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A prime analogue of the Erdös–Pomerance conjecture for elliptic curves

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Abstract. Let $E/\mathbb{Q}$ be an elliptic curve of rank ≥ 1 and $b \in E(\mathbb{Q})$ a rational point of infinite order. For a prime p of good reduction, let $g_b(p)$ be the order of the cyclic group generated by the reduction $\bar{b}$ of b modulo p. We denote by $\omega(g_b(p))$ the number of distinct prime divisors of $g_b(p)$. Assuming the GRH, we show that the normal order of $\omega(g_b(p))$ is $\log\log p$. We also prove conditionally that there exists a normal distribution for the quantity

$$\frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}}.$$

The latter result can be viewed as an elliptic analogue of a conjecture of Erdös and Pomerance about the distribution of $\omega(f_a(n))$, where a is a natural number > 1 and $f_a(n)$ the order of a modulo n.

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1. Introduction

For $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, let $\omega(n)$ denote the number of distinct prime divisors of n. The Turán Theorem is about the second moment of $\omega(n)$ [23]; it states that for $x \in \mathbb{R}$, $x > 1$,

$$\sum_{n \leq x} (\omega(n) - \log\log x)^2 \ll x \log\log x.$$

Turán's result implies an earlier theorem of Hardy and Ramanujan [8], which states that for any $\varepsilon > 0$

$$\#\{n \leq x \mid n \text{ satisfies } |\omega(n) - \log\log n| > \varepsilon \log\log n\}$$

is $o(x)$ as $x \to \infty$. In other words, the normal order of $\omega(n)$ is $\log\log n$. The significance of the '$\log\log n$' term is that it is about $\sum_{p \leq n} \frac{\omega(p)}{p}$ where p runs over primes.

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The idea behind Turán's proof was essentially probabilistic. Further development of probabilistic ideas led Erdös and Kac [5] to prove a remarkable refinement of the Turán Theorem, namely, the existence of a normal distribution for $\omega(n)$. More precisely, they proved that for $\gamma \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{1}{x} \#\Big\{n \le x \;\Big|\; n \text{ satisfies } \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \le \gamma\Big\} = G(\gamma) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\gamma} e^{\frac{-t^2}{2}}\, dt.$$

The theorem of Erdös and Kac opened a door to the study of probabilistic number theory. In the early 1960s and subsequently the 1970s, the theory was refined by many authors, culminating in a generalized Erdös–Kac theorem proved independently by Kubilius [10] and Shapiro [20]. Their result is applicable to what are called 'strongly additive functions'. The interested reader can find a comprehensive treatment of it in the monograph of Elliott [3].

We can also consider functions that are not strongly additive, say the Euler's φ-function. Using the same principle of the work of Kubilius and Shapiro, the issue of $\omega(\varphi(n))$ devolves upon the estimation of the sums

$$\sum_{p\le x} \omega(p-1) \quad \text{and} \quad \sum_{p\le x} \omega^2(p-1),$$

where p denotes a rational prime. Sums of this type were estimated by Haselgrove [9] and Erdös and Pomerance [6]. They proved that

$$\sum_{p\le x} \omega(p-1) = \pi(x)\log\log x + O\big(\pi(x)\big)$$

and

$$\sum_{p\le x} \omega^2(p-1) = \pi(x)(\log\log x)^2 + O\big(\pi(x)\log\log x\big),$$

where $\pi(x)$ is the number of rational primes $\le x$. Applying partial summation, we can derive from the above equalities that

$$\sum_{p\le n} \frac{\omega(p-1)}{p} = \frac{1}{2}(\log\log n)^2 + O\big(\log\log n\big)$$

and

$$\sum_{p\le n} \frac{\omega^2(p-1)}{p} = \frac{1}{3}(\log\log n)^3 + O\big((\log\log n)^2\big).$$

As a consequence we have the following result of Erdös and Pomerance [6], which states that

$$\lim_{x\to\infty} \frac{1}{x} \#\Big\{n \le x \;\Big|\; n \text{ satisfies } \frac{\omega(\varphi(n)) - \frac{1}{2}(\log\log n)^2}{\frac{1}{\sqrt{3}}(\log\log n)^{3/2}} \le \gamma\Big\} = G(\gamma).$$

