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The asymptotic estimates and Hasse principle for multidimensional Waring's problem ☆

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Motivated by the asymptotic estimates and Hasse principle for multidimensional Waring's problem via the circle method, we prove for the first time that the corresponding singular series is bounded below by an absolute positive constant without any nonsingular local solubility assumption. The number of variables we need is near-optimal. By proving a more general uniform density result over certain complete discrete valuation rings with finite residue fields, we also establish uniform lower bounds for both singular series and singular integral in $\mathbb{F}_q[t]$. We thus obtain asymptotic formulas and the Hasse principle for multidimensional Waring's problem in $\mathbb{F}_q[t]$ via a variant of the circle method.

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1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots, \}$. When $s, k \in \mathbb{N} \setminus \{0\}$ with $k \geq 2$, let $R_{s,k}(n)$ denote the number of representations of the positive integer n as the sum of at most s k^{th} powers of positive integers. In 1920s, Hardy and Littlewood (see [16–18]) obtained an asymptotic formula for $R_{s,k}(n)$. More precisely, they proved that when $s \geq (k-2)2^{k-1} + 5$, then

$$R_{s,k}(n) = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_{s,k}(n) n^{s/k-1} + o(n^{s/k-1}), \tag{1.1}$$

where $\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy$, and $\mathfrak{S}_{s,k}(n)$ is the singular series which satisfies $1 \ll \mathfrak{S}_{s,k}(n) \ll 1$. Hardy and Littlewood introduced the notation $G(k)$ for the least integer s such that $R_{s,k}(n) > 0$ for all sufficiently large n. Thus their result implies that

$$G(k) \leq (k-2)2^{k-1} + 5.$$

Various improvements on the upper bounds for $G(k)$ were achieved by Davenport [6,7], Vinogradov [36], Vaughan [33,34], etc. For large k, Wooley [37, Corollary 1.2.1] obtained the best known upper bound

$$G(k) \leq k\big(\log k + \log\log k + O(1)\big).$$

It is important to the success of the asymptotic relation in (1.1) that the singular series satisfies $\mathfrak{S}_{s,k}(n) \gg 1$. Also, when $s \geq \max\{k+1, 4\}$, one can decompose $\mathfrak{S}_{s,k}(n)$ as a product of the local densities at all finite primes as follows

$$\mathfrak{S}_{s,k}(n) = \prod_p \Big(\lim_{h \to \infty} \lambda_{s,k}(h; p; n) \Big)$$

with

$$\lambda_{s,k}(h;p;n) = p^{h(1-s)} \cdot \mathrm{card}\{\mathbf{x}\,(\mathrm{mod}\,p^h) \,|\, x_1^k + \cdots + x_s^k \equiv n\,(\mathrm{mod}\,p^h)\},$$

where $\mathbf{x} = (x_1, \ldots, x_s)$. Let $\Gamma_1(k)$ be the least integer s with the property that, for every prime p, there is a positive number $C_{s,k}(p)$ such that whenever h is sufficiently large in terms of s, k and p, one has

$$\lambda_{s,k,d}(h;p;n) \geq C_{s,k}(p) \quad \text{for all } n.$$

Hardy and Littlewood (see [19,20]) showed that $\mathfrak{S}_{s,k}(n) \gg 1$ whenever $s \geq \max\{\Gamma_1(k),\, k+1,\, 4\}$. For $k > 2$, they showed that $\Gamma_1(k) = 4k$ when k is a power of 2 and that $\Gamma_1(k) \leq 2k$ otherwise.

One can consider a more refined question for $R_{s,k}(n)$. Let $\widetilde{G}(k)$ denote the least integer s for which the asymptotic formula (1.1) holds for n sufficiently large. The aforementioned work of Hardy and Littlewood naturally implies that $\widetilde{G}(k) \leq (k-2)2^{k-1} + 5$. Various improvements for $\widetilde{G}(k)$ were obtained by Vinogradov [35], Hua [21], Boklan [4], Vaughan [31,32], Wooley [38], etc. In 1995, Ford proved in [13] that

$$\widetilde{G}(k) \leq (1 + o(1))k^2 \log k.$$

Due to the recent progress on the efficient congruencing method introduced by Wooley [14,39–44] and the decoupling method developed by Bourgain, Demeter and Guth [5], the main conjecture of Vinogradov's mean value theorem has been proved. As a consequence of their results, one can largely sharpen the upper bounds for $\widetilde{G}(k)$. In particular, Wooley proved in [44, Corollary 14.7] that

$$\widetilde{G}(k) \leq k^2 - k + 2\lfloor\sqrt{2k+2}\rfloor - 1.$$

Let $d \in \mathbb{N}$ with $d \geq 2$. It is natural to ask the d-dimensional Waring's problem. In order to state our question more precisely, we now introduce some notation. Write

$$\mathcal{M} = \left\{(i_1, \ldots, i_d) \in \mathbb{N}^d \,\middle|\, i_1 + \cdots + i_d = k \right\} \quad \text{and} \quad \varrho = \mathrm{card}(\mathcal{M}) = \binom{k+d-1}{d-1}. \tag{1.2}$$

For positive integers P and $n_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{M}$), denote by $R_{s,k,d}(\mathbf{n};P)$ the number of solutions of the system of equations

$$x_{11}^{i_1} \cdots x_{1d}^{i_d} + \cdots + x_{s1}^{i_1} \cdots x_{sd}^{i_d} = n_{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M}) \tag{1.3}$$

with $x_{ij} \in \{1, 2, \ldots, P\}$. For simplicity, a monomial of the shape $x_1^{i_1} \cdots x_d^{i_d}$ will be abbreviated by $\mathbf{x}^{\mathbf{i}}$. For a semiring S and $n \in \mathbb{N} \setminus \{0\}$, write

$$S(\mathcal{M};n) = \left\{ \left(\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_n^{\mathbf{i}}\right)_{\mathbf{i} \in \mathcal{M}} \,\middle|\, \mathbf{x}_j \in S^d (1 \leq j \leq n) \right\} \tag{1.4}$$

and

$$S(\mathcal{M}) = \bigcup_{n=1}^{\infty} S(\mathcal{M}; n). \tag{1.5}$$

For every prime p, let $\mathbb{Z}_p$ denote the ring of p-adic integers. Let

$$\mathcal{D} = \bigcap_{p} \big(\mathbb{N}^{\varrho} \cap \mathbb{Z}_p(\mathcal{M})\big). \tag{1.6}$$

We note that for $\mathbf{n} \in \mathbb{N}^{\varrho}$, if $\mathbf{n} \notin \mathcal{D}$, i.e., $\mathbf{n} \notin \mathbb{Z}_p(\mathcal{M})$ for some prime p, then $R_{s,k,d}(\mathbf{n}; P) = 0$ for any positive integer P. Hence it suffices to consider only $\mathbf{n} \in \mathcal{D}$. From a heuristic argument of multidimensional variant of the circle method, in the system (1.3), whenever the number of variables sd is greater than the sum of degrees $k\varrho$, a conjectural asymptotic estimate of $R_{s,k,d}(\mathbf{n}; P)$ takes the following shape

$$R_{s,k,d}(\mathbf{n}; P) = \mathfrak{J}_{s,k,d}(\mathbf{n}; P)\mathfrak{S}_{s,k.d}(\mathbf{n})P^{sd-k\varrho} + o(P^{sd-k\varrho}), \tag{1.7}$$

where $\mathfrak{J}_{s,k,d}(\mathbf{n}; P)$ and $\mathfrak{S}_{s,k,d}(\mathbf{n})$ are the related (normalized) singular integral and singular series. To ensure that the first term in (1.7) dominates, one needs $\mathfrak{J}_{s,k,d}(\mathbf{n}; P) \gg 1$ and $\mathfrak{S}_{s,k,d}(\mathbf{n}) \gg 1$, where the implicit constants are independent of $\mathbf{n}$ and P. For a prime p, $h \in \mathbb{N} \setminus \{0\}$, and $\mathbf{n} = (n_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}} \in \mathbb{Z}_p(\mathcal{M})$, define

$$\lambda_{s,k,d}(h; p; \mathbf{n}) = p^{h(\varrho - sd)} \cdot \mathrm{card}\big\{\mathbf{x} \pmod{p^h} \mid \mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} \equiv n_{\mathbf{i}} \pmod{p^h} \ \ (\mathbf{i} \in \mathcal{M})\big\}. \tag{1.8}$$

The fabric of the circle method also suggests that for $s \geq ck\varrho$ with c a positive constant depending at most on d, the singular series should satisfy

$$\mathfrak{S}_{s,k,d}(\mathbf{n}) = \prod_{p} \Big(\lim_{h \to \infty} \lambda_{s,k,d}(h; p; \mathbf{n}) \Big) \gg 1 \quad \text{for all } \mathbf{n} \in \mathcal{D}. \tag{1.9}$$

Assuming additionally that for every $\mathbf{n} \in \mathcal{D}$ and every prime p, the system (1.3) has a nonsingular p-adic solution, Parsell [26] conjectured that the singular series would satisfy (1.9) (for more details, see [26, Section 9]). In this paper, we will prove Parsell's conjecture. We will also establish the existence of the nonsingular local solutions required in this conjecture. For a semiring S, let $\gamma(S; \mathcal{M})$ be the least integer ℓ such that

$$S(\mathcal{M}) = S(\mathcal{M}; \ell) \tag{1.10}$$

if such ℓ exists.

Theorem 1.1. *Let $k, d \in \mathbb{N}$ with $k, d \geq 2$. Define $\mathcal{M}$ and ϱ as in (1.2). Fix a prime p. Then the following hold.*

(1) *Let $\gamma(\mathbb{Z}_p; \mathcal{M})$ be defined by (1.10). Then $\gamma(\mathbb{Z}_p; \mathcal{M}) \leq 4k\varrho$.*

(2) *Whenever $s \geq \gamma(\mathbb{Z}_p; \mathcal{M}) + \varrho$, for every $\mathbf{n} \in \mathbb{Z}_p(\mathcal{M})$, the system (1.3) has a non-singular p-adic solution.*

(3) *Whenever* $s \geq \gamma(\mathbb{Z}_p; \mathcal{M}) + \varrho$ *and* h *is sufficiently large in terms of* s, k, d *and* p, *there is a constant* $C_{s,k,d}(p) > 0$, *such that*

$$\lambda_{s,k,d}(h; p; \mathbf{n}) \geq C_{s,k,d}(p) \quad \textit{for all } \mathbf{n} \in \mathbb{Z}_p(\mathcal{M}).$$

We remark here that the upper bound in Theorem 1.1(1) is independent of p and the lower bound in Theorem 1.1(3) is independent of $\mathbf{n}$. Such properties are essential for us to obtain the following uniform lower bound for the singular series $\mathfrak{S}_{s,k,d}(\mathbf{n})$ in (1.9) free of any assumption on the nonsingular local solubility of the system (1.3). In other words, we obtain a result that is stronger than Parsell's conjecture. Our bound on s is also near-optimal.

Theorem 1.2. *Let* $k, d \in \mathbb{N}$ *with* $k, d \geq 2$. *Define* $\mathcal{M}$ *and* ϱ *as in* (1.2). *Suppose that* $s \geq 4k\varrho + \varrho$. *For* $\mathbf{n} \in \mathcal{D}$ *with* $\mathcal{D}$ *defined by* (1.6), *one has*

$$1 \ll \prod_p \Big(\lim_{h \to \infty} \lambda_{s,k,d}(h; p; \mathbf{n}) \Big) \ll 1,$$

where the implicit constants depend at most on s, k *and* d, *but independent of the choice for* $\mathbf{n}$.

As in the standard argument as in [26, Section 8], the lower bound for singular integral holds on assuming the local solubility at ∞ as follows.

Definition 1.1. Let $s \in \mathbb{N} \setminus \{0\}$. Fix real numbers $\mu_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{M}$) with the property that the system

$$\boldsymbol{\eta}_1^{\mathbf{i}} + \cdots + \boldsymbol{\eta}_s^{\mathbf{i}} = \mu_{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M})$$

has a non-singular solution with $0 < \eta_{ij} < 1$. Let $\mathbf{n} = (n_{\mathbf{i}}) \in \mathbb{N}^{\varrho}$ and $P \in \mathbb{N}$. If there exist $\delta = \delta(s, k, \boldsymbol{\mu}) > 0$ and an absolute constant $\alpha \geq 0$ such that

$$|n_{\mathbf{i}} P^{-k} - \mu_{\mathbf{i}}| < \delta P^{-\alpha} \quad (\mathbf{i} \in \mathcal{M}).$$

Then we say that the tuple $\mathbf{n}$ is rescaled to $\boldsymbol{\mu}$ by (P, δ, α).

For the special case $d = 2$, we have $\mathcal{M} = \{(i, k-i) \,|\, 0 \leq i \leq k\}$ and so $\varrho = k + 1$. Arkhipov and Karatsuba [1] first investigated this case. Suppose that $s \geq ck^3 \log k$, where c is an absolute constant and fix real numbers $\mu_0, \ldots, \mu_k$ with the property as in Definition 1.1. The result [1, Theorem 1] of Arkhipov and Karatsuba states that there exist positive numbers $P_0 = P_0(s, k, \boldsymbol{\mu})$ and $\delta = \delta(s, k, \boldsymbol{\mu})$ such that, whenever $P > P_0$ and $\mathbf{n}$ is rescaled to $\boldsymbol{\mu}$ by (P, δ, α) with $\alpha = 0.5$, the asymptotic relation (1.7) holds, where $\mathfrak{J}_{s,k,2}(\mathbf{n}; P) \gg 1$ and $\mathfrak{S}_{s,k,2}(\mathbf{n}) \geq 0$. Since they fail to show $\mathfrak{S}_{s,k,2}(\mathbf{n}) \gg 1$, one cannot obtain the correct magnitude of $R_{s,k,2}(\mathbf{n}; P)$ from their result. This obstacle is surmounted by Theorem 1.2. In particular, we see that when $d = 2$, the system (1.3) satisfies the Hasse principle. In addition, when dealing with other variants of the circle method for multidimensional Waring's problem, we can provide via Theorem 1.2 the corresponding uniform estimates for singular series. For example, Theorem 1.2 allows us to improve Parsell's result in [26, Theorem 4] by removing the nonsingular local solubility assumption for the singular series.

Theorem 1.3. *Suppose that* $s \geq \frac{14}{3}k^2 \log k + \frac{10}{3}k^2 \log\log k + O(k^2)$ *and fix real numbers* $\mu_0, \ldots, \mu_k$ *with the property as in Definition 1.1. There exist positive numbers* $P_0 = P_0(s, k, \boldsymbol{\mu})$ *and* $\delta = \delta(s, k, \boldsymbol{\mu})$ *such that, whenever* $P > P_0$ *and* $\mathbf{n} \in \mathcal{D}$ *is rescaled to* $\boldsymbol{\mu}$ *by* (P, δ, α) *with* $\alpha = 0$, *one has*

$$R_{s,k,2}(\mathbf{n}; P) \gg P^{2s-k(k+1)}.$$

In particular, when $d = 2$, *the system* (1.3) *satisfies the Hasse principle.*

It is the first time that the asymptotic estimates and Hasse principle for the 2-dimensional Waring's problem can be established without any nonsingular local solubility assumption. For the higher dimensional cases, one can obtain a major arc estimate by combining Theorem 1.2 with the standard arguments in [26]. The minor arc estimate can be delivered through the work in [27] or [28]. Thus one can establish the asymptotic estimates and the Hasse principle for $R_{s,k,d}(\mathbf{n}; P)$ as desired.

Let $\mathbb{A} = \mathbb{F}_q[t]$ be the ring of polynomials over the finite field $\mathbb{F}_q$ of q elements whose characteristic is denoted by $\mathrm{char}(\mathbb{F}_q)$. Let $\mathbb{A}_\infty = \mathbb{F}_q[\![1/t]\!]$ be the ring of formal power series in $1/t$ over $\mathbb{F}_q$. We now consider a multidimensional analogue of Waring's problem in $\mathbb{A}$. For fixed $k, d \in \mathbb{N}$ with $k, d \geq 2$, define $\mathcal{M}$ and ϱ as in (1.2). For $P \in \mathbb{N}$, write

$$I_P = \{x \in \mathbb{A} \mid \deg x < P\}.$$

For $\mathbf{m} = \big(m_{\mathbf{i}}\big)_{\mathbf{i} \in \mathcal{M}}$ with $m_{\mathbf{i}} \in \mathbb{A}$ $(\mathbf{i} \in \mathcal{M})$ and $P \in \mathbb{N}$, let $R_{q,s,k,d}(\mathbf{m}; P)$ denote the number of solutions of the system

$$\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} = m_{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M})$$

with $\mathbf{x}_j \in I_P^d$ $(1 \leq j \leq s)$. Let $\mathbb{A}_\infty(\mathcal{M})$ be defined as in (1.5). Note that for every $x \in I_P$, $t^{1-P}x \in \mathbb{A}_\infty$. Since each $\mathbf{i} = (i_1, \ldots, i_d) \in \mathcal{M}$ satisfies $i_1 + \cdots + i_d = k$, whenever $x_1, \ldots, x_d \in I_P$, we have $t^{k(1-P)}\mathbf{x}^{\mathbf{i}} = (t^{1-P}x_1)^{i_1} \cdots (t^{1-P}x_d)^{i_d} \in \mathbb{A}_\infty(\mathcal{M})$. Therefore if $R_{q,s,k,d}(\mathbf{m}; P) > 0$, then $t^{k(1-P)}\mathbf{m} \in \mathbb{A}_\infty(\mathcal{M})$. We will show in Section 7 (see Corollary 7.1) that whenever $\mathrm{char}(\mathbb{F}_q) \nmid k$, for every $\mathbf{m} \in \mathbb{A}^\varrho$, there exists N such that $t^{k(1-N)}\mathbf{m} \in \mathbb{A}_\infty(\mathcal{M})$. Thus, for $\mathbf{m} \in \mathbb{A}^\varrho$, define

$$T(\mathbf{m}) = \min\{N \in \mathbb{N} \mid t^{k(1-N)}\mathbf{m} \in \mathbb{A}_\infty(\mathcal{M})\}. \tag{1.11}$$

It will turn out that whenever $\mathrm{char}(\mathbb{F}_q) \nmid k$, one has

$$0 \le T(\mathbf{m}) - \min\{N \in \mathbb{N} \mid k(N-1) \ge \deg m_{\mathbf{i}}\ (\mathbf{i} \in \mathcal{M})\} \le C,$$

where $C = C(q,k,d) > 0$ is a constant, independent of $\mathbf{m}$. Note that if $P \ge T(\mathbf{m})$, then $t^{k(1-P)}\mathbf{m} \in \mathbb{A}_\infty(\mathcal{M})$, and vice versa. Thus it suffices to consider $R_{q,s,k,d}(\mathbf{m};P)$ with $P \ge T(\mathbf{m})$. For the infinite place ∞, $\mathbf{m} \in \mathbb{A}^\varrho$ and $P \ge T(\mathbf{m})$, define

$$\lambda_{q,s,k,d}(P;\infty;\mathbf{m}) = q^{P(\varrho - sd)} \cdot \mathrm{card}\big\{\mathbf{x} \in I_P^{sd} \big| \deg\big(\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} - m_{\mathbf{i}}\big) < (k-1)P\ (\mathbf{i} \in \mathcal{M})\big\}.$$

Let $\mathcal{P}$ denote the set of all monic irreducibles in $\mathbb{A}$. For every $w \in \mathcal{P}$, let $\mathbb{A}_w$ denote the completion of $\mathbb{A}$ at the place w and let $\mathbb{A}_w(\mathcal{M})$ be defined as in (1.5). For $h \in \mathbb{N} \setminus \{0\}$, $w \in \mathcal{P}$ and $\mathbf{m} \in \mathbb{A}_w(\mathcal{M})$, define

$$\lambda_{q,s,k,d}(h;w;\mathbf{m}) = \big(q^{\deg w}\big)^{h(\varrho - sd)} \cdot \mathrm{card}\big\{\mathbf{x}\ (\mathrm{mod}\, w^h) \big| \mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} \equiv m_{\mathbf{i}}\ (\mathrm{mod}\, w^h)\ (\mathbf{i} \in \mathcal{M})\big\}.$$

In this paper, we also aim to establish the following asymptotic formula for $R_{q,s,k,d}(\mathbf{m};P)$ via a variant of the circle method in function fields.

Theorem 1.4. *Let $k,d \in \mathbb{N}$ with $k,d \ge 2$. Define $\mathcal{M}$ and ϱ as in (1.2). Suppose that $p = \mathrm{char}(\mathbb{F}_q) \nmid k$ and that $s \ge 2\vartheta k + 2\vartheta + 1$ where $\vartheta = \binom{k+d}{d} - \binom{[k/p]+d}{d}$. Then for $\mathbf{m} \in \mathbb{A}^\varrho$ with $P \ge T(\mathbf{m})$ defined by (1.11), there exists a positive number $P_0 = P_0(q,s,k,d)$ such that whenever $\mathbf{m} \in \mathbb{A}_w(\mathcal{M})$ for every $w \in \mathcal{P}$ and $P \ge P_0$, one has*

$$R_{q,s,k,d}(\mathbf{m};P) = C_{q,s,k,d}(\mathbf{m};P)\big(q^P\big)^{sd - \varrho k} + O\big((q^P)^{sd - \varrho k - \delta}\big),$$

where

$$C_{q,s,k,d}(\mathbf{m};P) = \lambda_{q,s,k,d}(P;\infty;\mathbf{m}) \prod_{w \in \mathcal{P}} \Big(\lim_{h \to \infty} \lambda_{q,s,k,d}(h;w;\mathbf{m})\Big)$$

satisfying $1 \ll C_{q,s,k,d}(\mathbf{m};P) \ll 1$. Here the implicit constants and δ depend at most on q,s,k and d, but independent of $\mathbf{m}$ and P.

Due to our recent work on polynomial analogue of multidimensional Vinogradov's mean values (see [23, Theorem 1.1]), the minor arc contribution can be treated similarly as in [23, Section 6]. In this paper we will focus on the major arc contribution with an emphasis on the estimates for singular series and singular integral. The main difficulty in this work is to show the validity of the corresponding uniform local density hypothesis for $\lambda_{q,s,k,d}(P;\infty;\mathbf{m})$ and $\lambda_{q,s,k,d}(h;w;\mathbf{m})$. For the asymptotic estimates in the 1-dimensional Waring's problem in function fields, we refer the interested readers to [22], [24] and [45].

For every $\varpi \in \mathcal{P} \cup \{\infty\}$, let $\mathbb{A}_\varpi$ denote the completion of $\mathbb{A}$ at ϖ. Instead of only considering the uniform local density hypothesis over every $\mathbb{A}_\varpi$, we will indeed study the hypothesis over a more general algebraic setting. Let K be a complete field with respect to the norm $|\cdot|$ associated to a discrete non-archimedean valuation. Let $\mathcal{O} = \{x \in K|\ |x| \le 1\}$, π a primitive element, and $F = \mathcal{O}/(\pi)$.

Definition 1.2 *(Uniform Local Density Hypothesis over $\mathcal{O}$).* For $k, d \in \mathbb{N}$ with $k, d \ge 2$, let $\mathcal{M}$ and ϱ be defined by (1.2). For $(\mathfrak{f}_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathcal{O}(\mathcal{M})$ and $h \in \mathbb{N} \setminus \{0\}$, define

$$\begin{aligned}\lambda_{s,k,d}(h;\pi;\mathfrak{f}) = (\mathrm{card}(F))^{h(\varrho - sd)} \cdot \mathrm{card}\big\{\mathbf{x} \ (\mathrm{mod}\, \pi^h) \bigm| \mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} \\ \equiv \mathfrak{f}_{\mathbf{i}} \ (\mathrm{mod}\, \pi^h) \ \ (\mathbf{i} \in \mathcal{M})\big\}.\end{aligned}$$

If there exists a nonnegative integer $u^* = u^*(s, k, d, \pi)$ such that whenever $h \ge u^*$, one has

$$\lambda_{s,k,d}(h;\pi;\mathfrak{f}) \ge (\mathrm{card}(F))^{u^*(\varrho - sd)} \quad \text{for all } \mathfrak{f} \in \mathcal{O}(\mathcal{M}),$$

then we say that the system of polynomials $\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}}$ $(\mathbf{i} \in \mathcal{M})$ satisfies the uniform local density hypothesis over $\mathcal{O}$.

For the case of $\mathcal{O} = \mathbb{Z}_p$, this definition is consistent with Theorem 1.1(3) on taking $C_{s,k,d}(p) = p^{u^*(\varrho - sd)}$ for some nonnegative integer $u^* = u^*(s, k, d, p)$. In what follows, for a semiring S and $k, n \in \mathbb{N} \setminus \{0\}$, write

$$S(k;n) = \big\{x_1^k + \cdots + x_n^k \bigm| x_j \in S (1 \le j \le n)\big\} \tag{1.12}$$

and

$$S(k) = \bigcup_{n=1}^{\infty} S(k;n). \tag{1.13}$$

Theorem 1.5. *Let $k, d \in \mathbb{N}$ with $k, d \ge 2$. Define $\mathcal{M}$ and ϱ as in (1.2). Let $\mathcal{O}$ be a complete discrete valuation ring with the finite residue field F and define $F(k)$ by (1.13). Let $\gamma(\mathcal{O};\mathcal{M})$ be defined by (1.10). Suppose that* $\mathrm{char}(F) \nmid k$.

(1) *One has*

$$\gamma(\mathcal{O};\mathcal{M}) \le (k+1)\varrho.$$

Further, when $F \ne F(k)$, one has $k \ge 3$ and a better upper bound:

$$\gamma(\mathcal{O};\mathcal{M}) \le \begin{cases} \sqrt{(72/13)k}(\varrho - d/2), & \text{if } k \ge 7, \\ 2\sqrt{k+1}(\varrho - d/2), & \text{if } 3 \le k \le 6. \end{cases}$$

(2) *If* $s \geq \gamma(\mathcal{O};\mathcal{M}) + \varrho$, *then the system of polynomials* $\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}}$ $(\mathbf{i} \in \mathcal{M})$ *satisfies the uniform local density hypothesis over* $\mathcal{O}$.

It is natural to consider *local obstructions*; i.e., the elements in $\mathcal{O}^\varrho \setminus \mathcal{O}(\mathcal{M})$. We also give the following sufficient condition to ensure that no local obstruction exists.

Theorem 1.6. *Let* $k, d \in \mathbb{N}$ *with* $k, d \geq 2$. *Define* $\mathcal{M}$ *and* ϱ *as in* (1.2). *Let* $\mathcal{O}$ *be a complete discrete valuation ring with the finite residue field* F *and define* $F(k)$ *by* (1.13). *Suppose that* $\mathrm{char}(F) \nmid k$ *and* $\mathrm{card}(F) > (k-1)^2$. *Then one has* $F = F(k)$ *and* $\mathcal{O}(\mathcal{M}) = \mathcal{O}^\varrho$.

We now go back to the polynomial ring $\mathbb{A}$. On taking $\mathcal{O}$ to be $\mathbb{A}_\varpi$, the completion of $\mathbb{A}$ at the place $\varpi \in \mathcal{P} \cup \{\infty\}$, the following corollary is a direct consequence of Theorem 1.5.

Corollary 1.1. *Let* $k, d \in \mathbb{N}$ *with* $k, d \geq 2$. *Define* $\mathcal{M}$ *and* ϱ *as in* (1.2). *Suppose that* $\mathrm{char}(\mathbb{F}_q) \nmid k$. *Then the following hold.*

(1) *For each* $\varpi \in \mathcal{P} \cup \{\infty\}$, *one has* $\gamma(\mathbb{A}_\varpi;\mathcal{M}) \leq (k+1)\varrho$.

(2) *For each* $\varpi \in \mathcal{P} \cup \{\infty\}$, *if* $s \geq \gamma(\mathbb{A}_\varpi;\mathcal{M}) + \varrho$, *then the system of polynomials* $\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}}$ $(\mathbf{i} \in \mathcal{M})$ *satisfies the uniform local density hypothesis over* $\mathbb{A}_\varpi$.

