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A tale of topology in quantum gravity

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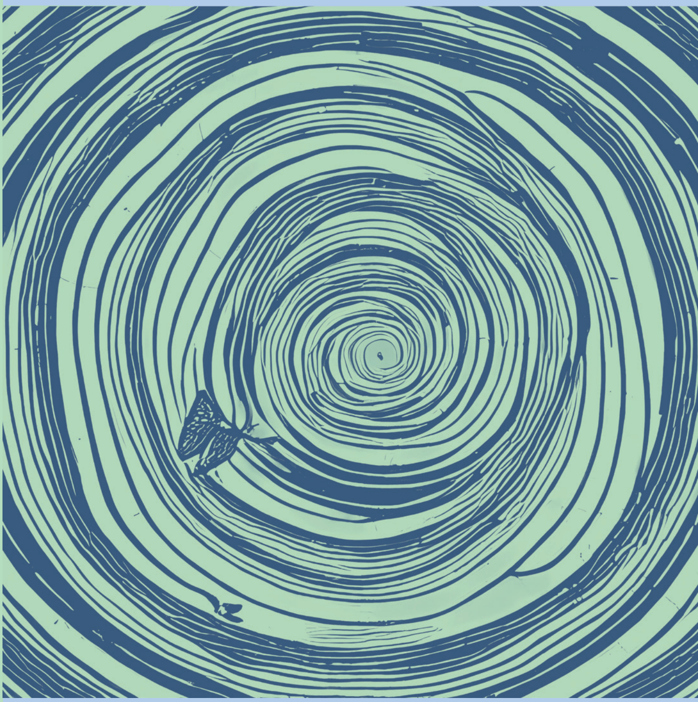

JEREMY VAN DER HEIJDEN

Chaos, wormholes, and holography

A Tale of Topology in Quantum Gravity

Jeremy van der Heijden

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Chaos, wormholes, and holography

A Tale of Topology in Quantum Gravity

Academisch Proefschrift

ter verkrijging van de graad van doctor

aan de Universiteit van Amsterdam

op gezag van de Rector Magnificus

prof. dr. ir. P.P.C.C. Verbeek

ten overstaan van een door het College voor Promoties

ingestelde commissie,

in het openbaar te verdedigen in de Aula der Universiteit

op vrijdag 20 oktober 2023, te 14:00 uur

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Faculteit der Natuurwetenschappen, Wiskunde en Informatica

Publications

THIS THESIS IS BASED ON THE FOLLOWING PUBLICATIONS:

Presented in Chapter 3:

[1] B. Post, J. van der Heijden and E. Verlinde, "A universe field theory for JT gravity," *JHEP* **05** (2022) 118, `arXiv:2201.08859 [hep-th]`

Presented in Chapter 4:

[2] A. Altland, B. Post, J. Sonner, J. van der Heijden and E. Verlinde, "Quantum chaos in 2D gravity," *SciPost Phys.* **15** (2023) 064, `arXiv:2204.07583 [hep-th]`

Presented in Chapter 5:

[3] J. de Boer, R. Espíndola, B. Najian, D. Patramanis, J. van der Heijden and C. Zukowski, "Virasoro entanglement Berry phases," *JHEP* **03** (2022) 179, `arXiv:2111.05345 [hep-th]`

Presented in Chapter 6:

[4] B. Czech, J. de Boer, R. Espíndola, B. Najian, J. van der Heijden and C. Zukowski, "Changing states in holography: From modular Berry curvature to the bulk symplectic form," *Phys. Rev. D* **108** no. 6, (2023) 066003, `arXiv:2305.16384 [hep-th]`

Contribution of the author to the publications:

[1] The author has participated in all conceptual discussions, and contributed to most of the computations in the paper. Moreover, the author came up with the main research question. The writing of the paper was jointly done with Boris Post.

[2] The author has participated in most conceptual discussions, and contributed to many of the computations in the paper. In particular, the author was responsible for the computations and ideas in Section 4, and also wrote a first draft of that section. Moreover, the author checked all the computations in the rest of the paper.

[3] The author has participated in all conceptual discussions, and contributed to all computations in the paper. In particular, the author performed all computations in Section 3 and 5, and also wrote the first draft of these sections. The author also made the crucial observation that led to discovery of the subtleties associated with the dual space of the Virasoro algebra, and came up with a definition of the entanglement wedge symplectic form.

[4] The author has participated in all conceptual discussions, and was responsible for all computations in the paper (except for the computation of the bulk symplectic form in section 4.1.2). The author also wrote the first draft of the entire paper based on these computations (except for the introduction).

"*It is a profound and necessary truth that the deep things in science are not found because they are useful; they are found because it was possible to find them.*"

– Robert Oppenheimer

"*If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.*"

– John von Neumann

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In our thoughts, we can visit a black hole without ever getting into trouble. That we can explore the universe in this way is one of the crazy miracles of human imagination, and in my opinion one of the most beautiful aspects of doing research in theoretical physics. However, spending much time with your own thoughts can also be a difficult and lonesome endeavor. For this reason, having people around you – to explain your ideas to, get inspired by or simply to take your mind of things – is very important. I would like to thank those who were around me, and without whom this thesis would never have come to be.

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Jeremy van der Heijden
June 2023

Contents

1 Introduction

The universe is a strange place. With the discovery of the atom, and subsequent discoveries of even smaller particles like quarks and gluons that make up atomic nuclei, physics has followed a reductionist line of thought: It aims to break things into smaller parts to find the most fundamental constituents. These elementary particles, as they are called, are the fundamental building blocks that constitute matter around us, and mediate the forces that act upon them. They include matter particles, like electrons, and force carrying particles, like photons.

Our everyday experiences do not prepare us well for how the universe operates on the smallest scales. The theory that governs the behavior of elementary particles is quantum mechanics. Its strange and sometimes bewildering predictions – from its inherent indeterminism to wave-particle duality and the possibility of quantum entanglement – have puzzled physicists over the years. However, time after time we have found that these predictions agree very well with experiment; one could even argue that quantum mechanics is one of the most successful theories, in the sense of predictive power, that science has ever produced.

The modern framework for dealing with Nature at subatomic scales – one could say 'the language of theoretical physics' – is quantum field theory (QFT). This framework trades particles for quantum fields as its most fundamental constituent: In this description particles are a derived concept that come about as local excitations in some field that permeates all of space. Over the years, one of the main goals of theoretical physics has been to write the physical laws that govern matter and forces in the language of QFT.

There are four fundamental forces in Nature: electromagnetism, the strong force, the weak force, and gravity. Of all four, gravity is probably the most well-known. It makes things fall down, and keeps our planet in orbit. It is all the more surprising that we still do not have a good understanding of how gravity works at the smallest of scales. Electromagnetism that deals amongst other things with light and chemical properties of matter finds its fundamental description in quantum electrodynamics (QED), and the strong force that keeps the quarks and gluons in the atomic nucleus together has a similar description in terms of quantum electrodynamics (QCD). Also the weak force, a transformative force that plays an important role in the chain of nuclear reactions that powers our Sun, has a quantum description.

Quantizing gravity is a different matter. In 1915, Albert Einstein revolutionized our understanding of the physical world with his theory of general relativity (GR). Einstein's theory says that spacetime (the combined notion of space and time) is dynamical: It can curve or bend under the influence of matter, an effect that we experience as gravity. This provides a deep relationship between the gravitational force and the geometry of spacetime, and shows that gravity plays a somewhat special role when compared to the other three forces.

To combine quantum mechanics and gravity into a theory of quantum gravity requires an understanding of the fundamental building blocks of spacetime itself. It is one of the biggest mysteries in modern theoretical physics, and it has been for quite some time. Luckily, Nature has left us some breadcrumbs to follow. One such important clue involves one of the most dramatic objects of all: black holes.

1.1 Black holes

Black holes are places where the gravitational attraction becomes so strong that nothing can escape; not even light. They are characterized by an event horizon, a boundary that demarcates the point of no return and hides everything on the inside from view. Quite dramatically, all things that fall into a black hole inevitably reach a singularity, a point with infinite curvature and zero size.

Einstein's equations relate the spacetime geometry in terms of the metric tensor $g_{\mu\nu}$ to the distribution of mass-energy in terms of the stress energy tensor $T_{\mu\nu}$. They are usually written in terms of the Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \ , \tag{1.1.1}$$

where $R_{\mu\nu}$ is the Ricci curvature and R is the Ricci scalar associated to the metric. Einstein's equations now relate geometry to mass-energy via

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{1.1.2}$$

where G is Newton's constant and Λ is the cosmological constant that determines the curvature of our universe as a whole.

Long before they were observed in Nature, black holes were predicted by GR. For example, the Schwarzschild geometry is a solution to the field equations (1.1.2) in flat[1] (i.e., $\Lambda = 0$) four-dimensional space without matter, that describes a black hole with mass M:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \ , \tag{1.1.3}$$

where the function $f(r)$ is given by

$$f(r) = 1 - \frac{2MG}{r} \ . \tag{1.1.4}$$

The location of its event horizon r_h, and therefore the size of the black hole, is determined by the equation $f(r_h) = 0$. Using (1.1.4) it is given by $r_h = 2MG$. The geometry has a curvature singularity at $r = 0$ that cannot be removed by a coordinate transformation.

Besides being solutions to the classical field equations, black holes turn out to be excellent probes for the apparent dichotomy between general relativity and quantum mechanics. Following upon an important observation of Jacob Bekenstein [5, 6] and others [7–11] that black holes satisfy equations that are very reminiscent of the laws of thermodynamics, Stephen Hawking [12] showed that black holes are not really black when one includes quantum effects at the event horizon: They carry away some of their mass in the form of radiation. The Hawking effect is fundamentally due to the entanglement structure of the vacuum state: The spontaneous creation of virtual particles at the event horizon leads to an effective Hawking radiation when one of the particles falls into the black hole, and its partner escapes to infinity. Just like ordinary radiating objects, black holes therefore have a temperature.

Given that black holes have a temperature, it is also possible to associate an entropy to them. Let us for concreteness consider the Schwarzschild solution (1.1.3) which has a temperature given by

$$T = \frac{1}{8\pi MG} \ . \tag{1.1.5}$$

The first law of thermodynamics now says that

$$dS = \frac{dM}{T} = 8\pi MG dM \ . \tag{1.1.6}$$

[1] Nothing prevents us from studying solutions with a different number of dimensions, or with different values of the cosmological constant. Most of this thesis will actually be devoted to Anti-de Sitter space, a universe of constant negative curvature.

It relates the change in entropy dS to the change in total energy, in this case the change in mass dM of the black hole. Integrating this relation, it follows that

$$S = 4\pi G M^2 \ . \tag{1.1.7}$$

Using the fact that the radius of the Schwarzschild black hole is twice its mass, we find the Bekenstein-Hawking formula for the entropy of a black hole:

$$S_{\mathrm{BH}} = \frac{A}{4G} \ . \tag{1.1.8}$$

A surprising feature of the Bekenstein-Hawking formula is that the entropy of a black hole scales with the area of the event horizon. This result (which holds for black holes in general) suggests that the information of a gravitating region can somehow be encoded on its boundary surface.

For its apparent simplicity, (1.1.8) has had profound implications for our understanding of black holes. Usually in thermodynamics, entropy is statistical in nature: It counts certain microscopic degrees of freedom that lead to the given macroscopic state. Bekenstein [5,6] already proposed that the area of a black hole should represent a statistical entropy that counts black hole microstates. However, in the case of black holes it is not so clear what those fundamental degrees of freedom are. A proper theory of quantum gravity should account for the entropy (1.1.8) in terms of the counting of some microscopic degrees of freedom. In fact, one of the great achievements of string theory, a candidate theory for quantum gravity, was to correctly account for the microstates of some specific types of black holes [13]. It is still an open problem, however, to explain the statistical entropy of generic black holes.

A related issue involves the black hole information problem. Hawking radiation causes a black hole to slowly evaporate. In Hawking's calculation [14] the resulting radiation does not seem to contain any specific details about the formation process, for example, about the stuff that was thrown into the black hole to begin with. Therefore, naively it seems that information gets destroyed when a black hole evaporates. However, in quantum mechanics such information loss cannot arise, because of unitary time evolution. We expect that the tension between quantum mechanical unitarity and information loss should be resolved in a full theory of quantum gravity.

1.2 Holography

The observation that the Bekenstein-Hawking entropy does not scale with the *volume* of the region (as is usually the case in statistical systems), but with the horizon *area* has important implications. It suggests that the degrees of freedom of quantum gravity live in one dimension fewer. This led 't Hooft [15] and Susskind [16] to propose the holographic principle: A theory of quantum gravity in $d+1$ dimensions can be equivalently described by a quantum field theory in d dimensions without gravity. In some way, quantum gravity behaves like a hologram, adding one extra dimension that emerges from some underlying quantum description.

The most explicit realization of the holographic principle is the AdS/CFT correspondence [17, 18] (see [19] for a review). In its most general form, it provides a duality between gravity on $(d+1)$-dimensional Anti-de Sitter space (AdS) and a conformal field theory (CFT) in d dimensions. The CFT is a special type of QFT that exhibits conformal symmetry. The AdS spacetime is usually represented as a solid cylinder, with the CFT living on the compact boundary, as is depicted in Figure 1.1. The correspondence is a weak/strong duality: A strongly coupled CFT is dual to a weakly coupled gravity theory, and vice versa. The archetypal examples of AdS/CFT come from string theory: the most famous one involves a duality between type IIB string theory on $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) with gauge group $SU(N)$ on the four-dimensional boundary of AdS_5. In the large $N \to \infty$ limit and for strong SYM coupling, the gravitational theory reduces to classical GR.

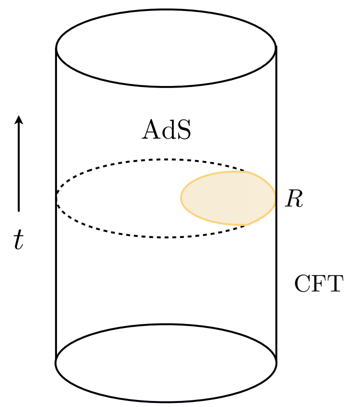


Figure 1.1: *The AdS/CFT correspondence. A subregion R in the 'boundary' CFT, has a corresponding geometric region in the 'bulk' AdS space, that is determined by the Ryu-Takayanagi (RT) surface: the entanglement wedge (depicted in yellow).*

One can think about the AdS/CFT correspondence as a dictionary: One can use it to translate quantities in one theory to the other, and vice versa. Each physical quantity in the gravitational 'bulk' theory should have an equivalent 'boundary' quantity that is dual to it. Much of the research in AdS/CFT is about uncovering interesting parts of the holographic dictionary.

On a formal level, the correspondence provides an isomorphism between Hilbert spaces:

$$\mathcal{H}_{\text{QG}} \longleftrightarrow \mathcal{H}_{\text{CFT}} \; , \tag{1.2.1}$$

i.e., a one-to-one mapping between states in quantum gravity and states in the CFT. Since we do not (yet) have an independent description of the quantum gravity Hilbert space, one can interpret the AdS/CFT correspondence in (1.2.1) as giving a definition of quantum gravity in terms of a microscopic CFT.

First, note that the symmetries on both sides match. To be explicit, the isometries of pure AdS_{d+1} constitute the conformal group $SO(2, d)$, which is also the symmetry group of a CFT_d in Lorentzian signature. For this reason, the vacuum state of the CFT (that is invariant under the full conformal group) gets mapped to the pure AdS spacetime. One can also consider excited states in the CFT, by acting with some operator on the vacuum. The corresponding geometries look like AdS space close to the boundary, but can have a different interior: They are asymptotically AdS spacetimes. An important class of examples are black hole geometries that correspond to states in the CFT at finite temperature.

Another important entry in the holographic dictionary is entanglement entropy. In quantum mechanics the entropy of a given quantum state, encoded in a density matrix ρ, is computed by the von Neumann entropy

$$S = -\text{tr}\,\rho \log \rho \; . \tag{1.2.2}$$

As an example consider a system of two qubits that are maximally entangled (i.e., constitute an Einstein-Podolsky-Rosen (EPR) pair):

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle + |10\rangle \right) \; . \tag{1.2.3}$$

The reduced density matrix (where we trace out the subsystem corresponding to one of the two qubits) is proportional to the identity $\rho = \frac{1}{2}\mathbb{1}$ operator, so that the entanglement entropy, using (1.2.2), is non-zero: $S = \log 2$. Therefore, the two qubits that constitute $|\Psi\rangle$ are entangled in a non-trivial way.

Given a CFT (or more generally any QFT) in a state $|\Psi\rangle$, one can associate a density matrix to a subregion R by tracing out the complement region $\overline{R}$: The

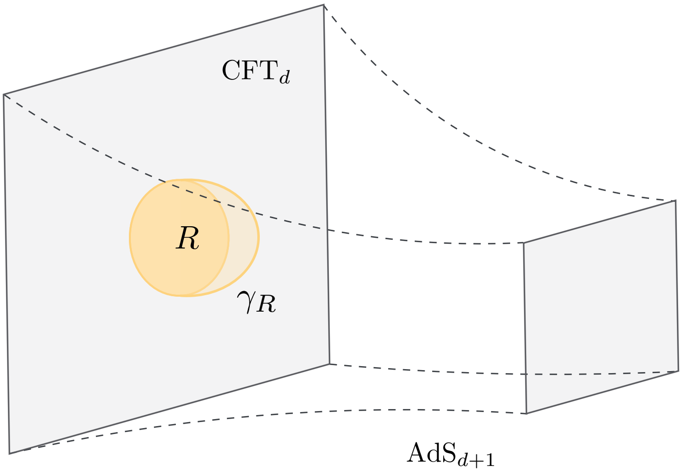


Figure 1.2: *The entanglement entropy $S(R)$ associated to some subregion R in the CFT is computed by the area of the RT surface γ_R that extends into the holographic AdS direction.*

reduced density matrix is given by

$$\rho = \mathrm{tr}_{\bar{R}}|\Psi\rangle\langle\Psi| \ . \tag{1.2.4}$$

Using (1.2.2) one can now associate an entanglement entropy $S(R)$ to a subregion R. The holographic interpretation of this quantity is now given by the Ryu-Takayanagi (RT) formula [20]

$$S(R) = \frac{A(\gamma_R)}{4G} \ , \tag{1.2.5}$$

where $A(\gamma_R)$ denotes the area of some spacelike codimension-2 extremal surface γ_R in the bulk, the so-called RT surface, that is homologous to R and shares the same boundary $\partial R = \partial\gamma_R$ (See Figure 1.2). The RT formula therefore gives a geometric interpretation to the boundary entanglement in terms of the area of some surface. The RT formula (1.2.5) looks very similar to the Bekenstein-Hawking formula (1.1.8) for black holes. There is a covariant generalization of the RT formula due to Hubeny, Rangamani and Takayanagi (HRT) [21], and an extension that includes quantum corrections due to the entropy of bulk fields, the so-called quantum extremal surface (QES) prescription [22].

The AdS/CFT correspondence sheds light on some of the major problems associated with black holes. It tells us, for example, that no information loss is expected for black holes in AdS, since an ordinary quantum field theory has unitary time evolution. This provides a (somewhat formal) answer to the information problem for evaporating AdS black holes. Similarly, the discrete spectrum of black hole microstates is readily explained from the boundary perspective: A CFT on a compact space exhibits a discrete spectrum. The true problem that remains is therefore to account for these issues, say information recovery or a discrete spectrum, from a bulk geometric perspective by including certain non-perturbative effects. An interesting approach towards such a description that will be investigated in this thesis amounts to a careful treatment of the gravitational path integral that defines quantum gravity as a 'sum over geometries.'