In [6], Erdös and Pomerance also proposed the following question. Let a be a positive integer > 1. For any natural number n coprime to a, let $f_a(n)$ denote the order of a modulo n. Thus $f_a(n)$ is a divisor of $\varphi(n)$. Based on the belief that the difference between $\omega(\varphi(n))$ and $\omega(f_a(n))$ is 'small on average', Erdös and Pomerance conjectured that

$$\lim_{x\to\infty} \frac{1}{x} \#\Big\{ n \le x \,\Big|\, n \text{ satisfies } (a,n) = 1 \text{ and } \frac{\omega(f_a(n)) - \frac{1}{2}(\log\log n)^2}{\frac{1}{\sqrt{3}}(\log\log n)^{3/2}} \le \gamma \Big\} = \frac{\varphi(a)}{a} G(\gamma).$$

The conjecture remains open until today. Even a conditional result was only obtained recently by Murty and Saidak [17] under the assumption of the GRH (i.e., the Riemann Hypothesis for all Dedekind zeta functions of number fields). Later Li and Pomerance [13] also provided an alternative proof of the same result. The difficulty of this conjecture lies in the intervention of the distribution of primes in the non-abelian extensions $\mathbb{Q}(\zeta_q, \sqrt[q]{a})$ where q varies over rational primes and ζ_q is a primitive q-th root of unity.

Let us recall that $f_a(n)$ is the least common multiple of $\{f_a(p^\gamma) \mid p^\gamma \parallel n\}$ where p^γ is the exact power of p which divides n. Also $f_a(p^\gamma)$ divides $p^{\gamma-1} f_a(p)$. Thus similarly to the case of $\omega(\varphi(n))$, to study the conjecture of Erdös and Pomerance, it is sufficient to estimate the sums

$$\sum_{p\le x} \omega(f_a(p)) \quad \text{and} \quad \sum_{p\le x} \omega^2(f_a(p)).$$

Under the assumption of the GRH, Murty and Saidak proved that

$$\sum_{p\le x} \omega(f_a(p)) = \pi(x) \log\log x + O\big(\pi(x)\big)$$

and

$$\sum_{p\le x} \omega^2(f_a(p)) = \pi(x)(\log\log x)^2 + O\big(\pi(x)\log\log x\big).$$

A conditional result of the conjecture follows.

In [17], Murty and Saidak also proved the following 'prime analogue' of the Erdös–Pomerance conjecture:

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\Big\{ p \le x \,\Big|\, p \text{ satisfies } (a,p) = 1 \text{ and } \frac{\omega(f_a(p)) - \log\log p}{\sqrt{\log\log p}} \le \gamma \Big\} = G(\gamma).$$

In a sense, as we see from [17, §5, §7], there is not much difference between the study of $\omega(f_a(n))$ and $\omega(f_a(p))$, as the main technical difficulty of both problems depends on the study of $\omega(i_a(p))$, where $i_a(p) = (p-1)/f_a(p)$.

The purpose of this paper is to formulate an analogous Erdös–Pomerance conjecture for elliptic curves and provide a conditional proof of it. Let $E/\mathbb{Q}$ be an elliptic curve of rank ≥ 1. Let $b \in E(\mathbb{Q})$ be a rational point of infinite order. For a prime p of good reduction, let $g_b(p)$ be the order of $\langle \bar{b} \rangle$, the cyclic group generated by the reduction $\bar{b}$ of b modulo p. The function $g_b(p)$ can be viewed as an elliptic analogue of $f_a(p)$. Thus, an analogous formulation of the conjecture of Erdös and Pomerance for elliptic curves is that there exists a normal distribution for the quantity

$$\frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}}.$$

We prove the following result.

Theorem 1. *Let $E/\mathbb{Q}$ be an elliptic curve of rank ≥ 1 and $b \in E(\mathbb{Q})$ a rational point of infinite order. For a prime p of good reduction, let $\langle \bar{b} \rangle$ be the cyclic group generated by the reduction $\bar{b}$ of b modulo p and $g_b(p)$ its order. Assuming the* GRH*, we have*

$$\sum_{\substack{p \leq x \\ p \text{ of good reduction}}} \big(\omega(g_b(p)) - \log\log x\big)^2 \ll \pi(x) \log\log x.$$

As a direct consequence of Theorem 1 we have

Corollary 2. *Assuming the* GRH*, the normal order of $\omega(g_b(p))$ is* $\log\log p$.

The following theorem is an analogous result of Murty and Saidak for elliptic curves.

