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Planning for the unexpected: working within symbolically structured environments

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What is your image of an ideal mathematics classroom culture? What kinds of conversation would be taking place? What kinds of tasks would be on offer? What would be the role of the teacher? Take a moment to write down or collect your thoughts.

We begin this chapter with two classroom examples, including some of the thinking behind their planning, that we offer to provoke thinking about what can be planned and what must remain unexpected, if a teacher aims to "create the conditions in which creative and independent work can take place" (Banwell, Saunders and Tahta, 1986, p.18). These examples were chosen, one from primary and one from secondary school, to allow the raising of issues related to planning for the unexpected. The phrase 'symbolically structured environment', from the title, is one we will elaborate on after the two examples. It links to common features we have recently become aware of, in the teaching of some key figures of mathematics education from the 20th century (for example, Gattegno, 1974; Papert, 1980).

We invite you to read the following two accounts and then consider similarities and differences. We also invite you to engage in the tasks themselves.

Account 1: Pick's Theorem (Secondary)

Pick's Theorem is used to determine the area of a polygon on a square dot grid. It connects three features of polygons. Students can work with multiple relationships, for example, by fixing one feature and varying the other two. The Theorem states that, if A is the area of the shape and I is the number of dots on the inside, and O is the number of dots on the perimeter, then $A = \frac{1}{2} O + I - 1$. An entry into work on the Theorem is described in the seminal book "Starting Points" by Banwell, Saunders and Tahta (1986). The classroom account below is reconstructed from work that took place in a year 8 class (age 12-13) in an English school where Alf used to work. In the school's departmental write-up of the task, the reasons for doing it were as follows:

Possible mathematical content:

- Distinguishing area from perimeter, finding the areas of shapes without counting squares, finding the areas of triangles as half of a rectangle, finding the areas of compound complex shapes.
- Opportunities for mathematical thinking.
- Making predictions about the areas of '8 dot shapes', '9 dot shapes' and so on, making conjectures, testing conjectures, finding counter-example, expressing conjectures using algebra, finding relationships between 3 variables. (Department Scheme of Work).

Work on this task would typically last over a 3 or 4-week period (that is, around 10 hours class time, along with related homework tasks each week). An overall aim for the year in this school was explicitly stated to students as being about supporting them in ‘becoming a mathematician’ and ‘thinking mathematically’. The language of conjecture, counter-example and proof was introduced by teachers and used in an explicit way, with the aim that students would use these notions to structure their own work on mathematics.

The teacher (and we are drawing here primarily on Alf’s experience of working on this task in school) draws on the board (see Figure 1a) two shapes and says: “These are both 8-dot shapes. Someone come and draw me another, different, 8-dot shape”.

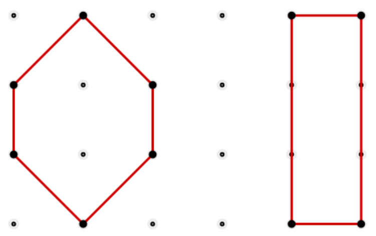


Figure 1a: Two 8-dot shapes

Students come to the board and, without comment about why, the teacher indicates if the shape is 8-dot or not.

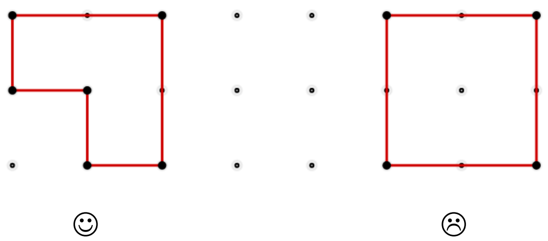
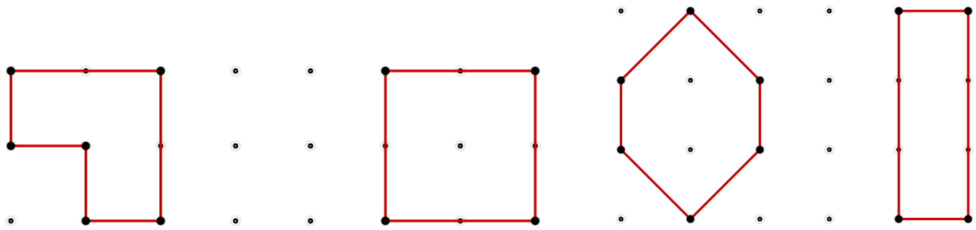