Since the residue field of $\mathbb{A}_w$ with $w \in \mathcal{P}$ is $\mathbb{A}/(w)$ of cardinality $q^{\deg w}$, it follows from Theorem 1.6 that when $\mathrm{char}(\mathbb{F}_q) \nmid k$ and $q^{\deg w} > (k-1)^2$, we have

$$\mathbb{A}^\varrho \subseteq \mathbb{A}_w^\varrho = \mathbb{A}_w(\mathcal{M}).$$

Let $\mathcal{P}_0 = \{w \in \mathcal{P} \mid q^{\deg w} \leq (k-1)^2\}$. Hence whenever $\mathbf{m} \in \mathbb{A}^\varrho \cap \mathbb{A}_w(\mathcal{M})$ for every $w \in \mathcal{P}_0$, the asymptotic formula in Theorem 1.4 implies $R_{q,s,k,d}(\mathbf{m};P) \to \infty$ as $P \to \infty$. We thus establish the Hasse principle for multidimensional Waring's system in function fields.

Theorem 1.7. *Let* $k, d \in \mathbb{N}$ *with* $k, d \geq 2$. *Define* $\mathcal{M}$ *and* ϱ *as in* (1.2). *Suppose that* $p = \mathrm{char}(\mathbb{F}_q) \nmid k$ *and that* $s \geq 2\vartheta k + 2\vartheta + 1$ *where* $\vartheta = \binom{k+d}{d} - \binom{[k/p]+d}{d}$. *Let* $\mathbf{m} \in \mathbb{A}^\varrho \cap \mathbb{A}_w(\mathcal{M})$ *for every* $w \in \mathcal{P}_0$. *Then the system of equations* $\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} = m_{\mathbf{i}}$ $(\mathbf{i} \in \mathcal{M})$ *has infinitely many solutions in* $\mathbb{A}$.

By applying Corollary 1.1 within another variant of the circle method introduced in [24], we will improve the lower bound for s in Theorem 1.7 in our future work. Let $\mathbb{A}(\mathcal{M})$ be defined by (1.5). The above theorem implies that

$$\mathbb{A}(\mathcal{M}) = \bigcap_{w \in \mathcal{P}_0} \Big(\mathbb{A}^\varrho \cap \mathbb{A}_w(\mathcal{M})\Big).$$

In order to consider the solutions counted by $R_{q,s,k,d}(\mathbf{m};P)$ with the box I_P^{sd} as small as possible, a multidimensional analogue of restricted Waring's problem in function fields concerns $R_{q,s,k,d}(\mathbf{m}) = R_{q,s,k,d}(\mathbf{m}; T(\mathbf{m}))$ for $\mathbf{m} \in \mathbb{A}(\mathcal{M})$. Thus we can derive an asymptotic formula for $R_{q,s,k,d}(\mathbf{m})$ from Theorem 1.4.

Theorem 1.8 *(Restricted multidimensional Waring's problem). Let $k, d \in \mathbb{N}$ with $k, d \geq 2$. Define $\mathcal{M}$ and ϱ as in (1.2). Suppose that $p = \mathrm{char}(\mathbb{F}_q) \nmid k$ and that $s \geq 2\vartheta k + 2\vartheta + 1$ where $\vartheta = \binom{k+d}{d} - \binom{[k/p]+d}{d}$. Then for $\mathbf{m} \in \mathbb{A}(\mathcal{M})$ with $T = T(\mathbf{m})$ defined by (1.11), there exists a positive number $P_0 = P_0(q, s, k, d)$ such that whenever $T \geq P_0$, one has*

$$R_{q,s,k,d}(\mathbf{m}) = C_{q,s,k,d}(\mathbf{m})\big(q^T\big)^{sd-\varrho k} + O\big((q^T)^{sd-\varrho k-\delta}\big),$$

where $1 \ll C_{q,s,k,d}(\mathbf{m}) \ll 1$. Here the implicit constants and δ depend at most on q, s, k and d, but independent of $\mathbf{m}$.

For the 1-dimensional restricted Waring's problem, Liu and Wooley [24] chose the least box I_P^s by using the notion of exceptional polynomials. We will show in Section 7 that the 1-dimensional analogue of (1.11) delivers the same least box as the former (see Proposition 7.4).

We then end this section by applying Theorem 1.3 and Theorem 1.6 to obtain the Hasse principle for the two-dimensional analogue of the classical Waring's problem. This result can be also extended to higher dimensions.

Corollary 1.2. *Suppose that $s \geq \frac{14}{3}k^2 \log k + \frac{10}{3}k^2 \log\log k + O(k^2)$ and fix real numbers $\mu_0, \ldots, \mu_k$ with the property as in Definition 1.1. Let $\mathbf{n} = (n_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{N}^{\varrho} \cap \mathbb{Z}_p(\mathcal{M})$ for every prime p with $p|k$ or $p \leq (k-1)^2$. Then there exist positive numbers $P_0 = P_0(s, k, \boldsymbol{\mu})$ and $\delta = \delta(s, k, \boldsymbol{\mu})$ such that, whenever $P > P_0$ and $\mathbf{n}$ is rescaled to $\boldsymbol{\mu}$ by (P, δ, α) with $\alpha = 0$, the system of equations $\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} = n_{\mathbf{i}}$ $(\mathbf{i} \in \mathcal{M})$ has a solution in $\mathbb{N}$.*

In Section 2, we aim to prove Theorem 1.1 and Theorem 1.2. In Sections 3-6, we develop several results to establish Theorem 1.5, and we prove Theorem 1.6 in Section 7. In Sections 8-9, we apply Corollary 1.1 to investigate the uniform lower bounds for singular series and singular integral required in a multidimensional analogue of restricted Waring's problem in function fields. We then establish the asymptotic formula Theorem 1.4 in Section 10. Finally, in Section 11, we will return to Theorem 1.5 and discuss some special cases to improve our result.

To conclude this section, we describe briefly the main difficulties in establishing Theorem 1.1 and Theorem 5.1 as well as our new ideas to overcome them. For a ring $\mathcal{O}$, to solve a system of equations such as (1.3), the standard way is to start with a given $\mathbf{n} \in \mathcal{O}(\mathcal{M})$, and then find solutions for the corresponding equations. However, instead of a single $\mathbf{n}$, we consider the set $\mathcal{O}(\mathcal{M})$ of all possible $\mathbf{n}$ which are soluble. It turns out that $\mathcal{O}(\mathcal{M})$ is an $\mathcal{O}(k)$-module (see Lemma 3.1). Moreover, if $\mathcal{O}$ is a discrete valuation ring, we prove the unexpected property that $\mathcal{O}(k)$ is a local ring (see Theorem 5.1). Then by applying some module theory, we obtain a near-optimal upper bound of the number of generators of $\mathcal{O}(\mathcal{M})$ over $\mathcal{O}(k)$. This allows us to establish the asymptotic estimates and

Hasse principle for the 2-dimensional Waring's problem without any nonsingular local solubility assumption. The same results hold for d-dimensional Waring's problem. We also intend to use this technique to study similar problems for more general systems.

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2. Proofs of Theorem 1.1 and Theorem 1.2

In what follows, for each prime p, let $\mathbb{Z}_p$ denote the ring of p-adic integers and write $|\cdot|_p$ for the usual p-adic norm, normalized with $|p|_p = p^{-1}$. Let $\mathcal{M}$ and ϱ be defined as in (1.2) and let $\mathbb{Z}_p(\mathcal{M})$ be defined as in (1.5). For $s \in \mathbb{N} \setminus \{0\}$, write

$$\Phi_{s,\mathbf{i}}(\mathbf{x}) = \mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M})$$

and let $\boldsymbol{\Phi}_s = \left(\Phi_{s,\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{M}}$. For a prime p, $h \in \mathbb{N} \setminus \{0\}$ and $\mathbf{n} = (n_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{Z}_p(\mathcal{M})$, define $\lambda_{s,k,d}(h; p; \mathbf{n})$ as in (1.8).

Definition 2.1. Let $n, r \in \mathbb{N}$ with $1 \le n \le r$. Let $\boldsymbol{\varphi} = (\varphi_i)_{1\le i\le n}$ with each $\varphi_i \in \mathbb{Z}_p[x_1, \dots, x_r]$ $(1 \le i \le n)$. For $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Z}_p^r$, denote by $|\Delta(\boldsymbol{\varphi}; \mathbf{a})|$ the maximal value of the determinants of all $n \times n$ submatrices of the Jacobian matrix $\left(\partial\varphi_i/\partial x_j\right)_{n\times r}$ when taking $x_1 = a_1, \dots, x_r = a_r$. If $|\Delta(\boldsymbol{\varphi}; \mathbf{a})|_p \ne 0$, we write $|\Delta(\boldsymbol{\varphi}; \mathbf{a})|_p = p^{-v}$ for some $v = v(\boldsymbol{\varphi}; \mathbf{a}; \mathbb{Z}_p) \in \mathbb{N}$, and say that the pair $(\boldsymbol{\varphi}; \mathbf{a})$ has a *nonsingular p-adic weight of v*. For any $S \subseteq \mathbb{Z}_p$, we further define

$$|\Delta(\boldsymbol{\varphi}; S)|_p = \max\left\{|\Delta(\boldsymbol{\varphi}; \mathbf{a})|_p \,\middle|\, \mathbf{a} \in S^r\right\}.$$

Lemma 2.1. *Let $\mathbf{n} = (n_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{Z}_p(\mathcal{M})$. Suppose that the system*

$$\Phi_{s,\mathbf{i}}(\mathbf{x}) = n_{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M})$$

has a solution $\mathbf{a} \in \mathbb{Z}_p^{sd}$ with $|\Delta(\boldsymbol{\Phi}_s; \mathbf{a})|_p = p^{-v_0}$, where $v_0 = v_0(\boldsymbol{\Phi}_s; \mathbf{a}; \mathbb{Z}_p) \in \mathbb{N}$. Then whenever $h \ge 2v_0 + 1$, one has

$$\lambda_{s,k,d}(h; p; \mathbf{n}) \ge p^{(2v_0+1)(\varrho - sd)}.$$

Proof. It follows from the standard Hensel-type arguments (for example, see [26, Lemma 9.9]). □

We observe that the above lower bound for $\lambda_{s,k,d}(h; p; \mathbf{n})$ depends on the nonsingular weight v_0 of the pair $(\boldsymbol{\Phi}_s; \mathbf{a})$ and thus on $\mathbf{n}$. To obtain a uniform lower bound for all

$\lambda_{s,k,d}(h;p;\mathbf{n})$, when $\mathbf{n}$ runs over $\mathbb{Z}_p(\mathcal{M})$, we need to find a nonsingular weight of v_0 which is independent of $\mathbf{n}$.

Lemma 2.2. *For $k \in \mathbb{N} \setminus \{0\}$, let $\nu = \nu(k) = 4k$ and let $\mathbb{Z}_p(k;\nu)$ be defined by (1.12). Then the following hold.*

(1) $\mathbb{Z}_p = \mathbb{Z}_p(k;\nu)$.

(2) $\mathbb{Z}_p(\mathcal{M})$ *is a submodule of the* $\mathbb{Z}_p$*-module* $\mathbb{Z}_p^\varrho$.

Proof. (1) Since $\mathbb{Z}_p(k;\nu) = \{x_1^k + \cdots + x_\nu^k \,|\, x_1, \ldots, x_\nu \in \mathbb{Z}_p\}$, it suffices to show that $\mathbb{Z}_p \subseteq \mathbb{Z}_p(k;\nu)$. Let $a \in \mathbb{Z}_p$. Let p^τ be the highest power of p dividing k. Take

$$v = \begin{cases} \tau + 1, & p > 2, \\ \tau + 2, & p = 2. \end{cases}$$

By [8, Lemmas 5.5-5.6], whenever $h \geq v$, the congruence

$$x_1^k + x_2^k + \cdots + x_\nu^k \equiv a \pmod{p^h}$$

is soluble with $x_1, \ldots, x_\nu$ not all divisible by p. Take $h = 2v$. Then there exist $a_1, a_2, \ldots, a_\nu \in \{1, \ldots, p^h\}$ with $(a_1, p) = 1$ such that

$$a_1^k + a_2^k + \cdots + a_\nu^k \equiv a \pmod{p^h}.$$

Let $f(x) = x^k + a_2^k + \cdots + a_\nu^k - a$. Then $|f(a_1)|_p \leq p^{-h} = p^{-2v}$. Also, $|f'(a_1)|_p = |ka_1^{k-1}|_p = |k|_p = p^{-\tau}$. Since $2v > 2\tau$, we have $|f(a_1)|_p < |f'(a_1)|_p^2$. By Hensel's Lemma [15, Lemma 5.9], there exists $b \in \mathbb{Z}_p$ such that $f(b) = 0$, namely,

$$b^k + a_2^k + \cdots + a_\nu^k = a.$$

Therefore $a \in \{x_1^k + \cdots + x_\nu^k \,|\, x_1, \ldots, x_\nu \in \mathbb{Z}_p\}$.

(2) In view of the definition of $\mathbb{Z}_p(\mathcal{M})$, it is closed under addition of the $\mathbb{Z}_p$-module $\mathbb{Z}_p^\varrho$. Let $a_1, \ldots, a_d, c \in \mathbb{Z}_p$. By Part (1), c can be decomposed as a sum of ν k-th powers x^k with $x \in \mathbb{Z}_p$. Since $i_1 + \cdots + i_d = k$ for every $\mathbf{i} = (i_1, \ldots, i_d) \in \mathcal{M}$, we have

$$x^k(a_1^{i_1} \cdots a_d^{i_d}) = (xa_1)^{i_1} \cdots (xa_d)^{i_d} \quad (\mathbf{i} \in \mathcal{M}).$$

It follows that $\mathbb{Z}_p(\mathcal{M})$ is closed under scalar multiplication by the elements in $\mathbb{Z}_p$. This completes the proof of the lemma. □

For $\varrho = \mathrm{card}(\mathcal{M})$, let $\mathbf{\Psi} = \big(\Psi_{\mathbf{i}}\big)_{\mathbf{i}\in\mathcal{M}}$ with $\Psi_{\mathbf{i}} = \mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_\varrho^{\mathbf{i}}$ for each $\mathbf{i} \in \mathcal{M}$. Write $u_p = u_p(\mathbf{\Psi};\mathbb{Z}_p)$ for the nonnegative integer defined by

$$|\Delta(\mathbf{\Psi};\mathbb{Z}_p)|_p = p^{-u_p}. \tag{2.1}$$

For every $n \in \mathbb{N} \setminus \{0\}$, let $\mathbb{Z}_p(\mathcal{M}; n)$ be defined by (1.4).

Proposition 2.1. *Suppose that $\mathbb{Z}_p(\mathcal{M}) = \mathbb{Z}_p(\mathcal{M}; \ell)$ for some positive integer ℓ. Suppose also that $s \geq \ell + \varrho$. Let u_p be defined as in (2.1). Let $\mathbf{n} = (n_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}} \in \mathbb{Z}_p(\mathcal{M})$. Then the following hold.*
(1) The system

$$\Phi_{s,\mathbf{i}}(\mathbf{z}) = n_{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M})$$

has a solution $\mathbf{a}^* \in \mathbb{Z}_p^{sd}$ with

$$|\Delta(\mathbf{\Phi}_s; \mathbf{a}^*)|_p \geq p^{-u_p}.$$

(2) Whenever $h \geq 2u_p + 1$, one has

$$\lambda_{s,k,d}(h; p; \mathbf{n}) \geq p^{(2u_p+1)(\varrho - sd)} \quad \text{for all } \mathbf{n} \in \mathbb{Z}_p(\mathcal{M}).$$

In other words, the system of polynomials $\mathbf{\Phi}_s = \left(\Phi_{s,\mathbf{i}}\right)_{\mathbf{i} \in \mathcal{M}}$ satisfies the uniform local density hypothesis over $\mathbb{Z}_p$.

Proof. Let $(n_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}} \in \mathbb{Z}_p(\mathcal{M})$. Take $\widetilde{\mathbf{a}}_1, \ldots, \widetilde{\mathbf{a}}_\varrho \in \mathbb{Z}_p^d$ such that

$$|\Delta(\mathbf{\Psi}; \widetilde{\mathbf{a}})|_p = |\Delta(\mathbf{\Psi}; \mathbb{Z}_p)|_p = p^{-u_p}.$$

Write $\widetilde{n}_{\mathbf{i}} = n_{\mathbf{i}} - (\widetilde{\mathbf{a}}_1^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_\varrho^{\mathbf{i}})$ for each $\mathbf{i} \in \mathcal{M}$. Since $(n_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}} \in \mathbb{Z}_p(\mathcal{M})$ and $\mathbb{Z}_p(\mathcal{M})$ is a module over $\mathbb{Z}_p$, then $\widetilde{n}_{\mathbf{i}} = n_{\mathbf{i}} + (-1)(\widetilde{\mathbf{a}}_1^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_\varrho^{\mathbf{i}}) \in \mathbb{Z}_p(\mathcal{M})$. Then there exist $\mathbf{a}_1, \ldots, \mathbf{a}_\ell \in \mathbb{Z}_p^d$ such that

$$\widetilde{n}_{\mathbf{i}} = \mathbf{a}_1^{\mathbf{i}} + \cdots + \mathbf{a}_\ell^{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M}).$$

Therefore

$$\begin{aligned} n_{\mathbf{i}} &= \widetilde{\mathbf{a}}_1^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_\varrho^{\mathbf{i}} + \widetilde{n}_{\mathbf{i}} \\ &= \widetilde{\mathbf{a}}_1^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_\varrho^{\mathbf{i}} + \mathbf{a}_1^{\mathbf{i}} + \cdots + \mathbf{a}_\ell^{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M}). \end{aligned}$$

Take $s \geq \ell + \varrho$. Then the system

$$\Phi_{s,\mathbf{i}}(\mathbf{z}) = n_{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M})$$

has a solution $\mathbf{a}^* \in \mathbb{Z}_p^{sd}$ with

$$|\Delta(\mathbf{\Phi}_s; \mathbf{a}^*)|_p \geq |\Delta(\mathbf{\Phi}_s; \widetilde{\mathbf{a}})|_p \geq p^{-u_p}.$$

By application of Lemma 2.1 for all $h \geq 2u_p + 1$, we have

$$\lambda_{s,k,d}(h;p;\mathbf{n}) \geq p^{(2u_p+1)(\varrho-sd)}.$$

Since u_p is independent of $\mathbf{n}$, this completes the proof of the proposition. □

It remains to find a positive integer ℓ such that $\mathbb{Z}_p(\mathcal{M}) = \mathbb{Z}_p(\mathcal{M};\ell)$.

Lemma 2.3. *For $k \in \mathbb{N} \setminus \{0\}$, let $\nu = \nu(k) = 4k$. Let $\Omega_p = \{(\mathbf{a}^{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \,|\, \mathbf{a} \in \mathbb{Z}_p^d\}$. Suppose that there exists a finite subset $B \subseteq \Omega_p$ such that the $\mathbb{Z}_p$-module $\mathbb{Z}_p(\mathcal{M})$ can be generated by B over $\mathbb{Z}_p$. Let $\ell = \nu \cdot \mathrm{card}(B)$. Then $\mathbb{Z}_p(\mathcal{M}) = \mathbb{Z}_p(\mathcal{M};\ell)$.*

Proof. Write $\varkappa = \mathrm{card}(B)$ and $B = \{(\mathbf{b}_j^{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \,|\, \mathbf{b}_j \in \mathbb{Z}_p^d,\ 1 \leq j \leq \varkappa\}$. Let $(n_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{Z}_p(\mathcal{M})$. Then there exist $c_1, \ldots, c_\varkappa \in \mathbb{Z}_p$ such that

$$n_{\mathbf{i}} = \sum_{j=1}^{\varkappa} c_j \mathbf{b}_j^{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M}).$$

By Lemma 2.2, every c_j can be decomposed as a sum of ν kth powers in $\mathbb{Z}_p$. More specifically, there exist $c_{j,1}, \ldots, c_{j,\nu} \in \mathbb{Z}_p$ such that

$$\sum_{j=1}^{\varkappa} c_j \mathbf{b}_j^{\mathbf{i}} = \sum_{j=1}^{\varkappa} (c_{j,1}^k + \cdots + c_{j,\gamma}^k)\mathbf{b}_j^{\mathbf{i}} = \sum_{j=1}^{\varkappa} \Big((c_{j,1}\mathbf{b}_j)^{\mathbf{i}} + \cdots + (c_{j,\nu}\mathbf{b}_j)^{\mathbf{i}}\Big) \quad (\mathbf{i} \in \mathcal{M}),$$

where the last equality holds because $i_1 + \cdots + i_d = k$ for every $\mathbf{i} = (i_1, \ldots, i_d) \in \mathcal{M}$. Therefore

$$n_{\mathbf{i}} = \sum_{j=1}^{\varkappa} \Big((c_{j,1}\mathbf{b}_j)^{\mathbf{i}} + \cdots + (c_{j,\nu}\mathbf{b}_j)^{\mathbf{i}}\Big) \quad (\mathbf{i} \in \mathcal{M}).$$

This completes the proof of the lemma. □

Proposition 2.2. *Suppose that R is a local ring with the maximal ideal $\mathfrak{m}$ and that M is a finitely generated R-module. Then the elements $x_1, \ldots, x_m \in M$ generate the R-module M if and only if $x_1 + \mathfrak{m}M, \ldots, x_m + \mathfrak{m}M$ span the vector space $M/\mathfrak{m}M$ over $R/\mathfrak{m}$.*

Proof. This is [2, Proposition 2.8]. □

Proposition 2.3. *Let $\ell = 4k\varrho$. Then $\mathbb{Z}_p(\mathcal{M}) = \mathbb{Z}_p(\mathcal{M};\ell)$.*

Proof. We first notice that $\mathbb{Z}_p$ is a complete discrete valuation ring, and thus a local ring and a principal ideal domain. Since $\mathbb{Z}_p^\varrho$ is a free $\mathbb{Z}_p$-module of rank ϱ, the submodule $\mathbb{Z}_p(\mathcal{M})$ has a generating set of cardinality not exceeding ϱ. Let $\varkappa_p$ denote the dimension of the vector space $\mathbb{Z}_p(\mathcal{M})/p\mathbb{Z}_p(\mathcal{M})$ over $\mathbb{F}_p$. Then $\varkappa_p \leq \varrho$. Let

$$\Omega_p = \{(\mathbf{a}^{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \,|\, \mathbf{a} \in \mathbb{Z}_p^d\}.$$

Since $\varkappa_p \leq \varrho$ and the vector space $\mathbb{Z}_p(\mathcal{M})/\big(p\mathbb{Z}_p(\mathcal{M})\big)$ can be generated by $\Omega_p + \big(p\mathbb{Z}_p(\mathcal{M})\big)$ over $\mathbb{Z}_p/p\mathbb{Z}_p$, there exist ϱ elements in $\Omega_p + \big(p\mathbb{Z}_p(\mathcal{M})\big)$ which generate $\mathbb{Z}_p(\mathcal{M})/\big(p\mathbb{Z}_p(\mathcal{M})\big)$. By Proposition 2.2, there exist ϱ elements in Ω_p forming a generating set of the $\mathbb{Z}_p$-module $\mathbb{Z}_p(\mathcal{M})$. Then the proposition follows from Lemma 2.3 immediately. □

On combining Proposition 2.3 with Proposition 2.1, we deduce the following theorem immediately.

Theorem 2.1. *Let $\gamma(\mathbb{Z}_p;\mathcal{M})$ be defined by (1.10). Let u_p be defined as in (2.1). Suppose that $s \geq \gamma(\mathbb{Z}_p;\mathcal{M}) + \varrho$. Then the following hold.*
(1) One has

$$\gamma(\mathbb{Z}_p;\mathcal{M}) \leq 4k\varrho.$$

(2) Let $\mathbf{n} = (n_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{Z}_p(\mathcal{M})$. The system

$$\Phi_{s,\mathbf{i}}(\mathbf{z}) = n_{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M})$$

has a solution $\mathbf{a}^* \in \mathbb{Z}_p^{sd}$ with

$$|\Delta(\mathbf{\Phi}_s;\mathbf{a}^*)|_p \geq p^{-u_p}.$$

(3) Whenever $h \geq 2u_p + 1$, one has

$$\lambda_{s,k,d}(h;p;\mathbf{n}) \geq p^{(2u_p+1)(\varrho - sd)} \quad \textit{for all } \mathbf{n} \in \mathbb{Z}_p(\mathcal{M}).$$

In other words, the system of polynomials $\mathbf{\Phi}_s = \big(\Phi_{s,\mathbf{i}}\big)_{\mathbf{i}\in\mathcal{M}}$ satisfies the uniform local density hypothesis over $\mathbb{Z}_p$.

We remark that the proof of Theorem 2.1 can be applied to any nonempty subset of $\mathcal{M}$. For convenience of future reference, we state the result in the following theorem. Let $\widetilde{\mathcal{M}}$ be a nonempty set of $\mathcal{M}$, $\widetilde{\varrho} = \mathrm{card}(\widetilde{M})$, $\widetilde{\mathbf{\Phi}}_s = \big(\Phi_{s,\mathbf{i}}\big)_{\mathbf{i}\in\widetilde{\mathcal{M}}}$ and $\widetilde{\mathbf{\Psi}} = \big(\Psi_{\mathbf{i}}\big)_{\mathbf{i}\in\widetilde{\mathcal{M}}}$ with $\Psi_{\mathbf{i}} = \mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_{\widetilde{\varrho}}^{\mathbf{i}}$ for each $\mathbf{i} \in \widetilde{\mathcal{M}}$. Write $\widetilde{u}_p = \widetilde{u}_p(\mathbf{\Psi};\mathbb{Z}_p)$ for the nonnegative integer defined by

$$|\Delta(\widetilde{\mathbf{\Psi}};\mathbb{Z}_p)|_p = p^{-\widetilde{u}_p}.$$

Theorem 2.2. *Define $\gamma(\mathbb{Z}_p;\widetilde{\mathcal{M}})$ to be the least integer ℓ for which*

$$\mathbb{Z}_p(\widetilde{\mathcal{M}}) = \left\{ \left(\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_\ell^{\mathbf{i}}\right)_{\mathbf{i}\in\widetilde{\mathcal{M}}} \,\middle|\, \mathbf{x}_j \in \mathbb{Z}_p^d \ (1 \leq j \leq \ell) \right\}.$$

Suppose $s \geq \gamma(\mathbb{Z}_p; \widetilde{\mathcal{M}}) + \widetilde{\varrho}$. Then the following hold.
(1) One has

$$\gamma(\mathbb{Z}_p; \widetilde{\mathcal{M}}) \leq 4k\widetilde{\varrho}.$$

(2) Let $\mathbf{n} = (n_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{Z}_p(\mathcal{M})$. The system

$$\Phi_{s,\mathbf{i}}(\mathbf{z}) = n_{\mathbf{i}} \quad (\mathbf{i} \in \widetilde{\mathcal{M}})$$

has a solution $\mathbf{b} \in \mathbb{Z}_p^{sd}$ with

$$|\Delta(\widetilde{\mathbf{\Phi}}_s; \mathbf{b})|_p \geq p^{-\widetilde{u}_p}.$$

(3) Whenever $h \geq 2\widetilde{u}_p + 1$, one has

$$\lambda_{s,k,d}(h; p; \widetilde{\mathbf{n}}) \geq p^{(2\widetilde{u}_p+1)(\varrho-sd)} \quad \text{for all } \widetilde{\mathbf{n}} \in \mathbb{Z}_p(\widetilde{\mathcal{M}}).$$

In other words, the system of polynomials $\widetilde{\mathbf{\Phi}}_s = \big(\Phi_{s,\mathbf{i}}\big)_{\mathbf{i}\in\widetilde{\mathcal{M}}}$ satisfies the uniform local density hypothesis over $\mathbb{Z}_p$.

