UNIVERSIDAD DE MURCIA

ESCUELA INTERNACIONAL DE DOCTORADO

TESIS DOCTORAL

On geometric and functional Grünbaum type inequalities

Desigualdades geométricas y funcionales de tipo Grünbaum

D. Francisco Marín Sola

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UNIVERSIDAD DE
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A Marta, Carmen y Pepe, porque sin ellos no hubiera sido posible.

A Fuen, porque sin ella no tendría sentido.

Contents

Resumen

El estudio de las desigualdades de tipo Grünbaum ha sido un campo de investigación muy fructífero durante los últimos años. Sus orígenes podemos encontrarlos en el trabajo [3] de Ascoli, publicado en los años treinta, y posteriormente generalizado a dimensión arbitraria por Grünbaum en el artículo [16]. La raíz de este resultado se encuentra en una pregunta natural relativa a los cuerpos convexos: ¿podemos asegurar la existencia de un punto "a priori" del interior de un cuerpo convexo, tal que al cortar este último por dicho punto resulten dos subcuerpos con una cantidad reseñable del volumen total?

A la hora de intentar responder a esta pregunta uno queda naturalmente conducido a la noción de centroide (también conocido como centro de masas) de un cuerpo convexo. Así, para un conjunto compacto K cualquiera, no necesariamente convexo, con volumen positivo $\mathrm{vol}(K)$ (es decir, con medida de Lebesgue n-dimensional positiva) el centroide de dicho conjunto es el punto covariante afín definido como

$$\mathrm{g}(K) := \frac{1}{\mathrm{vol}(K)} \int_K x \, \mathrm{d}x.$$

Volviendo a la pregunta anterior, la desigualdad de Grünbaum asegura entonces que para todo cuerpo convexo K, con centroide en el origen, se tiene que

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{n}{n+1}\right)^n, \tag{$*$}$$

donde $K^- = K \cap \{x \in \mathbb{R}^n : \langle x, u \rangle \leq 0\}$ y $K^+ = K \cap \{x \in \mathbb{R}^n : \langle x, u \rangle \geq 0\}$ representan las partes de K divididas por el hiperplano $H = \{x \in \mathbb{R}^n : \langle x, u \rangle = 0\}$, para un vector unitario u. La igualdad se cumple, fijado u, si y solo si K es un cono en la dirección u, es decir, la envoltura convexa de $\{x\} \cup (K \cap (y + H))$, para ciertos $x, y \in \mathbb{R}^n$.

Aunque escape a los objetivos de este trabajo hacer una lista exhaustiva, merece la pena destacar algunos de los trabajos publicados en los últimos años relacionados con la desigualdad de Grünbaum. Por un lado, podemos encontrar extensiones al caso de secciones [12, 29] y proyecciones [33] de cuerpos convexos, y generalizaciones al contexto analítico de funciones *log-cóncavas* [28] (véase también [21, Lemma 5.4] y [8, Lemma 2.2.6]) y funciones p-cóncavas [29], con $p > 0$. Por otro lado, el estudio de desigualdades de tipo Grünbaum en el contexto de secciones de cuerpos convexos, posteriormente generalizadas para *quermassintegrales* en [32], pueden encontrarse en [11, 23].

A grandes rasgos, esta tesis está dedicada al estudio de generalizaciones y extensiones de la desigualdad de Grünbaum, desde una perspectiva tanto geométrica como funcional (así como desde el enfoque propio de la teoría de la medida).

De forma más precisa, este trabajo comienza con un primer capítulo introductorio donde recopilamos algunas definiciones y resultados que serán utilizados más tarde. Así, la primera sección de este capítulo está dedicada tanto a establecer la notación usada como a recordar algunas nociones importantes, tales como la suma de Minkowski, la noción de cuerpo convexo, etc. Además de esto, algunos resultados importantes como la desigualdad de Brunn-Minkowski o el principio de concavidad de Brunn serán establecidos, así como el enunciado preciso de la desigualdad de Grünbaum (y otros resultados relacionados que involucran al centroide, bien de un cuerpo convexo, bien de una función con cierta concavidad). Finalmente, a lo largo de la última sección de este capítulo incluiremos algunas nociones clave como las de función cóncava y función p-cóncava, para después finalizar el capítulo recogiendo algunos resultados funcionales que usaremos posteriormente.

Volviendo al resultado que nos ocupa, la clave de su demostración original recae sobre el resultado clásico conocido como *principio de concavidad de Brunn.* Este garantiza que, para cualquier cuerpo convexo $K \subset \mathbb{R}^n$ y un hiperplano H, la función $f : H^\perp \longrightarrow \mathbb{R}_{\geq 0}$ dada por $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$ es $(1/(n-1))$-cóncava. En otras palabras, para cualquier hiperplano H dado, la función de los volúmenes seccionales f elevada a $(1/(n-1))$ es cóncava en su soporte. A pesar de que esta propiedad no puede ser en general mejorada, podríamos encontrar fácilmente ejemplos para los cuales la función f satisface uan concavidad más fuerte, para un hiperplano H adecuado. Por tanto, por un lado, es natural preguntarse por una versión mejorada de la desigualdad de Grünbaum $(*)$ para una familia de cuerpos convexos K tales que (existe un hiperplano H para el cual) f es p-cóncava, con $(1/(n-1)) < p$. Por otro lado, atendiendo a lo anterior, podríamos esperar una posible extensión para conjuntos compactos, no necesariamente convexos, para los cuales f es p-cóncava (para un cierto hiperplano H), con $p < (1/(n-1))$.

El Capítulo 2 está dedicado a este problema. Para ello, como evidencia la prueba original de Grünbaum, es clave caracterizar los casos extremos de la desigualdad que buscamos. De hecho, observando que el caso de igualdad en $(*)$ viene dado por conos, es decir, conjuntos para los cuales la función f es $\big(1/(n-1)\big)$-afín (en otras palabras, tales que $f^{1/(n-1)}$ es una función afín), los conjuntos de revolución asociados a una función p-afín, que denotaremos como C_p, emergen como candidatos naturales para los casos extremales de esa desigualdad que buscamos. De esta forma, la primera sección de este capítulo está dedicada a un estudio de los conjuntos C_p. Específicamente, obtenemos que, para todo $p \in (-\infty, -1] \cup [0, +\infty)$, si C_p está centrado (es decir, tiene centroide en el origen) entonces

$$\frac{\mathrm{vol}(C_p^-)}{\mathrm{vol}(C_p)} \geq \left(\frac{p+1}{2p+1}\right)^{(p+1)/p}.$$

Además, terminaremos la sección estudiando el caso de $p \in (-1/2, 0)$.

Usando la información obtenida en la sección anterior, tal y como se establece en los objetivos del capítulo, a lo largo de la segunda sección del capítulo probaremos una extensión de la desigualdad clásica de Grünbaum ($*$) al caso de conjuntos compactos con una función de los volúmenes seccionales f p-cóncava (abordando los casos en los que $p > 0$ y $p = 0$) para un hiperplano dado. Así, obtenemos que para todo conjunto compacto K con centroide en el origen (y cuya función f satisface las premisas anteriores) se tiene que

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{p+1}{2p+1}\right)^{(p+1)/p}. \tag{\dagger}$$

Por otro lado, también mostramos que, bajo la hipótesis de concavidad más débil posible, es decir, f siendo *quasi-cóncava*, añadiendo la condición adicional de que esta sea monótona podemos obtener una desigualdad para K del mismo tipo. Finalmente, abordamos el caso en el que p es negativo probando que la concavidad límite para un resultado de este tipo es la log-concavidad.

Sabiendo de la relación de las funciones log-cóncavas y p-cóncavas con la geometría de los cuerpo convexos, parece natural esperar una forma funcional de la desigualdad (†). De hecho, en [29, Corollary 7], a partir de un resultado para funciones p-cóncavas más general, los autores dan una respuesta positiva a esta pregunta para el caso $p > 0$. Sin embargo, teniendo en cuenta la relación entre la desigualdad de Grünbaum y la desigualdad de Brunn-Minkowski, podríamos esperar que la desigualdad de Borell-Brascamp-Lieb desempeñase un rol importante en la prueba de un resultado de este tipo.

En el Capítulo 3 daremos una prueba sencilla de la forma funcional de la desigualdad de Grünbaum usando inducción en la dimensión y la desigualdad de Borell-Brascamp-Lieb. Concretamente obtenemos que para cualquier función centrada p-cóncava $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$, con $p \geq 0$, se tiene que

$$\int_{H^-} f(x)\,\mathrm{d}x \geq \left(\frac{np+1}{(n+1)p+1}\right)^{(np+1)/p} \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x$$

para cualquier hiperplano H.

A este respecto, en la primera sección del capítulo recogemos la prueba del caso uno dimensional, basándose esta en un comparación de (una cierta potencia de) la función $x \mapsto \int_a^x f(t)\,\mathrm{d}t$ con su recta tangente en un punto. Además, mostraremos que como consecuencia de esto último podemos obtener una extensión de (†) para cualquier medida. Finalmente, en la segunda sección del capítulo, completamos la prueba de este resultado en dimensión arbitraria usando la desigualdad de Borell-Brascamp-Lieb junto con un argumento estándar para funciones p-cóncavas.

En este punto, atendiendo a lo que sucede en el ámbito de las operaciones de cuerpos convexos, donde la L_p*-suma* es extendida a la suma *Orlicz* con respecto a una función convexa y estrictamente creciente $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ con $\phi(0) = 0$, podría ser natural contemplar la posibilidad de una desigualdad de tipo Grünbaum para los conjuntos compactos K tales que (existe un hiperplano H para el cual) $\phi \circ f$ es cóncava, siendo ϕ una función de clase *Orlicz* (convexa, estrictamente creciente y con $\phi(0) = 0$). Además, podría esperarse obtener una familia de desigualdades dependiendo de la función ϕ, que recuperasen las estudiadas en el Capítulo 2 cuando $\phi(t) = t^p$ con $p > 0$, o $\phi(t) = \log(t)$ si $p = 0$, obteniendo, de esta forma, otra extensión de (∗).

Sabiendo lo anterior, es importante mencionar que debemos asumir ciertas restricciones para la función ϕ. Como mencionábamos en el Capítulo 2, la log-concavidad representa un caso límite para una generalización de la desigualdad de Grünbaum de este tipo, si consideramos el rango completo de $p \in \mathbb{R}$ en el caso de conjuntos compactos con una función f p-cóncava. Por tanto, sería interesante encontrar las condiciones adecuadas para esas familia general de funciones ϕ que nos permita obtener la desigualdad buscada. A lo largo del Capítulo 4 abordaremos este problema siguiendo una estrategia similar a la adoptada en el Capítulo 2.

Con este objetivo en mente, como mencionábamos antes, caracterizar los conjuntos extremales de esa posible desigualdad es clave. Por tanto, teniendo en cuanta que a lo largo del Capítulo 2 trabajábamos con funciones p-afines, parece natural abordar el caso que ocupa este capítulo usando los conjuntos de revolución generados por una función ϕ-afín, que denotaremos en este caso como C_ϕ. Así, la primera sección de este capítulo está dedicada a un estudio exhaustivo de esta familia de conjuntos donde, bajo ciertas condiciones técnicas para la función φ, establecemos que el ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$ de estos conjuntos (asumiendo que tienen su centroide en el origen) depende esencialmente solo de la función ϕ.

Explotando este resultado, a lo largo de la segunda sección del capítulo extendemos (†) (y en consecuencia (∗)) al caso de conjuntos compactos cuya función de los volúmenes seccionales f es ϕ-cóncava. Finalmente, en la tercera sección de este capítulo probamos algunas extensiones ϕ-cóncavas de otros resultados de tipo Grünbaum que podemos encontrar en la literatura. En concreto, a pesar de que seguimos la misma estrategia, incluimos en esta sección las pruebas de una extensión al caso ϕ-cóncavo de [23, Theorem 3.1] y [11, Lemma 1 and Theorem 2].

Volviendo al resultado original de Grünbaum (∗), uno podría dar una interpretación ligeremente distinta de este, como mencionábamos anteriormente. En base a esta, dicho resultado asegura que, para cualquier cuerpo convexo, siempre existe un punto contenido en su interior (el centroide) de forma que cuando se corta este mediante un hiperplano a través de dicho punto, obtenemos dos subcuerpos con una porción relativamente "grande" del volumen total. A partir de esta observación, surge la siguiente pregunta: ¿existe una familia de puntos, potencialmente conteniendo al centroide, que compartan una propiedad similar? Además de esto, ¿hay algún otro punto, digamos especial o reconocido en la literatura, con una característica de esta índole?

En el Capítulo 5 abordamos estas preguntas. Para ello, exploramos varios aspectos; en primer lugar, presentamos dos ejemplos de esos citado puntos "especiales" para los cuales no es posible obtener una desigualdad de tipo Grünbaum. Por otro lado, probaremos que considerando el punto medio en la dirección del vector normal unitario u del hiperplano H en cuestión, es decir, el punto $\big[(a+b)/2\big] \cdot u$, podemos establecer que, si K es un conjunto compacto tal que $(a+b)/2 = 0$ (y cuya función $f : [a, b] \longrightarrow \mathbb{R}_{\geq 0}$ satisface las mismas premisas que en (†)), entonces

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{1}{2}\right)^{(p+1)/p}.$$

Atendiendo a este hecho, en la primera sección del capítulo introduciremos la familia de puntos uniparamétrica (asociada a un hiperplano con vector normal unitario u, con función f de los volúmenes de las secciones mediante hiperplanos paralelos a este) dada por $\mathrm{g}_r \cdot u$, siendo

$$\mathrm{g}_r := \frac{\int_a^b t f(t)^r \,\mathrm{d}t}{\int_a^b f(t)^r \,\mathrm{d}t}, \qquad (‡)$$

para cada $r \geq 0$. Claramente, dicha familia contiene tanto al punto medio (en la dirección u) como al centroide del cuerpo convexo (casos $r = 0$ y $r = 1$, respectivamente), siendo esta la potencial familia de puntos para dar respuesta a la pregunta planteada previamente.

De esta forma, la segunda sección del capítulo está dedicada a probar una desigualdad de tipo Grünbaum que involucra a esta familia de puntos $\mathrm{g}_r \cdot u$. Para ello, lo abordaremos desde una perspectiva funcional, es decir, reescribiendo (‡) como los siguientes α-centroides de una función cóncava de una variable $h : [a, b] \longrightarrow \mathbb{R}_{\geq 0}$, dados por

$$\mathrm{g}_\alpha(h) := \frac{\int_a^b t h(t)^\alpha \,\mathrm{d}t}{\int_a^b h(t)^\alpha \,\mathrm{d}t}.$$

Con esta definición, obtendremos que cualquier función cóncava satisface una desigualdad de tipo Grünbaum funcional considerando estos puntos. Concretamente, probamos que si $\beta \leq \alpha$ entonces

$$\frac{\int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t}{\int_a^b h(t)^\beta \,\mathrm{d}t} \geq \left(\frac{\beta+1}{\alpha+2}\right)^{\beta+1},$$

mientras que si $\alpha \leq \beta$

$$\frac{\int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t}{\int_a^b h(t)^\beta \,\mathrm{d}t} \geq \left(\frac{\alpha+1}{\alpha+2}\right)^{\beta+1}.$$

Finalmente, veremos que a partir de este resultado podemos obtener (además de varias de las desigualdades contempladas en capítulos previos) otros dos resultados clásicos de la literatura, en apariencia no conectados entre sí, subrayando así algunas de las aplicaciones encontradas.

Los resultados originales contenidos en esta tesis se pueden encontrar en los artículos de investigación [1, 24, 25, 26].

Preface

The study of Grünbaum type inequalities has demonstrated to be a very prolific and interesting topic during the last years. Its origin goes back to a classical work by Ascoli [3] from the 30's, lately generalized to higher dimensions by Grünbaum [16], and published in 1960. It concerns a very natural question involving convex bodies: can one ensure the existence of an "a priori" point within the interior of a convex body in such a way that cutting it through this point results in two parts both having a remarkable portion of the total volume?

Trying to figure out an answer to the previous question one is led to the centroid (also known as the center of mass) of a convex body. For a compact set K, not necessarily convex, with positive volume $\mathrm{vol}(K)$ (i.e., with positive n-dimensional Lebesgue measure) the centroid of K is the affine-covariant point defined as

$$\mathrm{g}(K) := \frac{1}{\mathrm{vol}(K)} \int_K x \,\mathrm{d}x.$$

Grünbaum's inequality then asserts that any convex body with centroid at the origin satisfies that

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{n}{n+1}\right)^n, \tag{§}$$

where $K^- = K \cap \{x \in \mathbb{R}^n : \langle x, u\rangle \leq 0\}$ and $K^+ = K \cap \{x \in \mathbb{R}^n : \langle x, u\rangle \geq 0\}$ represent the parts of K which are split by the hyperplane $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$, for any given unit vector u. Equality holds, for a fixed u, if and only if K is a cone in the direction u, i.e., the convex hull of $\{x\} \cup \big(K \cap (y + H)\big)$, for some $x, y \in \mathbb{R}^n$.

Although it is not our intention to make an exhaustive list, it is worth mentioning that Grünbaum's result has been studied in many other contexts. Some examples which underscore its relevance are the extensions to the case of sections [12, 29] and projections [33] of convex bodies, the generalizations to the analytic setting of *log-concave* functions [28] (see also [21, Lemma 5.4] and [8, Lemma 2.2.6]) and p-concave functions [29], for $p > 0$. Other Grünbaum type inequalities involving volumes of sections of convex bodies through their centroid, later generalized to the case of classical and dual *quermassintegrals* in [32], can be found in [11, 23].

Roughly speaking, this thesis is devoted to the study of generalizations and extensions of Grünbaum's inequality from both a geometric and a functional (and measure theoretical) point of view.

More precisely, this dissertation starts with an introductory first chapter, where we collect some definitions and results that will be needed later on, both concerning convex bodies and concave functions. Thus, the first section is devoted to establishing the notation and recalling some important notions such as that of Minkowski addition, convex body, etc. Moreover, some important inequalities such as the Brunn-Minkowski inequality and Brunn's concavity principle are collected, together with the precise statement of Grünbaum's inequality (and other related results involving the centroid of either a convex body or a function with certain concavity). Finally, along the last section of this chapter we recall the notion of concave function, jointly with the definition of p-concave function, to conclude the chapter by stating some functional results that will be used throughout the thesis.

Going back to Grünbaum's inequality, the underlying key fact in the original proof of (§) is the so-called *Brunn's concavity principle*. This result ensures that for any compact convex set $K \subset \mathbb{R}^n$ and any hyperplane H, then the function $f : H^{\perp} \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$ is $(1/(n-1))$-concave. In other words, for any given hyperplane H, the cross-sections volume function f to the power $1/(n-1)$ is concave on its support. Although this property cannot be in general enhanced, one can easily find compact convex sets for which f satisfies a stronger concavity, for a suitable hyperplane H; thus, on the one hand, it is natural to wonder about a possible enhanced version of Grünbaum's inequality (1.2) for the family of those compact convex sets K such that (there exists a hyperplane H for which) f is p-concave, with $1/(n-1) < p$. On the other hand, one could expect to extend this inequality to compact sets K, not necessarily convex, for which f is p-concave (for some hyperplane H), with $p < 1/(n-1)$.

In Chapter 2, we address the aforementioned problem and delve into its intricacies. A crucial aspect, as evident from Grünbaum's original proof, lies in characterizing the extremal cases to establish the desired inequality. Indeed, observing that the equality case in Grünbaum's result is characterized by cones, that is, those sets for which f is $\big(1/(n-1)\big)$-affine (i.e., such that $f^{1/(n-1)}$ is an affine function), the sets of revolution associated now to p-affine functions, which will be denoted as C_p, emerge as natural candidates to be the extremal sets, in some sense, for the inequalities we seek. Consequently, the first section of this chapter is dedicated to a thorough examination of the sets C_p. More precisely, we get that, for any $p \in (-\infty, -1] \cup [0, +\infty)$, if C_p is centred then

$$\frac{\mathrm{vol}(C_p^-)}{\mathrm{vol}(C_p)} \geq \left(\frac{p+1}{2p+1}\right)^{(p+1)/p}.$$

Furthermore, an investigation of the case $p \in (-1/2, 0)$ is also given.

By exploiting the information gathered during this analysis, we extend Grünbaum's inequality (§) to encompass the case of compact sets with a p-concave cross-sections volume function (both the case of $p > 0$ and $p = 0$) with respect to a given hyperplane. Indeed, if K is a centred compact set satisfying the previous premise, it is proved that

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{p+1}{2p+1}\right)^{(p+1)/p}. \tag{¶}$$

Moreover, we also show that, under the weakest possible concavity assumption, i.e., the cross-section volume function being *quasi-concave*, monotonicity is enough to ensure such an inequality. Lastly, we provide a brief discussion on the scenario where p is negative, highlighting that the limit concavity assumption for this type of result is the log-concavity (i.e., $p = 0$).

Attending to the interplay between log-concave and p-concave functions and the geometry of convex sets, it seems natural to expect a functional form of Grünbaum's inequality (¶). In fact, in [29, Corollary 7] the authors provide a positive answer to this question when $p > 0$, with a sharp constant, obtaining it from a more general result for p-concave functions. Nevertheless, taking into account its connection with the Brunn-Minkowski inequality, one would claim that the Borell-Brascamp-Lieb inequality (1.9) should play a relevant role in the proof of such an analytic result.

In Chapter 3, we give a simpler proof of the functional form of Grünbaum's inequality by using induction on the dimension and the Borell-Brascamp-Lieb inequality (1.9). More precisely, we obtain that for any centred p-concave function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$, with $p \geq 0$, then

$$\int_{H^-} f(x)\,\mathrm{d}x \geq \left(\frac{np+1}{(n+1)p+1}\right)^{(np+1)/p} \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x$$

for any hyperplane H.

To this regard, the first section of this chapter collects the proof of the one-dimensional case, which arises from a quite direct comparison of (a suitable power of) the function $x \mapsto \int_a^x f(t)\,\mathrm{d}t$ with its tangent line at one point of its graph. Moreover, we show that, as a consequence of the latter, an extension of (¶) for general measures follows. Then the second section is devoted to the proof of the n-dimensional case, which is shown by using the Borell-Brascamp-Lieb inequality together with a standard argument for p-concave functions.

At this stage, following the principles observed in the realm of convex body operations, where L_p-*addition* is extended to *Orlicz*-sums with respect to a convex and strictly increasing function $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ with $\phi(0) = 0$, it is natural to contemplate the possibility of a Grünbaum type inequality for the family of those compact sets K such that (there exists a hyperplane H for which) $\phi \circ f$ is concave, with ϕ being an *Orlicz*-class function (convex, strictly increasing and $\phi(0) = 0$).

Furthermore, one can expect to derive a family of inequalities based on a function ϕ, which can recover the ones studied in Chapter 2 when $\phi(t) = t^p$ for $p > 0$, or $\phi(t) = \log(t)$. This would lead to a further generalization of (§). However, it is important to note that certain assumptions regarding the function ϕ must be made. As mentioned in Chapter 2, the log-concave case represents the limiting assumption of concavity for this generalization of Grünbaum's inequality, when considering the full-range of $p \in \mathbb{R}$ in the case of compact sets with a p-concave cross-sections volume function. Therefore, it would be interesting to find the suitable conditions for such a general family of functions ϕ's that permit to obtain these inequalities. In Chapter 4 we address this problem by following a similar approach to the one used in Chapter 2.

With this objective in mind, as mentioned earlier, characterizing the extremal sets of the desired inequality is crucial. Thus, since we want to obtain a family of inequalities recovering (¶), observing that the compact sets are compared with sets of revolution given by a p-affine function with $p \geq 0$, it seems natural to work now with sets of revolution associated to a ϕ-affine function, denoted as C_ϕ. Therefore, the first section of this chapter provides a comprehensive study of this family of sets. Under certain technical assumptions for the function ϕ, we establish that the ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$ for these sets (assuming that they have their centroid at the origin) depends essentially only on the function ϕ.

