the project. They also thank the referee for providing helpful comments.

and [**10**]). To recall the conjecture for the cuve (x, x^3), let $\mathbf{a} = \{a(n)\}_{n\in\mathbb{Z}}$ and

$$E\mathbf{a}\,(\alpha,\beta) := \sum_{|n|\leq N} a(n)e\left(\alpha n^3 + \beta n\right) \quad \text{with} \quad \alpha,\beta\in\mathbb{R}.$$

A weaker version of the conjecture can be stated as follows.

Conjecture 1.1. *For each $p\in[1,\infty]$ and $\epsilon>0$, there exists a constant $C_{p,\epsilon}>0$, such that for all $N\in\mathbb{N}$ and all sequence $\mathbf{a}=\{a(n)\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$, one has*

$$\|E\mathbf{a}\|_{L^p(\mathbb{T}^2)} \leq C_{p,\epsilon}N^{\epsilon}\left(1+N^{\frac{1}{2}-\frac{4}{p}}\right)\|\mathbf{a}\|_{\ell^2(\mathbb{Z})}.$$

By proving the bound for $p=6$, Bourgain [**5**] established Conjecture 1.1 for the cases $1\leq p\leq 6$. Little further progress was made before Hu and Li [**8**] proved the case $p=14$. Lai and Ding [**10**] extended the range to $p\geq 12$. Recently, Hughes and Wooley [**9**] proved the bound for $p=10$ and hence established the conjecture for $p\geq 10$. For more details about the discrete restriction conjecture and its relation to KdV equations, we refer interested readers to [**8**], [**9**] and [**10**].

Given a set $\mathcal{S}\subset\mathbb{Z}$, let $\mathcal{S}(N)=\mathcal{S}\cap[-N,N]$ and $S=|\mathcal{S}(N)|$. For $s\in\mathbb{N}=\{1,2,\cdots\}$, we denote by $J_s(\mathcal{S}(N))$ the number of solutions to the system of equations

$$\sum_{i=1}^{s}\left(x_i^3-y_i^3\right)=0$$
$$\sum_{i=1}^{s}(x_i-y_i)=0,$$

with $\mathbf{x},\mathbf{y}\in\mathcal{S}(N)^s$. In order to prove Conjecture 1.1 with $p=10$, Hughes and Wooley studied in [**9**] the quantity $J_5(\mathcal{S}(N))$ for any set $\mathcal{S}$. In particular, they proved that there exists a positive constant κ such that

$$J_5(\mathcal{S}(N)) \ll N\exp\left(\kappa\frac{\log N}{\log\log N}\right)S^5.$$

The bound for $J_5(\mathcal{S}(N))$ can be improved if additional structures are employed on the set $\mathcal{S}$. In this paper, we consider the cases of ellipsephic sets.

The term *ellipsephic set* was introduced by Biggs in [**3**] and [**4**]. She used the terminology to mimic the word *ellipséphique* used in the French mathematical literature to denote integers with missing digits (for example, see [**1**] and [**2**]). Let ℓ be a positive integer and $\mathcal{A}_\ell \subset \{0, 1, \cdots, \ell-1\}$ with $2 \le |\mathcal{A}_\ell| \le \ell - 1$. In other words, $\mathcal{A}_\ell$ contains at least two elements of $\{0, 1, \cdots, \ell-1\}$, but at most $(\ell-1)$ elements from the set. We say $\mathcal{A}$ is an *ellipsephic set in the base* ℓ if for any $n \in \mathcal{A}$,

$$n = \sum_i a_i \ell^i \quad \text{with} \quad a_i \in \mathcal{A}_\ell.$$

An ellipsephic set $\mathcal{A}$ is said to be *with small digit sumsets* if there exists some constant $K > 0$, such that $|\mathcal{A}_\ell + \mathcal{A}_\ell| \le K|\mathcal{A}_\ell|$. For example, if $\ell \ge 3$ and $\mathcal{A}_\ell = \{0, 1\}$, then $\mathcal{A}_\ell + \mathcal{A}_\ell \in \{0, 1, 2\}$ and we have $K = 3/2$. In this paper, we will show that if $\mathcal{A}$ is an ellipsephic set with small digit sumsets, then we can obtain the following bound for $J_5(\mathcal{A}(N))$.