Theorem 3. *Let $E/\mathbb{Q}$, b, and $g_b(p)$ be defined as in Theorem* 1*. Let $\gamma \in \mathbb{R}$. Assuming the* GRH*, we have*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\Big\{ p \leq x \;\Big|\; p \text{ is of good reduction and } \frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}} \leq \gamma \Big\} = G(\gamma).$$

Thus, we obtain an elliptic analogue of a conjecture of Erdös and Pomerance in terms of primes.

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Notation. For $x \in \mathbb{R}, x > 0$, let $f(x)$ and $g(x)$ be two functions of x. If $g(x)$ is positive and there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$, we write either $f(x) \ll g(x)$ or $f(x) = O(g(x))$. If both $f(x)$ and $g(x)$ are positive, we use $f(x) \asymp g(x)$ to denote that $f(x) = O(g(x))$ and $g(x) = O(f(x))$. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, we write $f(x) = o(g(x))$. Also, we use $\bar{\mathbb{Q}}$ and $\bar{\mathbb{F}}_p$ to denote some fixed algebraic closures of $\mathbb{Q}$ and $\mathbb{F}_p$ respectively.

2. Preliminaries

We first recall some theorems about elliptic curves that will be needed later. Let $E/\mathbb{Q}$ be an elliptic curve of rank ≥ 1. For a prime $l \in \mathbb{N}$, we denote by $E[l]$ the l-torsion points. By adjoining to $\mathbb{Q}$ the coordinates of the l-torsion points, we obtain $\mathbb{Q}(E[l])$, a finite Galois extension of $\mathbb{Q}$. Since

$$E[l] \cong (\mathbb{Z}/l\mathbb{Z}) \times (\mathbb{Z}/l\mathbb{Z})$$

(see [21, Corollary 6.4]), by choosing a basis, we have a natural injection

$$\Phi_l : \mathrm{Gal}\big(\mathbb{Q}(E[l])/\mathbb{Q}\big) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/l\mathbb{Z}).$$

In the following discussion we will abuse our notation by identifying an element $\gamma \in \mathrm{Gal}\big(\mathbb{Q}(E[l])/\mathbb{Q}\big)$ with its image $\Phi_l(\gamma) \in \mathrm{GL}_2(\mathbb{Z}/l\mathbb{Z})$.

Let $b \in E(\mathbb{Q})$ be a rational point of infinite order. We denote by $l^{-1}b$ the set of elements $v \in E(\bar{\mathbb{Q}})$ such that

$$[l]\, v = \underbrace{v + v + \cdots + v}_{l \text{ times}} = b.$$

Define $L_l = \mathbb{Q}(E[l], l^{-1}b)$, which is a finite extension of $\mathbb{Q}(E[l])$. We have the following theorem.

Theorem 4 (Bachmakov [1]). *For a prime l, the Galois group $\mathrm{Gal}(L_l/\mathbb{Q}(E[l]))$ can be identified with a subgroup of $E[l]$ and is equal to $E[l]$ for all but finitely many l.*

The group $\mathrm{GL}_2(\mathbb{Z}/l\mathbb{Z})$ acts naturally on $E[l]$ by matrix multiplication. We denote this action by $*$ and we see that it induces a semidirect product $E[l] \rtimes \mathrm{GL}_2(\mathbb{Z}/l\mathbb{Z})$. Let G_l be the Galois group $\mathrm{Gal}(L_l/\mathbb{Q})$. From Theorem 4, for all but finitely many l, we have

$$G_l \cong E[l] \rtimes \mathrm{Gal}\big(\mathbb{Q}(E[l])/\mathbb{Q}\big),$$

which is a subgroup of $E[l] \rtimes \mathrm{GL}_2(\mathbb{Z}/l\mathbb{Z})$.

An element $(\tau, \gamma) \in G_l$ acts on $E[l]$ and $l^{-1}b$ as follows: let $v_0 \in l^{-1}b$ be a fixed element; for $u \in E[l]$ and $v \in l^{-1}b$ we have

- $(\tau, \gamma) \cdot u := \gamma * u$.
- $(\tau, \gamma) \cdot v := v_0 + \gamma * (v - v_0) + \tau$.