1.3 Path integrals

Let us briefly explain the relation between path integrals and states in QFT, before going into quantum gravity. The Euclidean path integral computes a transition amplitude

$$\langle\phi_1|e^{-\beta H}|\phi_2\rangle = \int_{\phi(0)=\phi_2}^{\phi(\beta)=\phi_1} D\phi\, e^{-S_{\mathrm{E}}[\phi]} \ , \tag{1.3.1}$$

where $S_{\mathrm{E}}[\phi]$ is the Euclidean action of the theory. The fields $\phi_{1,2}$ are boundary conditions for the fields ϕ at imaginary time $t_E = 0, \beta$ respectively. When the QFT lives on a plane $\mathbb{R}^d$, the Euclidean manifold M that is integrated over has the topology of a strip $\mathbb{R}^d \times [0,\beta]$, see Figure 1.3.

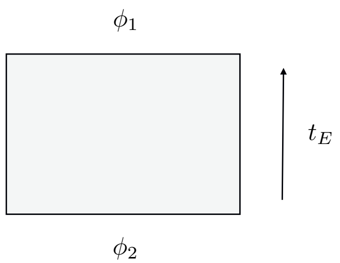


Figure 1.3: *The Euclidean path integral in QFT. The path integral is computed on a Euclidean strip $M = \mathbb{R}^d \times [0,\beta]$ with boundary conditions $\phi(0) = \phi_2$ and $\phi(\beta) = \phi_1$.*

Equivalently, one can think about (1.3.1) as defining the matrix elements of the (non-normalized) thermal density matrix $\rho = e^{-\beta H}$ at temperature $T = 1/\beta$. The thermal partition function is given by

$$Z(\beta) = \operatorname{tr} e^{-\beta H} = \sum_i \langle\phi_i|e^{-\beta H}|\phi_i\rangle \ , \tag{1.3.2}$$

so it can be expressed in terms of the Euclidean path integral using (1.3.1). The sum over states can be implemented by imposing periodic boundary conditions on the strip, gluing the two sides of the Euclidean manifold into a cylinder. The thermal partition function can therefore be computed via the Euclidean path integral on an infinitely long cylinder $\mathbb{R}^d \times S^1$ of period β.

Figure 1.4: *The thermal partition function $Z(\beta)$. The trace over boundary conditions amounts to gluing both sides of the strip into a Euclidean cylinder with period β.*

To define the path integral in QFT we fix the Euclidean manifold M, and integrate over all the fields ϕ on M. In quantum gravity on the other hand, we expect that we have to integrate over all possible spacetime geometries as well. The gravitional path integral can therefore be formally expressed as:

$$Z = \int DgD\phi\, e^{-S_{\mathrm{E}}[g,\phi]} \ . \tag{1.3.3}$$

where the action is given by the Euclidean Einstein-Hilbert action with possible boundary terms S_{bdy}, and suitable matter content S_{matter}:

$$S_{\mathrm{E}}[g,\phi] = -\frac{1}{16\pi G}\int \sqrt{g}(R-2\Lambda) + S_{\mathrm{matter}} + S_{\mathrm{bdy}} \ . \tag{1.3.4}$$

In analogy with QFT, we define the thermal partition function $Z(\beta)$ in quantum gravity as the path integral with boundary conditions such that the Euclidean time coordinate is periodic with period β at infinity:

$$t_E \sim t_E + \beta \ . \tag{1.3.5}$$

Of course, we do not know how to compute, or even properly define this Euclidean integral. In practice, the approach is to approximate the answer by means of a saddle point analysis, where one evaluates the action at its classical solution:

$$Z(\beta) \approx \exp\left(-S_E[g_c,\phi_c] + \cdots\right) \ . \tag{1.3.6}$$

This is the semiclassical approximation to the path integral, and the $\cdots$ denote higher loop corrections.

For example, in the case of the Schwarschild geometry (1.1.3) The Euclidean solution can be obtained from the original geometry by sending $t \to -it_E$:

$$ds^2 = f(r)dt_E^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \ . \tag{1.3.7}$$

The horizon $r_h = 2MG$ now simply corresponds to the origin of a disk in polar coordinates with angular coordinate $t_E \sim t_E + \beta$. To see this, we approximate the function $f(r)$ near the horizon r_h via a Taylor expansion:

$$f(r) \approx \frac{4\pi}{\beta}(r - r_h) \ , \quad \text{where} \quad \beta = \frac{4\pi}{f'(r_h)} \ . \tag{1.3.8}$$

This leads to the following Euclidean metric near the horizon:

$$ds^2 \approx \frac{4\pi}{\beta}(r - r_h)dt_E^2 + \frac{\beta}{4\pi}\frac{dr^2}{(r - r_h)} + r^2 d\Omega^2 \ . \tag{1.3.9}$$

Introducing the coordinates ρ and θ defined by

$$\rho = \sqrt{\frac{\beta}{\pi}(r - r_h)} \ , \quad \theta = \frac{2\pi}{\beta} t_E \ , \tag{1.3.10}$$

the metric looks like a two-dimensional plane in polar coordinates:

$$ds^2 \approx d\rho^2 + \rho^2 d\theta^2 + \ldots \tag{1.3.11}$$

Requiring that the geometry is smooth (and does not have a conical singularity), one has to impose that $\theta \sim \theta + 2\pi$. As a consequence, t_E also needs to be periodic with period β given in (1.3.8). This gives a quick (and arguably somewhat mysterious) derivation of the formula for the Hawking temperature (1.1.5). After a careful analysis of the Euclidean action when evaluated on the Euclidean Schwarzschild solution [23], one finds that the thermal partition function is given by:

$$Z(\beta) \approx e^{-\frac{\beta^2}{16\pi G}} \ . \tag{1.3.12}$$

This indeed reproduces the usual thermodynamic relations (e.g., (1.1.8)) for the Schwarzschild black hole.

Finally, let us come back to the AdS/CFT correspondence. Black holes in quantum gravity are dual to thermal states in the CFT. The mapping between thermodynamic quantities on both sides is now simply that they should be equal. In short,

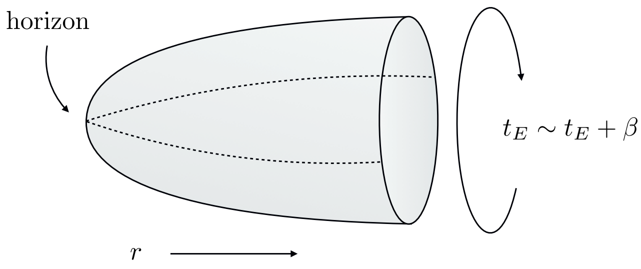


Figure 1.5: *The Euclidean black hole. The 'cigar'-shaped geometry exhibits a thermal circle at infinity with period β, and caps off at the horizon.*

we have the relation between partition functions[2]:

$$Z_{\mathrm{CFT}}(\beta) = Z_{\mathrm{grav}}(\beta) \ . \tag{1.3.13}$$

The left-hand side of (1.3.13) is the usual thermal partition function in the CFT, and the right-hand side is given by the Euclidean path integral construction that was described above. It is usually evaluated in a semiclassical way by relating it to the Euclidean on-shell action. In principle, on should, however, include all quantum corrections. While this is not feasible in general, we will see that the situation in certain models for two-dimensional gravity is much better, and that it will be possible to compute the gravitational path integral exactly.

1.4 What to expect?

This thesis broadly cover three topics, divided into five chapters. The first topic is a study of the gravitational path integral in the context of a simple two-dimensional model for quantum gravity, presented in Chapter 2 and 3. The second topic, discussed in Chapter 4, is holographic quantum chaos. The third involves the study of geometric phases in AdS/CFT, in Chapter 5 and 6. The overarching theme is that all topics deal with aspects of topology change and the AdS/CFT correspondence.

Let me now go into a bit more detail and present some of the more specialized background that is useful later, and anticipate some of the important ideas that will follow.

[2]This is a special example of the more general dictionary that relates path integral generating functions on both sides of the duality.

1.4.1 Euclidean wormholes

In recent years, there has been much interest in holographic setups on spacetimes with multiple boundaries. The general picture to have in mind is a gravitational theory on some Euclidean AdS bulk spacetime M, together with a collection of quantum theories living on the disconnected boundaries of M (see Figure 1.6). A component of such a manifold with disconnected boundaries but connected interior is called a spacetime or Euclidean wormhole[3]. In this case, the bulk manifold has a non-trivial topology.

Developments in two-dimensional gravity [24–26] have pointed towards the importance of such higher topologies in the Euclidean path integral. Most notably, it was demonstrated that the unitary Page curve [27,28] for the Hawking radiation of an evaporating black hole in AdS can be obtained by including Euclidean 'replica' wormholes in the computation [29,30]. The basic ingredient is the 'replica trick' on n copies of the same black hole. Before the Page time, the dominant contribution to the Rényi entropy comes from n disconnected spacetimes, whereas after the Page time spacetime wormholes which connect the n copies dominate. The analytic continuation $n \to 1$ then results in the Page curve for the Hawking radiation. This begs the question: What can we learn from the gravitational path integral if we allow for more general topologies?

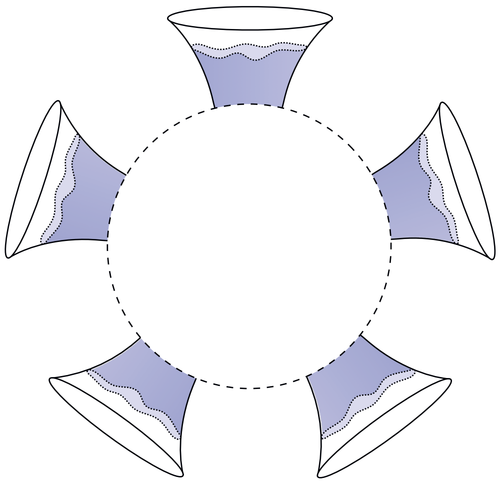


Figure 1.6: *Multi-boundary holographic setup.*

[3] The qualifier 'spacetime' indicates that the wormhole extends in space and time, distinguishing it from the more familiar Einstein-Rosen (ER) bridge, which only extends in the spatial directions.

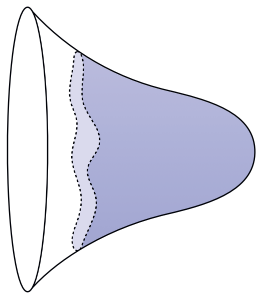


Figure 1.7: *The Euclidean black hole in JT gravity. The geometry is given by a hyperbolic disk, which is cut off at some distance from the boundary circle. The dashed line represents the 'wiggles' due to the boundary dynamics of the Schwarzian theory.*

A nice playground to investigate this question is in two dimensions, where we have a good handle on the different spacetimes that can arise in the sum over topologies. We will focus on Jackiw-Teitelboim (JT) gravity [31,32] with negative cosmological constant, a theory of two-dimensional gravity that has a metric $g_{\mu\nu}$ and a scalar field $\phi : M \to \mathbb{R}$, called the dilaton. A nice feature of JT gravity is that its Euclidean path integral can be given a precise mathematical definition, and evaluates as a sum over topologies in terms of some expansion parameter e^{-S_0}. Each topology is specified by two integers (g, n), where g denotes the genus of the surface and n the number of boundaries of spacetime, and contributions from higher topologies are suppressed by the Euler characteristic

$$\chi = 2 - 2g - n \ . \tag{1.4.1}$$

For example, in the case that M has a single boundary with regulated length β the path integral splits into topological classes of the form:

$$Z(\beta) = \int D\phi Dg \, e^{-S_{\mathrm{JT}}} \sim \sum_{g=0}^{\infty} e^{(1-2g)S_0} Z_g(\beta) \ . \tag{1.4.2}$$

The residual path integral $Z_g(\beta)$ over surfaces with g handles can be computed exactly as a function of G. It involves two distinct contributions: a boundary graviton mode due to the boundary conditions that we impose, sometimes called 'wiggly boundary', and an integral over the internal bulk moduli space of metrics that are allowed on the surface of given topology. Because $S_0 \sim 1/G$, the Euclidean wormholes are non-perturbative effects in gravity.

The Euclidean black hole solution in JT has the topology of a disk with thermal circle (see Figure 1.7), and it turns out that the quantum dynamics are governed by a boundary graviton mode that is described by the Schwarzian theory on the circle [33]. The disk contribution to the path integral $Z_0(\beta)$ is therefore given by the Schwarzian path integral, which can be solved exactly [34, 35]. Following (1.3.13), the result should have the interpretation of a thermal partition function in a putative microscopic theory, described by some Hamiltonian H:

$$Z_0(\beta) = \operatorname{tr} e^{-\beta H} = \int_0^\infty dE \rho(E)\, e^{-\beta E} \ , \tag{1.4.3}$$

where $\rho(E)$ is the density of states. If we assume that JT gravity has a quantum mechanical dual with a finite spectrum $\{E_i\}$ that counts the microstates of the black hole, its density of states should be a sum of delta function:

$$\rho(E) = \sum_i \delta(E - E_i) \ . \tag{1.4.4}$$

However, the computation of the Schwarzian density of states shows that $\rho(E)$ is a continuous function. Therefore, the Euclidean disk geometry in JT gravity does not seem to result in a discrete spectrum. How do we resolve this apparent tension?

Another confusing aspect of the Euclidean path integral in JT involves the presence of spacetime wormholes. Assuming that the dual quantum mechanical theory is local in the sense that two distant asymptotic boundary theories should be uncorrelated (which is true in the standard AdS/CFT dictionary) we expect that the full gravitational path integral should be a product of path integrals of disconnected spacetimes, each with a single boundary dual. An immediate contradiction arises when one allows Euclidean wormholes in the sum over bulk topologies: we should then include both disconnected and connected contributions, which leads to non-factorization.

For example, in the case of the two asymptotic boundaries the connected contribution is given by a Euclidean wormhole $Z_{\text{conn}}(\beta_1, \beta_2)$, as depicted in Figure 1.8. The full path integral therefore takes the form

$$Z(\beta_1, \beta_2) \sim e^{2S_0} Z_0(\beta_1) Z_0(\beta_2) + Z_{\text{conn}}(\beta_1, \beta_2) + \dots , \tag{1.4.5}$$

which leads to an explicit non-factorization $Z(\beta_1, \beta_2) \neq Z(\beta_1)Z(\beta_2)$. Therefore, a natural question arises: How do we interpret connected bulk contribution in an underlying microscopic theory?

Both these issues, a continuous density of states and non-factorization, might be resolved (or at least given an interpretation) if we relax our notion of holographic

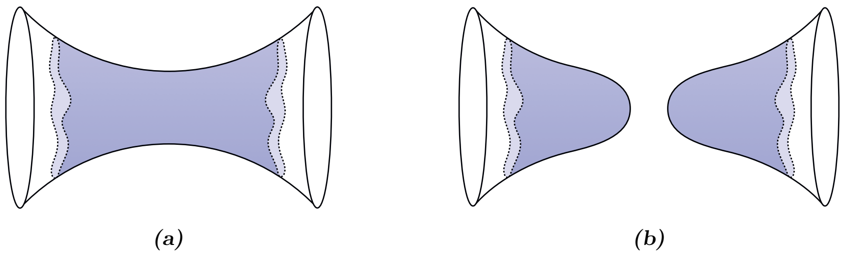


Figure 1.8: *The connected (a) and disconnected (b) contributions to the JT gravity path integral with two boundaries. By including the connected contribution we obtain a non-factorized answer.*

duality. It may be true that the bulk gravitational theory is not dual to a single well-defined quantum theory, but instead to a statistical ensemble of boundary theories. The partition function of the boundary theory would then be a random variable, and the bulk path integral computes averages over these observables in some model-dependent statistical ensemble, denoted by $\langle \cdots \rangle = \langle \cdots \rangle_{\text{ensemble}}$. In particular, ensemble averaging smoothens out the discrete density of states into a continuous function

$$\rho(E) = \left\langle \sum_i \delta(E - E_i) \right\rangle , \tag{1.4.6}$$

and the lack of factorization is simply the statement that the random boundary observables are correlated. In other words, Euclidean wormholes represent statistical correlation. In particular, the two-sided spacetime wormhole corresponds to a variance in the ensemble:

$$Z_{\text{grav}}(\beta_1, \beta_2) = \langle Z(\beta_1) Z(\beta_2) \rangle \neq \langle Z(\beta_1) \rangle \langle Z(\beta_2) \rangle . \tag{1.4.7}$$

The most clean realization of this idea was presented in [36], where the holographic dual of JT gravity was argued to be a (double-scaled) random matrix integral. The precise statement is that the topological expansion of the Euclidean path of JT agrees with the genus expansion of certain correlation functions of macroscopic loop operators in the matrix integral, if we identify the expansion parameter e^{S_0} with the size L of the random matrix via some double-scaling procedure. This relation will be explained in much more detail in Chapter 2.

The first part of this thesis aims to understand this surprising duality, and the appearance of a matrix integral dual, from a different perspective. The topological expansion in wormholes is reminiscent of the expansion that appears in string theory, where strings can merge and combine resulting in different world-sheet topologies. In our setting, the role of the two-dimensional string world-sheet is

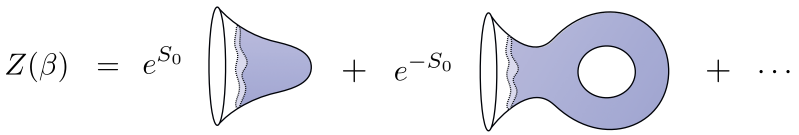


Figure 1.9: *The Euclidean path integral in JT as a sum over topologies.*

played by the JT universe. Inspired by topological string theory, which is a special type of string theory that is somewhat under control, we have constructed a quantum field theory that reproduces the spacetime wormholes as correlations functions of some fundamental scalar field Φ (not to be confused with the dilaton).

The relevant theory that describes the dynamics of this scalar field is the Kodaira-Spencer (KS) field theory, that was originally studied by Robbert Dijkgraaf and Cumrun Vafa [37] as the closed string field theory for the B-model topological string [38–40]. The intricacies of the splitting and joining of strings, represented by a cubic vertex in the KS theory, precisely reproduce the non-perturbative spacetime wormhole contributions as a perturbative expansion in the KS field theory coupling constant. Following the terminology 'string field theory', we have coined the term 'universe field theory' for this description. The non-trivial topologies are then given an interpretation in terms of little 'baby universes' parting from (and later recombining with) their larger 'parent universe.'

On a technical level, our computation amounts to a careful study of certain recursion relations associated to the moduli space of Riemann surfaces, that govern the topology change in JT, and a demonstration that these can be realized as Dyson-Schwinger equations for the universe field theory. One of the interesting features of our framework is that it allows for the study of more general observables; in string theory it is actually very natural to consider extended objects on which open strings can end: D-branes. In the KS theory these are represented by vertex operators. Using some vertex operator calculus in the universe field theory we are able to reproduce the non-decaying behavior of the spectral form factor, a particular analytically continued two-point function, which strongly hints at a connection between JT gravity and quantum chaos.

1.4.2 Holographic quantum chaos

A very interesting and fruitful research topic involves the interplay between quantum chaos and quantum gravity (see for example [41–43]). In the seminal work [36] it was shown that Euclidean JT gravity is dual to a (double-scaled) matrix integral, demonstrating a very concrete model for holographic quantum chaos. Specifically, this duality provides an interesting relation between Euclidean wormholes and quantum chaos.

A quantum chaotic system has energy levels that are well-modeled by random matrix theory [44]. A generic matrix integral is given by

$$Z = \int dH \, \exp(-L \, \mathrm{Tr} \, V(H)) \, , \tag{1.4.8}$$

where dH is the flat measure on the space of $L \times L$ Hermitian matrices and $V(H)$ is some potential function. A characteristic feature of matrix models (and thus quantum chaos) is eigenvalue repulsion: Individual eigenvalues tend to stay away from each other. This makes the spectrum of a chaotic theory very rigid with a typical level spacing $\Delta \sim e^{-S}$ (as opposed to an integrable theory that can have many degeneracies in the spectrum).