Theorem 1.2. *Let $\mathcal{A}$ be an ellipsephic set with $\mathcal{A}_\ell = \mathcal{A} \cap [0, \ell-1]$. Write $\mathcal{A}(N) = \mathcal{A} \cap [-N, N]$ with $A = |\mathcal{A}(N)|$. Suppose that $|\mathcal{A}_\ell + \mathcal{A}_\ell| \le K|\mathcal{A}_\ell|$ for some constant $K > 0$. Then there exists a positive constant κ such that*

$$J_5(\mathcal{A}(N)) \ll K^4 N^{4\log_\ell K} \exp\left(\kappa \frac{\log N}{\log\log N}\right) A^6.$$

We recall that for a general set $\mathcal{S}(N)$ with $S = |\mathcal{S}(N)|$, Hughes and Wooley proved in [**9**, Theorem 2.1] that

$$J_5(\mathcal{S}(N)) \ll N \exp\left(\kappa \frac{\log N}{\log\log N}\right) S^5.$$

To compare the above result with Theorem 1.2, we first notice that the set $\mathcal{S}$ ($\mathcal{S}_\ell$ and S, respectively) in [**9**] plays the role of $\mathcal{A}$ ($\mathcal{A}_\ell$ and A, respectively) in our setting. Write r for both $|\mathcal{S}_\ell|$ and $|\mathcal{A}_\ell|$ with $2 \le r \le \ell - 1$. Then $A \le r^{\log_\ell N + 1} = rN^{\log_\ell r}$. It follows that if $4\log_\ell K + \log_\ell r < 1$, i.e., $K^4 r < \ell$, then

$$K^4 N^{4\log_\ell K} A \le rK^4 N^{4\log_\ell K + \log_\ell r} = o_{r,K}(N).$$

Hence the bound for ellipsephic sets in Theorem 1.2 is sharper in this case. The condition $K^4 r < \ell$ is often satisfied. For example, if $\ell \ge 3$ and $\mathcal{A}_\ell \in \{0, 1\}$, then $r = 2$ and $K = 3/2$. In this case, $K^4 r < \ell$ provided that $\ell > 2 \cdot (3/2)^4$, i.e., $\ell \ge 11$.

Using his efficient congruencing method, Wooley proved in [**14**] that for a general set $\mathcal{S}$,

$$J_3(\mathcal{S}(N)) \ll N^{\epsilon} S^3$$

for any $\epsilon > 0$. By trivially bounding the additional four variables by S^4, this gives us

$$J_5(\mathcal{S}(N)) \ll N^{\epsilon} S^7.$$

Let $r = |\mathcal{S}_\ell|$. Then $S \leq rN^{\log_\ell r}$. We notice that the above bound is stronger than the $N^{1+\epsilon}S^5$ bound in [**9**, Theorem 2.1] provided that $S^2 = o(N)$, which holds if $2\log_\ell r < 1$, i.e., $r^2 < \ell$. By noticing the set $\mathcal{S}$ ($\mathcal{S}_\ell$ and S, respectively) here plays the role of $\mathcal{A}$ ($\mathcal{A}_\ell$ and A, respectively) in Theorem 1.2, we can also compare the nested efficient congruencing bound $N^{\epsilon}S^7$ with the $K^4N^{4\log_\ell K+\epsilon}A^6$ bound in Theorem 1.2. We see that if $4\log_\ell K < \log_\ell r$, i.e., $K^4 < r$, then

$$K^4 N^{4\log_\ell K} = o(rN^{\log_\ell r}),$$

so $K^4N^{4\log_\ell K} = o(S)$. Hence the bound in Theorem 1.2 is sharper in this case. The conditions $K^4 < r$ and $r^2 < \ell$ are often satisfied. For example, if $\mathcal{A} = \{0, 1, \cdots, 13\}$, then $\mathcal{A} + \mathcal{A} \in \{0, 1, \cdots, 26\}$. We have $K = 27/14$ and $r = 14$. By taking $\ell > 14^2$, we have $K^4 < r < \ell^{1/2}$.