Exploiting this result, in the second section of this chapter, we extend (¶) (and consequently (§)) to the case of a compact set with a ϕ-concave cross-sections volume function. Finally, the third section of Chapter 4 focuses on discussing potential ϕ-concave extensions of Grünbaum-type results found in the literature. More precisely, although we follow a similar approach, we include a proof of the ϕ-concave extension of [23, Theorem 3.1] and [11, Lemma 1 and Theorem 2].

Returning to Grünbaum's original result §, we can interpret it from a slightly different perspective. It asserts that for any convex body, there always exists a point within the set (the centroid) in such a way that cutting the body through a hyperplane passing by this point yields two sub-bodies with a substantial proportion of the total volume. This observation prompts a natural inquiry: can we identify a family of points, potentially including the centroid, that share this property? Moreover, are there additional special points that possess similar intriguing characteristics, warranting further investigation?

In Chapter 5, we delve into the questions raised earlier and provide a comprehensive approach to addressing them. In doing so, we explore various aspects. Firstly, we present two examples of *special* points where a Grünbaum-type inequality is not feasible, shedding light on the limitations of certain points in this context. Conversely, we demonstrate that by considering the midpoint in the direction of the unit normal vector u of a given hyperplane, i.e., the point $\big[(a+b)/2\big]\cdot u$, we can establish that for any compact set K with $(a+b)/2 = 0$ (and whose cross-sections volume function $f : [a, b] \longrightarrow \mathbb{R}_{\geq 0}$ satisfies the same assumptions as in (¶)) one has the following inequality:

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{1}{2}\right)^{(p+1)/p}.$$

This insight leads us to define a family of points to investigate further. Indeed, in the first section of this chapter we define a uniparametric family of powered centroids (associated to a hyperplane with normal unit vector u, and with cross-sections volume function f) given by $\mathrm{g}_r \cdot u$, where

$$\mathrm{g}_r := \frac{\int_a^b t f(t)^r \,\mathrm{d}t}{\int_a^b f(t)^r \,\mathrm{d}t}, \tag{$\|$}$$

for any $r \geq 0$. Clearly, such a family encompasses both the midpoint (in the direction u) and the centroid (cases of $r = 0$ and $r = 1$, respectively), and serves as a potential family of points to answer the previously posed question.

Thus, the second section of this chapter is devoted to proving a sharp Grünbaum type inequality involving the points $\mathrm{g}_r \cdot u$. To accomplish this, we adopt a functional approach, namely, by rewriting ($\|$) as the following α-powered centroids of a concave function $h : [a.b] \longrightarrow \mathbb{R}_{\geq 0}$, given by

$$\mathrm{g}_\alpha(h) := \frac{\int_a^b t h(t)^\alpha \,\mathrm{d}t}{\int_a^b h(t)^\alpha \,\mathrm{d}t}.$$

Taking into account this definition, we show that any concave function satisfies a (functional) Grünbaum type inequality when considering these points. More precisely, it is proven that, if $\beta \leq \alpha$ then

$$\frac{\int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t}{\int_a^b h(t)^\beta \,\mathrm{d}t} \geq \left(\frac{\beta+1}{\alpha+2}\right)^{\beta+1},$$

whereas if $\alpha \leq \beta$

$$\frac{\int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t}{\int_a^b h(t)^\beta \,\mathrm{d}t} \geq \left(\frac{\alpha+1}{\alpha+2}\right)^{\beta+1}.$$

Therefore, we provide a solid foundation for our investigation into the defined family of points ($\|$). Finally, we show that, from the latter result, one may derive (besides various inequalities appearing in the previous chapters) two classical results from the literature, in principle not connected among them, highlighting in this way some of the applications we have found so far.

The original results which are contained in this dissertation can be found in the papers [1, 24, 25, 26].

Chapter 1

Preliminaries

This chapter is devoted to collecting some definitions, properties, and results of convex bodies and concave functions that will be used throughout this dissertation.

1.1 Notation and definitions

We shall work with the following standard notation. We use $\mathbb{R}^n$ to denote the n-dimensional Euclidean space with standard scalar product $\langle\cdot,\cdot\rangle$. We will denote by e_i the i-th canonical unit vector, we represent by B_n the n-dimensional Euclidean (closed) unit ball and by $\mathbb{S}^{n-1}$ its boundary. Given a vector $u\in\mathbb{S}^{n-1}$, an orthonormal basis of $\mathbb{R}^n$ $(u_1,u_2,\dots,u_n)$ with $u_1=u$, and $x\in\mathbb{R}^n$, we will denote by $[x]_1$ the first coordinate of x with respect to this basis. Moreover, for a hyperplane $H=\{x\in\mathbb{R}^n:\langle x,u\rangle=c\}$, $c\in\mathbb{R}$, we represent by $H^-=\{x\in\mathbb{R}^n:\langle x,u\rangle\leq c\}$ and $H^+=\{x\in\mathbb{R}^n:\langle x,u\rangle\geq c\}$ the halfspaces bounded by H.

The Grassmannian of k-dimensional vector subspaces of $\mathbb{R}^n$ is denoted by $\mathrm{G}(n,k)$, and for $H\in\mathrm{G}(n,k)$, the orthogonal projection of a subset $M\subset\mathbb{R}^n$ onto H is represented by $M|H$, whereas the orthogonal complement of H is denoted by $H^\perp$. Moreover, the positive hull of M will be written by $\operatorname{pos} M$ whereas the convex hull of it will be represented by $\operatorname{conv} M$.

Definition 1.1. *Let $M\subset\mathbb{R}^n$ be a measurable set. The* volume *of M, denoted as* $\mathrm{vol}(M)$, *is the (n-dimensional) Lebesgue measure of M.*

Moreover, the k-dimensional Lebesgue measure of M (whenever M is measurable) is written as $\mathrm{vol}_k(M)$. When integrating, as usual, $\mathrm{d}x$ will stand for $\mathrm{dvol}(x)$, and we set $\kappa_n=\mathrm{vol}(B_n)$.

Definition 1.2. *Let* $A, B \subset \mathbb{R}^n$. *The* Minkowski *(vectorial)* addition *of* A *and* B *is defined as*

$$A + B = \{a + b : a \in A,\ b \in B\}$$

(see Figure 1.1).

This operation of sets clearly preserves convexity and compactness.

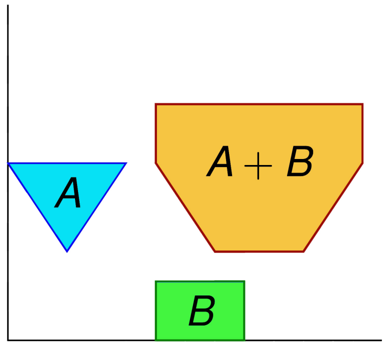

Figure 1.1: Minkowski sum.

Definition 1.3. *A non-empty set* $K \subset \mathbb{R}^n$ *is convex if for any two points* $x, y \in K$ *the segment determined by them, i.e., the set* $[x, y] := \{(1-\lambda)x + \lambda y : 0 \le \lambda \le 1\}$, *is contained in* K. *In other words,* $(1-\lambda)x + \lambda y \in K$ *for all* $0 \le \lambda \le 1$.

Definition 1.4. *A non-empty set* $K \subset \mathbb{R}^n$ *is said to be a convex body if it is a compact convex set.*

Two convex bodies $K, L \subset \mathbb{R}^n$ are said to be *homothetic* if $K = \lambda L + t$ with $t \in \mathbb{R}^n$ and $\lambda \ge 0$.

1.2 The Brunn-Minkowski and Grünbaum's inequalities

Relating the volume with the Minkowski addition of convex bodies, in terms of their volumes, one is led to the famous Brunn-Minkowski inequality (for extensive survey articles on this and related inequalities we refer the reader to [4, 13]; for a general reference on Brunn-Minkowski theory, we also refer to the updated monograph [31]). One form of it is the following one:

Theorem A. *Let* $K, L \subset \mathbb{R}^n$ *be convex bodies and* $\lambda \in (0, 1)$, *then*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/n} \ge (1-\lambda)\mathrm{vol}(K)^{1/n} + \lambda \mathrm{vol}(L)^{1/n}, \tag{1.1}$$

with equality, if $\mathrm{vol}(K)\mathrm{vol}(L) > 0$, *if and only if* K *and* L *are homothetic.*

Although this result is also true for the more general case of measurable sets, since our dissertation is mostly focused on convex bodies, we will use the above version. Moreover, when dealing with convex bodies, the proof of Brunn-Minkowski's inequality relies on the following result concerning sections of convex bodies (see e.g. [8, Section 1.2.1] and also [27, Theorem 12.2.1]):

Theorem B (Brunn's concavity principle)**.** *Let $K \subset \mathbb{R}^n$ be a convex body and let F be a k-dimensional subspace. Then, the function $f : F^{\perp} \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-k}\big(K \cap (x + F)\big)^{1/k}$ is concave.*

Another definition that will be crucial along this work is the following one.

Definition 1.5. *Let $K \subset \mathbb{R}^n$ be a compact set with positive volume $\mathrm{vol}(K)$ (i.e., with positive n-dimensional Lebesgue measure). The centroid of K is the affine-covariant point*

$$\mathrm{g}(K) := \frac{1}{\mathrm{vol}(K)} \int_K x \, \mathrm{d}x.$$

As we have mentioned during the introduction, Grünbaum's inequality asserts that, given a convex body $K \subset \mathbb{R}^n$ with centroid at the origin, one can find a lower bound for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ depending only on the dimension of K, where K^- denotes the intersection of K with a halfspace bounded by a hyperplane passing through its centroid. Its statement is the following:

Theorem C. *Let $H = \{x \in \mathbb{R}^n : \langle x, u \rangle = 0\}$, $u \in \mathbb{S}^{n-1}$, be a hyperplane and, given a convex body $K \subset \mathbb{R}^n$ with non-empty interior, let $K^- = K \cap H^-$. If K has centroid at the origin then*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{n}{n+1}\right)^n. \tag{1.2}$$

Equality holds, for a fixed $u \in \mathbb{S}^{n-1}$, if and only if K is a cone in the direction u, i.e., the convex hull of $\{x\} \cup (K \cap (y + H))$, for some $x, y \in \mathbb{R}^n$.

Similar in spirit to Grünbaum's inequality is a classical inequality, attributed to Minkowski for $n = 2, 3$ and Radon for general n, which bounds the distance from $\mathrm{g}(K)$ to a supporting hyperplane of the convex body K (see [5, p. 57-58]). This result asserts that when K has centroid at the origin then $K \subset -nK$, a fact that is equivalent to the following statement:

Theorem D. *Let $K \subset \mathbb{R}^n$ be a convex body with non-empty interior and let H be a hyperplane. If K has centroid at the origin then*

$$\frac{\mathrm{vol}_1\big(K^- | H^{\perp}\big)}{\mathrm{vol}_1\big(K | H^{\perp}\big)} \geq \frac{1}{n+1}. \tag{1.3}$$

Another inequality of this type, but now involving volume sections instead of projections, is the following inequality (1.4). It was shown (independently) by Makai Jr. and Martini [23], and later by Fradelizi [11], who further proved this result when considering sections by planes of arbitrary dimension.

Theorem E ([11, 23])**.** *Let $K \subset \mathbb{R}^n$ be a convex body with non-empty interior, let H be a hyperplane and let $f : [a,b] \longrightarrow \mathbb{R}_{\geq 0}$ be the function given by $f(t) = \mathrm{vol}_{n-1}\bigl(K \cap (tu + H)\bigr)$. If K has centroid at the origin then*

$$\frac{f(0)}{\|f\|_\infty} \geq \left(\frac{n}{n+1}\right)^{n-1}. \tag{1.4}$$

For the sake of simplicity, from now on we shall consider $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$, for a given direction $u \in \mathbb{S}^{n-1}$ that is extended to an orthonormal basis $(u_1, u_2, \dots, u_n)$ of $\mathbb{R}^n$, with $u_1 = u$. Also, a non-negative measurable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ satisfying that

$$\int_{\mathbb{R}^n} x f(x)\,\mathrm{d}x = 0$$

will be referred to as a centred function.

Furthermore, for a compact set $K \subset \mathbb{R}^n$ with non-empty interior, we shall write

$$K(t) = K \cap (tu + H) \tag{1.5}$$

for any $t \in \mathbb{R}$, and the function $f : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(t) = \mathrm{vol}_{n-1}\bigl(K(t)\bigr)$ will denote the cross-sections volume function of K. We observe that if $K|H^\perp \subset [au, bu]$, Fubini's theorem implies (if $a \leq 0$) that

$$\mathrm{vol}(K) = \int_a^b f(t)\,\mathrm{d}t \quad \text{and} \quad \mathrm{vol}(K^-) = \int_a^0 f(t)\,\mathrm{d}t. \tag{1.6}$$

Moreover, we notice that by Fubini's theorem, we have

$$[\mathrm{g}(K)]_1 = \frac{1}{\mathrm{vol}(K)} \int_a^b t f(t)\,\mathrm{d}t \tag{1.7}$$

and so, $a < [\mathrm{g}(K)]_1 < b$ (cf. (1.6)).

1.3 Concave functions and related results

Given a convex body $K \subset \mathbb{R}^n$, the *Brunn concavity principle* implies that the function $f(t) = \mathrm{vol}_{n-1}\bigl(K(t)\bigr)$ is $\bigl(1/(n-1)\bigr)$-concave. In this regard, we recall that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is said to be p-concave, for $p \in \mathbb{R} \cup \{\pm\infty\}$, if

$$f\bigl((1-\lambda)x + \lambda y\bigr) \geq M_p\bigl(f(x), f(y), \lambda\bigr)$$

for all $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$ and any $\lambda \in (0,1)$. Here M_p is the *p-mean* of two positive numbers a, b:

$$M_p(a,b,\lambda) = \begin{cases} \big((1-\lambda)a^p + \lambda b^p\big)^{1/p}, & \text{if } p \neq 0, \pm\infty, \\ a^{1-\lambda}b^\lambda & \text{if } p = 0, \\ \max\{a,b\} & \text{if } p = \infty, \\ \min\{a,b\} & \text{if } p = -\infty. \end{cases}$$

We observe that if $p > 0$, then f is p-concave if and only if f^p is concave on its support $\{x \in \mathbb{R}^n : f(x) > 0\}$ and hence 1-concavity is nothing but concavity (on the support) in the usual sense, namely, that

$$f\big((1-\lambda)x + \lambda y\big) \geq (1-\lambda)f(x) + \lambda f(y)$$

for all x, y in the support of f. A 0-concave function is usually referred to as *log-concave* whereas a $(-\infty)$-concave function is called *quasi-concave*.

When $p \geq 0$, we can also see the notion of p-concavity as a particular case of the following one (this concept has already appeared in the literature; see, e.g., [20]):

Definition 1.6. *Let $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function with $\varphi := \phi^{-1}$ (defined on $\phi(\mathbb{R}_{\geq 0})$). We say that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is ϕ-concave if*

$$f\big((1-\lambda)x + \lambda y\big) \geq \varphi\Big((1-\lambda)\phi\big(f(x)\big) + \lambda\phi\big(f(y)\big)\Big)$$

for all $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$. In other words, the function f is ϕ-concave if and only if $\phi \circ f$ is concave on the support of f.

Note that if $f : [a,b] \longrightarrow \mathbb{R}_{\geq 0}$ is ϕ-concave, then the superlevel set $\{s \in \mathbb{R} : f(s) > t\}$ is convex for all $t \geq 0$. And so, from now on we will assume, without loss of generality, that $f(t) > 0$ for all $t \in (a,b)$. Furthermore, we may suppose, if needed, that the support of f is actually $[a,b]$ (since, when integrating, $[a,b]$ and (a,b) are indistinguishable). And, more generally, whenever we deal with either a concave/convex function or a ϕ-concave function f (and then, in particular, a p-concave function, for some $p \geq 0$), we will always exclude the trivial case of $f \equiv 0$, i.e., we will assume that it is a non-zero function.

Additionally, in the following, we will assume that f is upper semicontinuous (and the same will apply for any function $f : K \longrightarrow \mathbb{R}_{\geq 0}$ defined on any convex body $K \subset \mathbb{R}^n$ such that $\phi \circ f$ is concave). Indeed, otherwise, we would work with its upper closure, which is determined via the closure of the superlevel sets of f (see [30, page 14 and Theorem 1.6]), and thus they have the same Lebesgue measure to those of f. In particular, this will imply that the maximum of such an f is attained.

A classical tool when dealing with concave functions is the well-known *Jensen's inequality.* For a reference on it, we recommend the classical texts [9, 17]. We collect here the version of it that we will use throughout this dissertation.

Theorem F (Jensen's inequality)**.** *Let μ be a probability measure on $\mathbb{R}^n$, let $K \subset \mathbb{R}^n$ be a compact set, and let $f : K \longrightarrow \mathbb{R}_{\geq 0}$ be a measurable and bounded function. If $g : [a,b] \longrightarrow \mathbb{R}_{\geq 0}$ is a concave function with $[a,b] \supset f(K)$ then*

$$g\left(\int_K f(x)\,\mathrm{d}\mu(x)\right) \geq \int_K g\big(f(x)\big)\,\mathrm{d}\mu(x). \tag{1.8}$$

We conclude this chapter by collecting here the following result, originally proved in [6] and [7] (see also [13] for a detailed presentation), which can be regarded as the functional counterpart of the Brunn-Minkowski inequality (1.1).

Theorem G (The Borell-Brascamp-Lieb inequality)**.** *Let $\lambda \in (0,1)$. Let $-1/n \leq p \leq \infty$ and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be measurable functions, with positive integrals, such that*

$$h\big((1-\lambda)x + \lambda y\big) \geq M_p\big(f(x), g(y), \lambda\big)$$

for all $x, y \in \mathbb{R}^n$ such that $f(x)g(y) > 0$. Then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq M_q\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x, \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x, \lambda\right), \tag{1.9}$$

where $q = p/(np+1)$.

Chapter 2

On Grünbaum type inequalities for the volume

Despite its apparent simplicity, Grünbaum's inequality (1.2) turns out to be a property of convex bodies (or functions) satisfied in many different contexts. Indeed, Grünbaum's result was extended to the case of sections [12, 29] and projections [33] of convex bodies, and generalized to the analytic setting of *log-concave* functions [28] (see also [21, Lemma 5.4] and [8, Lemma 2.2.6]) and p-concave functions [29], for $p > 0$. Other Grünbaum type inequalities involving volumes of sections of convex bodies through their centroid, later generalized to the case of classical and dual *quermassintegrals* in [32], can be found in [11, 23].

In this chapter, given a compact set $K \subset \mathbb{R}^n$ of positive volume, we deal with the problem of showing that fixing a hyperplane H, one can find a sharp lower bound for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ depending on the concavity nature of the function that gives the volumes of cross-sections (parallel to H) of K. When K is convex, this inequality recovers the above-mentioned result by Grünbaum. To this respect, we will also prove that the log-concave case is the limit concavity assumption for such a generalization of Grünbaum's inequality. The original content of this chapter can be found in [26].

2.1 Natural candidates for the extremal cases

From Brunn's concavity principle (Theorem B) we have that, for any given hyperplane H, the cross-sections volume function f to the power $1/(n-1)$ is concave on its support, which is equivalent (due to the convexity of K) to the well-known *Brunn-Minkowski inequality* (see (1.1)). Although this property cannot be in general enhanced, one can easily find convex bodies for which f satisfies a stronger concavity, for a suitable hyperplane H; similarly, the Brunn-Minkowski inequality can be refined when dealing with restricted families of sets (see e.g. [18, 19] and the references therein). Thus, on the one hand, it is natural to wonder about a possible enhanced version of Grünbaum's inequality (1.2) for the family of those convex bodies K such that (there exists a hyperplane H for which) f is p-concave, with $1/(n-1) < p$. On the other hand, one could expect to extend this inequality to compact sets K, not necessarily convex, for which f is p-concave (for some hyperplane H), with $p < 1/(n-1)$.

Remark 2.1. *Let us note that the concavity nature of the cross-sections volume function f may depend on the choice of the hyperplane H. Indeed, given $H_1 = \{x \in \mathbb{R}^3 : \langle x, \mathrm{e}_1\rangle = 0\}$ and $H_2 = \{x \in \mathbb{R}^3 : \langle x, \mathrm{e}_2\rangle = 0\}$, and considering the set*

$$C_1 = \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \in [0,1],\, x_2^2 + x_3^2 \leq r(x_1)^2\right\}$$

of radius $r(t) = t^{1/2}$, we have $\mathrm{vol}_2\big(C_1 \cap (t\mathrm{e}_1 + H_1)\big) = \kappa_2 t$ and

$$\begin{aligned}\mathrm{vol}_2\big(C_1 \cap (t\mathrm{e}_2 + H_2)\big) &= \mathrm{vol}_2\big(\{x \in \mathbb{R}^3 : x_1 \in [t^2, 1],\, x_3^2 \leq r(x_1)^2 - t^2\}\big)\\ &= \frac{4}{3}(1-t^2)^{3/2},\end{aligned}$$

for any $t \in [0,1]$. Therefore, the function $f_1 : H_1^{\perp} \longrightarrow \mathbb{R}_{\geq 0}$ defined by $\mathrm{vol}_2\big(C_1 \cap (x + H_1)\big)$ is 1-concave whereas the function $f_2 : H_2^{\perp} \longrightarrow \mathbb{R}_{\geq 0}$ given by $\mathrm{vol}_2\big(C_1 \cap (x + H_2)\big)$ is not 1-concave.

Observing that the equality case in Grünbaum's inequality (1.2) is characterized by cones, that is, those sets for which f is $(1/(n-1))$-affine (i.e., such that $f^{1/(n-1)}$ is an affine function), the following sets of revolution, associated to p-affine functions, arise as natural candidates to be the extremal sets, in some sense, of these inequalities that we are seeking.

Definition 2.1. *Let $p \in \mathbb{R}$ and let $c, \gamma, \delta > 0$ be fixed. Then*

i) *if $p \neq 0$, let $g_p : I \longrightarrow \mathbb{R}_{\geq 0}$ be the non-negative function given by $g_p(t) = c(t+\gamma)^{1/p}$, where $I = [-\gamma, \delta]$ if $p > 0$ and $I = (-\gamma, \delta]$ if $p < 0$;*

ii) *if $p = 0$, let $g_0 : (-\infty, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ be the non-negative function defined by $g_0(t) = ce^{\gamma t}$.*

Let $u \in \mathbb{S}^{n-1}$ be fixed. By C_p we denote the set of revolution whose section by the hyperplane $\{x \in \mathbb{R}^n : \langle x, u\rangle = t\}$ is an $(n-1)$-dimensional ball of radius $\big(g_p(t)/\kappa_{n-1}\big)^{1/(n-1)}$ with axis parallel to u (see Figure 2.1). (We warn the reader that, in the following, we will use the word "radius" to refer to such a generating function $\big(g_p(t)/\kappa_{n-1}\big)^{1/(n-1)}$ of the set C_p, for short.)

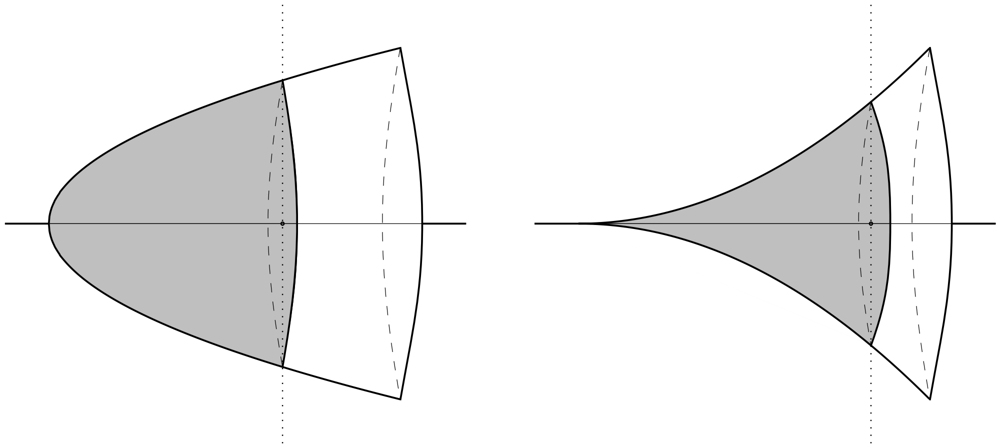

Figure 2.1: Sets C_p in $\mathbb{R}^3$, with centroid at the origin, and C_p^- (coloured), for $p = 1$ (left) and $p = 1/4$ (right).