Notice that since $[l]v = [l]v_0 = b$, $(v - v_0) \in E[l]$. Thus, $\gamma * (v - v_0)$ is well defined. Also, since both $(v - v_0)$ and τ are in $E[l]$, for $v \in l^{-1}b$, we have

$$[l]\big((\tau, \gamma) \cdot v\big) = [l]v_0 = b.$$

Thus, (τ, γ) is a well-defined action on the set $l^{-1}b$. Moreover, for $v \in l^{-1}b$, we have

$$(\tau, \gamma) \cdot v = v \quad \text{if and only if} \quad (\gamma - I) * (v_0 - v) = \tau,$$

where I is the 2×2 identity matrix.

Let p be a prime of good reduction. We denote by $\overline{E}$ the reduction of E modulo p. Let $\overline{E}(\mathbb{F}_p)$ be the set of rational points of $\overline{E}$ defined over the finite field $\mathbb{F}_p$. Let $b \in E(\mathbb{Q})$ be a rational point of infinite order and $\bar{b} \in \overline{E}(\mathbb{F}_p)$ the reduction of b modulo p. Let $\langle \bar{b} \rangle$ be the cyclic group generated by $\bar{b}$, which is a subgroup of $\overline{E}(\mathbb{F}_p)$. We denote by $g_b(p)$ the order of $\langle \bar{b} \rangle$. Thus $g_b(p)$ is a divisor of $\#\overline{E}(\mathbb{F}_p)$. We write

$$\#\overline{E}(\mathbb{F}_p) = g_b(p) \cdot i_b(p),$$

where $i_b(p)$ is the index of $\langle \bar{b} \rangle$ in $\overline{E}(\mathbb{F}_p)$. Let Δ be the discriminant of E. For $p \nmid l\Delta$, Lang and Trotter [12] gave a condition on the Frobenius element $(\tau_p, \gamma_p) \in G_l$ in order that $l \mid i_b(p)$. We review their arguments below.

Notice that $l \mid i_b(p)$ implies that $l \mid \#\overline{E}(\mathbb{F}_p)$. Since

$$\operatorname{tr} \gamma_p \equiv p + 1 - \#\overline{E}(\mathbb{F}_p) \pmod{l}$$

and

$$\det \gamma_p \equiv p \pmod{l}$$

(see [22, p. 172]), if $l \mid \#\overline{E}(\mathbb{F}_p)$, we have

$$1 - \operatorname{tr} \gamma_p + \det \gamma_p \equiv 0 \pmod{l}.$$

Thus $\gamma_p \in \mathrm{Gal}(\mathbb{Q}(E[l])/\mathbb{Q}) \subseteq \mathrm{GL}_2(\mathbb{Z}/l\mathbb{Z})$ has an eigenvalue 1.

We consider first the case when $\gamma_p = I$. We recall that the cyclic group generated by $\pi_p \colon x \mapsto x^p$ is dense in $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. The group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ acts on $w \in \overline{E}(\bar{\mathbb{F}}_p)$ coordinatewise. Thus for $w \in \overline{E}(\bar{\mathbb{F}}_p)$ we have

$$\pi_p \cdot w = w \quad \text{if and only if} \quad w \in \overline{E}(\mathbb{F}_p).$$

Let $w_1 \in E(\mathbb{Q}(E[l]))$. The Frobenius element $\gamma_p \in \mathrm{Gal}(\mathbb{Q}(E[l])/\mathbb{Q})$ acts on w_1 coordinatewise. This action is compatible with π_p in the following sense: let $\bar{w}_1 \in \overline{E}(\bar{\mathbb{F}}_p)$ be the reduction of w_1 modulo p; we have

$$\overline{\gamma_p \cdot w_1} = \pi_p \cdot \bar{w}_1.$$

Thus for $\gamma_p = I$ we have

$$\bar{w}_1 = \overline{\gamma_p \cdot w_1} = \pi_p \cdot \bar{w}_1.$$

It follows that $\bar{w}_1 \in \overline{E}(\mathbb{F}_p)$. Let $\overline{E}[l]$ denote the reduction of $E[l]$ modulo p. Since $E[l] \subseteq E(\mathbb{Q}(E[l]))$, the above argument shows that

$$\overline{E}(\mathbb{F}_p) \supseteq \overline{E}[l] \cong (\mathbb{Z}/l\mathbb{Z}) \times (\mathbb{Z}/l\mathbb{Z}), \quad \text{provided that } p \nmid l\Delta$$

(see [21, Corollary 6.4]). Consider the subgroup $\langle \bar{b} \rangle$ in $\overline{E}(\mathbb{F}_p)$. Since $\langle \bar{b} \rangle$ is cyclic, it can not contain two $(\mathbb{Z}/l\mathbb{Z})$ factors. Thus, at least one of $(\mathbb{Z}/l\mathbb{Z})$ factors of $\overline{E}(\mathbb{F}_p)$ is contained in $\overline{E}(\mathbb{F}_p)/\langle \bar{b} \rangle$. Since $i_b(p)$ is the order of $\overline{E}(\mathbb{F}_p)/\langle \bar{b} \rangle$, we have $l \mid i_b(p)$. We conclude that for $\gamma_p = I$, l is a divisor of $i_b(p)$.