Theorem 1.1 follows from Theorem 2.1 immediately. We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. For a prime p, and $\mathbf{n} \in \mathcal{D}$, we may deduce from the work of Parsell [27] that whenever $s \geq \varrho k + k + 1$, we have

$$\prod_p \Big(\lim_{h\to\infty} \lambda_{s,k,d}(h; p; \mathbf{n})\Big) \ll 1 \quad \text{for all } \mathbf{n} \in \mathcal{D}.$$

On recalling Theorem 2.1, for $s \geq 4k\varrho + \varrho$, we have

$$\lim_{h\to\infty} \lambda_{s,k,d}(h; p; \mathbf{n}) \geq p^{(2u_p+1)(\varrho-sd)} \quad \text{for all } \mathbf{n} \in \mathcal{D}.$$

On combining this with the arguments in [26, Section 9], we have

$$1 \ll \prod_p \Big(\lim_{h\to\infty} \lambda_{s,k,d}(h; p; \mathbf{n})\Big) \ll 1 \quad \text{for all } \mathbf{n} \in \mathcal{D},$$

where the implicit constants are independent of the choice for $\mathbf{n}$. $\square$

3. Uniform local density hypothesis over complete discrete valuation rings

To consider the multidimensional analogue of Waring's problem in number fields or function fields, it is also necessary to ask whether the corresponding local densities have a uniform lower bound. In this section, we aim to extend the strategy in Section 2 to more general complete discrete valuation rings. It will allows our result to cover both finite places and infinite place of function fields. Let K be a complete field with respect to the norm $|\cdot|$ associated to a discrete non-archimedean valuation. Let $\mathcal{O} = \{x \in K| \, |x| \leq 1\}$, π a primitive element, $\mathfrak{n} = \pi\mathcal{O}$ and $F = \mathcal{O}/\mathfrak{n}$, where F is a finite field. Let $\mathcal{M}$ and ϱ be defined as in (1.2). Let $\mathcal{O}(\mathcal{M})$ and $\mathcal{O}(k)$ be defined as in (1.5) and (1.13) respectively. We see from Lemma 2.2 that $\mathbb{Z}_p = \mathbb{Z}_p(k)$. However, it might happen that $\mathcal{O} \neq \mathcal{O}(k)$. In this case, $\mathcal{O}(\mathcal{M})$ is not necessarily a module over $\mathcal{O}$. We therefore have to replace $\mathcal{O}$ by $\mathcal{O}(k)$ as in (1.13) and study the properties of $\mathcal{O}(k)$. First of all, $\mathcal{O}(k)$ is a ring.

Lemma 3.1. *Let $\mathcal{O}(k)$ be defined by (1.13). Then $\mathcal{O}(k)$ is subring of $\mathcal{O}$ with $-1 \in \mathcal{O}(k)$. In addition, $\mathcal{O}(\mathcal{M})$ is a module over $\mathcal{O}(k)$.*

Proof. Since $\mathcal{O}(k)$ is closed under addition and multiplication of $\mathcal{O}$, it suffices to show that $-1 \in \mathcal{O}(k)$. To see this, we divide into two cases.

When $\mathrm{char}(\mathcal{O}) > 0$, Let $p_1 = \mathrm{char}(F)$ and so $-1 = (p_1 - 1)(1^k) \in \mathcal{O}(k)$.

When $\mathrm{char}(\mathcal{O}) = 0$, then $\mathcal{O}$ can be viewed as an extension form some $\mathbb{Z}_p$ and so $-1 \in \mathcal{O}(k)$ by Lemma 2.2.

Thus $\mathcal{O}(k)$ is a subring of $\mathcal{O}$ and $\mathcal{O}(\mathcal{M})$ is a module over $\mathcal{O}(k)$. □

Definition 3.1. Let $n, r \in \mathbb{N}$ with $1 \leq n \leq r$. Let $\boldsymbol{\varphi} = (\varphi_i)_{1 \leq i \leq n}$ with each $\varphi_i \in \mathcal{O}[x_1, \ldots, x_r]$ $(1 \leq i \leq n)$. For $\mathbf{a} = (a_1, \ldots, a_r) \in \mathcal{O}^r$, denote by $|\Delta(\boldsymbol{\varphi}; \mathbf{a})|$ the maximal value of the determinants of all $n \times n$ submatrices of the Jacobian matrix $\big(\partial\varphi_i/\partial x_j\big)_{n \times r}$ when taking $x_1 = a_1, \ldots, x_r = a_r$. If $|\Delta(\boldsymbol{\varphi}; \mathbf{a})| \neq 0$, we may write $|\Delta(\boldsymbol{\varphi}; \mathbf{a})| = |\pi|^u$ for some integer $v = v(\boldsymbol{\varphi}; \mathbf{a}; \mathcal{O})$, and say that the pair $(\boldsymbol{\varphi}; \mathbf{a})$ has *a nonsingular weight of* v. For any $S \subseteq \mathcal{O}^r$, we further define

$$|\Delta(\boldsymbol{\varphi}; S)| = \max\big\{|\Delta(\boldsymbol{\varphi}; \mathbf{a})| \,\big|\, \mathbf{a} \in S\big\}.$$

For $s \in \mathbb{N} \setminus \{0\}$, write

$$\Phi_{s,\mathbf{i}}(\mathbf{x}) = \mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M})$$

and let $\boldsymbol{\Phi}_s = \big(\Phi_{s,\mathbf{i}}\big)_{\mathbf{i} \in \mathcal{M}}$. For $h \in \mathbb{N} \setminus \{0\}$ and $\mathfrak{f} = (\mathfrak{f}_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}} \in \mathcal{O}(\mathcal{M})$, define $\lambda_{s,k,d}(h; \pi; \mathfrak{f})$ as in Definition 1.2.

Lemma 3.2. *Let $(\mathfrak{f}_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}} \in \mathcal{O}(\mathcal{M})$. Suppose that the system*

$$\Phi_{s,\mathbf{i}}(\mathbf{x}) = \mathfrak{f}_{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M})$$

has a solution $\mathbf{a} \in \mathcal{O}^{sd}$ *with* $|\Delta(\boldsymbol{\Phi}_s;\mathbf{a})| = |\pi|^{v_0}$*, where* $v_0 = v_0(\boldsymbol{\Phi}_s;\mathbf{a};\mathcal{O}) \in \mathbb{N}$*. Then whenever* $h \geq 2v_0+1$*, one has*

$$\lambda_{s,k,d}(h;\pi;\mathfrak{f}) \geq (\mathrm{card}(F))^{(2v_0+1)(\varrho-sd)}.$$

Proof. It follows from the standard Hensel-type arguments (for example, see [26, Lemma 9.9]). □

We then establish the existence of the nonsingular solution required in Lemma 3.2.

Lemma 3.3. *Let* $\boldsymbol{\psi} = (\psi_i)_{1\leq i\leq n}$ *with each* $\varphi_i \in \mathcal{O}[x_1,\ldots,x_n]$ $(1 \leq i \leq n)$*. Let* $\Delta(\boldsymbol{\psi};\mathbf{x})$ *be the Jacobian of* $\boldsymbol{\psi}$*. suppose that* $\mathbf{a} = (a_1,\ldots,a_n) \in \mathcal{O}^n$ *satisfies*

$$\left|\psi_j(\mathbf{a})\right| < \left|\Delta(\boldsymbol{\psi};\mathbf{a})\right|^2 \quad (1 \leq j \leq n).$$

Then there exists a unique $\mathbf{b} = (b_1,\ldots,b_n) \in \mathcal{O}^n$ *such that*

$$\psi_j(\mathbf{b}) = 0 \quad (1 \leq j \leq n) \quad \text{ and } \quad \left|b_i - a_i\right| < \left|\Delta(\boldsymbol{\psi};\mathbf{a})\right| \quad (1 \leq i \leq n).$$

Proof. This is [15, Proposition 5.20]. □

Lemma 3.4. *Let* $\xi_{\mathbf{i}} \in \mathcal{O}\backslash\{0\}$ $(\mathbf{i} \in \mathcal{M})$*. For every* $\mathbf{i} \in \mathcal{M}$*, we let* $\phi_{\mathbf{i}}(\mathbf{z})$ *denote the polynomial* $\Phi_{s,\mathbf{i}}(\mathbf{x}) - \xi_{\mathbf{i}}$ *with* $\mathbf{x} = (\mathbf{x}_1,\ldots,\mathbf{x}_s)$ *replaced by* $\mathbf{z} = (z_1,\ldots,z_{sd})$*. Let* $\boldsymbol{\phi} = (\phi_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}}$*. Suppose that* $\boldsymbol{\eta} \in \mathcal{O}^{sd}$ *satisfies that*

$$|\phi_{\mathbf{i}}(\boldsymbol{\eta})| < \left|\Delta(\boldsymbol{\phi};\boldsymbol{\eta})\right|^2 \quad (\mathbf{i} \in \mathcal{M}).$$

Then there exists $\widetilde{\boldsymbol{\eta}} \in \mathcal{O}^{sd}$ *such that*

$$\phi_{\mathbf{i}}(\widetilde{\boldsymbol{\eta}}) = 0 \quad (\mathbf{i} \in \mathcal{M}) \quad \text{ and } \quad |\Delta(\boldsymbol{\phi};\widetilde{\boldsymbol{\eta}})| \geq |\Delta(\boldsymbol{\phi};\boldsymbol{\eta})|.$$

Proof. Suppose that $\left|\Delta(\boldsymbol{\phi};\boldsymbol{\eta})\right| = |\pi|^u$. Thus at the point $\boldsymbol{\eta}$ there exist columns $i_1,\ldots,i_\varrho$ in the Jacobian matrix of $(\phi_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}}$ forming a submatrix whose determinant has value $|\pi|^u$. We then regard $\phi_{\mathbf{i}}(\mathbf{z})$ as a polynomial in ϱ variables $z_{i_1},\ldots,z_{i_\varrho}$ after substituting $z_i = \eta_i$ for $i \notin \{i_1,\ldots,i_\varrho\}$. By applying Lemma 3.3, we obtain $\widetilde{\eta}_{i_1},\ldots,\widetilde{\eta}_{i_\varrho} \in \mathcal{O}$ such that

$$\phi_{\mathbf{i}}(\widetilde{\boldsymbol{\eta}}) = 0 \ (\mathbf{i} \in \mathcal{M})$$

and

$$\widetilde{\eta}_j \equiv \eta_j \ (\mathrm{mod}\, \pi^{u+1}) \ (j = i_1,\ldots,i_\varrho).$$

Therefore

$$\Delta(\boldsymbol{\phi}; \widetilde{\eta}_{i_1}, \ldots, \widetilde{\eta}_{i_\varrho}) \equiv \Delta(\boldsymbol{\phi}; \eta_{i_1}, \ldots, \eta_{i_\varrho}) \pmod{\pi^{u+1}}.$$

On setting $\widetilde{\eta}_j = \eta_j$ for $j \notin \{i_1, \ldots, i_\varrho\}$, we find that $|\Delta(\boldsymbol{\phi}; \widetilde{\boldsymbol{\eta}})| \geq |\Delta(\boldsymbol{\phi}; \boldsymbol{\eta})|$. □

In what follows, for every $\mathbf{i} \in \mathcal{M}$, let $\Psi_{\mathbf{i}}(\mathbf{z})$ denote the polynomial $\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_\varrho^{\mathbf{i}}$ with $(z_1, \ldots, z_{d\varrho}) = (x_{11}, \ldots, x_{d1}, \ldots, x_{1\varrho}, \ldots, x_{d\varrho})$. Write $\boldsymbol{\Psi} = (\Psi_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}}$. For every $n \in \mathbb{N} \setminus \{0\}$, let $\mathcal{O}(\mathcal{M}; n)$ be defined by (1.4). By viewing $\mathcal{O}(\mathcal{M})$ as a module over $\mathcal{O}(k)$, we deduce the following criterion from standard arguments as in Proposition 2.1.

Proposition 3.1. *Suppose that there exists a positive integer ℓ such that*

$$\mathcal{O}(\mathcal{M}) = \mathcal{O}(\mathcal{M}; \ell). \tag{3.1}$$

Suppose also that $|\Delta(\boldsymbol{\Psi}; \mathcal{O})| \neq 0$. Let $u_0 \in \mathbb{N}$ and $r_0 \in \mathbb{N}$ such that

$$|\Delta(\boldsymbol{\Psi}; \mathcal{O})| = |\pi|^{u_0} \quad \text{and} \quad r_0 \geq 2u_0 + 1. \tag{3.2}$$

Then the following hold.
(1) *Suppose that $(\mathfrak{f}_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathcal{O}^\varrho$ and $(\mathfrak{g}_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathcal{O}(\mathcal{M})$ satisfy that*

$$\mathfrak{f}_{\mathbf{i}} \equiv \mathfrak{g}_{\mathbf{i}} \pmod{\pi^{r_0}} \quad (\mathbf{i} \in \mathcal{M}).$$

Whenever $s \geq \ell + \varrho$, the system

$$\Phi_{s,\mathbf{i}}(\mathbf{z}) = \mathfrak{f}_{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M})$$

has a solution $\mathbf{a}^* \in \mathcal{O}^{sd}$ with

$$|\Delta(\boldsymbol{\Phi}_s; \mathbf{a}^*)| \geq |\pi|^{u_0}.$$

(2) *Whenever $s \geq \ell + \varrho$, the system of polynomials $\boldsymbol{\Phi}_s = (\Phi_{s,\mathbf{i}})_{\mathbf{i}\in\mathcal{M}}$ satisfies the uniform local density hypothesis over $\mathcal{O}$.*

Proof. (1) Take $\widetilde{\mathbf{a}}_1, \ldots, \widetilde{\mathbf{a}}_\varrho \in \mathcal{O}^d$ such that

$$|\Delta(\boldsymbol{\Psi}; \widetilde{\mathbf{a}})| = |\Delta(\boldsymbol{\Psi}; \mathcal{O})| = |\pi|^{u_0}.$$

Write $\widetilde{\mathfrak{f}_{\mathbf{i}}} = \mathfrak{f}_{\mathbf{i}} - (\widetilde{\mathbf{a}}_1^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_\varrho^{\mathbf{i}})$ for each $\mathbf{i} \in \mathcal{M}$. Since $-1 \in \mathcal{O}(k)$ and $\mathfrak{f}_{\mathbf{i}} \equiv \mathfrak{g}_{\mathbf{i}} \pmod{\pi^{r_0}}$ $(\mathbf{i} \in \mathcal{M})$ where $(\mathfrak{g}_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathcal{O}(\mathcal{M})$, then there exist $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathcal{O}^d$ such that

$$\widetilde{\mathfrak{f}_{\mathbf{i}}} = \mathfrak{f}_{\mathbf{i}} + (-1)(\widetilde{\mathbf{a}}_1^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_\varrho^{\mathbf{i}}) \equiv \sum_{j=1}^{n} \mathbf{a}_j^{\mathbf{i}} \pmod{\pi^{r_0}} \quad (\mathbf{i} \in \mathcal{M}).$$

By the assumption, there exist $\widetilde{\mathbf{a}}_{\varrho+1}, \ldots, \widetilde{\mathbf{a}}_{\varrho+\ell} \in \mathcal{O}^d$ such that

$$\sum_{j=1}^{n} \mathbf{a}_j^{\mathbf{i}} = \widetilde{\mathbf{a}}_{\varrho+1}^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_{\varrho+\ell}^{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M}).$$

Therefore

$$\begin{aligned}\mathfrak{f}_{\mathbf{i}} &= \widetilde{\mathbf{a}}_1^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_{\varrho}^{\mathbf{i}} + \widetilde{\mathfrak{f}}_{\mathbf{i}} \\ &\equiv \widetilde{\mathbf{a}}_1^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_{\varrho}^{\mathbf{i}} + \widetilde{\mathbf{a}}_{\varrho+1}^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_{\varrho+\ell}^{\mathbf{i}} \pmod{\pi^{r_0}} \quad (\mathbf{i} \in \mathcal{M}).\end{aligned}$$

Since $r_0 \geq 2u_0 + 1$ and $s \geq \ell + \varrho$ and hence there exist $\widetilde{\mathbf{a}}_{\varrho+1}, \dots, \widetilde{\mathbf{a}}_s \in \mathcal{O}^d$ such that

$$\mathfrak{f}_{\mathbf{i}} \equiv \widetilde{\mathbf{a}}_1^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_{\varrho}^{\mathbf{i}} + \widetilde{\mathbf{a}}_{\varrho+1}^{\mathbf{i}} + \cdots + \widetilde{\mathbf{a}}_{s}^{\mathbf{i}} \pmod{\pi^{r_0}} \quad (\mathbf{i} \in \mathcal{M}).$$

Thus

$$|\Phi_{s,\mathbf{i}}(\widetilde{\mathbf{a}}) - \mathfrak{f}_{\mathbf{i}}| \leq |\pi|^{r_0} < |\pi|^{2u_0} \quad (\mathbf{i} \in \mathcal{M})$$

and

$$|\Delta(\mathbf{\Phi}_s; \widetilde{\mathbf{a}})| \geq |\Delta(\mathbf{\Psi}; \widetilde{\mathbf{a}})| = |\pi|^{u_0}.$$

We can deduce from Lemma 3.4 that the system

$$\Phi_{s,\mathbf{i}}(\mathbf{z}) = \mathfrak{f}_{\mathbf{i}} \quad (\mathbf{i} \in \mathcal{M})$$

has a solution $\mathbf{a}^* \in \mathcal{O}^{sd}$ with

$$|\Delta(\mathbf{\Phi}_s; \mathbf{a}^*)| \geq |\Delta(\mathbf{\Phi}_s; \widetilde{\mathbf{a}})| \geq |\pi|^{u_0}.$$

(2) In combination of Lemma 3.2 with Part (1), for all $h \geq 2u_0 + 1$, we have

$$\lambda_{s,k,d}(h; \pi; \mathfrak{f}) \geq (\mathrm{card}(F))^{(2u_0+1)(\varrho - sd)}.$$

Since u_0 is independent of $\mathfrak{f}$, this completes the proof of the proposition. □

By carrying out similar argument in Lemma 2.3, we may obtain the following criterion for seeking the positive integer ℓ required in Proposition 3.1.

Proposition 3.2. *Suppose that there exists a positive integer θ^* such that $\mathcal{O}(k) = \mathcal{O}(k; \theta^*)$ with $\mathcal{O}(k; \theta^*)$ defined by (1.12). Let $\Omega = \{(\mathbf{a}^{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}} \,|\, \mathbf{a} \in \mathcal{O}^d\}$. Suppose also that there exist μ^* elements in Ω which form a generating set of the $\mathcal{O}(k)$-module $\mathcal{O}(\mathcal{M})$. Let $\ell^* = \theta^* \mu^*$. Then*

$$\mathcal{O}(\mathcal{M}) = \mathcal{O}(\mathcal{M}; \ell^*).$$

Let $\gamma(\mathcal{O};\mathcal{M})$ be defined by (1.10). Then

$$\gamma(\mathcal{O};\mathcal{M}) \le \theta^* \mu^*.$$

In order to find θ^* and μ^*, we need to extract more algebraic properties of $\mathcal{O}$, $\mathcal{O}(k)$ and $\mathcal{O}(\mathcal{M})$.

Definition 3.2. Let S be a semiring. For $k, n \in \mathbb{N} \setminus \{0\}$, let $S(k;n)$ and $S(k)$ be defined by (1.12) and (1.13) respectively. Let $\gamma(S;k)$ be the least integer u for which $S(k) = S(k;u)$ if such u exists.

Thus the positive integer θ^* in Proposition 3.2 exists if and only if $\gamma(\mathcal{O};k) \le \theta^*$. To this end, our idea is motivated by the following facts.

Lemma 3.5. *For $k \in \mathbb{N}$ with $k \ge 2$, one has*
(1) *$F(k)$ is a subfield of F.*
(2) *$\gamma(F;k) \le k$.*

Proof. (1) and (2) are [30, Lemma 1] and [30, Theorem 1] respectively. □

By applying Hensel-type arguments as in [29, Lemma 1], we can extend the result about Waring's problem in the finite field F to the complete valuation ring $\mathcal{O}$.

Proposition 3.3. *Suppose that $\mathrm{char}(F) \nmid k$. Consider the surjective homomorphism $f_{\mathcal{O}}$ from $\mathcal{O}$ to $\mathcal{O}/\mathfrak{n} = F$. The following hold.*
(1) *One has*

$$\mathcal{O}(k) = f_{\mathcal{O}}^{-1}\big(F(k)\big).$$

(2) *$\gamma(\mathcal{O};k) \le \gamma(F;k) + 1$.*
(3) *$\mathcal{O}(k)$ is a local ring with the maximal ideal $\mathfrak{n}$. In addition,*

$$\mathcal{O}(k)/\mathfrak{n} = F(k).$$

(4) *Suppose that $\mathcal{O}(\mathcal{M})$ is a finitely generated module over $\mathcal{O}(k)$. Let μ be the minimal number of generators of $\mathcal{O}(\mathcal{M})$. Then μ is equal to the positive integer μ^* defined in Proposition 3.2. More precisely, there exist μ elements in Ω which form a minimal generating set of the $\mathcal{O}(k)$-module $\mathcal{O}(\mathcal{M})$. In addition,*

$$\gamma(\mathcal{O};\mathcal{M}) \le \gamma(\mathcal{O};k)\mu.$$

Proof. When the characteristic of $\mathcal{O}$ is 0, Parts (1) and (2) have been proved in [29, Lemma 1] (use $\gamma(F,k)$ in place of k). The same proof technique can apply to the case when the characteristic of $\mathcal{O}$ is positive. It remains to show Part (3) and Part (4).

(3) It follows from Part (1) that $\mathfrak{n} = f_{\mathcal{O}}^{-1}(0)$ and thus an ideal of $\mathcal{O}(k)$. Moreover, we have

$$\mathcal{O}(k)/\mathfrak{n} = F(k).$$

Therefore Lemma 3.5(1) implies that $\mathfrak{n}$ is a maximal ideal of $\mathcal{O}(k)$. Let $x \in \mathfrak{n}$. Then $1 + x$ is a unit in $\mathcal{O}$ and so there exists $y \in \mathcal{O}$ such that $(1 + x)y = 1$. We then get $y = 1 - xy \in 1 + \mathfrak{n}$. Since $1 + \mathfrak{n} = (1 + \mathfrak{n})^k \in F(k)$, we deduce from Part (1) that $y \in \mathcal{O}(k)$. Therefore, $1 + x$ is a unit in $\mathcal{O}(k)$. By [2, Proposition 1.6], we conclude that $\mathcal{O}(k)$ is a local ring with the maximal ideal $\mathfrak{n}$.
(4) Note that $\mathcal{O}(\mathcal{M})/\big(\mathfrak{n}\mathcal{O}(\mathcal{M})\big)$ is a vector space over $F(k)$. By Proposition 2.2, μ is equal to the dimension of the vector space $\mathcal{O}(\mathcal{M})/\big(\mathfrak{n}\mathcal{O}(\mathcal{M})\big)$. Let

$$\Omega = \{(\mathbf{a}^{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \,|\, \mathbf{a} \in \mathcal{O}^d\}.$$

Thus there exist μ elements in $\Omega + \big(\mathfrak{n}\mathcal{O}(\mathcal{M})\big)$ which form a basis of $\mathcal{O}(\mathcal{M})/\big(\mathfrak{n}\mathcal{O}(\mathcal{M})\big)$. By Proposition 2.2, there exist μ elements in Ω forming a minimal generating set of the $\mathcal{O}(k)$-module $\mathcal{O}(\mathcal{M})$. □

On combining Lemma 3.5(2) with Proposition 3.3(2), we have

$$\gamma(\mathcal{O}; k) \le k + 1.$$

In Section 4, we aim to refine this upper bound when $F \ne F(k)$. In Section 5, we will show that $\mathcal{O}(\mathcal{M})$ is a finitely generated module over $\mathcal{O}(k)$ and estimate the minimal number of generators of $\mathcal{O}(\mathcal{M})$ via the Noetherian module theory. In Section 6, to prove Theorem 1.5, it remains to show that $|\Delta(\boldsymbol{\Psi}; \mathcal{O})| \ne 0$ as required in Proposition 3.1.

4. Refinements over finite fields

In this section, let K, $\mathcal{O}$, π and F be defined as in Section 3. Let $p_1 = \mathrm{char}(F)$. We focus on the case when $F \ne F(k)$. For convenience, since $p_1 = \mathrm{char}(F)$, we may consider F as a finite extension of $\mathbb{F}_{p_1}$, and define

$$\kappa = [F : F(k)], \quad \sigma = [F(k) : \mathbb{F}_{p_1}], \quad \tau = [F : \mathbb{F}_{p_1}] \tag{4.1}$$

and

$$\mathcal{L} = \left\{ l \in \mathbb{N} \,\middle|\, l < \tau,\ l|\tau, \text{ and } \frac{p_1^\tau - 1}{p_1^l - 1}\middle| k \right\}. \tag{4.2}$$

Our work stems from the following fact.

Proposition 4.1. *Let $k \in \mathbb{N}$ with $k \ge 2$. One has $\kappa = 1$ if and only if $\mathcal{L} = \emptyset$.*

Proof. This is [3, Theorem G]. □

Our goal is to improve the upper bound for $\gamma(F;k)$ from k to $O(k^{1/2}/\kappa)$ when $\kappa > 1$. In this case, the above proposition implies that $\mathcal{L} \neq \emptyset$.

Theorem 4.1. *Let $k \in \mathbb{N}$ with $k \geq 2$. Define κ, σ, τ and $\mathcal{L}$ as in* (4.1) and (4.2). *Suppose that $\kappa > 1$. For $l \in \mathcal{L}$, let F_l denote the subfield of F with $[F_l : \mathbb{F}_{p_1}] = l$, and let*

$$k_l = k(p_1^l - 1)/(p_1^\tau - 1).$$

Then the following hold.
(1) *For every $l \in \mathcal{L}$, one has $F(k) = F_l(k_l)$ and $\gamma(F;k) = \gamma(F_l;k_l)$.*
(2) *One has $\sigma = \min\{l \,|\, l \in \mathcal{L}\}$ and $F_\sigma = F(k) = F_\sigma(k_\sigma)$.*
(3) *One has*

$$\gamma(F;k) \leq k_\sigma = k/(1 + p_1^\sigma + \cdots + p_1^{\sigma(\kappa-1)})$$

and

$$\kappa < \log_{p_1^\sigma} k + 1.$$

(4) *One has*

$$\mathrm{card}(F) \leq \min\left\{ \left(\frac{k}{k_l} - 1\right)^2 \,\middle|\, l \in \mathcal{L} \right\}.$$

Proof. (1) Consider the norm map N_{F/F_l} of the extension of finite fields F/F_l. For every $x \in F$, $N_{F/F_l}(x) = x^{(p_1^\tau - 1)/(p_1^l - 1)} = x^{k/k_l}$. Thus for every $x \in F$, we have $x^k = (x^{k/k_l})^{k_l} \in F_l(k_l)$. Hence $F(k) \subseteq F_l(k_l)$. Also, since the norm map is surjective, we have $F_l(k_l) \subseteq F(k)$. Therefore,

$$F(k) = F_l(k_l).$$

For any $x_1, \ldots, x_n \in F$, since

$$x_1^k + \cdots + x_n^k = \left(x_1^{k/k_l}\right)^{k_l} + \cdots + \left(x_n^{k/k_l}\right)^{k_l},$$

it follows from the surjectiveness of the norm map that

$$\gamma(F;k) = \gamma(F_l;k_l).$$

(2) Since $\kappa > 1$, we have $\sigma = \tau/\kappa < \tau$. Let a_0 be a generator of $F^\times$. Then $a_0^k \in F(k)$. Since $\mathrm{card} F(k) = p_1^\sigma$ and $\mathrm{card}(F) = p_1^\tau$, we have

$$a_0^{k(p_1^\sigma-1)}=1$$

and thus

$$p_1^{\tau-1}\big|((p_1^\sigma-1)k).$$

Therefore $\sigma\in\mathcal{L}$. By Part (1), $F_\sigma=F(k)=F_\sigma(k_\sigma)$. In addition, for every $l\in\mathcal{L}$, we have

$$F_\sigma=F(k)=F_l(k_l)\subseteq F_l$$

and thus

$$\sigma\le l.$$

Therefore

$$\sigma=\min\{l\,|\,l\in\mathcal{L}\}.$$

(3) By Part (2) and Lemma 3.5, we have

$$\gamma(F;k)=\gamma(F_\sigma;k_\sigma)\le k_\sigma.$$

Since $\tau=\kappa\sigma$, we have

$$k_\sigma=k(p_1^\sigma-1)/(p_1^\tau-1)=k/(1+p_1^\sigma+\cdots+p_1^{\sigma(\kappa-1)}).$$

Note that

$$k_\sigma\le k/(1+p_1^{\sigma(\kappa-1)}).$$

Thus

$$\kappa<\log_{p_1^\sigma}k+1.$$

(4) Let $l\in\mathcal{L}$. Then there exists $n\in\mathbb{N}$ with $n\ge 2$ such that $\tau=ln$. Thus

$$\frac{k}{k_l}=\frac{p_1^\tau-1}{p_1^l-1}=p_1^{l(n-1)}+\cdots+p_1^l+1.$$

Then we have

$$p_1^{l(n-1)}\le\frac{k}{k_l}-1.$$

Therefore

$$\mathrm{card}(F) = p_1^\tau = \frac{k}{k_l}(p_1^l - 1) + 1 = \frac{k}{k_l}\left(\left(p_1^{l(n-1)}\right)^{\frac{1}{n-1}} - 1\right) + 1 \le \frac{k}{k_l}\left(\left(\frac{k}{k_l} - 1\right)^{\frac{1}{n-1}} - 1\right) + 1.$$

Since $n \ge 2$, we have

$$\mathrm{card}(F) \le \left(\frac{k}{k_l} - 1\right)^2.$$

This completes the proof of the theorem. □

We now discuss further improvement on $\gamma(F; k)$.

