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A generalization of Meshulam's theorem on subsets of finite abelian groups with no 3-term arithmetic progression (II)

Yu-Ru Liu [a], Craig V. Spencer [b], Xiaomei Zhao [c]

[a] Department of Pure Mathematics, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

[b] Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506, United States

[c] Department of Mathematics, Huazhong Normal University, Wuhan, Hubei, 430079, China

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ABSTRACT

Let $G \simeq \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_N\mathbb{Z}$ be a finite abelian group with $k_i | k_{i-1}$ $(2 \le i \le N)$. For a matrix $Y = (a_{i,j}) \in \mathbb{Z}^{R\times S}$ satisfying $a_{i,1} + \cdots + a_{i,S} = 0$ $(1 \le i \le R)$, let $D_Y(G)$ denote the maximal cardinality of a set $A \subseteq G$ for which the equations $a_{i,1}x_1 + \cdots + a_{i,S}x_S = 0$ $(1 \le i \le R)$ are never satisfied simultaneously by distinct elements $x_1, \ldots, x_S \in A$. Under certain assumptions on Y and G, we prove an upper bound of the form $D_Y(G) \le |G|(C/N)^{\gamma}$ for positive constants C and γ.

1. Introduction

Let G be a finite abelian group, and let $D_3(G)$ denote the maximal cardinality of a subset $A \subseteq G$ which does not contain a 3-term arithmetic progression. Let $k \in \mathbb{N} = \{1, 2, \ldots\}$ with $\gcd(2, k) = 1$. In his fundamental paper [9], Roth proved that $D_3(\mathbb{Z}/k\mathbb{Z}) = O(k/\log\log k)$. His result was later improved by Heath-Brown [6] and Szemerédi [11] to $D_3(\mathbb{Z}/k\mathbb{Z}) = O(k/(\log k)^{\alpha})$ for some small positive constant $\alpha > 0$. Recently, Bourgain [2] proved that $D_3(\mathbb{Z}/k\mathbb{Z}) = O(k(\log\log k)^2/(\log k)^{2/3})$, which provides the best bound currently known. For a general finite abelian group G of odd order, Brown and Buhler [1] and Frankl et al. [3] showed that $D_3(G) = o(|G|)$. In [8], Meshulam considered the case where G has many constituents, and he proved that $D_3(G) \le 2|G|/c(G)$, where $c(G)$ denotes the number of constituents of G. By combining Meshulam's result with Bourgain's bound, one can follow the proof of [8, Corollary 1.3] to obtain that $D_3(G) = O(|G|/(\log|G|)^{\beta})$, where β is any positive constant with $\beta < 2/5$. By adapting Bourgain's argument in [2] to a general finite abelian group G of odd order, one should in fact be able to prove that $D_3(G) = O(|G|/(\log|G|)^{\beta})$, where β is any positive constant with $\beta < 2/3$. In [7], the first two authors of this paper generalized Meshulam's result to give an upper

E-mail addresses: yrliu@math.uwaterloo.ca (Y.-R. Liu), cvs@math.ksu.edu, craigvspencer@gmail.com (C.V. Spencer), x8zhao@gmail.com (X. Zhao).

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bound for subsets of finite abelian groups which avoid non-trivial solutions to a linear equation of the form $r_1x_1 + r_2x_2 + \cdots + r_sx_s = 0$. In this paper, we follow the approaches of [7] and [10] to further generalize Meshulam's result by investigating the solutions of a system of equations.

Given a finite abelian group G, we can write

$$G \simeq \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_M\mathbb{Z},$$

where $\mathbb{Z}/k_i\mathbb{Z}$ is a non-trivial cyclic group of order k_i $(1 \le i \le M)$ and $k_i | k_{i-1}$ $(2 \le i \le M)$. We denote by $c(G) = M$ the number of constituents of G and by $a(G) = k_1$ the annihilator of G. For $R, S \in \mathbb{N}$ with $S \ge 2R + 1$, let $Y = (a_{i,j})$ be an $R \times S$ matrix whose elements are integers. Let $L \in \mathbb{N}$ with $L \ge R$. We say that the group G is *L-coprime to Y* if there exist L columns of Y such that:

- any R of these L columns form a matrix of determinant coprime to $a(G)$,
- after removing any $L - R + 1$ of these L columns from Y, we can find two disjoint sets of R columns which form matrices of determinant coprime to $a(G)$.