On the other hand, if γ_p has an eigenvalue 1 and $\gamma_p \neq I$, $\overline{E}(\mathbb{F}_p)$ can not contain a $(\mathbb{Z}/l\mathbb{Z}) \times (\mathbb{Z}/l\mathbb{Z})$ factor. Hence, the l-torsion points of $\overline{E}(\mathbb{F}_p)$, which is the kernel of the map $\gamma_p - I \colon E[l] \to E[l]$, form a cyclic subgroup. In other words, the l-primary part of $\overline{E}(\mathbb{F}_p)$ is of the form $\mathbb{Z}/l^\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{N}$. Write

$$\overline{E}(\mathbb{F}_p) \cong \mathbb{Z}/l^\alpha\mathbb{Z} \times H,$$

where H is an abelian group with $(|H|, l) = 1$. We will abuse our notation by identifying an element in $\overline{E}(\mathbb{F}_p)$ with its image in $\mathbb{Z}/l^\alpha\mathbb{Z} \times H$. For $\bar{b} \in \overline{E}(\mathbb{F}_p)$, without loss of generality, we can assume that either $\bar{b} = (0, h)$ or $\bar{b} = (l^\beta, h)$ where $h \in H$ and $\beta \geq 0$.

Case 1. Suppose $\bar{b} = (0, h)$. Since $(|H|, l) = 1$, the element $\bar{b}_l = (0, l^{-1}h) \in \overline{E}(\mathbb{F}_p)$ is well defined and $[l]\bar{b}_l = \bar{b}$.

Case 2. Suppose $\bar{b} = (l^\beta, h)$. If $\beta = 0$, the order of the cyclic group $\langle b \rangle$ is divisible by l^α, i.e., $l \nmid i_b(p)$. Hence, if $l \mid i_b(p)$, it implies that $\beta \geq 1$. Choosing $\bar{b}_l = (l^{\beta-1}, l^{-1}h) \in \overline{E}(\mathbb{F}_p)$, we have $[l]\bar{b}_l = \bar{b}$.

We conclude that if γ_p has an eigenvalue 1, $\gamma_p \neq 1$ and $l \mid i_b(p)$, there exists $\bar{b}_l \in \overline{E}(\mathbb{F}_p)$ such that $[l]\bar{b}_l = \bar{b}$. Let $b_l \in \overline{E}(\overline{\mathbb{Q}})$ such that the reduction of b_l modulo p is $\bar{b}_l$. Since $[l]\bar{b}_l = \bar{b}$, it follows that $b_l \in l^{-1}b$. Moreover, since $\bar{b}_l \in E(\mathbb{F}_p)$, we have

$$(\tau_p, \gamma_p) \cdot b_l = b_l,$$

which is equivalent to

$$(\gamma_p - I) * (v_0 - b_l) = \tau_p,$$

i.e., $\tau_p \in \mathrm{Im}(\gamma_p - I)$.

Define a subset S_l of G_l as follows: an element (τ, γ) of G_l belongs to S_l if it satisfies one of the two following conditions:

(1) $\gamma = I$ or

(2) γ has an eigenvalue 1, $\ker((\gamma - I)\colon E[l] \to E[l])$ is cyclic, and $\tau \in \mathrm{Im}(\gamma - I)$.

Notice that S_l is a union of conjugacy classes of G_l. Combining all the above discussions, we obtain the following result of Lang and Trotter.

Theorem 5 (Lang and Trotter [12]). *Let $i_b(p)$ be the index of the cyclic group $\langle \bar{b} \rangle$ in $\bar{E}(\mathbb{F}_p)$. For a prime $l \in \mathbb{N}$, $p \nmid l\Delta$, the following two statements are equivalent:*

(1) $l \mid i_b(p)$.

(2) $(\tau_p, \gamma_p) \in S_l$.