Eigenvalue repulsion manifests itself in correlation functions. A useful diagnostic is the so-called spectral form factor. Let us first introduce the observable

$$Z(\beta; t) = \mathrm{tr} \, e^{-\beta H - itH} \, . \tag{1.4.9}$$

This quantity can be obtained from the thermal partition function $Z(\beta)$ by an analytic continuation $\beta \to \beta + it$. For late times t, (1.4.9) fluctuates erratically (with mean zero) due to the complex oscillatory factor. The size of these fluctuations is captured by the quantity

$$G(\beta; t) \equiv \Big| \frac{Z(\beta; t)}{Z(\beta)} \Big|^2 = \frac{1}{Z(\beta)^2} \sum_{n,m} e^{-\beta(E_n + E_m) - it(E_n - E_m)} \, . \tag{1.4.10}$$

The spectral form factor (at finite temperature) is now defined as the averaged $g(\beta; t) = \langle G(\beta; t) \rangle$, where in the case of a random matrix theory the average is taken with respect to (1.4.8). It has the effect of smoothing out the erratic fluctuations, but leaving a small non-zero contribution at very late times, the so-called 'plateau.' After $t = 0$ the curve starts dipping below the plateau value, after which it exhibits a linear growth (the 'ramp') before reaching the plateau. This 'dip-ramp-plateau' structure of the spectral form factor is a characteristic feature of quantum chaotic theories (that are well-described by (1.4.8)).

A quick way to see the size of the plateau effect is by computing the long time average of the fluctuations:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \, G(\beta; t) = \frac{1}{Z(\beta)^2} \sum_E \mathcal{N}_E^2 \, e^{-2\beta E} \, , \tag{1.4.11}$$

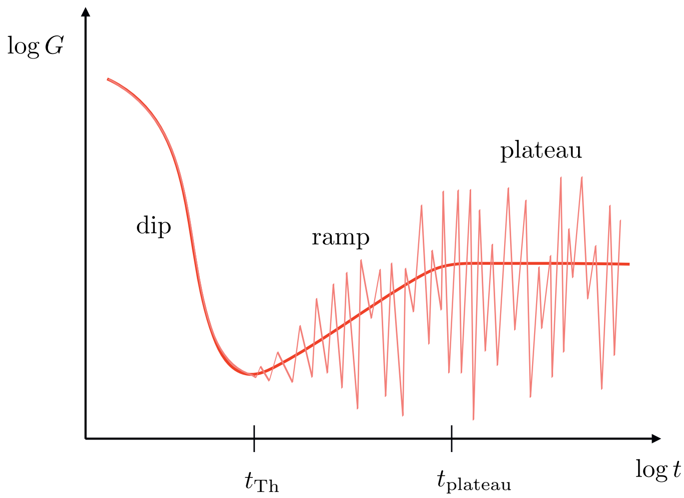


Figure 1.10: *The spectral form factor. The erratic fluctuations correspond to a single realization, while the averaged version is smoothed out (depicted with the thick red curve). It has a characteristic 'dip-ramp-plateau' shape, with corresponding timescales called the Thouless time t_{Th} and the plateau time t_{plateau}.*

where $\mathcal{N}_E$ denotes the degeneracy of the energy E. Using that the partition function scales as $Z(\beta) \sim e^{cS}$ for some constant c, and assuming that the spectrum is non-degenerate one finds that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, G(\beta;t) = \frac{Z(2\beta)}{Z(\beta)^2} \sim e^{-cS} \ . \tag{1.4.12}$$

Since $S \sim 1/G$, the size of the plateau is a non-perturbative effect of the order of the typical level spacing.

Certain two-point functions that exhibit a non-decaying behavior at late times (e.g., the spectral form factor) were argued [45, 46] to be excellent probes for a discrete spectrum in gravity. Indeed, at early times one can replace the discrete sum over states in (1.4.10) by a continuous density: This is the semi-classical gravity approximation leading to a decay that continues forever. However, at very late times the discreteness of the spectrum becomes important, and a non-zero value (1.4.12) remains. For this reason, it would be very interesting to understand the full 'dip-ramp-plateau' structure of the spectral form factor from a bulk computation.

Euclidean wormholes are argued to be responsible for (part of) this behavior. In particular, it was shown [26] that the connected contribution with two boundaries provides a universal contribution to the spectral form factor that leads to a linear growth. This gives a 'bulk' explanation of the ramp. A gravitational explanation of the full quantum chaotic behavior (including the non-perturbative plateau in the spectral from factor) was until recently lacking. This thesis will provide such an answer.

In the chaos community, it is a well-known fact that at late times (after the so-called Thouless time t_{Th}) all quantum chaotic theories fall into certain universality classes, and are described by a non-linear σ-model modeled on some supersymmetric coset [47, 48]. Concretely, such a σ-model reduces to a finite-dimensional matrix integral of Kontsevich type, which is called a 'flavor matrix model.' In recent work [49] Alexander Altland and Julian Sonner revisited these techniques in the context of holographic quantum chaos, which they describe in terms of a symmetry breaking principle, so-called causal symmetry breaking. In Chapter 4, we will show how the symmetry breaking principle and the corresponding flavor matrix integrals can be realized in our framework of the universe field theory, in terms of a dynamical theory of non-compact branes in KS theory. This gives an explicit geometrical representation of quantum chaos at late 'plateau' times.

The inspiration for how everything weaves together is again found in string theory: Since JT gravity is the analogue of a closed string, there is a dual open string description that knows about non-perturbative physics; it involves D-branes. From the universe field theory perspective the appearance of the matrix integral description is therefore quite natural: it arises from an open/closed duality, and the non-linear σ-model for quantum chaos is precisely the open string description. A comprehensive diagram displaying all the relations (Figure 4.1) can be found in Chapter 4.

This gives a very nice (and I believe deep) conceptual understanding of where the chaotic behavior in quantum gravity comes from. Although the explanation of the plateau in terms of D-brane calculus was already anticipated in earlier work [36], our universe field theory allows for a treatment of these dynamical branes in a controlled setting, and an explicit calculation of their partition function. An interesting consequence of our work is that it provides a late-time bridge between the microscopic SYK model and JT gravity, since they are both described by the same flavor matrix integral [50].

1.4.3 Modular Berry phases

Another interesting example of the interplay between topology and quantum gravity is the study of modular Berry phases in the context of AdS/CFT. One of the exciting challenges in holography is to understand the emergence of space and time from the CFT. The notion of quantum entanglement plays a key role in this relation [51], an idea that is nicely captured by the slogan 'ER=EPR' [52]. It turns out that one can understand the entanglement spectrum of the CFT, abstractly, in terms of a parallel transport problem for modular Hamiltonians, to which one associates some geometric phase, the modular Berry curvature [53, 54].

In quantum mechanics, Berry phases appear as geometric phases associated to changes in a state. For concreteness consider a pure state $|\psi\rangle$ in some Hilbert space $\mathcal{H}$. We can deform the state by some unitary (e.g., by turning on some external magnetic field)

$$|\psi(\lambda)\rangle = U(\lambda)|\psi\rangle \ . \tag{1.4.13}$$

The Berry connection is defined as

$$A = i\,\langle\psi(\lambda)|\,\delta\,|\psi(\lambda)\rangle = i\,\langle\psi|\,U^\dagger\delta U\,|\psi\rangle \ . \tag{1.4.14}$$

After making a closed loop γ in parameter space, the state has picked up a geometric phase $|\psi(\lambda_f)\rangle = e^{i\theta(\gamma)}|\psi\rangle$ which is given by

$$\theta(\gamma) = \oint_\gamma A \ = \int_B F \ , \tag{1.4.15}$$

where we have used Stokes' theorem to write it in terms of the flux of the Berry curvature $F = dA$ through a region B with the property $\partial B = \gamma$.

The notion of Berry phases can also be studied in the context of holography. While (1.4.14) holds true for pure states, a similar geometric quantity can be associated to a parallel transport of modular Hamiltonians. Let us anticipate some of the ideas that go into the construction, relegating a more complete description to the main text. The idea is to consider a global state $|\psi\rangle$ and a subregion A on a time slice of the CFT. The reduced density matrix is given by

$$\rho_A = \mathrm{tr}_{\bar{A}}\,|\psi\rangle\langle\psi| \ . \tag{1.4.16}$$

The modular Hamiltonian H_{mod} is defined by

$$\rho_A = e^{-H_{\mathrm{mod}}}/(\mathrm{tr}\ e^{-H_{\mathrm{mod}}}) \ . \tag{1.4.17}$$

We now modify the system by some auxiliary parameter λ (e.g., by changing the location of the interval A or the global state $|\psi\rangle$ of the system). This leads to a one-parameter family of modular Hamiltonians $H_{\mathrm{mod}}(\lambda)$. Diagonalizing the modular Hamiltonians as

$$H_{\mathrm{mod}} = U^\dagger \Delta U \ , \tag{1.4.18}$$

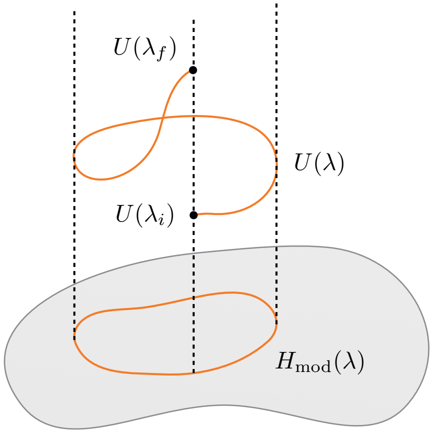


Figure 1.11: *A parallel transport problem for modular Hamiltonians. The space of modular Hamiltonians (gray shape) is the base space of some fiber bundle, where the vertical fibers are zero mode spaces. A generic closed loop $H_{\text{mod}}(\lambda)$ in the base space (indicated with orange), lifts to a non-closed curve in the fiber bundle, providing a non-trivial 'Berry phase.'*

where Δ is a diagonal matrix of eigenvalues, and taking a derivative with respect to λ one finds

$$\dot{H}_{\text{mod}} = [\dot{U}^\dagger U, H_{\text{mod}}] + U^\dagger \dot{\Delta} U \ . \tag{1.4.19}$$

This equation exhibits a redundancy due to the presence of modular zero modes:

$$U \to \tilde{U} = UV \ , \tag{1.4.20}$$

which are defined in terms of operators that commute with the modular Hamiltonian:

$$V = e^{-i\sum_i s_i Q_i} \ , \quad \text{where} \quad [Q_i, H_{\text{mod}}] = 0 \ . \tag{1.4.21}$$

The zero mode frame redundancy leads to a Berry transport problem for operators. One can think about the zero mode ambiguity as a gauge symmetry in space of modular Hamiltonians. After performing the parallel transport around a closed loop, the operator $\dot{U}^\dagger U$ that appears in (1.4.19) has a definite value, but U itself may differ by a modular zero mode:

$$U(\lambda_f) = U(\lambda_i) e^{-i\sum_i \alpha_i Q_i} \ . \tag{1.4.22}$$

It is also possible to associate a curvature F to this transport problem as well,

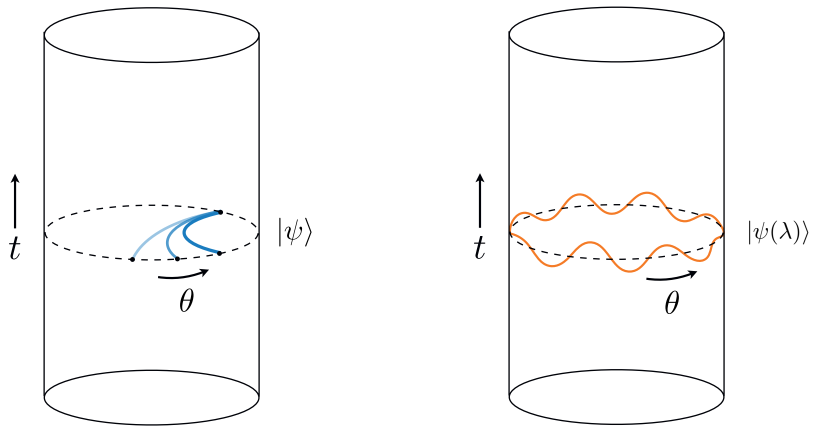


Figure 1.12: *Two types of parallel transport problems in AdS/CFT. Left: shape-changing Berry transport, where the shape of the boundary subregion is changed, and the global state $|\psi\rangle$ is kept fixed. Right: state-changing Berry transport, where the location of the subregion is kept fixed, and the global state $|\psi\rangle$ is changed (indicated by a wiggly orange line.) Both lead to a one-parameter family of modular Hamiltonians.*

by considering parallel transport along an infinitesimal loop. This is the modular Berry curvature.

In the context of holography, one can now ask the following question: What is the bulk dual for a general transport problem on the boundary? In the final part of this thesis, we make a significant contributions towards answering this question by studying a fairly general class of state-changing transport problems. First, we use Virasoro excitation in the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$ to get a bulk description in terms of geometries with a backreacted cosmic brane inserted at the RT surface. Then, we go beyond the special case of symmetry-based transport, and compute the Berry curvature associated to general coherent state deformations in arbitrary dimensions. In both cases, we establish that the Berry curvature is holographically related to a bulk symplectic form associated to the entanglement wedge. Although previous discussions (e.g., [55]) are restricted to the full boundary, we have proposed an explicit definition of the symplectic form associated to a subregion.

2 Holography in two dimensions

Holography is a powerful tool to study quantum gravity in AdS spacetimes. To make computations tractable, it is often useful to work in certain simplified low-dimensional models. For example, two-dimensional gravity is a toy model for gravity with a single space and single time dimension. It allows for an exact (non-perturbative) result for many quantities in the theory, such as the Euclidean path integral. However, by virtue of the Gauss-Bonnet theorem, the usual Einstein-Hilbert action does not generate any dynamics for the gravitational field in two dimensions: It is a topological invariant. Therefore, to get something non-trivial one has to construct slightly more involved models.

One important model, that has recently received a lot of attention, is Jackiw-Teitelboim (JT) gravity. It has been solved completely on a quantum level. The theory incorporates finite energy excitations that break the AdS_2 asymptotics [56]. These small deviations from the AdS_2 geometry are called 'nearly' AdS_2 geometries. In the context of the $\mathrm{nAdS}_2/\mathrm{nCFT}_1$ correspondence, the quantum mechanical dual is still under investigation. Famously, it was shown that the Sachdev-Ye-Kitaev (SYK) model [57–59] has a low-energy description in terms of the Schwarzian theory that also governs JT gravity [33, 60–62], arguing for its interpretation as a microscopic description of quantum black holes.

Besides being a testing bed for certain difficult questions in higher-dimensional gravity, JT gravity also has applications in more realistic physical problems. For example, the AdS_2 geometry and corresponding JT dynamics arise naturally as part of the near-horizon description of certain near-extremal black holes in four dimensions (see, e.g., [63]). Our findings are therefore also relevant for these higher-dimensional black hole settings.

In this chapter, I will present the relevant background that is useful for understanding the rest of this thesis. In Section 2.1, I will give the precise definition of the Euclidean path integral in JT gravity as a 'sum over all topologies.' I will argue that the Euclidean wormholes that arise are an indication that the dual description shares many features with random matrix theory. For this reason, I will describe the basics of random matrix theory in Section 2.2, and its proposed holographic relation to JT gravity.

2.1 The path integral in JT gravity

2.1.1 A topological expansion

The classical action of Euclidean JT gravity on a two-dimensional surface M is given by

$$S_{\rm JT}[g,\phi] = -S_0\,\chi(M) - \frac{1}{2}\int_M d^2x\sqrt{g}\,\phi(R+2) - \int_{\partial M} du\sqrt{h}\,\phi(K-1)\ . \quad (2.1.1)$$

The field content constitutes a two-dimensional metric $g_{\mu\nu}$ and a scalar field ϕ, the dilaton. The first term in the action is purely topological: It is the Euler characteristic $\chi(M)$ of the manifold M. The second term can be used to derive the bulk equations of motion

$$R+2=0\ , \qquad (\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^\alpha\nabla_\alpha)\phi + g_{\mu\nu}\phi = 0\ . \quad (2.1.2)$$

We assume that the boundary ∂M consists of a disjoint union of one-dimensional circles,

$$\partial M = S^1\cup\cdots\cup S^1\ , \quad (2.1.3)$$

so a choice of boundary conditions is required. A natural such choice involves Dirichlet boundary conditions where we fix the fields near the boundary as

$$g|_{\partial M} = \frac{1}{\epsilon^2}\ , \qquad \phi|_{\partial M} = \frac{\phi_r}{\epsilon}\ . \quad (2.1.4)$$

The holographic renormalization parameter ϵ defines a cut-off surface in the bulk which approaches the boundary in the limit $\epsilon\to 0$. The length of the thermal circle is taken to be β/ϵ and the boundary conditions (2.1.4) give rise to a graviton mode [33] that parametrizes the 'boundary wiggles' of the cut-off surface.

With these boundary conditions it is possible to compute the Euclidean path integral of JT gravity exactly. Doing the integral over ϕ along an imaginary contour in field space puts a delta function in the integral over the metrics, enforcing the equation of motion $R+2=0$. In two dimensions, this equation can be solved easily. The solutions are precisely the Riemann surfaces with negative Euler number $\chi = 2-2g-n<0$, together with the hyperbolic disk and the annulus. When we take the number n of asymptotic boundary components to be fixed, this reasoning gives the following expansion for the path integral into topologically distinct sectors

$$Z(\beta_1, \ldots, \beta_n) = \sum_{g=0}^{\infty} e^{S_0(2-2g-n)} Z_{g,n}(\beta_1, ..., \beta_n) \ . \tag{2.1.5}$$

This is an expansion in the genus (i.e., number of handles) of the surface. The expansion parameter[1] $(e^{S_0})^\chi$ comes from the topological term in the action (2.1.1), so higher genus surfaces are suppressed. Having integrated out the ϕ-field there is a residual path integral over the space of metrics on a surface of fixed topology, and an integral for the boundary degrees of freedom. Symbolically, we have

$$Z_{g,n}(\beta_1, ..., \beta_n) = \int d(\text{moduli})_{g,n} \int d(\text{bdy wiggles})\, e^{-S_{\text{bdy}}} \ . \tag{2.1.6}$$

The remainder of this section will be devoted to evaluating this expression. After careful analysis of the integration measure, we will find that the connected contribution to the JT path integral is given by

$$Z^{\mathsf{c}}_{g,n}(\beta_1, \ldots, \beta_n) = \int_0^\infty \prod_{i=1}^{n} d\ell_i \ell_i \, Z_{\mathsf{trumpet}}(\beta_i, \ell_i)\, V_{g,n}(\ell_1, \ldots, \ell_n) \ , \tag{2.1.7}$$

where we have have written the residual path integral as a bulk contribution $V_{g,n}$, which computes the volume of the geometric moduli, and a term Z_{trumpet} coming from the path integral over the boundary degrees of freedom.

The expression for the path integral (2.1.7) holds for surfaces with $\chi < 0$; the path integral on the disk and the annulus are computed independently.

2.1.2 The disk and trumpet partition function

Let us first consider the disk partition function, where we have a single asymptotic boundary of renormalized length β. The disk topology admits a hyperbolic metric of the form

$$ds^2 = d\rho^2 + \sinh^2 \rho \, d\theta^2 \ , \tag{2.1.8}$$

where ρ and θ are respectively radial and angular coordinates on the disk. According to the boundary conditions defined in (2.1.4), the circumference of the cut-off (which depends on ϵ) is fixed, but the location of the cut-off surface is allowed to fluctuate (See Figure 2.1).

[1]The parameter S_0 is referred to as the extremal entropy as it corresponds to the entropy of a four-dimensional extremal black hole with throat region $\text{AdS}_2 \times S_2$ and near-horizon dynamics described by JT gravity.