Figure 1b: Student offers of 8-dot shapes

The distinction the teacher needs students to make, in setting up this task, is that shapes are labelled by adding up the number of dots inside and on the outside (perimeter). Rather than try to explain this (which would inevitably lead to confusion), the teacher invites students to do something (publicly draw some shapes), not knowing what they will draw, but knowing he will give them feedback. He continues inviting new shapes to be drawn until students can explain what makes a shape an “8-dot shape”. The right-hand shape in Figure 1b is classified as “9-dot”.

The teacher now sets up a structure for the task that, whenever students draw a shape, they need to write next to it, I (for the number of dots ‘inside’), O (for the number of dots on the ‘outside’) and A (for the area of the shape). Students have met the concept of ‘area’ before but need reminding about it. The class work together, finding the three values (I, O, A) for all the shapes on the board.



$O = 8$
$I = 0$
$A = 3$

$O = 8$
$I = 1$
$A = 4$

$O = 6$
$I = 2$
$A = 4$

$O = 8$
$I = 0$
$A = 3$

Figure 1c: Information needed for each shape drawn

Having collected all the information the teacher says: "Okay, look at all the shapes we have drawn here, what do you notice? What is the same? What is different? Can anyone make a prediction? Or ask a question?". (Names below are pseudonyms, the dialogue is reconstructed from real events. Punctuation is used to give a sense of phrasing.)

Jordan: "They all have straight lines"
Teacher: "Nice, and that is one of the rules for this task – all the shapes you draw must have straight lines"
Alice: "The area is 3 or 4"
Teacher: "Right, for all the shapes we have here, the area is 3 or 4. So, could someone turn that into a question or a challenge?"
Mike: "Are the areas always 3 or 4?"
Teacher: "Lovely, so let's have that as a challenge – can you find an 8-dot shape where the area is not 3 or 4" (writing this on the board)
Abi: "If the inside is zero, the area is 3"
Teacher: "So, we have our first conjecture on this project. Abi, can you say that again and I am going to write it down, and remember this is just for 8-dot shapes that we are looking at"
Abi repeats and the teacher writes on the board: 'Abi's conjecture: for 8-dot shapes if $I = 0$, then $A = 3$'
Teacher: "So, how could we test Abi's conjecture?"

The students in the class, by this point in the year, are familiar with what a 'conjecture' means and that they can 'work' on conjectures by trying out if the prediction is correct, for different examples (and examples that do not work are called 'counter-examples', which prompt the need to re-work the conjecture). The teacher cannot know what the students will notice, but is confident that there will be some things noticed that allow tasks to be set up for the class, related to the problem. It does not really matter what particular things are said, so long as they can lead to questions, challenges or conjectures. The teacher is alert to how things said could become things for others to work on (for example, with Alice and Abi's comments above). The classroom culture is one where students are invited to follow their own ideas (within the parameters of a task) but where there is also always a suggestion to fall back on (in this case, to work on testing Abi's conjecture).

There seems to be something powerful in using students' names to label conjectures. Perhaps there is a message here that mathematics is, ultimately, a human invention and that students can be creators within the subject. At this stage in the lesson, the teacher sets up a period of independent or paired work for the students. Throughout the work on the task there are times when the class reconvenes to discuss what they have noticed. During this first phase of independent work, the teacher sets up a way of collecting results (see Table 1) and students come up to the board to add to it, whenever they find a new shape.