In other words, one may speculate whether, among all compact sets K with centroid at the origin such that f is p-concave (for some hyperplane H), C_p gives the infimum for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$. We note that, in this way, we would have a general family of inequalities depending on a real parameter p (with extremal sets varying continuously on it), and having Grünbaum's inequality (1.2) as the particular case $p = 1/(n-1)$.

As mentioned, these sets C_p associated to (cross-sections volume) functions that are p-affine (see Definition 2.1) seem to be possible extremal sets of such expected inequalities. So, we start by computing the ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$ for the sets C_p.

Lemma 2.1.1 ([26])**.** *Let $p \in (-\infty, -1) \cup [0, \infty)$ and let $H \in \mathrm{G}(n, n-1)$ be a hyperplane with unit normal vector $u \in \mathbb{S}^{n-1}$. Let g_p and C_p, with axis parallel to u, be as in Definition 2.1, for any fixed $c, \gamma, \delta > 0$. If C_p has centroid at the origin then*

$$\frac{\mathrm{vol}\big(C_p^-\big)}{\mathrm{vol}(C_p)} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} \tag{2.1}$$

where, if $p = 0$, the above identity must be understood as

$$\frac{\mathrm{vol}\big(C_0^-\big)}{\mathrm{vol}(C_0)} = \lim_{p\to 0^+} \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} = e^{-1}. \tag{2.2}$$

Proof. First we assume that $p \neq 0$ and show (2.1). On the one hand, by Fubini's theorem, we get

$$\mathrm{vol}(C_p) = \int_{-\gamma}^{\delta} g_p(t)\,\mathrm{d}t = \frac{c\,p(\delta+\gamma)^{(p+1)/p}}{p+1}.$$

On the other hand, from (1.7), we have

$$\begin{aligned}\big[\mathrm{g}(C_p)\big]_1 &= \frac{1}{\mathrm{vol}(C_p)}\int_{-\gamma}^{\delta} t g_p(t)\,\mathrm{d}t = \frac{p+1}{p(\delta+\gamma)^{(p+1)/p}}\int_0^{\delta+\gamma} (s-\gamma)s^{1/p}\,\mathrm{d}s \\ &= \frac{(p+1)(\delta+\gamma)}{2p+1} - \gamma.\end{aligned}$$

Therefore, from the hypothesis $\mathrm{g}(C_p) = 0$, we obtain that $\gamma/(\delta+\gamma) = (p+1)/(2p+1)$, and hence

$$\frac{\mathrm{vol}\big(C_p^-\big)}{\mathrm{vol}(C_p)} = \frac{1}{\mathrm{vol}(C_p)} \int_{-\gamma}^{0} g_p(t)\,\mathrm{d}t = \left(\frac{\gamma}{\delta+\gamma}\right)^{(p+1)/p} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p},$$

as desired.

Now we assume that $p = 0$ and show (2.2). Again, by Fubini's theorem and (1.7), respectively, we get

$$\mathrm{vol}(C_0) = \int_{-\infty}^{\delta} g_0(t)\,\mathrm{d}t = \frac{ce^{\gamma\delta}}{\gamma}$$

and

$$\big[\mathrm{g}(C_0)\big]_1 = \frac{1}{\mathrm{vol}(C_0)} \int_{-\infty}^{\delta} t g_0(t)\,\mathrm{d}t = \delta - \frac{1}{\gamma}.$$

In particular, $\mathrm{g}(C_0) = 0$ implies that $\delta = 1/\gamma$, and hence

$$\frac{\mathrm{vol}\big(C_0^-\big)}{\mathrm{vol}(C_0)} = \frac{1}{\mathrm{vol}(C_0)} \int_{-\infty}^{0} g_0(t)\,\mathrm{d}t = e^{-1}.$$

This concludes the proof. □

Notice that the value $\big((p+1)/(2p+1)\big)^{(p+1)/p}$ obtained in (2.1) is not defined for $p \in [-1, -1/2]$. However, although the above expression makes also sense for any $p \in (-1/2, 0)$, the corresponding sets C_p present remarkable differences with those of the range $p \geq 0$, as we will see next. So, we will study this case separately.

To this aim, let $p \in (-1/2, 0)$ and let $\varepsilon > 0$ be fixed. Let $C_{p,\varepsilon}$ be the set of revolution, with axis parallel to u, of radius $\big(g_{p,\varepsilon}(t)/\kappa_{n-1}\big)^{1/(n-1)}$ associated to the non-negative p-affine function $g_{p,\varepsilon} : [-\gamma+\varepsilon, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ given by $g_{p,\varepsilon}(t) = c(t+\gamma)^{1/p}$, for some $c, \gamma, \delta > 0$ (for our purpose we may assume that $\gamma > \varepsilon$).

On the one hand, by Fubini's theorem, we get

$$\mathrm{vol}(C_{p,\varepsilon}) = \int_{-\gamma+\varepsilon}^{\delta} g_p(t)\,\mathrm{d}t = \frac{c\,p\left((\delta+\gamma)^{(p+1)/p} - \varepsilon^{(p+1)/p}\right)}{p+1}.$$

Then we notice first that $\mathrm{vol}(C_{p,\varepsilon}) \to \infty$ as $\varepsilon \to 0^+$. On the other hand, from (1.7), we have

$$\begin{aligned}
\big[\mathrm{g}(C_{p,\varepsilon})\big]_1 &= \frac{1}{\mathrm{vol}(C_{p,\varepsilon})} \int_{-\gamma+\varepsilon}^{\delta} t g_p(t)\,\mathrm{d}t \\
&= \frac{p+1}{p\left((\delta+\gamma)^{(p+1)/p} - \varepsilon^{(p+1)/p}\right)} \int_{\varepsilon}^{\delta+\gamma} (s-\gamma) s^{1/p}\,\mathrm{d}s \\
&= \frac{(p+1)\alpha(\varepsilon)}{2p+1} - \gamma,
\end{aligned}$$

where

$$\alpha(\varepsilon) = \frac{(\delta+\gamma)^{(2p+1)/p} - \varepsilon^{(2p+1)/p}}{(\delta+\gamma)^{(p+1)/p} - \varepsilon^{(p+1)/p}}.$$

We note that $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, and moreover that $\alpha(\varepsilon) > 0$ because of the direct relation $-\gamma + \varepsilon \leq \left[\mathrm{g}(C_{p,\varepsilon})\right]_1$ jointly with $(p+1)/(2p+1) > 0$. Hence, we get

$$\begin{aligned}\frac{\mathrm{vol}\big(C_{p,\varepsilon} \cap \big(\mathrm{g}(C_{p,\varepsilon}) + H\big)^+\big)}{\mathrm{vol}(C_{p,\varepsilon})} &= \frac{1}{\mathrm{vol}(C_{p,\varepsilon})} \int_{-\gamma+(p+1)\alpha(\varepsilon)/(2p+1)}^{\delta} g_p(t)\,\mathrm{d}t \\ &= \frac{(\delta+\gamma)^{(p+1)/p} - \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} \alpha(\varepsilon)^{(p+1)/p}}{(\delta+\gamma)^{(p+1)/p} - \varepsilon^{(p+1)/p}}.\end{aligned}$$

Therefore, although $\lim_{\varepsilon\to 0^+} \mathrm{vol}(C_{p,\varepsilon}) = \infty$, we have

$$\lim_{\varepsilon\to 0^+} \frac{\mathrm{vol}\big(C_{p,\varepsilon} \cap \big(\mathrm{g}(C_{p,\varepsilon}) + H\big)^+\big)}{\mathrm{vol}(C_{p,\varepsilon})} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p}. \tag{2.3}$$

Thus, the value $\big(p+1)/(2p+1)\big)^{(p+1)/p}$ is asymptotically attained by the sets $C_{p,\varepsilon}$. The main difference with the case $p \geq 0$ is that it is now reached by the part obtained by the positive halfspace (see (2.3)) bounded by the hyperplane passing through their centroid, whereas in the case $p \geq 0$ the above value is attained by the part lying at the negative halfspace, i.e., one has

$$\frac{\mathrm{vol}\big(C_p \cap \big(\mathrm{g}(C_p) + H\big)^-\big)}{\mathrm{vol}(C_p)} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p}$$

(see Lemma 2.1.1).

2.2 Main results

Grünbaum's inequality (1.2) can also be expressed by saying that if K is a convex body, of positive volume, with centroid at the origin, then

$$\min\left\{\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)}, \frac{\mathrm{vol}(K^+)}{\mathrm{vol}(K)}\right\} \geq \left(\frac{n}{n+1}\right)^n.$$

To this regard, we start this section by showing that, if the cross-sections volume function f associated to a compact set K is increasing in the direction of the normal vector of H, then the above minimum is attained at $\mathrm{vol}(K^-)/\mathrm{vol}(K)$, independently of the concavity nature of f.

Proposition 2.2.1 ([26]). *Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let $H \in \mathrm{G}(n, n-1)$ be a hyperplane, with unit normal vector $u \in \mathbb{S}^{n-1}$, such that the function $f : H^\perp \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x+H)\big)$ is quasi-concave with $f(bu) = \max_{x\in H^\perp} f(x)$, where $[au, bu] = K|H^\perp$. Then*

$$\frac{\mathrm{vol}(K^+)}{\mathrm{vol}(K)} \geq \frac{1}{2}.$$

Proof. Let $g : [-\gamma, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ be the constant function given by $g(t) = f(0)$, where

$$\gamma = \frac{1}{f(0)} \int_a^0 f(t)\,\mathrm{d}t \quad \text{and} \quad \delta = \frac{1}{f(0)} \int_0^b f(t)\,\mathrm{d}t. \tag{2.4}$$

Since f is quasi-concave with $f(b) = \max_{t\in\mathbb{R}} f(t)$, f is increasing on $[a, b]$ and thus (from (2.4)) we have $a \leq -\gamma < 0 < b \leq \delta$. Hence, since $\mathrm{g}(K) = 0$ (and using (1.7)), from (2.4) we get

$$\begin{aligned} f(0)\,\frac{\gamma^2 - \delta^2}{2} &= -\int_{-\gamma}^{\delta} t g(t)\,\mathrm{d}t = \int_a^b t f(t)\,\mathrm{d}t - \int_{-\gamma}^{\delta} t g(t)\,\mathrm{d}t \\ &= \int_a^{-\gamma} (t+\gamma) f(t)\,\mathrm{d}t + \int_{-\gamma}^{0} (t+\gamma)\big(f(t) - g(t)\big)\,\mathrm{d}t \\ &\quad + \int_0^b (t-b)\big(f(t) - g(t)\big)\,\mathrm{d}t + \int_b^{\delta} (t-b)\big(-g(t)\big)\,\mathrm{d}t \leq 0, \end{aligned}$$

which yields $\gamma \leq \delta$, or equivalently $\mathrm{vol}(K^-) \leq \mathrm{vol}(K^+)$. This concludes the proof. □

We are now ready to show the following result, which constitutes the main aim of this chapter.

Theorem 2.2.1 ([26])**.** *Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let H be a hyperplane such that the function $f : H^{\perp} \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$ is p-concave, for some $p \in [0, \infty)$. If $p > 0$ then*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} \tag{2.5}$$

with equality if and only if $\sigma_{H^{\perp}}(K) = C_p$. If $p = 0$ then

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq e^{-1}. \tag{2.6}$$

The inequality is sharp; that is, the quotient $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ comes arbitrarily close to e^{-1}.

Proof. First we assume that $p > 0$ and show (2.5). We assert that there exists a (p-affine) function $g_p : [-\gamma, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ given by $g_p(t) = c(t + \gamma)^{1/p}$, for some $\gamma, \delta, c > 0$, such that $g_p(0) = f(0)$,

$$\int_{-\gamma}^0 g_p(t)\,\mathrm{d}t = \int_a^0 f(t)\,\mathrm{d}t \quad \text{and} \quad \int_0^{\delta} g_p(t)\,\mathrm{d}t = \int_0^b f(t)\,\mathrm{d}t. \tag{2.7}$$

Indeed, taking

$$\gamma = \frac{p+1}{p f(0)} \int_a^0 f(t)\,\mathrm{d}t, \quad c = \frac{f(0)}{\gamma^{1/p}} \quad \text{and} \quad \delta = \left(\frac{p+1}{pc} \int_a^b f(t)\,\mathrm{d}t\right)^{p/(p+1)} - \gamma,$$

elementary computations show (2.7). We also note that, since

$$\gamma = \left(\frac{p+1}{pc} \int_a^0 f(t)\,\mathrm{d}t\right)^{p/(p+1)},$$

we actually have $\delta > 0$.

In other words, for the set of revolution C_p of radius $(g_p(t)/\kappa_{n-1})^{1/(n-1)}$, we have $C_p(0) = \sigma_{H^\perp}\bigl(K(0)\bigr)$,

$$\mathrm{vol}\bigl(C_p^-\bigr) = \mathrm{vol}(K^-) \quad \text{and} \quad \mathrm{vol}\bigl(C_p^+\bigr) = \mathrm{vol}(K^+). \tag{2.8}$$

And thus, in particular, $\mathrm{vol}(C_p) = \mathrm{vol}(K)$.

From the concavity of f^p, together with the relations $g_p(0) = f(0)$ and (2.7), we get on the one hand that $-\gamma \leq a < 0 < \delta \leq b$. On the other hand, defining the functions $\bar{f}, \bar{g}_p : [-\gamma, b] \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$\bar{f}(t) = \begin{cases} f(t) & \text{if } t \in [a,b], \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \bar{g}_p(t) = \begin{cases} g_p(t) & \text{if } t \in [-\gamma,\delta], \\ 0 & \text{otherwise}, \end{cases}$$

we may conclude that there exists $x_0 \in [a,0)$ such that $\bar{f}(t) \geq \bar{g}_p(t)$ for all $t \in [x_0, 0] \cup [\delta, b]$ and $\bar{f}(t) \leq \bar{g}_p(t)$ otherwise (see Figure 2.2). Hence, since $\mathrm{g}(K) = 0$ (and using (1.7)), from (2.7) we have

$$\begin{aligned}
&-\int_{-\gamma}^{\delta} t g_p(t)\,\mathrm{d}t = \int_a^b t f(t)\,\mathrm{d}t - \int_{-\gamma}^{\delta} t g_p(t)\,\mathrm{d}t = \int_{-\gamma}^{b} t\bigl(\bar{f}(t) - \bar{g}_p(t)\bigr)\,\mathrm{d}t \\
&= \int_{-\gamma}^{0} t\bigl(\bar{f}(t) - \bar{g}_p(t)\bigr)\,\mathrm{d}t + \int_0^b t\bigl(\bar{f}(t) - \bar{g}_p(t)\bigr)\,\mathrm{d}t \\
&= \int_{-\gamma}^{0} (t - x_0)\bigl(\bar{f}(t) - \bar{g}_p(t)\bigr)\,\mathrm{d}t + \int_0^b (t-\delta)\bigl(\bar{f}(t) - \bar{g}_p(t)\bigr)\,\mathrm{d}t \geq 0,
\end{aligned}$$

with equality if and only if $f = g_p$. Thus, we have $[\mathrm{g}(C_p)]_1 \leq 0$, and equality holds if and only if $f = g_p$. Then, from (2.8) and Lemma 2.1.1,

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} = \frac{\mathrm{vol}\bigl(C_p^-\bigr)}{\mathrm{vol}(C_p)} \geq \frac{\mathrm{vol}\bigl(C_p \cap \bigl(\mathrm{g}(C_p) + H\bigr)^-\bigr)}{\mathrm{vol}(C_p)} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p},$$

with equality if and only if $f = g_p$, that is, if and only if $\sigma_{H^\perp}(K) = C_p$.

Now we assume that $p = 0$ and show (2.6). We assert that there exists an exponential function $g_0 : (-\infty, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ given by $g_0(t) = ce^{\gamma t}$, for some $\gamma, \delta, c > 0$, such that $g_0(0) = f(0)$,

$$\int_{-\infty}^{0} g_0(t)\,\mathrm{d}t = \int_a^0 f(t)\,\mathrm{d}t \quad \text{and} \quad \int_0^{\delta} g_0(t)\,\mathrm{d}t = \int_0^b f(t)\,\mathrm{d}t. \tag{2.9}$$

Straightforward computations show that the above relations are equivalent to take

$$c = f(0), \quad \gamma = f(0)\left(\int_a^0 f(t)\,\mathrm{d}t\right)^{-1} \quad \text{and} \quad \delta = \frac{1}{\gamma}\log\left(\frac{\gamma}{f(0)}\int_a^b f(t)\,\mathrm{d}t\right);$$

note that, indeed, $\delta > 0$.

Again, the set of revolution C_0 of radius $(g_0(t)/\kappa_{n-1})^{1/(n-1)}$ satisfies that $C_0(0) = \sigma_{H^\perp}\bigl(K(0)\bigr)$,

$$\mathrm{vol}\bigl(C_0^-\bigr) = \mathrm{vol}(K^-) \quad \text{and} \quad \mathrm{vol}\bigl(C_0^+\bigr) = \mathrm{vol}(K^+), \tag{2.10}$$

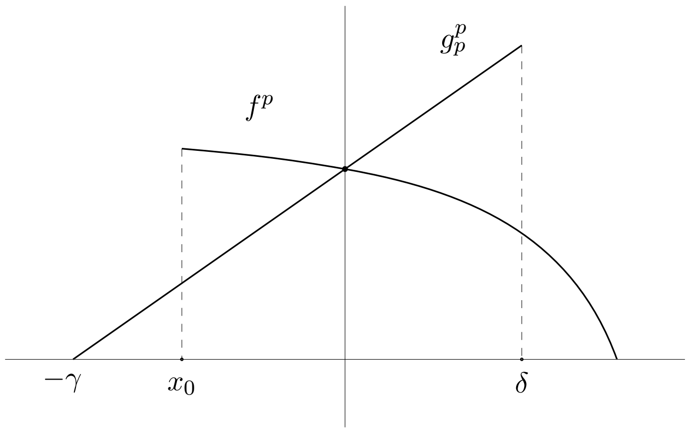

Figure 2.2: Relative position of the functions f^p and g_p^p.

and thus, in particular, $\mathrm{vol}(C_0) = \mathrm{vol}(K)$.

Now the concavity of $\log f$, jointly with the relations $g_0(0) = f(0)$ and (2.9), implies that ($g_0(t) \geq f(t)$ for all $t \in [0, \delta]$ and so) $\delta \leq b$. Moreover, for the functions $\bar{f}, \bar{g}_0 : (-\infty, b] \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$\bar{f}(t) = \begin{cases} f(t) & \text{if } t \in [a, b], \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \bar{g}_0(t) = \begin{cases} g_0(t) & \text{if } t \in (-\infty, \delta], \\ 0 & \text{otherwise}, \end{cases}$$

we conclude that there exists $x_0 \in [a, 0)$ such that $\bar{f}(t) \geq \bar{g}_0(t)$ for all $t \in [x_0, 0] \cup [\delta, b]$ and $\bar{f}(t) \leq \bar{g}_0(t)$ otherwise (cf. Figure 2.2). Arguing as in the case $p > 0$, using (2.9) and $\mathrm{g}(K) = 0$, we have that $[\mathrm{g}(C_0)]_1 \leq 0$. Then, from (2.10) and Lemma 2.1.1,

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} = \frac{\mathrm{vol}\big(C_0^-\big)}{\mathrm{vol}(C_0)} \geq \frac{\mathrm{vol}\big(C_0 \cap \big(\mathrm{g}(C_0) + H\big)^-\big)}{\mathrm{vol}(C_0)} = e^{-1}.$$

Finally, we notice that if we consider an unbounded set L such that $\sigma_{H^\perp}(L) = C_0$, for a given $g_0 : (-\infty, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ of the form $g_0(t) = ce^{\gamma t}$, with $\gamma, \delta, c > 0$, and so that $\int_{-\infty}^{\delta} t g_0(t) \, \mathrm{d}t = 0$, then $\mathrm{vol}(L^-)/\mathrm{vol}(L) = e^{-1}$ (cf. (2.2)). Hence, considering $K_a = L \cap \{x \in \mathbb{R}^n : \langle x, u \rangle \geq a\}$, $a < \delta$, we have $[\mathrm{g}(K_a)]_1 \to 0$ and $\mathrm{vol}\big(K_a^-\big)/\mathrm{vol}(K_a) \to e^{-1}$, as $a \to -\infty$. This proves the final statement of the theorem. □

Note that the "limit case" $p = \infty$ in Theorem 2.2.1 is also trivially fulfilled. Indeed, if f is ∞-concave then f is constant on $[a, b]$ and thus $0 = [\mathrm{g}(K)]_1 = b + a$ (see (1.7)), which yields that $a = -b$ and hence

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} = \frac{1}{2} = \lim_{p \to \infty} \left(\frac{p+1}{2p+1}\right)^{(p+1)/p}.$$

Remark 2.2. *Grünbaum's inequality* (1.2)*, jointly with its equality case, is collected in the case* $p = 1/(n-1)$ *of Theorem 2.2.1. Indeed, on the one hand, Theorem B implies that the cross-sections volume function* f *is* $(1/(n-1))$*-concave, and thus* (2.5) *yields* (1.2)*. On the other hand, regarding* *the equality case of* (1.2)*, we note that the fact that f is $(1/(n-1))$-affine, combined with the convexity of K jointly with the equality case of the Brunn-Minkowski inequality* (1.1)*, implies that K must be a cone in the direction of the normal vector of H.*

Remark 2.3. *We point out that Theorem 2.2.1 can be obtained from recent involved results in the functional setting (more precisely, the case $p > 0$ is derived from [29, Theorem 1] whereas the case $p = 0$ follows from [28, Theorem in p. 746] -see also [21, Lemma 5.4] and [8, Lemma 2.2.6]). Our goal here is to provide a simpler geometric (and unified) proof, inspired by the role of Brunn's concavity principle and comparing (the original bodies) with the sets C_p, in the spirit of Grünbaum's approach in [16].*

2.3 A note on the case of negative concavity

Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin, such that its cross-sections volume function f is p-concave, with respect to a given hyperplane H. Moreover, if $p \in (-\infty, -1) \cup (-1/2, \infty)$, we write for short

$$\alpha_p := \left(\frac{p+1}{2p+1}\right)^{(p+1)/p},$$

where, if $p = 0$, α_0 is the value that is obtained "by continuity", that is,

$$\alpha_0 = \lim_{p \to 0} \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} = e^{-1}.$$

In Theorem 2.2.1 we have shown that, whenever $p \geq 0$, α_p is a sharp lower bound for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$, as a consequence of the fact that $[\mathrm{g}(C_p)]_1 \leq 0$ for the (suitable) set C_p such that $\mathrm{vol}(C_p^-) = \mathrm{vol}(K^-)$ and $\mathrm{vol}(C_p^+) = \mathrm{vol}(K^+)$. In the following result we point out that, even for certain negative values of p, these two conditions are equivalent.

Corollary 2.3.1 ([26])**.** *Let $p \in (-\infty, -1) \cup [0, \infty)$ and let $H \in \mathrm{G}(n, n-1)$ be a hyperplane with unit normal vector $u \in \mathbb{S}^{n-1}$. Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. If C_p, given as in Definition 2.1, with axis parallel to u, is such that*

$$\mathrm{vol}(C_p^-) = \mathrm{vol}(K^-) \quad \textit{and} \quad \mathrm{vol}(C_p^+) = \mathrm{vol}(K^+),$$

then the following assertions are equivalent:

(a) $\mathrm{vol}(K^-)/\mathrm{vol}(K) \geq \alpha_p$*;*

(b) $[\mathrm{g}(C_p)]_1 \leq 0$.