Lemma 4.1. *Suppose that F is of cardinality p_1^τ. Let $k^* = \gcd(k, p_1^\tau - 1)$. Then*

$$\gamma(F; k) \le k^*.$$

Proof. Suppose that a is a generator of the cyclic group $F^\times$. Then the order of a is $p_1^\tau - 1$. Since $k^* = \gcd(k, p_1^\tau - 1)$, then $k^* = \gcd(k^*, p_1^\tau - 1)$. Thus a^k and a^{k^*} have the same order so that they generate the same subgroup. Therefore $F(k) = F(k^*)$. It then follows from Lemma 3.5 that

$$\gamma(F; k) \le k^*. \quad \square$$

We then begin to consider the case when $\kappa = [F : F(k)] > 1$.

Proposition 4.2. *Suppose that $\kappa = [F : F(k)] = 2$. Then $k \ge 3$ and*

$$\gamma(F; k) \le (k + 1)^{1/2} - 1.$$

Proof. We deduce from Theorem 4.1 that

$$\gamma(F; k) = \gamma(F_\sigma; k_\sigma),$$

where $k_\sigma = k/(1+p_1^\sigma)$, and $F_\sigma = F(k) = F_\sigma(k_\sigma)$ with $[F_\sigma : \mathbb{F}_{p_1}] = \sigma$. Thus $k \ge 1+p_1^\sigma \ge 3$. Let $k_* = \gcd(k_\sigma, p_1^\sigma - 1)$. We then obtain from Lemma 4.1 that

$$\gamma(F, k) = \gamma(F_\sigma, k_\sigma) \le k_*.$$

In view of the definition of k_*, we obtain

$$k_* \le k_\sigma \quad \text{and} \quad k_* \le p_1^\sigma - 1.$$

Thus

$$k_*(k_* + 2) \le k_\sigma(p_1^\sigma + 1) = k,$$

and so

$$(k_* + 1)^2 \le k + 1.$$

This completes the proof of the proposition. $\square$

Proposition 4.3. *Suppose that* $\kappa = [F : F(k)] \ge 3$. *Then* $k \ge 7$. *In addition, one has*

$$\gamma(F; k) \le \sqrt{(18/13)k}/\kappa.$$

Proof. We first observe from Theorem 4.1(3) that

$$\frac{k}{k_\sigma} = 1 + p_1^\sigma + \cdots + p_1^{\sigma(\kappa-1)}.$$

Since $p_1 \ge 2$, $\sigma \ge 1$ and $\kappa \ge 3$, we have $k \ge 7k_\sigma$. By Theorem 4.1(1) and (2), we have

$$\gamma(F; k) = \gamma(F_\sigma; k_\sigma),$$

where $F_\sigma = F(k)$ with $\operatorname{card}\big(F(k)\big) = p_1^\sigma$. Let $k_* = \gcd(k_\sigma, p_1^\sigma - 1)$. We then deduce from Lemma 4.1 that

$$\gamma(F, k) = \gamma(F_\sigma, k_\sigma) \le k_*.$$

In view of the definition of k_*, we obtain

$$k_* \le k_\sigma \quad \text{and} \quad k_* \le p_1^\sigma - 1.$$

Thus

$$k_*^2 \le k_\sigma(p_1^\sigma - 1) = \frac{(p_1^\sigma - 1)k}{1 + p_1^\sigma + \cdots + p_1^{\sigma(\kappa-1)}} = \frac{(p_1^\sigma - 1)^2 k}{p_1^{\sigma\kappa} - 1}.$$

For $x \in \mathbb{N} \setminus \{0\}$, define

$$f_\kappa(x) = \frac{x^2}{(x+1)^\kappa - 1}.$$

Let $M_\kappa = \sup\big\{f_\kappa(x) \,|\, x \in \mathbb{N} \setminus \{0\}\big\}$. Then

$$\gamma(F; k) \le k_* \le \big(f_\kappa(p_1^\sigma - 1)k\big)^{1/2} \le M_\kappa^{1/2} k^{1/2}.$$

For $\kappa = 3$, we have

$$f_3(x) = \frac{x^2}{x^3 + 3x^2 + 3x}$$

and thus $M_3 = 2/13$. Therefore, when $\kappa = 3$, we find

$$\gamma(F;k) \le \sqrt{2/13}k^{1/2} = \sqrt{18/13}k^{1/2}/\kappa.$$

For $\kappa \ge 4$, we now claim that $M_\kappa \le (2^\kappa - 1)^{-1}$. Since $(x+1)^\kappa - 1 = \binom{\kappa}{\kappa}x^\kappa + \binom{\kappa}{\kappa-1}x^{\kappa-1} + \cdots + \binom{\kappa}{3}x^3 + \binom{\kappa}{2}x^2 + \binom{\kappa}{1}x$, we have

$$\frac{1}{f_\kappa(x)} = \frac{(x+1)^\kappa - 1}{x^2} = \binom{\kappa}{\kappa}x^{\kappa-2} + \cdots + \binom{\kappa}{3}x + \binom{\kappa}{2} + \binom{\kappa}{1}x^{-1}.$$

For $\kappa \ge 4$ and $x \ge 1$, because $\binom{\kappa}{\kappa-1}x^{\kappa-2} + \binom{\kappa}{1}x^{-1} \ge \binom{\kappa}{\kappa-1}x + \binom{\kappa}{1}x^{-1} \ge \binom{\kappa}{\kappa-1} + \binom{\kappa}{1}$, we then find

$$\frac{1}{f_\kappa(x)} \ge \binom{\kappa}{\kappa} + \binom{\kappa}{\kappa-1} + \cdots + \binom{\kappa}{3} + \binom{\kappa}{2} + \binom{\kappa}{1} = 2^\kappa - 1.$$

Therefore

$$M_\kappa \le \frac{1}{2^\kappa - 1}$$

and so

$$\gamma(F;k) \le k_* \le (M_\kappa k)^{1/2} \le \frac{k^{1/2}}{(2^\kappa - 1)^{1/2}}.$$

For $x \ge 4$, define $g(x) = x/(2^x - 1)^{1/2}$. Since $\sup_{x \ge 4} g(x) \le 4/\sqrt{15}$, for $\kappa \ge 4$, we have

$$\gamma(F;k) \le \frac{k^{1/2}}{\kappa} \cdot \frac{\kappa}{(2^\kappa - 1)^{1/2}} \le (4/\sqrt{15})k^{1/2}/\kappa.$$

This completes the proof of the proposition. □

Corollary 4.1. *Suppose that* $\operatorname{char}(F) \nmid k$. *Let* $\kappa = [F : F(k)]$. *Then the following hold.*
(1) *If* $\kappa = 1$, *then*

$$\gamma(\mathcal{O};k) \le k + 1.$$

(2) *If* $\kappa = 2$, *then* $k \ge 3$ *and*

$$\gamma(\mathcal{O};k) \le \sqrt{k+1}.$$

(3) *If* $\kappa \ge 3$, *then* $k \ge 7$ *and*

$$\gamma(\mathcal{O};k) \le \sqrt{(18/13)k}/\kappa + 1 \le \sqrt{(72/13)k}/\kappa.$$

Proof. On recalling Proposition 3.3(2), we have

$$\gamma(\mathcal{O}; k) \leq \gamma(F; k) + 1.$$

Part (1) and Part (2) then follow from Lemma 3.5(2) and Proposition 4.2 respectively. To deduce Part (3) from Proposition 4.3, it suffices to notice that

$$1 \leq \gamma(F; k) \leq \sqrt{(18/13)k}/\kappa$$

and $2\sqrt{18/13} = \sqrt{72/13}$. This completes the proof of the corollary. □

5. Multidimensional Waring's problem over complete discrete valuation rings

Let K, $\mathcal{O}$, π and F be defined as in Section 3. Throughout this section, we always assume that $\mathrm{char}(F) \nmid k$. It is a natural observation from Proposition 3.3(3) that $\mathcal{O}(k) = \mathcal{O}$ if and only if $F(k) = F$. When $\mathcal{O}(k) \neq \mathcal{O}$, since $\mathcal{O}(k)$ is a subring of $\mathcal{O}$, one can regard $\mathcal{O}$ as an $\mathcal{O}(k)$-module, and ask the minimal number of generators of $\mathcal{O}$ over $\mathcal{O}(k)$. In particular, to investigate the minimal number of generators of $\mathcal{O}(\mathcal{M})$ over $\mathcal{O}(k)$, it is also necessary to seek similar Noetherian properties about $\mathcal{O}^n$ for $n \in \mathbb{N}$ with $n \geq 1$.

We now introduce some notations and concepts in the theory of Noetherian modules.

Definition 5.1. Let R be a ring and M be an R-module. Given a submodule N of M, let $\mu(N) = \mu_R(N)$ to be the minimal number of generators of N as an R-module. Define

$$\mathfrak{v}(M) = \mathfrak{v}_R(M) := \sup\{\mu(N) | N \subset M, \text{ a submodule}\}.$$

For the special case $M = R$, submodules are ideals. Then

$$\mathfrak{v}(R) := \sup\{\mu(I) | I \subset R, \text{ an ideal}\}.$$

We are interested in the bound for $\mathfrak{v}(M)$. In general, $\mathfrak{v}(M)$ is not a finite number.

Definition 5.2. Let R be a ring and M be an R-module. We say that M is Noetherian if every R-submodule of M is finitely generated. In particular, if $M = R$, R is a Noetherian ring.

Our first goal in this section is to show that the $\mathcal{O}(k)$-module $\mathcal{O}$ is Noetherian and $\mathfrak{v}(\mathcal{O}) = [F : F(k)]$. To this end, we now summarize some basic facts about the $\mathcal{O}(k)$-module $\mathcal{O}$.

Lemma 5.1. *Let $\kappa = [F : F(k)]$. One has $\mathcal{O}/\mathfrak{n} \cong F$ as vector spaces over $F(k)$. For $b_1, \ldots, b_\kappa \in \mathcal{O}$, the following are equivalent.*

(1) $b_1 + \mathfrak{n}, \ldots, b_\kappa + \mathfrak{n}$ *span* $\mathcal{O}/\mathfrak{n}$.

(2) $b_1 + \mathfrak{n}, \ldots, b_\kappa + \mathfrak{n}$ *form a basis of* $\mathcal{O}/\mathfrak{n}$.

(3) $b_1 + \mathfrak{n}, \ldots, b_\kappa + \mathfrak{n} \in \mathcal{O}/\mathfrak{n}$ *are linearly independent over* $\mathcal{O}(k)/\mathfrak{n}$. *Namely, given* $c_1, \ldots, c_\kappa \in \mathcal{O}(k)$, *whenever* $c_1 b_1 + \cdots + c_\kappa b_\kappa \in \mathfrak{n}$, *then* $c_1, \ldots, c_\kappa \in \mathfrak{n}$.

(4) $b_1, \ldots, b_\kappa$ *form a generating set of the* $O(k)$*-module* $\mathcal{O}$.

Proof. First off, note that (1)⇔(2)⇔(3) and (4)⇒(1) are elementary facts in linear algebra and the module theory. (1)⇒(4) is a direct consequence of Proposition 2.2. This completes the proof. □

Remark. For $a_1, \ldots, a_r \in \mathcal{O}$, if $a_1 + \mathfrak{n}, \ldots, a_r + \mathfrak{n}$ span $\mathcal{O}/\mathfrak{n}$, then $r \geq \kappa$. Thus

$$\mu(\mathcal{O}) = \kappa.$$

Lemma 5.2. *Let* M *be an* $\mathcal{O}(k)$*-module. Then* $\mathfrak{n}M$ *is an* $\mathcal{O}$*-module.*

Proof. Since the multiplication of $\mathcal{O}$ on $\mathfrak{n}M$ satisfies the axioms of modules, it is enough to show that $\mathfrak{n}M$ is closed under the multiplication of $\mathcal{O}$. From the fact that $\mathfrak{n}$ is an ideal of $\mathcal{O}$,

$$\mathcal{O} \cdot (\mathfrak{n}M) = (\mathcal{O} \cdot \mathfrak{n})M = \mathfrak{n}M.$$

Thus $\mathfrak{n}M$ is closed under the multiplication of $\mathcal{O}$ and $\mathfrak{n}M$ is an $\mathcal{O}$-module. □

It is worth a remark that $\mathfrak{n}M$ is also an $\mathcal{O}(k)$-submodule of M. In addition, $M/\mathfrak{n}M$ is a module over $\mathcal{O}(k)/\mathfrak{n}$ and thus a vector space over $F(k)$.

Lemma 5.3. *Let* $I \subset \mathcal{O}$ *be an* $\mathcal{O}(k)$*-module. Then there exists* $l \in \mathbb{N}$ *with* $l \geq 1$ *such that*

$$I \subset \mathfrak{n}^{l-1} \quad \textit{and} \quad \mathfrak{n}I = \mathfrak{n}^l$$

with the convention $\mathfrak{n}^0 = \mathcal{O}$.

Proof. By the previous lemma, $\mathfrak{n}I$ is an $\mathcal{O}$-module and therefore an ideal of $\mathcal{O}$. Since $\mathcal{O}$ is a discrete valuation ring with the maximal ideal $\mathfrak{n}$, there exists $l \in \mathbb{N}$ such that

$$\mathfrak{n}I = \mathfrak{n}^l = \mathcal{O} \cdot \pi^l.$$

For all $a \in I$, there exists $a_I \in \mathcal{O}$ such that

$$\pi \cdot a = a_I \cdot \pi^l.$$

It implies that $a = a_I \cdot \pi^{l-1} \in \mathcal{O} \cdot \pi^{l-1} = \mathfrak{n}^{l-1}$; i.e.,

$$I \subset \mathfrak{n}^{l-1}.$$

This completes the proof. □

Proposition 5.1. *Let $I \subset \mathcal{O}$ be an $\mathcal{O}(k)$-submodule of $\mathcal{O}$. Let κ_I be the dimension of the vector space $I/(\mathfrak{n}I)$ over $\mathcal{O}(k)/\mathfrak{n}$. Then*

$$\mu(I) = \kappa_I \leq \kappa.$$

Proof. From Lemma 5.3, there exists $l \in \mathbb{N}$ with $l \geq 1$ such that

$$I \subset \mathfrak{n}^{l-1} \quad \text{and} \quad \mathfrak{n}I = \mathfrak{n}^l.$$

Thus $I/\mathfrak{n}^l$ is a subspace of $\mathfrak{n}^{l-1}/\mathfrak{n}^l$. Note that $\mathfrak{n}^{l-1}/\mathfrak{n}^l$ is isomorphic to $\mathcal{O}/\mathfrak{n}$ as vector spaces over $\mathcal{O}(k)/\mathfrak{n}$. We thus have

$$\kappa_I \leq \kappa.$$

We now choose a subset $\{b_1, \ldots, b_{\kappa_I}\}$ of $\mathcal{O}$ such that $\{b_1\pi^{l-1} + \mathfrak{n}^l, \ldots, b_{\kappa_I}\pi^{l-1} + \mathfrak{n}^l\}$ is a basis of $I/\mathfrak{n}^l$. We claim that $\{b_1\pi^{l-1}, \ldots, b_{\kappa_I}\pi^{l-1}\}$ is a generating set of I as a module over $\mathcal{O}(k)$. We then have

$$\mu(I) \leq \kappa_I.$$

Given $a \in I$, there exist $c_1, \ldots, c_{\kappa_I} \in \mathcal{O}(k)$ such that

$$a + \mathfrak{n}^l = \sum_{i=1}^{\kappa_I} c_i b_i \pi^{l-1} + \mathfrak{n}^l.$$

Then there exists $\widetilde{a} \in \mathfrak{n}$ such that

$$a = \sum_{i=1}^{\kappa_I} c_i b_i \pi^{l-1} + \widetilde{a}\pi^{l-1}.$$

Since $\widetilde{a} \in \mathfrak{n}$, we then have $\widetilde{a} \in \mathcal{O}(k)$. In addition, since $b_1\pi^{l-1} \notin \mathfrak{n}^l$, then $b_1 \notin \mathfrak{n}$ and so b_1 is a unit in $\mathcal{O}$. Thus

$$\widetilde{a} = (\widetilde{a}b_1^{-1})b_1$$

where $\widetilde{a}b_1^{-1} \in \mathfrak{n}$. Therefore

$$\begin{aligned} a &= \sum_{i=1}^{\kappa_I} c_i b_i \pi^{l-1} + \widetilde{a}\pi^{l-1} \\ &= \sum_{i=1}^{\kappa_I} c_i b_i \pi^{l-1} + (\widetilde{a} b_1^{-1}) b_1 \pi^{l-1} \\ &= (c_1 + \widetilde{a} b_1^{-1}) \cdot (b_1 \pi^{l-1}) + \sum_{j=2}^{\kappa_I} c_j \cdot (b_j \pi^l). \end{aligned}$$

It remains to show that $\mu(I) \geq \kappa_I$. Suppose that $a_1, \ldots, a_r \in I$ generate I as an $\mathcal{O}(k)$-module. Then $a_1 + \mathfrak{n}I, \ldots, a_r + \mathfrak{n}$ span the vector space $I/(\mathfrak{n}I)$ over $\mathcal{O}(k)/\mathfrak{n}$. Thus $r \geq \kappa_I$ and so $\mu(I) \geq \kappa_I$. This completes the proof. □

Theorem 5.1. *Let $\kappa = [F : F(k)]$. The $\mathcal{O}(k)$-module $\mathcal{O}$ is Noetherian and*

$$\mathfrak{v}(\mathcal{O}) = \kappa$$

In particular, $\mathcal{O}(k)$ is a Noetherian ring.

Proof. Let $I \subset \mathcal{O}$ be an $\mathcal{O}(k)$-submodule of $\mathcal{O}$. It follows from Proposition 5.1 that

$$\mu(I) \leq \kappa.$$

Thus

$$\mathfrak{v}(\mathcal{O}) \leq \kappa$$

Since $\mathfrak{v}(\mathcal{O}) \geq \mu(\mathcal{O}) = \kappa$, we have

$$\mathfrak{v}(\mathcal{O}) = \kappa.$$

This completes the proof. □

The goal of the remainder of this section is to extend the previous Noetherian properties to $\mathcal{O}^n$. More precisely, we aim to prove that

$$\mathfrak{v}(\mathcal{O}^n) = \kappa n.$$

Noetherian modules enjoy many nice properties under various operations of modules. However, we need the quantitative version of those properties.

Lemma 5.4. *Let M be an R-module. Suppose that L and N are submodules of M with $\mu((L+N)/N) < \infty$ and $\mu(L \cap N) < \infty$. Then*

$$\mu((L+N)/N) \leq \mu(L) \leq \mu((L+N)/N) + \mu(L \cap N).$$

In particular, suppose further that $N \subseteq L$. Then

$$\mu(L/N) \leq \mu(L) \leq \mu(L/N) + \mu(N).$$

Proof. Let $x_1, \ldots, x_m \in L$ with $m = \mu((L+N)/N)$ such that $x_1 + N, \ldots, x_m + N$ generate $(L+N)/N$, and let $y_1, \ldots, y_l \in L \cap N$ with $l = \mu(L \cap N)$ generate $L \cap N$. For any $x \in L$, we have

$$x \equiv r_1 x_1 + \cdots + r_m x_m \pmod N$$

for some $r_i \in R$, so $x - \sum r_i x_i \in L \cap N$. Therefore

$$x - \sum r_i x_i = \sum s_j y_j$$

with $s_j \in R$, so $x = \sum r_i x_i + \sum s_j y_j$. It proves that L is generated by $x_1, \ldots, x_m, y_1, \ldots, y_l$, and

$$\mu(L) \leq m + l = \mu((L+N)/N) + \mu(L \cap N).$$

Note that the reduction of the generators of L modulo N will generate $(L+N)/N$ as an R-module. Thus, we have

$$\mu((L+N)/N) \leq \mu(L).$$

It completes the proof. □

Proposition 5.2. *Let M be an R-module and N be a submodule. Then M is Noetherian if and only if N and M/N are Noetherian. Moreover,*

$$\mathfrak{v}(M) \leq \mathfrak{v}(M/N) + \mathfrak{v}(N).$$

Proof. If M is Noetherian, in view of definition, N is Noetherian. In addition, every submodule of M/N has the form $\overline{L}/N$ where $\overline{L}$ is a submodule of M with $N \subseteq \overline{L} \subseteq M$. It follows from Lemma 5.4 that

$$\mu(\overline{L}/N) \leq \mu(\overline{L}) < \infty.$$

Thus M/N is Noetherian.

For another direction, let L be a submodule of M. Then the image of L in M/N is $(L+N)/N$. Since N and M/N are Noetherian, we have $\mu((L+N)/N) < \infty$ and $\mu(L \cap N) < \infty$. Then Lemma 5.4 implies that

$$\mu(L) \leq \mu((L+N)/N) + \mu(L \cap N) < \infty.$$

Thus M is Noetherian and satisfies

$$\mathfrak{v}(M) \leq \mathfrak{v}(M/N) + \mathfrak{v}(N).$$

It completes the proof. □

As a corollary of the previous proposition, we have

Corollary 5.1. *Let M and N be Noetherian R-modules. Then their direct sum $M \oplus N$ is a Noetherian R-module and*

$$\mathfrak{v}(M \oplus N) \leq \mathfrak{v}(M) + \mathfrak{v}(N). \tag{5.1}$$

Proof. Take the submodule $M \oplus 0$ of $M \oplus N$ and consider the exact sequence

$$0 \to M \cong M \oplus 0 \to M \oplus N \to (M \oplus N)/(M \oplus 0) \cong N \to 0.$$

By applying Proposition 5.2, we have

$$\mathfrak{v}(M \oplus N) \leq \mathfrak{v}(M) + \mathfrak{v}(N). \quad \square$$

Remark. It might happen that

$$\mathfrak{v}(M \oplus N) < \mathfrak{v}(M) + \mathfrak{v}(N).$$

For example, when we take $R = \mathbb{Z}$, $M = \mathbb{Z}/(3)$, and $N = \mathbb{Z}/(5)$, then $M \oplus N = \mathbb{Z}/(15)$. Since $\mathfrak{v}(M) = \mathfrak{v}(N) = \mathfrak{v}(M \oplus N) = 1$, we have

$$\mathfrak{v}(M \oplus N) < \mathfrak{v}(M) + \mathfrak{v}(N).$$

However, in the special case when R is a local ring, the equality in (5.1) always holds.

Proposition 5.3. *Suppose that R is a local ring with the maximal ideal $\mathfrak{m}$. Let M and N be Noetherian R-modules. Then their direct sum $M \oplus N$ is a Noetherian R-module and*

$$\mathfrak{v}(M \oplus N) = \mathfrak{v}(M) + \mathfrak{v}(N).$$

In particular, for $n \in \mathbb{N}$ with $n \geq 1$,

$$\mathfrak{v}(M^n) = n \cdot \mathfrak{v}(M).$$

Proof. By Corollary 5.1, it suffices to show that

$$\mathfrak{v}(M \oplus N) \geq \mathfrak{v}(M) + \mathfrak{v}(N).$$

If $\mathfrak{v}(M) = \infty$ or $\mathfrak{v}(N) = \infty$, inequality trivially holds. We now assume that $\mathfrak{v}(M) < \infty$ and $\mathfrak{v}(N) < \infty$. Choose $M' \oplus N'$, where $M' \subseteq M$ and $N' \subseteq N$ with $\mu(M') = \mathfrak{v}(M)$ and $\mu(N') = \mathfrak{v}(N)$. Proposition 2.2 implies that

$$\mu(M') = \dim_{R/\mathfrak{m}}(M'/\mathfrak{m}M'), \quad \mu(N') = \dim_{R/\mathfrak{m}}(N'/\mathfrak{m}N'),$$

and

$$\mu(M' \oplus N') = \dim_{R/\mathfrak{m}} \big((M' \oplus N')/(\mathfrak{m}(M' \oplus N'))\big).$$

Since $(M' \oplus N')/(\mathfrak{m}(M' \oplus N'))$ is isomorphic to $(M'/\mathfrak{m}M') \oplus (N'/\mathfrak{m}N')$ as $R/\mathfrak{m}$ vector spaces, we have

$$\mu(M' \oplus N') = \mu(M') + \mu(N') = \mathfrak{v}(M) + \mathfrak{v}(N).$$

Therefore,

$$\mathfrak{v}(M \oplus N) \geq \mathfrak{v}(M) + \mathfrak{v}(N).$$

This completes the proof. □

By Proposition 3.3(3), $\mathcal{O}(k)$ is a local ring. On combining Theorem 5.1 with Proposition 5.3, we deduce the following result.

Theorem 5.2. *Let $n \in \mathbb{N}$ with $n \geq 1$. One has*

$$\mathfrak{v}(\mathcal{O}^n) = \kappa n.$$

In view of the definition of $\mathfrak{v}(\mathcal{O}^n)$, "every" $\mathcal{O}(k)$-submodule of $\mathcal{O}^n$, as an $\mathcal{O}(k)$-module, can be generated by κn elements.

Since $\mathcal{O}$ is a Noetherian $\mathcal{O}(k)$-module, $\mathcal{O}^\varrho$ is Noetherian. Thus, the $\mathcal{O}(k)$-submodule $\mathcal{O}(\mathcal{M})$ of $\mathcal{O}^\varrho$ is also Noetherian.

Proposition 5.4. *Let $\kappa = [F : F(k)]$. Then*

$$\mu\big(\mathcal{O}(\mathcal{M})\big) \leq \kappa(\varrho - d + \kappa^{-1} d).$$

Proof. Let $\mathbf{i} \in \mathcal{M}$, define

$$\mathcal{O}(\mathbf{i}) = \bigcup_{n=1}^{\infty} \{\, \mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_n^{\mathbf{i}} \,|\, \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathcal{O}^d \}$$

Given an integer j, $1 \leq j \leq d$, let $\mathbf{k}_j = (\mathbf{k}_{j,i})_{1 \leq i \leq d} \in \mathcal{M}$ be the index in $\mathcal{M}$ such that $\mathbf{k}_{j,j} = k$. Then $\mathcal{O}(\mathbf{k}_j) = \mathcal{O}(k)$. There is a natural projection $f_{\mathbf{k}}$ such that

$$f_{\mathbf{k}} : \mathcal{O}(\mathcal{M}) \to \sum_{j=1}^{d} \mathcal{O}(\mathbf{k}_j) \cong \mathcal{O}(k)^d$$

By restricting to $\mathbf{k}_j$ coordinates, the kernel of $f_{\mathbf{k}}$, $\ker f_{\mathbf{k}}$, is a submodule of $\sum_{\substack{\mathbf{i}\in\mathcal{M},\mathbf{i}\neq\mathbf{k}_j \\ 1\leq j\leq d}} \mathcal{O}(\mathbf{i})$.