8-dot shapes

O	I	A
8	0	3
6	2	4
8	0	3
7	1	3.5

Table 1: A table of results

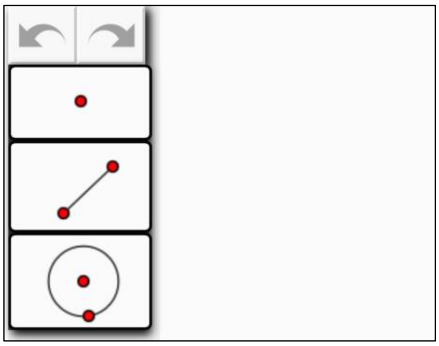
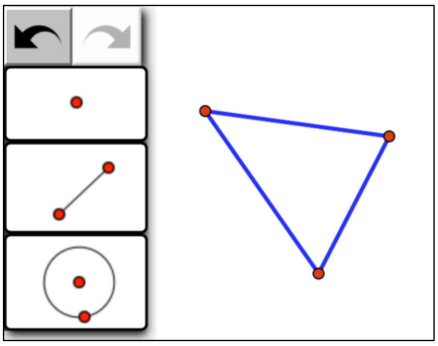
At some point, this table will be organised to support the further noticing of patterns and making of conjectures. Again, the teacher cannot know what shapes students will try out, but can plan to collect results in this way. The structure of the table invites questions such as, what is the greatest number of dots you can create 'inside', with an 8-dot shape?

Once the class have worked on 8-dot shapes for a while, the teacher invites students to try out a different number of dots of their own choosing, following the same pattern of collecting results in a table. All these tables are made visible to all students (what were called 'common boards' in the school) and this visibility invites further noticing of patterns and relationships. Some students get to a general relationship between the three variables.

Account 2: Triangles (Primary)

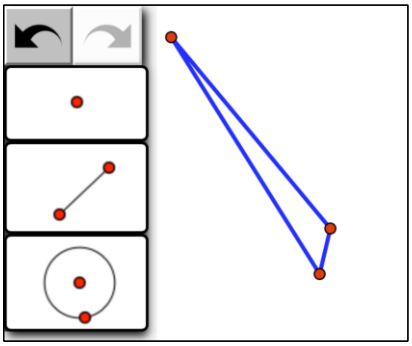
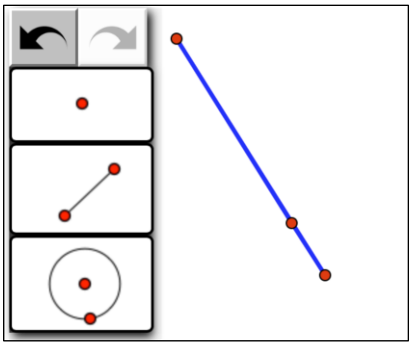
The second task we invite you to read about took place in a grade 2 (aged 7-8) classroom in Canada; the teacher (and we are drawing here primarily on Nathalie's experience of working on this task in school) wants to work on the topic of triangles. Knowing that children of this age often have a prototypical image of triangle, as an equilateral triangle sitting on one of its sides, the teacher wants to engage the children in coordinating the descriptions of triangles with various configurations of triangles. The goal, after all, is to engage in geometric reasoning, and not in the kind of shape identification and classification that one might pursue in a biology class. Since dynamic geometric software provides an easy and precise way of generating different configurations of triangles, she invites the children to tell her which of the tools she could use to create a triangle (in the left of Figure 2a, there are three tools: point, circle and segment. The children quickly identify the segment tool as useful and one child instructs the teachers to "make three sides that are connected." The teacher creates the three-sided shape shown in Figure 2b. The children protest: "that's not a triangle. It's upside down." The teacher responds: "but you told me to make three sides that were connected and that's what I did." The children are silent. The teacher adds, "You didn't say it had to be right side up." Still more silence.

The teacher announces, "I'm going to move one of the points. Watch what happens." The teacher begins to drag the vertex of the triangle on the screen, producing a range of triangles including one that looks more like 'the triangle' the children are accustomed to seeing. Many children make exclamations of surprise. They are riveted to the screen. One child comments, "You can stretch it out!" Another adds, "It's a triangle." And a third says, "Every shape can be a triangle but it just has to have three corners."

Figures: 2a Sketchpad tools; 2b A triangle

Up to this point, the teacher could predict that the children would not count the upside-down three-sided shape as a triangle. She knew that she could make it look more like ‘their’ triangle. She had also been in many other classrooms, with children ranging from kindergarten to grade 6, in which the act of dragging the triangle had eventually helped the children to coordinate their descriptions of triangles (it has three sides) with configurations of triangles that initially looked strange. Sometimes the children would quickly accept an ‘upside-down triangle’, but were more reticent with a really long and skinny triangle (Figure 2c). It could take some more time to get to the kind of definitional statement quoted above. Even though the actual examples might change, as well as the children’s comments and reasoning, the teacher’s plan was that the children could identify non-prototypical three-sided shapes as triangles.