Proof. From Lemma 2.1.1, we have

$$\frac{\mathrm{vol}\big(C_p \cap \big(\mathrm{g}(C_p) + H\big)^-\big)}{\mathrm{vol}(C_p)} = \alpha_p.$$

Moreover, by hypothesis, we get

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} = \frac{\mathrm{vol}\big(C_p \cap H^-\big)}{\mathrm{vol}(C_p)}.$$

Therefore, the result now follows from the fact that, for any $x, y \in \mathbb{R}^n$ such that $\{x, y\}|H^\perp \subset C_p|H^\perp$, $\mathrm{vol}\big(C_p \cap (x+H)^-\big) \leq \mathrm{vol}\big(C_p \cap (y+H)^-\big)$ if and only if $[x]_1 \leq [y]_1$. □

Furthermore, we show that Theorem 2.2.1 cannot be extended to the range of $p \in (-\infty, -1)$. In fact, we prove a more general result:

Proposition 2.3.1 ([26]). *Let $p \in (-\infty, -1)$. There exists no positive constant β_p such that*

$$\min\left\{\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)}, \frac{\mathrm{vol}(K^+)}{\mathrm{vol}(K)}\right\} \geq \beta_p$$

for all compact sets $K \subset \mathbb{R}^n$ with non-empty interior and with centroid at the origin, for which there exists $H \in \mathrm{G}(n, n-1)$ such that $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x+H)\big)$, $x \in H^\perp$, is p-concave.

Proof. By Lemma 2.1.1, for any $q \in (-\infty, -1)$ we have $\alpha_q = \mathrm{vol}\big(C_q^-\big)/\mathrm{vol}(C_q)$, provided that C_q has centroid at the origin. Since $\alpha_q \to 1$ as $q \to -1^-$, we obtain

$$\lim_{q \to -1^-} \min\left\{\frac{\mathrm{vol}\big(C_q^-\big)}{\mathrm{vol}(C_q)}, \frac{\mathrm{vol}\big(C_q^+\big)}{\mathrm{vol}(C_q)}\right\} = \lim_{q \to -1^-} (1 - \alpha_q) = 0.$$

The proof is now concluded from the fact that any q-concave function is also p-concave, whenever $q > p$. □

We conclude the chapter by showing that the statement of Theorem 2.2.1 cannot be extended to the range of $p \in (-1/2, 0)$ either. To this aim, note that if $p < q$ are parameters for which β_p and β_q are such sharp lower bounds for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ (i.e., in the cases in which f is respectively p-concave and q-concave) then $\beta_p \leq \beta_q$, because every q-concave function is also p-concave. We notice however that, if $p \in (-1/2, 0)$, the value obtained by C_p is not α_p but $1 - \alpha_p$ (cf. (2.3)), and then $1 - \alpha_p \geq 1 - \alpha_0 > 1/2$ for any $p \in (-1/2, 0)$ whereas $\alpha_p \leq 1/2$ for all $p \geq 0$.

Therefore, this fact (jointly with the case in which $p \in (-\infty, -1)$, collected in Proposition 2.3.1) gives that $[0, \infty]$ is the largest subset of the real line (with respect to set inclusion) for which C_p provides us with the infimum value for the ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$, among all compact sets with (centroid at the origin and) p-concave cross-sections volume function. However, since α_p is increasing in the parameter p on $(-1/2, \infty)$, and $\alpha_p \to 0$ as $p \to (-1/2)^+$, it is still possible to expect α_p to be a lower bound for $\min\big\{\mathrm{vol}(K^-)/\mathrm{vol}(K), \mathrm{vol}(K^+)/\mathrm{vol}(K)\big\}$. Unfortunately, we do not know so far whether this issue has a positive answer or not.

Chapter 3

(A simple proof of) The functional form of Grünbaum's inequality

The interplay between log-concave and p-concave functions and the geometry of convex sets is one of the main topics of high interest in Convexity, as it can be seen by means of many different works in the literature of the last decades.

In this regard, it is natural to expect a functional form of Grünbaum's inequality (1.2). Furthermore, and taking into account its connection with the Brunn-Minkowski inequality (1.1), one would claim that the Borell-Brascamp-Lieb inequality (1.9) should play a relevant role in the possible proof of such an analytic result. A feasible statement would be the following: there exists some positive constant $C(n,p)$, depending on the dimension n and the degree of concavity p, such that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is a p-concave function, for some $p \in [0,\infty)$, with compact support, and that is centred, then

$$\int_{H^-} f(x)\,\mathrm{d}x \geq C(n,p) \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \tag{3.1}$$

for any hyperplane H. Moreover, one would expect that (3.1) recovers (1.2) when considering as f the characteristic function of a convex body $K \subset \mathbb{R}^n$ with centroid at the origin.

In [29, Corollary 7] the authors provide a positive answer to this question when $p > 0$, with the sharp constant

$$C(n,p) = \left(\frac{np+1}{(n+1)p+1}\right)^{(np+1)/p}.$$

Furthermore, in the log-concave case (i.e., when $p = 0$), we have the tight constant $C(n,0) = e^{-1}$ (due to Lovász and Vempala [21, Lemma 5.4] -see also [8, Lemma 2.2.6]) and the work [28]).

In this chapter, we give a simpler (and unified) proof of the above-mentioned functional form of Grünbaum's inequality by using induction on the dimension and the Borell-Brascamp-Lieb inequality (1.9), where the one-dimensional case arises from a quite direct comparison of (a suitable power of) the function $x \mapsto \int_a^x f(t)\,\mathrm{d}t$ with its tangent line at one point of its graph. For the proof of the latter, we follow the main ideas of the proof of [21, Lemma 5.4] (see also [8, Lemma 2.2.6]), in which the authors show the corresponding log-concave case ($p = 0$). The original content of this chapter can be found in [25].

3.1 The one-dimensional case

This section is devoted to the proof of the one-dimensional case of the functional form of Grünbaum's inequality. As we have said before, our approach is based on the main ideas of [21, Lemma 5.4] (see also [8, Lemma 2.2.6]). We include here the proof in full detail for the sake of completeness.

Theorem 3.1.1 ([25]). *Let $f : [a, b] \longrightarrow \mathbb{R}_{\geq 0}$ be a p-concave function, for some $p \in (0, \infty)$. If f is centred then*

$$\int_a^0 f(t)\,\mathrm{d}t \geq \left(\frac{p+1}{2p+1}\right)^{(p+1)/p} \int_a^b f(t)\,\mathrm{d}t, \tag{3.2}$$

with equality if and only if f is p-affine.

Proof. First we observe that since $[a, b]$ is the support of f, jointly with the fact that $\{t \in \mathbb{R} : f(t) > 0\}$ is convex, we have that $f(t) > 0$ for all $t \in (a, b)$ and so the hypothesis of being centred, namely that

$$\int_a^b t f(t)\,\mathrm{d}t = 0,$$

yields $a < 0 < b$.

Now, let $F : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ be the function given by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Then, from the Borell-Brascamp-Lieb inequality (1.9), F is a q-concave function, for $q = p/(p+1)$, and we have that $F(x) = 0$ for all $x \leq a$ and $F(x) = F(b)$ for all $x \geq b$. Moreover, since f is continuous in (a, b) (due to the fact that every concave function is continuous in the interior of its domain), from the fundamental theorem of calculus we have that

$$F'(x) = f(x) \tag{3.3}$$

for all $x \in (a, b)$. Hence, by applying integration by parts we get

$$\int_a^b F(x)\,\mathrm{d}x = b\int_a^b f(t)\,\mathrm{d}t - \int_a^b t f(t)\,\mathrm{d}t = b\int_a^b f(t)\,\mathrm{d}t, \tag{3.4}$$

where in the last equality we have used that f is centred.

Thus F^q is concave and differentiable on (a,b), and then its tangent at $x=0$, which is given by the function $h_q:\mathbb{R}\longrightarrow\mathbb{R}$ defined by

$$h_q(x)=F(0)^q(q\alpha x+1),$$

for $\alpha=F'(0)/F(0)>0$, lies above its graph. Then, $F(x)^q\le h_q(x)$ for all $x\in(a,b)$ and further

$$F(x)^q\le h_q(x) \tag{3.5}$$

for all $x\in\big(-1/(q\alpha),\infty\big)$.

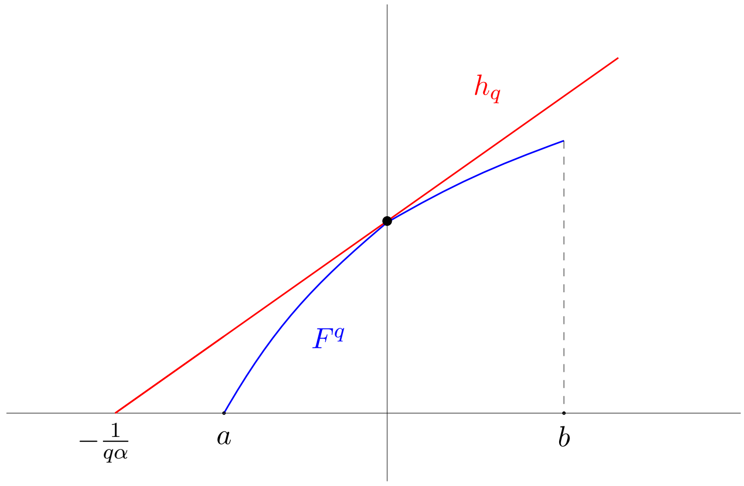

Figure 3.1: The functions F^q and h_q.

Hence, from (3.4) and (3.5), and taking into account that $F(x)<F(b)$ for all $x<b$ and $F(x)=F(b)$ for any $x\ge b$, we have

$$\begin{aligned} b\int_a^b f(t)\,\mathrm{d}t &= \int_a^b F(t)\,\mathrm{d}t=\int_{-1/(q\alpha)}^b F(t)\,\mathrm{d}t\\ &=\int_{-1/(q\alpha)}^{1/\alpha}F(t)\,\mathrm{d}t+\int_{1/\alpha}^b F(t)\,\mathrm{d}t\\ &\le\int_{-1/(q\alpha)}^{1/\alpha}h_q(t)^{1/q}\,\mathrm{d}t+\left(b-\frac{1}{\alpha}\right)\int_a^b f(t)\,\mathrm{d}t\\ &=\frac{F(0)}{\alpha}(q+1)^{1/q}+\left(b-\frac{1}{\alpha}\right)\int_a^b f(t)\,\mathrm{d}t. \end{aligned}$$

Therefore,

$$\int_a^0 f(t)\,\mathrm{d}t=F(0)\ge\left(\frac{1}{q+1}\right)^{1/q}\int_a^b f(t)\,\mathrm{d}t=\left(\frac{p+1}{2p+1}\right)^{(p+1)/p}\int_a^b f(t)\,\mathrm{d}t,$$

and so (3.2) follows. Furthermore, equality holds if and only if

$$F(x) = h_q(x)^{1/q}$$

for all $x \in [a,b]$ (with $a = -1/(q\alpha)$ and $b = 1/\alpha$), which is equivalent (cf. (3.3)) to the fact that

$$f(x) = (1/q)h_q(x)^{1/q-1} = (1/q)h_q(x)^{1/p}$$

for all $x \in [a,b]$ (with $a = -1/(q\alpha)$ and $b = 1/\alpha$). So, since h_q is an affine function, the latter implies that there is equality in (3.2) if and only if f is p-affine. This concludes the proof. □

Remark 3.1. *In [21, Lemma 5.4] (see also [8, Lemma 2.2.6]) it is shown what would be the corresponding case $p = 0$ of the previous result. That is, one has that if $f : [a,b] \longrightarrow \mathbb{R}_{\geq 0}$ is a log-concave function, that is centred, then*

$$\int_a^0 f(t)\,\mathrm{d}t \geq e^{-1} \int_a^b f(t)\,\mathrm{d}t. \tag{3.6}$$

Moreover, the inequality is sharp, in the sense that $\int_a^0 f(t)\,\mathrm{d}t / \int_a^b f(t)\,\mathrm{d}t$ comes arbitrarily close to e^{-1} (when f is log-affine and $a \to -\infty$).

Given a measure μ on $\mathbb{R}^n$ and a compact set K of positive measure $\mu(K)$, the μ-centroid of K is given by

$$\mathrm{g}_\mu(K) := \frac{1}{\mu(K)} \int_K x\,\mathrm{d}\mu(x).$$

Furthermore, in the following we will use the notation (cf. (1.5))

$$K(t) = \{x \in \mathbb{R}^{n-1} : (x,t) \in K\},$$

for any $t \in \mathbb{R}$.

The above result allows us to extend Theorem 2.2.1 to the more general setting of general measures.

Corollary 3.1.1 ([25])**.** *Let $\mu = \mu_{n-1} \times \mu_1$ be a product measure on $\mathbb{R}^n$ such that μ_{n-1} is a locally finite measure on $\mathbb{R}^{n-1}$ and μ_1 is the measure on $\mathbb{R}$ defined by $\mathrm{d}\mu_1(t) = \phi(t)\,\mathrm{d}t$, where $\phi : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ is an s-concave function, for some $s \in (0,\infty)$, supported on $[a,b]$.*

Let $K \subset \mathbb{R}^n$ be a compact set of positive measure $\mu(K)$ with μ-centroid at the origin and let $H = \{x \in \mathbb{R}^n : \langle x, \mathrm{e}_n \rangle = 0\}$. If the function $f : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(t) = \mu_{n-1}\big(K(t)\big)$ is p-concave, for some $p \in (0,\infty)$, then

$$\frac{\mu(K^-)}{\mu(K)} \geq \left(\frac{q+1}{2q+1}\right)^{(q+1)/q},$$

where $q = (ps)/(p+s)$.

Proof. By Fubini's theorem we have that

$$\mu(K^-) = \int_{K^-} \mathrm{d}\mu(x) = \int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \chi_K(y,t)\,\mathrm{d}\mu_{n-1}(y)\,\mathrm{d}\mu_1(t) = \int_a^0 f(t)\phi(t)\,\mathrm{d}t \tag{3.7}$$

and, analogously, also that

$$\mu(K) = \int_a^b f(t)\phi(t)\,\mathrm{d}t. \tag{3.8}$$

Now, using that $\mathrm{g}_\mu(K) = 0$, by Fubini's theorem we also get

$$\begin{aligned} 0 &= \int_K \langle x, \mathrm{e}_n \rangle\,\mathrm{d}\mu(x) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} t\,\chi_K(y,t)\,\mathrm{d}\mu_{n-1}(y)\,\mathrm{d}\mu_1(t) \\ &= \int_a^b t f(t)\phi(t)\,\mathrm{d}t. \end{aligned} \tag{3.9}$$

Notice also that since f is p-concave and ϕ is s-concave then the function $f \cdot \phi$ is q-concave for $q = (1/p + 1/s)^{-1}$ (see e.g. [2, Lemma 1.2.4]). Thus, taking into account (3.7), (3.8) and (3.9), the result now follows from Theorem 3.1.1. □

3.2 The general case

We conclude this chapter by showing the n-dimensional case of the functional form of Grünbaum's inequality.

Theorem 3.2.1 ([25]). *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be a p-concave function, for some $p \in [0, \infty)$, with compact support, and let H be a hyperplane. If f is centred then*

$$\int_{H^-} f(x)\,\mathrm{d}x \geq \left(\frac{np+1}{(n+1)p+1} \right)^{(np+1)/p} \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x, \tag{3.10}$$

where, if $p = 0$, the above identity must be understood as

$$\lim_{p \to 0^+} \left(\frac{np+1}{(n+1)p+1} \right)^{(np+1)/p} = e^{-1}.$$

Proof. The case $n = 1$ is collected in Theorem 3.1.1 (see also Remark 3.1 for the case $p = 0$), and so we suppose that $n \geq 2$.

Assuming without loss of generality that $u = \mathrm{e}_1$, let $g : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ be the function given by

$$g(t) = \int_{\mathbb{R}^{n-1}} f(t, y)\,\mathrm{d}y.$$

Since f is p-concave with compact support, there exist $a, b \in \mathbb{R}$ such that $[a, b]$ is the support of g. Moreover, from the Borell-Brascamp-Lieb inequality (1.9), g is q-concave, for $q = p/\big((n-1)p + 1\big)$.

On the one hand, by Fubini's theorem we have

$$\int_{H^-} f(x)\,\mathrm{d}x = \int_{-\infty}^{0}\int_{\mathbb{R}^{n-1}} f(t,y)\,\mathrm{d}y\,\mathrm{d}t = \int_a^0 g(t)\,\mathrm{d}t,$$

and also

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = \int_a^b g(t)\,\mathrm{d}t.$$

On the other hand, since f is centred, from Fubini's theorem we get

$$\int_a^b tg(t)\,\mathrm{d}t = \int_{\mathbb{R}^n} \langle x, \mathrm{e}_1\rangle\, f(x)\,\mathrm{d}x = 0.$$

Thus, using that g is q-concave and centred, if $p > 0$ (and thus $q > 0$) we may apply Theorem 3.1.1 to obtain that

$$\begin{aligned}\int_{H^-} f(x)\,\mathrm{d}x = \int_a^0 g(t)\,\mathrm{d}t &\geq \left(\frac{q+1}{2q+1}\right)^{(q+1)/q}\int_a^b g(t)\,\mathrm{d}t\\ &= \left(\frac{np+1}{(n+1)p+1}\right)^{(np+1)/p}\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x.\end{aligned}$$

Finally, if $p = 0$ (and hence $q = 0$ as well) we may apply (3.6) to obtain

$$\int_{H^-} f(x)\,\mathrm{d}x = \int_a^0 g(t)\,\mathrm{d}t \geq e^{-1}\int_a^b g(t)\,\mathrm{d}t = e^{-1}\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x.$$

This concludes the proof. □

Observe that if $f = \chi_K$ is the characteristic function of a convex body $K \subset \mathbb{R}^n$ with centroid at the origin, we have that f is centred on the one hand, and that f is ∞-concave on the other hand, and hence p-concave for any $p > 0$. Thus, (3.10) holds for all $p > 0$ and then, taking limits as $p \to \infty$ in both sides, (1.2) infers.

Chapter 4

On ϕ-concave extensions of Grünbaum type inequalities

In Chapter 2 we showed that Grünbaum's inequality (1.2) can be enhanced in terms of the concavity nature of the function $f : H^{\perp} \longrightarrow \mathbb{R}_{\geq 0}$, asssociated to a convex body $K \subset \mathbb{R}^n$ and a hyperplane H, given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$, and furthermore that this result may be extended to the case of compact sets, not necessarily convex, for which such an f (for some hyperplane H) satisfies certain concavity assumption.

At this point, in the spirit of what happens in the setting of operations of convex bodies, in which the L_p-*addition* (see [10], [22] and [31, Sect. 9.1]) is extended to the *Orlicz*-sum (see [14] and [15]) with respect to a convex and strictly increasing function $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ with $\phi(0) = 0$, it is natural to wonder about a possible Grünbaum type inequality for the family of those compact sets K such that (there exists a hyperplane H for which) $\phi \circ f$ is concave, with ϕ being an *Orlicz*-class function (convex, strictly increasing and $\phi(0) = 0$). Moreover, one could expect to obtain a family of inequalities depending on a function ϕ that may recover (2.5) when $\phi(t) = t^p$ if $p > 0$, and (2.6) if $\phi(t) = \log(t)$, getting therefore a further generalization of (1.2).

Throughout this chapter, we will prove that if the cross-sections volume function of a compact set $K \subset \mathbb{R}^n$ (of positive volume) with respect to some hyperplane H passing through its centroid is ϕ-concave, then one can find a sharp lower bound for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$. Moreover, we will study other related results for ϕ-concave functions (and involving the centroid). The results collected in this chapter can be found in [24].

4.1 Some preliminary results

We start this section by showing that if the cross-sections volume function f (of a compact set $K \subset \mathbb{R}^n$ with non-empty interior) is *ϕ-concave* in the direction of the normal vector of the hyperplane H, with ϕ being an Orlicz-class function, then $4/9$ is a lower bound for $\mathrm{vol}(K^-)/\mathrm{vol}(K)$. More precisely, we have:

Proposition 4.1.1 ([24])**.** *Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let H be a hyperplane such that the function $f : H^\perp \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x+H)\big)$ is ϕ-concave, for some Orlicz-class function $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$. Then*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \frac{4}{9}.$$

Proof. Let $\varphi := \phi^{-1}$ and let $s_1 = \phi(t_1)$ and $s_2 = \phi(t_2)$, for some $t_1, t_2 \in \mathbb{R}_{\geq 0}$. Thus, for every $\lambda \in (0,1)$,

$$\begin{aligned}\varphi\big((1-\lambda)s_1 + \lambda s_2\big) &= \varphi\big((1-\lambda)\phi(t_1) + \lambda\phi(t_2)\big) \geq \varphi\Big(\phi\big((1-\lambda)t_1 + \lambda t_2\big)\Big) \\ &= (1-\lambda)t_1 + \lambda t_2 = (1-\lambda)\varphi(s_1) + \lambda\varphi(s_2),\end{aligned}$$

because ϕ is strictly increasing and convex. Hence φ is a concave function on its support.

Furthermore, using that φ is concave together with the fact that the function $\phi \circ f$ is concave, we deduce that, for all $\lambda \in (0,1)$ and any $x, y \in \mathbb{R}^n$ with $f(x)f(y) > 0$,

$$\begin{aligned}f\big((1-\lambda)x + \lambda y\big) &\geq \varphi\Big((1-\lambda)\phi\big(f(x)\big) + \lambda\phi\big(f(y)\big)\Big) \\ &\geq (1-\lambda)f(x) + \lambda f(y)\end{aligned}$$

and thus f is concave on its support.

Finally, using Theorem 2.2.1 with $p = 1$, we obtain that

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \frac{4}{9},$$

as we wanted to see. □

Remark 4.1. *Although some assumptions on such a function ϕ must be assumed, as we have seen in Proposition 4.1.1, the case of ϕ being an* Orlicz-*class function is deduced directly from Theorem 2.2.1. Therefore, within the rest of chapter, we will focus on the case of $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R} \cup \{-\infty\}$ being a strictly increasing function.*

Since the aim of the chapter is to obtain a family of inequalities recovering (2.5) and (2.6), observing that in [26] (and also in Grünbaum's original proof collected in [16]) the compact sets are compared with sets of revolution given by a p-affine function with $p \geq 0$, it seems natural to work now with sets of revolution associated to a ϕ-affine function. This leads us to the following definition.

Definition 4.1. *Let $c, \delta > 0$ and $\gamma \in \mathbb{R}$ be fixed and $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function satisfying that $\lim_{t\to\infty} \phi(t) = \infty$. Then, setting $\varphi := \phi^{-1}$, $g_\phi : (-\gamma + \phi(0)/c, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ is a non-negative function given by $g_\phi(t) = \varphi\bigl(c(t+\gamma)\bigr)$.*

Let $u \in \mathbb{S}^{n-1}$ be fixed. By C_ϕ we denote the set of revolution whose section by the hyperplane $\{x \in \mathbb{R}^n : \langle x, u\rangle = t\}$ is an $(n-1)$-dimensional ball of radius $\bigl(g_\phi(t)/\kappa_{n-1}\bigr)^{1/(n-1)}$ with axis parallel to u. (We would like to warn the reader that, in the following, we shall use the word "radius" for such a generating function $\bigl(g_\phi(t)/\kappa_{n-1}\bigr)^{1/(n-1)}$ of the set C_ϕ, for short.)

Therefore, we shall start by computing the value of the ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$ for the above-mentioned sets C_ϕ.

Lemma 4.1.1 ([24, Lemma 2.1])**.** *Let $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R} \cup \{-\infty\}$ be a strictly increasing function and let $H \in \mathrm{G}(n, n-1)$ be a hyperplane with unit normal vector $u \in \mathbb{S}^{n-1}$. Let g_ϕ and C_ϕ, with axis parallel to u, be as in Definition 4.1, for any fixed $c, \delta > 0$ and $\gamma \in \mathbb{R}$. Then, if C_ϕ has centroid at the origin and $\varphi := \phi^{-1}$ satisfies that $\int_{\phi(0)}^0 \varphi(s)\,\mathrm{d}s < \infty$ and $\left|\int_{\phi(0)}^0 s\varphi(s)\,\mathrm{d}s\right| < \infty$, we have*

$$\frac{\mathrm{vol}\bigl(C_\phi^-\bigr)}{\mathrm{vol}(C_\phi)} = \frac{\int_{\phi(0)}^{x_\varphi} \varphi(s)\mathrm{d}s}{\int_{\phi(0)}^{R} \varphi(s)\,\mathrm{d}s},$$

where $x_\varphi = \frac{\int_{\phi(0)}^{R} s\varphi(s)\,\mathrm{d}s}{\int_{\phi(0)}^{R} \varphi(s)\,\mathrm{d}s}$ and $R = c(\delta+\gamma)$.