We claim that $f_{\mathbf{k}}$ is surjective. Since the image of $f_{\mathbf{k}}$, $\mathrm{image} f_{\mathbf{k}}$, is an $\mathcal{O}(k)$-module, by symmetry, it is enough to show that $(1, 0, 0, \dots, 0) \in \mathcal{O}(k)^d$ is contained in $\mathrm{image} f_{\mathbf{k}}$. Now just choose $\mathbf{x}_1 \in \mathcal{O}^d$ with $x_{1,1} = 1, x_{1,2} = \cdots = x_{1,d} = 0$. Then

$$(\mathbf{x}_1^{\mathbf{k}_1}, \dots, \mathbf{x}_1^{\mathbf{k}_d}) = (1, 0, 0, \dots, 0).$$

Thus the claim is true.

We then deduce from Lemma 5.4 that

$$\mu\big(\mathcal{O}(\mathcal{M})\big) \leq \mu\big(\ker f_{\mathbf{k}}\big) + \mu(\mathrm{image} f_{\mathbf{k}}) = \mu\big(\ker f_{\mathbf{k}}\big) + \mu\big(\mathcal{O}(k)^d\big) \leq \mathfrak{v}\big(\mathcal{O}^{\varrho-d}\big) + d = \kappa(\varrho - d) + d. \quad \square$$

Theorem 5.3. *One has*

$$\gamma(\mathcal{O}; \mathcal{M}) \leq (k+1)\varrho.$$

In particular, when $F \neq F(k)$, one has $k \geq 3$ and

$$\gamma(\mathcal{O}; \mathcal{M}) \leq \begin{cases} \sqrt{(72/13)k}(\varrho - d/2), & \text{if } k \geq 7, \\ 2\sqrt{k+1}(\varrho - d/2), & \text{if } 3 \leq k \leq 6. \end{cases}$$

Proof. Let $\kappa = [F : F(k)]$. In combination of Proposition 5.4 with Proposition 3.3(4), we get

$$\gamma(\mathcal{O}; \mathcal{M}) \leq \gamma(\mathcal{O}; k)\kappa(\varrho - d + \kappa^{-1} d).$$

When $\kappa = 1$, we obtain from Corollary 4.1(1) that

$$\gamma(\mathcal{O}; \mathcal{M}) \leq (k+1)\varrho.$$

When $\kappa = 2$, Corollary 4.1(2) implies that $k \geq 3$ and

$$\gamma(\mathcal{O}; k)\kappa \leq 2\sqrt{k+1}$$

and so

$$\gamma(\mathcal{O}; \mathcal{M}) \leq 2\sqrt{k+1}(\varrho - d/2).$$

When $\kappa \geq 3$, Corollary 4.1(3) implies that $k \geq 7$ and

$$\gamma(\mathcal{O};k)\kappa \le \sqrt{(72/13)k},$$

and hence

$$\gamma(\mathcal{O};\mathcal{M}) \le \sqrt{(72/13)k}(\varrho - d/2).$$

Since $2\sqrt{k+1} < \sqrt{(72/13)k}$ for all $k \ge 7$, on combining the above estimates, the theorem follows. $\square$

6. Proof of Theorem 1.5

Let K, $\mathcal{O}$, π and F be defined as in Section 3. Let $\mathcal{M}$ and ϱ be defined as in (1.2). In what follows, for every $\mathbf{i} \in \mathcal{M}$, let $\Psi_{\mathbf{i}}(\mathbf{z})$ denote the polynomial $\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_\varrho^{\mathbf{i}}$ with $(z_1, \ldots, z_{d\varrho}) = (x_{11}, \ldots, x_{d1}, \ldots, x_{1\varrho}, \ldots, x_{d\varrho})$. Write $\mathbf{\Psi} = \left(\Psi_{\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{M}}$. Let $|\Delta(\mathbf{\Psi};\mathcal{O})|$ be defined as in Proposition 3.1. In order to prove Theorem 1.5, based on Proposition 3.1(2), Proposition 3.3 and Theorem 5.3, it remains to show the existence of nonsingularity

$$|\Delta(\mathbf{\Psi};\mathcal{O})| \ne 0.$$

By contrapositive law, our approach starts with the singularity of the system the pair $\mathbf{\Psi}$ over the residue field F, via Dedekind's Lemma.

Lemma 6.1 *(Dedekind's Lemma). Let G be a group and L a field. Let $\tau_1, \ldots, \tau_n$ be distinct group homomorphisms from G to $L^\times$. Then the τ_i are linearly independent over L; that is, if $\sum_i c_i\tau_i(g) = 0$ for all $g \in G$, where the $c_i \in L$, then all $c_i = 0$.*

Proof. This is [25, Lemma 2.12]. $\square$

Consider the group $(F^\times)^d$. For every $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, define a mapping $\tau_{\mathbf{n}}$ from $(F^\times)^d$ to $F^\times$ as follows:

$$\tau_{\mathbf{n}}(a_1, \ldots, a_d) = a_1^{n_1} \cdots a_d^{n_d} = \mathbf{a}^{\mathbf{n}}.$$

Then $\tau_{\mathbf{n}}$ is a group homomorphism.

Lemma 6.2. *Let $\mathbf{n} = (n_1, \ldots, n_d), \mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$. Suppose that $n_v \ne m_v$ for some v with $1 \le v \le d$. If $\tau_{\mathbf{n}} = \tau_{\mathbf{m}}$, then*

$$\mathrm{card}(F^\times) \le |n_v - m_v|.$$

Proof. Without loss of generality, we may assume that $n_1 > m_1$. For every $\mathbf{a} = (a, 1, \ldots, 1)$ with $a \in \mathbb{F}^\times$, since $\tau_{\mathbf{n}} = \tau_{\mathbf{m}}$, we have

$$\mathbf{a}^{\mathbf{n}} = \mathbf{a}^{\mathbf{m}},$$

that is,

$$a^{n_1} = a^{m_1}.$$

Thus every element in $F^\times$ is a root of the polynomial $x^{n_1-m_1} = 1$, and hence

$$\operatorname{card}(F^\times) \le n_1 - m_1.$$

This completes the proof of the lemma. $\square$

For every $n \ge 1$, write

$$\operatorname{Jac}\big((\mathbf{x}^{\mathbf{i}})_{\mathbf{i}\in\mathcal{M},1\le l\le d}; \mathbf{z}_1, \dots, \mathbf{z}_n\big) = \bigg(\Big(\frac{\partial \mathbf{x}^{\mathbf{i}}}{\partial x_l}\Big)_{\mathbf{i}\in\mathcal{M},1\le l\le d}(\mathbf{z}_1); \dots; \Big(\frac{\partial \mathbf{x}^{\mathbf{i}}}{\partial x_l}\Big)_{\mathbf{i}\in\mathcal{M},1\le l\le d}(\mathbf{z}_n)\bigg).$$

For $\mathbf{c} = (c_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in F^\varrho \setminus \{\mathbf{0}\}$ and $1 \le l \le d$, let

$$\varphi_l(\mathbf{c}, \mathbf{x}) = \sum_{\mathbf{i}\in\mathcal{M}} c_{\mathbf{i}} i_l \mathbf{x}^{\mathbf{i}} x_l^{-1}.$$

Lemma 6.3. *Suppose* $\operatorname{char}(F) \nmid k$. *Let* $\mathbf{c} = (c_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in F^\varrho \setminus \{\mathbf{0}\}$. *There exists* $v \in \mathbb{N}$ *with* $1 \le v \le d$ *such that* $\varphi_v(\mathbf{c}, \mathbf{x})$ *is a nonzero polynomial in* $F[\mathbf{x}]$.

Proof. Let $p_1 = \operatorname{char}(F)$. Define

$$\mathcal{M}_1 = \big\{\mathbf{i} \in \mathcal{M} \,\big|\, p_1 \nmid i_1\big\} \text{ and } \mathcal{M}_l = \big\{\mathbf{i} \in \mathcal{M} \,\big|\, p_1|i_1, \dots, p_1|i_{l-1}, p_1 \nmid i_l\big\}\ (2 \le l \le d).$$

Since $p_1 \nmid k$, then $\mathcal{M}$ is a disjoint union of $\mathcal{M}_1, \dots, \mathcal{M}_d$. Also, define $\mathcal{M}_l' = \{\mathbf{i} \in \mathcal{M}_l \,|\, c_{\mathbf{i}} \ne 0\}$ $(1 \le l \le d)$. Since the $c_{\mathbf{i}}$ are not all zero, there must exist some l such that $\mathcal{M}_l'$ is nonempty. Let $v = \min\{l \,|\, 1 \le l \le d,\, \mathcal{M}_l' \ne \emptyset\}$. For each $\mathbf{i} \in \mathcal{M}_v$, since $p_1 \nmid i_v$ and $\partial\mathbf{x}^{\mathbf{i}}/\partial x_v = i_v \mathbf{x}^{\mathbf{i}} x_v^{-1}$, we have

$$\sum_{\mathbf{i}\in\mathcal{M}_v} c_{\mathbf{i}} \frac{\partial \mathbf{x}^{\mathbf{i}}}{\partial x_v} = \sum_{\mathbf{i}\in\mathcal{M}_v'} c_{\mathbf{i}} i_v \mathbf{x}^{\mathbf{i}} x_v^{-1} \ne 0$$

in $F[\mathbf{x}]$. By the minimality of v, for any $\mathbf{i} \in \mathcal{M}_l$ with $l < v$, $c_{\mathbf{i}} = 0$ and so

$$\sum_{\substack{\mathbf{i}\in\mathcal{M}_l\\ l<v}} c_{\mathbf{i}} \frac{\partial \mathbf{x}^{\mathbf{i}}}{\partial x_v} = 0$$

in $F[\mathbf{x}]$. For $l > v$, $\mathbf{i} \in \mathcal{M}_l$ implies that $p|i_v$ and hence $\partial\mathbf{x}^{\mathbf{i}}/\partial x_v = 0$. Thus

$$\varphi_v(\mathbf{c},\mathbf{x}) = \sum_{\mathbf{i}\in\mathcal{M}} c_{\mathbf{i}} \frac{\partial \mathbf{x}^{\mathbf{i}}}{\partial x_v} = \sum_{\mathbf{i}\in\mathcal{M}'_v} c_{\mathbf{i}} \frac{\partial \mathbf{x}^{\mathbf{i}}}{\partial x_v} \neq 0.$$

This completes the proof of the lemma. □

Lemma 6.4. *Suppose that* $\mathrm{char}(F) \nmid k$ *and* $|\Delta(\mathbf{\Psi};\mathcal{O})| < 1$. *Then* $\mathrm{card}(F) \leq k$.

Proof. We first claim that when $|\Delta(\mathbf{\Psi};\mathcal{O})| < 1$, for every $n \geq \varrho$ and every choice of $\mathbf{z}_1,\ldots,\mathbf{z}_n \in F^d$, one has

$$\mathrm{rk}\,\mathrm{Jac}\big((\mathbf{x}^{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}};\mathbf{z}_1,\ldots,\mathbf{z}_n\big) < \varrho.$$

To see this, suppose that there exist $\mathbf{z}_1,\ldots,\mathbf{z}_n \in F^d$ with $n \geq \varrho$ such that

$$\mathrm{rk}\,\mathrm{Jac}\big((\mathbf{x}^{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}};\mathbf{z}_1,\ldots,\mathbf{z}_n\big) = \varrho.$$

Thus there exist ϱ tuples, say $\mathbf{z}_1,\ldots,\mathbf{z}_\varrho$, such that

$$\mathrm{rk}\,\mathrm{Jac}\big((\mathbf{x}^{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}};\mathbf{z}_1,\ldots,\mathbf{z}_\varrho\big) = \varrho.$$

This means that

$$|\Delta(\mathbf{\Psi};\mathcal{O})| = |\pi|^0 = 1,$$

contradicting that $|\Delta(\mathbf{\Psi};\mathcal{O})| < 1$.

We now divide into two cases.

Case 1: Suppose that $\mathrm{card}(F^d) \geq \varrho$. Take $n = \mathrm{card}(F^d)$ and list the elements in F^d as $\mathbf{a}_1,\ldots,\mathbf{a}_n$. We then have

$$\mathrm{rk}\,\mathrm{Jac}\big((\mathbf{x}^{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}};\mathbf{a}_1,\ldots,\mathbf{a}_n\big) < \varrho.$$

Thus there exist $c_{\mathbf{i}} \in F$ ($\mathbf{i} \in \mathcal{M}$), not all zero, such that for all $1 \leq l \leq d$ and $1 \leq j \leq n$,

$$\sum_{\mathbf{i}\in\mathcal{M}} c_{\mathbf{i}} \frac{\partial \mathbf{x}^{\mathbf{i}}}{\partial x_l}(\mathbf{a}_j) = 0. \tag{6.1}$$

By Lemma 6.3, there exists $v \in \mathbb{N}$ with $1 \leq v \leq d$ such that

$$\varphi_v(\mathbf{c},\mathbf{x}) = \sum_{\mathbf{i}\in\mathcal{M}} c_{\mathbf{i}} \frac{\partial \mathbf{x}^{\mathbf{i}}}{\partial x_v}$$

is a nonzero polynomial in $F[\mathbf{x}]$. By (6.1), for all $\mathbf{a} \in F^d$, we have

$$\varphi_v(\mathbf{c},\mathbf{a}) = 0.$$

Write

$$\mathcal{N} = \bigcup_{l=1}^{d} \{(i_1, \dots, i_l - 1, \dots, i_d) \,|\, (i_1, \dots, i_l, \dots, i_d) \in \mathcal{M}, \mathrm{char}(F) \nmid i_l\}.$$

It then follows from Lemma 6.1 that the group homomorphisms $\tau_{\mathbf{n}}$, where $\mathbf{n}$ runs over $\mathcal{N}$, are not distinct. For every $\mathbf{n} \in \mathcal{N}$, in view of the definition, we have $|\mathbf{n}| = k - 1$. By Lemma 6.2, we can conclude that

$$\mathrm{card}(F^{\times}) \leq k - 1.$$

Case 2: Suppose that $\mathrm{card}(F^d) < \varrho$. On recalling that

$$\varrho = \binom{k+d-1}{d-1} = \frac{(k+d-1)\cdots(k+1)}{(d-1)!},$$

we get

$$\frac{\varrho}{k^d} = \frac{k+1}{k^2} \prod_{j=2}^{d-1} \frac{k+j}{kj}.$$

Note that $kj - k - j + 1 = (k-1)(j-1)$, when $j \geq 2$, since $k \geq 2$, we have

$$kj - k - j + 1 \geq 1$$

and so $kj \geq k + j$. Thus

$$\frac{\varrho}{k^d} \leq \frac{k+1}{k^2} \leq 1.$$

Hence

$$\mathrm{card}(F) < \varrho^{1/d} \leq k.$$

On combining the two cases, we have

$$\mathrm{card}(F) \leq k.$$

This completes the proof of the lemma. □

Lemma 6.5. *Suppose that $|\Delta(\mathbf{\Psi}; \mathcal{O})| = 0$. Then for any complete discrete valuation ring $\widetilde{\mathcal{O}}$ with the same characteristic as $\mathcal{O}$, one has*

$$|\Delta(\mathbf{\Psi}; \widetilde{\mathcal{O}})| = 0.$$

Proof. Recall that for $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_\varrho) \in \mathcal{O}^{d\varrho}$, we define $|\Delta(\boldsymbol{\Psi}; \mathbf{a})|$ to be the maximal value of the determinants of all $\varrho \times \varrho$ submatrices of the Jacobian matrix of the system $\boldsymbol{\Psi}$ at the point $\mathbf{a}$. Also, recall that

$$|\Delta(\boldsymbol{\Psi}; \mathcal{O})| = \max\left\{|\Delta(\boldsymbol{\Psi}; \mathbf{a})| \,\middle|\, \mathbf{a} \in \mathcal{O}^{d\varrho}\right\}.$$

Suppose that $|\Delta(\boldsymbol{\Psi}; \mathcal{O})| = 0$. Then $|\Delta(\boldsymbol{\Psi}; \mathbf{a})| = 0$ for all $\mathbf{a} \in \mathcal{O}^{d\varrho}$. Note that every element of the Jacobian matrix of the system $\boldsymbol{\Psi}$ takes the form of $\partial \mathbf{x}_j^{\mathbf{i}} / \partial x_{j,l} = i_l x_{j,1}^{i_1} \cdots x_{j,l}^{i_l - 1} \cdots x_{j,d}^{i_d}$ with $i_1 + \cdots + i_d = k$, $1 \leq j \leq \varrho$ and $1 \leq l \leq d$. Thus the determinant of every $\varrho \times \varrho$ submatrix is a homogeneous polynomial in $\mathbf{x}_1, \dots, \mathbf{x}_\varrho$. Therefore $|\Delta(\boldsymbol{\Psi}; \mathbf{a})| = 0$ for all $\mathbf{a} \in K^{d\varrho}$. Since K is an infinite field, the determinant of every $\varrho \times \varrho$ submatrix must be the zero polynomial in $K[\mathbf{x}_1, \dots, \mathbf{x}_\varrho]$. We now consider two cases according to the characteristic of $\mathcal{O}$.

Case 1: Suppose that $\mathrm{char}(\mathcal{O}) = p_1$. Since the elements in the Jacobian matrix of the system $\boldsymbol{\Psi}$ lie in $\mathbb{F}_{p_1}[\mathbf{x}_1, \dots, \mathbf{x}_\varrho]$, we may regard the determinant of every $\varrho \times \varrho$ submatrix as the zero polynomial in $\mathbb{F}_{p_1}[\mathbf{x}_1, \dots, \mathbf{x}_\varrho]$ and so in $\widetilde{\mathcal{O}}[\mathbf{x}_1, \dots, \mathbf{x}_\varrho]$ where $\widetilde{\mathcal{O}}$ is any complete discrete valuation ring of characteristic p_1. Thus

$$|\Delta(\boldsymbol{\Psi}; \widetilde{\mathcal{O}})| = 0.$$

Case 2: Suppose that $\mathrm{char}(\mathcal{O}) = 0$. Then the elements in the Jacobian matrix of the system $\boldsymbol{\Psi}$ lie in $\mathbb{Z}[\mathbf{x}_1, \dots, \mathbf{x}_\varrho]$. Thus we may regard the determinant of every $\varrho \times \varrho$ submatrix as the zero polynomial in $\mathbb{Z}[\mathbf{x}_1, \dots, \mathbf{x}_\varrho]$ and so in $\widetilde{\mathcal{O}}[\mathbf{x}_1, \dots, \mathbf{x}_\varrho]$ with $\mathrm{char}\widetilde{\mathcal{O}} = 0$. Hence

$$|\Delta(\boldsymbol{\Psi}; \widetilde{\mathcal{O}})| = 0.$$

On combining the two cases, the lemma follows. □

We are now prepared to prove the existence of nonsingular weight.

Proposition 6.1. *Let $k, d \in \mathbb{N}$ with $k, d \geq 2$. Suppose that $\mathrm{char}(F) \nmid k$. Then one has*

$$|\Delta(\boldsymbol{\Psi}; \mathcal{O})| \neq 0.$$

Proof. Suppose that $|\Delta(\boldsymbol{\Psi}; \mathcal{O})| = 0$. By Lemma 6.5, for any complete discrete valuation ring $\widetilde{\mathcal{O}}$ with the same characteristic as $\mathcal{O}$, we have

$$|\Delta(\boldsymbol{\Psi}; \widetilde{\mathcal{O}})| = 0.$$

When $\mathrm{char}(\mathcal{O}) = p_1$ with $\mathrm{char}(F) \nmid k$, we may choose $\widetilde{\mathcal{O}}$ with residue field $\widetilde{F}$ satisfying that $\mathrm{char}(\widetilde{F}) = p_1$ and $\mathrm{card}\widetilde{F} > k$. This contradicts Lemma 6.4.

When $\mathrm{char}(\mathcal{O}) = 0$, we may take $\widetilde{\mathcal{O}} = \mathbb{Z}_{\widetilde{p}}$ with $\widetilde{p} > k$. Since the residue field is $\mathbb{F}_{\widetilde{p}}$, Lemma 6.4 implies that

$$\widetilde{p} \leq k,$$

a contradiction again. We then finish the proof of this proposition. □

We are now in a position to prove Theorem 1.5.

Proof of Theorem 1.5. On combining Proposition 3.1 with Theorem 5.3 and Proposition 6.1, the result follows immediately. □

7. Local obstructions

Let K, $\mathcal{O}$, π and F be defined as in Section 3. The uniform local density hypothesis works only for elements in $\mathcal{O}(\mathcal{M})$. We will refer to the elements in $\mathcal{O}^{\varrho} \setminus \mathcal{O}(\mathcal{M})$ as *local obstructions*. In this section, we aim to show that no local obstruction exists, except for the case when $\mathcal{O}$ has small residue field F. In what follows, we define $F(\mathcal{M})$ as in (1.5).

Proposition 7.1. *Suppose that* $\mathrm{char}(F) \nmid k$. *Let* $u_0 \in \mathbb{N}$ *and* $r_0 \in \mathbb{N}$ *be defined as in* (3.2). *The following hold.*
(1) *Let* $\mathfrak{f} = (\mathfrak{f}_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}} \in \mathcal{O}^{\varrho}$. *Then* $\mathfrak{f} \in \mathcal{O}(\mathcal{M})$ *if and only if there exists* $\mathfrak{g} = (\mathfrak{g}_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}} \in \mathcal{O}(\mathcal{M})$ *such that*

$$\mathfrak{f}_{\mathbf{i}} \equiv \mathfrak{g}_{\mathbf{i}} \pmod{\pi^{r_0}} \quad (\mathbf{i} \in \mathcal{M}).$$

(2) *Suppose that* $u_0 = 0$. *Then* $\mathcal{O}(\mathcal{M}) = \mathcal{O}^{\varrho}$ *if and only if* $F(\mathcal{M}) = F^{\varrho}$.

Proof. (1) follows from Proposition 3.1(1) immediately. By putting $u_0 = 0$ and taking $r_0 = 1$, (2) holds as a special case of (1). □

Lemma 7.1. *Let* $c_{\mathbf{i}} \in F$ *for each* $\mathbf{i} \in \mathcal{M}$. *Suppose that* $\sum_{\mathbf{i} \in \mathcal{M}} c_{\mathbf{i}} \mathbf{a}^{\mathbf{i}} = 0$ *for all* $\mathbf{a} \in F^d$. *If* $\mathrm{card}(F) > k$, *then* $c_{\mathbf{i}} = 0$ *for all* $\mathbf{i} \in \mathcal{M}$.

Proof. Suppose that the $c_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{M}$) are not all zero. Let $\mathcal{N}_0 = \mathcal{M}$ and $\mathcal{N}_l = \{\mathbf{i} \in \mathcal{M} \mid i_1 = 0, \ldots, i_l = 0\}$ for $1 \leq l \leq d-1$. Let $\varphi_l(\mathbf{x}) = \sum_{\mathbf{i} \in \mathcal{N}_l} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ for $0 \leq l \leq d-1$. We now claim that for each l with $0 \leq l \leq d-2$, if $\varphi_l(\mathbf{a}) = 0$ for all $\mathbf{a} \in F^d$ and the $c_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{N}_l$) are not all zero, then $\varphi_{l+1}(\mathbf{a}) = 0$ for all $\mathbf{a} \in F^d$ and the $c_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{N}_{l+1}$) are not all zero. By repeatedly applying this claim, we have $\varphi_{d-1}(\mathbf{a}) = 0$ for all $\mathbf{a} \in F^d$ and the $c_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{N}_{d-1}$) are not all zero. Since $\mathcal{N}_{d-1} = \{\mathbf{i} \in \mathcal{M} \mid i_1 = 0, \ldots, i_{d-1} = 0\} = \{(0, \ldots, 0, k)\}$, we have $\varphi_{d-1}(\mathbf{a}) = c_{(0,\ldots,0,k)} a_1^0 \cdots a_{d-1}^0 a_d^k = 0$ for all $a_1, \ldots, a_d \in F$ and $c_{(0,\ldots,0,k)} \neq 0$. This gives a contradiction.

It remains to show the claim. For all $\mathbf{a} \in F^d$, since $\varphi_l(\mathbf{a}) = 0$ we have

$$\varphi_{l+1}(\mathbf{a}) = \varphi_{l+1}(\mathbf{a}) - \varphi_l(\mathbf{a}) = -\sum_{\mathbf{i}\in\mathcal{N}_l\setminus\mathcal{N}_{l+1}} c_{\mathbf{i}}\mathbf{a}^{\mathbf{i}}.$$

For each $\mathbf{i} = (i_1, \dots, i_d) \in \mathcal{N}_l \setminus \mathcal{N}_{l+1}$, we have $i_1 = \cdots = i_l = 0$ and $i_{l+1} > 0$. For all $\mathbf{a} = (a_1, \dots, a_d) \in F^d$, since

$$\varphi_{l+1}(a_1, \dots, a_d) = \sum_{\mathbf{i}\in\mathcal{N}_{l+1}} c_{\mathbf{i}} a_1^0 \cdots a_{l+1}^0 a_{l+2}^{i_{l+2}} \cdots a_d^{i_d} = \sum_{\mathbf{i}\in\mathcal{N}_{l+1}} c_{\mathbf{i}} 0^0 \cdots 0^0 a_{l+2}^{i_{l+2}} \cdots a_d^{i_d},$$

it follows that

$$\varphi_{l+1}(a_1, \dots, a_d) = \varphi_{l+1}(0, \dots, 0, a_{l+2}, \dots, a_d) = -\sum_{\mathbf{i}\in\mathcal{N}_l\setminus\mathcal{N}_{l+1}} c_{\mathbf{i}} 0^0 \cdots 0^{i_{l+1}} a_{l+2}^{i_{l+2}} \cdots a_d^{i_d} = 0.$$

Assume that the $c_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{N}_{l+1}$) are all zero. Then the coefficients $c_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{N}_l \setminus \mathcal{N}_{l+1}$) are not all zero. On viewing $-\varphi_{l+1}(\mathbf{x})$ as a linear combination of the group homomorphisms $\mathbf{x}^{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{N}_l \setminus \mathcal{N}_{l+1}$) from $(F^\times)^d$ to $F^\times$, it follows from Lemma 6.1 that there exist distinct $\mathbf{i}, \mathbf{j} \in \mathcal{N}_l \setminus \mathcal{N}_{l+1}$ such that $\mathbf{a}^{\mathbf{i}} = \mathbf{a}^{\mathbf{j}}$ for all $\mathbf{a} \in (F^\times)^d$. By Lemma 6.2, we have

$$\operatorname{card}(F^\times) \le |i_{l+1} - j_{l+1}| \le k - 1,$$

contradicting that $\operatorname{card}(F) > k$. This completes the proof of the lemma. □

Proposition 7.2. *Suppose that* $\operatorname{card}(F) > k$. *Then* $F(\mathcal{M}) = F^\varrho$ *if and only if* $F(k) = F$.

Proof. Suppose that $F(\mathcal{M}) = F^\varrho$. Let $a \in F$ and $(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in F^\varrho$ where

$$a_{\mathbf{i}} = \begin{cases} a, & \text{if } \mathbf{i} = (k, 0, \dots, 0), \\ 0, & \text{otherwise.} \end{cases}$$

Since $F^\varrho = F(\mathcal{M})$, there exist $\mathbf{a}_j = (a_{j,1}, \dots, a_{j,d}) \in F^d$ $(1 \le j \le n)$ such that

$$a_{\mathbf{i}} = \sum_{j=1}^{n} a_{j,1}^{i_1} \cdots a_{j,d}^{i_d} \quad (\mathbf{i} \in \mathcal{M}).$$

When $\mathbf{i} = (k, 0, \dots, 0)$, we get

$$a = \sum_{j=1}^{n} a_{j,1}^{k} \in F(k).$$

Thus $F \subseteq F(k)$. In combination with $F(k) \subseteq F$, we have $F(k) = F$.