In this case, we denote by $\mathcal{I}_Y(G; L)$ the set of indices of these L columns. The L-coprimality condition on Y is essential for the arguments of this paper. In order to study systems of higher complexity, one could use higher-order Fourier analysis (see, for example, [4,5]).

Let $Y = (a_{i,j}) \in \mathbb{Z}^{R \times S}$ satisfy $a_{i,1} + \cdots + a_{i,S} = 0$ $(1 \le i \le R)$. Consider the system of equations

$$a_{i,1}x_1 + \cdots + a_{i,S}x_S = 0 \quad (1 \le i \le R). \tag{1}$$

Let $D_Y(G)$ denote the maximal cardinality of a set $A \subseteq G$ for which the equations in (1) are never satisfied simultaneously by *distinct* elements $x_1, \ldots, x_S \in A$, and let $|G|$ denote the cardinality of G. For $L, N \in \mathbb{N}$ with $L \ge R$, we denote by $d_Y(N; L)$ the supremum of $D_Y(G)|G|^{-1}$ as G ranges over all finite abelian groups with $c(G) \ge N$ that are L-coprime to Y. In this paper, we prove the following theorem.

Theorem 1. *For $R, S \in \mathbb{N}$ with $S \ge 2R+1$, let $Y = (a_{i,j}) \in \mathbb{Z}^{R\times S}$ satisfy $a_{i,1}+\cdots+a_{i,S} = 0$ $(1 \le i \le R)$. For $L \in \mathbb{N}$ with $L \ge R$, there exists an effectively computable constant $C = C(Y; L) > 1$ such that for $N \in \mathbb{N}$, we have*

$$d_Y(N; L) \le \left(\frac{C}{N}\right)^{\frac{L-R+1}{R}}.$$

We note that in the special case when $L = R$, the above conditions on G and Y are analogous to Conditions 1 and 2 in [10]. Hence, Theorem 1 is more general than the finite abelian group analogue of Roth's result in [10]. Also, in the special case when $R = 1$ and $L = S - 2$, we can derive [7, Theorem 1] from Theorem 1 (see Remark 1). In particular, if $Y = (1, -2, 1)$ (thus $L = R = 1$ and G is of odd order), by [7, Remark 6], the constant C in Theorem 1 can be taken to be 2. Thus, Theorem 1 implies Meshulam's result on subsets of finite abelian groups with no 3-term arithmetic progression [8, Theorem 1.2].

We conclude this section by recalling some properties of character sums of finite abelian groups. Let $\hat{G}$ denote the character group of G. For $g \in G$, we have

$$|G|^{-1} \sum_{\chi \in \hat{G}} \chi(g) = \begin{cases} 1, & \text{if } g = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $R \in \mathbb{N}$, the character group of G^R is equivalent to the product of R copies of $\hat{G}$, and we denote it by $\hat{G}^R$. Thus, for $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_R) \in \hat{G}^R$ and $(g_1, \ldots, g_R) \in G^R$, we have

$$|G|^{-R} \sum_{\boldsymbol{\chi} \in \hat{G}^R} \chi_1(g_1) \cdots \chi_R(g_R) = \prod_{i=1}^{R} \left(|G|^{-1} \sum_{\chi_i \in \hat{G}} \chi_i(g_i) \right)$$
$$= \begin{cases} 1, & \text{if } g_j = 0 \quad (1 \le j \le R), \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

In what follows, we will write $\mathbf{1}$ for the trivial character $(1, \ldots, 1) \in \hat{G}^R$ and $\Gamma(G)$ for $\hat{G}^R \setminus \{\mathbf{1}\}$.