Proof. On the one hand, by Fubini's theorem, we obtain

$$\mathrm{vol}(C_\phi) = \int_{-\gamma+\phi(0)/c}^{\delta} \varphi\bigl(c(t+\gamma)\bigr)\,\mathrm{d}t = \frac{1}{c}\int_{\phi(0)}^{c(\delta+\gamma)} \varphi(s)\,\mathrm{d}s.$$

Therefore, from the hypothesis $g(C_\phi) = 0$, we get that

$$0 = \bigl[\mathrm{g}(C_p)\bigr]_1 = \frac{1}{\mathrm{vol}(C_\phi)}\int_{-\gamma+\phi(0)/c}^{\delta} t\varphi\bigl(c(t+\gamma)\bigr)\,\mathrm{d}t$$

if and only if

$$0 = \int_{-\gamma+\phi(0)/c}^{\delta} t\varphi\bigl(c(t+\gamma)\bigr)\,\mathrm{d}t = \frac{1}{c}\int_{\phi(0)}^{c(\delta+\gamma)} \Bigl(\frac{s}{c} - \gamma\Bigr)\varphi(s)\,\mathrm{d}s.$$

Hence, for $R = c(\delta+\gamma)$,

$$x_\varphi = \frac{\int_{\phi(0)}^{R} s\varphi(s)\,\mathrm{d}s}{\int_{\phi(0)}^{R} \varphi(s)\,\mathrm{d}s} = c\gamma.$$

Finally, we have that

$$\frac{\mathrm{vol}\bigl(C_\phi^-\bigr)}{\mathrm{vol}(C_\phi)} = \frac{\int_{-\gamma+\phi(0)/c}^{0} \varphi\bigl(c(t+\gamma)\bigr)\,\mathrm{d}t}{\int_{-\gamma+\phi(0)/c}^{\delta} \varphi\bigl(c(t+\gamma)\bigr)\,\mathrm{d}t} = \frac{\int_{\phi(0)}^{c\gamma} \varphi(s)\,\mathrm{d}s}{\int_{\phi(0)}^{R} \varphi(s)\,\mathrm{d}s} = \frac{\int_{\phi(0)}^{x_\varphi} \varphi(s)\,\mathrm{d}s}{\int_{\phi(0)}^{R} \varphi(s)\,\mathrm{d}s}.$$

This concludes the proof. □

4.2 The case of the volume

Once the possible extremal cases have been studied in Section 4.1, we are in conditions to prove one of the main theorems of the chapter. Although the proof we collect here exploits the original idea used by Grünbaum in [16] (and follows similar steps to that of [26, Theorem 1.1]), we include here the details to make reading easier.

Theorem 4.2.1 ([24]). *Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let H be a hyperplane such that the function $f : H^\perp \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$ is ϕ-concave, for some strictly increasing function $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R} \cup \{-\infty\}$ with $\lim_{t\to\infty} \phi(t) = \infty$, $\int_{\phi(0)}^0 \varphi(s)\,\mathrm{d}s < \infty$ and $\left|\int_{\phi(0)}^0 s\varphi(s)\,\mathrm{d}s\right| < \infty$, where $\varphi := \phi^{-1}$. Then, if $F : (\phi(0), \infty) \longrightarrow \mathbb{R}_{>0}$ is the function given by $F(t) = \int_{\phi(0)}^t \varphi(s)\,\mathrm{d}s$, we have that*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \frac{\int_{\phi(0)}^{x_\varphi} \varphi(s)\mathrm{d}s}{\int_{\phi(0)}^{R} \varphi(s)\,\mathrm{d}s}, \tag{4.1}$$

where $x_\varphi = \frac{\int_{\phi(0)}^R s\varphi(s)\,\mathrm{d}s}{\int_{\phi(0)}^R \varphi(s)\,\mathrm{d}s}$ and $R = F^{-1}\left(F\Big(\phi\big(f(0)\big)\Big)\frac{\mathrm{vol}(K)}{\mathrm{vol}(K^-)}\right)$.

Proof. First, we set the function $g_\phi : (-\gamma + \phi(0)/c, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ given by $g_\phi(t) = \varphi\big(c(t+\gamma)\big)$ with $c, \delta > 0$ and $\gamma \in \mathbb{R}$ such that $g_\phi(0) = f(0)$,

$$\int_{-\gamma+\phi(0)/c}^0 g_\phi(t)\,\mathrm{d}t = \int_a^0 f(t)\,\mathrm{d}t \quad \text{and} \quad \int_0^\delta g_\phi(t)\,\mathrm{d}t = \int_0^b f(t)\,\mathrm{d}t. \tag{4.2}$$

Indeed, if we set the function $F : \big(\phi(0), \infty\big) \longrightarrow \mathbb{R}_{>0}$ as $F(t) = \int_{\phi(0)}^t \varphi(s)\,\mathrm{d}s$, since $\varphi(t) > 0$ for all $t > \phi(0)$ (note that φ is defined on $\big(\phi(0), \infty\big)$ because $\lim_{t\to\infty} \phi(t) = \infty$), then F is a strictly increasing function and so there exists $F^{-1} : \mathbb{R}_{>0} \longrightarrow \big(\phi(0), \infty\big)$. Hence, if $\phi\big(f(0)\big) \neq 0$, taking

$$\gamma = \frac{\phi\big(f(0)\big)}{F\big(\phi(f(0))\big)} \int_a^0 f(t)\,\mathrm{d}t, \quad c = \frac{\phi\big(f(0)\big)}{\gamma} \quad \text{and} \\ \delta = \frac{1}{c}F^{-1}\left(c\int_a^b f(t)\,\mathrm{d}t\right) - \gamma, \tag{4.3}$$

elementary computations show (4.2) (we also note that, since F^{-1} is strictly increasing and

$$\gamma = \frac{1}{c}F^{-1}\left(c\int_a^0 f(t)\,\mathrm{d}t\right),$$

we actually have $\delta > 0$), whereas if $\phi\big(f(0)\big) = 0$, taking $\gamma = 0$,

$$c = \frac{F(0)}{\int_a^0 f(t)\,\mathrm{d}t} \quad \text{and} \quad \delta = \frac{1}{c}F^{-1}\left(c\int_a^b f(t)\,\mathrm{d}t\right), \tag{4.4}$$

we also have (4.2). And note that, in any case, $-\gamma + \phi(0)/c < \delta$.

In other words, for the set of revolution C_ϕ of radius $(g_\phi(t)/\kappa_{n-1})^{1/(n-1)}$, we have $\mathrm{vol}_{n-1}\big(C_\phi(0)\big) = \mathrm{vol}_{n-1}\big(K(0)\big)$,

$$\mathrm{vol}\big(C_\phi^-\big) = \mathrm{vol}(K^-) \quad \text{and} \quad \mathrm{vol}\big(C_\phi^+\big) = \mathrm{vol}(K^+). \tag{4.5}$$

And thus, in particular, $\mathrm{vol}(C_\phi) = \mathrm{vol}(K)$.

From the concavity of $\phi \circ f$, together with $g_\phi(0) = f(0)$ and thus $\big(\phi \circ g_\phi\big)(0) = \big(\phi \circ f\big)(0)$, the fact that $\phi \circ g_\phi$ is affine and (4.2), we get on the one hand that $-\gamma + \phi(0)/c \leq a < 0 < \delta \leq b$. On the other hand, defining the functions $\bar{f}, \bar{g}_\phi : (-\gamma + \phi(0)/c, b] \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$\bar{f}(t) = \begin{cases} f(t) & \text{if } t \in [a,b], \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \bar{g}_\phi(t) = \begin{cases} g_\phi(t) & \text{if } t \in (-\gamma + \phi(0)/c, \delta], \\ 0 & \text{otherwise,} \end{cases}$$

we may conclude that there exists $x_0 \in [a, 0)$ such that $\bar{f}(t) \geq \bar{g}_\phi(t)$ for all $t \in [x_0, 0] \cup [\delta, b]$ and $\bar{f}(t) \leq \bar{g}_\phi(t)$ otherwise. Hence, since $\mathrm{g}(K) = 0$, from (4.2) we have

$$\begin{aligned} -\int_{-\gamma+\phi(0)/c}^{\delta} t g_\phi(t)\,\mathrm{d}t &= \int_a^b t f(t)\,\mathrm{d}t - \int_{-\gamma+\phi(0)/c}^{\delta} t g_\phi(t)\,\mathrm{d}t \\ &= \int_{-\gamma+\phi(0)/c}^{b} t\big(\bar{f}(t) - \bar{g}_\phi(t)\big)\,\mathrm{d}t \\ &= \int_{-\gamma+\phi(0)/c}^{0} t\big(\bar{f}(t) - \bar{g}_\phi(t)\big)\,\mathrm{d}t + \int_0^b t\big(\bar{f}(t) - \bar{g}_\phi(t)\big)\,\mathrm{d}t \\ &= \int_{-\gamma+\phi(0)/c}^{0} (t - x_0)\big(\bar{f}(t) - \bar{g}_\phi(t)\big)\,\mathrm{d}t + \int_0^b (t-\delta)\big(\bar{f}(t) - \bar{g}_\phi(t)\big)\,\mathrm{d}t \geq 0, \end{aligned}$$

with equality if and only if $f = g_\phi$ almost everywhere. Thus, we have $[\mathrm{g}(C_\phi)]_1 \leq 0$, and then,

$$\frac{\mathrm{vol}\big(C_\phi^-\big)}{\mathrm{vol}(C_\phi)} \geq \frac{\mathrm{vol}\big(C_\phi \cap \big(\mathrm{g}(C_\phi) + H\big)^-\big)}{\mathrm{vol}(C_\phi)}.$$

Finally, from (4.5) and Lemma 4.1.1 we get that

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} = \frac{\mathrm{vol}\big(C_\phi^-\big)}{\mathrm{vol}(C_\phi)} \geq \frac{\mathrm{vol}\big(C_\phi \cap \big(\mathrm{g}(C_\phi) + H\big)^-\big)}{\mathrm{vol}(C_\phi)} = \frac{\int_{\phi(0)}^{x_\varphi} \varphi(s)\mathrm{d}s}{\int_{\phi(0)}^{R} \varphi(s)\,\mathrm{d}s},$$

where from (4.2) (see (4.3) and (4.4)), $R = F^{-1}\left(F\Big(\phi\big(f(0)\big)\Big)\frac{\mathrm{vol}(K)}{\mathrm{vol}(K^-)}\right)$. □

Remark 4.2. *Inequality (4.1) is (asymptotically) sharp:*

- *if the function $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ satisfies that $\phi(0) > -\infty$, then there is equality if and only if $f = g_\phi$ almost everywhere.*

- *Otherwise, that is, if $\phi(0) = -\infty$, considering the set C_ϕ with centroid at the origin and, for simplicity, $\delta = 1$, from Lemma 4.1.1 we know that*

$$\frac{\mathrm{vol}(C_\phi^-)}{\mathrm{vol}(C_\phi)} = \frac{\int_{-\infty}^{x_\varphi} \varphi(s)\mathrm{d}s}{\int_{-\infty}^{R} \varphi(s)\,\mathrm{d}s}.$$

Hence, if we consider $K_a = C_\phi \cap \{x \in \mathbb{R}^n : \langle x, u\rangle \geq a\}$, $a < 1$, we have $[\mathrm{g}(K_a)]_1 \to 0$ and

$$\lim_{a\to-\infty} \frac{\mathrm{vol}(K_a^-)}{\mathrm{vol}(K_a)} = \frac{\int_{-\infty}^{x_\varphi} \varphi(s)\mathrm{d}s}{\int_{-\infty}^{R} \varphi(s)\,\mathrm{d}s}.$$

Remark 4.3. *On the one hand,* (2.5) *is collected in the case of f being ϕ-concave with $\phi(t) = t^p$, $p \in (0,\infty)$. In this case, $\varphi(t) = t^{1/p}$ and a straightforward computation shows that*

$$x_\varphi = \frac{\int_0^{c(\delta+\gamma)} s^{(p+1)/p}\,\mathrm{d}s}{\int_0^{c(\delta+\gamma)} s^{1/p}\,\mathrm{d}s} = \frac{p+1}{2p+1} c(\delta+\gamma).$$

Thus,

$$\frac{\int_{\phi(0)}^{x_\varphi} \varphi(s)\mathrm{d}s}{\int_{\phi(0)}^{R} \varphi(s)\,\mathrm{d}s} = \frac{\int_0^{x_\varphi} s^{1/p}\mathrm{d}s}{\int_0^{c(\delta+\gamma)} s^{1/p}\,\mathrm{d}s} = \left(\frac{p+1}{2p+1}\right)^{(p+1)/p}.$$

On the other hand, (2.6) *can be recovered when f is ϕ-concave with $\phi(t) = \log(t)$. Concerning this case, considering by convention* $\log(0) = -\infty$*, we have that*

$$x_\varphi = \frac{\int_{-\infty}^{c(\delta+\gamma)} se^s\,\mathrm{d}s}{\int_{-\infty}^{c(\delta+\gamma)} e^s\,\mathrm{d}s} = c(\delta+\gamma) - 1.$$

Hence,

$$\frac{\int_{\phi(0)}^{x_\varphi} \varphi(s)\mathrm{d}s}{\int_{\phi(0)}^{R} \varphi(s)\,\mathrm{d}s} = \frac{\int_{-\infty}^{x_\varphi} e^s\,\mathrm{d}s}{\int_{-\infty}^{c(\delta+\gamma)} e^s\,\mathrm{d}s} = e^{-1},$$

as we expected.

4.3 A note on ϕ-concave functions

Throughout this section, our focus will be on ϕ-concave functions and their behavior with respect to the centroid. To this regard, we start by showing that, exploiting the approach used in [23, Theorem 3.1] we can also prove, in the same more general setting of ϕ-concave functions (for some unbounded strictly increasing function ϕ), a Grünbaum type inequality for the volume of cross-sections of compact sets with centroid at the origin, which is furthermore a consequence of the following more general result for *centred functions*:

Theorem 4.3.1 ([24]). *Let $f : [a, b] \longrightarrow \mathbb{R}_{\geq 0}$ be a non-negative (and not identically zero) centred function. Then, if f is ϕ-concave for some strictly increasing function $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R} \cup \{-\infty\}$ with $\lim_{t\to\infty} \phi(t) = \infty$, $\int_{\phi(0)}^{0} \varphi(s)\,\mathrm{d}s < \infty$ and $\left|\int_{\phi(0)}^{0} s\varphi(s)\,\mathrm{d}s\right| < \infty$, where $\varphi := \phi^{-1}$, we have*

$$\frac{f(0)}{|f|_\infty} \geq \frac{\varphi(x_\varphi)}{\varphi(R)}, \tag{4.6}$$

where $x_\varphi = \frac{\int_{\phi(0)}^{R} s\varphi(s)\,\mathrm{d}s}{\int_{\phi(0)}^{R} \varphi(s)\,\mathrm{d}s}$ and $R = \phi\big(|f|_\infty\big)$.

Proof. On the one hand, we can assume, without loss of generality, that $|f|_\infty = f(t_0)$ with $0 < t_0 \leq b$ and $f(0) < f(t_0)$ (otherwise there is nothing to prove).

On the other hand, since f is centred, we get that

$$\int_a^b tf(t)\,\mathrm{d}t = 0,$$

and hence,

$$\int_a^0 (-t)f(t)\,\mathrm{d}t = \int_0^b tf(t)\,\mathrm{d}t \geq \int_0^{t_0} tf(t)\,\mathrm{d}t. \tag{4.7}$$

Now, let $g_\phi : (-\gamma + \phi(0)/c, t_0] \longrightarrow \mathbb{R}_{\geq 0}$ be the function given by $g_\phi(t) = \varphi\big(c(t+\gamma)\big)$ such that $g_\phi(0) = f(0)$ and $g_\phi(t_0) = f(t_0)$. Thus, if $\phi\big(f(0)\big) \neq 0$,

$$\gamma = t_0 \left(\frac{\phi\big(f(0)\big)}{\phi\big(f(t_0)\big) - \phi\big(f(0)\big)} \right) \quad \text{and} \quad c = \frac{\phi\big(f(0)\big)}{\gamma}.$$

Otherwise, when $\phi\big(f(0)\big) = 0$ we have that $\gamma = 0$ and $c = \phi\big(f(t_0)\big)/t_0$. Note that, in any case, we get that $c > 0$ because ϕ is strictly increasing.

Notice also that since $\phi \circ g_\phi$ is an affine function and $\phi \circ f$ is a concave function, and taking into account that $g_\phi(0) = f(0)$ and $g_\phi(t_0) = f(t_0)$, we have that $\phi\big(g_\phi(t)\big) \geq \phi\big(f(t)\big)$ for all $t \in [a, 0]$ and $\phi\big(g_\phi(t)\big) \leq \phi(f(t))$ for all $t \in [0, t_0]$. Thus, using that φ is increasing, we get that $g_\phi(t) \geq f(t)$ for all $t \in [a, 0]$ and $g_\phi(t) \leq f(t)$ for all $t \in [0, t_0]$, and hence, from (4.7), we get

$$\begin{aligned} \int_{-\gamma+\phi(0)/c}^{0} (-t)g_\phi(t)\,\mathrm{d}t \geq \int_a^0 (-t)g_\phi(t)\,\mathrm{d}t &\geq \int_a^0 (-t)f(t)\,\mathrm{d}t \\ &\geq \int_0^{t_0} tf(t)\,\mathrm{d}t \geq \int_0^{t_0} tg_\phi(t)\,\mathrm{d}t. \end{aligned}$$

Therefore,

$$0 \geq \int_{-\gamma+\phi(0)/c}^{t_0} tg_\phi(t)\,\mathrm{d}t = \frac{1}{c}\left(\frac{1}{c}\int_{\phi(0)}^{c(t_0+\gamma)} s\varphi(s)\,\mathrm{d}s - \gamma \int_{\phi(0)}^{c(t_0+\gamma)} \varphi(s)\,\mathrm{d}s \right),$$

which implies, taking into account that $R=\phi\big(f(t_0)\big)$, that

$$x_\varphi=\frac{\int_{\phi(0)}^{R} s\varphi(s)\,\mathrm{d}s}{\int_{\phi(0)}^{R}\varphi(s)\,\mathrm{d}s}\leq c\gamma.$$

Finally, putting all these facts together and taking into account that φ is strictly increasing, we conclude that

$$\frac{f(0)}{f(t_0)}=\frac{g_\phi(0)}{g_\phi(t_0)}=\frac{\varphi(c\gamma)}{\varphi\big(c(t_0+\gamma)\big)}\geq\frac{\varphi(x_\varphi)}{\varphi(R)}.$$

This concludes the proof. □

Remark 4.4. *Note that, if we consider a compact set with centroid at the origin, then the cross-sections volume function is centred as a consequence of Fubini's theorem. And so, if the cross-sections volume function is ϕ-concave (where ϕ is a function in the conditions of Theorem 4.3.1), then* (4.6) *holds.*

We end this chapter by showing the suitable extension of [11, Lemma 1 and Theorem 2] to the setting of ϕ-concave functions. Although we exploit the techniques there used, we include here all the details of the proofs for the sake of completeness.

Lemma 4.3.1 ([24, Lemma 3.1]). *Let $K\subset\mathbb{R}^n$ be a convex body with non-empty interior. If $f:K\longrightarrow\mathbb{R}_{\geq 0}$ is a non-negative (and not identically zero) function such that for a given strictly increasing function $\phi:\mathbb{R}_{\geq 0}\longrightarrow\mathbb{R}_{\geq 0}$ the function $\phi\circ f$ is concave, then for every interior point x_1 of K there exists a hyperplane $H\in\mathrm{G}(n,n-1)$ such that*

$$f(x_1)=\max_{x\in K\cap H} f(x).$$

Proof. First, since x_1 is an interior point of K we know that $f(x_1)>0$. Now, if we define the set $\tilde{K}=\big\{(x,t)\in\mathbb{R}^n\times\mathbb{R}: x\in K,\ 0\leq t\leq\phi\big(f(x)\big)\big\}$, considering that $\phi\circ f$ is concave, then $\tilde{K}$ is a convex body. Thus, taking into account that the point $\big(x_1,\phi\big(f(x_1)\big)\big)$ belongs to the boundary of $\tilde{K}$, we have that there exists a support hyperplane $\tilde{H}$ passing through that point, i.e, there exists a point $(y,s)\in\mathbb{R}^n\times\mathbb{R}$ such that, for every $(x,t)\in\tilde{K}$,

$$\langle x-x_1,y\rangle+\Big(t-\phi\big(f(x_1)\big)\Big)s\leq 0.$$

Furthermore, knowing that $(x_1,0)\in\tilde{K}$, we have that $\phi\big(f(x_1)\big)s\geq 0$ and hence, since $\phi(0)=0$ (and $f(x_1)>0$) together with the fact that ϕ is strictly increasing, we get that $s\geq 0$. Moreover, if $s=0$, then every $x\in K$ satisfies that $\langle x,y\rangle\leq\langle x_1,y\rangle$ and thus, $H=\{x\in\mathbb{R}^n:\langle x,y\rangle=\langle x_1,y\rangle\}$ would separate x_1 and K, which is a contradiction with the fact that K is convex and x_1 belongs to its interior.

Therefore, $s>0$ and for every $x\in K\cap H$ we have that $\phi\big(f(x)\big)-\phi\big(f(x_1)\big)\leq 0$ $\big($since $\big(x,\phi\big(f(x)\big)\big)\in\tilde{K}\big)$. Then, using again that ϕ is strictly increasing, we obtain that $f(x)\leq f(x_1)$, as we wanted to see. □

Theorem 4.3.2 ([24]). *Let $K\subset\mathbb{R}^n$ be a convex body with non-empty interior. Let $f:K\longrightarrow\mathbb{R}_{\geq 0}$ be a non-negative (and not identically zero) function such that $\phi\circ f$ is concave, for some strictly increasing function $\phi:\mathbb{R}_{\geq 0}\longrightarrow\mathbb{R}_{\geq 0}$. Then, writing $\varphi:=\phi^{-1}$, we have*

$$\frac{\phi\big(f(x_f)\big)}{\phi(|f|_\infty)}\geq 1-\frac{I^n(\varphi,R)}{I^{n-1}(\varphi,R)},$$

where $I^n(\varphi,R)=\int_{\phi(0)}^R(1-s/R)^n\varphi(s)\,\mathrm{d}s$, $x_f=\frac{\int_K xf(x)\,\mathrm{d}x}{\int_K f(x)\,\mathrm{d}x}$ and $R=\phi\big(|f|_\infty\big)$.

Proof. First, if we consider the point x_f previously defined, using Lemma 4.3.1 we get that there exists a hyperplane H such that $f(x_f)=\max_{K\cap H}f(x)$. Therefore, without loss of generality, we can assume that $0\in K$ and $|f|_\infty=f(0)>f(x_f)$. Then $0\notin H$, which implies that $H=\{x\in\mathbb{R}^n:\langle x,u\rangle=1\}$ for some $u\in\mathbb{R}^n$, $u\neq 0$.

Now, we define the sets $D_0=K\cap H$, $D=\mathrm{pos}D_0$ and the function $\Psi:\mathbb{R}^n\longrightarrow\mathbb{R}$ given by

$$\Psi(rx_0)=\begin{cases}\varphi\big((1-r)\phi(f(0))+r\phi(f(x_0))\big) & \text{if}\quad (1-r)\big(\phi\circ f\big)(0)\\ & \qquad +r\big(\phi\circ f\big)(x_0)\geq\phi(0),\\ 0 & \text{otherwise,}\end{cases}$$

for any $x_0\in D_0$ and all $r\geq 0$, and $\Psi(x)=0$ if $x\notin D$.