Conversely, suppose that $F(k) = F$. Let A be the matrix over F formed by the vectors $\Omega_1 = \{(a_1^{i_1} \cdots a_d^{i_d})_{\mathbf{i}\in\mathcal{M}} \mid a_1, \ldots, a_d \in F\}$. Assume that $\operatorname{rk} A < \varrho$. Then there exist $c_{\mathbf{i}} \in F$, not all zero, such that for all $\mathbf{a} \in F^d$, one has

$$\sum_{\mathbf{i}\in\mathcal{M}} c_{\mathbf{i}}\mathbf{a}^{\mathbf{i}} = 0,$$

contradicting Lemma 7.1. Thus $\operatorname{rk} A = \varrho$ and so we can find a basis of the vector space F^ϱ from the set Ω_1. Since $\Omega_1 \subseteq F(\mathcal{M})$, $F(\mathcal{M})$ contains a basis of F^ϱ. Recall that $F(\mathcal{M})$ is a vector space over $F(k)$. When $F(k) = F$, we have $F(\mathcal{M}) = F^\varrho$. This completes the proof of the proposition. □

Lemma 7.2. *Suppose that* $\operatorname{char}(F) \nmid k$. *Let* $u_0 \in \mathbb{N}$ *be defined as in* (3.2)*. Suppose that* $u_0 > 0$. *Then* $\operatorname{card}(F) \le k$.

Proof. If $u_0 > 0$, then $|\Delta(\boldsymbol{\Psi}; \mathcal{O})| = |\pi|^{u_0} < 1$. The result follows from Lemma 6.4 immediately. □

Proposition 7.3. *Suppose that* $\operatorname{char}(F) \nmid k$ *and* $\operatorname{card}(F) > k$. *Then* $\mathcal{O}(\mathcal{M}) = \mathcal{O}^\varrho$ *if and only if* $F(k) = F$.

Proof. It follows from Lemma 7.2 that $u_0 = 0$. Thus Proposition 7.1(3) implies that $\mathcal{O}(\mathcal{M}) = \mathcal{O}^\varrho$ if and only if $F(\mathcal{M}) = F^\varrho$. We then conclude from Proposition 7.2 that $\mathcal{O}(\mathcal{M}) = \mathcal{O}^\varrho$ if and only if $F(k) = F$. □

If $\operatorname{card}(F) > k$ and $\mathcal{O}(\mathcal{M}) \neq \mathcal{O}^\varrho$, then $F(k) \neq F$. Next, we aim to discuss the size of the residue field F in this case.

Lemma 7.3. *Suppose that* $F(k) \neq F$. *Then* $\operatorname{card}(F) \le (k-1)^2$.

Proof. This follows from Theorem 4.1(4) immediately. □

We are now in a position to prove Theorem 1.6.

Proof of Theorem 1.6. When $k \ge 3$ and $\operatorname{card}(F) > (k-1)^2$, by Lemma 7.3, we have $F(k) = F$. Since $(k-1)^2 > k$ for all $k \ge 3$, it thus follows from Proposition 7.3 that $\mathcal{O}(\mathcal{M}) = \mathcal{O}^\varrho$. When $k = 2$, it follows from Corollary 4.1 that $F(k) = F$. Since $\operatorname{char}(F) \nmid k$, we have $\operatorname{card}(F) \ge \operatorname{char}(F) \ge 3 > k$. By applying Proposition 7.3 again, we get $\mathcal{O}(\mathcal{M}) = \mathcal{O}^\varrho$. This completes the proof of the theorem. □

We now end this section by applying Proposition 7.1 to show the existence of P defined in (1.11).

Corollary 7.1. *Let* $\mathbf{m} = (m_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{A}^{\varrho}$ *and* $T_0 = C + \min\{N \in \mathbb{N} \mid k(N-1) \geq \deg m_{\mathbf{i}}\ (\mathbf{i} \in \mathcal{M})\}$ *where* $C = 1 + 2u_0(\boldsymbol{\Psi}; \mathbb{A}_\infty)$*. Suppose that* $\mathrm{char}(\mathbb{F}_q) \nmid k$*. Then* $t^{k(1-T_0)}\mathbf{m} \in \mathbb{A}_\infty(\mathcal{M})$*. In addition, define* $T = T(\mathbf{m})$ *as in* (1.11)*. Then* $T \leq T_0$ *and*

$$T \geq \min\{N \in \mathbb{N} \mid k(N-1) \geq \deg m_{\mathbf{i}}\ (\mathbf{i} \in \mathcal{M})\}.$$

Proof. In view of definitions of T_0 and C, for each $\mathbf{i} \in \mathcal{M}$, we have

$$\mathrm{ord}\left(t^{k(1-T_0)} m_{\mathbf{i}}\right) = k(1-T_0) + \mathrm{ord}\, m_{\mathbf{i}} \leq -kC,$$

and so

$$t^{k(1-T_0)} m_{\mathbf{i}} \equiv 0 \pmod{(t^{-1})^C}.$$

It thus follows from Proposition 7.1 that $t^{k(1-T_0)}\mathbf{m} \in \mathbb{A}_\infty(\mathcal{M})$. For each $\mathbf{i} \in \mathcal{M}$, we have

$$k(T-1) \geq \deg m_{\mathbf{i}}\ (\mathbf{i} \in \mathcal{M}).$$

Therefore

$$T \geq \min\{N \in \mathbb{N} \mid k(N-1) \geq \deg m_{\mathbf{i}}\ (\mathbf{i} \in \mathcal{M})\}. \quad \square$$

Remark. This relation actually is consistent with the 1-dimensional case considered in [24]. More precisely, for $m \in \mathbb{A}$, let $c(m)$ denote the leading coefficient of m. We say that m is exceptional if $k | \deg m$ and $c(m) \notin \mathbb{F}_q(k)$. Let $R = R_k(m) + 1$ where

$$R_k(m) = \begin{cases} \lceil (\deg m)/k \rceil, & \text{if } m \text{ is not exceptional,} \\ (\deg m)/k + 1, & \text{if } m \text{ is exceptional.} \end{cases}$$

The following proposition implies that

$$R = \min\{N \in \mathbb{N} \mid t^{k(1-N)} m \in \mathbb{A}_\infty(k)\}.$$

Proposition 7.4. *Let* $m \in \mathbb{A}$ *and* $N \in \mathbb{N}$*. Suppose that* $\mathrm{char}(\mathbb{F}_q) \nmid k$*. Then the following are equivalent.*

(1) $t^{k(1-N)} m \in \mathbb{A}_\infty(k)$.
(2) $t^{k(1-N)} m \in \mathbb{A}_\infty$ *and* $t^{k(1-N)} m \pmod{t^{-1}} \in \mathbb{F}_q(k)$.
(3) *When* m *is not exceptional,* $N \geq \lceil (\deg m)/k \rceil + 1$*; otherwise,* $N \geq \lceil (\deg m)/k \rceil + 2$*.*

Proof. (1)⇔(2) It follows from Proposition 3.3(1) when $\mathcal{O} = \mathbb{A}_\infty = \mathbb{F}_q[\![1/t]\!]$ and $\pi = t^{-1}$ with residue field $F = \mathbb{F}_q$.

(2)⇔(3) First off, note that $t^{k(1-N)} m \in \mathbb{A}_\infty$ if and only if $k(1-N) + \deg m \leq 0$, namely,

$$N \geq \lceil (\deg m)/k \rceil + 1.$$

It suffices to show that when $N \geq \lceil (\deg m)/k \rceil + 1$, $t^{k(1-N)}m \pmod{t^{-1}} \notin \mathbb{F}_q(k)$ if and only if m is exceptional and $N = \lceil (\deg m)/k \rceil + 1$. Recall that $c(m)$ denotes the leading coefficient of m. We then have

$$t^{k(1-N)}m \pmod{t^{-1}} = \begin{cases} 0, & N = \lceil (\deg m)/k \rceil + 2, \\ 0, & N = \lceil (\deg m)/k \rceil + 1, k \nmid \deg m, \\ c(m), & N = \lceil (\deg m)/k \rceil + 1, k \mid \deg m. \end{cases}$$

Thus $t^{k(1-N)}m \pmod{t^{-1}} \notin \mathbb{F}_q(k)$ if and only if m and N satisfy the following three conditions simultaneously: $c(m) \notin \mathbb{F}_q(k)$, $N = \lceil (\deg m)/k \rceil + 1$ and $k \mid \deg m$. On recalling that m is defined to be exceptional when $c(m) \notin \mathbb{F}_q(k)$ and $k \mid \deg m$, we can conclude that $t^{k(1-N)}m \pmod{t^{-1}} \notin \mathbb{F}_q(k)$ if and only if m is exceptional and $N = \lceil (\deg m)/k \rceil + 1$. This completes the proof of the proposition. □

8. Local densities at finite places and singular series

We begin with introducing a standard additive character for function fields, used throughout Sections 8, 9 and 10. First off, write $\mathbb{K} = \mathbb{F}_q(t)$ be the field of fractions of $\mathbb{A}$ and let $\mathbb{K}_\infty = \mathbb{F}_q((1/t))$ be the completion of $\mathbb{K}$ at ∞. We may write each $\alpha \in \mathbb{K}_\infty$ as $\alpha = \sum_{i \leq v} a_i t^i$ for some $v \in \mathbb{Z}$ and $a_i = a_i(\alpha) \in \mathbb{F}_q$ $(i \leq v)$, where a_{-1} is often referred to as the residue of α, denoted by $\operatorname{res}\alpha$. If $a_v \neq 0$, we define $\operatorname{ord}\alpha = v$ and write $\langle \alpha \rangle = q^{\operatorname{ord}\alpha}$. We adopt the convention that $\operatorname{ord} 0 = -\infty$. Let $p = \operatorname{char}(\mathbb{F}_q)$ and let $\operatorname{tr} : \mathbb{F}_q \to \mathbb{F}_p$ denote the familiar trace map. There is a non-trivial additive character $e_q : \mathbb{F}_q \to \mathbb{C}^\times$ defined by taking $e_q(a) = e^{2\pi i \operatorname{tr}(a)/p}$ for each $a \in \mathbb{F}_q$. This character induces a map $e : \mathbb{K}_\infty \to \mathbb{C}^\times$ by defining $e(\alpha) = e_q(\operatorname{res}\alpha)$ for each element $\alpha \in \mathbb{K}_\infty$. Let $\mathcal{M}$ and ϱ be defined as in (1.2). Let $s \in \mathbb{N} \setminus \{0\}$. For $\boldsymbol{\alpha} = (\alpha_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}}$, $\boldsymbol{\nu} = (\nu_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}}$, and $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_s)$ where $\mathbf{x}_j = (x_{j,1}, \ldots, x_{j,d})$, write

$$G(\boldsymbol{\alpha}; \mathbf{x}; \boldsymbol{\nu}) = \sum_{\mathbf{i} \in \mathcal{M}} \alpha_{\mathbf{i}} \big(\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} - \nu_{\mathbf{i}} \big). \tag{8.1}$$

In this section, we assume that $\operatorname{char}(\mathbb{F}_q) \nmid k$. In addition, for $X \in \mathbb{R}$, let $\widehat{X} = q^X$. For each $w \in \mathcal{P}$, let $\mathbb{A}_w$ denote the completion of $\mathbb{A}$ at the place w and define $\mathbb{A}_w(\mathcal{M})$ by (1.5). For $h \in \mathbb{N}$, $w \in \mathcal{P}$ and $\mathbf{m} \in \mathbb{A}_w(\mathcal{M})$, recall that

$$\lambda_{q,s,k,d}(h; w; \mathbf{m}) = \langle w \rangle^{h(\varrho - sd)} \cdot \operatorname{card}\big\{ \mathbf{x} \pmod{g} \big| \, \mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} \equiv m_{\mathbf{i}} \pmod{g} \ (\mathbf{i} \in \mathcal{M}) \big\}.$$

To prove that when h goes to ∞, the limit of $\lambda_{q,s,k,d}(h; w; \mathbf{m})$ exists, we start with the following congruences and rational exponential sums. For monic $g \in \mathbb{A}$ and $\mathbf{m} = (m_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}} \in \mathbb{A}^{\varrho}$, let

$$M(g;\mathbf{m}) = \mathrm{card}\big\{\mathbf{x}\,(\mathrm{mod}\,g)\big|\,\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} \equiv m_{\mathbf{i}}\ (\mathrm{mod}\,g)\ (\mathbf{i} \in \mathcal{M})\big\}. \tag{8.2}$$

For simplicity, given $\mathbf{a} = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{A}^{\varrho}$ and $g \in \mathbb{A}$, denote by $(\mathbf{a}, g)$ the monic greatest common divisor of g and all the $a_{\mathbf{i}}$. For monic $g \in \mathbb{A}$ and $\mathbf{m} \in \mathbb{A}^{\varrho}$, define

$$S(g;\mathbf{m}) = \langle g\rangle^{-sd} \sum_{\substack{\mathbf{a}\ (\mathrm{mod}\ g) \\ (\mathbf{a},g)=1}} \sum_{\mathbf{x}\ (\mathrm{mod}\ g)} e\left(\frac{G(\mathbf{a};\mathbf{x};\mathbf{m})}{g}\right). \tag{8.3}$$

Therefore, for $h \in \mathbb{N}$, $w \in \mathcal{P}$ and $\mathbf{m} \in \mathbb{A}^{\varrho} \cap \mathbb{A}_w(\mathcal{M})$, we have

$$\lambda_{q,s,k,d}(h;w;\mathbf{m}) = \langle w\rangle^{h(\varrho-sd)} \cdot M(w^h;\mathbf{m}). \tag{8.4}$$

To continue, some preliminaries are now required.

Lemma 8.1. *The exponential function $e : \mathbb{K}_\infty \to \mathbb{C}^\times$ has the following properties.*

(1) *e is a continuous function.*
(2) *$e(\alpha + \beta) = e(\alpha)e(\beta)$.*
(3) *$e(x) = 1$, if $x \in \mathbb{A}$.*
(4) *If $x, g \in \mathbb{A}$ with $g \neq 0$, then*

$$\frac{1}{\langle g\rangle} \sum_{a (\mathrm{mod}\ g)} e\left(\frac{xa}{g}\right) = \begin{cases} 1, & \text{if } g \mid x, \\ 0, & \text{if } g \nmid x. \end{cases}$$

(5) *For monic $g \in \mathbb{A}$ and $\mathbf{m} \in \mathbb{A}^{\varrho}$, one has*

$$\sum_{\substack{g_1 | g \\ g_1 \text{monic}}} S(g_1;\mathbf{m}) = \langle g\rangle^{\varrho - sd} M(g;\mathbf{m}).$$

(6) *For monic $g \in \mathbb{A}$ and $\mathbf{m} \in \mathbb{A}^{\varrho}$, for any small positive number $\epsilon > 0$, one has*

$$S(g;\mathbf{m}) \ll \langle g\rangle^{\varrho - \frac{s}{2k} + \epsilon}.$$

Proof. The first four items are part of [22, Lemma 1]. It remains to show that last two parts.

(5) Let g be a monic polynomial in $\mathbb{A}$. For every $\mathbf{i} \in \mathcal{M}$, by Part (4), we have

$$\langle g\rangle^{-1} \sum_{a_{\mathbf{i}}\ (\mathrm{mod}\ g)} e\left(\frac{a_{\mathbf{i}}\big(\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} - m_{\mathbf{i}}\big)}{g}\right) = \begin{cases} 1, & \text{if } \mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} \equiv m_{\mathbf{i}}\,(\mathrm{mod}\,g), \\ 0, & \text{otherwise.} \end{cases}$$

On recalling (8.1) and (8.2), it then follows from Part (2) that

$$M(g;\mathbf{m}) = \sum_{\mathbf{x} \,(\mathrm{mod}\, g)} \prod_{\mathbf{i}\in\mathcal{M}} \langle g\rangle^{-1} \sum_{a_{\mathbf{i}} \,(\mathrm{mod}\, g)} e\left(\frac{a_{\mathbf{i}}\left(\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} - m_{\mathbf{i}}\right)}{g}\right)$$
$$= \langle g\rangle^{-\varrho} \sum_{\substack{\mathbf{x} \,(\mathrm{mod}\, g)\\ \mathbf{a} \,(\mathrm{mod}\, g)}} e\left(\frac{G(\mathbf{a};\mathbf{x};\mathbf{m})}{g}\right).$$

From Parts (2) and (3), we then have

$$M(g;\mathbf{m}) = \langle g\rangle^{-\varrho} \sum_{\substack{g_1|g\\ g_1 \text{ monic}}} \sum_{\substack{\mathbf{a} \,(\mathrm{mod}\, g)\\ (\mathbf{a},g)=g_1}} \sum_{\mathbf{x} \,(\mathrm{mod}\, g)} e\left(\frac{G(\mathbf{a};\mathbf{x};\mathbf{m})}{g}\right)$$
$$= \langle g\rangle^{-\varrho} \sum_{\substack{g_1|g\\ g_1 \text{ monic}}} \sum_{\substack{\mathbf{b} \,(\mathrm{mod}\, g/g_1)\\ (\mathbf{b},g/g_1)=1}} \langle g_1\rangle^{sd} \sum_{\mathbf{x} \,(\mathrm{mod}\, g/g_1)} e\left(\frac{G(\mathbf{b};\mathbf{x};\mathbf{m})}{g/g_1}\right)$$
$$= \langle g\rangle^{-\varrho} \sum_{\substack{g_1|g\\ g_1 \text{ monic}}} \langle g/g_1\rangle^{sd} \sum_{\substack{\mathbf{b} \,(\mathrm{mod}\, g_1)\\ (\mathbf{b},g_1)=1}} \sum_{\mathbf{x} \,(\mathrm{mod}\, g_1)} e\left(\frac{G(\mathbf{b};\mathbf{x};\mathbf{m})}{g_1}\right)$$
$$= \langle g\rangle^{sd-\varrho} \sum_{\substack{g_1|g\\ g_1 \text{ monic}}} \langle g_1\rangle^{-sd} \sum_{\substack{\mathbf{b} \,(\mathrm{mod}\, g_1)\\ (\mathbf{b},g_1)=1}} \sum_{\mathbf{x} \,(\mathrm{mod}\, g_1)} e\left(\frac{G(\mathbf{b};\mathbf{x};\mathbf{m})}{g_1}\right).$$

In view of (8.3), we thus find

$$\sum_{\substack{g_1|g\\ g_1 \text{monic}}} S(g_1;\mathbf{m}) = \langle g\rangle^{\varrho-sd} M(g;\mathbf{m}).$$

(6) Let $\mathbf{a} = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{A}^{\varrho}$ and $g \in \mathbb{A}\setminus\{0\}$. If $\mathrm{char}(\mathbb{F}_q) \nmid k$ and $(\mathbf{a},g) = 1$, it then follows from [47, Corollary 1.1] that

$$\sum_{\mathbf{z} \,(\mathrm{mod}\, g)} e\left(\frac{1}{g}\sum_{\mathbf{i}\in\mathcal{M}} a_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}\right) \ll \langle g\rangle^{d-1/(2k)+\epsilon},$$

where $\mathbf{z} \in \mathbb{A}^d$ runs through all congruence classes modulo g. We then note that

$$\left|\sum_{\mathbf{x} \,(\mathrm{mod}\, g)} e\left(\frac{G(\mathbf{a};\mathbf{x};\mathbf{m})}{g}\right)\right| = \left|\sum_{\mathbf{x} \,(\mathrm{mod}\, g)} e\left(\frac{G(\mathbf{a};\mathbf{x};\mathbf{0})}{g}\right)\right| = \left|\sum_{\mathbf{z} \,(\mathrm{mod}\, g)} e\left(\frac{\sum_{\mathbf{i}\in\mathcal{M}} a_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}}{g}\right)\right|^{s}.$$

When $(\mathbf{a},g) = 1$, we have

$$\left|\sum_{\mathbf{x} \,(\mathrm{mod}\, g)} e\left(\frac{G(\mathbf{a};\mathbf{x};\mathbf{m})}{g}\right)\right| \ll \langle g\rangle^{sd-\frac{s}{2k}+\epsilon}.$$

On recalling that $S(g;\mathbf{m})$ is defined by (8.3), we then find

$$\big|S(g;\mathbf{m})\big| \ll \langle g\rangle^{\varrho-\frac{s}{2k}+\epsilon}.$$

This completes the proof of the lemma. □

Proposition 8.1. *Let $w \in \mathcal{P}$ and $\mathbf{m} \in \mathbb{A}^{\varrho} \cap \mathbb{A}_w(\mathcal{M})$. Suppose that $s \geq 2k(\varrho+1)+1$. Define*

$$\chi(w;\mathbf{m}) = \chi_{q,s,k,d}(w;\mathbf{m}) = \lim_{h\to\infty} \lambda_{q,s,k,d}(h;w;\mathbf{m}).$$

Then the following hold.

(1) *The limit $\chi(w;\mathbf{m})$ exists and*

$$\chi(w;\mathbf{m}) = \sum_{h=0}^{\infty} S(w^h;\mathbf{m}).$$

(2) *One has*

$$\big|\chi(w;\mathbf{m})-1\big| \ll \langle w\rangle^{\varrho-\frac{s}{2k}+\epsilon}.$$

(3) *There exists a constant $c = c(s,k,d;w)$ such that*

$$\chi(w;\mathbf{m}) \geq \langle w\rangle^{c(\varrho-sd)}.$$

Proof. For $h \in \mathbb{N}$, on recalling (8.4) and taking $g = w^h$, it follows from Lemma 8.1(5) that

$$\lambda_{q,s,k,d}(h;w;\mathbf{m}) = \langle w^h\rangle^{\varrho-sd} \cdot M(w^h;\mathbf{m}) = \sum_{l=0}^{h} S(w^l;\mathbf{m}).$$

Thus

$$\chi(w;\mathbf{m}) = \sum_{h=0}^{\infty} S(w^h;\mathbf{m}).$$

On taking $g = w^h$, when $s \geq 2k(\varrho+1)+1$, by Lemma 8.1(6), we get

$$S(w^h;\mathbf{m}) \ll \langle w^h\rangle^{\varrho-\frac{s}{2k}+\epsilon} \quad (h \in \mathbb{N}).$$

We therefore obtain the absolute convergence of the series

$$\chi(w;\mathbf{m}) = \sum_{h=0}^{\infty} S(w^h;\mathbf{m})$$

and

$$|\chi(w;\mathbf{m})-1|\leq\sum_{h=1}^{\infty}|S(w^h;\mathbf{m})|\ll\langle w\rangle^{\varrho-\frac{s}{2k}+\epsilon}.$$

Since $s\geq 2k(\varrho+1)+1\geq(k+2)\varrho$, it follows from Corollary 1.1 that there exists a constant $c=c(s,k,d;w)$ such that

$$\chi(w;\mathbf{m})=\lim_{h\to\infty}\lambda_{q,s,k,d}(h;w;\mathbf{m})\geq\langle w\rangle^{c(\varrho-sd)}.$$

This completes the proof of the proposition. □

In the remainder of this section, we aim to transform the infinite product of local densities at finite places to singular series.

Lemma 8.2. *Fix $\mathbf{m}\in\mathbb{A}^{\varrho}$. Then the function $S(g;\mathbf{m})$ is multiplicative with respect to g.*

Proof. Suppose that g_1 and g_2 are monic polynomials in $\mathbb{A}$ with $(g_1,\ g_2)=1$. Thus

$$S(g_1g_2;\mathbf{m})=\langle g_1g_2\rangle^{-sd}\sum_{\substack{\mathbf{a}\ (\mathrm{mod}\ g_1g_2)\\(\mathbf{a},g_1g_2)=1}}\ \sum_{\mathbf{x}\ (\mathrm{mod}\ g_1g_2)}e\left(\frac{G(\mathbf{a};\mathbf{x};\mathbf{m})}{g_1g_2}\right).$$

As $\mathbf{b}_i$ runs through $\mathbb{A}^{\varrho}(\mathrm{mod}\,g_i)$ with $(\mathbf{b}_i,g_i)=1$ $(i=1,2)$, by the Chinese Remainder Theorem, $\big(g_2\mathbf{b}_1+g_1\mathbf{b}_2\big)$ runs through

$$\big\{\mathbf{a}\,(\mathrm{mod}\,g_1g_2)\big|\,(\mathbf{a},g_1g_2)=1\big\}.$$

Therefore,

$$S(g_1g_2;\mathbf{m})=\langle g_1g_2\rangle^{-sd}\sum_{\substack{\mathbf{b}_1\ (\mathrm{mod}\ g_1)\\(\mathbf{b}_1,g_1)=1}}\ \sum_{\substack{\mathbf{b}_2\ (\mathrm{mod}\ g_2)\\(\mathbf{b}_2,g_2)=1}}\ \sum_{\mathbf{x}\ (\mathrm{mod}\ g_1g_2)}e\left(\frac{G(g_2\mathbf{b}_1+g_1\mathbf{b}_2;\mathbf{x};\mathbf{m})}{g_1g_2}\right).$$

Note that

$$G(g_2\mathbf{b}_1+g_1\mathbf{b}_2;\mathbf{x};\mathbf{m})=G(g_2\mathbf{b}_1;\mathbf{x};\mathbf{m})+G(g_1\mathbf{b}_2;\mathbf{x};\mathbf{m})=g_2G(\mathbf{b}_1;\mathbf{x};\mathbf{m})+g_1G(\mathbf{b}_2;\mathbf{x};\mathbf{m}).$$

We then have

$$e\left(\frac{G(g_2\mathbf{b}_1+g_1\mathbf{b}_2;\mathbf{x};\mathbf{m})}{g_1g_2}\right)=e\left(\frac{G(\mathbf{b}_1;\mathbf{x};\mathbf{m})}{g_1}\right)e\left(\frac{G(\mathbf{b}_2;\mathbf{x};\mathbf{m})}{g_2}\right).$$

As $\mathbf{y},\mathbf{z}$ run through $\mathbb{A}^{sd}(\mathrm{mod}\,g_i)$ $(i=1,2)$ respectively, by the Chinese Remainder Theorem, $\big(g_2\mathbf{y}+g_1\mathbf{z}\big)$ runs through $\mathbb{A}^{sd}(\mathrm{mod}\,g_1g_2)$. Since

$$e\left(\frac{G(\mathbf{b}_1; g_2\mathbf{y} + g_1\mathbf{z}; \mathbf{m})}{g_1}\right) = e\left(\frac{G(\mathbf{b}_1; g_2\mathbf{y}; \mathbf{m})}{g_1}\right)$$

and

$$e\left(\frac{G(\mathbf{b}_2; g_2\mathbf{y} + g_1\mathbf{z}; \mathbf{m})}{g_2}\right) = e\left(\frac{G(\mathbf{b}_2; g_1\mathbf{z}; \mathbf{m})}{g_2}\right),$$

it follows that

$$\begin{aligned} &S(g_1g_2; \mathbf{m}) \\ &= \langle g_1g_2\rangle^{-sd} \sum_{\substack{\mathbf{b}_1 \pmod{g_1} \\ (\mathbf{b}_1, g_1)=1}} \sum_{\substack{\mathbf{b}_2 \pmod{g_2} \\ (\mathbf{b}_2, g_2)=1}} \sum_{\substack{\mathbf{y} \pmod{g_1} \\ \mathbf{z} \pmod{g_2}}} e\left(\frac{G(\mathbf{b}_1; g_2\mathbf{y}; \mathbf{m})}{g_1}\right) e\left(\frac{G(\mathbf{b}_2; g_1\mathbf{z}; \mathbf{m})}{g_2}\right). \end{aligned}$$

In view of the hypothesis that $(g_1, g_2) = 1$, we find

$$S(g_1g_2; \mathbf{m}) = \langle g_1g_2\rangle^{-sd} \sum_{\substack{\mathbf{y} \pmod{g_1} \\ \mathbf{b}_1 \pmod{g_1} \\ (\mathbf{b}_1, g_1)=1}} e\left(\frac{G(\mathbf{b}_1; \mathbf{y}; \mathbf{m})}{g_1}\right) \sum_{\substack{\mathbf{z} \pmod{g_2} \\ \mathbf{b}_2 \pmod{g_2} \\ (\mathbf{b}_2, g_2)=1}} e\left(\frac{G(\mathbf{b}_2; \mathbf{z}; \mathbf{m})}{g_2}\right).$$

Thus

$$S(g_1g_2; \mathbf{m}) = S(g_1; \mathbf{m})S(g_2; \mathbf{m}).$$

This completes the proof of the lemma. □

Proposition 8.2. *Suppose that $s \geq 2k(\varrho + 1) + 1$. Whenever $\mathbf{m} \in \mathbb{A}^{\varrho} \cap \mathbb{A}_w(\mathcal{M})$ for every $w \in \mathcal{P}$, one has*

$$\prod_{w\in\mathcal{P}} \chi(w; \mathbf{m}) = \sum_{g \text{ monic}} S(g; \mathbf{m}),$$

both of which converge absolutely.