2. Proof of Theorem 1

For $R, S \in \mathbb{N}$ with $S \geq 2R+1$, let $Y = (a_{i,j}) \in \mathbb{Z}^{R\times S}$ satisfy $a_{i,1} + \cdots + a_{i,S} = 0$ $(1 \leq i \leq R)$. For $L, N \in \mathbb{N}$ with $L \geq R$, let G be a finite abelian group with $c(G) \geq N$ that is L-coprime to Y. Let $D_Y(G)$ and $d_Y(N; L)$ be defined as in Section 1. For convenience, in what follows, we will write $D(G)$ in place of $D_Y(G)$ and $d(N)$ in place of $d_Y(N; L)$. For a set $A \subseteq G$, let $T(A) = T_Y(A)$ denote the number of solutions of (1) with $x_i \in A$ $(1 \leq i \leq S)$. For $1 \leq j \leq S$ and $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_R) \in \hat{G}^R$, define

$$F_j(\boldsymbol{\chi}) = F_j(\boldsymbol{\chi}; A) = \sum_{x\in A} \chi_1(a_{1,j}x)\cdots\chi_R(a_{R,j}x) = \sum_{x\in A} \chi_1^{a_{1,j}}\cdots\chi_R^{a_{R,j}}(x).$$

Then by (2), we have

$$\begin{aligned} T(A) &= |G|^{-R} \sum_{\boldsymbol{\chi}\in\hat{G}^R} F_1\cdots F_S(\boldsymbol{\chi}) \\ &= |G|^{-R} F_1\cdots F_S(\mathbf{1}) + |G|^{-R} \sum_{\boldsymbol{\chi}\in\Gamma(G)} F_1\cdots F_S(\boldsymbol{\chi}). \end{aligned} \tag{3}$$

Before proving Theorem 1, we will need to obtain bounds on $T(A)$ and the contribution of the non-trivial characters.

Lemma 2. *Let G be a finite abelian group. For $R \in \mathbb{N}$, let $Z \in \mathbb{Z}^{R\times R}$ satisfy $\gcd(\det Z, a(G)) = 1$, where $\det Z$ denotes the determinant of Z. For $\mathbf{x} \in G^R$, we have $Z\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$.*

Proof. For a finite abelian group G, we can write $G \simeq \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_M\mathbb{Z}$ with $k_i | k_{i-1}$ $(2 \leq i \leq M)$. For $\mathbf{x} \in G^R$, we have $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_M$ with $\mathbf{x}_i \in (\mathbb{Z}/k_i\mathbb{Z})^R$ $(1 \leq i \leq M)$. Then $Z\mathbf{x} = \mathbf{0}$ is equivalent to $Z\mathbf{x}_i = \mathbf{0}$ $(1 \leq i \leq M)$. Fix $i \in \mathbb{N}$ with $1 \leq i \leq M$. Since $\gcd(\det Z, a(G)) = 1$ and $k_i | a(G)$, Z is invertible over the ring $\mathbb{Z}/k_i\mathbb{Z}$. Hence $Z\mathbf{x}_i = \mathbf{0}$ if and only if $\mathbf{x}_i = \mathbf{0}$. Thus, $Z\mathbf{x} = \mathbf{0}$ is equivalent to $\mathbf{x} = \mathbf{0}$. □

Lemma 3. *For $Y = (a_{i,j}) \in \mathbb{Z}^{R\times S}$ and $L \in \mathbb{N}$ with $L \geq R$, suppose that G is a finite abelian group that is L-coprime to Y. Suppose that $A \subseteq G$ for which the equations in (1) are never satisfied simultaneously by distinct elements $x_1, \ldots, x_S \in A$. Then we have*

$$T(A) \leq C_1|A|^{S-R-1},$$

where $C_1 = C_1(Y) = \binom{S}{2}$.

Proof. We have

$$T(A) = \text{card}\,\{\mathbf{x} \in A^S \mid Y\mathbf{x} = \mathbf{0}\},$$

where $\text{card}\,\{V\}$ denotes the cardinality of a set V. Since $A \subseteq G$ for which the equations in (1) are never satisfied simultaneously by distinct elements $x_1, \ldots, x_S \in A$, whenever $Y\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} = (x_1, \ldots, x_S) \in A^S$, there exist distinct elements $i, j \in \{1, \ldots, S\}$ with $x_i = x_j$. Fix one of the $C_1 = \binom{S}{2}$ choices of $\{i, j\}$. We consider two cases.