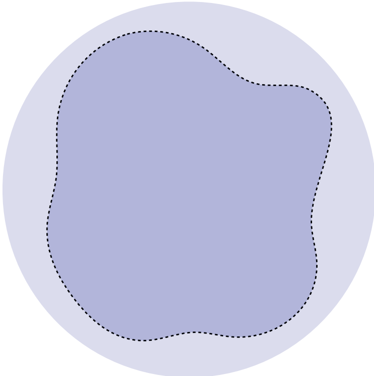


Figure 2.1: *The disk geometry in JT gravity. The dashed curve $(\rho(u),\theta(u))$ encloses a cut-out shape of the hyperbolic disk (dark purple), and describes the 'wiggles' due to the boundary dynamics of the Schwarzian theory.*

The action for these fluctuations follows from the extrinsic curvature term in (2.1.1). In the limit $\epsilon \to 0$, the boundary action was shown [33] to reduce to a Schwarzian theory for the reparametrizations $u \mapsto \theta(u)$ of the angular coordinate:

$$\int_{\partial M} du\sqrt{h}\,\phi(K-1) \quad\longrightarrow\quad \frac{2\pi\phi_r}{\beta}\int_0^{2\pi} du\,\{\tan\frac{\theta(u)}{2},u\}\ , \tag{2.1.9}$$

where the Schwarzian derivative of some function $f = f(u)$ is defined by

$$\{f,u\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\ . \tag{2.1.10}$$

The resulting path integral can be written as an integral over the coadjoint orbit

$$\mathcal{O} = \mathrm{diff}(S^1)/SL(2,\mathbb{R})\ . \tag{2.1.11}$$

From the bulk perspective, the reparametrizations $\theta = \theta(u) \in \mathcal{O}$ are precisely the diffeomorphisms of the circle modulo the ones which can be extended into the bulk: The $SL(2,\mathbb{R})$ group moves the cut-out shape around, but does not change the global geometry of the hyperbolic disk. It acts as

$$\tan\frac{\theta}{2} \to \frac{a\tan\frac{\theta}{2}+b}{c\tan\frac{\theta}{2}+d}\ , \qquad \text{where} \qquad ad-bc=1\ . \tag{2.1.12}$$

The Schwarzian derivative (2.1.10) is indeed invariant under transformations of this type. Hence, the configurations which are related to each other by the action

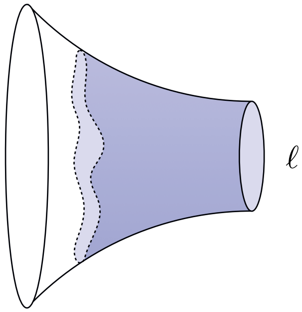


Figure 2.2: *The trumpet geometry.*

of $SL(2,\mathbb{R})$ should be made equivalent by dividing it out.

The resulting integral over the boundary wiggles is therefore given by

$$Z_{0,1}(\beta) = \int d(\text{bdy wiggles})\, e^{-S_{\text{bdy}}} = \int_{\mathcal{O}} d\theta\, e^{-S_{\text{schw}}[\theta(u)]} \ . \tag{2.1.13}$$

This path integral was shown [34,35,63,64] to be one-loop exact[2] and evaluates to

$$Z_{0,1}(\beta) = \frac{1}{4\pi^{1/2}\beta^{3/2}} e^{\pi^2/\beta} \ . \tag{2.1.14}$$

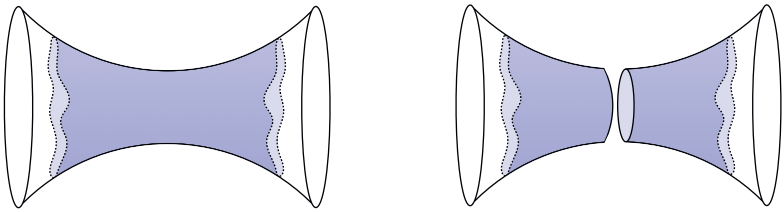
For the path integral over more general surfaces there is also a contribution coming from the boundary: Near each of the boundary circles we can remove an annulus by cutting along a geodesic of length ℓ. The resulting geometry has the shape of a trumpet, as depicted in Figure 2.2. The JT trumpet has two boundaries (one of renormalized length β and one of finite length ℓ), but one only needs a single boundary term in the action.

The full isometry group of the disk is now broken to a $U(1) \subset SL(2,\mathbb{R})$, where the $U(1)$ rotates the 'funnel' end of the trumpet. The corresponding coadjoint orbit is therefore given by

$$\mathcal{O} = \text{diff}(S^1)/U(1) \ . \tag{2.1.15}$$

Analogously to the disk computation, the extrinsic curvature term gives a (twisted) version of the Schwarzian theory, and the boundary path integral is one-loop exact.

[2]One can see this, for example, by using a localization argument following Alekseev and Shatashvili [65,66] who used a generalization the Duistermaat-Heckman formula [67] to compute such orbital integrals.



(a) *(b)*

Figure 2.3: *The double trumpet geometry (a) can be obtained from gluing two trumpets along a common geodesic of length ℓ (b). The resulting moduli space involves two parameters: the length ℓ of the geodesic and the relative twist τ before gluing.*

A similar computation shows that

$$Z_{\mathsf{trumpet}}(\beta,\ell) = \frac{1}{\sqrt{4\pi\beta}} e^{-\ell^2/(4\beta)} \ . \tag{2.1.16}$$

For example, one can use (2.1.16) to compute the path integral $Z_{0,2}(\beta_1,\beta_2)$ on a geometry with two asymptotic boundaries that are connected by a 'double trumpet'. We simply have to 'glue' two such trumpets, with asymptotic boundaries of renormalized lengths β_1 and β_2 respectively, along a common geodesic boundary of length ℓ (the 'throat' of the wormhole). In the JT path integral one should integrate over these distinct geometries. The internal moduli space amounts to an integration over the length ℓ, as well as an integral over a relative twist parameter τ. The twist parameter measures the length that a segment of a curve would travel along the geodesic boundary after a relative twist between the two trumpets has been made, before gluing them. The length and twist provide the following 'gluing measure'

$$d(\text{moduli})_{0,2} = d\ell \wedge d\tau \tag{2.1.17}$$

It turns out that not all values of τ lead to a distinct geometry: Geometries that can be obtained from one another by a full twist of the boundary are related by a diffeomorphism, and should therefore be viewed as equivalent. Such a full twist of the internal geodesic boundary is called a Dehn twist. Identifying $\tau \sim \tau + \ell$ is an example of dividing out the mapping class group (that is generated by such Dehn twists). In general, on more complicated surfaces the mapping class group acts highly non-trivially, but in the simple case of the double trumpet it amounts to restricting the domain of integration to $0 \leq \tau < \ell$. A straightforward computation shows that

$$Z^{\mathsf{c}}_{0,2}(\beta_1,\beta_2) = \int_0^\infty d\ell \int_0^\ell d\tau Z_{\mathsf{trumpet}}(\beta_1,\ell) Z_{\mathsf{trumpet}}(\beta_2,\ell) = \frac{1}{2\pi}\frac{\sqrt{\beta_1\beta_2}}{\beta_1+\beta_2} \ . \tag{2.1.18}$$

2.1.3 Weil-Petersson volumes

Let us now evaluate the general formula (2.1.6) in the case that spacetime has Euler characteristic $\chi < 0$. We have already seen how to deal with the boundary degrees of freedom: In the neighborhood of each asymptotic boundary we cut out a trumpet along some geodesic, and perform an additional integral over the wiggles given by (2.1.16). The bulk spacetime that remains is a compact Riemann surface $M = \Sigma_{g,n}$ with geodesic boundaries. One is now instructed to integrate over all possible metrics on this surface, up to diffeomorphism. In two dimensions this computation is under control: Holding the the geodesic lengths $\ell_1, ..., \ell_n$ fixed, we have to do a finite-dimensional integral over the moduli space.

Abstractly, the moduli space is defined as the space of Riemann surfaces with g handles and n geodesic boundaries, up to diffeomorphism. Let us first define Teichmüller space $\mathcal{T}(\Sigma_{g,n})$ as the space of hyperbolic metrics up to small isometries, i.e., isometries connected to the identity. To describe it more explicitly, we use a particular decomposition of the surface $\Sigma_{g,n}$ into simpler building blocks. For a surface of genus $g \geq 2$ with n geodesic boundaries there is a decomposition into $2g-2+n$ copies of a pair-of-pants geometry. This pair-of-pants surface has genus zero and admits a hyperbolic metric with three geodesic boundaries of lengths ℓ_1, ℓ_2, ℓ_3. It turns out that the hyperbolic structure on this surface is uniquely specified by fixing the hyperbolic lengths of the three geodesic boundaries. Therefore, the internal moduli of $\Sigma_{g,n}$ come from some additional freedom in combining the individual building blocks.

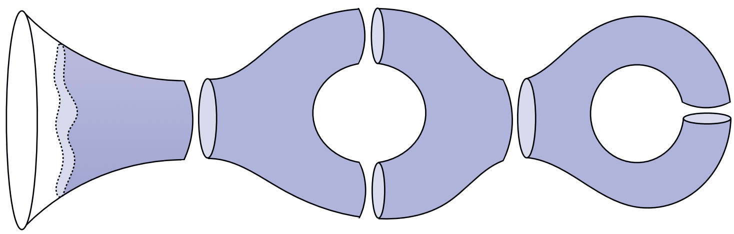


Figure 2.4: *The pair-of-pants decomposition. A Riemann surface of genus $g = 2$ with $n = 1$ boundary can be decomposed into $2g - 2 + n = 3$ copies of the pair-of-pants geometry. Note that this decomposition is by no means unique.*

Similar to the double trumpet computation, the different ways of combing the pieces are parametrized by a set of length and twist coordinates. For example, by changing the length $\tilde{\ell}_i$ of some internal gluing boundary we obtain a different hyperbolic structure on the glued surface. One could also imagine twisting one of the boundary components before gluing back, which leads another degree of freedom. Therefore, instead of the $3g-3+n$ internal boundary lengths $\tilde{\ell}_i$ we require some extra $3g-3+n$ parameters τ_i which specify the twist along the gluing curves. This gives a homeomorphism of Teichmüller space $\mathcal{T}(\Sigma_{g,n})$ with $\mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$.

The length and twist parameters $\left(\tilde{\ell}_1, ..., \tilde{\ell}_{3g-3+n}, \tau_1, ..., \tau_{3g-3+n}\right)$ form a local coordinate system known as Fenchel-Nielsen coordinates. The external boundaries are assumed to have fixed length. Moreover, the twist parameter τ takes values in $\mathbb{R}$ so both clockwise and counterclockwise rotations are allowed. Given a choice of pair-of-pants decomposition there is a natural symplectic form such that length and twist parameters are dual variables. The Weil-Petersson (WP) symplectic form is given by

$$\Omega_{\mathrm{WP}} = \frac{1}{2} \sum_{i=1}^{3g-3+n} d\tilde{\ell}_i \wedge d\tau_i \ . \tag{2.1.19}$$

Note that the pair-of-pants decomposition we used in the definition of the symplectic form is by no means unique: There are different ways of decomposing a given surface into pair-of-pants building blocks. However, it turns out that Ω_{WP} is independent of the pants decomposition [68]. Therefore, one can think about the moduli space integral in the JT path integral as defined by the measure

$$d(\mathrm{moduli})_{g,n} = \frac{1}{k!} \Omega_{\mathrm{WP}}^k \tag{2.1.20}$$

where $k = 3g-3+n$. It is the generalization of (2.1.17).

There is a subtlety associated to the expression in (2.1.20). Naively integrating over the full space of length and twist parameters overcounts the degrees of freedom. Instead, one should be careful about the correct notion of diffeomorphism invariance in the gravitational path integral. The definition we used above involved only diffeomorphisms in the path component of the identity. There is however a larger class of diffeomorphisms, the so-called mapping classes, which provide a finer equivalence relation on the space of metrics. Formally, the mapping class group $\mathrm{MCG}_{g,n}$ of a surface $M = \Sigma_{g,n}$ is defined via the short exact sequence

$$0 \to \mathrm{Diff}_0(\Sigma_{g,n}) \to \mathrm{Diff}(\Sigma_{g,n}) \to \mathrm{MCG}_{g,n} \to 0 \ . \tag{2.1.21}$$

It consists of the large diffeomorphisms of the space time which are not continuously connected to the identity. In analogy with the situation for the torus, they are also referred to as modular transformations. We now define the moduli space of Riemann surfaces as the quotient[3]

$$\mathcal{M}_{g,n} \equiv \mathcal{T}(\Sigma_{g,n})/\mathrm{MCG}_{g,n} \ . \tag{2.1.22}$$

We will often write $\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(\ell_1, ..., \ell_n)$ to make the dependence on the external boundary lengths manifest.

For now it is sufficient to mention that to get a sensible (finite) answer we should only integrate over the fundamental domain $\mathcal{M}_{g,n}$ by modding out the action of the mapping class group elements. Importantly, the WP symplectic form is still defined on this domain. This gives the proper diffeomorphism invariant definition of the space of all hyperbolic metrics which should be used in the gravitational path integral. Integrating with respect to (2.1.20) one obtains the symplectic volumes of the moduli space

$$V_{g,n}(\ell_1, ..., \ell_n) = \mathrm{vol}(\mathcal{M}_{g,n}(\ell_1, ..., \ell_n)) \ , \tag{2.1.23}$$

the so-called Weil-Petersson (WP) volumes. This explains how the expression (2.1.7) arises from a path integral computation.

We would like to stress that the above computations do not involve any saddle-point approximations. Instead, the quantity $Z_{g,n}(\beta_1, ..., \beta_n)$ expresses an exact path integral involving all possible off-shell configurations. In fact, the only configuration which is a true classical solution of the theory (solving the equations of motions (2.1.2) for both the metric $g_{\mu\nu}$ and for the dilaton ϕ) is the hyperbolic disk.

2.1.4 Recursion relations

The problem of computing the WP volumes, and thereby the Euclidean path integral in JT gravity, directly from the moduli space definition (2.1.23) is very hard. It was therefore all the more surprising that Maryam Mirzakhani [69, 70] was able to derive a set of recursion relations for the WP volumes, which could be used to compute them up to arbitrary high order in the genus and number of boundaries of the Riemann surface.

Let us for concreteness consider a (bordered) Riemann surface $M = \Sigma_{g,n}$. The idea is, roughly speaking, that by stripping off a pair-of-pants from M the moduli space integral reduces to an integral over WP volumes of 'smaller' surfaces, i.e., with larger Euler characteristic χ. When one removes a pair-of-pants geometry from M, three distinct things can happen: Either it leaves the surface connected,

[3]The action of $\mathrm{MCG}_{g,n}$ on a given surface extends to a properly discontinuous action on the Teichmüller space $\mathcal{T}(\Sigma_{g,n})$, so that the quotient defines an orbifold.

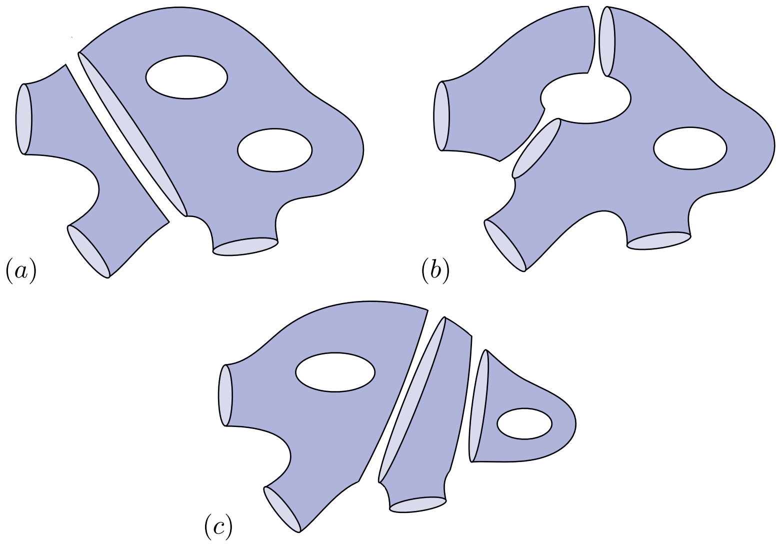


Figure 2.5: *Removing a pair-of-pants from a surface Σ can lead to three distinct scenarios. (a) The stripped surface Σ' has one boundary less than Σ, but the genus stays the same. (b) The surface Σ' has one boundary more than Σ, but the genus goes down by one. (c) The surface Σ' splits into two disconnected pieces, such that the genera and external boundaries of Σ get distributed among the two parts.*

in which case it can leave behind one or two boundaries, or it separates the surface into two disconnected pieces. The three outcomes are depicted in Figure 2.5.

One would perhaps expect that the computation of the WP volumes is similar to the double-trumpet computation (2.1.18), where we split the cylinder into two trumpets and computed the partition function using a simple gluing integral over the internal modulus ℓ. However, that case was really special: The MCG acts in a simple way by identifying $\tau \sim \tau + \ell$. In general, however, the MCG acts highly non-trivially, and we have to be more careful. The problem is that the different decompositions in Figure 2.5 are not invariant under the MCG. To describe the quotient, one would have to find a fundamental domain for the MCG, but this is a hard problem.

From a physical point of view this statement amounts to the fact that in a quantum gravitational theory it is not natural to fix a particular time slice: One rather wants to include a sum over all possible slices. By choosing a particular decomposition we necessarily break some of the diffeomorphism invariance in the path integral over the full manifold: The diffeomorphism group of the full spacetime is larger than the combined diffeomorphism groups of the separate pieces. Therefore, the

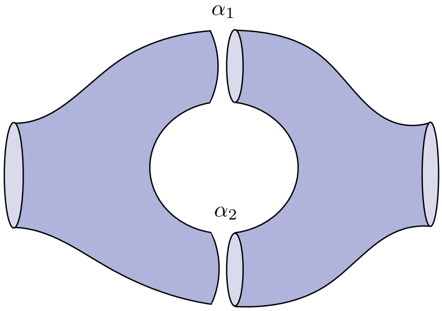


Figure 2.6: *A 'time slice' in the surface $\Sigma_{1,2}$. It gets cut along two internal geodesics α_1, α_2, and decomposes into two copies of the pair-of-pants geometry. The moduli space of the full surface does not factorize nicely.*

gluing prescription for gravity has to be rather non-trivial.

As an illustrative example, we consider the surface $\Sigma_{1,2}$ and a decomposition as given in Figure 2.6, where we have written α_1 and α_2 for the two dividing cycles. Accordingly, we have decomposed

$$\Sigma_{1,2} = \Sigma_{0,3} \cup_{\alpha_1,\alpha_2} \Sigma_{0,3} \ , \tag{2.1.24}$$

in terms of two copies of the pair-of-pants geometry. Naively cutting open the path integral would lead to a formula of the form

$$V_{1,2}(\ell_1,\ell_2) \stackrel{!}{=} \int_0^\infty d\ell\,\ell \int_0^\infty d\ell'\,\ell'\,V_{0,3}(\ell_1,\ell,\ell')V_{0,3}(\ell,\ell',\ell_2) \ , \tag{2.1.25}$$

where we integrate over the internal lengths $\ell = \ell(\alpha_1), \ell' = \ell(\alpha_2)$.

The problem with (2.1.25) is that the geometric moduli of a spacetime with handle do not factorize in the way described. In fact, the right-hand side is divergent: By naively cutting open the manifold we have broken part of this diffeomorphism invariance that makes the left-hand side finite. As comparison, one would not have this issue in a purely topological quantum field theory: For example, in the topological BF theory[4] on some surface $\Sigma_{1,2}$ with two external holonomies V_1, V_2 we have the following identity:

$$V_{1,2}^{\mathrm{BF}}(V_1,V_2) = \int dU_1 \int dU_2\, V_{0,3}^{\mathrm{BF}}(V_1,U_1,U_2)V_{0,3}^{\mathrm{BF}}(V_2,U_1^{-1},U_2^{-1}) \ . \tag{2.1.26}$$

[4]One can write JT gravity without large diffeomorphisms in terms of topological BF theory, by mapping the metric g to a gauge field A and the dilaton Φ to the scalar B in the adjoint representation of a suitable non-compact gauge group. See for example [71].

This captures the idea that we can cut along a non-contractible cycle and insert a complete set of states to split the Euclidean path integral into smaller building blocks (see, e.g., [72]). However, when large diffeomorphisms are included the moduli space cannot be expressed in terms of the moduli spaces of the building blocks in such a simple way. Instead, one needs a correction to account for the sum over modular images.