Figures: 2c A stretched triangle; 2d Triangle or line?

Now she starts to move one of the vertices around again and this time it lands *on* the opposite side of the triangle. She stops moving it. The children giggle. One child says, “that is *not* a triangle.” Another points out that there are still three corners (see Figure 2d). To this statement, yet another child notes, “but there is no inside”. To each of these comments, the teacher nods. By deciding to collapse the triangle in this way, she is braced for the unexpected. Some children react strongly to this degenerate case; already the long, skinny triangle was a stretch, but this monster? Others want to know the right answer, is it, or is it not a triangle? New ideas come up that haven’t been mentioned before, like the inside of the triangle. Does every triangle have an inside, or do we really only count the three sides? One child proposes that it should indeed be a triangle, but we are only seeing it head-on and it’s actually three-dimensional. What is unpredictable is the valence of the discussion, that is, the emotional reaction the children have, to this shape; whether they accept it into the family of triangles or want to expel it. What is also unpredictable is the direction that the conversation will take. Unlike the upside-down triangle, this degenerate triangle is both a triangle and not a triangle. The definition in play—having three corners—does not discriminate. Indeed, in a pencil and paper environment, this triangle never even comes up for consideration. No plastic shapes have been made that look like this either. And if the children were working with strings or sticks, the configuration wouldn't come up either because the sides would have to be lifted off the plane.

In allowing it to emerge, the teacher is drawing attention to the indeterminacy of mathematics itself, and to the fact that sometimes, there is no right answer. If the discussion cannot be resolved by logical means alone, then considerations of aesthetics, in relation to choices, might need to be allowed through the classroom door. Maybe, since the point can be dragged almost anywhere else and still form a triangle, this flat object should also be counted as a triangle. That would satisfy a penchant for continuity and inclusiveness. Maybe it shouldn't count as a triangle; after all, there's nothing you can do with this triangle: you can't measure its area, you can't find its point of balance, except perhaps abstractly. This somewhat utilitarian view cares less about inclusiveness and more about shapes you can actually do something with. If it's a three-dimensional shape, then are all triangles also to be thought of as being three-dimensional? Does this open up a new path of mathematical activity, or make the current path much too complicated? To work productively with the unpredictability inherent in this task, the teacher must not only listen to the children's ideas, but also help them articulate reasons that may be both logical and aesthetic, both about what they already know about triangles and what they value in mathematics.

This kind of planning for the unexpected might expose children to aspects of mathematics they do not usually encounter. It exposes the very contingent nature of mathematics; to the fact that any definition will have counter-examples and even aspects of vagueness; to the way in which what initially gets dismissed as nonsense (negative and irrational numbers, parallel lines that meet at a point, quantities bigger than infinity) can later become common sense.

Reflections

Looking back over these two descriptions of classroom tasks, what do you notice? What is the same? What is different? What has the teacher planned? What is unexpected and what is not?

Some of the similarities we see are as follows:

- there appears to be some inevitability, and therefore predictability, in student responses (for example, the noticing of patterns in Pick's Theorem; or the engagement in considering the degenerate triangle) even though the details of responses are unknowable.
- there is some definitional work taking place in both examples; with the triangles, the definition is central to the task, for Pick's Theorem, definitions (for example, of '8-dot shape') are 'held' by the teacher, leading to questions from students.
- we see some of the engagement of students, in both tasks, as arising from the need for them to grapple with deliberately ambiguous definitions ('what is an 8-dot shape?', 'what is a triangle?'). There is perhaps a contrast here to images of mathematics teaching that might value clarity and unambiguous definition. The ambiguity of definition seems significant in creating the space for discussion.
- students' thinking is made visible to each other with support from the teacher (either through language or writing).

- the situations given to students provide them with immediate feedback related to their actions (in Pick's Theorem, in the form of the three numbers they generate for each shape; in the Triangle task, via the way the shape responds to moves of a vertex).