Let $\phi(x)$ be a polynomial with integer coefficients and $\deg(\phi) \geq 3$. We can generalize Theorem 1.2 from the curve (x, x^3) to the curve $(x, \phi(x))$. Let $J_{5,\phi}(\mathcal{A}(N))$ denote the number of solutions to the system of equations

$$\sum_{i=1}^{5} (\phi(x_i) - \phi(y_i)) = 0$$
$$\sum_{i=1}^{5} (x_i - y_i) = 0,$$

with $\mathbf{x}, \mathbf{y} \in \mathcal{A}(N)^5$. We will prove the following result.

Theorem 1.3. *Let $\mathcal{A}$ be an ellipsephic set with $\mathcal{A}_\ell = \mathcal{A} \cap [0, \ell - 1]$. Write $\mathcal{A}(N) = \mathcal{A} \cap [-N, N]$ with $A = |\mathcal{A}(N)|$. Suppose that $|\mathcal{A}_\ell + \mathcal{A}_\ell| \leq K|\mathcal{A}_\ell|$. Then for* *any* $\epsilon > 0$*, we have*

$$J_{5,\phi}(\mathcal{A}(N)) \ll K^4 N^{4\log_\ell K+\epsilon} A^6.$$

We assume here and throughout that the implicit constant in the symbol $\ll$ may depend on ε, s, k, and the coefficients of ϕ.

Let $|\mathcal{A}_\ell| = r$ with $2 \le r \le \ell - 1$. Similar to the remarks after Theorem 1.2, the result in Theorem 1.3 is sharper than the general bounds in [2, Theorem 3.4] and the corresponding nested efficient congruencing estimates, provided that $K^4 < r < \ell^{1/2}$.

We now restrict our attention to the case when ℓ is a prime. A subset $\mathcal{R} \subset \mathbb{N} \cup \{0\}$ is called a E_2^*-set if for any $n \in \mathbb{N}$, we have

$$\#\left\{(a_1, a_2) \in \mathcal{R}^2 : a_1 + a_2 = n\right\} \ll n^\epsilon \tag{1.1}$$

for any $\epsilon > 0$. Let $\mathcal{R}_\ell = \mathcal{R} \cap [0, \ell-1]$ and suppose that $2 \le |\mathcal{R}_\ell| \le \ell - 1$. Given a prime ℓ and a E_2^*-set $\mathcal{R}$, a set $\mathcal{E} = \mathcal{E}_\ell^{\mathcal{R}}$ is called a $(\ell, 2)^*$-ellipsephic set if

$$\mathcal{E} = \left\{ n = \sum_i a_i \ell^i : a_i \in \mathcal{R}_\ell \text{ for all } i \right\}.$$

In this setting, Biggs [**4**, Theorem 1.2] proved that

$$J_{5,\phi}(\mathcal{E}(N)) \ll N^\epsilon E^5,$$

where $E = |\mathcal{E}(N)|$. Her bound is essentially optimal as we get $J_{5,\phi}(\mathcal{E}(N)) \gg E^5$ from the diagonal solutions. She also obtained similar bounds for general E_t^*-sets with $t \ge 2$.

The optimal result of Biggs and Theorem 1.3 are applied to sets of different nature. To illustrate their difference, we first notice that the set $\mathcal{E}$ ($\mathcal{E}_\ell$ and E, respectively) in [**4**] plays the role of $\mathcal{A}$ ($\mathcal{A}_\ell$ and A, respectively) in our setting. In [**11**], Landau proved that the set of squares satisfies the condition (1.1). Write r for both $|\mathcal{E}_\ell|$ and $|\mathcal{A}_\ell|$. Since the set of squares is sparse, the set $\mathcal{E}_\ell + \mathcal{E}_\ell$ could be of size r^2 for sufficiently large ℓ. On the other hand, if an ellipsephic set $\mathcal{A}$ satisfies $|\mathcal{A}_\ell + \mathcal{A}_\ell| < Kr$, then Theorem 1.3 provides meaningful improvement only if $K^4 < r$. This condition $K < r^{1/4}$ is not always satisfied for large r if we take an ellipsephic set $\mathcal{E}$ with square digits since $\mathcal{E}_\ell + \mathcal{E}_\ell$ could be of size r^2. Thus one can say the result of Biggs provides useful estimates for "large K," while Theorem 1.3 is meaningful for "small K."