- *Case* 1: Suppose that $\{i, j\} \cap \mathcal{I}_Y(G; L) = \emptyset$. Since G is L-coprime to Y, by Lemma 2, we have

$$\text{card}\,\{\mathbf{x} \in A^S \mid x_i = x_j \text{ and } Y\mathbf{x} = \mathbf{0}\} \leq |A|^{S-R-1}.$$

- *Case* 2: Suppose that $\{i, j\} \cap \mathcal{I}_Y(G; L) \neq \emptyset$. Without loss of generality, we may assume that $j \in \mathcal{I}_Y(G; L)$. Since G is L-coprime to Y, we can find two disjoint R-element subsets U and V of $\{1, \ldots, S\} \setminus \{j\}$ such that the columns of Y indexed by either set form a matrix of determinant coprime to $a(G)$. Since $(U \cup V) \cap \{i, j\} \subseteq \{i\}$ and $U \cap V = \emptyset$, without loss of generality, we may assume that $U \cap \{i, j\} = \emptyset$. It now follows from Lemma 2 that

$$\text{card}\,\{\mathbf{x} \in A^S \mid x_i = x_j \text{ and } Y\mathbf{x} = \mathbf{0}\} \leq |A|^{S-R-1}.$$

On recalling the definition of C_1 and combining Cases 1 and 2, the lemma follows. □

Lemma 4. *Let $Y \in \mathbb{Z}^{R\times S}$ satisfy $a_{i,1} + \cdots + a_{i,S} = 0$ $(1 \le i \le R)$. For $L, N \in \mathbb{N}$ with $L \ge R$, let G be a finite abelian group with $c(G) \ge N$ that is L-coprime to Y. Suppose that $A \subseteq G$ for which the equations in (1) are never satisfied simultaneously by distinct elements $x_1, \ldots, x_S \in A$. Then we have*

$$\sup_{\chi\neq 1}\left|\sum_{x\in A}\chi(x)\right| \le d(N-1)|G| - |A|.$$

Proof. This proof can be carried out in the same way as the proof of [7, Lemma 3]. To do this, in the proof of [7, Lemma 3], we set $r_i = -1$, and we replace the condition that G is coprime to $\mathbf{r}$ with the condition that G is L-coprime to Y. We also change the notion of non-trivial solutions in [7] to solutions with distinct coordinates. Finally, we replace the linear equation $r_1x_1 + \cdots + r_sx_s = 0$ with the system of Eq. (1). □

Lemma 5. *For $Y = (a_{i,j}) \in \mathbb{Z}^{R\times S}$ and $L \in \mathbb{N}$ with $L \ge R$, suppose that G is a finite abelian group that is L-coprime to Y. Let*

$$Q = Q_Y(G; L) = \{B \subseteq \mathcal{I}_Y(G; L) \mid |B| = L - R + 1\}.$$

For $B \in Q$, let

$$\Gamma_B = \Gamma_{B,Y}(G; L) = \{\boldsymbol{\chi} = (\chi_1, \ldots, \chi_R) \in \hat{G}^R \mid \chi_1^{a_{1,j}} \cdots \chi_R^{a_{R,j}} \neq 1 \ (j \in B)\}.$$

Then we have

$$\Gamma(G) \subseteq \bigcup_{B\in Q}\Gamma_B.$$

Proof. Let $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_R) \in \Gamma(G)$. Select any R columns indexed by $\{l_1, \ldots, l_R\} \subseteq \mathcal{I}_Y(G; L)$, and we denote by $Z = (a_{i,l_j})_{1\le i,j\le R}$ the matrix formed by these columns. Suppose that $\chi_1^{a_{1,l_i}} \cdots \chi_R^{a_{R,l_i}} = 1$ for every $i \in \{1, \ldots, R\}$. Let ρ be an isomorphism from $\hat{G}$ to G. It follows that for $1 \le i \le R$,

$$0 = \rho(1) = \rho(\chi_1^{a_{1,l_i}} \cdots \chi_R^{a_{R,l_i}}) = a_{1,l_i}\rho(\chi_1) + \cdots + a_{R,l_i}\rho(\chi_R).$$

Write $\rho(\boldsymbol{\chi}) = (\rho(\chi_1), \ldots, \rho(\chi_R))$. Then the above equation is equivalent to having $\rho(\boldsymbol{\chi})Z = \mathbf{0}$. Since G is L-coprime to Y, we have $\gcd(\det Z, a(G)) = 1$. By Lemma 2, we have $\rho(\boldsymbol{\chi}) = \mathbf{0}$. It follows that $\boldsymbol{\chi} = \mathbf{1}$, contradicting the fact that $\boldsymbol{\chi} \in \Gamma(G)$.