What else did you notice?

In terms of what was planned, we consider each account in turn. For Pick's Theorem, we suggest that what had been planned in advance of the lesson included:

1. a process all students will follow (attending to 8-dot shapes only at first; drawing shapes against a criterion (for example, having 8 dots); finding O, I, A; recording results in a table; looking for patterns).
2. symbols linked to that process (values for O, I, A).
3. a way of collecting results so that everyone can see (common boards for O, I, A, for 8-dot shapes, 9-dot shapes, and so on).
4. complex mathematical relationships (between 3 variables).
5. a starting point that is accessible to all.

In the triangle task, we observe the following plans:

1. the digital environment to use (in this case Geometer's Sketchpad) that allows actions of transformation (moving the triangle vertices).
2. an image to work with, the image acting as a symbol for a triangle.
3. a way of sharing reasoning (norms for discussion).
4. a digital environment and set of tools that are structured to allow access to the whole of Euclidean geometry.
5. a task that challenges pre-conceptions.

We have deliberately ordered these features of planning to draw out parallels. (1) is about an environment that embodies some mathematical constraints and relationships. (2) is about the way relationships are symbolised. (3) points to the different ways that students' work is made visible to each other. (4) is about the importance of the complexity of the mathematical environment, within which a smaller subset of possible relationships is considered as a starting point (5).

And, what was unexpected? We observe:

- the teacher in the 'Triangle' lesson cannot have known what arguments students would bring to the 'degenerate' case, nor where the balance of views would lie.
- the teacher for 'Pick's Theorem' cannot have known what students will notice and therefore what particular conjectures will be worked on by students.

What can allow, then, for both the planning of a lesson and the unexpectedness of student responses, that seems to be so crucial to their engagement in mathematical thinking? We see a key role for the mathematics that is offered. In the next section, we set out our thinking about how tasks can be designed to allow interplay between planning and the unexpected. We conclude this chapter by reflecting on the role of the teacher in planning for the unexpected.

The role of the mathematics: symbolically structured environments

In both of these examples, we'd like to highlight the ways in which the teacher's planning for the unexpected has something important to do with the discipline of mathematics. In a sense mathematics itself is a structure with defined symbols and constraints (and thus it is planned) that can nevertheless produce unexpected things. Mathematicians have ways of producing the unexpected such as changing axioms, shifting dimensions, finding counter-examples, building analogies. In doing so, there is a certain arbitrariness in what actually occurs. In the same way, for the Pick's theorem example, it does not matter what the conjectures are; it matters only that they emerge from the constraints of the 'game' that was offered. In the triangle example, the nature of the arbitrariness is a little different. It seems to relate to the nature of geometric definitions, which the mathematician Coxeter (1987) compared to dictionary entries: when you look up one word, you find in its definition another word that needs to be looked up, which gives a definition of still another word, and so on, *without a real starting point on which all definitions depend*.

In thinking about the environment in which the two examples occurred, and in comparing them to similar kinds of situations, we have found it useful to think of them as "symbolically structured environments" (SSEs) that are both constrained (by mathematical rules and norms) and generative. We provide a list of features of SSEs and describe how they relate to each of the examples.

(a) Symbols are offered to stand for actions or distinctions (unlike a resource such as, say, Dienes blocks, where symbols relate directly to objects).

> Triangle: the task is all about what distinctions come under the label/symbol 'triangle' and about acting on the geometric objects of points and segments.
>
> Pick's: labels of I and O are used to describe distinctions students make about shapes. The label for A is introduced by the teacher, initially linked, perhaps, to splitting a shape into squares.

(b) Symbol use is governed by mathematical rules or constraints embedded in the structuring of the environment (rules for symbol use do not need to be memorised but can be enacted and corrected, if needed, with feedback from the SSE).

> Triangle: the way points move is constrained by the mathematics inherent in Euclidean space (they cannot all of a sudden be split, or move into a hyperbolic plane). A further norm at play is the relation between the particular and the general—there is little interest in defining one particular shape and instead there is a propensity to think in terms of classes of shapes (in this case, classes of triangles).
>
> Pick's: the constraint of creating shapes on a square dot grid ensures that there are relationships to be found between the three values that are used on each shape.