The idea of Mirzakhani for making the decomposition invariant under the MCG was to write the constant function ℓ on the Teichmüller space as a weighted sum over MCG orbits of simple closed curves which bound a pair-of-pants. Instead of summing over all ways to strip off a pair-of-pants, we can use this 'resolution of the identity' to first pick a single pair-of-pants and then sum over orbits of the MCG action, weighted by some functions on $\mathcal{T}_{g,n}$. Mirzakhani called this 'resolution of the identity' the generalized McShane identity[5]. Integrating the sum leads to an integral over the moduli space of the stripped surface. This establishes a recursion relation for the volumes $V_{g,n}$.

The starting point for Mirzakhani's recursion will be two pieces of 'initial data': the volume of the moduli space of a pair-of-pants and of a torus with one boundary,

$$V_{0,3}(\ell_1,\ell_2,\ell_3) = 1 \;, \qquad V_{1,1}(\ell) = \frac{1}{24}\left(\ell^2 + 4\pi^2\right) \;. \tag{2.1.27}$$

From these initial data it is possible to compute all $V_{g,n}$ recursively. The result, obtained by Mirzakhani, from integrating the generalized McShane identity is:

$$\begin{aligned}
\ell_1 V_{g,n}(\boldsymbol{\ell}) = &\sum_{j=2}^{n} \int_0^\infty dl\, l\, \mathcal{F}_2(\ell_1,\ell_j,l)\, \underbrace{V_{g,n-1}(l,\widehat{\boldsymbol{\ell}_j})}_{(a)} \\
&+ \frac{1}{2}\int_0^\infty dl\, l \int_0^\infty dl'\, l'\, \mathcal{F}_1(\ell_1,l,l')\Big[\underbrace{V_{g-1,n+1}(l,l',\boldsymbol{\ell}_I)}_{(b)} \\
&\qquad + \sum_{\substack{g_1+g_2=g\\ J_1\sqcup J_2=I}} \underbrace{V_{g_1,1+|J_1|}(l,\boldsymbol{\ell}_{J_1})V_{g_2,1+|J_2|}(l',\boldsymbol{\ell}_{J_2})}_{(c)}\Big] \;.
\end{aligned} \tag{2.1.28}$$

We have used multi-index notation $\boldsymbol{\ell}_I = (\ell_2,\ldots,\ell_n)$ and $I = (i_2,\ldots i_n)$ with length $|I| = n-1$. The sum in the last term of is over all ways to partition I into subsets J_1 and J_2, and over all distributions of the genus g into a sum $g_1 + g_2$. We have

[5]This name refers to a similar identity on the Teichmüller space of Riemann surfaces with marked points discovered by McShane [73], generalized by Mirzakhani to the case were the marked points are blown up to geodesic boundaries.

labelled each terms by the corresponding situation in Figure 2.5. The functions $\mathcal{F}_1$ and $\mathcal{F}_2$ are determined by the hyperbolic geometry of a pair-of-pants. Explicitly, they are given by

$$\mathcal{F}_1(l_1, l_2, l_3) \equiv -2 \log \left(\frac{1 + e^{-(l_3+l_2+l_1)/2}}{1 + e^{-(l_3+l_2-l_1)/2}} \right) , \tag{2.1.29}$$

and

$$\mathcal{F}_2(l_1, l_2, l_3) \equiv \frac{1}{2} \left(\mathcal{F}_1(l_1, l_2, l_3) + \mathcal{F}_1(l_1, -l_2, l_3) \right) . \tag{2.1.30}$$

A useful interpretation of the recursion relation (2.1.28) is in terms of the splitting and joining of closed strings. The right-hand-side describes the different ways the string can split into two strings ($\mathcal{F}_1$), or join with another string ($\mathcal{F}_2$). From this point of view, the JT gravity universes play the role of a string world-sheet, and the recursive structure resembles that of a Dyson-Schwinger equation in the corresponding string field theory. We will make this analogy precise in Chapter 3.

Having described the bulk contributions in terms of WP-volumes, the final step in the computation of the JT path integral is to add the trumpets. To each geodesic boundary, we glue a trumpet in the same way that we did for the cylinder, by integrating over the length and twist coordinates τ_i and ℓ_i at each boundary. Since neither Z_{trumpet} nor $V_{g,n}$ depends on τ_i, the integral over the twist parameter just gives a factor of ℓ_i for each trumpet. Taking these things into account, we obtain (2.1.7). This finishes our computation of the Euclidean path integral in JT gravity.

2.2 Random matrix theory

Let us now give some more background on random matrix theory, and its relation to the path integral in JT gravity. For a more extensive account of matrix integrals one can consult the recent review [74], and references therein. An important technique that will be highlighted is the topological recursion formalism, that can be used to solve the matrix integral recursively.

2.2.1 Eigenvalue repulsion

A Hermitian matrix integral is given by

$$Z = \int dH \, \exp(-L \, \mathrm{tr} \, V(H)) , \tag{2.2.1}$$

where dH is the flat measure on the space of $L \times L$ Hermitian matrices and $V(H)$ is some potential function. The above formula should be interpreted as giving a probability measure

$$p(H) \equiv \frac{1}{Z}\exp(-L\,\mathrm{tr}\,V(H)) \ , \tag{2.2.2}$$

for some ensemble of random matrices. One can also consider different types of matrix ensembles, where one, for example, considers the space of symmetric matrices, instead of Hermitians ones. This would lead to the Gaussian Orthogonal Ensemble (GOE), instead of the Gaussian Unitary Ensemble (GUE).

Notice that the above probability distribution (2.2.2) is invariant under transformations of the form $H \mapsto UHU^\dagger$ for a unitary matrix U. This enables us to express the probability measure in terms of the eigenvalues of H. Suppose that we have diagonalized H via some unitary $H = U\Lambda U^\dagger$, where $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_L)$ is a diagonal matrix. We would like to compute the measure under the change of coordinates $H \mapsto (\Lambda, U)$. Written in matrix elements we have the following relation

$$dM_{ij} = (\lambda_j - \lambda_i)dV_{ij} + \delta_{ij}d\lambda_i \ , \tag{2.2.3}$$

where $dV = U^\dagger dU$ is anti-Hermitian. The Jacobian of the coordinate transformation is now readily computed to be given by the square of the Vandermonde determinant:

$$\Delta^2(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)^2 \ . \tag{2.2.4}$$

After integrating out the the space of unitaries dU (which gives a finite volume that can be absorbed in the normalization constant) one obtains a joint probability distribution for the eigenvalues

$$p(\lambda_1, \ldots, \lambda_L) = \Delta^2(\lambda)\exp\left(-L\sum_{i=1}^{L} V(\lambda_i)\right) . \tag{2.2.5}$$

Note that a single eigenvalue is correlated to all others through the Vandermonde determinant $\Delta^2(\lambda)$. This terms acts like logarithmic potential that is responsible for a repulsive force between two eigenvalues when they get very close. The logarithmic interaction prevents the eigenvalues to localize as $L \to \infty$ and gives a continuous spectrum that is spread out in some way. Eugene Wigner famously computed the resulting density of states in the Gaussian case with potential

$$V(H) = \frac{1}{2}H^2 \ . \tag{2.2.6}$$

In that case, the spectral density of a single eigenvalue satisfies the following limiting behaviour:

$$\rho(\lambda) = \frac{1}{L}\left\langle \sum_{i=1}^{L} \delta(\lambda - \lambda_i) \right\rangle \longrightarrow \rho_0(\lambda) = \frac{1}{\pi}\sqrt{4-\lambda^2}\ , \tag{2.2.7}$$

as $L \to \infty$. This is known as Wigner's semicircular[6] law.

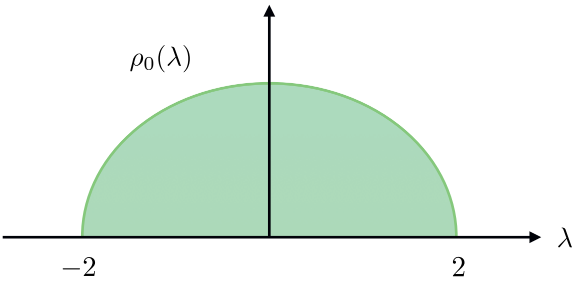


Figure 2.7: *Wigner's semicircular law.*

2.2.2 The resolvent trick

The result (2.2.7) can be obtained using a trick involving some complex analysis. We consider the resolvent function which is defined as

$$R(x) \equiv \left\langle \frac{1}{L}\mathrm{tr}\left(\frac{1}{x-H}\right) \right\rangle\ . \tag{2.2.8}$$

The expectation value is taken with respect to the ensemble (2.2.2). One can write equivalently

$$R(x) = \int_{\mathbb{R}} d\lambda\, \frac{\rho(\lambda)}{x-\lambda}\ , \tag{2.2.9}$$

which has a good $L \to \infty$ limit. The right-hand side of (2.2.9) is the Stieltjes transform of the spectral density $\rho(\lambda)$ and, by general arguments, it defines a multi-valued function on the complex plane with a branch cut along the support of $\rho(\lambda)$. For large $x \to \infty$ it contains the information of all the moments of the density via a geometric series expansion.

If we know the spectral density we can compute the resolvent through the relation

[6] Unfortunately, the shape of the density function is actually a semi-ellipse and not a semi-circle.

given in (2.2.9). The converse is true as well. We have

$$R(x \pm i\epsilon) = \int_{\mathbb{R}} d\lambda \, \frac{\rho(\lambda)(x-\lambda)}{(x-\lambda)^2+\epsilon^2} \mp i \int_{\mathbb{R}} d\lambda \, \frac{\rho(\lambda)\epsilon}{(x-\lambda)^2+\epsilon^2} \; . \tag{2.2.10}$$

Taking the limit $\epsilon \downarrow 0$, it follows that the real part of $R(x \pm i\epsilon)$ becomes precisely the principal value of the integral $R(x)$, while the imaginary part picks out the spectral density[7]. To be precise

$$\lim_{\epsilon \downarrow 0} R(x \pm i\epsilon) = \mathrm{Pr} \int_{\mathbb{R}} \frac{\rho(\lambda)}{x-\lambda} d\lambda \mp i\pi\rho(x) \; . \tag{2.2.11}$$

Therefore, we obtain the formula

$$\rho(x) = \frac{1}{2\pi i} \lim_{\epsilon \downarrow 0} \left(R(x-i\epsilon) - R(x+i\epsilon) \right) \; . \tag{2.2.12}$$

To summarize, the behavior of the resolvent along the branch cut specifies the spectral density.

We will now show that the resolvent satisfies an algebraic equation from which it can be solved exactly. The arguments are easily generalized to arbitrary matrix potentials $V(H)$, but for simplicity we will assume that we work in the Gaussian model. The idea is to solve the $L \to \infty$ model via a stationary phase approximation where the integral localizes to the extrema of the effective potential

$$W \equiv \frac{1}{2} \sum_{i=1}^{L} \lambda_i^2 - \frac{2}{L} \sum_{i<j} \log(\lambda_i - \lambda_j) \; . \tag{2.2.13}$$

The stationary point are solutions to the equation

$$\frac{\partial W}{\partial \lambda_i} = \lambda_i - \frac{2}{L} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = 0 \; , \tag{2.2.14}$$

Multiplying this expression by $\frac{1}{L}\frac{1}{x-\lambda_i}$ and summing over i one obtains

$$\frac{1}{L} \sum_{i=1}^{L} \frac{\lambda_i}{x-\lambda_i} = \frac{2}{L^2} \sum_{i=1}^{L} \sum_{j \neq i} \frac{1}{x-\lambda_i} \frac{1}{\lambda_i - \lambda_j} \; . \tag{2.2.15}$$

The left-hand side can be modified by adding and subtracting a factor x in the

[7] Here we use the expression $\lim_{\epsilon \downarrow 0} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2} = \delta(x)$ for the nascent delta function.

numerator, while the right-hand side is simplified by observing that

$$\left(\frac{1}{x-\lambda_i}-\frac{1}{x-\lambda_j}\right)\frac{1}{\lambda_i-\lambda_j}=\frac{1}{x-\lambda_i}\frac{1}{x-\lambda_j}\ . \tag{2.2.16}$$

The result of these manipulations is

$$\frac{1}{L}\sum_{i=1}^{L}\frac{x}{x-\lambda_i}-1=\left(\frac{1}{L}\sum_{i=1}^{L}\frac{1}{x-\lambda_i}\right)^2-\frac{1}{L}\frac{d}{dx}\left(\frac{1}{L}\sum_{i=1}^{L}\frac{1}{x-\lambda_i}\right)\ . \tag{2.2.17}$$

Because the derivative term is subleading in $1/L$ it vanishes when taking the limit $L\to\infty$, and by taking expectation values on both sides we end up with an algebraic equation for the resolvent:

$$R(x)^2-xR(x)+1=0\quad\longrightarrow\quad R(x)=\frac{x}{2}\pm\sqrt{\frac{x^2}{4}-1}\ . \tag{2.2.18}$$

As a last step we can compute the spectral density via (2.2.12), which precisely reproduces Wigner's semicircular law as given in (2.2.7).

The above stationary phase procedure for obtaining the resolvent can be easily generalized to matrix ensembles with arbitrary potentials $V(H)$. If we go through similar steps, it is not hard to check that the resolvent takes the general form

$$R(x)=\frac{V'(x)}{2}\pm\sqrt{\left(\frac{V'(x)}{2}\right)^2-P(x)}\ , \tag{2.2.19}$$

where $P(x)$ is some polynomial of degree $d-2$ where d is the degree of $V(x)$. It is defined as the limit[8]

$$P(x)=\lim_{L\to\infty}\left\langle\frac{1}{L}\sum_{i=1}^{L}\frac{V'(x)-V'(\lambda_i)}{x-\lambda_i}\right\rangle\ . \tag{2.2.20}$$

The above results are obtained in the strict $L\to\infty$ limit, but it turns out that the geometry of the saddle point solution contains more information about the matrix model, in the sense that one can use it to compute correlation functions recursively, using a technique called topological recursion.

[8] In practice, however, one hardly uses the above formula when trying to compute $P(x)$. Instead, one uses the asymptotic behaviour of $R(x)$ for large $x\to\infty$, which fixes this polynomial.

2.2.3 Topological recursion

It is possible to completely solve the matrix model in a $1/L$-expansion due to a set of loop equations [75] that provide relations between correlation functions. These loop equations are obtained through an integration by parts inside the matrix integral, and can be used to efficiently compute correlation functions order by order in a large L expansion. The structure of the loop equations in a Hermitian matrix model were streamlined [76] into the framework of topological recursion [77].

We are mostly interested in (connected) correlation functions of resolvent operators

$$R(x_1, \ldots, x_n) \equiv \frac{1}{L^n} \left\langle \mathrm{tr} \left(\frac{1}{x_1 - H} \right) \cdots \mathrm{tr} \left(\frac{1}{x_n - H} \right) \right\rangle \ , \tag{2.2.21}$$

and their corresponding $1/L$-expansion [78] in terms of 't Hooft diagrams [79]:

$$R(x_1, \ldots, x_n) = \sum_{g=0}^{\infty} \frac{R_{g,n}(x_1, \ldots, x_n)}{L^{2g+n-2}} \ . \tag{2.2.22}$$

Note that the resolvent (2.2.19) is a multi-valued function (due to the presence of the square root) and should therefore be understood as living on some algebraic curve $\mathscr{S}$, the so-called spectral curve. In the case at hand, it is given by the following algebraic equation

$$\mathscr{S}: \quad y^2 = \frac{V'(x)^2}{4} - P(x) \ . \tag{2.2.23}$$

More abstractly, we have a compact Riemann surface with two analytical functions x and y on an open domain, that satisfy a relation of the form $H(x, y) = 0$, which defines the spectral curve $\mathscr{S}$. It is a branched cover of the spectral x-plane, with branch points a_i defined by $dx(a_i) = 0$. The covering map corresponds to the projection to the x-axis, which for our purposes is two-to-one. The two 'sheets' of the double cover are exchanged in the neighborhood of a branch point by a local involution.

The geometry of the spectral curve, combined with the associated Bergmann kernel $\mathscr{B}$, are the input for the topological recursion. The Bergmann kernel $\mathscr{B}$ of $\mathscr{S}$ is defined by the property that, in local coordinates z, w, it is the unique bilinear differential with a double pole at $z = w$ and no other poles. We will consider the case in which $\mathscr{S}$ is (topologically) the Riemann sphere

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \ . \tag{2.2.24}$$

In that case, the Bergmann kernel is simply given by[9]

$$\mathscr{B}(z_1,z_2)=\frac{dz_1\,dz_2}{(z_1-z_2)^2}\ . \tag{2.2.25}$$

The topological recursion associates to the spectral curve data, a set of so-called symplectic invariants $\omega_{g,n}$ on the spectral curve. They are defined recursively, starting from the initial data

$$\omega_{0,1}(z)\equiv y(z)dx(z)\ ,\qquad \omega_{0,2}(z_1,z_2)\equiv\mathscr{B}(z_1,z_2)\ . \tag{2.2.26}$$

The recursion kernel is defined by

$$\mathcal{K}(z_0,z)\equiv\frac{\frac{1}{2}\int_{\tilde{z}}^{z}\mathscr{B}(z_0,\,\cdot\,)}{\big(y(z)-y(\tilde{z})\big)dx(z)}\ . \tag{2.2.27}$$

Here, the coordinates z,w and the involution $z\to\widetilde{z}$ are defined locally near a branch point a_i. Moreover, $\frac{1}{dx(z)}$ denotes the contraction with the vector field $\left(\frac{dx}{dz}\right)^{-1}\partial_z$. The notation $\int\mathscr{B}(z_0,\,\cdot\,)$ means that we integrate only with respect to the second argument. This makes $\mathcal{K}(z_0,z)$ a tensor of the type $dz_0\otimes\partial_z$. In other words, when acting on a multilinear differential $dz_1\otimes\cdots\otimes dz_n$, it removes a factor of dz and tensors with dz_0.

The topological recursion then produces multi-differentials of the form

$$\omega_{g,n}(z_1,\ldots,z_n)=\mathcal{W}_{g,n}(z_1,\ldots,z_n)\,dz_1\otimes\cdots\otimes dz_n\ , \tag{2.2.28}$$

which are defined recursively by taking residues at the branch points,

$$\begin{aligned}\omega_{g,n+1}(z_0,z_I)=\sum_i\operatorname*{Res}_{z\to a_i}\mathcal{K}(z_0,z)\Big[&\underbrace{\omega_{g-1,n+2}(z,\widetilde{z},z_I)}_{(b)}\\ &+\sum_{\substack{g_1+g_2=g\\ J_1\sqcup J_2=I}}{}^{'}\underbrace{\omega_{g_1,1+|J_1|}(z,z_{J_1})\,\omega_{g_2,1+|J_2|}(\widetilde{z},z_{J_2})}_{(a)+(c)}\Big]\ .\end{aligned} \tag{2.2.29}$$

We denote $z_J=(z_j)_{j\in J}$ and the sum in the last term goes over all ways to

[9] If $\mathscr{S}$ is a higher genus Riemann surface, one additionally needs to specify a basis of non-contractible A- and B-cycles on $\mathscr{S}$, and require that the periods of the Bergmann kernel $\mathscr{B}$ vanish on the A-cycles. This requirement ensures that $\mathscr{B}$ is unique: If there were two Bergmann kernels, their difference would be holomorphic and therefore constant; vanishing of the A-periods then implies that the constant is zero.

partition the multi-index $I = (i_1, \ldots, i_n)$ into subsets J_1 and J_2, and over all ways to distribute g into $g_1 + g_2$. The prime indicates that terms involving $(g, n) = (0, 1)$ should be excluded from the summation.

The precise relation between the Hermitian matrix model and the topological recursion is that the latter with the choice of spectral curve given by (2.2.23) computes terms in the genus expansion of the resolvents correlation functions if we indentify:

$$\mathcal{W}_{g,n}(z_1, \ldots, z_n) = 2^n z_1 \cdots z_n R_{g,n}(z_1^2, \ldots, z_n^2) \ . \tag{2.2.30}$$

Therefore, the topological recursion can be used, in principle, to solve the matrix integral recursively in the large L expansion.