Proof. It follows from Proposition 8.1 that

$$\sum_{w\in\mathcal{P}} \left|\chi(w; \mathbf{m}) - 1\right| \ll \sum_{w\in\mathcal{P}} \langle w\rangle^{\varrho - \frac{s}{2k} + \epsilon} \ll 1.$$

We then obtain that the infinite product $\prod_{w\in\mathcal{P}} \chi(w; \mathbf{m})$ converges absolutely. In combination of the absolute convergence with Proposition 8.1 and Lemma 8.2, we find

$$\prod_{w\in\mathcal{P}} \chi(w; \mathbf{m}) = \prod_{w\in\mathcal{P}} \left(1 + S(w; \mathbf{m}) + S(\mathbf{m}; w^2) + \cdots\right) = \sum_{g \text{ monic}} S(g; \mathbf{m}).$$

This completes the proof of the proposition. □

We now introduce the *singular series* to be

$$\mathfrak{S}(\mathbf{m}) = \mathfrak{S}_{q,s,k,d}(\mathbf{m}) = \sum_{g \text{ monic}} S(g;\mathbf{m}).$$

Also for $Q \in \mathbb{N}$ with $Q > 0$, we define

$$\mathfrak{S}(\mathbf{m};Q) = \mathfrak{S}_{q,s,k,d}(\mathbf{m};Q) = \sum_{\substack{\langle g\rangle \le \widehat{Q} \\ g \text{ monic}}} S(g;\mathbf{m}).$$

Next, we establish the upper and lower bounds of the singular series.

Proposition 8.3. *Suppose that* $s \ge 2k(\varrho+1)+1$.

(1) *Whenever* $\mathbf{m} \in \mathbb{A}^{\varrho}$, *one has*

$$\left|\mathfrak{S}(\mathbf{m})\right| \le \sum_{g \text{ monic}} \left|S(g;\mathbf{m})\right| \ll 1.$$

In addition, for any ϵ *with* $0 < \epsilon < 1/(2k)$, *one has*

$$\left|\mathfrak{S}(\mathbf{m}) - \mathfrak{S}(\mathbf{m};Q)\right| \le \sum_{\substack{\langle g\rangle > \widehat{Q} \\ g \text{ monic}}} \left|S(g;\mathbf{m})\right| \ll \widehat{Q}^{1+\varrho-\frac{s}{2k}+\epsilon}.$$

(2) *Whenever* $\mathbf{m} \in \mathbb{A}^{\varrho} \cap \mathbb{A}_w(\mathcal{M})$ *for every* $w \in \mathcal{P}$, *one has*

$$\mathfrak{S}(\mathbf{m}) = \prod_{w\in\mathcal{P}} \Big(\lim_{h\to\infty} \lambda_{q,s,k,d}(h;w;\mathbf{m}) \Big) \gg 1.$$

Proof. (1) On recalling Lemma 8.1(6), we have

$$\left|S(g;\mathbf{m})\right| \ll \langle g\rangle^{\varrho-\frac{s}{2k}+\epsilon},$$

which implies that

$$\left|\mathfrak{S}(\mathbf{m};Q)\right| \le \sum_{h=0}^{Q} \sum_{\substack{\text{ord } g = h \\ g \text{ monic}}} \left|S(g;\mathbf{m})\right| \ll \sum_{h=0}^{Q} q^{h+h(\varrho-\frac{s}{2k}+\epsilon)} = \sum_{h=0}^{Q} q^{h(1+\varrho-\frac{s}{2k}+\epsilon)}.$$

Note that if $s \ge 2k(\varrho+1)+1$, we obtain $1+\varrho-\frac{s}{2k}+\epsilon < 0$ for any ϵ with $0 < \epsilon < 1/(2k)$. It follows that

$$\left|\mathfrak{S}(\mathbf{m})\right| \le \sum_{g \text{ monic}} \left|S(g;\mathbf{m})\right| \ll \sum_{h=0}^{\infty} q^{h(1+\varrho-\frac{s}{2k}+\epsilon)} \ll 1$$

and

$$\left|\mathfrak{S}(\mathbf{m})-\mathfrak{S}(\mathbf{m};Q)\right| \le \sum_{\substack{\langle g\rangle>\widehat{Q}\\ g \text{ monic}}} \left|S(g;\mathbf{m})\right| \ll \widehat{Q}^{1+\varrho-\frac{s}{2k}+\epsilon}.$$

(2) By carrying over a standard argument as in [24, Lemma 5.2], there exists a constant $C^*=C^*(q,s,k,d)$ such that

$$\frac{1}{2} < \prod_{\substack{w\in\mathcal{P}\\ \operatorname{ord} w>C^*}} \chi(w;\mathbf{m}) < \frac{3}{2}.$$

By Proposition 8.1, there exists a nonnegative integer $c=c(q,s,k,d)$, independent of $\mathbf{m}$, such that

$$\prod_{\substack{w\in\mathcal{P}\\ \operatorname{ord} w\le C^*}} \chi(w;\mathbf{m}) \ge \prod_{\substack{w\in\mathcal{P}\\ \operatorname{ord} w\le C^*}} \langle w\rangle^{c(\varrho-sd)}.$$

On combining the above estimates, we have

$$\prod_{w\in\mathcal{P}} \chi(w;\mathbf{m}) \gg 1.$$

In combination with Proposition 8.2, this completes the proof of the proposition. □

Remark. We actually only need to apply the uniform local density hypothesis for w with $\operatorname{ord} w \le C^*$.

9. Local density at ∞ and singular integral

Let $\mathcal{M}$ and ϱ be defined as in (1.2). Let $s\in\mathbb{N}\setminus\{0\}$. Recall that for $\mathbf{m}\in\mathbb{A}^\varrho$ and $P\ge T(\mathbf{m})$ with $T(\mathbf{m})$ defined by (1.11), $\lambda_{q,s,k,d}(P;\infty;\mathbf{m})$ is defined to be

$$\widehat{P}^{\varrho-sd}\cdot \operatorname{card}\left\{\mathbf{x}\in I_P^{sd} \,\middle|\, \operatorname{ord}\left(\mathbf{x}_1^{\mathbf{i}}+\cdots+\mathbf{x}_s^{\mathbf{i}}-m_{\mathbf{i}}\right)<(k-1)P \ (\mathbf{i}\in\mathcal{M})\right\}.$$

In this section, we still assume that $\operatorname{char}(\mathbb{F}_q)\nmid k$. To be prepared, recall that $\mathbb{A}_\infty=\mathbb{F}_q[\![1/t]\!]$ and $\mathbb{A}_\infty(\mathcal{M})$ is defined by (1.5). Thus $\mathbb{A}_\infty=\{\alpha\in\mathbb{K}_\infty \,|\, \operatorname{ord}\alpha\le 0\}$.

Proposition 9.1. *Let $\mathbf{m}=(m_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}}\in\mathbb{A}^\varrho$ and $P\in\mathbb{N}$ with $P\ge T(\mathbf{m})$. Suppose that $s\ge(k+2)\varrho$ and P is sufficiently large in terms of s,k,d and q. Then there exists $\widetilde{c}=\widetilde{c}(q,s,k,d)>0$ such that*

$$\lambda_{q,s,k,d}(P;\infty;\mathbf{m})\ge q^{\widetilde{c}(\varrho-sd)}.$$

Proof. We now need to take $K = \mathbb{F}_q((1/t))$ for the completion of $\mathbb{F}_q(t)$ at ∞ and $\pi = t^{-1}$. Since $P \geq T(\mathbf{m})$, we have $t^{k(1-P)}\mathbf{m} \in \mathbb{A}_\infty(\mathcal{M})$. On recalling Definition 1.2, for $h > 0$, we have

$$\begin{aligned}&\lambda_{s,k,d}(h; t^{-1}; t^{k(1-P)}\mathbf{m})\\ =&q^{h(\varrho - sd)} \cdot \mathrm{card}\big\{\mathbf{y} \ (\mathrm{mod}\, t^{-h}) \ \big|\ \mathbf{y}_1^{\mathbf{i}} + \cdots + \mathbf{y}_s^{\mathbf{i}} \equiv t^{k(1-P)} m_{\mathbf{i}} \ (\mathrm{mod}\, t^{-h}) \ (\mathbf{i} \in \mathcal{M})\big\}.\end{aligned}$$

Since $t^{k(1-P)}\mathbf{m} \in \mathbb{A}_\infty(\mathcal{M})$ and $\mathrm{char}(\mathbb{F}_q) \nmid k$, by Corollary 1.1, when $s \geq (k+2)\varrho$, there exists a constant $u^* = u^*(s, k, d; t^{-1})$ such that $h \geq u^*$, we have

$$\lambda_{s,k,d}(h; t^{-1}; t^{k(1-P)}\mathbf{m}) \geq q^{u^*(\varrho - sd)}.$$

Recall that $I_P = \{x \in \mathbb{A} \mid \deg x < P\}$. Write $L_P = \{t^{1-P}x \mid x \in I_P\}$ and let $Q = P - k + 1$. On making a change of variables by $\mathbf{y} = t^{1-P}\mathbf{x}$, since $-Q = (k-1)P + k(1-P) - 1$, we have

$$\begin{aligned}&\mathrm{card}\big\{\mathbf{x} \in I_P^{sd} \ \big|\ \mathrm{ord}\,(\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} - m_{\mathbf{i}}) < (k-1)P \ (\mathbf{i} \in \mathcal{M})\big\}\\ =&\mathrm{card}\big\{\mathbf{y} \in L_P^{sd} \ \big|\ \mathrm{ord}\,\big(\mathbf{y}_1^{\mathbf{i}} + \cdots + \mathbf{y}_s^{\mathbf{i}} - t^{k(1-P)} m_{\mathbf{i}}\big) < (k-1)P + k(1-P) \ (\mathbf{i} \in \mathcal{M})\big\}\\ =&\mathrm{card}\big\{\mathbf{y} \ (\mathrm{mod}\, t^{-P}) \ \big|\ \mathbf{y}_1^{\mathbf{i}} + \cdots + \mathbf{y}_s^{\mathbf{i}} - t^{k(1-P)} m_{\mathbf{i}} \equiv 0 \ (\mathrm{mod}\, t^{-Q}) \ (\mathbf{i} \in \mathcal{M})\big\}\\ =&q^{(k-1)sd}\mathrm{card}\big\{\mathbf{y} \ (\mathrm{mod}\, t^{-Q}) \ \big|\ \mathbf{y}_1^{\mathbf{i}} + \cdots + \mathbf{y}_s^{\mathbf{i}} - t^{k(1-P)} m_{\mathbf{i}} \equiv 0 \ (\mathrm{mod}\, t^{-Q}) \ (\mathbf{i} \in \mathcal{M})\big\}.\end{aligned}$$

Thus on taking $h = Q$, we get

$$\begin{aligned}&\lambda_{s,k,d}(Q; t^{-1}; t^{k(1-P)}\mathbf{m})\\ =&q^{Q(\varrho - sd)} \cdot \mathrm{card}\big\{\mathbf{y} \ (\mathrm{mod}\, t^{-Q}) \ \big|\ \mathbf{y}_1^{\mathbf{i}} + \cdots + \mathbf{y}_s^{\mathbf{i}} \equiv t^{k(1-P)} m_{\mathbf{i}} \ (\mathrm{mod}\, t^{-Q}) \ (\mathbf{i} \in \mathcal{M})\big\}\\ =&q^{Q(\varrho - sd)} \cdot q^{(1-k)sd} \cdot \mathrm{card}\big\{\mathbf{x} \in I_P^{sd} \ \big|\ \mathrm{ord}\,(\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} - m_{\mathbf{i}}) < (k-1)P \ (\mathbf{i} \in \mathcal{M})\big\}\\ =&q^{Q(\varrho - sd)} \cdot q^{(1-k)sd} \cdot q^{-P(\varrho - sd)} \cdot \lambda_{q,s,k,d}(P; \infty; \mathbf{m}).\end{aligned}$$

Since $Q = P - k + 1$, we get

$$\lambda_{s,k,d}(Q; t^{-1}; t^{k(1-P)}\mathbf{m}) = q^{(1-k)\varrho}\lambda_{q,s,k,d}(P; \infty; \mathbf{m}).$$

Therefore, when P is sufficiently large, we have

$$\lambda_{s,k,d}(Q; t^{-1}; t^{k(1-P)}\mathbf{m}) \geq q^{u^*(\varrho - sd)},$$

and so

$$\lambda_{q,s,k,d}(P; \infty; \mathbf{m}) = q^{(k-1)\varrho}\lambda_{s,k,d}(Q; t^{-1}; t^{k(1-P)}\mathbf{m}) \geq q^{(k-1)\varrho}q^{u^*(\varrho - sd)}.$$

This completes the proof of the proposition. □

To proceed, we now introduce some analytic properties for the exponential function e. Let $\mathbb{T} = \{\alpha \in \mathbb{K}_\infty \mid \langle \alpha \rangle < 1\}$. Given any Haar measure $d\alpha$ on $\mathbb{K}_\infty$, we normalize it in such a manner that $\int_{\mathbb{T}} 1\, d\alpha = 1$. In what follows, for $\ell \in \mathbb{Z}$, define

$$J_\ell = \{\alpha \in \mathbb{K}_\infty \mid \langle \alpha \rangle < \widehat{\ell}\}.$$

For $Q \in \mathbb{N}$, recall that $I_Q = \{x \in \mathbb{A} \mid \langle x \rangle < \widehat{Q}\}$ and write

$$t^{-Q} I_Q = \{t^{-Q} x \mid x \in I_Q\}.$$

For $\boldsymbol{\alpha} = (\alpha_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}}$, write

$$\operatorname{ord}(\boldsymbol{\alpha}) = \max_{\mathbf{i} \in \mathcal{M}} \operatorname{ord}(\alpha_{\mathbf{i}}).$$

Lemma 9.1. *The exponential function $e : \mathbb{K}_\infty \to \mathbb{C}^\times$ satisfies the following properties.*

(1) *If $Q \in \mathbb{N}$ and $x \in \mathbb{A}$, then*

$$\widehat{Q} \int_{J_{-Q}} e(x\alpha) d\alpha = \begin{cases} 1, & \text{if } x \in I_Q, \\ 0, & \textit{otherwise}. \end{cases}$$

(2) *Let $\ell \in \mathbb{Z}$ and $S \subseteq \mathbb{K}_\infty$ be measurable. If $f : \mathbb{K}_\infty \longrightarrow \mathbb{C}$ is integrable, then*

$$q^\ell \int_{t^{-\ell} X} f(t^\ell \alpha) d\alpha = \int_X f(\beta) d\beta.$$

(3) *Suppose that $Q \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}} \in \mathbb{K}_\infty^{\varrho}$ such that $\operatorname{ord}(\boldsymbol{\alpha}) < Q$. Then*

$$\sum_{\mathbf{x} \in (t^{-Q} I_Q)^{sd}} e\big(G(\boldsymbol{\alpha}; \mathbf{x}; \mathbf{m})\big) = \widehat{Q}^{sd} \int_{\mathbb{T}^{sd}} e\big(G(\boldsymbol{\alpha}; \mathbf{x}; \mathbf{m})\big) d\mathbf{x}.$$

Proof. The first two parts are proved in [22, Lemma 1] and [12, Equation (2.16)] respectively.

(3) Let $G(\boldsymbol{\alpha}; \mathbf{y}; \mathbf{0})$ be defined by (8.1). It then follows from [48, Lemma 3.2] that

$$\sum_{\mathbf{y} \in (t^{-Q} I_Q)^{sd}} e\big(G(\boldsymbol{\alpha}; \mathbf{y}; \mathbf{0})\big) = \widehat{Q}^{sd} \int_{\mathbb{T}^{sd}} e\big(G(\boldsymbol{\alpha}; \mathbf{y}; \mathbf{0})\big) d\mathbf{y}.$$

In view of the definition (8.1), we observe

$$e\big(G(\boldsymbol{\alpha}; \mathbf{y}; \mathbf{m})\big) = e\big(G(\boldsymbol{\alpha}; \mathbf{y}; \mathbf{0})\big) e(-\boldsymbol{\alpha} \cdot \mathbf{m}),$$

where $\boldsymbol{\alpha} \cdot \mathbf{m} = \sum_{\mathbf{i} \in \mathcal{M}} \alpha_{\mathbf{i}} m_{\mathbf{i}}$. Therefore,

$$\sum_{\mathbf{y}\in(t^{-Q}I_Q)^{sd}} e\big(G(\boldsymbol{\alpha};\mathbf{y};\mathbf{m})\big) = \widehat{Q}^{sd}\int_{\mathbb{T}^{sd}} e\big(G(\boldsymbol{\alpha};\mathbf{y};\mathbf{m})\big)d\mathbf{y}. \quad \square$$

Lemma 9.2. *Let $\mathbf{m}\in\mathbb{A}^{\varrho}$ and $P\geq T(\mathbf{m})$ with $T(\mathbf{m})$ defined by (1.11). Let $G(\boldsymbol{\beta};\mathbf{x};\mathbf{m})$ be defined by (8.1). Then*

$$\lambda_{q,s,k,d}(P;\infty;\mathbf{m}) = \widehat{P}^{\varrho k-sd}\int_{J^{\varrho}_{-(k-1)P}}\sum_{\mathbf{x}\in I_P^{sd}} e\Big(G(\boldsymbol{\beta};\mathbf{x};\mathbf{m})\Big)d\boldsymbol{\beta}.$$

Proof. By Lemma 9.1(1), on taking $Q=(k-1)P$, for every $\mathbf{i}\in\mathcal{M}$ and $\mathbf{x}\in I_P^{sd}$, we have

$$\widehat{P}^{k-1}\int_{J_{-(k-1)P}} e\big(\beta_{\mathbf{i}}(\mathbf{x}_1^{\mathbf{i}}+\cdots+\mathbf{x}_s^{\mathbf{i}}-m_{\mathbf{i}})\big)d\beta_{\mathbf{i}} = \begin{cases}1, & \text{if } (\mathbf{x}_1^{\mathbf{i}}+\cdots+\mathbf{x}_s^{\mathbf{i}}-m_{\mathbf{i}})\in I_{(k-1)P},\\ 0, & \text{otherwise.}\end{cases}$$

Thus

$$\widehat{P}^{(k-1)\varrho}\int_{J^{\varrho}_{-(k-1)P}} e\Big(G(\boldsymbol{\beta};\mathbf{x};\mathbf{m})\Big)d\boldsymbol{\beta} = \prod_{\mathbf{i}\in\mathcal{M}}\widehat{P}^{(k-1)}\int_{J_{-(k-1)P}} e\big(\beta_{\mathbf{i}}(\mathbf{x}_1^{\mathbf{i}}+\cdots+\mathbf{x}_s^{\mathbf{i}}-m_{\mathbf{i}})\big)d\beta_{\mathbf{i}}.$$

Therefore

$$\begin{aligned}\lambda_{q,s,k,d}(P;\infty;\mathbf{m}) &= \widehat{P}^{\varrho-sd}\mathrm{card}\big\{\mathbf{x}\in I_P^{sd}\;\big|\;(\mathbf{x}_1^{\mathbf{i}}+\cdots+\mathbf{x}_s^{\mathbf{i}}-m_{\mathbf{i}})\in I_{(k-1)P}\ (\mathbf{i}\in\mathcal{M})\big\}\\ &= \widehat{P}^{\varrho-sd}\cdot\widehat{P}^{(k-1)\varrho}\sum_{\mathbf{x}\in I_P^{sd}}\int_{J^{\varrho}_{-(k-1)P}} e\Big(G(\boldsymbol{\beta};\mathbf{x};\mathbf{m})\Big)d\boldsymbol{\beta}\\ &= \widehat{P}^{\varrho k-sd}\int_{J^{\varrho}_{-(k-1)P}}\sum_{\mathbf{x}\in I_P^{sd}} e\Big(G(\boldsymbol{\beta};\mathbf{x};\mathbf{m})\Big)d\boldsymbol{\beta}.\end{aligned}$$

This completes the proof. $\square$

Lemma 9.3. *Let $\ell\in\mathbb{Z}$ and $Q\in\mathbb{N}$ with $\ell\leq(1-k)Q$. Let $\mathbf{m}\in\mathbb{A}^{\varrho}$ and let $G(\boldsymbol{\beta};\mathbf{x};\mathbf{m})$ be defined by (8.1). Then*

$$\int_{J^{\varrho}_{\ell}}\sum_{\mathbf{x}\in I_Q^{sd}} e\Big(G(\boldsymbol{\beta};\mathbf{x};\mathbf{m})\Big)d\boldsymbol{\beta} = \widehat{Q}^{sd-\varrho k}\int_{J^{\varrho}_{\ell+kQ}}\left(\int_{\mathbb{T}^{sd}} e\big(G(\boldsymbol{\alpha};\mathbf{x};t^{-kQ}\mathbf{m})\big)d\mathbf{x}\right)d\boldsymbol{\alpha}.$$

In particular, for $P\geq T(\mathbf{m})$ with $T(\mathbf{m})$ defined by (1.11), one has

$$\lambda_{q,s,k,d}(P;\infty;\mathbf{m}) = \int_{J^{\varrho}_{P}}\left(\int_{\mathbb{T}^{sd}} e\Big(G(\boldsymbol{\alpha};\mathbf{x};t^{-kP}\mathbf{m})\Big)d\mathbf{x}\right)d\boldsymbol{\alpha}.$$

Proof. On recalling that the definition of $G(\boldsymbol{\alpha};\mathbf{x};\mathbf{m})$ in (8.1), we find from Lemma 8.1(1), (2), and Lemma 9.1(2) that

$$\begin{aligned}\int_{J_\ell^\varrho} e\big(G(\boldsymbol{\beta};\mathbf{x};\mathbf{m})\big)d\boldsymbol{\beta} &= \int_{J_\ell^\varrho} \prod_{\mathbf{i}\in\mathcal{M}} e\Big(\beta_{\mathbf{i}}\big(\mathbf{x}_1^{\mathbf{i}}+\cdots+\mathbf{x}_s^{\mathbf{i}}-m_{\mathbf{i}}\big)\Big)d\boldsymbol{\beta}\\ &= \prod_{\mathbf{i}\in\mathcal{M}} \int_{J_\ell} e\Big(\beta_{\mathbf{i}}\big(\mathbf{x}_1^{\mathbf{i}}+\cdots+\mathbf{x}_s^{\mathbf{i}}-m_{\mathbf{i}}\big)\Big)d\beta_{\mathbf{i}}\\ &= \prod_{\mathbf{i}\in\mathcal{M}} \widehat{Q}^{-k} \int_{J_{\ell+kQ}} e\Big(t^{-kQ}\alpha_{\mathbf{i}}\big(\mathbf{x}_1^{\mathbf{i}}+\cdots+\mathbf{x}_s^{\mathbf{i}}-m_{\mathbf{i}}\big)\Big)d\alpha_{\mathbf{i}}\\ &= \widehat{Q}^{-\varrho k} \int_{J_{\ell+kQ}^\varrho} e\Big(G(t^{-kQ}\boldsymbol{\alpha};\mathbf{x};\mathbf{m})\Big)d\boldsymbol{\alpha}.\end{aligned}$$

On letting $\mathbf{y}=t^{-Q}\mathbf{x}$, we see that

$$\begin{aligned}\sum_{\mathbf{x}\in I_Q^{sd}} e\big(G(t^{-kQ}\boldsymbol{\alpha};\mathbf{x};\mathbf{m})\big) &= \sum_{\mathbf{x}\in I_Q^{sd}} e\Big(\sum_{\mathbf{i}\in\mathcal{M}} t^{-kQ}\alpha_{\mathbf{i}}\big(\mathbf{x}_1^{\mathbf{i}}+\cdots+\mathbf{x}_s^{\mathbf{i}}-m_{\mathbf{i}}\big)\Big)\\ &= \sum_{\mathbf{y}\in(t^{-Q}I_Q)^{sd}} e\big(G(\boldsymbol{\alpha};\mathbf{y};t^{-kQ}\mathbf{m})\big).\end{aligned}$$

Thus

$$\begin{aligned}\int_{J_\ell^\varrho} \sum_{\mathbf{x}\in I_Q^{sd}} e\big(G(\boldsymbol{\beta};\mathbf{x};\mathbf{m})\big)d\boldsymbol{\beta} &= \widehat{Q}^{-\varrho k} \int_{J_{\ell+kQ}^\varrho} \sum_{\mathbf{x}\in I_Q^{sd}} e\big(G(t^{-kQ}\boldsymbol{\alpha};\mathbf{x};\mathbf{m})\big)d\boldsymbol{\alpha}\\ &= \widehat{Q}^{-\varrho k} \int_{J_{\ell+kQ}^\varrho} \sum_{\mathbf{y}\in(t^{-Q}I_Q)^{sd}} e\big(G(\boldsymbol{\alpha};\mathbf{y};t^{-kQ}\mathbf{m})\big)d\boldsymbol{\alpha}.\end{aligned}$$

For $\boldsymbol{\alpha}\in J_{\ell+kQ}^\varrho$, we have

$$\operatorname{ord}(\boldsymbol{\alpha}) < \ell+kQ \le (1-k)Q+kQ = Q.$$

It follows from the above argument and Lemma 9.1(3) that

$$\int_{J_\ell^\varrho} \sum_{\mathbf{x}\in I_Q^{sd}} e\big(G(\boldsymbol{\beta};\mathbf{x};\mathbf{m})\big)d\boldsymbol{\beta} = \widehat{Q}^{sd-\varrho k} \int_{J_{\ell+kQ}^\varrho} \bigg(\int_{\mathbb{T}^{sd}} e\big(G(\boldsymbol{\alpha};\mathbf{y};t^{-kQ}\mathbf{m})\big)d\mathbf{y}\bigg)d\boldsymbol{\alpha}.$$

By Lemma 9.2, on taking $\ell=-(k-1)P$ and $Q=P$, we obtain

$$\lambda_{q,s,k,d}(P;\infty;\mathbf{m}) = \int\limits_{J_P^{\varrho}} \Big(\int\limits_{\mathbb{T}^{sd}} e\Big(G(\boldsymbol{\alpha};\mathbf{x};t^{-kP}\mathbf{m})\Big)d\mathbf{x}\Big)d\boldsymbol{\alpha}.$$

This completes the proof of the lemma. □

Definition 9.1. For $\mathbf{m} \in \mathbb{A}^{\varrho}$ and $P \geq T(\mathbf{m})$ with $T(\mathbf{m})$ defined by (1.11), define the singular integral to be

$$\mathfrak{J}(\mathbf{m};P) = \mathfrak{J}_{q,s,k,d}(\mathbf{m};P) = \int\limits_{\mathbb{K}_{\infty}^{\varrho}} \Big(\int\limits_{\mathbb{T}^{sd}} G(\boldsymbol{\alpha};\mathbf{x};t^{-kP}\mathbf{m})d\mathbf{x}\Big)d\boldsymbol{\alpha}.$$

For $Q \in \mathbb{N}$, define

$$\mathfrak{J}(\mathbf{m};P;Q) = \int\limits_{J_Q^{\varrho}} \Big(\int\limits_{\mathbb{T}^{sd}} e\Big(G(\boldsymbol{\alpha};\mathbf{x};t^{-kP}\mathbf{m})\Big)d\mathbf{x}\Big)d\boldsymbol{\alpha}.$$