(c) Symbols or actions can be immediately linked to their inverse (symbols are not taught in isolation and gain meaning from links to other symbols).

> Triangle: an action or transformation of the triangle can be undone.
>
> Pick's: as a result of putting results in tables, students quickly move from finding the area of shapes they have drawn, to the inverse challenge of looking for shapes with particular areas (to fill in missing rows of the table).

(d) Complexity can be constrained, while still engaging with a mathematically integral, whole environment (the starting point can become more or less complex, contingent on learners).

> Triangle: only three tools are considered initially and the most simple of closed shapes that can be created.

Pick’s: the teacher constrains attention to only 8-dot shapes initially, before opening up the space to other numbers of dots (but constraining students to focus on a specific number at a time).

(e) Novel symbolic moves can be made (creative symbol use does not have to arrive late in the learning process).

Triangle: moving the third point onto the line connecting the other two; but there is also the potential of working with four-sided instead of three-sided shapes, etc.

Pick’s: new symbolic relationships are noticed by students; the actual shape of the 8-dot polygons can vary enormously (common polygons, concave ones, etc).

As alluded to in the Introduction, we see these principles as characteristic of some of the work of Caleb Gattegno (1974), who developed a curriculum around the structured use of Cuisenaire rods, and Seymour Papert (1980), who developed ‘microworlds’ such as Logo, within which children could experience mathematical relationship in a creative and independent manner.

How do you find more of these kinds of spaces? We first address this question by taking each example and trying to vary it a little. For the triangle example, an important trigger involves the tension between precision and ambiguity in geometric vocabulary. How could this be used in a different space? Consider a situation in which students are asked to draw the diagonals of various polygons. If these polygons are regular, they might propose a certain relation between the number of sides of the polygon and the number of diagonals (Figure 3a). But the teacher might then propose the concave polygon in Figure 3b. How many diagonals does it have?

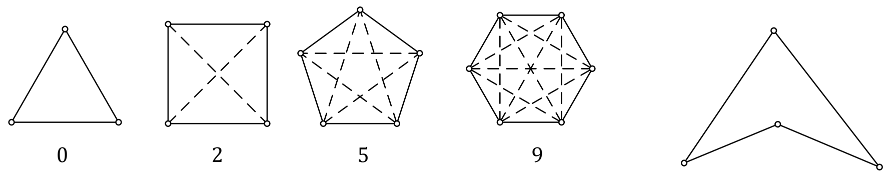


Figure 3. (a) Diagonals of regular polygons. (b) A concave quadrilateral.

The direction that the investigation will take depends on how students want to define a diagonal. The teacher might be able to predict that the students will assume all diagonals have to be “inside” the polygon and therefore be challenged by the concave polygon. Perhaps they make a conjecture that extends the diagonals of regular polygons. The novel symbolic moves thus relate to the way diagonals are drawn and defined. The complexity arises from the opening up of the terrain to non-regular and concave polygons.

For the Pick’s example, an important trigger relates to the establishing of a certain entry into the terrain that then has multiple potential directions. The environment was initially constrained to 8-dot polygons, rather than to all polygons. The first few shapes gave a sense of the potential variation, while also hinting at an aspect of invariance or pattern. We might thus be able to imagine a similar kind of unexpectedness in planning a task where students are asked to type the following fractions into the “colour calculator”[1]: ¼, 5/6 and 8/7. The

[1] The colour calculator is available on-line at http://wayback.cecm.sfu.ca/cgi-bin/ColorCalc/n/maths.cgi/11m/?Eqn=&RawEqn=&nDigits=100&ColourPlotWidth=10&Base=10&do=activity&

display is a representation of the decimal expansion of any fraction, with the digits 0 to 9 having a unique colour (see key, in Figure 4).