Another important ingredient of the proof of Theorems 1 and 3 is the Chebotarev density theorem. Let $L/\mathbb{Q}$ be a finite Galois extension of degree n_L and discriminant d_L. We denote by G the Galois group of $L/\mathbb{Q}$ and C a union of conjugacy classes of G. Let $\sigma_p \in G$ be a Frobenius element. Define

$$\pi_C\big(x, L/\mathbb{Q}\big) = \#\big\{p \le x \mid p \text{ is an unramified prime in } L/\mathbb{Q} \text{ and } \sigma_p \subseteq C\big\}.$$

We have

Theorem 6 (Lagarias and Odlyzko [11], Serre [19]). *Assuming the* GRH *for the Dedekind zeta function of L, we have*

$$\pi_C\big(x, L/\mathbb{Q}\big) = \frac{|C|}{|G|}\,\mathrm{li}\,x + O\left(|C|\,x^{\frac{1}{2}}\left(\frac{\log|d_L|}{n_L} + \log x\right)\right),$$

where $\mathrm{li}\,x = \int_2^x \frac{dt}{\log t}$.

The following theorem is useful for estimating the error term in the Chebotarev density theorem.

Theorem 7 (Serre [19]). *Let $L/\mathbb{Q}$ be a finite Galois extension of degree n_L and discriminant d_L. We have*

$$\frac{n_L}{2} \sum_{q \text{ ramified}} \log q \le \log|d_L| \le (n_L - 1) \sum_{q \text{ ramified}} \log q + n_L \log n_L,$$

where the sum is over all primes q that are ramified in L.

3. Prime divisors of $i_b(p)$

We recall that $i_b(p)$ is the index of $\langle\bar{b}\rangle$ in $\overline{E}(\mathbb{F}_p)$. In this section, we consider the number of distinct prime divisors of $i_b(p)$. The following lemma is essential for the proof of Theorems 1 and 3. We use the notation $\sum'$ to denote the sum over primes of good reduction.

Lemma 8. *Assuming the* GRH*, we have*

$$\sum_{p\le x}{}' \omega^2(i_b(p)) \ll \pi(x).$$

Proof. Let $y = x^\delta$ with $0 < \delta < 1$ (a choice of δ will be made later). Define a truncation function ω_y of ω as follows:

$$\omega_y(i_b(p)) = \#\{l \le y \mid l \text{ is a prime and } l \mid i_b(p)\}.$$

For a prime $p \le x$, since

$$i_b(p) \le \#\overline{E}(\mathbb{F}_p) \le (p + 2\sqrt{p} + 1) \le 3x,$$

it follows that

$$\omega(i_b(p)) = \omega_y(i_b(p)) + O(1).$$

Hence we have

$$\begin{aligned}\sum_{p\le x}{}' \omega^2(i_b(p)) &= \sum_{p\le x}{}' \big(\omega_y(i_b(p)) + O(1)\big)^2 \ll \sum_{p\le x}{}' \omega_y^2(i_b(p)) + O\big(\pi(x)\big)\\ &= \sum_{\substack{l_1,l_2\le y\\ l_1\ne l_2}} \sum_{\substack{p\le x\\ l_1l_2\mid i_b(p)}}{}' 1 + \sum_{l\le y} \sum_{\substack{p\le x\\ l\mid i_b(p)}}{}' 1 + O\big(\pi(x)\big),\end{aligned}$$

where l_1, l_2, and l are rational primes. Consider the sum

$$\sum_{l\le y} \sum_{\substack{p\le x\\ l\mid i_b(p)}}{}' 1.$$

Applying Theorems 5, 6 and 7 for all but finitely many primes l, under the GRH we have

$$\begin{aligned}&\#\{p \le x \mid p \text{ satisfies } l \mid i_b(p)\}\\ &\quad = \operatorname{li} x \cdot \frac{|S_l|}{|G_l|} + O\Big(|S_l| \cdot x^{\frac{1}{2}} \cdot \big(\sum_{q \text{ ramified}} \log q + + \log n_l + \log x\big)\Big),\end{aligned}$$

where the sum is over all primes q that are ramified in L_l and $n_l = |G_l|$.