We will prove Theorem 1.2 in Section 2 and Theorem 1.3 in Section 3. The key idea of our paper is to make use of the fact that ellipsephic sets with small digit sumsets have fewer solutions of linear equations. More precisely, we can bound elements of the form $2\mathcal{A} - 2\mathcal{A}$ for ellipsephic sets $\mathcal{A}$ with small digit sumsets more efficiently than general sets (see Lemma 2.1). We will highlight this idea with variations of Theorem 1.3 at the end of the paper.

2. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following lemma.

LEMMA 2.1. *Let $\mathcal{A}$ be an ellipsephic set with $\mathcal{A}_\ell = \mathcal{A} \cap [0, \ell-1]$. Write $\mathcal{A}(N) = \mathcal{A} \cap [-N, N]$ with $A = |\mathcal{A}(N)|$. Suppose that $|\mathcal{A}_\ell + \mathcal{A}_\ell| \leq K|\mathcal{A}_\ell|$. Then*

$$|2\mathcal{A} - 2\mathcal{A}| \ll K^4 N^{4 \log_\ell K} A.$$

Proof: Since $\mathcal{A}_\ell \subset [0, \ell-1]$, by viewing $\mathcal{A}_\ell$ as a subset of the abelian group $\mathbb{Z}_{4\ell-3} = \{2-2\ell, \cdots, -1, 0, 1, \cdots, 2\ell-2\}$, we have $\mathcal{A}_\ell + \mathcal{A}_\ell \subset \mathbb{Z}_{4\ell-3}$. By the Plünnecke-Ruzsa inequality [**12**], [**13**], we have $|2\mathcal{A}_\ell - 2\mathcal{A}_\ell| \leq K^4|\mathcal{A}_\ell|$. Since $\mathcal{A} \subset [-N, N]$, each element of $\mathcal{A}$ is formed of $\leq \log_\ell N + 1$ digits, each of which is in $\mathcal{A}_\ell$. Hence, each element of $2\mathcal{A} - 2\mathcal{A}$ corresponds to at least one element of $(2\mathcal{A}_\ell - 2\mathcal{A}_\ell)^{\log_\ell N+1}$. Thus we have

$$\begin{aligned} |2\mathcal{A} - 2\mathcal{A}| &\leq |2\mathcal{A}_\ell - 2\mathcal{A}_\ell|^{\log_\ell N+1} \\ &\leq K^{4(\log_\ell N+1)} |\mathcal{A}_\ell|^{\log_\ell N+1} \\ &= K^4 N^{4 \log_\ell K} A. \end{aligned}$$

□

Remark Let $\mathcal{A}$ be an ellipsephic set satisfying the conditions in Lemma 2.1. For $m,n\in\mathbb{N}$, using the same argument as the above proof, we can show that

$$|m\mathcal{A}-n\mathcal{A}|\ll K^{m+n}N^{(m+n)\log_\ell K}A.$$

Let

$$E\mathbb{1}_{\mathcal{A}(N)}(\alpha,\beta):=\sum_{n\in\mathcal{A}(N)}e\left(\alpha n^3+\beta n\right).$$