Since we can find an element k such that $\chi_1^{a_{1,k}} \cdots \chi_R^{a_{R,k}} \neq 1$ amongst any R-element subset of $\mathcal{I}_Y(G; L)$, it follows that there are at least $L - R + 1$ values $k \in \mathcal{I}_Y(G; L)$ with $\chi_1^{a_{1,k}} \cdots \chi_R^{a_{R,k}} \neq 1$. That is, there exists $B \subseteq \mathcal{I}_Y(G; L)$ with $|B| = L - R + 1$ such that $\boldsymbol{\chi} \in \Gamma_B$. This completes the proof of the lemma. □

Lemma 6. *Let $Y \in \mathbb{Z}^{R\times S}$ satisfy $a_{i,1} + \cdots + a_{i,S} = 0$ $(1 \le i \le R)$. For $L, N \in \mathbb{N}$ with $L \ge R$, let G be a finite abelian group with $c(G) \ge N$ that is L-coprime to Y. Suppose that $A \subseteq G$ for which the equations in (1) are never satisfied simultaneously by distinct elements $x_1, \ldots, x_S \in A$. Then we have*

$$|G|^{-R}\sum_{\boldsymbol{\chi}\in\Gamma(G)}|F_1\cdots F_S(\boldsymbol{\chi})| \le C_2(d(N-1)|G| - |A|)^{L-R+1}|A|^{S-L-1},$$

where $C_2 = C_2(Y; L) = \binom{L}{L-R+1}$.

Proof. Let Q and Γ_B $(B \in Q)$ be defined as in Lemma 5. We have

$$|G|^{-R}\sum_{\boldsymbol{\chi}\in\Gamma_B}|F_1\cdots F_S(\boldsymbol{\chi})| \le \left(\sup_{\boldsymbol{\chi}\in\Gamma_B}\prod_{j\in B}|F_j(\boldsymbol{\chi})|\right)\cdot|G|^{-R}\sum_{\boldsymbol{\chi}\in\hat{G}^R}\prod_{j\notin B}|F_j(\boldsymbol{\chi})|.$$

By Lemma 4, we see that for $j \in B$,

$$\sup_{\chi \in \Gamma_B} |F_j(\chi)| \le d(N-1)|G| - |A|.$$

Since G is L-coprime to Y, there are two disjoint R-element subsets U and V of $\{1, \ldots, S\} \setminus B$ such that the columns of Y indexed by either set form a matrix of determinant coprime to $a(G)$. Let Z be an $R \times R$ matrix formed by the columns indexed by U (or V). Note that since $\gcd(\det Z, a(G)) = 1$, by Lemma 2, for $\mathbf{y}_1, \mathbf{y}_2 \in A^R$, we have $Z\mathbf{y}_1 = Z\mathbf{y}_2$ if and only if $\mathbf{y}_1 = \mathbf{y}_2$. Then by (2), we have

$$|G|^{-R} \sum_{\chi \in \hat{G}^R} \left| \prod_{\substack{j \in U \\ (\text{or } j \in V)}} F_j(\chi) \right|^2 = \operatorname{card}\{(\mathbf{y}_1, \mathbf{y}_2) \in A^R \times A^R \mid Z\mathbf{y}_1 = Z\mathbf{y}_2\} = |A|^R.$$

On combining the above equality with the Cauchy–Schwarz inequality, we see that

$$\begin{aligned} |G|^{-R} \sum_{\chi \in \hat{G}^R} \prod_{j \notin B} |F_j(\chi)| &\le |A|^{S-|B|-2R} \cdot |G|^{-R} \sum_{\chi \in \hat{G}^R} \left| \prod_{j \in U} F_j(\chi) \right| \left| \prod_{j \in V} F_j(\chi) \right| \\ &\le |A|^{S-|B|-2R} \left(|G|^{-R} \sum_{\chi \in \hat{G}^R} \left| \prod_{j \in U} F_j(\chi) \right|^2 \right)^{\frac{1}{2}} \left(|G|^{-R} \sum_{\chi \in \hat{G}^R} \left| \prod_{j \in V} F_j(\chi) \right|^2 \right)^{\frac{1}{2}} \\ &= |A|^{S-|B|-R}. \end{aligned}$$