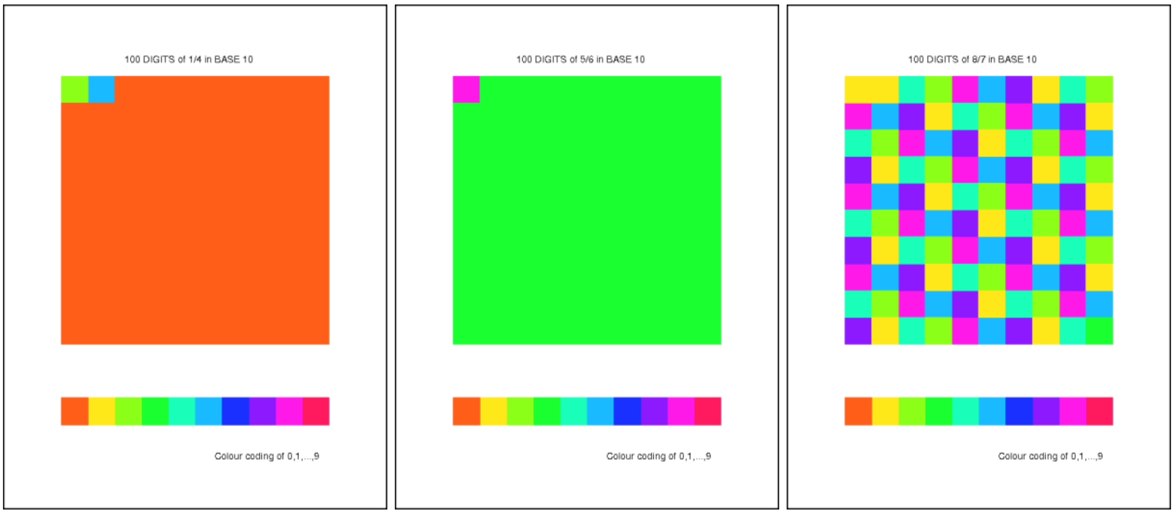


Figure 4: The colour calculator

The instructions give a sense of the potential variance (the numerator and the denominator can change, the numerator can be bigger than the denominator) and the output provides a hint at what possible patterns might look like. Asking the students to write down the colour or set of colours that repeats in each case and asking them to try a fourth fraction might lead to different conjecture. If a student tries 5/7, the following conjecture might be made: if I use 7 in the denominator, I always get a repeating patter of 6 colours. If the student tries instead ¾, the following conjecture might be made: with 4 in the denominator there are always three different colours. Upon more exploration, more conjectures arise and more similarities and differences are noticed. The concept of fraction is subordinated to the exploration of pattern, to the generation of conjectures, just as the concept of area was subordinated in the Pick's example. While the students' prior mathematical experience might help the teacher expect certain kinds of noticings, these are not determined in advance.

The ideas above are not so much 'tasks', as environments ('microworlds' in the language of Seymour Papert, 'spaces' in the language of Laurinda Brown) which are structured by the mathematics they embody. The start of work is not necessarily a 'problem' or 'challenge'. Rather, there is an entry into an environment that provides learners with things to do and a mechanism through which to gain immediate (symbolic) feedback from that doing. One advantage we observe from students working within such structured environments is that it is possible for them to develop, quickly, sophisticated symbol use. We conjecture the reason for this symbolic facility is that symbols are introduced and used to represent distinctions, actions or transformations made by students and, furthermore, that some definitional ambiguity may be an important in terms of provoking engagement with these symbols. Symbols are introduced and used in complexes or groups, allowing the development of awareness about how the symbols relate to each other.

However, we also recognise that there is a sophisticated and delicate role for the teacher, in terms of devolving the task sufficiently to the students to occasion their enthusiasm and

activity=calc. A similar interface is available here: http://www.sineofthetimes.org/experiments-with-a-color-calculator/

commitment while at the same time constraining work sufficiently to ensure a focus on mathematical relationships. In the final section of this chapter, we return to our two examples to consider in more detail the role of the teacher, working with a symbolically structured environment.

The role of the teacher

In the two accounts, the teacher focus is not, we would argue, on the details of the students' mathematics. What we mean by this is that while the students are becoming energised by patterns they notice related to Pick's Theorem, the teacher's attention is on whether their offers can be turned into a question or conjecture that others could use to guide their work; or, while students become energised by their own sense of what it means to be a triangle, the teacher's attention is on the form of talk taking place and whether students are reasoning and hearing the reasoning of others.