In the case of elliptic curves without complex multiplication (non-CM) Serre [18] proved that for all but finitely many primes l,

$$\mathrm{Gal}(\mathbb{Q}(E[l])/\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Z}/l\mathbb{Z}).$$

Hence, for all but finitely many l, we have

$$|G_l| \asymp l^6 \quad \text{and} \quad |S_l| \asymp l^4.$$

In the case of elliptic curves with complex multiplication (CM), from [7, p. 35–37], we have

$$|G_l| \asymp l^4 \quad \text{and} \quad |S_l| \asymp l^2.$$

It is well known that q is ramified in L_l if and only if $q \mid l\Delta$ (see [2]). Hence, assuming the GRH, we have

$$\sum_{l\le y} \sum_{\substack{p\le x \\ l \mid i_b(p)}}{}' 1 \ll \sum_{l\le y}\left(\frac{\pi(x)}{l^2} + O\big(l^4 x^{\frac{1}{2}} \log(l^6 x\Delta)\big)\right) \ll \pi(x) + O\big(x^{\frac{1}{2}+5\delta+\varepsilon}\big),$$

where $\varepsilon > 0$ is arbitrarily small. Choosing $\delta = \frac{1}{11}$, we have

$$\sum_{l\le y} \sum_{\substack{p\le x \\ l \mid i_b(p)}}{}' 1 \ll \pi(x).$$

Consider the sum

$$\sum_{\substack{l_1,l_2\le y \\ l_1\ne l_2}} \sum_{\substack{p\le x \\ l_1 l_2 \mid i_b(p)}}{}' 1.$$

The group homomorphisms

$$E[l_1 l_2] \to E[l_1] \times E[l_2] \quad \text{and} \quad \mathrm{GL}_2(\mathbb{Z}/l_1 l_2\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/l_1\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/l_2\mathbb{Z}),$$

which are induced by reduction modulo l_1 and l_2 respectively, are indeed isomorphisms. Moreover, these maps are compatible with the actions defined in Section 2. Since $|S_l|/|G_l| \asymp 1/l^2$, by Theorems 5, 6 and 7 we have

$$\begin{aligned}\sum_{\substack{l_1,l_2\le y \\ l_1\ne l_2}} \sum_{\substack{p\le x \\ l_1 l_2 \mid i_b(p)}}{}' 1 &\ll \sum_{\substack{l_1,l_2\le y \\ l_1\ne l_2}} \left(\frac{\pi(x)}{(l_1 l_2)^2} + O\big((l_1 l_2)^4 x^{\frac{1}{2}} \log(l_1^6 l_2^6 x\Delta)\big)\right) \\ &\ll \pi(x) + O\big(x^{\frac{1}{2}+10\delta+\varepsilon}\big),\end{aligned}$$

where $\varepsilon \to 0$ as $x \to \infty$. Choosing $\delta = \frac{1}{21}$, we have

$$\sum_{\substack{l_1,l_2\le y \\ l_1\ne l_2}} \sum_{\substack{p\le x \\ l_1 l_2 \mid i_b(p)}}{}' 1 \ll \pi(x).$$

It follows that

$$\sum_{p\le x}{}' \omega^2(i_b(p)) \ll \pi(x).$$

This completes the proof of Lemma 8. □

4. A Turán analogue of $\omega(g_b(p))$

In this section, we provide a proof of Theorem 1 which states that under the GRH, we have

$$\sum_{p\le x}{}' \big(\omega(g_b(p)) - \log\log x\big)^2 \ll \pi(x)\log\log x.$$

Our proof is a combination of Lemma 8 with the following theorem.

Theorem 9 (Miri and Murty [16], Liu [14]). *Let $E/\mathbb{Q}$ be an elliptic curve. We have (assuming the GRH if E is non-CM)*

$$\sum_{p\le x}{}' \big(\omega(\#\bar{E}(\mathbb{F}_p)) - \log\log x\big)^2 \ll \pi(x)\log\log x.$$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Since

$$\#\bar{E}(\mathbb{F}_p) = g_b(p)\cdot i_b(p),$$

we have

$$\omega(\#\bar{E}(\mathbb{F}_p)) \ge \omega(g_b(p)) \ge \omega(\#\bar{E}(\mathbb{F}_p)) - \omega(i_b(p)).$$

It follows that

$$\begin{aligned}\sum_{p\le x}{}' \big(\omega(g_b(p)) - \log\log x\big)^2 &= \sum_{p\le x}{}' \big(\omega(\#\bar{E}(\mathbb{F}_p)) + O\big(\omega(i_b(p))\big) - \log\log x\big)^2 \\ &\ll \sum_{p\le x}{}' \big(\omega(\#\bar{E}(\mathbb{F}_p)) - \log\log x\big)^2 + \sum_{p\le x}{}' \omega^2(i_b(p)).\end{aligned}$$