There is a structural similarity between the topological recursion (2.2.29) and Mirzakhani's recursion relations. The terms appearing on the right-hand side of the topological recursion are packaged in a way similar to Mirzakhani's recursion, if we identify g and n with the genus and number of boundaries, respectively. To make the comparison more transparent, we have labelled the terms by the three scenarios (a), (b) and (c) which are depicted in Figure 2.5. The contact term (a), which corresponds to the joining of two 'JT universes', is incorporated in the topological recursion (2.2.29) as the $(g_1, g_2) = (g, 0)$ term of the primed sum. At each recursion step, this term is the only one that contains the Bergmann kernel $\mathscr{B} = \omega_{0,2}$.

There is a special case in which the topological recursion is indeed equivalent to Mirzakhani's recursion. The relevant spectral curve, that we will refer to as the JT spectral curve, is given by

$$\mathscr{S}_{\mathrm{JT}}: \quad x(z) = z^2 \ , \quad y(z) = \frac{1}{4\pi} \sin(2\pi z) \ . \tag{2.2.31}$$

The function $x(z) = z^2$ gives $\mathscr{S}_{\mathrm{JT}}$ the structure of a branched double cover of the spectral x-plane. There is a single[10] branch point at $z = 0$, since $dx = 2z dz$. The branch point $z = 0$ gets mapped to $x = 0$ on the spectral plane. The involution that exchanges the sheets of the double cover is simply $z \to -z$.

To be precise, it was proven [80] that a Laplace transform of Mirzakhani's recursion gives the topological recursion in the case that the spectral curve is given by (2.2.31). The relation between the WP volumes and the symplectic invariants for

[10]Since the branch cut extends along the whole half-line, there is also a branch point at ∞. However, we will only need to know the local behavior of $y(z)$ near the branch point at $z = 0$, because the topological invariants are defined as residues at the origin.

general (g, n) is the following multi-Laplace transform:[11]

$$\mathcal{W}_{g,n}(z_1, \dots, z_n) = 2^{-\delta_{g,1}\delta_{n,1}} \int_0^\infty \prod_{i=1}^n d\ell_i \, \ell_i \, e^{-z_i \ell_i} V_{g,n}(\ell_1, \dots, \ell_n) \ . \tag{2.2.32}$$

It should be noted that the curve $H(x, y) = 0$ is not algebraic, since $\omega(x)^2$ is not a finite polynomial[12], so $\mathscr{S}_{\mathrm{JT}}$ is a non-compact Riemann surface. However, since it can be parametrized by a single variable $z \in \mathbb{P}^1$ one could possibly add a point at ∞, and argue that $\mathscr{S}_{\mathrm{JT}}$ effectively has genus zero. However, the notion of 'genus' for such non-compact Riemann surfaces is somewhat vague, and another interesting interpretation of $\mathscr{S}_{\mathrm{JT}}$ is as a Riemann surface of *infinite* genus of which infinitely many A-cycles have been pinched. This interpretation can be justified when we compare to the $(2, p)$-minimal string theory with p an odd integer, studied for example in [81]. In that case, the spectral curve was found to be an odd power $y(z) \sim z^p$, so that $H(x, y) = 0$ describes a compact Riemann surface of genus p. We can thus quite possibly see the JT spectral curve $y(z) \sim \sin(2\pi z)$, being an odd power series in z, as a 'infinite linear combination' of $(2, p)$-minimal models, as was recently advocated in [82, 83].

The reason that we effectively see a genus zero spectral curve is that all the A-cycles have been pinched to points, at the zeroes of the $\sin(2\pi z)$ where the two sheets of the double cover meet. Non-perturbative effects may cause the zeroes of $y(z)$ to 'open up', adding small corrections to the right-hand side of (2.2.31) and thereby un-pinching the A-cycles. Since $\sin(2\pi z)$ has infinitely many zeroes, this un-pinching renders $\mathscr{S}_{\mathrm{JT}}$ a genus $p \to \infty$ Riemann surface. The $p \to \infty$ limit of minimal string theory was recently studied more thoroughly in [84], where many quantities were matched to quantities in JT gravity.

2.2.4 Double-scaling and JT gravity

The observation that the recursions relations for the WP volumes are equivalent to the topological recursion relations that govern Hermitian one-matrix models led [36] to propose the holographic duality that JT gravity is dual to a double-scaled matrix integral. Since the Euclidean path integral of JT gravity can be expressed in terms of the WP volumes they argued that one we should identify JT gravity with some matrix model having the leading order spectral density

$$\rho_0(\lambda) = \frac{1}{2\pi} \sinh(2\pi\sqrt{\lambda}) \ . \tag{2.2.33}$$

[11] The factor of $\frac{1}{2}$ is related to the extra $\mathbb{Z}_2$-symmetry of the one-holed torus $(g, n) = (1, 1)$ [69].
[12] Note that $\omega(x)$ has an essential singularity at ∞.

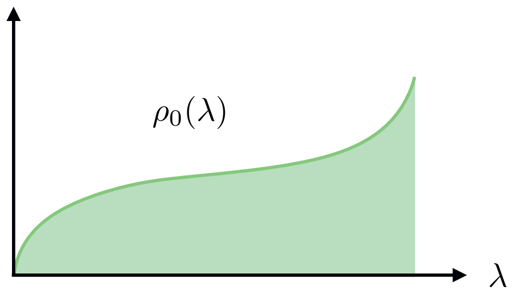


Figure 2.8: *The Schwarzian density of states.*

However, the above spectral density is unbounded, and does not arise from some Hermitian matrix integral with polynomial potential. Instead, one should interpret (2.2.33) as coming from some double-scaling procedure. The idea of double-scaling is to take some 'conventional' matrix integral with normalized density $\rho_0(\lambda)$, and take a double limit where $L \to \infty$, while simultaneously adjusting the potential in such a way that some combination of parameters is kept fixed [85–87].

A nice example to illustrate the general idea is the Airy spectral curve (see, e.g., [88, 89] for a discussion of the Airy matrix integral and its relation to pure topological gravity), that is obtained from the Gaussian ensemble via a double-scaling procedure. Let us start with the rescaled potential

$$V(H) = \frac{8}{c^2} H^2 \ , \tag{2.2.34}$$

that depends on some external parameter c. Shifting the individual eigenvalues by c we find the total eigenvalue density

$$\rho_0^{\text{total}}(\lambda) = \frac{8L}{\pi c^{3/2}} \sqrt{\lambda(1 - \lambda/c)} \ . \tag{2.2.35}$$

If we now take the double-scaling limit, where

$$c, L \to \infty \quad \text{with} \quad Lc^{-3/2} = e^{S_0}/8 \quad \text{fixed} \ , \tag{2.2.36}$$

the resulting density of states becomes

$$\rho_0^{\text{total}}(\lambda) = \frac{e^{S_0}}{\pi} \sqrt{\lambda} \ . \tag{2.2.37}$$

By double-scaling we zoom in on the tail of the eigenvalue distribution. Moreover, it effectively replaces the size of the matrix L in the $1/L$-expansion of the matrix integral by the parameter e^{S_0}.

Using (2.2.32), one finds that the path integral in JT gravity computes correlation functions in the double-scaled matrix integral of certain macroscopic loop operators

$$Z(\beta) = \mathrm{tr} e^{-\beta H} = \sum_{i=1}^{L} e^{-\beta\lambda_i} \ , \tag{2.2.38}$$

that are related to the resolvent observables by a similar Laplace transformation. This provides the schematic duality:

$$\text{Euclidean path integral of JT} \quad \cong \quad \text{double-scaled matrix integral} \ ,$$

where the symbol $\cong$ indicated that the path integral contribution of a spacetime with genus g and n boundaries in the e^{-S_0} genus expansion is mapped to the corresponding coefficient in the perturbative $1/L$-expansion of the double-scaled matrix integral correlation function:

$$Z_{g,n}(\beta_1, \ldots, \beta_n) \quad \longleftrightarrow \quad \langle Z(\beta_1) \ldots Z(\beta_n) \rangle_{g,n} \ . \tag{2.2.39}$$

Note that the higher order terms in gravity are non-perturbative, as they get multiplied by a factor $\sim e^{-1/G}$, while from the matrix integral perspective they are perturbative.

2.2.5 Spectral form factor and wormholes

We end this section with a discussion of the spectral form factor from a gravitational perspective. At early times, it should be well approximated by the disconnected contribution of two separate disks:

$$Z^{\mathsf{c}}_{0,1}(\beta + it) Z^{\mathsf{c}}_{0,1}(\beta - it) \sim \frac{1}{(\beta^2 + t^2)^{3/2}} e^{\frac{2\pi^2 \beta}{\beta^2 + t^2}} \ , \tag{2.2.40}$$

which decays to zero as $t \to \infty$. However, this is not the full answer: At some time, the connected contribution should become important. Recall that the double-trumpet partition function in JT is given by

$$Z^{\mathsf{c}}_{0,2}(\beta_1, \beta_2) = \frac{1}{2\pi} \frac{\sqrt{\beta_1 \beta_2}}{\beta_1 + \beta_2} \ . \tag{2.2.41}$$

By analytically continuing this expression to $\beta_1 = \beta + it$ and $\beta_2 = \beta - it$ we find that for large t

$$Z_{0,2}(\beta + it, \beta - it) = \frac{\sqrt{\beta^2 + t^2}}{4\pi\beta} \sim \frac{t}{4\pi\beta} \ , \tag{2.2.42}$$

which represents the linear ramp behavior that is characteristic of quantum chaotic theories at late times [26]. This surprising result is at the heart of the relation between Euclidean wormholes and quantum chaos: By including non-perturbative wormhole contributions to the computation, it is possible to reproduce the linear ramp behavior that is typical for a quantum chaotic spectrum at late times.

However, this is not the complete answer: We know that at very late times (of the order of an inverse typical level spacing) the spectral form factor stabilizes to a small non-zero value. It is has not been clear how this plateau arises from a similar computation in JT gravity, if it is possible at all. From the perspective of the matrix integral the plateau is a non-perturbative effect,[13] so this suggests that one should look for certain doubly non-perturbative effects in JT gravity. Finding a geometric interpretation of these effects is one of the main goals of Chapter 3 and Chapter 4.

[13]It manifests itself in the 'sine-kernel' [90] of the density-density correlator.

3 A field theory for baby universes

3.1 Introduction

Recently, the role of topology change in quantum gravity has found some renewed interest. In particular, questions about the definition of the gravitational path integral (GPI) and what it can tell us about the microscopic properties of gravity have resurfaced. Heuristically, the GPI is a recipe for any theory of quantum gravity that instructs us to sum over all fields of the theory, including the metric, weighted by the gravitational action. It has been a long-standing debate whether different topologies of the spacetime manifold should be included in this procedure or not, but recent developments have shown that a great deal can be learned when we do. For example, it was shown [29, 30] that one can obtain the Page curve for the entanglement entropy of Hawking radiation of an evaporating black hole by adding non-trivial topologies called 'replica wormholes' to the gravitational path integral.

How to interpret these non-trivial topologies from a microscopic point of view is still an open question, but developments of the past years have led to the following intuition: While semiclassical gravity is a low-energy effective description of some UV complete theory, the gravitational path integral still has access to some of the UV data, but only in an *averaged* sense. The non-trivial topologies now probe certain statistical correlations within the model-dependent average. Although the general mechanism is not very well-understood, this idea has been concretely realized in some controlled settings. Let us highlight two viewpoints that have been influential:

Matrix models. In 2-dimensional Euclidean Jackiw-Teitelboim (JT) gravity [31, 32] the relevant averaging procedure has been identified by Saad, Shenker and Stanford [36] in terms of a double-scaled matrix integral. Instead of a single well-defined boundary quantum system described by a Hamiltonian H, it was argued that the bulk JT gravity theory is dual to an ensemble of boundary theories,

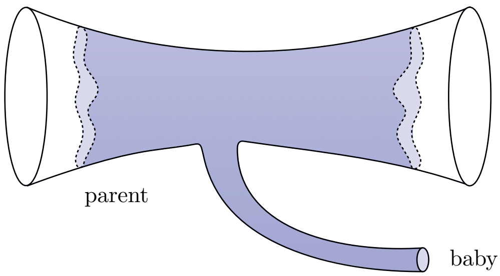


Figure 3.1: *Baby universes. A pictorial representation of a baby universe splitting off from some parent universe.*

whose Hamiltonians are random matrices drawn from some probability distribution. Each boundary theory is characterized by a partition function:

$$Z(\beta_i) = \operatorname{Tr} e^{-\beta_i H}, \quad i = 1\,, \ldots, n\,, \tag{3.1.1}$$

where the inverse temperature β_i corresponds to the (renormalized) length of the i-th boundary. This partition function becomes a random variable in an ensemble defined by a matrix integral $\langle\cdots\rangle_{\mathsf{MM}}$. The spacetime wormhole connecting n boundaries now computes the n-th connected correlation function of the random boundary partition function

$$\mathcal{Z}_{\text{wormhole}}(\beta_1, \ldots, \beta_n) = \langle \operatorname{Tr} e^{-\beta_1 H} \cdots \operatorname{Tr} e^{-\beta_n H} \rangle^{\mathsf{c}}_{\mathsf{MM}}\,, \tag{3.1.2}$$

after taking some suitable double-scaling limit of the matrix model. See, for example, [82–84, 91–99] for some related work on ensemble averaging in JT gravity, including the generalization to JT supergravity, non-perturbative effects, conical defect geometries and its relation to minimal strings and Liouville theory.

Baby universes. An interesting interpretation of the ensemble average is given by Marolf and Maxfield [100], building upon earlier ideas on spacetime wormholes [101–103]. Roughly speaking, a theory of dynamical gravity, where spacetime itself is allowed to change its topology, is most clearly formulated in a *third-quantized* picture. This means that on top of the usual rules of quantum field theory we apply another quantization to account for the dynamics of the topology change. The quantum mechanical system consisting of states labeled by these topologically distinct universes is referred to as the Hilbert space of baby universes, since in Lorentzian signature such geometries can be viewed as modeling the emission and absorption of auxiliary baby universes [104]. See Figure 3.1. Given this quantum mechanical Hilbert space, one can define certain boundary creation operators which move you from one configuration to another. The ensemble now comes from a decomposition into *α-states*, which are eigenstates obtained from simultaneously diagonalizing these operators.

In this chapter, we present a framework that naturally incorporates both viewpoints, in the case of JT gravity. Using intuition from string theory, where one can describe the topological expansion for the splitting and joining of closed strings in terms of a *string field theory*, we introduce a quantum field theory for the non-perturbative splitting and joining of baby universes. This effective description lives on an auxiliary space $\mathscr{S}_{\mathrm{JT}}$ called the *spectral curve*. The geometry of this space is determined by the leading order density of states and is given by

$$\mathscr{S}_{\mathrm{JT}}: \quad y^2 - \frac{1}{(4\pi)^2}\sin^2(2\pi\sqrt{x}) = 0 \ , \tag{3.1.3}$$

where $x, y \in \mathbb{C}$. This curve can be uniformized by a single complex coordinate z using

$$x(z) = z^2 \ , \quad y(z) = \frac{1}{4\pi}\sin(2\pi z) \ . \tag{3.1.4}$$

In the string field theory analogy, the spectral curve should be viewed as defining the target space geometry in which the JT universes (the equivalent of string world-sheets or 'JT strings') propagate. The quantum field theory living on the spectral curve now corresponds to the closed string field theory, in the sense that it describes the splitting and joining of JT universes by a cubic interaction vertex. Following [105], where a useful analogy with world-line gravity is presented (see also [106]), we use the term *universe field theory* for this description.

We will show that our universe field theory is the 2-dimensional Kodaira-Spencer (KS) theory of complex structure deformations of $\mathscr{S}_{\mathrm{JT}}$, originally found by Dijkgraaf and Vafa [37]. It is obtained as a dimensional reduction of the topological B-model closed string field theory [107] to the spectral curve. This shows that JT gravity can be understood in terms of the well-established topological string theory framework (see [108] where a similar statement was made). However, the interpretation from the gravity point of view is fundamentally different: the *perturbative* expansion in the string coupling constant λ corresponds to the *non-perturbative* genus expansion in JT gravity via the identification

$$\lambda = e^{-S_0} \ , \tag{3.1.5}$$

where S_0 is proportional to $1/G_N$. Hence, higher loop corrections to the universe field theory amplitudes correspond to non-perturbative wormhole configurations on the gravity side.

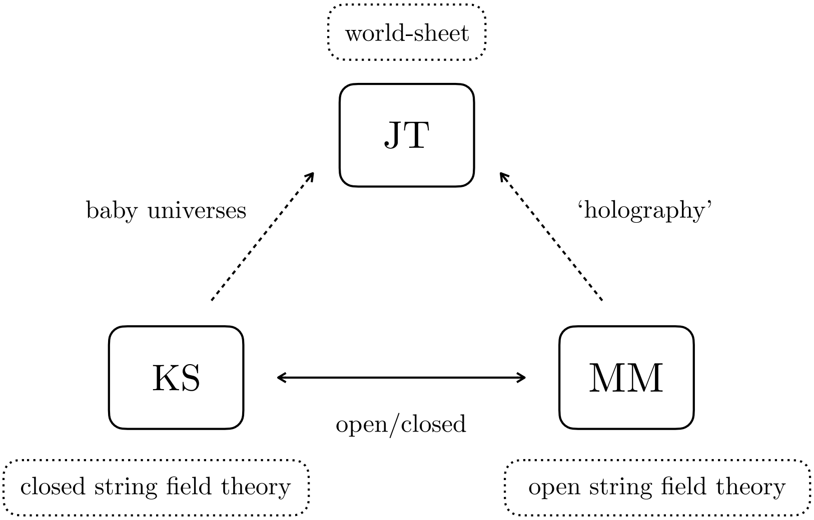


Figure 3.2: *A triangle of relations between JT, KS and MM, together with their interpretation in topological string theory. Dashed arrows depict perturbative expansions. The term 'holograpy' refers to the JT/matrix integral correspondence [36] that relates the Euclidean path integral to the double-scaled matrix integral.*

The relation with KS theory connects the JT/matrix integral correspondence to earlier work on the relation between matrix integrals and topological string theory, e.g., [38, 39]. It also nicely agrees with the viewpoint [109–112] that JT gravity is equivalent to the world-sheet topological gravity [88, 89, 113–115]. The formulation in terms of KS theory is in some ways more transparent than the matrix integral, as it is formulated directly in the double-scaling limit. Moreover, it makes the embedding in topological string theory manifest and thus provides many useful tools to study non-perturbative aspects of gravity.