On combining the above three inequalities, we have

$$|G|^{-R} \sum_{\chi \in \Gamma_B} |F_1 \cdots F_S(\chi)| \le (d(N-1)|G| - |A|)^{L-R+1} |A|^{S-L-1}.$$

By Lemma 5, $\Gamma(G) \subseteq \bigcup_{B \in Q} \Gamma_B$. Since $|\mathcal{I}_Y(G; L)| = L$, we have $|Q| = \binom{L}{L-R+1} = C_2$. It follows that

$$|G|^{-R} \sum_{\chi \in \Gamma(G)} |F_1 \cdots F_S(\chi)| \le C_2 (d(N-1)|G| - |A|)^{L-R+1} |A|^{S-L-1}.$$

This completes the proof of the lemma. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. This statement will follow by induction. Since $d(N) \le 1$ and $C > 1$, we trivially have that $d(N) \le (\frac{C}{N})^{\frac{L-R+1}{R}}$ whenever $N \le C$. Let $N > C$, and assume that $d(N-1) \le (\frac{C}{N-1})^{\frac{L-R+1}{R}}$. Let G be a finite abelian group with $c(G) \ge N$ that is L-coprime to Y. Suppose that $A \subseteq G$ for which $|A| = D(G)$ and the equations in (1) are never satisfied simultaneously by distinct elements $x_1, \ldots, x_S \in A$. By (3), we have

$$|G|^{-R} |F_1(\mathbf{1}) \cdots F_S(\mathbf{1})| - |G|^{-R} \sum_{\chi \in \Gamma(G)} |F_1 \cdots F_S(\chi)| \le T(A).$$

On applying Lemmas 3 and 6, there exist computable constants $C_1, C_2 > 0$ such that

$$|G|^{-R} |A|^S - C_2 (d(N-1)|G| - |A|)^{L-R+1} |A|^{S-L-1} \le C_1 |A|^{S-R-1}.$$

Let $d^*(G) = |A||G|^{-1}$. We have

$$d^*(G)^S - C_1 d^*(G)^{S-R-1} |G|^{-1} - C_2 (d(N-1) - d^*(G))^{L-R+1} d^*(G)^{S-L-1} \le 0. \tag{4}$$

We consider two cases.

- *Case* 1: Suppose that $d^*(G)^S - C_1 d^*(G)^{S-R-1}|G|^{-1} \le \frac{1}{2} d^*(G)^S$. Since $c(G) \ge N$, we have $|G| \ge 2^N$, and hence

  $$d^*(G) \le (2C_1)^{\frac{1}{R+1}} |G|^{-\frac{1}{R+1}} \le (2C_1)^{\frac{1}{R+1}} 2^{-\frac{N}{R+1}}.$$

  For $x > 0$, the function $2^{-\frac{x}{R+1}} x^{\frac{L-R+1}{R}}$ obtains its maximum of $(\frac{(R+1)(L-R+1)}{\mathrm{Re}\log 2})^{\frac{L-R+1}{R}}$ when $x = \frac{(R+1)(L-R+1)}{R\log 2}$. Thus, provided that $C \ge \frac{(R+1)(L-R+1)}{\mathrm{Re}\log 2}(2C_1)^{\frac{R}{(R+1)(L-R+1)}}$, we have

  $$d^*(G) \le (C/N)^{\frac{L-R+1}{R}}.$$

- *Case* 2: Suppose that $d^*(G)^S - C_1 d^*(G)^{S-R-1}|G|^{-1} > \frac{1}{2} d^*(G)^S$. We can deduce from (4) that