Then, on the one hand, taking into account that $\phi\circ f$ is concave we get

$$(\phi\circ\Psi)(rx_0)\leq(\phi\circ f)(rx_0)\quad\text{for all}\quad 0\leq r\leq 1 \tag{4.8}$$

and any $x_0\in D_0$.

On the other hand, for any $x_0\in D_0$ and $r\geq 1$ such that $rx_0\in K$, and taking $\lambda=1/r$, we have that

$$\begin{aligned}\big(\phi\circ f\big)(x_0)&=\big(\phi\circ f\big)\big((1-\lambda)0+\lambda(rx_0)\big)\\ &\geq(1-\lambda)\big(\phi\circ f\big)(0)+\lambda\big(\phi\circ f\big)(rx_0).\end{aligned}$$

Hence,

$$r\big(\phi\circ f\big)(x_0)\geq(r-1)\big(\phi\circ f\big)(0)+\big(\phi\circ f\big)(rx_0),$$

and thus

$$(1-r)\big(\phi\circ f\big)(0)+r\big(\phi\circ f\big)(x_0)\geq\big(\phi\circ f\big)(rx_0),$$

implying (using that $f(rx_0)\geq 0$) that

$$(1-r)\big(\phi\circ f\big)(0)+r\big(\phi\circ f\big)(x_0)\geq\phi(0)$$

for all $r\geq 1$ such that $rx_0\in K$. Taking into account the latter (and that $\phi\circ f$ is concave), when $r\geq 1$ and $rx_0\in K$ we have

$$\begin{aligned}\big(\phi\circ\Psi\big)(rx_0)&=(1-r)\big(\phi\circ f\big)(0)+r\big(\phi\circ f\big)(x_0)\\ &\geq(1-r)\big(\phi\circ f\big)(0)+r(1-\lambda)\big(\phi\circ f\big)(0)+r\lambda\big(\phi\circ f\big)\left(\frac{x_0}{\lambda}\right)\end{aligned}$$

for every $\lambda \in (0,1]$. Hence, if we set $\lambda = 1/r$, this yields

$$\big(\phi \circ \Psi\big)(rx_0) \geq \big(\phi \circ f\big)(rx_0) \quad \text{for all} \quad r \geq 1 \tag{4.9}$$

such that $rx_0 \in K$ (for $x_0 \in D_0$).

Then, using that ϕ is strictly increasing, from (4.8) and (4.9) we get that $\Psi \leq f$ in $K \cap H^-$ and $\Psi \geq f$ in $K \cap H^+$.

Moreover, on the one hand, if we define the point x_Ψ as

$$x_\Psi = \frac{\int_D x\Psi(x)\,\mathrm{d}x}{\int_D \Psi(x)\,\mathrm{d}x},$$

we obtain that

$$\langle x_\Psi - x_f, u\rangle = \frac{\int_D \langle x - x_f, u\rangle\, \Psi(x)\,\mathrm{d}x}{\int_D \Psi(x)\,\mathrm{d}x} = \frac{\int_D (\langle x, u\rangle - 1)\Psi(x)\,\mathrm{d}x}{\int_D \Psi(x)\,\mathrm{d}x},$$

which yields, using the previous two relations, that $(\langle x, u\rangle - 1)\Psi(x) \geq (\langle x, u\rangle - 1)f(x)$ for all $x \in K$. Hence, since the point x_f belongs to H, we deduce that

$$\int_D (\langle x, u\rangle - 1)\Psi(x)\,\mathrm{d}x \geq \int_K (\langle x, u\rangle - 1)\Psi(x)\,\mathrm{d}x \geq \int_K (\langle x, u\rangle - 1)f(x)\,\mathrm{d}x = 0,$$

and thus we have that $\langle x_\Psi, u\rangle \geq 1$.

On the other hand, for all $x \in D$ with $\Psi(x) \neq 0$ we have

$$\begin{aligned}\Psi(x) &= \varphi\left((1 - \langle x, u\rangle)\phi\big(f(0)\big) + \langle x, u\rangle\, \phi\left(f\left(\frac{x}{\langle x, u\rangle}\right)\right)\right)\\ &= \varphi\left(\phi\big(f(0)\big)\left(1 - \frac{\langle x, u\rangle}{\psi(x)}\right)\right),\end{aligned}$$

where $\psi(x) = \frac{\phi(f(0))}{\phi(f(0)) - \phi\left(f\left(\frac{x}{\langle x,u\rangle}\right)\right)}$. Then, using Fubini's Theorem, we obtain that

$$\begin{aligned}\int_D \langle x, u\rangle\, \Psi(x)\,\mathrm{d}x &= \int_D \langle x, u\rangle\, \varphi\left(\phi\big(f(0)\big)\left(1 - \frac{\langle x, u\rangle}{\psi(x)}\right)\right)\mathrm{d}x\\ &= \int_0^\infty \int_{\{x\in\mathbb{R}^n : \langle x,u\rangle = 0\}} \chi_D(z + tu)t\varphi\left(\phi\big(f(0)\big)\left(1 - \frac{t}{\psi(z + tu)}\right)\right)\mathrm{d}z\,\mathrm{d}t,\end{aligned}$$

where, doing the change of variables $z = ty$ and using that $\psi(tx) = \psi(x)$, we deduce again from Fubini's theorem (jointly with the definition of Ψ) that

$$\begin{aligned}&\int_D \langle x, u\rangle \Psi(x)\,\mathrm{d}x\\ &\quad= \int_0^\infty \int_{D_0} t^n \varphi\left(\phi\big(f(0)\big)\left(1 - \frac{t}{\psi(w)}\right)\right)\mathrm{d}w\,\mathrm{d}t\\ &\quad= \int_{D_0} \int_0^{\psi(w)\left(1 - \phi(0)/\phi(f(0))\right)} t^n \varphi\left(\phi\big(f(0)\big)\left(1 - \frac{t}{\psi(w)}\right)\right)\mathrm{d}t\,\mathrm{d}w.\end{aligned}$$

Hence, doing the change of variables $s = \phi\big(f(0)\big)\big(1 - t/\psi(w)\big)$, we get

$$\begin{aligned}
&\int_D \langle x, u\rangle \Psi(x)\,\mathrm{d}x \\
&\qquad = \frac{1}{\phi\big(f(0)\big)} \int_{D_0} \int_{\phi(0)}^{\phi(f(0))} \psi(w)^{n+1} \left(1 - \frac{s}{\phi\big(f(0)\big)}\right)^n \varphi(s)\,\mathrm{d}s\,\mathrm{d}w \\
&\qquad = \frac{1}{\phi\big(f(0)\big)} \int_{D_0} \psi(w)^{n+1}\,\mathrm{d}w \left(\int_{\phi(0)}^{\phi(f(0))} \left(1 - \frac{s}{\phi\big(f(0)\big)}\right)^n \varphi(s)\mathrm{d}s \right) \\
&\qquad = \frac{1}{\phi\big(f(0)\big)} I^n(\varphi, R) \int_{D_0} \psi(w)^{n+1}\,\mathrm{d}w.
\end{aligned}$$

Furthermore, using the same technique, it is easy to see that

$$\int_D \Psi(x)\,\mathrm{d}x = \frac{1}{\phi\big(f(0)\big)} I^{n-1}(\varphi, R) \int_{D_0} \psi(w)^n\,\mathrm{d}w$$

and then,

$$1 \le \langle x_\Psi, u\rangle = \frac{I^n(\varphi, R)\int_{D_0}\psi(w)^{n+1}\,\mathrm{d}w}{I^{n-1}(\varphi, R)\int_{D_0}\psi(w)^n\,\mathrm{d}w} \le \frac{I^n(\varphi, R))}{I^{n-1}(\varphi, R)} \max_{D_0} \psi.$$

Finally, using the previous lemma, for every point $x \in D_0$ we have

$$\psi(x) = \frac{\phi\big(f(0)\big)}{\phi\big(f(0)\big) - \phi\big(f(x)\big)} \le \frac{\phi\big(f(0)\big)}{\phi\big(f(0)\big) - \phi\big(f(x_f)\big)}$$

and hence,

$$1 \le \langle x_\Psi, u\rangle \le \frac{I^n(\varphi, R)\phi\big(f(0)\big)}{I^{n-1}(\varphi, R)\Big(\phi\big(f(0)\big) - \phi\big(f(x_f)\big)\Big)}.$$

The latter yields

$$\phi\big(f(x_f)\big) \ge \left(1 - \frac{I^n(\varphi, R)}{I^{n-1}(\varphi, R)}\right) \phi\big(f(0)\big),$$

as we wanted to see. □

Chapter 5

Cutting the body through other points

One way of interpreting Grünbaum's result (1.2) is that, given any convex body, we can always find a point within the set (the centroid) in such a way that cutting the body through a hyperplane passing by this point results in two sets with a significant proportion of the total volume. Considering this, a natural question arises: is there a family of points, potentially including the centroid, also exhibiting this property? Furthermore, are there other special points with similar properties worth exploring?

Along this chapter, together with the study of some other cases, we will extend Grünbaum's result (1.2) (and Theorem 2.2.1) to the case in which the hyperplane H passes by any of the points lying in a whole uniparametric family of r-powered centroids associated to K (depending on a real parameter $r \geq 0$), by proving a more general functional result on concave functions. The original content of this chapter is collected in [1].

5.1 A first approach

When dealing with potential candidates for those points we have mentioned before (that is, that ensure a big amount of volume for both sets that are obtained when cutting the original body by a hyperplane passing through them), one could first think about the incenter and the circumcenter of a convex body. We start this section by showing that it is not possible to get an anologue of Grünbaum's inequality 1.2 if we assume that one of these points is at the origin instead of its centroid.

On the one hand, regarding the case of the circumcenter, if we consider the planar set K_t given by the convex hull of a right triangle with hypotenuse at the y-axis and opposite vertex placed at e_1, and the point $(-t,0)$, for $t>0$ arbitrarily small (see Figure 5.1), then we clearly obtain that the circumcenter of such a set lies at the origin (for any t small enough). However, we have that

$$\frac{\mathrm{vol}(K_t^-)}{\mathrm{vol}(K_t)} = \frac{t}{1+t},$$

and hence, we get $\lim_{t\to 0^+}\mathrm{vol}(K_t^-)/\mathrm{vol}(K_t)=0$.

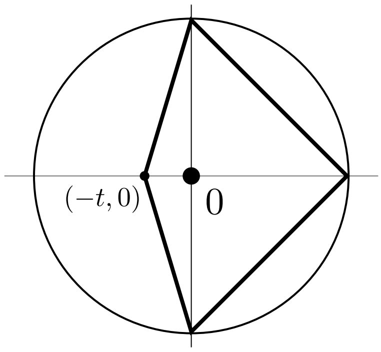

Figure 5.1: A set with circumcenter at the origin but with a small ratio of volumes.

On the other hand, considering the planar set $K_t=\mathrm{conv}\big(B_2\cup\{(t,0)\}\big)$ (see Figure 5.2), it is clear that the incenter of K_t is at the origin for every $t>1$. Therefore, K_t^- does not depend on t, and we clearly have that $\lim_{t\to\infty}\mathrm{vol}(K_t^-)/\mathrm{vol}(K_t)=0$.

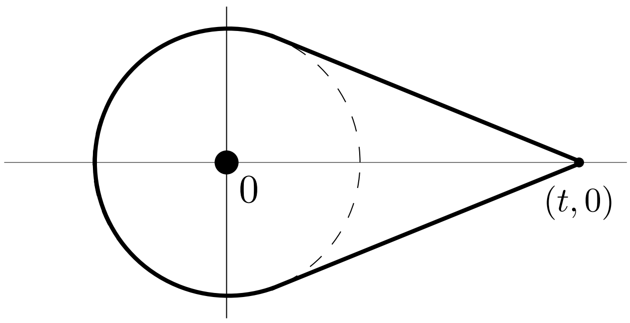

Figure 5.2: A set with incenter at the origin but with a small ratio of volumes.

As we have seen, one can easily find "special" points for which it is not possible to get an analogue of Theorem 2.2.1. Now, by considering the midpoint in the direction u of K, instead of its centroid, that is, the point $\big[(a+b)/2\big]\cdot u$, we will show that a positive answer for the previously posed question is possible. This is the content of the following result.

Proposition 5.1.1 ([1]). *Let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior and with midpoint, with respect to some direction $u \in \mathbb{S}^{n-1}$, at the origin. Let $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$ be the hyperplane with normal vector u and assume that the function $f : H^\perp \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$ is p-concave, for some $p \in (0, \infty)$. Then*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{1}{2}\right)^{(p+1)/p}.$$

Proof. In the proof of Theorem 2.2.1 it is shown that there exists a non-negative p-affine function $g_p : [-\gamma, \delta] \longrightarrow \mathbb{R}_{\geq 0}$ given by $g_p(t) = c(t+\gamma)^{1/p}$, for some $\gamma, \delta, c > 0$, such that $g_p(0) = f(0)$,

$$\int_{-\gamma}^{0} g_p(t)\,\mathrm{d}t = \int_{a}^{0} f(t)\,\mathrm{d}t \quad \text{and} \quad \int_{0}^{\delta} g_p(t)\,\mathrm{d}t = \int_{0}^{b} f(t)\,\mathrm{d}t,$$

and further that $-\gamma \leq a < 0 < \delta \leq b$.

Here, since K has its midpoint (w.r.t. u) at the origin, we have that $a = -b$ and then we get $-\gamma + \delta \leq 0$. Thus

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} = \frac{\int_{-\gamma}^{0} g_p(t)\,\mathrm{d}t}{\int_{-\gamma}^{\delta} g_p(t)\,\mathrm{d}t} \geq \frac{\int_{-\gamma}^{(-\gamma+\delta)/2} g_p(t)\,\mathrm{d}t}{\int_{-\gamma}^{\delta} g_p(t)\,\mathrm{d}t} = \left(\frac{1}{2}\right)^{(p+1)/p},$$

as desired. □

At this point, having in mind Theorem 2.2.1 and Proposition 5.1.1, to figure out such a possible family of points we notice that, fixed a unit direction $u \in \mathbb{S}^{n-1}$, the corresponding components w.r.t. u of both the centroid and the midpoint have a similar nature. Indeed, the component of $\mathrm{g}(K)$ w.r.t. u is given by (see (1.7))

$$[\mathrm{g}(K)]_1 = \frac{1}{\mathrm{vol}(K)} \int_a^b t f(t)\,\mathrm{d}t = \frac{\int_a^b t f(t)^1\,\mathrm{d}t}{\int_a^b f(t)^1\,\mathrm{d}t},$$

whereas the corresponding component of the midpoint is

$$\frac{a+b}{2} = \frac{\int_a^b t f(t)^0\,\mathrm{d}t}{\int_a^b f(t)^0\,\mathrm{d}t}.$$

Thus, with the above-mentioned aim in mind, it seems reasonable to consider the points $\mathrm{g}_r \cdot u$, where

$$\mathrm{g}_r := \frac{\int_a^b t f(t)^r\,\mathrm{d}t}{\int_a^b f(t)^r\,\mathrm{d}t} \tag{5.1}$$

for any $r \geq 0$. In fact, following the same idea as in [11, Theorem 3], we can get a first result concerning this family of points. The statement reads as follows.

Corollary 5.1.1 ([1]). *Let $r \in (0,\infty)$ and let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior having the point $\mathrm{g}_r \cdot u$, with respect to some direction $u \in \mathbb{S}^{n-1}$, at the origin. Let $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$ be the hyperplane with normal vector u and assume that the function $f : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(t) = \mathrm{vol}_{n-1}\big(K \cap (tu + H)\big)$ is p-concave, for some $p \in (0,\infty)$. Then*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{p}{2p+r}\right)^{(p+1)/p}. \tag{5.2}$$

We will derive Corollary 5.1.1 as a simple application of the following (slightly more general) functional result.

Proposition 5.1.2 ([1]). *Let $K \subset \mathbb{R}^n$ be a convex body. Let $g : K \longrightarrow \mathbb{R}_{\geq 0}$ be a concave function and let $f : K \longrightarrow \mathbb{R}_{\geq 0}$ be a p-concave function, with $p > 0$. Then*

$$g\left(\frac{\int_K x f(x)\,\mathrm{d}x}{\int_K f(x)\,\mathrm{d}x}\right) \geq \left(\frac{p}{(n+1)p+1}\right)|g|_\infty. \tag{5.3}$$

Proof. Let μ be the probability measure whose density function is given by

$$\mathrm{d}\mu(x) = \frac{f(x)}{\int_K f(x)\,\mathrm{d}x}\mathrm{d}x.$$

Since g is concave, using Jensen's inequality (1.8) we get that

$$g\left(\frac{\int_K x f(x)\,\mathrm{d}x}{\int_K f(x)\,\mathrm{d}x}\right) = g\left(\int_K x\,\mathrm{d}\mu(x)\right) \geq \int_K g(x)\,\mathrm{d}\mu(x) = \int_0^{|g|_\infty} \mu\big(\{x \in K : g(x) \geq t\}\big)\,\mathrm{d}t,$$

where in the last identity we have used Fubini's theorem. Now, since the density of μ, with respect to the Lebesgue measure, is p-concave, from the Borell-Brascamp-Lieb inequality (1.9), we have that the function $\varphi(t) := \mu\big(\{x \in K : g(x) \geq t\}\big)$ is $\big(p/(np+1)\big)$-concave. Hence, and taking into account that $\varphi(0) = 1$ and $\varphi\big(|g|_\infty\big) \geq 0$, we may assure that $\varphi(t)^{p/(np+1)} \geq (1 - t/|g|_\infty)$ for all $t \in [0, |g|_\infty]$. So, by integrating we obtain that

$$\int_0^{|g|_\infty} \mu\big(\{x \in K : g(x) \geq t\}\big)\,\mathrm{d}t \geq \int_0^{|g|_\infty} (1 - t/|g|_\infty)^{(np+1)/p}\,\mathrm{d}t = \left(\frac{p}{(n+1)p+1}\right)|g|_\infty,$$

from where the result immediately follows. □

We conclude this section by showing Corollary 5.1.1.

Proof of Corollary 5.1.1. Denoting by $[a,b]$ the support of f, let $\bar{f}, g : [a,b] \longrightarrow \mathbb{R}_{\geq 0}$ be the functions given by

$$g(t) = \mathrm{vol}\big(K\cap(tu+H)^-\big)^{p/(p+1)} = \left(\int_a^t f(s)\,\mathrm{d}s\right)^{p/(p+1)} \quad \text{and} \quad \bar{f}(t) = \mathrm{vol}_{n-1}\big(K\cap(tu+H)\big)^r = f(t)^r.$$

By hypothesis, it is clear that $\bar{f}$ is (p/r)-concave, whereas from the Borell-Brascamp-Lieb inequality (1.9), we have that g is concave on $[a, b]$. So, from Proposition 5.1.2 applied to the functions $\bar{f}$ and g, and taking into account that $\mathrm{g}_r = 0$, we get that

$$\mathrm{vol}(K^-) = g\left(\frac{\int_a^b t\bar{f}(t)\,\mathrm{d}t}{\int_a^b \bar{f}(t)\,\mathrm{d}t}\right)^{(p+1)/p} \geq \left(\frac{p}{2p+r}\right)^{(p+1)/p} |g|_\infty^{(p+1)/p} = \left(\frac{p}{2p+r}\right)^{(p+1)/p} \mathrm{vol}(K),$$

as desired. □

5.2 General Grünbaum type inequalities involving a uniparametric family of points

During this section, we will show that the uniparametric class of points given by $\{\mathrm{g}_r \cdot u : r \geq 0\}$, where g_r is given by (5.1), allows us to extend Grünbaum's inequality (or more generally Theorem 2.2.1) to the case in which one replaces the classical centroid by any of them. Moreover, two tighter inequalities improving the one that was shown in Corollary 5.1.1 will be obtained. More precisely, we will enhance the constant given by (5.2), namely, $\big(p/(2p+r)\big)^{(p+1)/p}$, by other two constants (depending on whether r is either less than one or greater than one). Furthermore, these new constants do fit well with (2.5), since, in fact, they all coincide when $r = 1$, that is, when one considers the centroid of the set. The precise statement of our main (geometric) result reads as follows.

Theorem 5.2.1 ([1])**.** *Let $r \in [0, \infty)$ and let $K \subset \mathbb{R}^n$ be a compact set with non-empty interior having the point $\mathrm{g}_r \cdot u$, with respect to some direction $u \in \mathbb{S}^{n-1}$, at the origin. Let $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$ be the hyperplane with normal vector u and assume that the function $f : H^\perp \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$ is p-concave, for some $p \in (0, \infty)$. If $r \geq 1$ then*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{p+1}{2p+r}\right)^{(p+1)/p},$$

whereas if $0 \leq r \leq 1$ then

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)} \geq \left(\frac{p+r}{2p+r}\right)^{(p+1)/p}.$$

Notice that the cases $r = 1$ and $r = 0$ correspond to Theorem 2.2.1 and Proposition 5.1.1, respectively. Taking into account that, once a unit direction $u \in \mathbb{S}^{n-1}$ is fixed, the above geometric results are reduced to the study of one variable functions with certain concavity, here we deal with the corresponding functional counterpart of these statements (from which the latter result will be obtained as a consequence of such an equivalent functional one). To this aim, first we need to define the notion of functional α-centroid: given a non-negative function $h : [a, b] \longrightarrow [0, \infty)$ with positive integral, for any $\alpha > 0$ we will write

$$\mathrm{g}_\alpha(h) := \frac{\int_a^b t h(t)^\alpha \,\mathrm{d}t}{\int_a^b h(t)^\alpha \,\mathrm{d}t}. \tag{5.4}$$

Now, the statement of our main result reads as follows.

Theorem 5.2.2 ([1])**.** *Let $h : [a, b] \longrightarrow [0, \infty)$ be a non-negative concave function, and let $\alpha, \beta > 0$. If $\beta \le \alpha$ then*

$$\frac{\int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t}{\int_a^b h(t)^\beta \,\mathrm{d}t} \ge \left(\frac{\beta+1}{\alpha+2}\right)^{\beta+1}, \tag{5.5}$$

whereas if $\alpha \le \beta$ then

$$\frac{\int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t}{\int_a^b h(t)^\beta \,\mathrm{d}t} \ge \left(\frac{\alpha+1}{\alpha+2}\right)^{\beta+1}. \tag{5.6}$$

Remark 5.1. *We observe that Theorem 5.2.1 is directly obtained from the previous result by just taking $h = f^p$, $\beta = 1/p$ and $\alpha = r\beta$ (where the case $r = 0$ is derived when doing $\alpha \to 0^+$). Moreover, Theorem 5.2.2 can be shown from Theorem 5.2.1 by just considering the set of revolution K associated to the radius function $r = (1/\kappa_{n-1}) f^{1/(n-1)}$, $f = h^\beta$, $p = 1/\beta$ and $r = \alpha/\beta$. Therefore, in fact, both results (Theorems 5.2.1 and 5.2.2) are equivalent.*

We would like also to point out that, apart from the already mentioned Theorem 5.2.1, and thus in particular Theorem 2.2.1 and Grünbaum's inequality, both Theorems D and E can be derived as direct applications of Theorem 5.2.2.

Indeed, on the one hand, applying Theorem 5.2.2 with $h = f^{1/(n-1)}$, which is concave because of *Brunn's concavity principle* (see e.g. [8, Section 1.2.1] and also [27, Theorem 12.2.1]), and taking $\alpha = n - 1$ and $\beta \to 0^+$, one gets $b/(b - a) \ge 1/(n + 1)$, which is exactly (1.3).

On the other hand, applying Theorem 5.2.2 with $h = f^{1/(n-1)}$, $\alpha = n - 1$ and taking $\beta \to \infty$ (by raising first both sides of (5.6) to the power $1/\beta$), one has

$$\left(\frac{\max_{t\in[0,b]} f(t)}{|f|_\infty}\right)^{1/(n-1)} \ge \frac{n}{n+1}. \tag{5.7}$$

Since we may assume without loss of generality that $|f|_\infty > \max_{t\in[0,b]} f(t)$ (considering otherwise the function $t \mapsto f(-t)$), we then get $\max_{t\in[0,b]} f(t) = f(0)$ (since f is $(1/(n-1))$-concave and thus quasi-concave), and therefore (5.7) is nothing but (1.4).