More importantly, it gives another explanation for why the random matrix ensemble of [36] arises in the study of JT gravity, namely as the dual open string field theory [116, 117] through a version of the open/closed duality [118, 119]. Therefore, we provide evidence for the claim that the JT/matrix integral correspondence is a special example of a more standard open/closed duality in topological string theory. This suggests the triangle Figure 3.2 of relations between JT gravity, the matrix model (MM), and KS theory. We now outline the dictionary between the KS theory and JT gravity:

Observables. The basic field in KS theory is a $\mathbb{Z}_2$-twisted chiral boson $\mathcal{J}(z) = \partial\Phi(z)$ which parametrizes the complex structure deformations of the spectral curve. The KS field theory contains a cubic interaction that is localized on a contour around the branch point $z = 0$:

$$S_{\text{int}} = \lambda \oint \frac{dz}{2\pi i} \frac{\Phi(z)}{\omega(z)} T(z) \ , \tag{3.1.6}$$

where $T(z) = \frac{1}{2}(\mathcal{J}\mathcal{J})(z)$ is the holomorphic stress tensor, and $\omega = y(z)dx(z)$ is a holomorphic $(1,0)$-form that encodes the complex structure of $\mathscr{S}_{\text{JT}}$. We will show that the n-point function of $\mathcal{J}(z)$ in the KS theory, after an inverse Laplace transform, computes the all-genus gravitational path integral for JT gravity with n asymptotic boundaries:

$$\mathcal{Z}_{\mathsf{JT}}(\beta_1, \ldots, \beta_n) = \int_{c-i\infty}^{c+i\infty} \prod_{i=1}^{n} \frac{dz_i}{2\pi i} e^{\beta_i z_i^2} \langle \mathcal{J}(z_1) \cdots \mathcal{J}(z_n) \rangle_{\mathsf{KS}} \ . \tag{3.1.7}$$

The renormalized boundary length β_i of the i-th boundary has the interpretation of a fixed temperature in the boundary Schwarzian theory [33, 60–62, 120]. Note that the left-hand side of (3.1.7) only makes sense as a perturbative expansion in λ^{2g-2+n}, where $\lambda = e^{-S_0}$ and g is the genus of the spacetime wormhole, while the right-hand side is a correlator in a well-defined Euclidean non-gravitational QFT. Expanding the interaction vertex (3.1.6) and doing Wick contractions gives a matching expansion in $\lambda^{-\chi}$, where χ is the Euler number of the diagram. Thus, the right-hand side provides a non-perturbative completion[1] of the topological expansion of the gravitational path integral in JT gravity.

Coming back to the discussion of ensemble averaging, we see that (3.1.7) expresses the gravitational path integral as an 'average' $\langle \cdots \rangle_{\mathsf{KS}}$ of the following boundary operators:

$$Z(\beta) = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} e^{\beta z^2} \mathcal{J}(z) \ . \tag{3.1.8}$$

Recursion relations. The argument for the identification (3.1.7) is the universal recursive structure present in both descriptions. Computing the JT gravity path integral amounts to the computation of Weil-Petersson volumes $V_{g,n}(\boldsymbol{\ell})$ of the moduli space of bordered Riemann surfaces. These volumes can be found recursively, as was discovered by Maryam Mirzakhani [69, 70], by iteratively 'strip-

[1] This completion is by no means unique (for example, in defining the KS theory itself one already has to make a choice of integration contour, similar to what happens for matrix models). We will show that the KS theory contains brane-like objects, that reproduce correctly certain non-perturbative effects characteristic of matrix models (like the 'plateau' of the density-density correlator).

ping off' 3-holed spheres in a modular invariant way. Mirzakhani's recursion is related via a Laplace transform to the topological recursion relations of Eynard and Orantin [76, 77, 80] for double-scaled matrix models. In this chapter, we identify yet another recursion relation: We will show that the Schwinger-Dyson (SD) equations for $\Phi(z)$ in the KS theory imply the topological recursion relations. The SD equations can be expressed as a differential equation for the generating functional of connected correlation functions $W_{\mathsf{KS}}[\mu_{\mathcal{J}}] = -\log Z_{\mathsf{KS}}[\mu_{\mathcal{J}}]$, where $\mu_{\mathcal{J}}(z)$ is a source field for $\mathcal{J}(z)$. They take the following form:

$$\frac{\delta W_{\mathsf{KS}}}{\delta \mu_{\mathcal{J}}(z_0)}\Big|_{\chi<0} = \frac{\lambda}{4} \oint_{\gamma} \frac{dz}{2\pi i} \frac{\mathsf{G}(z_0, z)}{\omega(z)} \left[\frac{\delta^2 W_{\mathsf{KS}}}{\delta \mu_{\mathcal{J}}(z) \delta \mu_{\mathcal{J}}(z)} + \frac{\delta W_{\mathsf{KS}}}{\delta \mu_{\mathcal{J}}(z)} \frac{\delta W_{\mathsf{KS}}}{\delta \mu_{\mathcal{J}}(z)} \right] . \tag{3.1.9}$$

The details of this equation will be explained in Section 3.2.2. In particular, we show that expanding both sides in powers of λ gives the topological recursion for the symplectic invariants $\omega_{g,n}(z_1, \ldots, z_n)$, which are identified with connected correlation functions $\langle \mathcal{J}(z_1) \cdots \mathcal{J}(z_n) \rangle^{(g)}_{\mathsf{KS,c}}$. Using the map (3.1.7) at fixed genus g, this gives a recursion relation between contributions from spacetime wormholes to the full GPI. In Appendix A.2, we show that these recursion relations can be recast as a Virasoro constraint [114] in the oscillator formalism of the KS theory.

Non-perturbative effects. The topological string perspective provides a natural setting to study non-perturbative effects due to D-branes. The spectral curve can be embedded in a 6-dimensional Calabi-Yau manifold, which defines the target space of the JT string:

$$uv - y^2 + \frac{1}{(4\pi)^2} \sin^2(2\pi\sqrt{x}) = 0 . \tag{3.1.10}$$

This geometry has non-compact submanifolds $u = 0$ and $v = 0$, which can be wrapped by *branes* and *anti-branes* respectively [40]. In the KS theory, these branes can be described by a pair of complex fermions $\psi(E) = e^{\Phi(E)}$ and $\psi^\dagger(E) = e^{-\Phi(E)}$ in terms of the coordinate $E = -x$ on $\mathscr{S}_{\mathrm{JT}}$. We will identify the dual observables in JT gravity to be universes with fixed energy boundaries [121], ending on branes in the target space geometry (3.1.10). Here, we do not fix the length of the boundary metric, but we fix the dilaton and its normal derivative, which corresponds to a fixed energy E in the Schwarzian theory. The fixed energy boundary conditions are related to the asymptotically AdS_2 boundary conditions by a Legendre transform. In fact, we will extend the dictionary (3.1.7) for n boundaries with energies $E_1, \ldots, E_n$ to

$$\mathcal{Z}_{\mathsf{JT}}(E_1, \ldots, E_n) = \int_{c-i\infty}^{c+i\infty} \prod_{i=1}^{n} \frac{d\beta_i}{\beta_i} e^{\beta_i E_i} \langle Z(\beta_1) \cdots Z(\beta_n) \rangle_{\mathsf{KS}} . \tag{3.1.11}$$

Using (3.1.8), the KS observables appearing on the right-hand side of (3.1.11) can be rewritten in terms of the discontinuity of the boson Φ across the branch cut:

$$Z(E) \equiv \int_{c-i\infty}^{c+i\infty} \frac{d\beta}{\beta} e^{\beta E} Z(\beta) = \operatorname{disc}\Phi(E) = \int^{E} dE'\, \rho(E')\ . \tag{3.1.12}$$

At the last equality, the discontinuity of Φ is rewritten as an integrated density of states operator $\rho(E)$, to make clear that $Z(E)$ represents a microcanonical partition function, in the same way that $Z(\beta)$ is a canonical partition function (3.1.1) in the boundary theory. We will show that non-perturbative corrections to density correlators can be computed from insertions of the 'energy brane' operators $e^{\pm\Omega(E)}$, where $\Omega(E) \equiv 2\pi i Z(E)$. In particular, we retrieve the universal sine-kernel for the connected part of the density-density amplitude:

$$\langle \rho_{\rm np}(E_1)\rho_{\rm np}(E_2)\rangle^{\mathsf{c}}_{\mathsf{KS}} \approx -\frac{1}{\pi^2(E_1-E_2)^2}\sin^2\left(\pi\, e^{S_0}\int_{E_2}^{E_1}\rho_0(E')dE'\right)\ . \tag{3.1.13}$$

The remainder of this chapter is organized as follows:

- In Section 3.2, we introduce the KS field theory, and present a detailed derivation of the SD equations, which characterize the KS correlation functions up to all orders in perturbation theory.
- In Section 3.3, we make the connection to JT gravity. In particular, we show that the SD equations of the universe field theory coincide with the topological recursion relations, with a choice of initial conditions given by the JT spectral curve. Therefore, the diagrams of the KS theory are in one-to-one correspondence with the JT universes in the asymptotic expansion of the GPI.
- We generalize the setup in Section 3.4 to include fixed energy boundaries in JT. On the KS side of the duality we interpret these boundaries as attached to D-branes in the Calabi-Yau target space, which allows us to explore non-perturbative physics. In particular, we show how to derive the sine-kernel in the density-density correlator.
- We conclude with a discussion and a list of open questions in Section 3.5.
- Some KS calculations have been relegated to Appendix A.1. In Appendix A.2 we have worked out the relation with topological gravity in more detail, which gives another perspective on the KS/JT duality in terms of the oscillator algebra of a twisted boson, the baby universe Hilbert space and the Virasoro constraints.

3.2 A universe field theory

In Section 3.2.1 we present the universe field theory that describes dynamical topology change in JT gravity. We will argue that this is the 2-dimensional KS theory on the JT spectral curve. The precise identification follows from matching the topological recursion relations for JT gravity with the SD equations for the KS field theory, which will be derived in Section 3.2.2.

3.2.1 Kodaira-Spencer theory on the spectral curve

The KS theory on the spectral curve has the following path integral representation:

$$Z_{\mathsf{KS}}[\mu_\Phi, \mu_{\mathcal{J}}] = \int [d\mathcal{J}][d\Phi] \exp\left[-S_{\mathsf{KS}}[\Phi, \mathcal{J}] - \int_{\mathscr{S}_{\mathrm{JT}}} \mu_\Phi \Phi - \int_{\mathscr{S}_{\mathrm{JT}}} \mu_{\mathcal{J}} \mathcal{J} \right] , \quad (3.2.1)$$

where μ_Φ and $\mu_{\mathcal{J}}$ are external source fields, and the action is given by

$$S_{\mathsf{KS}}[\Phi, \mathcal{J}] = \int_{\mathscr{S}_{\mathrm{JT}}} \left[\frac{1}{2} \partial\Phi \wedge \overline{\partial}\Phi - \mathcal{J} \wedge \overline{\partial}\Phi \right] + \oint_\gamma \left[\frac{\omega\, \Phi}{\lambda} + \frac{\lambda}{2} \frac{\Phi}{\omega} \mathcal{J}^2 \right] . \qquad (3.2.2)$$

Let us explain all the terms appearing in this action. First of all, the action consists of a ‘bulk’ and a ‘boundary’ contribution: the bulk integral is over the JT gravity spectral curve $\mathscr{S}_{\mathrm{JT}}$ given in (3.1.3). We will use a uniformizing coordinate z as in (3.1.4). In particular, the relation $x = z^2$ shows that in terms of the variable z the KS theory is defined on a branched double cover of the spectral x-plane, with a branch point at $z = 0$. The boundary integral is over a closed curve γ encircling the branch point, which does not enclose any other poles or zeroes of the holomorphic $(1,0)$-form:

$$\omega = \omega(z)dz = y(z)dx(z) \ . \qquad (3.2.3)$$

We have used complex differential notation in the sense of Dolbeault cohomology, so that for example $\partial\Phi = \partial\Phi(z)dz$, and $d = \partial + \overline{\partial}$. We will always distinguish form fields and ordinary fields by writing fields with their argument and form fields without. For example, $\mathcal{J} = \mathcal{J}(z)dz$ is a $(1,0)$-form field, while $\mathcal{J}(z)$ is a function of the local coordinate z on the spectral curve. To further ease our notation, we define the integral $\int_{\mathscr{S}_{\mathrm{JT}}}$ to include a factor of $\frac{i}{2}$ to make the action real. This factor arises from the usual relation $d^2z = \frac{i}{2} dz \wedge d\overline{z}$. Similarly, we define the contour integral $\oint_\gamma$ to include a factor of $\frac{1}{2\pi i}$ to make the boundary action real. We will also often drop the wedge product when it is clear from the context.

Having set the notation, we go on to analyze the field content of the theory. There are two dynamical bosonic fields $\Phi = \Phi(z)$ and $\mathcal{J} = \mathcal{J}(z)dz$. We do not explicitly write the anti-holomorphic dependence on $\bar{z}$, but on the level of the path integral Φ and $\mathcal{J}$ are not necessarily chiral. For now, this is just a notational convenience, but we will see that on-shell Φ and $\mathcal{J}$ will be chiral fields. The source fields μ_Φ and $\mu_{\mathcal{J}}$ are $(1,1)$ and $(0,1)$-form fields, respectively. The holomorphic $(1,0)$-form ω appears in the boundary contribution to the action, and it serves to give the chiral boson $\mathcal{J}(z)$ a vacuum expectation value. The term proportional to $\frac{\Phi}{\omega}\mathcal{J}^2$ is the most interesting: This cubic interaction term encodes all the non-trivial dynamics of the splitting and joining of baby universes.

The action (3.2.2) was first written down by Dijkgraaf and Vafa [37] in the context of topological string theory. There, it was obtained by reducing the 6-dimensional KS theory of the closed string B-model developed in [107] to a chiral boson on a Riemann surface. For this reason, we have labeled the action by KS, for 'Kodaira-Spencer'. This chiral boson perspective was used, for example, in the 're-modeling the B-model' program of [122]. See also [39, 40, 123, 124] for more work on the relation between topological strings, matrix models and integrable systems.

As first observed in [37], the SD equations of the KS theory agree with the topological recursion relations. Our interpretation of this result (in addition to being more explicit in analyzing the SD equation) is novel in the sense that it replaces the worldsheet theory that underlies the topological string, by a 2-dimensional gravity that one likes to study in its own right, which we take to be JT gravity. In that sense, KS becomes a field theory for gravitational baby universes. In the next subsection, we will explain the origin of the universe field theory action (3.2.2) in topological string theory.

Topological string theory origin of S_{KS}

As stated in the introduction, we will embed the spectral curve into a non-compact Calabi-Yau manifold in the following way:

$$\mathsf{CY}: \quad uv = H(x,y) \ , \quad u,v \in \mathbb{C} \ , \tag{3.2.4}$$

where $H(x,y)$ is given by:

$$H(x,y) \equiv y^2 - \frac{1}{(4\pi)^2}\sin^2(2\pi\sqrt{x}) \ . \tag{3.2.5}$$

The submanifolds where u or v vanish correspond to the spectral curve $\mathscr{S}_{\mathrm{JT}}$: $H(x,y) = 0$, and CY can be viewed as a fiber bundle over the spectral curve. The defining relation (3.2.4) shows that CY has three complex dimensions, and the complex structure of CY is encoded in the holomorphic $(3,0)$-form:

$$\Omega_{\mathsf{CY}} = \frac{1}{u}\, du \wedge dx \wedge dy \ . \tag{3.2.6}$$

The KS field theory on CY describes deformations of the complex structure such that the cohomology class of Ω_{CY} is unchanged. Upon reduction of the theory to the base Riemann surface $\mathscr{S}_{\mathrm{JT}}$, this translates to complex structure deformations of $\mathscr{S}_{\mathrm{JT}}$ such that the holomorphic $(1,0)$-form $\omega = y\, dx$ is preserved. To see this, consider a 3-cycle $\widetilde{C}$ in CY. For a Calabi-Yau modeled on a Riemann surface, there is a one-to-one correspondence between 3-cycles in CY and 1-cycles on the Riemann surface [124]. Explicitly, a 3-cycle $\widetilde{C}$ can be made by fibering an S^1 over a disk D, whose boundary ∂D is a non-trivial 1-cycle C on the Riemann surface. Computing a period of Ω_{CY} on $\widetilde{C}$ then reduces to a period integral of ω on C:

$$\int_{\widetilde{C}} \Omega_{\mathsf{CY}} = \int_{\widetilde{C}} \frac{du \wedge dx \wedge dy}{u} = \frac{1}{2\pi i} \oint_{S^1} \frac{du}{u} \int_D dx \wedge dy = \int_C y\, dx \ . \tag{3.2.7}$$

At the last equality, we have evaluated the residue at $u = 0$, followed by an application of Stokes' theorem. The complex structure deformations of $\mathscr{S}_{\mathrm{JT}}$ are captured by deforming the $\overline{\partial}$ operator:

$$\overline{\partial} \to \overline{\partial} - \upmu\, \partial \ , \tag{3.2.8}$$

where $\upmu = \upmu_{\bar{z}}^{z}\, d\bar{z} \otimes \partial_z$ is a so-called Beltrami differential. In the deformed complex structure, a function f is holomorphic if and only if $(\overline{\partial} - \upmu\, \partial) f = 0$. As in the 6-dimensional KS theory [107], the 2-dimensional KS theory is the quantization of fluctuations of the complex structure such that the cohomology class of ω is unchanged. That is, we demand that there is a vector field ξ such that

$$\upmu = \overline{\partial} \xi \quad \text{and} \quad \delta_\xi \omega = d\Phi \ , \tag{3.2.9}$$

where Φ is the basic field of the KS action (3.2.2). Explicitly, the variation of ω under a diffeomorphism ξ is found by taking the Lie derivative in the direction of ξ:

$$\delta_\xi \omega \equiv \mathcal{L}_\xi \omega = d(\iota_\xi \omega) - \iota_\xi d\omega \ . \tag{3.2.10}$$

Now we use that ω is a holomorphic $(1,0)$-form, so that

$$d\omega = (\overline{\partial} + \partial)\omega = \overline{\partial}\omega = 0 \ . \tag{3.2.11}$$

Here, we used that $\partial\omega = 0$: there are no $(2,0)$-forms on a Riemann surface. Comparing (3.2.9) and (3.2.10) we conclude that $\iota_\xi \omega = \Phi$ up to a d-closed form, which we can conveniently write as:

$$\xi = \frac{\Phi}{\omega} \ . \tag{3.2.12}$$

So we see that the Beltrami differential μ depends on the field Φ. Imposing that $\delta_\xi \omega$ is holomorphic in the deformed complex structure implies:

$$(\overline{\partial} - \mu\,\partial)\delta_\xi \omega = (\overline{\partial} - \mu\,\partial)\partial\Phi = 0 \ . \qquad (3.2.13)$$

This should be implemented in the field theory as an equation of motion. So we see that the action should contain the term:

$$\frac{1}{2}\int_{\mathscr{S}_{\mathrm{JT}}} \partial\Phi \wedge (\overline{\partial} - \mu\,\partial)\Phi \ . \qquad (3.2.14)$$

This contains a kinetic term for Φ, as well as the interaction:

$$\int d^2 z\, \mu_{\bar{z}}^{z}\, T(z) \ , \qquad (3.2.15)$$

where we have written $T(z)$ for the stress tensor $T(z) \equiv \frac{1}{2}\partial\Phi(z)\partial\Phi(z)$. In the quantum theory, $T(z)$ is normal ordered in the usual way using a point-splitting regularization, i.e., by subtracting the divergent part of the OPE. Plugging in the expression for $\mu = \overline{\partial}\xi$ explains the origin of the cubic interaction in the KS action (3.2.2). Before going into the details, let us pause and give some more intuition for why we have found the interaction (3.2.15).

Consider the cartoon of our setup in Figure 3.3. We have drawn the (compactified)

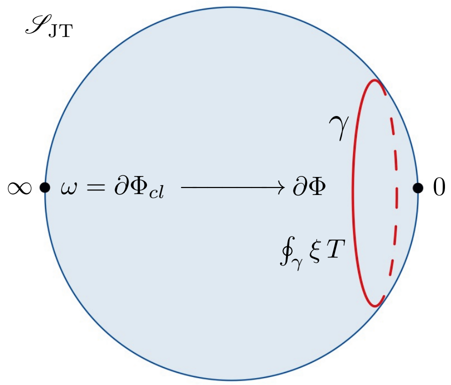


Figure 3.3: *Fixing the behaviour of ω at ∞ determines the classical value $\partial\Phi_{cl}$. As one moves away from infinity, quantum fluctuations of $\partial\Phi$ can appear which deform the complex structure. At the contour γ there is a coordinate change to the patch that covers* 0.

spectral curve $\mathscr{S}_{\mathrm{JT}}$ as a single Riemann sphere, by going to the covering space. The antiperiodicity is implemented by a twist field at 0 and ∞. The Riemann sphere is covered by two coordinate patches, with a transition function that determines the complex structure. Basically, the complex structure defines what we mean by a holomorphic function in each patch. In the left patch covering ∞, the complex structure is such that $\omega = ydx$ is holomorphic. We want to think of ω as the classical vacuum expectation value of the basic field $\partial\Phi$, set, for example, by some background gauge field. As we move away from infinity into the 'bulk' of $\mathscr{S}_{\mathrm{JT}}$, the 1-form ω is allowed to fluctuate:

$$\omega + \delta_\xi \omega = \partial\Phi_{cl} + \partial\Phi \ . \tag{3.2.16}$$

Classically, only negative frequencies are allowed in the mode expansion of the fluctuation $\partial\Phi$ (corresponding to positive powers of z) with the same behaviour at infinity as ω. But quantum mechanically, there can also be positive frequencies $\partial\Phi_+$, which are expanded in powers of z^{-1}. These modes fall off to 0 at $z \to \infty$, but they give non-zero contributions in the interior. Moving even further into the bulk, there is a coordinate change to the right patch, which is implemented by the operator

$$\oint_\gamma dz\, \xi(z) T(z) \ . \tag{3.2.17}$$

In the right coordinate patch, $\delta_\xi\omega$ is holomorphic in the deformed complex structure. Indeed, we will show in subsection 3.2.1 that after an integration by parts, the interaction (3.2.15) can be written as a contour integral (3.2.17). This gives a nice interpretation of the KS interaction as a kind of Φ-dependent coordinate transformation for the field $\partial\Phi$.