  $$d^*(G)^{L+1} < 2C_2(d(N-1) - d^*(G))^{L-R+1}.$$

  By setting $C_3 = (2C_2)^{-\frac{1}{L-R+1}}$, we have

  $$C_3 d^*(G)^{\frac{L+1}{L-R+1}} + d^*(G) < d(N-1).$$

  Assume that $C \ge \frac{C_4}{C_4-1}$, where $C_4 = (C_3+1)^{\frac{R}{L-R+1}}$. Since the function $x^{\frac{L+1}{R}}(x-1)^{-\frac{L-R+1}{R}} - x$ is decreasing for $x > 1$, when $N > C$, we have

  $$N^{\frac{L+1}{R}}(N-1)^{-\frac{L-R+1}{R}} - N \le C^{\frac{L+1}{R}}(C-1)^{-\frac{L-R+1}{R}} - C \le CC_3.$$

  On combining the above two inequalities with the induction hypothesis, we see that

  $$\begin{aligned} C_3 d^*(G)^{\frac{L+1}{L-R+1}} + d^*(G) &< (C/(N-1))^{\frac{L-R+1}{R}} \\ &\le C_3 (C/N)^{\frac{L+1}{R}} + (C/N)^{\frac{L-R+1}{R}}. \end{aligned}$$

  Since the function $C_3 x^{\frac{L+1}{L-R+1}} + x$ is increasing for $x > 0$, we have

  $$d^*(G) \le (C/N)^{\frac{L-R+1}{R}}.$$

On combining Cases 1 and 2, whenever $C \ge \max\{\frac{(R+1)(L-R+1)}{\mathrm{Re}\log 2}(2C_1)^{\frac{R}{(R+1)(L-R+1)}}, \frac{C_4}{C_4-1}\}$, we obtain

$$d(N) = \sup\{d^*(G) \mid c(G) \ge N \text{ and } G \text{ is } L\text{-coprime to } Y\} \le (C/N)^{\frac{L-R+1}{R}}.$$

This completes the proof of Theorem 1. □

Remark 1. Let $Y = (a_{i,j}) \in \mathbb{Z}^{R\times S}$ satisfy $a_{i,1} + \cdots + a_{i,S} = 0$ $(1 \le i \le R)$. For $L, N \in \mathbb{N}$ with $L \ge R$, let G be a finite abelian group with $c(N) \ge N$ that is L-coprime to Y. Following the notation in [7], we say that a solution $\mathbf{x} = (x_1, \ldots, x_S) \in G^S$ of (1) is *trivial* if $x_{j_1} = \cdots = x_{j_l}$ for some subset $\{j_1, \ldots, j_l\} \subseteq \{1, \ldots, S\}$ with $l \ge 2$ and $a_{i,j_1} + \cdots + a_{i,j_l} = 0$ $(1 \le i \le R)$. Otherwise, we say a solution $\mathbf{x}$ of (1) is *non-trivial*. Let $\tilde{D}(G) = \tilde{D}_Y(G)$ denote the maximal cardinality of a set $A \subseteq G$ for which (1) has no non-trivial solution with $x_j \in A$ $(1 \le j \le S)$. Since a solution $\mathbf{x}$ of (1) with distinct coordinates is also a non-trivial solution, we have $\tilde{D}(G) \le D(G)$. Thus, by Theorem 1, there exists a positive constant $C = C(Y; L)$ such that $\tilde{D}(G) \le |G|(C/N)^{\frac{L-R+1}{R}}$.

Remark 2. Let $Y = (a_{i,j}) \in \mathbb{Z}^{R\times S}$ satisfy $a_{i,1} + \cdots + a_{i,S} = 0$ $(1 \le i \le R)$, and let G be a finite abelian group that is R-coprime to Y. For $k \in \mathbb{N}$ and $G = \mathbb{Z}/k\mathbb{Z}$, Roth [10] proved that $D(\mathbb{Z}/k\mathbb{Z}) = O(k/(\log\log k)^{1/R^2})$. By combining his result with Theorem 1, the proof of [8, Corollary 1.3] yields that for a finite abelian group G, we have $D(G) = O(|G|/(\log\log|G|)^{1/R^2})$. By adapting Bourgain's method in [2], one can significantly improve Roth's bound for $D(\mathbb{Z}/k\mathbb{Z})$ by replacing the power of log log k with a power of log k. This would lead to a better bound for $D(G)$.

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