In order to make the reading easier, we shall divide the proof into several sections. Firstly, we will provide some general considerations and next we will explore two separate cases to distinguish whether $\beta < \alpha$ or $\alpha < \beta$.

5.2.1 Some preliminaries concerning the proof

Here we are going to slightly modify the approach followed in [33, Theorem 8] (which is equivalent to our case $\beta = n-1$) in order to cover all the cases for a general concave function h, which will allow us to show Theorem 5.2.2. To this aim, we split the proof into two steps, depending on whether $\beta < \alpha$ or $\alpha < \beta$. Note also that the case $\alpha = \beta$ is equivalent to the statement of Theorem 2.2.1 (by just taking $p := 1/\alpha = 1/\beta$ and $f = h^\alpha = h^\beta$). Before distinguishing whether $\beta < \alpha$ or $\alpha < \beta$, we make some general considerations.

We may assume, without loss of generality, that $a = 0$. Now, let $L \subset \mathbb{R}^2$ be the convex body

$$L := \big\{(x, y) \in \mathbb{R}^2 \,:\, 0 \le x \le b,\, 0 \le y \le h(x)\big\}$$

and notice that, from Fubini's theorem, we have

$$\mathrm{g}_\alpha(h) = \frac{\int_0^b t h(t)^\alpha \,\mathrm{d}t}{\int_0^b h(t)^\alpha \,\mathrm{d}t} = \frac{\int_L \langle x, \mathrm{e}_1\rangle \langle x, \mathrm{e}_2\rangle^{\alpha-1} \,\mathrm{d}x}{\int_L \langle x, \mathrm{e}_2\rangle^{\alpha-1} \,\mathrm{d}x}. \tag{5.8}$$

Let μ_β be the measure on $\mathbb{R}^2$ given by $\mathrm{d}\mu_\beta(x) = \langle x, \mathrm{e}_2\rangle^{\beta-1} \,\mathrm{d}x$. Then

$$\mathrm{g}_\alpha(h) = \frac{\int_L \langle x, \mathrm{e}_1\rangle \langle x, \mathrm{e}_2\rangle^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)}{\int_L \langle x, \mathrm{e}_2\rangle^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)}$$

and

$$\begin{aligned}\frac{\int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t}{\int_0^b h(t)^\beta \,\mathrm{d}t} &= \frac{\int_{\{x \in L \,:\, \langle x, \mathrm{e}_1\rangle \ge \mathrm{g}_\alpha(h)\}} \langle x, \mathrm{e}_2\rangle^{\beta-1} \,\mathrm{d}x}{\int_L \langle x, \mathrm{e}_2\rangle^{\beta-1} \,\mathrm{d}x} \\ &= \frac{\mu_\beta\{x \in L \,:\, \langle x, \mathrm{e}_1\rangle \ge \mathrm{g}_\alpha(h)\}}{\mu_\beta(L)}.\end{aligned}$$

We may assure that there exist $\gamma < \delta$ and $c > 0$ in such a way that the affine decreasing function $g : [\gamma, \delta] \longrightarrow [0, \infty)$ given by

$$g(t) = c(\delta - t)$$

satisfies

$$\begin{aligned}&\text{(i) } g\big(\mathrm{g}_\alpha(h)\big) = h\big(\mathrm{g}_\alpha(h)\big), \\ &\text{(ii) } \textstyle\int_\gamma^\delta g(t)^\beta \,\mathrm{d}t = \int_0^b h(t)^\beta \,\mathrm{d}t, \quad \text{and} \\ &\text{(iii) } \textstyle\int_{\mathrm{g}_\alpha(h)}^\delta g(t)^\beta \,\mathrm{d}t = \int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t.\end{aligned} \tag{5.9}$$

Indeed, taking

$$\delta = \frac{\beta+1}{h\big(\mathrm{g}_\alpha(h)\big)^\beta} \int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t + \mathrm{g}_\alpha(h),\ c = \frac{h\big(\mathrm{g}_\alpha(h)\big)}{\delta - \mathrm{g}_\alpha(h)} \text{ and } \gamma = \delta - \left(\frac{\beta+1}{c^\beta} \int_0^b h(t)^\beta \,\mathrm{d}t\right)^{1/(\beta+1)},$$

elementary computations show (5.9).

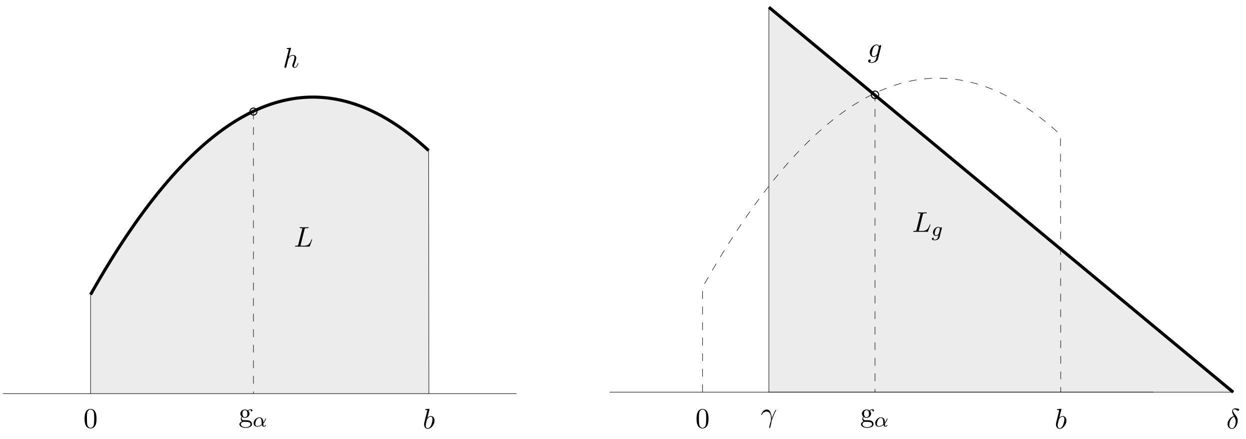

Figure 5.3: Sets L and L_g.

Then, denoting by $L_g \subset \mathbb{R}^2$ the triangle (see Figure 5.3) given by

$$L_g := \big\{(x, y) \in \mathbb{R}^2 \,:\, \gamma \le x \le \delta,\, 0 \le y \le g(x)\big\},$$

from (ii) and (iii) in (5.9) (by using Fubini's theorem) and the relative position of h and g (see Figure 5.4), we have that

$$\begin{aligned}
&\text{(i) } \mu_\beta(L) = \mu_\beta(L_g),\\
&\text{(ii) } \mu_\beta\big(\{x \in L : \langle x, \mathrm{e}_1\rangle \ge \mathrm{g}_\alpha(h)\}\big) = \mu_\beta\big(\{x \in L_g : \langle x, \mathrm{e}_1\rangle \ge \mathrm{g}_\alpha(h)\}\big),\\
&\text{(iii) } 0 \le \gamma \le \mathrm{g}_\alpha(h) \le b \le \delta.
\end{aligned} \tag{5.10}$$

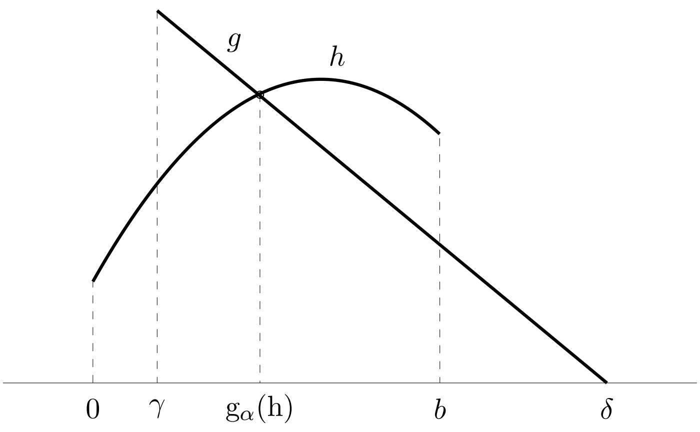

Figure 5.4: Relative position of the functions h and g.

Moreover, defining $g(t) := 0$ for all $t \in [0, \gamma]$ and $h(t) := 0$ for any $t \in [b, \delta]$, there exists $x_0 \in (\mathrm{g}_\alpha(h), b]$ such that $h(t) \ge g(t)$ for all $t \in [0, \gamma] \cup [\mathrm{g}_\alpha(h), x_0]$ and $h(t) \le g(t)$ otherwise (see Figure 5.4 -there, observe that x_0 coincides with b). Hence, on the one hand, for every $s \in [\mathrm{g}_\alpha(h), x_0]$

(the case of $s \geq x_0$ immediately follows) we get that

$$\int_s^b h(t)^\beta \,\mathrm{d}t = \int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t - \int_{\mathrm{g}_\alpha(h)}^s h(t)^\beta \,\mathrm{d}t \leq \int_{\mathrm{g}_\alpha(h)}^\delta g(t)^\beta \,\mathrm{d}t - \int_{\mathrm{g}_\alpha(h)}^s g(t)^\beta \,\mathrm{d}t = \int_s^\delta g(t)^\beta \,\mathrm{d}t.$$

On the other hand, for every $s \in [\gamma, \mathrm{g}_\alpha(h)]$ (again, the case of $s \leq \gamma$ immediately follows) we have that

$$\int_s^b h(t)^\beta \,\mathrm{d}t = \int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t + \int_s^{\mathrm{g}_\alpha(h)} h(t)^\beta \,\mathrm{d}t \leq \int_{\mathrm{g}_\alpha(h)}^\delta g(t)^\beta \,\mathrm{d}t + \int_s^{\mathrm{g}_\alpha(h)} g(t)^\beta \,\mathrm{d}t = \int_s^\delta g(t)^\beta \,\mathrm{d}t.$$

Therefore,

$$\mu_\beta\big(\{x \in L \,:\, \langle x, \mathrm{e}_1\rangle \geq s\}\big) = \int_s^b h(t)^\beta \,\mathrm{d}t \leq \int_s^\delta g(t)^\beta \,\mathrm{d}t = \mu_\beta\big(\{x \in L_g \,:\, \langle x, \mathrm{e}_1\rangle \geq s\}\big) \qquad (5.11)$$

for every $s \in [0, \delta]$.

5.2.2 The case $\beta < \alpha$

We devote this section to proving the first part of Theorem 5.2.2, namely, we show (5.5) provided that $\beta < \alpha$.

We will first prove that there exists a non-negative and concave function $\varphi : [\gamma, \delta] \longrightarrow [0, \|h\|_\infty]$ such that

$$\mathrm{g}_\alpha(h) \leq \frac{\int_{L_g} \langle x, \mathrm{e}_1\rangle \varphi(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)}{\int_{L_g} \varphi(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)}. \qquad (5.12)$$

To this aim, we consider the function $W : [0, \|h\|_\infty] \longrightarrow [0, \mu_\beta(L)]$ given by

$$W(s) = \mu_\beta\big(\{x \in L \,:\, \langle x, \mathrm{e}_2\rangle \geq s\}\big),$$

which is clearly both strictly decreasing and surjective. We may then define the non-negative function $w : [0, \|h\|_\infty] \longrightarrow [\gamma, \delta]$ that satisfies

$$W(s) = \mu_\beta\big(\{x \in L_g \,:\, \langle x, \mathrm{e}_1\rangle \geq w(s)\}\big)$$

for any $s \in [0, \|h\|_\infty]$. Indeed, since

$$W(s) = \frac{c^\beta(\delta - w(s))^{\beta+1}}{\beta(\beta+1)},$$

we get that

$$w(s) = \delta - \left(\frac{\beta(\beta+1)}{c^\beta} W(s)\right)^{1/(\beta+1)}.$$

Notice that, from the Borell-Brascamp-Lieb inequality (1.9), we have that the function $W^{1/(\beta+1)}$ is concave (since the density of μ_β, with respect to the Lebesgue measure, is $\big(1/(\beta-1)\big)$-concave).

Therefore, w is strictly increasing, surjective and convex, and then there exists the function $\varphi = w^{-1} : [\gamma, \delta] \longrightarrow [0, \|h\|_\infty]$, which is further (strictly increasing and) concave.

Now, we will start by bounding from above the right-hand side of (5.8). By using Fubini's theorem, (iii) in (5.10) and (5.11), we have

$$\begin{aligned}
&\frac{1}{\alpha-\beta}\int_L \langle x, \mathrm{e}_1\rangle\langle x, \mathrm{e}_2\rangle^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x) \\
&\quad= \int_L \int_0^{\langle x,\mathrm{e}_1\rangle} \mathrm{d}s_1 \int_0^{\langle x,\mathrm{e}_2\rangle} s_2^{\alpha-\beta-1}\,\mathrm{d}s_2\,\mathrm{d}\mu_\beta(x) \\
&\quad= \int_0^b \int_0^{\|h\|_\infty} s_2^{\alpha-\beta-1}\mu_\beta\big(L^+(\mathrm{e}_1, s_1)\cap L^+(\mathrm{e}_2, s_2)\big)\,\mathrm{d}s_2\,\mathrm{d}s_1 \\
&\quad\le \int_0^b \int_0^{\|h\|_\infty} s_2^{\alpha-\beta-1}\min\big\{\mu_\beta\big(L^+(\mathrm{e}_1, s_1)\big), \mu_\beta\big(L^+(\mathrm{e}_2, s_2)\big)\big\}\,\mathrm{d}s_2\,\mathrm{d}s_1 \\
&\quad\le \int_0^b \int_0^{\|h\|_\infty} s_2^{\alpha-\beta-1}\min\big\{\mu_\beta\big(L_g^+(\mathrm{e}_1, s_1)\big), \mu_\beta\big(L^+(\mathrm{e}_2, s_2)\big)\big\}\,\mathrm{d}s_2\,\mathrm{d}s_1 \\
&\quad\le \int_0^\delta \int_0^{\|h\|_\infty} s_2^{\alpha-\beta-1}\min\big\{\mu_\beta\big(L_g^+(\mathrm{e}_1, s_1)\big), \mu_\beta\big(L^+(\mathrm{e}_2, s_2)\big)\big\}\,\mathrm{d}s_2\,\mathrm{d}s_1.
\end{aligned} \tag{5.13}$$

So, on the one hand, from (5.13) we get

$$\begin{aligned}
&\frac{1}{\alpha-\beta}\int_L \langle x, \mathrm{e}_1\rangle\langle x, \mathrm{e}_2\rangle^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x) \\
&\quad\le \int_0^\delta \int_0^{\|h\|_\infty} s_2^{\alpha-\beta-1}\min\big\{\mu_\beta\big(L_g^+(\mathrm{e}_1, s_1)\big), \mu_\beta\big(L^+(\mathrm{e}_2, s_2)\big)\big\}\,\mathrm{d}s_2\,\mathrm{d}s_1 \\
&\quad= \int_0^\delta \int_0^{\|h\|_\infty} s_2^{\alpha-\beta-1}\min\Big\{\mu_\beta\big(L_g^+(\mathrm{e}_1, s_1)\big), \mu_\beta\Big(L_g^+\big(\mathrm{e}_1, w(s_2)\big)\Big)\Big\}\,\mathrm{d}s_2\,\mathrm{d}s_1 \\
&\quad= \int_0^\delta \int_0^{\|h\|_\infty} s_2^{\alpha-\beta-1}\mu_\beta\Big(L_g^+(\mathrm{e}_1, s_1)\cap L_g^+\big(\mathrm{e}_1, w(s_2)\big)\Big)\,\mathrm{d}s_2\,\mathrm{d}s_1 \\
&\quad= \frac{1}{\alpha-\beta}\int_{L_g} \langle x, \mathrm{e}_1\rangle\varphi(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x),
\end{aligned} \tag{5.14}$$

where in the last equality above we have used that $\langle x, \mathrm{e}_1\rangle \ge w(s_2)$ if and only if $\varphi(\langle x, \mathrm{e}_1\rangle) \ge s_2$.

On the other hand, since $\alpha - \beta > 0$, from Fubini's theorem we have that

$$\begin{aligned}
\frac{1}{\alpha-\beta}\int_L \langle x, \mathrm{e}_2\rangle^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x) &= \int_0^{\|h\|_\infty} s^{\alpha-\beta-1}\mu_\beta\big(\{x\in L : \langle x, \mathrm{e}_2\rangle \ge s\}\big)\,\mathrm{d}s \\
&= \int_0^{\|h\|_\infty} s^{\alpha-\beta-1}\mu_\beta\big(\{x\in L_g : \langle x, \mathrm{e}_1\rangle \ge w(s)\}\big)\,\mathrm{d}s \\
&= \int_0^{\|h\|_\infty} s^{\alpha-\beta-1}\mu_\beta\big(\{x\in L_g : \varphi(\langle x, \mathrm{e}_1\rangle) \ge s\}\big)\,\mathrm{d}s \\
&= \int_{L_g} \varphi(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x).
\end{aligned} \tag{5.15}$$

Hence, from (5.14) and (5.15) (and using (5.8)) we obtain (5.12), as desired.

Now we will prove that for any concave function $\varphi : [\gamma, \delta] \longrightarrow [0, \infty)$ we have that

$$\frac{\int_{L_g} \langle x, \mathrm{e}_1\rangle \varphi(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)}{\int_{L_g} \varphi(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)} \le \frac{\int_{L_g} \langle x, \mathrm{e}_1\rangle \left(\langle x, \mathrm{e}_1\rangle - \gamma\right)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)}{\int_{L_g} \left(\langle x, \mathrm{e}_1\rangle - \gamma\right)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)}. \tag{5.16}$$

To this aim, let $C_1 > 0$ be such that

$$\int_{L_g} \big(C_1(\langle x, \mathrm{e}_1\rangle - \gamma)\big)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x) = \int_{L_g} \varphi(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x). \tag{5.17}$$

Since the latter identity is equivalent (by Fubini's theorem) to

$$\int_\gamma^\delta \Big(\big(C_1(t-\gamma)\big)^{\alpha-\beta} - \varphi(t)^{\alpha-\beta}\Big) g(t)^\beta \,\mathrm{d}t = 0,$$

we may assert, taking into account that φ is concave, that there exists $t_0 \in (\gamma, \delta)$ such that

$$\begin{aligned} &\text{(i) } C_1(t-\gamma) \le \varphi(t) \text{ for every } \gamma \le t \le t_0, \quad \text{and} \\ &\text{(ii) } C_1(t-\gamma) \ge \varphi(t) \text{ for every } t_0 \le t \le \delta. \end{aligned} \tag{5.18}$$

Then, from (5.17) and (5.18) (and using Fubini's theorem), we get

$$\begin{aligned} &\beta\Big(\int_{L_g} \langle x, \mathrm{e}_1\rangle \varphi(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x) - \int_{L_g} \langle x, \mathrm{e}_1\rangle \big(C_1(\langle x, \mathrm{e}_1\rangle - \gamma)\big)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)\Big) \\ &= \int_\gamma^\delta t\Big(\varphi(t)^{\alpha-\beta} - \big(C_1(t-\gamma)\big)^{\alpha-\beta}\Big) g(t)^\beta \,\mathrm{d}t \\ &= \int_\gamma^{t_0} t\Big(\varphi(t)^{\alpha-\beta} - \big(C_1(t-\gamma)\big)^{\alpha-\beta}\Big) g(t)^\beta \,\mathrm{d}t \\ &+ \int_{t_0}^\delta t\Big(\varphi(t)^{\alpha-\beta} - \big(C_1(t-\gamma)\big)^{\alpha-\beta}\Big) g(t)^\beta \,\mathrm{d}t \\ &\le t_0 \int_\gamma^{t_0} \Big(\varphi(t)^{\alpha-\beta} - \big(C_1(t-\gamma)\big)^{\alpha-\beta}\Big) g(t)^\beta \,\mathrm{d}t \\ &+ t_0 \int_{t_0}^\delta \Big(\varphi(t)^{\alpha-\beta} - \big(C_1(t-\gamma)\big)^{\alpha-\beta}\Big) g(t)^\beta \,\mathrm{d}t \\ &= t_0 \int_\gamma^\delta \Big(\varphi(t)^{\alpha-\beta} - \big(C_1(t-\gamma)\big)^{\alpha-\beta}\Big) g(t)^\beta \,\mathrm{d}t \\ &= \beta t_0 \int_{L_g} \Big(\varphi(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta} - \big(C_1(\langle x, \mathrm{e}_1\rangle - \gamma)\big)^{\alpha-\beta}\Big) \,\mathrm{d}\mu_\beta(x) = 0. \end{aligned}$$

Thus, we have

$$\int_{L_g} \langle x, \mathrm{e}_1\rangle \varphi(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x) \le \int_{L_g} \langle x, \mathrm{e}_1\rangle \big(C_1(\langle x, \mathrm{e}_1\rangle - \gamma)\big)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x),$$

which, together with (5.17), yields (5.16).

Now, we will compute the right-hand side of (5.16). On the one hand,

$$\begin{aligned}\int_{L_g} (\langle x, \mathrm{e}_1\rangle - \gamma)^{\alpha-\beta}\, \mathrm{d}\mu_\beta(x) =& \frac{c^\beta}{\beta}\int_\gamma^\delta (t-\gamma)^{\alpha-\beta}(\delta-t)^\beta\, \mathrm{d}t\\ =& \frac{c^\beta}{\beta}(\delta-\gamma)^{\alpha+1}\int_0^1 s^{\alpha-\beta}(1-s)^\beta\, \mathrm{d}s\\ =& \frac{c^\beta}{\beta}(\delta-\gamma)^{\alpha+1}\frac{\Gamma(\alpha-\beta+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+2)},\end{aligned}$$

whereas, on the other hand, we obtain

$$\begin{aligned}\int_{L_g} \langle x, \mathrm{e}_1\rangle (\langle x, \mathrm{e}_1\rangle - \gamma)^{\alpha-\beta}\, \mathrm{d}\mu_\beta(x) &= \frac{c^\beta}{\beta}\int_\gamma^\delta t(t-\gamma)^{\alpha-\beta}(\delta-t)^\beta\, \mathrm{d}t\\ = \gamma\frac{c^\beta}{\beta}(\delta-\gamma)^{\alpha+1}\int_0^1 s^{\alpha-\beta}(1-s)^\beta\, \mathrm{d}s &+ \frac{c^\beta}{\beta}(\delta-\gamma)^{\alpha+2}\int_0^1 s^{\alpha-\beta+1}(1-s)^\beta\, \mathrm{d}s\\ = \frac{c^\beta}{\beta}(\delta-\gamma)^{\alpha+1}\frac{\Gamma(\alpha-\beta+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+2)}&\left(\gamma + (\delta-\gamma)\frac{\alpha-\beta+1}{\alpha+2}\right).\end{aligned}$$

Hence, we have

$$\frac{\int_{L_g} \langle x, \mathrm{e}_1\rangle (\langle x, \mathrm{e}_1\rangle - \gamma)^{\alpha-\beta}\, \mathrm{d}\mu_\beta(x)}{\int_{L_g} (\langle x, \mathrm{e}_1\rangle - \gamma)^{\alpha-\beta}\, \mathrm{d}\mu_\beta(x)} = \gamma + (\delta-\gamma)\frac{\alpha-\beta+1}{\alpha+2},$$

and therefore, this together with (5.12) and (5.16) yields

$$\mathrm{g}_\alpha(h) \le \gamma + (\delta-\gamma)\frac{\alpha-\beta+1}{\alpha+2} =: g_0.$$

Finally, the latter relation jointly with (ii) and (iii) in (5.9) gives us

$$\begin{aligned}\frac{\int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta\, \mathrm{d}t}{\int_a^b h(t)^\beta\, \mathrm{d}t} = \frac{\int_{\mathrm{g}_\alpha(h)}^\delta g(t)^\beta\, \mathrm{d}t}{\int_\gamma^\delta g(t)^\beta\, \mathrm{d}t} \ge \frac{\int_{g_0}^\delta g(t)^\beta\, \mathrm{d}t}{\int_\gamma^\delta g(t)^\beta\, \mathrm{d}t} &= \left(\frac{\delta-g_0}{\delta-\gamma}\right)^{\beta+1}\\ = \left(1-\frac{\alpha-\beta+1}{\alpha+2}\right)^{\beta+1} &= \left(\frac{\beta+1}{\alpha+2}\right)^{\beta+1},\end{aligned}$$

as desired. This finishes the proof of (5.5).