By the orthogonal relation of the exponential function, we see that

$$J_5(\mathcal{A}(N))=\int_{\mathbb{T}^2}\left|E\mathbb{1}_{\mathcal{A}(N)}(\alpha,\beta)\right|^{10}d\alpha d\beta,$$

where $\mathbb{T}=[0,1)$. Hence to prove Theorem 1, it is equivalent to show that there exists a positive constant κ such that

$$\int_{\mathbb{T}^2}\left|E\mathbb{1}_{\mathcal{A}(N)}(\alpha,\beta)\right|^{10}d\alpha d\beta\ll K^4N^{4\log_\ell K}\exp\left(\kappa\frac{\log N}{\log\log N}\right)A^6.$$

Proof of Theorem 1.2 Write $a=\mathbb{1}_{\mathcal{A}(N)}$. The tenth moment $\|Ea\|_{10}^{10}$ counts the number of solutions to the system of equations

$$\sum_{i=1}^{2}\left(x_i^3-y_i^3\right)=\sum_{i=3}^{5}\left(x_i^3-y_i^3\right)$$
$$\sum_{i=1}^{2}(x_i-y_i)=\sum_{i=3}^{5}(x_i-y_i),$$

with $\mathbf{x},\mathbf{y}\in\mathcal{A}(N)^5$. By the second equation above, we let

$$h:=x_1-y_1+x_2-y_2=x_3-y_3+x_4-y_4+x_5-y_5\in 2\mathcal{A}-2\mathcal{A}.$$

We now write $\|Ea\|_{10}^{10}$ as

$$\sum_{h\in2\mathcal{A}-2\mathcal{A}}\int_{\mathbb{T}}\int_{\mathbb{T}}|Ea(\alpha_1,\alpha_2)|^4e(-\alpha_2h)d\alpha_2\int_{\mathbb{T}}|Ea(\alpha_1,\alpha_3)|^6e(-\alpha_3h)d\alpha_3d\alpha_1.$$

By the triangle inequality and Lemma 2.1, this gives

$$\|Ea\|_{10}^{10}\le K^4N^{4\log_\ell K}A\int_{\mathbb{T}^3}|Ea(\alpha_1,\alpha_2)|^4|Ea(\alpha_1,\alpha_3)|^6d\alpha_1d\alpha_2d\alpha_3. \tag{2.1}$$

Let $c_t(k)$ denote the number of solutions of the simultaneous equations

$$\sum_{i=1}^{t}(x_i^3-y_i^3)=k\qquad\text{and}\qquad\sum_{i=1}^{t}(x_i-y_i)=0.$$

In the proof of [**9**, Theorem 2.1], Hughes and Wooley proved that

$$\int_{\mathbb{T}^3} |Ea(\alpha_1,\alpha_2)|^4 \, |Ea(\alpha_1,\alpha_3)|^6 \, d\alpha_1 d\alpha_2 d\alpha_3 \ll \sum_{|k|\le 4N^3} c_2(k)c_3(k)$$
$$\ll \exp\left(\kappa \frac{\log N}{\log\log N}\right) A^5.$$

It follows that

$$\|Ea\|_{10}^{10} \ll K^4 N^{4\log_\ell K} A \exp\left(\kappa \frac{\log N}{\log\log N}\right) A^5$$
$$\ll K^4 N^{4\log_\ell K} \exp\left(\kappa \frac{\log N}{\log\log N}\right) A^6.$$

□

Remark We see in Section 1 a remark after Theorem 1.2 that if $K^4 < r = |\mathcal{A}_\ell|$, then the bound of Theorem 1.2 is sharper than the bound derived from the efficient congruencing method. One can find examples to satisfy $K^m < r$ for all $m \in \mathbb{N}$, provided that r is sufficiently large. For example, for a large ℓ, if $\mathcal{A} = \{0, 1, \cdots, q\}$ with $2q < \ell$, then $\mathcal{A} + \mathcal{A} \in \{0, 1, \cdots, 2q\}$. We have

$$K = (2q+1)/(q+1) = 2 - 1/(q+1) < 2.$$

Hence by taking $q = 2^m - 1$, we get $K^m < r = q + 1$.