Combining Lemma 8 with Theorem 9 we obtain that under the GRH,

$$\sum_{p\le x}{}' \big(\omega(g_b(p)) - \log\log x\big)^2 \ll \pi(x)\log\log x.$$

This completes the proof of Theorem 1. □

5. An Erdös–Kac analogue of $\omega(g_b(p))$

In this section, we give a proof of Theorem 3. More precisely, under the GRH we prove that there exists a normal distribution for the quantity

$$\frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}}.$$

Our proof is dependent on the following theorem.

Theorem 10 (Liu [15]). *Let $E/\mathbb{Q}$ be an elliptic curve. We have* (*assuming the GRH if E is non-CM*)

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\Big\{ p \le x \;\Big|\; p \text{ is of good reduction and } \frac{\omega(\#\overline{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} \le \gamma \Big\} \\ = G(\gamma).$$

Proof of Theorem 3. As in the proof of Theorem 1, we have

$$\frac{\omega(\#\overline{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} \ge \frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}} \\ \ge \frac{\omega(\#\overline{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} - \frac{\omega(i_b(p))}{\sqrt{\log\log p}}.$$

For any $\varepsilon > 0$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, define the set

$$S(\varepsilon, \alpha, \beta) = \Big\{ p \;\Big|\; p \text{ is of good reduction, } \alpha < p \le \beta, \text{ and } \frac{\omega(i_b(p))}{\sqrt{\log\log p}} \ge \varepsilon \Big\}.$$

Let $N(\varepsilon, \alpha, \beta)$ be the cardinality of $S(\varepsilon, \alpha, \beta)$. We have

$$N(\varepsilon, 0, x) \le \pi(\sqrt{x}) + N(\varepsilon, \sqrt{x}, x).$$

Notice that

$$\sideset{}{'}\sum_{p\le x} \omega(i_b(p)) \ge \sum_{p\in S(\varepsilon,\sqrt{x},x)} \omega(i_b(p)) \ge N(\varepsilon, \sqrt{x}, x)\cdot \varepsilon\sqrt{\log\log x - \log 2}.$$

Since $\omega^2(i_b(p)) \ge \omega(i_b(p))$, Lemma 8 implies that

$$N(\varepsilon, \sqrt{x}, x) \ll \frac{\pi(x)}{\sqrt{\log\log x}} = o\big(\pi(x)\big).$$

It follows that

$$N(\varepsilon, 0, x) = o\big(\pi(x)\big).$$

Thus for $\gamma \in \mathbb{R}$ we obtain

$$\#\Big\{p \le x \,\Big|\, p \text{ is of good reduction and } \frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}} \le \gamma\Big\}$$
$$\le \#\Big\{p \le x \,\Big|\, p \text{ is of good reduction and } \frac{\omega(\#\bar{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} - \frac{\omega(i_b(p))}{\sqrt{\log\log p}} \le \gamma\Big\}$$
$$\le \#\Big\{p \le x \,\Big|\, p \text{ is of good reduction and } \frac{\omega(\#\bar{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} \le \gamma + \varepsilon\Big\} + o\big(\pi(x)\big).$$

Also we have

$$\#\Big\{p \le x \,\Big|\, p \text{ is of good reduction and } \frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}} \le \gamma\Big\}$$
$$\ge \#\Big\{p \le x \,\Big|\, p \text{ is of good reduction and } \frac{\omega(\#\bar{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} \le \gamma\Big\}.$$

Combine all of the above results with Theorem 10. As $x \to \infty$, for all $\varepsilon > 0$ we obtain

$$G(\gamma) \le \lim_{x\to\infty} \frac{1}{\pi(x)} \#\Big\{p \le x \,\Big|\, p \text{ is of good reduction and } \frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}} \le \gamma\Big\} \le G(\gamma + \varepsilon).$$

Since $G(\gamma)$ is a continuous function, for any $\varepsilon > 0$ we have

$$G(\gamma + \varepsilon) = G(\gamma) + O(\varepsilon).$$

Let $\varepsilon \to 0$. It follows that under the GRH,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\Big\{p \le x \,\Big|\, p \text{ is of good reduction and } \frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}} \le \gamma\Big\} = G(\gamma).$$

This completes the proof of Theorem 3. □

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