Lemma 9.4. *Whenever $s \geq 2k\varrho + 1$, there exist two constants $C = C(q,s,k,d) > 0$ and $\widetilde{C} = \widetilde{C}(q,s,k,d) > 0$ such that the following inequalities hold.*

(1) $\big|\mathfrak{J}(\mathbf{m};P)\big| \leq \int\limits_{\mathbb{K}_{\infty}^{\varrho}} \Big| \int\limits_{\mathbb{T}^{sd}} e\big(G(\boldsymbol{\alpha};\mathbf{x};t^{-kP}\mathbf{m})\big)d\mathbf{x}\Big| d\boldsymbol{\alpha} \leq C.$

(2) $\int\limits_{\mathbb{K}_{\infty}^{\varrho}\setminus J_Q^{\varrho}} \Big| \int\limits_{\mathbb{T}^{sd}} e\big(G(\boldsymbol{\alpha};\mathbf{x};t^{-kP}\mathbf{m})\big)d\mathbf{x}\Big| d\boldsymbol{\alpha} \leq \widetilde{C}\widehat{Q}^{-1/(4k\varrho)} \quad (Q \in \mathbb{N}).$

Proof. In view of definition, we have

$$\Big| \int\limits_{\mathbb{T}^{sd}} e\big(G(\boldsymbol{\alpha};\mathbf{x};t^{-kP}\mathbf{m})\big)d\mathbf{x}\Big| = \Big| \int\limits_{\mathbb{T}^{sd}} e\big(G(\boldsymbol{\alpha};\mathbf{x};\mathbf{0})\big)d\mathbf{x}\Big|.$$

Then the lemma follows at once from [48, Theorem 3.2]. □

Proposition 9.2. *Suppose that $s \geq 2k\varrho + 1$. For $\mathbf{m} \in \mathbb{A}^{\varrho}$ and $P \in \mathbb{N}$ with $P \geq T(\mathbf{m})$, one has*

$$\lambda_{q,s,k,d}(P;\infty;\mathbf{m}) = \mathfrak{J}(\mathbf{m};P;P) \ll 1$$

and

$$\lambda_{q,s,k,d}(P;\infty;\mathbf{m}) - \mathfrak{J}(\mathbf{m};P) \ll \widehat{P}^{-1/(4k\varrho)}.$$

Whenever P is sufficiently large in terms of s,k,d and q, one has

$$\lambda_{q,s,k,d}(P;\infty;\mathbf{m}) \gg 1.$$

Proof. Since $s \geq 2k\varrho + 1 \geq (k+2)\varrho$, by Proposition 9.1, the lower bound follows immediately. Lemma 9.3 implies that

$$\lambda_{q,s,k,d}(P;\infty;\mathbf{m}) = \mathfrak{J}(\mathbf{m};P;P).$$

By Lemma 9.4(2), we deduce that

$$\left|\mathfrak{J}(\mathbf{m};P) - \mathfrak{J}(\mathbf{m};P;P)\right| \ll \widehat{P}^{-1/(4k\varrho)}.$$

Thus

$$\mathfrak{J}(\mathbf{m};P;P) \ll 1$$

and

$$\lambda_{q,s,k,d}(P;\infty;\mathbf{m}) - \mathfrak{J}(\mathbf{m};P) \ll \widehat{P}^{-1/(4k\varrho)}.$$

This completes the proof of the proposition. □

10. The asymptotic formula in Theorem 1.4

In this section, we will prove Theorem 1.4 via the Hardy-Littlewood circle method. We first recall the following orthogonality relation established in [22, Lemma 1]

$$\int_{\mathbb{T}} e(x\alpha)\,d\alpha = \begin{cases} 1, & \text{when } x = 0, \\ 0, & \text{when } x \in \mathbb{A} \setminus \{0\}. \end{cases}$$

Therefore, for $n \in \mathbb{N} \setminus \{0\}$, $(x_1, \cdots, x_n) \in \mathbb{A}^n$, and $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in \mathbb{K}_\infty^n$, we have

$$\begin{aligned} \int_{\mathbb{T}^n} e(x_1\alpha_1 + \cdots + x_n\alpha_n)\,d\boldsymbol{\alpha} &= \prod_{i=1}^n \int_{\mathbb{T}} e(x_i\alpha_i)\,d\alpha_i \\ &= \begin{cases} 1, & \text{when } x_i = 0\,(1 \leq i \leq n), \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{10.1}$$

Let $\mathcal{M}$ and ϱ be defined as in (1.2). Let $s \in \mathbb{N} \setminus \{0\}$. For $\boldsymbol{\alpha} = (\alpha_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{T}^{\varrho}$ and $P \in \mathbb{N} \setminus \{0\}$, define

$$f(\boldsymbol{\alpha};P) = \sum_{\mathbf{x}\in I_P^d} e\left(\sum_{\mathbf{i}\in\mathcal{M}} \alpha_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}\right).$$

By (10.1), for $\mathbf{m} = \left(m_{\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{M}} \in \mathbb{A}^{\varrho}$ and $P \in \mathbb{N} \setminus \{0\}$, we have

$$R_{q,s,k,d}(\mathbf{m};P)=\int\limits_{\mathbb{T}^{\varrho}} f(\boldsymbol{\alpha};P)^s e(-\boldsymbol{\alpha}\cdot\mathbf{m})d\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha}\cdot\mathbf{m}=\sum_{\mathbf{i}\in\mathcal{M}}\alpha_{\mathbf{i}}m_{\mathbf{i}}$. To start the circle method, we divide $\mathbb{T}^{\varrho}$ into the Farey arcs defined as follows: given $\mathbf{a}=(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}}\in\mathbb{A}^{\varrho}$ and $g\in\mathbb{A}$ for which the monic common divisor of g and all the $a_{\mathbf{i}}$ is 1, denoted by $(\mathbf{a},g)=1$, we define the *Farey arc* $\mathfrak{M}(g,\mathbf{a})$ about $\mathbf{a}/g$ by

$$\mathfrak{M}(g,\mathbf{a})=\Big\{\boldsymbol{\alpha}\in\mathbb{T}^{\varrho}\;\Big|\;\langle g\alpha_{\mathbf{i}}-a_{\mathbf{i}}\rangle<\widehat{P}^{\frac{1}{2}-k}\;(\mathbf{i}\in\mathcal{M})\Big\}. \tag{10.2}$$

The set of *major arcs* $\mathfrak{M}$ is defined to be the union of all $\mathfrak{M}(g,\mathbf{a})$ with

$$\mathbf{a}\in\mathbb{A}^{\varrho},\; g\in\mathbb{A},\; g\,\text{monic},\; (\mathbf{a},g)=1,\;\text{and}\; 0\le\langle a_{\mathbf{i}}\rangle<\langle g\rangle\le\widehat{P}^{\frac{1}{2}}\;(\mathbf{i}\in\mathcal{M}). \tag{10.3}$$

The conditions (10.2) and (10.3) ensure that the arcs $\mathfrak{M}(g,\mathbf{a})$ comprising $\mathfrak{M}$ are disjoint. Furthermore, we write $\mathfrak{m}=\mathbb{T}^{\varrho}\setminus\mathfrak{M}$ for the complementary set of *minor arcs*.

We first consider the major arc contribution.

Proposition 10.1. *Let $\mathcal{M}$ and ϱ be defined as in (1.2). Suppose that* $\mathrm{char}(\mathbb{F}_q)\nmid k$ *and $s\ge 2k(\varrho+1)+1$. Then for $\mathbf{m}\in\mathbb{A}^{\varrho}$ with $P\ge T(\mathbf{m})$ defined by (1.11), there exists a positive number $P_0=P_0(q,s,k,d)$ such that whenever $\mathbf{m}\in\mathbb{A}_w(\mathcal{M})$ for every $w\in\mathcal{P}$ and $P\ge P_0$, one has*

$$\int\limits_{\mathfrak{M}} f(\boldsymbol{\alpha};P)^s e(-\boldsymbol{\alpha}\cdot\mathbf{m})d\boldsymbol{\alpha}=C_{q,s,k,d}(\mathbf{m};P)\widehat{P}^{sd-\varrho k}+O\big(\widehat{P}^{sd-\varrho k-1/(16k\varrho)}\big),$$

where

$$C_{q,s,k,d}(\mathbf{m};P)=\lambda_{q,s,k,d}(P;\infty;\mathbf{m})\prod_{w\in\mathcal{P}}\Big(\lim_{h\to\infty}\lambda_{q,s,k,d}(h;w;\mathbf{m})\Big).$$

In addition, one has $1\ll C_{q,s,k,d}(\mathbf{m};P)\ll 1$.

Proof. Suppose that $\boldsymbol{\alpha}=(\alpha_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}}\in\mathfrak{M}(g,\mathbf{a})\subseteq\mathfrak{M}$. Write $\boldsymbol{\beta}=\boldsymbol{\alpha}-\mathbf{a}/g$. Then $\langle\beta_{\mathbf{i}}\rangle<\langle g\rangle^{-1}\widehat{P}^{\frac{1}{2}-k}\;\big(\mathbf{i}\in\mathcal{M}\big)$. It follows from similar arguments as in [46, Lemma 3.3] that

$$f(\boldsymbol{\alpha};P)=\langle g\rangle^{-d}T(g,\mathbf{a})f(\boldsymbol{\beta};P),$$

where

$$T(g,\mathbf{a})=\sum_{\mathbf{x}\in I^d_{\mathrm{ord}\,g}} e\Big(\sum_{\mathbf{i}\in\mathcal{M}}\frac{a_{\mathbf{i}}}{g}\mathbf{x}^{\mathbf{i}}\Big).$$

Thus

$$\sum_{\substack{\mathbf{a}\in I^{\varrho}_{\mathrm{ord}\,g}\\(\mathbf{a},g)=1}}\int_{\mathfrak{M}(g,\mathbf{a})} f(\boldsymbol{\alpha};P)^s e\big(-\boldsymbol{\alpha}\cdot\mathbf{m}\big)d\boldsymbol{\alpha} = S^*(g;\mathbf{m})\int_{\mathcal{B}_g} f(\boldsymbol{\beta};P)^s e\big(-\boldsymbol{\beta}\cdot\mathbf{m}\big)d\boldsymbol{\beta},$$

where

$$S^*(g;\mathbf{m}) = \langle g\rangle^{-sd}\sum_{\substack{\mathbf{a}\in I^{\varrho}_{\mathrm{ord}\,g}\\(\mathbf{a},g)=1}} T(g,\mathbf{a})^s e\Bigg(-\sum_{\mathbf{i}\in\mathcal{M}}\frac{a_{\mathbf{i}}m_{\mathbf{i}}}{g}\Bigg)$$

and

$$\mathcal{B}_g = \big\{\boldsymbol{\beta} = (\beta_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{T}^{\varrho} \mid \langle\beta_{\mathbf{i}}\rangle < \langle g\rangle^{-1}\widehat{P}^{\frac{1}{2}-k}\ \big(\mathbf{i}\in\mathcal{M}\big)\big\}.$$

Note that

$$S^*(g;\mathbf{m}) = \langle g\rangle^{-sd}\sum_{\substack{\mathbf{a}(\mathrm{mod}\,g)\\(\mathbf{a},g)=1}} e\Bigg(\frac{G(\mathbf{a};\mathbf{x};\mathbf{m})}{g}\Bigg) = S(g;\mathbf{m}).$$

Then we have

$$\int_{\mathfrak{M}} f(\boldsymbol{\alpha};P)^s e\big(-\boldsymbol{\alpha}\cdot\mathbf{m}\big)d\boldsymbol{\alpha} = \sum_{\substack{\langle g\rangle<\widehat{P}^{1/2}\\ g \text{ monic}}} S(g;\mathbf{m})\int_{\mathcal{B}_g} f(\boldsymbol{\beta};P)^s e\big(-\boldsymbol{\beta}\cdot\mathbf{m}\big)d\boldsymbol{\beta}.$$

For monic $g\in\mathbb{A}$, when $P>0$, since

$$-\mathrm{ord}\,g + (1/2-k)P < (1-k)P,$$

by Lemma 9.3, we have

$$\int_{\mathcal{B}_g} f(\boldsymbol{\beta};P)^s e\big(-\boldsymbol{\beta}\cdot\mathbf{m}\big)d\boldsymbol{\beta} = \int_{\mathcal{B}_g}\sum_{\mathbf{x}\in I_P^{sd}} e\big(G(\boldsymbol{\beta};\mathbf{x};\mathbf{m})\big)d\boldsymbol{\beta} = \widehat{P}^{sd-\varrho k}\mathfrak{J}(\mathbf{m};g;P),$$

where

$$\mathfrak{J}(\mathbf{m};g;P) = \int_{\mathcal{C}_g}\Bigg(\int_{\mathbb{T}^{sd}} e\big(G(\boldsymbol{\alpha};\mathbf{y};t^{-kP}\mathbf{m})\big)d\mathbf{y}\Bigg)d\boldsymbol{\alpha}$$

with

$$\mathcal{C}_g = \big\{\boldsymbol{\alpha} = (\alpha_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}} \in \mathbb{K}_\infty^{\varrho} \mid \langle\alpha_{\mathbf{i}}\rangle < \langle g\rangle^{-1}\widehat{P}^{1/2}\ \big(\mathbf{i}\in\mathcal{M}\big)\big\}.$$

We then deduce that

$$\int_{\mathfrak{M}} f(\boldsymbol{\alpha}; P)^s e\big(-\boldsymbol{\alpha}\cdot\mathbf{m}\big)d\boldsymbol{\alpha} = \widehat{P}^{sd-\varrho k} \sum_{\substack{\langle g\rangle<\widehat{P}^{1/2}\\ g \text{ monic}}} S(g;\mathbf{m})\mathfrak{J}(\mathbf{m};g;P). \tag{10.4}$$

Note that $s \geq 2\varrho k+1$. It follows from Lemma 9.4 that

$$\mathfrak{J}(\mathbf{m};P) \ll 1 \tag{10.5}$$

and for monic $g \in \mathbb{A}$ with $\langle g\rangle \leq \widehat{P}^{1/2}$, since $(P/2) - \operatorname{ord} g \geq 0$, we get

$$-\mathfrak{J}(\mathbf{m};P) + \mathfrak{J}(\mathbf{m};g;P) \ll 1. \tag{10.6}$$

Also, for monic $g \in \mathbb{A}$ with $\langle g\rangle \leq \widehat{P}^{1/4}$, since $(P/2) - \operatorname{ord} g \geq P/4$ and we find from Lemma 9.4(2) that

$$-\mathfrak{J}(\mathbf{m};P) + \mathfrak{J}(\mathbf{m};g;P) \ll q^{-(P/2-\operatorname{ord} g)/(4k\varrho)} \leq \widehat{P}^{-1/(16k\varrho)}. \tag{10.7}$$

By Proposition 8.3, for $s \geq 2k(\varrho+1)+1$, on taking $\epsilon = 1/(4k)$, we obtain

$$\sum_{\substack{\langle g\rangle\leq\widehat{P}^{1/4}\\ g \text{ monic}}} \big|S(g;\mathbf{m})\big| \ll 1 \quad \text{and} \quad \sum_{\substack{\langle g\rangle>\widehat{P}^{1/4}\\ g \text{ monic}}} \big|S(g;\mathbf{m})\big| \ll \widehat{P}^{-1/(16k)}. \tag{10.8}$$

On combining (10.7) and the first equality in (10.8), we get

$$\sum_{\substack{\langle g\rangle\leq\widehat{P}^{1/4}\\ g \text{ monic}}} S(g;\mathbf{m})\big(-\mathfrak{J}(\mathbf{m};P)+\mathfrak{J}(\mathbf{m};g;P)\big) \ll \sum_{\substack{\langle g\rangle\leq\widehat{P}^{1/4}\\ g \text{ monic}}} \big|S(g;\mathbf{m})\big|\widehat{P}^{-1/(16k\varrho)} \ll \widehat{P}^{-1/(16k\varrho)}.$$

By applying (10.6) with the second equality in (10.8), we have

$$\sum_{\substack{\widehat{P}^{1/4}<\langle g\rangle<\widehat{P}^{1/2}\\ g \text{ monic}}} S(g;\mathbf{m})\big(-\mathfrak{J}(\mathbf{m};P)+\mathfrak{J}(\mathbf{m};g;P)\big) \ll \sum_{\substack{\langle g\rangle>\widehat{P}^{1/4}\\ g \text{ monic}}} |S(g;\mathbf{m})| \ll \widehat{P}^{-1/(16k)}.$$

Therefore

$$\sum_{\substack{\langle g\rangle<\widehat{P}^{1/2}\\ g \text{ monic}}} S(g;\mathbf{m})\big(-\mathfrak{J}(\mathbf{m};P)+\mathfrak{J}(\mathbf{m};g;P)\big) \ll \widehat{P}^{-1/(16k\varrho)}.$$

It then follows from the equality in (10.4) and the above estimate that

$$\int_{\mathfrak{M}} f(\boldsymbol{\alpha}; P)^s e\big(-\boldsymbol{\alpha}\cdot\mathbf{m}\big)d\boldsymbol{\alpha} = \widehat{P}^{sd-\varrho k} \sum_{\substack{\langle g\rangle<\widehat{P}^{1/2}\\ g \text{ monic}}} S(g;\mathbf{m})\mathfrak{J}(\mathbf{m};P) + O\big(\widehat{P}^{sd-\varrho k-1/(16k\varrho)}\big).$$

By (10.8), we have

$$\sum_{\substack{\langle g\rangle<\widehat{P}^{1/2}\\ g \text{ monic}}} S(g;\mathbf{m}) = \mathfrak{S}(\mathbf{m}) - \sum_{\substack{\langle g\rangle\geq\widehat{P}^{1/2}\\ g \text{ monic}}} S(g;\mathbf{m}) = \mathfrak{S}(\mathbf{m}) + O\big(\widehat{P}^{-1/(16k)}\big).$$

We thus deduce from (10.5) that

$$\int_{\mathfrak{M}} f(\boldsymbol{\alpha};P)^s e(-\boldsymbol{\alpha}\cdot\mathbf{m})d\boldsymbol{\alpha} = \mathfrak{J}(\mathbf{m};P)\mathfrak{S}(\mathbf{m})\widehat{P}^{sd-\varrho k} + O\big(\widehat{P}^{sd-\varrho k-1/(16k\varrho)}\big).$$

In combination of Proposition 8.3 with Proposition 9.2, the proposition follows. □

We then consider the minor arc contribution.

Proposition 10.2. *Suppose that* $p = \mathrm{char}(\mathbb{F}_q) \nmid k$. *Define*

$$\mathcal{R}' = \{\mathbf{j} = (j_1,\ldots,j_d) \in \mathbb{N}^d \,|\, \gcd(j_1,\ldots,j_d,p) = 1,\ 1 \leq j_1+\cdots+j_d \leq k\}.$$

Let $\vartheta = \mathrm{card}\mathcal{R}' = \binom{k+d}{d} - \binom{[k/p]+d}{d}$. *Suppose that* $s \geq 2\vartheta k + 2\vartheta + 1$. *Then one has*

$$\int_{\mathfrak{m}} |f(\boldsymbol{\alpha};P)|^s d\boldsymbol{\alpha} \ll \widehat{P}^{sd-\varrho k-\delta},$$

where $\delta = 1/(8\varrho\vartheta(k+1))$.

Proof. Let $J_{k\vartheta+\vartheta}(P)$ denote the number of solutions of the system

$$\mathbf{x}_1^{\mathbf{j}} + \cdots + \mathbf{x}_{k\vartheta+\vartheta}^{\mathbf{j}} = \mathbf{y}_1^{\mathbf{j}} + \cdots + \mathbf{y}_{k\vartheta+\vartheta}^{\mathbf{j}} \quad (\mathbf{j}\in\mathcal{R}'),$$

with $\mathbf{x}_j, \mathbf{y}_j \in I_P^d$ $(1 \leq j \leq k\vartheta+\vartheta)$. Let $\Theta = \sum_{\mathbf{j}\in\mathcal{R}'}(j_1+\cdots+j_d)$. By [23, Theorem 1.1], when $p \nmid k$ and $k \geq 2$, for each $\epsilon > 0$, one has

$$J_{k\vartheta+\vartheta}(P) \ll \big(\widehat{P}\big)^{2(k\vartheta+\vartheta)d-\Theta+\epsilon}.$$

Let $\mathcal{M}$ and ϱ be defined as in (1.2). Let $I_{k\vartheta+\vartheta}(P)$ denote the number of solutions of the system

$$\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_{k\vartheta+\vartheta}^{\mathbf{i}} = \mathbf{y}_1^{\mathbf{i}} + \cdots + \mathbf{y}_{k\vartheta+\vartheta}^{\mathbf{i}} \quad (\mathbf{i}\in\mathcal{M})$$

with $\mathbf{x}_j, \mathbf{y}_j \in I_P^d$ $(1 \leq j \leq k\vartheta+\vartheta)$. For $\mathbf{z} = (z_{\mathbf{j}})_{\mathbf{j}\in\mathcal{R}'} \in \mathbb{A}^\vartheta$, write $J_{k\vartheta+\vartheta}(P;\mathbf{z})$ for the number of solutions of the system

$$\big(\mathbf{x}_1^{\mathbf{j}} + \cdots + \mathbf{x}_{k\vartheta+\vartheta}^{\mathbf{j}}\big) - \big(\mathbf{y}_1^{\mathbf{j}} + \cdots + \mathbf{y}_{k\vartheta+\vartheta}^{\mathbf{j}}\big) = z_{\mathbf{j}} \quad (\mathbf{j}\in\mathcal{R}')$$

with $\mathbf{x}_j, \mathbf{y}_j \in I_P^d$ $(1 \leq j \leq k\vartheta+\vartheta)$. For $\mathbf{j}\in\mathcal{R}'$, write

$$I(P;\mathbf{j}) = \{x \in \mathbb{A} \,|\, \operatorname{ord} x < (j_1 + \cdots + j_d)P\}.$$

Then $J_{k\vartheta+\vartheta}(P;\mathbf{z}) = 0$ if $\mathbf{z} = (z_\mathbf{j})_{\mathbf{j}\in\mathcal{R}'} \notin \prod_{\mathbf{j}\in\mathcal{R}'} I(P;\mathbf{j})$. Otherwise, if $\mathbf{z} = (z_\mathbf{j})_{\mathbf{j}\in\mathcal{R}'} \in \prod_{\mathbf{j}\in\mathcal{R}'} I(P;\mathbf{j})$, by carrying out the argument in [23, Page 6, 2$^{\text{nd}}$ paragraph], we have

$$J_{k\vartheta+\vartheta}(P;\mathbf{z}) \le J_{k\vartheta+\vartheta}(P).$$

Since $p \nmid k$, we have $\mathcal{M} \subseteq \mathcal{R}'$, and hence

$$I_{k\vartheta+\vartheta}(P) = \sum_{\mathbf{z}} J_{k\vartheta+\vartheta}(P,\mathbf{z}) \le \widehat{P}^{\Theta-\varrho k} J_{k\vartheta+\vartheta}(P) \ll \widehat{P}^{2(k\vartheta+\vartheta)d-\varrho k+\epsilon},$$

where the summation is over $\mathbf{z} \in \prod_{\mathbf{j}\in\mathcal{R}'} I(P;\mathbf{j})$ with $z_\mathbf{i} = 0$ when $\mathbf{i} \in \mathcal{M}$. Thus

$$\int_{\mathbb{T}^\varrho} \left|f(\boldsymbol{\alpha};P)\right|^{2(k\vartheta+\vartheta)} d\boldsymbol{\alpha} = I_{k\vartheta+\vartheta}(P) \ll \widehat{P}^{2(k\vartheta+\vartheta)d-\varrho k+\epsilon}.$$

When $s \ge 2k\vartheta + 2\vartheta + 1$, we have

$$\int_{\mathfrak{m}} \left|f(\boldsymbol{\alpha};P)\right|^s d\boldsymbol{\alpha} \le \widehat{P}^{sd-(2k\vartheta+2\vartheta+1)d} \sup_{\boldsymbol{\alpha}\in\mathfrak{m}} \left|f(\boldsymbol{\alpha};P)\right| \int_{\mathbb{T}^\varrho} \left|f(\boldsymbol{\alpha};P)\right|^{2k\vartheta+2\vartheta} d\boldsymbol{\alpha}.$$

It follows from standard argument as in [23, Lemmas 6.1-6.2] that

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |f(\boldsymbol{\alpha};P)| \ll \widehat{P}^{d-1/(4\varrho\vartheta(k+1))+\epsilon}.$$

We therefore get

$$\begin{aligned}\int_{\mathfrak{m}} \left|f(\boldsymbol{\alpha};P)\right|^s d\boldsymbol{\alpha} &\ll \widehat{P}^{sd-(2k\vartheta+2\vartheta+1)d} \cdot \left(\widehat{P}^{d-1/(4\varrho\vartheta(k+1))+\epsilon}\right) \cdot \widehat{P}^{2(k\vartheta+\vartheta)d-\varrho k+\epsilon} \\ &\ll \widehat{P}^{sd-\varrho k-1/(8\varrho\vartheta(k+1))}.\end{aligned}$$

This completes the proof of the proposition. □

On recalling the definitions of $\mathcal{M}$ and $\mathcal{R}'$ with $\varrho = \operatorname{card}(\mathcal{M})$ and $\vartheta = \operatorname{card}(\mathcal{R}')$, when $\operatorname{char}(\mathbb{F}_q) \nmid k$, we have $k < \varrho < \vartheta$. Thus $2\vartheta k + 2\vartheta + 1 \ge 2k(\varrho + 1) + 1$. In combination of Proposition 10.1 with Proposition 10.2, Theorem 1.4 follows.

11. Further improvement on Theorem 1.5

In the proof of Theorem 1.5 we establish the following relation

$$\gamma(\mathcal{O};\mathcal{M}) \le \gamma(\mathcal{O};k)\kappa(\varrho - d + \kappa^{-1}d).$$

By Proposition 3.3, we have

$$\gamma(\mathcal{O}; k) \leq \gamma(F; k) + 1.$$

When $\kappa = [F : F(k)] = 1$, namely $F = F(k)$, recent work on Waring's problem in finite fields establish refined bounds for $\gamma(F; k)$ as follows.

Proposition 11.1. *Suppose that $F = F(k)$ with $\operatorname{card}\big(F(k)\big) = p^\sigma$. Let $k^* = \gcd(k, p^\sigma - 1)$. When $\sigma = 1$ and $p - 1 = k^*$ or $p - 1 = 2k^*$, one has*

$$\gamma(F; k) = k^*;$$

otherwise, one has

$$\gamma(F; k) \ll (k^*)^{1/2}.$$

Proof. The results are proved in [10, Theorem 4], [11, Theorem 1] and [9]. □

We now consider the case when $\kappa = [F; F(k)] > 1$. Theorem 4.1(1) and (2) imply that

$$\gamma(F; k) = \gamma(F_\sigma; k_\sigma),$$

where $k_\sigma = k/(1 + p^\sigma + \cdots + p^{\sigma(\kappa-1)})$, and $F_\sigma = F(k) = F_\sigma(k_\sigma)$. Thus Proposition 11.1 can be applied to bound $\gamma(F_\sigma, k_\sigma)$.

Proposition 11.2. *Suppose that $\kappa = [F : F(k)] > 1$ and $\operatorname{card}\big(F(k)\big) = p^\sigma$. Let $k_* = \gcd(k_\sigma, p^\sigma - 1)$. When $\sigma = 1$ and $p - 1 = k_*$ or $p - 1 = 2k_*$, one has*

$$\gamma(F; k) = \gamma(F_\sigma, k_\sigma) = k_*;$$

otherwise, one has

$$\gamma(F; k) = \gamma(F_\sigma, k_\sigma) \ll (k_*)^{1/2}.$$

On recalling Theorem 4.1(3), we may improve Theorem 1.5 in the latter case:

$$\gamma(\mathcal{O}; \mathcal{M}) \ll (k_*)^{1/2}(\log k)\varrho.$$

On carrying out similar arguments as in Proposition 4.2, one has

$$k_* \ll k^{1/2}.$$

Thus

$$\gamma(\mathcal{O}; \mathcal{M}) \ll k^{1/4}(\log k)\varrho.$$

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