3. Proof of Theorem 1.3

Let $\phi(x)$ be a polynomial with integer coefficients and $\deg(\phi) \ge 3$. Let

$$F\mathbb{1}_{\mathcal{A}(N)}(\alpha,\beta) := \sum_{n\in\mathcal{A}(N)} e\left(\alpha\phi(n) + \beta n\right).$$

By the orthogonal relation of the exponential function, we see that

$$J_{5,\phi}(\mathcal{A}(N)) = \int_{\mathbb{T}^2} \left|F\mathbb{1}_{\mathcal{A}(N)}(\alpha,\beta)\right|^{10} d\alpha d\beta.$$

Hence to prove Theorem 1.3, it is equivalent to show that for any $\epsilon > 0$, we have

$$\int_{\mathbb{T}^2} \left|F\mathbb{1}_{\mathcal{A}(N)}(\alpha,\beta)\right|^{10} d\alpha d\beta \ll K^4 N^{4\log_\ell K+\epsilon} A^6.$$

Proof of Theorem 1.3 Write $a = \mathbb{1}_{\mathcal{A}(N)}$. The tenth moment $\|Fa\|_{10}^{10}$ counts the number of solutions to the system of equations

$$\sum_{i=1}^{2}(\phi(x_i)-\phi(y_i)) = \sum_{i=3}^{5}(\phi(x_i)-\phi(y_i))$$
$$\sum_{i=1}^{2}(x_i-y_i) = \sum_{i=3}^{5}(x_i-y_i),$$

with $\mathbf{x}, \mathbf{y} \in \mathcal{A}(N)^5$. Using the same argument as the one in Theorem 1.2, we get

$$\|Fa\|_{10}^{10} \le K^4 N^{4\log_\ell K} A \int_{\mathbb{T}^3} |Fa(\alpha_1,\alpha_2)|^4 \, |Fa(\alpha_1,\alpha_3)|^6 \, d\alpha_1 d\alpha_2 d\alpha_3. \tag{3.1}$$

Let $c_t(k)$ counts the number of solutions of the simultaneous equations

$$\sum_{i=1}^{t}(\phi(x_i)-\phi(y_i)) = k \qquad \text{and} \qquad \sum_{i=1}^{t}(x_i-y_i) = 0,$$

with $\mathbf{x}, \mathbf{y} \in \mathcal{A}^t$. In the proof of [**9**, Theorem 3.4], Hughes and Wooley proved that there exists a constant C, depending on ϕ, such that

$$\begin{aligned}\int_{\mathbb{T}^3} |Fa(\alpha_1,\alpha_2)|^4 \, |Fa(\alpha_1,\alpha_3)|^6 \, d\alpha_1 d\alpha_2 d\alpha_3 &\le \sum_{|k|\le CN^k} c_2(k)c_3(k) \\ &\ll N^\epsilon A^5.\end{aligned}$$

It follows that

$$\|Fa\|_{10}^{10} \ll K^4 N^{4\log_\ell K} A N^\epsilon A^5 \ll K^4 N^{4\log_\ell K+\epsilon} A^6.$$

$\square$

The improved bounds in Theorem 1.2 and Theorem 1.3 come from the fact that we can bound elements of the form $2\mathcal{A} - 2\mathcal{A}$ for ellipsephic sets $\mathcal{A}$ with small digit sumsets more efficiently than general sets. To highlight the idea, we consider a variation of Theorem 1.3.

Given a set $\mathcal{S} \subset \mathbb{Z}$, let $\mathcal{S}(N) = \mathcal{S} \cap [-N, N]$ and $S = |\mathcal{S}(N)|$. Suppose that $|2\mathcal{S}(N) - 2\mathcal{S}(N)| \le P(S)$ for some function P of S. Let $\phi(x)$ be a polynomial with integer coefficients and $\deg(\phi) \ge 3$. Following the proof of Theorem 1.3, we have