5.2.3 The case $\alpha < \beta$

Now we show the second part of Theorem 5.2.2, namely, (5.6) provided that $\alpha < \beta$. We point out that here we use an approach similar to the one followed in Subsection 5.2.2, but with the main difference that we need to truncate the sets L and L_g due to certain integrability issues, since now the exponent of some functions under the integral sign (vanishing at some points of the domains of integration) is $\alpha - \beta < 0$.

We start by considering the function $W_1 : [0, \|h\|_\infty] \longrightarrow [0, \mu_\beta(L)]$ given by

$$W_1(s) = \mu_\beta\bigl(\{x \in L \,:\, \langle x, \mathrm{e}_2\rangle \leq s\}\bigr),$$

which is clearly both strictly increasing and surjective. We may then define the non-negative function $w_1 : [0, \|h\|_\infty] \longrightarrow [\gamma, \delta]$ that satisfies

$$W_1(s) = \mu_\beta\bigl(\{x \in L_g \,:\, \langle x, \mathrm{e}_1\rangle \geq w_1(s)\}\bigr)$$

for any $s \in [0, \|h\|_\infty]$. Indeed, since

$$W_1(s) = \frac{c^\beta\bigl(\delta - w_1(s)\bigr)^{\beta+1}}{\beta(\beta+1)},$$

we get that

$$w_1(s) = \delta - \left(\frac{\beta(\beta+1)}{c^\beta} W_1(s)\right)^{1/(\beta+1)}.$$

Note that, from the Borell-Brascamp-Lieb inequality (1.9), we have that the function $W_1^{1/(\beta+1)}$ is concave (since the density of μ_β, with respect to the Lebesgue measure, is $\bigl(1/(\beta-1)\bigr)$-concave). Therefore, w_1 is strictly decreasing, surjective and convex, and then there exists the function $\varphi_1 = w_1^{-1} : [\gamma, \delta] \longrightarrow [0, \|h\|_\infty]$, which is further (strictly decreasing and) concave.

Now, for any $0 < \varepsilon \leq \|h\|_\infty$ we define the sets

$$L_\varepsilon := \{x \in L \,:\, \langle x, \mathrm{e}_2\rangle \geq \varepsilon\}$$

and

$$L_{g,\varepsilon} := \{x \in L_g \,:\, \langle x, \mathrm{e}_1\rangle \leq w_1(\varepsilon)\}.$$

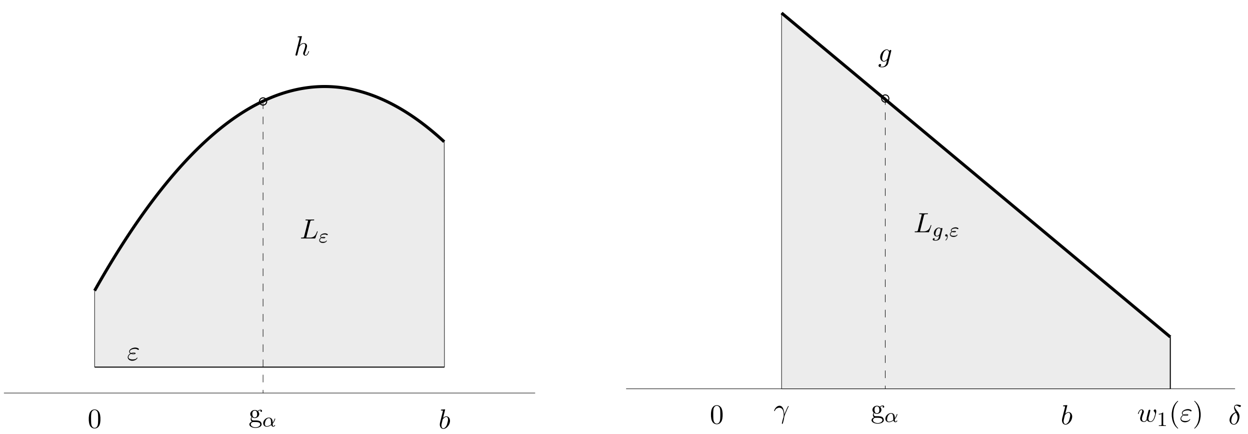

Figure 5.5: Sets L_ε and $L_{g,\varepsilon}$.

Notice that, from the definition of W_1 and w_1 (jointly with (i) in (5.10)) we have that

$$\begin{aligned}&\text{(i) } \mu_\beta(L_\varepsilon)=\mu_\beta(L_{g,\varepsilon}), \quad \text{and}\\ &\text{(ii) } \mu_\beta\big(\{x\in L_\varepsilon : \langle x,\mathrm{e}_2\rangle\le s\}\big)=\mu_\beta\big(\{x\in L_{g,\varepsilon} : \langle x,\mathrm{e}_1\rangle\ge w_1(s)\}\big),\\ &\text{for all } \varepsilon\le s\le\|h\|_\infty.\end{aligned}\tag{5.19}$$

We will first prove that, for any $0<\varepsilon\le\|h\|_\infty$, we have

$$\begin{aligned}\frac{\int_{L_\varepsilon}\langle x,\mathrm{e}_1\rangle\langle x,\mathrm{e}_2\rangle^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x)}{\int_{L_\varepsilon}\langle x,\mathrm{e}_2\rangle^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x)}&<\frac{\int_{L_{g,\varepsilon}}\langle x,\mathrm{e}_1\rangle\varphi_1(\langle x,\mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x)}{\int_{L_{g,\varepsilon}}\varphi_1(\langle x,\mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x)}\\ &\quad+\frac{b\delta\varepsilon^\alpha}{\int_{L_\varepsilon}\langle x,\mathrm{e}_2\rangle^\alpha\,\mathrm{d}x}.\end{aligned}\tag{5.20}$$

To this aim, we will observe that, denoting by $b_\varepsilon=\max\big\{x\in\mathbb{R}:(x,y)\in L_\varepsilon\big\}$, for any $0\le s\le b_\varepsilon$ we have that

$$\mu_\beta\big(\{x\in L_\varepsilon : \langle x,\mathrm{e}_1\rangle\ge s\}\big)\le\mu_\beta\big(\{x\in L_{g,\varepsilon} : \langle x,\mathrm{e}_1\rangle\ge s\}\big)+W_1(\varepsilon).\tag{5.21}$$

Indeed, taking into account that

$$\mu_\beta\big(\{x\in L_g : \langle x,\mathrm{e}_1\rangle\ge s\}\big)=\mu_\beta\big(\{x\in L_{g,\varepsilon} : \langle x,\mathrm{e}_1\rangle\ge s\}\big)+W_1(\varepsilon)$$

if $s\le w_1(\varepsilon)$ and that

$$\mu_\beta\big(\{x\in L_g : \langle x,\mathrm{e}_1\rangle\ge s\}\big)\le W_1(\varepsilon)=\mu_\beta\big(\{x\in L_{g,\varepsilon} : \langle x,\mathrm{e}_1\rangle\ge s\}\big)+W_1(\varepsilon)$$

if $s>w_1(\varepsilon)$, we get, from (iv) in (5.10), that

$$\begin{aligned}\mu_\beta\big(\{x\in L_\varepsilon : \langle x,\mathrm{e}_1\rangle\ge s\}\big)&\le\mu_\beta\big(\{x\in L : \langle x,\mathrm{e}_1\rangle\ge s\}\big)\\ &\le\mu_\beta\big(\{x\in L_g : \langle x,\mathrm{e}_1\rangle\ge s\}\big)\\ &\le\mu_\beta\big(\{x\in L_{g,\varepsilon} : \langle x,\mathrm{e}_1\rangle\ge s\}\big)+W_1(\varepsilon)\end{aligned}$$

for all $0\le s\le b_\varepsilon$, which shows (5.21). This, together with

$$W_1(\varepsilon)\le\int_0^b\int_0^\varepsilon y^{\beta-1}\,\mathrm{d}y\,\mathrm{d}x=\frac{b}{\beta}\varepsilon^\beta$$

and (ii) in (5.19), implies that

$$\begin{aligned}&\min\big\{\mu_\beta\big(L_\varepsilon^+(\mathrm{e}_1,s_1)\big),\mu_\beta\big(L_\varepsilon^-(\mathrm{e}_2,1/s_2)\big)\big\}\\ &\quad\le\min\Big\{\mu_\beta\big(L_{g,\varepsilon}^+(\mathrm{e}_1,s_1)\big)+W_1(\varepsilon),\mu_\beta\Big(L_{g,\varepsilon}^+\big(\mathrm{e}_1,w_1(1/s_2)\big)\Big)\Big\}\\ &\quad\le W_1(\varepsilon)+\min\Big\{\mu_\beta\big(L_{g,\varepsilon}^+(\mathrm{e}_1,s_1)\big),\mu_\beta\Big(L_{g,\varepsilon}^+\big(\mathrm{e}_1,w_1(1/s_2)\big)\Big)\Big\}\\ &\quad\le\frac{b}{\beta}\varepsilon^\beta+\min\Big\{\mu_\beta\big(L_{g,\varepsilon}^+(\mathrm{e}_1,s_1)\big),\mu_\beta\Big(L_{g,\varepsilon}^+\big(\mathrm{e}_1,w_1(1/s_2)\big)\Big)\Big\}\end{aligned}$$

for all $0 \leq s_1 \leq b_\varepsilon$ and all $0 < 1/s_2 \leq \|h\|_\infty$.

So, defining $w_1(s) := \gamma$ if $s \geq \|h\|_\infty$, and taking into account that $b_\varepsilon \leq b \leq \delta$ (by (iii) in (5.10)), from the fact that $\alpha < \beta$ and using Fubini's theorem we have on the one hand that

$$\begin{aligned}
&\frac{1}{\beta-\alpha}\int_{L_\varepsilon} \langle x, \mathrm{e}_1\rangle \langle x, \mathrm{e}_2\rangle^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x) \\
&\quad= \int_{L_\varepsilon}\int_0^{\langle x,\mathrm{e}_1\rangle} \mathrm{d}s_1 \int_0^{1/\langle x,\mathrm{e}_2\rangle} s_2^{\beta-\alpha-1}\,\mathrm{d}s_2\,\mathrm{d}\mu_\beta(x) \\
&\quad= \int_0^{b_\varepsilon}\int_0^{1/\varepsilon} s_2^{\beta-\alpha-1}\mu_\beta\big(L_\varepsilon^+(\mathrm{e}_1,s_1)\cap L_\varepsilon^-(\mathrm{e}_2,1/s_2)\big)\,\mathrm{d}s_2\,\mathrm{d}s_1 \\
&\quad< \int_0^{b_\varepsilon}\int_0^{1/\varepsilon} s_2^{\beta-\alpha-1}\min\big\{\mu_\beta\big(L_\varepsilon^+(\mathrm{e}_1,s_1)\big),\mu_\beta\big(L_\varepsilon^-(\mathrm{e}_2,1/s_2)\big)\big\}\,\mathrm{d}s_2\,\mathrm{d}s_1 \\
&\quad\leq \int_0^{\delta}\int_0^{1/\varepsilon} s_2^{\beta-\alpha-1}\frac{b}{\beta}\varepsilon^\beta\,\mathrm{d}s_2\,\mathrm{d}s_1 \\
&\quad+ \int_0^{\delta}\int_0^{1/\varepsilon} s_2^{\beta-\alpha-1}\min\Big\{\mu_\beta\big(L_{g,\varepsilon}^+(\mathrm{e}_1,s_1)\big),\mu_\beta\Big(L_{g,\varepsilon}^+\big(\mathrm{e}_1,w_1(1/s_2)\big)\Big)\Big\}\,\mathrm{d}s_2\,\mathrm{d}s_1 \\
&\quad= \frac{b\delta}{\beta-\alpha}\varepsilon^\alpha + \int_0^{\delta}\int_0^{1/\varepsilon} s_2^{\beta-\alpha-1}\mu_\beta\Big(L_{g,\varepsilon}^+(\mathrm{e}_1,s_1)\cap L_{g,\varepsilon}^+\big(\mathrm{e}_1,w_1(1/s_2)\big)\Big)\,\mathrm{d}s_2\,\mathrm{d}s_1 \\
&\quad= \frac{b\delta}{\beta-\alpha}\varepsilon^\alpha + \frac{1}{\beta-\alpha}\int_{L_{g,\varepsilon}} \langle x,\mathrm{e}_1\rangle\varphi_1(\langle x,\mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x),
\end{aligned} \tag{5.22}$$

where in the last equality above we have also used that $\langle x, \mathrm{e}_1\rangle \geq w_1(1/s_2)$ if and only if $1/\varphi_1(\langle x, \mathrm{e}_1\rangle) \geq s_2$ (since φ_1 is decreasing).

On the other hand, since $\alpha < \beta$, from Fubini's theorem (jointly with (ii) in (5.19)) we have that

$$\begin{aligned}
\frac{1}{\beta-\alpha}\int_{L_\varepsilon}\langle x,\mathrm{e}_2\rangle^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x) &= \int_{L_\varepsilon}\int_0^{1/\langle x,\mathrm{e}_2\rangle} s^{\beta-\alpha-1}\,\mathrm{d}s\,\mathrm{d}\mu_\beta(x) \\
&= \int_0^{1/\varepsilon} s^{\beta-\alpha-1}\mu_\beta\big(L_\varepsilon^-(\mathrm{e}_2,1/s)\big)\,\mathrm{d}s \\
&= \int_0^{1/\varepsilon} s^{\beta-\alpha-1}\mu_\beta\Big(L_{g,\varepsilon}^+\big(\mathrm{e}_1,w_1(1/s)\big)\Big)\,\mathrm{d}s \\
&= \frac{1}{\beta-\alpha}\int_{L_{g,\varepsilon}}\varphi_1(\langle x,\mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x),
\end{aligned} \tag{5.23}$$

where in the last equality above we have used again that $\langle x, \mathrm{e}_1\rangle \geq w_1(1/s)$ if and only if $1/\varphi_1(\langle x, \mathrm{e}_1\rangle) \geq s$. Hence, from (5.22) and (5.23), we obtain (5.20), as desired.

Now we will prove that for any concave function $\varphi_1 : [\gamma, \delta] \longrightarrow [0,\infty)$ we have that

$$\frac{\int_{L_{g,\varepsilon}}\langle x,\mathrm{e}_1\rangle\varphi_1(\langle x,\mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x)}{\int_{L_{g,\varepsilon}}\varphi_1(\langle x,\mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x)} \leq \frac{\int_{L_{g,\varepsilon}}\langle x,\mathrm{e}_1\rangle\,(\delta-\langle x,\mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x)}{\int_{L_{g,\varepsilon}}(\delta-\langle x,\mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x)}. \tag{5.24}$$

To this aim, let $C_1 > 0$ be such that

$$\int_{L_{g,\varepsilon}} \big(C_1(\varepsilon)(\delta - \langle x, \mathrm{e}_1\rangle)\big)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x) = \int_{L_{g,\varepsilon}} \varphi_1(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x). \tag{5.25}$$

Since the latter identity is equivalent (by Fubini's theorem) to

$$\int_\gamma^{w_1(\varepsilon)} \Big(\big(C_1(\varepsilon)(\delta - t)\big)^{\alpha-\beta} - \varphi_1(t)^{\alpha-\beta}\Big)\, g^\beta(t)\,\mathrm{d}t = 0,$$

we may assert, taking into account that φ_1 is concave, that there exists $t_0(\varepsilon) \in (\gamma, \delta)$ such that

$$\begin{aligned} &\text{(i) } C_1(\varepsilon)(\delta - t) \geq \varphi_1(t) \text{ for every } \gamma \leq t \leq t_0(\varepsilon), \quad \text{and} \\ &\text{(ii) } C_1(\varepsilon)(\delta - t) \leq \varphi_1(t) \text{ for every } t_0(\varepsilon) \leq t \leq w_1(\varepsilon). \end{aligned} \tag{5.26}$$

Then, from (5.25) and (5.26) (taking into account that $\alpha < \beta$), and using Fubini's theorem, we obtain

$$\begin{aligned} &\int_{L_{g,\varepsilon}} \langle x, \mathrm{e}_1\rangle \varphi_1(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x) - \int_{L_{g,\varepsilon}} \langle x, \mathrm{e}_1\rangle \big(C_1(\varepsilon)(\delta - \langle x, \mathrm{e}_1\rangle)\big)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x) \\ &\quad = \frac{1}{\beta}\int_\gamma^{w_1(\varepsilon)} t\Big(\varphi_1(t)^{\alpha-\beta} - \big(C_1(\varepsilon)(\delta - t)\big)^{\alpha-\beta}\Big) g^\beta(t)\,\mathrm{d}t \\ &\quad = \frac{1}{\beta}\int_\gamma^{t_0(\varepsilon)} t\Big(\varphi_1(t)^{\alpha-\beta} - \big(C_1(\varepsilon)(\delta - t)\big)^{\alpha-\beta}\Big) g^\beta(t)\,\mathrm{d}t \\ &\quad + \frac{1}{\beta}\int_{t_0}^{w_1(\varepsilon)} t\Big(\varphi_1(t)^{\alpha-\beta} - \big(C_1(\varepsilon)(\delta - t)\big)^{\alpha-\beta}\Big) g^\beta(t)\,\mathrm{d}t \\ &\quad \leq \frac{t_0(\varepsilon)}{\beta}\int_\gamma^{t_0(\varepsilon)} \Big(\varphi_1(t)^{\alpha-\beta} - \big(C_1(\varepsilon)(\delta - t)\big)^{\alpha-\beta}\Big)\, g^\beta(t)\,\mathrm{d}t \\ &\quad + \frac{t_0(\varepsilon)}{\beta}\int_{t_0(\varepsilon)}^{w_1(\varepsilon)} \Big(\varphi_1(t)^{\alpha-\beta} - \big(C_1(\varepsilon)(\delta - t)\big)^{\alpha-\beta}\Big)\, g^\beta(t)\,\mathrm{d}t \\ &\quad = \frac{t_0(\varepsilon)}{\beta}\int_\gamma^{w_1(\varepsilon)} \Big(\varphi_1(t)^{\alpha-\beta} - \big(C_1(\varepsilon)(\delta - t)\big)^{\alpha-\beta}\Big)\, g^\beta(t)\,\mathrm{d}t \\ &\quad = t_0(\varepsilon)\int_{L_{g,\varepsilon}} \Big(\varphi_1(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta} - \big(C_1(\varepsilon)(\delta - \langle x, \mathrm{e}_1\rangle)\big)^{\alpha-\beta}\Big)\,\mathrm{d}\mu_\beta(x) = 0. \end{aligned}$$

Thus, we have

$$\int_{L_{g,\varepsilon}} \langle x, \mathrm{e}_1\rangle \varphi_1(\langle x, \mathrm{e}_1\rangle)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x) \leq \int_{L_{g,\varepsilon}} \langle x, \mathrm{e}_1\rangle \big(C_1(\varepsilon)(\delta - \langle x, \mathrm{e}_1\rangle)\big)^{\alpha-\beta}\,\mathrm{d}\mu_\beta(x),$$

which, together with (5.25), yields (5.24).

Hence, from (5.20) and (5.24), for every $0 < \varepsilon \leq \|h\|_\infty$ we get

$$\frac{\int_{L_\varepsilon} \langle x, \mathrm{e}_1 \rangle \langle x, \mathrm{e}_2 \rangle^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)}{\int_{L_\varepsilon} \langle x, \mathrm{e}_2 \rangle^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)} < \frac{\int_{L_{g,\varepsilon}} \langle x, \mathrm{e}_1 \rangle \left(\delta - \langle x, \mathrm{e}_1 \rangle\right)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)}{\int_{L_{g,\varepsilon}} \left(\delta - \langle x, \mathrm{e}_1 \rangle\right)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)} + \frac{b\delta\varepsilon^\alpha}{\int_{L_\varepsilon} \langle x, \mathrm{e}_2 \rangle^\alpha \,\mathrm{d}x}.$$

Now, taking limits as $\varepsilon \to 0^+$ in the above inequality, we have that the left-hand side, namely,

$$\frac{\int_{L_\varepsilon} \langle x, \mathrm{e}_1 \rangle \langle x, \mathrm{e}_2 \rangle^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)}{\int_{L_\varepsilon} \langle x, \mathrm{e}_2 \rangle^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)} = \frac{\int_{L_\varepsilon} \langle x, \mathrm{e}_1 \rangle \langle x, \mathrm{e}_2 \rangle^{\alpha-1} \,\mathrm{d}x}{\int_{L_\varepsilon} \langle x, \mathrm{e}_2 \rangle^{\alpha-1} \,\mathrm{d}x},$$

tends to

$$\frac{\int_L \langle x, \mathrm{e}_1 \rangle \langle x, \mathrm{e}_2 \rangle^{\alpha-1} \,\mathrm{d}x}{\int_L \langle x, \mathrm{e}_2 \rangle^{\alpha-1} \,\mathrm{d}x} = \mathrm{g}_\alpha(h)$$

as $\varepsilon \to 0^+$ (see (5.8)). Furthermore, the first term in the right-hand side,

$$\frac{\int_{L_{g,\varepsilon}} \langle x, \mathrm{e}_1 \rangle \left(\delta - \langle x, \mathrm{e}_1 \rangle\right)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)}{\int_{L_{g,\varepsilon}} \left(\delta - \langle x, \mathrm{e}_1 \rangle\right)^{\alpha-\beta} \,\mathrm{d}\mu_\beta(x)} = \frac{\frac{c^\beta}{\beta} \int_\gamma^{w_1(\varepsilon)} t(\delta - t)^\alpha \,\mathrm{d}t}{\frac{c^\beta}{\beta} \int_\gamma^{w_1(\varepsilon)} (\delta - t)^\alpha \,\mathrm{d}t},$$

tends to

$$\mathrm{g}_\alpha(g) = \frac{\int_\gamma^\delta t(\delta - t)^\alpha \,\mathrm{d}t}{\int_\gamma^\delta (\delta - t)^\alpha dt} = \left(\delta - \frac{(\alpha+1)(\delta - \gamma)}{\alpha + 2}\right)$$

for $\varepsilon \to 0^+$, whereas the second term in the right-hand side,

$$\frac{b\delta\varepsilon^\alpha}{\int_{L_\varepsilon} \langle x, \mathrm{e}_2 \rangle^\alpha \,\mathrm{d}x},$$

clearly tends to

$$\frac{0}{\int_L \langle x, \mathrm{e}_2 \rangle^\alpha \,\mathrm{d}x} = 0$$

for $\varepsilon \to 0^+$. Therefore, we have that

$$\mathrm{g}_\alpha(h) \leq \mathrm{g}_\alpha(g).$$

Finally, the latter relation jointly with (ii) and (iii) in (5.9) gives us

$$\frac{\int_{\mathrm{g}_\alpha(h)}^b h(t)^\beta \,\mathrm{d}t}{\int_0^b h(t)^\beta \,\mathrm{d}t} = \frac{\int_{\mathrm{g}_\alpha(h)}^\delta g(t)^\beta \,\mathrm{d}t}{\int_\gamma^\delta g(t)^\beta \,\mathrm{d}t} \geq \frac{\int_{\mathrm{g}_\alpha(g)}^\delta g(t)^\beta \,\mathrm{d}t}{\int_\gamma^\delta g(t)^\beta \,\mathrm{d}t} = \left(\frac{\delta - \mathrm{g}_\alpha(g)}{\delta - \gamma}\right)^{\beta+1} = \left(\frac{\alpha+1}{\alpha+2}\right)^{\beta+1} = \left(\frac{\alpha+1}{\alpha+2}\right)^{\beta+1},$$

as desired. This finishes the proof of (5.6), and hence also that of Theorem 5.2.2.

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