A requirement of such a focus is that the teacher is able to listen to the detail and subtlety of student contributions. This is no easy task (see Coles and Scott, 2015 for a story of learning and change in relation to listening and the unexpected). In our examples, the teacher is not listening out for particular responses (in the sense of a 'right' answer), but perhaps for particular *kinds* of example (the right 'kind' of answer). Familiarity with the environment seems key, in being able to focus attention on the *kinds* of things students say, without needing to worry overly about what might happen next. There is a paradox here. It is hard to work contingently within a symbolically structured environment that is new to you. And yet, the only way to become more comfortable teaching within such an environment is to use it. The teachers in our examples were both experienced at working within those environments, meaning they could take on the role of orchestrating rather than directing events (for example, provoking students with the degenerate triangle or knowing when to move students on to looking at other dot shapes and when not).

Having skills to manage classroom talk appears to be a need, for the teacher, across both our examples. Starting to teach, it can seem like an impossibility to generate productive classroom talk. We can split the issue of managing talk into two elements: how to generate some response from students; and, how to respond to those responses.

To take the second one first, we are perhaps straying into the realms of the unsayable. A salutary but powerful experience can be to audio record your interactions with students and listen to them at your leisure. *Did you hear what was said? What interpretations might be possible of what a student meant, other than the one you assumed?*

In terms of how to generate student responses, we first invite you to think back over your teaching. *What are some questions or tasks that generated a buzz of response or conversation? Are there any common features to what you did, or what students were considering?*

Having contrasting examples for students to compare is one suggestion for generating energetic responses (see Brown and Coles, 2000). (In a similar way, we have tried to provide contrasting examples in this chapter, hoping to generate some energetic responses from readers!) From contrasting examples, it is possible to ask 'what is the same and what is different?' (ibid) which is a question that can be answered at any level from the most straightforward to the most sophisticated. We hypothesise that a SSE supports students and teachers in answering this question within a mathematical discourse, using the symbols and actions of the constrained space and allowing a creative interplay of the unexpected and the planned.

Further reading

Banwell, C., Saunders, K., & Tahta, D. (1986). *Starting Points: for teaching mathematics in middle and secondary schools*. St Albans, UK: Tarquin Publishers. This book contains a collection of rich mathematical tasks that have been used extensively around the world. There are also suggestions around ways of working. The book is out of print but is a classic text that stands the test of time and is as relevant now as when it was published.

Brown, L., & Coles, A. (2000). Same/different: A 'natural' way of learning mathematics. *Proceedings of 24th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 113 - 120. Available at: https://eric.ed.gov/?id=ED452032 This article exemplifies the use of the teaching strategy of setting up tasks where the starting point for students is to answer the question “What is the same, what is different?”.

Coles, A., & Scott, H. (2015). Planning for the unexpected in the mathematics classroom: teacher and student change. *Research in mathematics education, 17*(2), 121-138. This article charts the change, over a year, of a teacher (Scott) and one of her students, drawing out parallels and implications.

Coxeter, H. S. M. (1987). *Projective geometry* (2nd ed.). New York: Springer-Verlag. Coxeter was a geometer, whose writings on mathematics combine rigour and aesthetics. We are currently working on ways in which projective geometry might be offered to adolescents in school.

Gattegno, C. (1974). *The common sense of teaching mathematics*. NY: Educational Solutions. Gattegno inspired mathematics teachers across the globe; in his own teaching he claimed to be able to teach the 5 year secondary curriculum (to mastery) in 18 months. This is one of his most accessible texts.

Papert, S. (1980). *Mindstorms: Children, Computers and Powerful Ideas*. London: Harvester Press. Papert’s writing about Turtle Geometry (using the progamming language Logo (one of whose heirs is Scratch)) proposed a constructionist approach to mathematics education where children learn by *making*. His vision was of a mathematics education in which students could be creative inventors at the same time as becoming skilled in technique and in problem solving (through processes such as bricolage and debugging).

Sinclair, N. (2001). The aesthetics *is* relevant. *For the learning of mathematics, 21*(1), 25-32. This article provides examples of the colour calculator at work in a middle school mathematics setting, highlighting the role of aesthetics in prompting the children’s problem posing, guiding their problem solving and evaluating their solutions.