Theorem 3.1. *Let $\mathcal{S}(N) \subset \mathbb{Z} \cap [-N, N]$ with $|\mathcal{S}(N)| = S$. Suppose that $|2\mathcal{S}(N) - 2\mathcal{S}(N)| \leq P(S)$ for some function P of S. Then for any $\epsilon > 0$, we have*

$$J_{5,\phi}(\mathcal{S}(N)) \ll P(S) N^{\epsilon} S^5.$$

For a set $\mathcal{S}(N) \subset \mathbb{Z} \cap [-N, N]$, suppose that $|\mathcal{S}(N) + \mathcal{S}(N)| \leq KS$ for some constant K (by Freiman's theorem [**6**] [**7**], sets satisfying this condition are contained in a generalized arithmetic progression). By the Plünnecke-Ruzsa inequality [**12**], [**13**], we have $|2\mathcal{S}(N) - 2\mathcal{S}(N)| \leq K^4 S$. Hence as a direct consequence of Theorem 3.1, we have

Corollary 3.2. *Let $\mathcal{S}(N) \subset \mathbb{Z} \cap [-N, N]$ with $|\mathcal{S}(N)| = S$. Suppose that $|\mathcal{S}(N) + \mathcal{S}(N)| \leq KS$ for some constant K. Then for any $\epsilon > 0$, we have*

$$J_{5,\phi}(\mathcal{S}(N)) \ll K^4 N^{\epsilon} S^6.$$

References

[1] K. Aloui, Sur les entiers ellipséphiques: somme des chiffres et répartition dans les classes de congruences. *Period. Math. Hungar.* **70** (2015), no. 2, 171–208.

[2] K. Aloui, C. Mauduit, and M. Mkaouar, Somme des chiffres et répartition dans les classes de congruences. pour les palindromes ellipséphiques. *Acta Math. Hungar.* **151** (2017), no. 2, 409–445.

[3] K. D. Biggs, Efficient congruencing in ellipsephic sets: the quadratic case. *Acta Arith.* **200** (2021), no. 4, 331–348.

[4] K. D. Biggs, Efficient congruencing in ellipsephic sets: the general case, 2019. arXiv: 1912.04351.

[5] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part II: The KDV-equation. *Geometric and Functional Analysis* **3**, no. 3 (1993): 209–62.

[6] G. A. Fridman, Addition of finite sets, *Soviet Mathematics. Doklady* **5**: 1366–1370.

[7] G. A. Freiman, Foundations of a Structural Theory of Set Addition (in Russian), *Kazan: Kazan Gos. Ped. Inst.*: p. 140.

[8] Y. Hu, and X. Li. Discrete Fourier restriction associated with KdV equations. *Analysis and PDE* **6**, no. 4 (2013): 859–92.

[9] K. Hughes and T. D. Wooley, Discrete restriction for (x, x^3) and related topics, *Int. Math. Res. Not. IMRN*, (2021): 20 pages

[10] X. Lai and Y. Ding. A note on the discrete Fourier restriction problem. *Proceedings of the American Mathematical Society* **146**, no. 9 (2018): 3839–46.

[11] E. Landau. Über die Anzahl der Gitterpunkte in geweissen Bereichen. *Nachr. Ges. Wiss. Goettingen, Math.-Phys. Kl.* **1912** (1912), 687-770.

[12] H. Plünnecke, Eine zahlentheoretische Anwendung der Graphtheorie, *Journal für die reine und angewandte Mathematik.* **243** (1970): 171–183.

[13] I. Ruzsa, An application of graph theory to additive number theory, *Scientia, Series A* **3** (1989): 97–109.

[14] T. D. Wooley, Nested efficient congruencing and relatives of Vinogradov's mean value theorem, Proc. London Math. Soc. 118 (2019), no. 4, 942–1016.

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213.

E-mail address: `tanders2@andrew.cmu.edu`

Department of Mathematics, Purdue University, 150 N. University St., West Lafayette, IN, 47907, U.S.A

E-mail address: `hu776@purdue.edu`

Department of Pure Mathematics, University of Waterloo, 200 University Ave. West, Waterloo, ON, N2L 3G1, CANADA

E-mail address: `yrliu@uwaterloo.ca`

Department of Pure Mathematics, University of Waterloo, 200 University Ave. West, Waterloo, ON, N2L 3G1, CANADA

E-mail address: `alan.talmage@uwaterloo.ca`