The free theory

We will first analyze the free bosonic theory, i.e., without the interaction induced by the complex structure deformation. Consider the free action

$$S^{(0)}_{\mathsf{KS}}[\Phi, \mathcal{J}] = \int_{\mathscr{S}_{\mathrm{JT}}} \left[\frac{1}{2}\, \partial\Phi \wedge \overline{\partial}\Phi - \mathcal{J} \wedge \overline{\partial}\Phi \right] \ . \tag{3.2.18}$$

The equation of motion for $\mathcal{J}$ forces Φ to be chiral:

$$\overline{\partial}\Phi = 0 \quad \text{on-shell} \ . \tag{3.2.19}$$

This chirality constraint for Φ reflects the fact that the classical value of $\partial\Phi$ is ω, which is holomorphic. However, $\mathcal{J}$ is not merely a Lagrange multiplier: it is a dynamical field. In fact, the classical equation of motion for Φ shows that:

$$\mathcal{J} = \partial\Phi \quad \text{on-shell} \ . \tag{3.2.20}$$

This identification holds up to holomorphic forms. Now we see why we have used the notation $\mathcal{J}$: on-shell, it plays the role of the holomorphic current $\partial\Phi$. The monodromy properties of $\partial\Phi$ around the branch point follow from the fact that the 1-form ω is *odd* in z. The variation under a diffeomorphism $\delta_\xi\omega$ should preserve the parity under the involution $z \to -z$ around the branch point, and so we conclude that $\partial\Phi$ should also be odd. This shows that we are dealing with a $\mathbb{Z}_2$-twisted chiral boson on the spectral curve.

Let us now compute the two-point functions of the free theory. Consider the free partition function including the sources:

$$Z_{\mathsf{KS}}^{(0)} = \frac{1}{Z_{\mathsf{KS}}^{(0)}[0]} \int [d\mathcal{J}][d\Phi] \exp\left[-S_{\mathsf{KS}}^{(0)}[\Phi,\mathcal{J}] - \int_{\mathscr{S}_{\mathrm{JT}}} (\mu_\Phi \Phi + \mu_{\mathcal{J}} \mathcal{J})\right] . \tag{3.2.21}$$

Since this is a Gaussian integral in Φ and $\mathcal{J}$, we can solve it using functional determinants. The determinants cancel against the normalization $Z_{\mathsf{KS}}^{(0)}[0]$. In Appendix A.1, we show in detail how to compute the functional integral, which gives the result:

$$\log Z_{\mathsf{KS}}^{(0)} = \int d^2z \int d^2w \left[\frac{\mu_{\mathcal{J}}(z)\mathsf{B}(z,w)\mu_{\mathcal{J}}(w)}{2} + \mu_{\mathcal{J}}(z)\mathsf{G}(z,w)\mu_\Phi(w)\right] , \tag{3.2.22}$$

where we have defined

$$\mathsf{B}(z,w) = \frac{1}{(z-w)^2} + \frac{1}{(z+w)^2} , \tag{3.2.23}$$

$$\mathsf{G}(z,w) = \frac{1}{z-w} - \frac{1}{z+w} . \tag{3.2.24}$$

Defining connected correlation functions as functional derivatives of $\log Z_{\mathsf{KS}}^{(0)}[\mu_\Phi,\mu_{\mathcal{J}}]$, we find that the only non-zero two-point functions are:

$$\langle \mathcal{J}(z)\Phi(w)\rangle_0^{\mathsf{c}} = \left.\frac{\delta^2 \log Z_{\mathsf{KS}}^{(0)}}{\delta\mu_{\mathcal{J}}(z)\delta\mu_\Phi(w)}\right|_{\mu=0} = \mathsf{G}(z,w) , \tag{3.2.25}$$

$$\langle \mathcal{J}(z)\mathcal{J}(w)\rangle_0^{\mathsf{c}} = \left.\frac{\delta^2 \log Z_{\mathsf{KS}}^{(0)}}{\delta\mu_{\mathcal{J}}(z)\delta\mu_{\mathcal{J}}(w)}\right|_{\mu=0} = \mathsf{B}(z,w) . \tag{3.2.26}$$

In particular, we see that there are no contractions of Φ with itself. This will be an important fact, when we make the connection to the topological recursion in Section 3.3.1. It can be seen as a result of taking operator insertions inside correlation functions on-shell: since $\overline{\partial}\Phi = 0$ on-shell, Φ only contains the negative frequencies (positive powers of z), and so it does not have a two-point function in the vacuum.

However, we stress that we derived this fact purely in the functional formalism, without making reference to mode expansions of Φ. Moreover, the functions $\mathsf{B}(z,w)$ and $\mathsf{G}(z,w)$ agree with the standard two-point functions of a free $\mathbb{Z}_2$-twisted chiral boson, as is explicitly verified in Appendix A.2.2.

The interacting theory

Up till now, we have not given our bosonic fields a vacuum expectation value, although we argued that we want to think of ω as the classical value of $\partial\Phi$. We can incorporate this shift in ω by noticing that:

$$\int_{\mathscr{S}_{\mathrm{JT}}} \omega \wedge \overline{\partial}\Phi = \oint_{\gamma} \omega\, \Phi \ . \tag{3.2.27}$$

So we can simply shift $\mathcal{J}$ by ω and integrate by parts, using that $\overline{\partial}\omega = 0$:

$$\begin{aligned}\int_{\mathscr{S}_{\mathrm{JT}}} &\left[\frac{1}{2}\,\partial\Phi \wedge \overline{\partial}\Phi - (\mathcal{J}-\omega)\wedge\overline{\partial}\Phi\right] \\ &= \int_{\mathscr{S}_{\mathrm{JT}}} \left[\frac{1}{2}\,\partial\Phi \wedge \overline{\partial}\Phi - \mathcal{J}\wedge\overline{\partial}\Phi\right] + \oint_{\gamma}\omega\,\Phi \ . \end{aligned} \tag{3.2.28}$$

This does not change the e.o.m. for Φ, since the identification $\partial\Phi = \mathcal{J}$ only holds up to a $\overline{\partial}$-closed form. The shift of $\mathcal{J}$ by ω also does not affect connected correlation functions, except for the classical one-point function:

$$\langle \mathcal{J}(z)\rangle_0 = \omega(z) \ . \tag{3.2.29}$$

The boundary integral in (3.2.27) is along a contour γ that encircles the branch point at $z = 0$. We can use a similar argument to show that the interaction term localizes to the branch point. Plugging in our expression for the Beltrami differential, and writing the stress tensor as $T = T(z)\,dz \otimes dz$, the interaction that implements the complex structure deformation is written as:

$$S_{\mathrm{int}} = \int_{\mathscr{S}_{\mathrm{JT}}} \mu \cdot T = \int_{\mathscr{S}_{\mathrm{JT}}} \overline{\partial}\xi \cdot T \ . \tag{3.2.30}$$

The dot $\cdot$ is shorthand for contracting $dw\partial_z = \delta_z^w$ in the second tensor factor and then integrating the (1,1)-form $\mu_{\bar{z}}^{z} T(z) d\bar{z} \wedge dz$ coming from the first tensor factor. In perturbation theory, the stress tensor remains holomorphic:

$$\overline{\partial}T = 0 \ . \tag{3.2.31}$$

We can thus integrate by parts in a region V where $\xi(z)T(z)$ is holomorphic. Notice that the vector field $\xi = \frac{\Phi}{\omega}$ has poles at the branch point $z = 0$ and at the other zeroes of $\omega(z) \sim z\sin(2\pi z)$. The zeroes of $\omega(z)$ different from the branch point correspond to the 'pinched cycles' of the Riemann surface $\mathscr{S}_{\mathrm{JT}}$. We take V such that its boundary is a collection of contours γ_i surrounding the zeroes of ω, including the branch point, and use Stokes' theorem:

$$S_{\mathrm{int}} = \int_V \overline{\partial}(\xi \cdot T) = \int_V d(\xi \cdot T) = \sum_i \oint_{\gamma_i} \xi \cdot T \ . \tag{3.2.32}$$

Now we argue that only the contribution from the branch point gives a non-zero result inside correlation functions. To see this, recall that the spectral curve is a branched double cover of the spectral plane, with a twist field $\sigma(0)$ inserted at the branch point. As explained in Appendix A.2.2, the twisted vacuum is related to the conformally invariant free boson vacuum by $|\sigma\rangle = \sigma(0)\,|0\rangle$. For any contour γ_j which does *not* surround the branch point, the operator $\oint_{\gamma_j} \xi \cdot T$ commutes with $\sigma(0)$ because their operator product is trivial (γ_j never gets close to 0). It then annihilates the untwisted vacuum:

$$\oint_{\gamma_j} \xi \cdot T\,|\sigma\rangle = \sigma(0) \oint_{\gamma_j} \xi \cdot T\,|0\rangle = 0 \ . \tag{3.2.33}$$

To see why the untwisted vacuum gets annihilated, let ζ be a local coordinate around the j-th zero of $\omega(z)$. Then, expand the stress tensor in even powers of ζ, the field Φ in odd positive powers of ζ, and $\omega(z)^{-1}$ in powers of ζ^{2i-2}, $i \geq 0$. Working out the contour integral shows that only stress tensor modes $L_{n\geq -1}$ appear in the operator $\oint \xi \cdot T$. Since the untwisted vacuum $|0\rangle$ is conformally invariant, it gets annihilated by $\{L_{-1}, L_0, L_1\}$. Moreover, the normal ordering of the stress tensor T ensures that the $L_{n>0}$ contain only annihilation operators to the right of the creation operators. So we conclude that $L_n\,|0\rangle = 0$ for all $n \geq -1$. Therefore, the exponential of the interaction collapses to a single contribution from the contour γ that does surround the branch point:

$$e^{-\widehat{S}_{\mathrm{int}}}\,|\sigma\rangle = e^{-\sum_i \oint_{\gamma_i} \xi\cdot T}\,|\sigma\rangle = e^{-\oint_{\gamma} \xi\cdot T}\,|\sigma\rangle \ . \tag{3.2.34}$$

This argument can easily be generalized to spectral curves with multiple branch points and twist operators. In that case, the interaction term will localize to a sum over contributions from the branch points only.

With this localization argument, we arrive at the action of the universe field theory (3.2.2). Indeed, we can write the stress tensor on-shell as $T = \frac{1}{2}\mathcal{J}^2$, with the point-splitting regularization

$$T(z) = \frac{1}{2}\lim_{w\to z}\left(\mathcal{J}(w)\mathcal{J}(z) - \frac{1}{(z-w)^2}\right) , \tag{3.2.35}$$

and plugging this into (3.2.32) we see that the interaction term is:

$$S_{\text{int}} = \frac{1}{2}\oint_{\gamma} \frac{\Phi}{\omega}\mathcal{J}^2 \ . \tag{3.2.36}$$

Rescaling ω by λ gives the interaction a coupling constant. The fact that the interaction is localized to a contour around $z = 0$ ensures that the theory is free of UV divergences which normally crop up when adding an irrelevant deformation to a CFT. When doing conformal perturbation theory and expanding the exponential of S_{int}, the contours can be chosen to be non-intersecting so that operators are never inserted at the same point [125]. In some sense, we can think of (3.2.36) as a 'topological' interaction: The contour γ can be deformed at will, as long as it does not cross or enclose the other zeroes of $\omega(z)$.

Most notably, the interaction is *cubic* in the fields Φ and $\mathcal{J} = \partial\Phi$. We will argue that this cubic vertex represents the pair-of-pants that is used as a building block in constructing hyperbolic surfaces, which are the relevant geometries in JT gravity. The Feynman diagrams of $\mathcal{J}$-correlators are then to be viewed as the 'skeletons' of the spacetime wormholes. The usual rules of summing over all possible diagrams then ensure the modular invariance of the GPI. In the next subsection, we will establish a recursion relation between the diagrams of KS theory, which will be matched to the topological recursion for JT gravity in Section 3.3.

3.2.2 Schwinger-Dyson equations

The Schwinger-Dyson (SD) equations in a quantum field theory can be seen as the quantum version of the equations of motion. They are usually derived by requiring that the measure is invariant under an infinitesimal linear shift in the field variable, or equivalently, that the functional integral of a total functional derivative is zero. This gives a set of differential equations for n-point functions, which are sometimes taken as a definition of the theory.

In our case, we will have a SD equation for both Φ and $\mathcal{J}$. The SD equation for $\mathcal{J}$ just imposes the quantum version of the chiral constraint. The SD equation for $\Phi \to \Phi + \delta\Phi$ is the most interesting equation: We will show that it is directly equivalent to the topological recursion. The starting point will be the full interacting partition function including sources in (3.2.1). Imposing that the path integral is invariant gives the SD equation:

$$\frac{1}{Z_{\mathsf{KS}}[0]}\int [d\mathcal{J}][d\Phi]\,\delta_\Phi \exp\left[-S_{\mathsf{KS}}[\Phi,\mathcal{J}] - \int_{\mathscr{S}_{\text{JT}}} \mu_\Phi \Phi - \int_{\mathscr{S}_{\text{JT}}} \mu_{\mathcal{J}}\mathcal{J}\right] = 0 \ . \tag{3.2.37}$$

Here, we have written the functional variation $\delta_\Phi(\dots) = \frac{\delta}{\delta\Phi}(\dots)\delta\Phi$. This variation brings down two terms from the exponential:

$$\left\langle \delta_\Phi S_{\mathsf{KS}} + \int_{\mathscr{S}_{\mathrm{JT}}} \mu_\Phi \, \delta\Phi \right\rangle_{\mu_\Phi,\, \mu_{\mathcal{J}}} = 0 \; . \tag{3.2.38}$$

To make the variational problem well-defined, we need to specify boundary conditions for the field variation $\delta\Phi$. Since Φ is odd, we will also impose that $\delta\Phi$ is odd. We further impose the regularity condition at the boundary that $\delta\Phi(z) \to 0$ as $z \to 0$. So in particular, $\delta\Phi$ cannot have a pole at the branch point. Summarizing, we demand that:

$$\delta\Phi\big|_\gamma = \text{odd and analytic} \; . \tag{3.2.39}$$

Let us now compute the variation of the action, carefully treating the surface and boundary contributions:

$$\delta_\Phi S_{\mathsf{KS}} = \int_{\mathscr{S}_{\mathrm{JT}}} \big(- \partial\overline{\partial}\Phi + \overline{\partial}\mathcal{J} \big) \delta\Phi \, + \oint_\gamma \left[\frac{\omega \, \delta\Phi}{\lambda} - \mathcal{J}\delta\Phi + \frac{\lambda}{2} \frac{\mathcal{J}^2}{\omega} \, \delta\Phi \right] \; . \tag{3.2.40}$$

Notice that the first term inside the boundary integral vanishes, because both ω and $\delta\Phi|_\gamma$ are holomorphic. The second term in the boundary integral came from an integration by parts in the bulk integral. Having separated the bulk and boundary contributions to the variation in (3.2.38), both should vanish separately. The SD equation in the bulk becomes:

$$\left\langle -\partial\overline{\partial}\Phi + \overline{\partial}\mathcal{J} + \mu_\Phi \right\rangle_{\mu_\Phi,\, \mu_{\mathcal{J}}} = 0 \; , \tag{3.2.41}$$

since the bulk variation $\delta\Phi$ was arbitrary. This just gives the quantum version of the classical e.o.m., $\mathcal{J} = \partial\Phi$. By writing Φ and $\mathcal{J}$ as functional derivatives of the partition function, we can obtain the bulk SD equation in arbitrary n-point functions. The more interesting condition is the SD equation for the boundary term:

$$\left\langle \oint_\gamma \frac{dz}{2\pi i} \left(\frac{\lambda}{2} \frac{\mathcal{J}^2(z)}{\omega(z)} - \mathcal{J}(z) \right) \delta\Phi(z) \right\rangle_{\mu_\Phi,\, \mu_{\mathcal{J}}} = 0 \; . \tag{3.2.42}$$

At the boundary, the variation $\delta\Phi(z)|_\gamma$ is an arbitrary odd and analytic function. This means that in the Laurent expansion of the integrand all the terms with even negative powers of z should vanish. The projection to the even negative powers of z is done precisely with the free two-point function:

$$\mathsf{G}(z_0, z) = \frac{1}{z_0 - z} - \frac{1}{z_0 + z} = \big\langle \mathcal{J}(z_0)\Phi(z) \big\rangle_0^{\mathsf{c}} \; . \tag{3.2.43}$$

Using this projection, the SD equation (3.2.42) becomes

$$\frac{1}{2}\oint_\gamma \frac{dz}{2\pi i}\mathsf{G}(z_0,z)\left\langle \frac{\lambda}{2}\frac{\mathcal{J}^2(z)}{\omega(z)} - \mathcal{J}(z)\right\rangle_{\mu_\Phi,\,\mu_\mathcal{J}} = 0\ . \tag{3.2.44}$$

The second term is just the even and singular part of $\mathcal{J}(z)$. So the requirement that Φ is odd (which followed from the parity of ω) automatically allows us to treat $\mathcal{J}(z)$ as an even function, and hence $\mathcal{J} = \mathcal{J}(z)dz$ is odd in z. This was already manifest in our on-shell identification $\mathcal{J}(z) = \partial\Phi(z)$, but now we see that also in the quantum theory the structure of the SD equation gives $\mathcal{J}(z)$ the right properties of a twisted boson on the spectral curve.

We can now use the source field $\mu_\mathcal{J}$ to write $\mathcal{J}(z)$ as a functional derivative of the KS partition function. We turn off the source for Φ, since it has disappeared from the boundary SD equation. In terms of the free energy

$$W_{\mathsf{KS}}[\mu_\mathcal{J}] = \log Z_{\mathsf{KS}}[\mu_\mathcal{J}]\ , \tag{3.2.45}$$

the SD equation (3.2.44) becomes the following functional differential equation:

$$\frac{\delta W_{\mathsf{KS}}}{\delta\mu_\mathcal{J}(z_0)}\bigg|_{\chi<0} = \frac{\lambda}{4}\oint_\gamma \frac{dz}{2\pi i}\,\frac{\mathsf{G}(z_0,z)}{\omega(z)}\left[\frac{\delta^2 W_{\mathsf{KS}}}{\delta\mu_\mathcal{J}(z)\delta\mu_\mathcal{J}(z)} + \frac{\delta W_{\mathsf{KS}}}{\delta\mu_\mathcal{J}(z)}\frac{\delta W_{\mathsf{KS}}}{\delta\mu_\mathcal{J}(z)}\right]\ . \tag{3.2.46}$$

The fact that we should pick the negative powers of z in $\mathcal{J}$ has been denoted by $\chi < 0$. Furthermore, there is an explicit normal ordering prescription through the point-splitting regularization for $T = \frac{1}{2}\mathcal{J}^2$.

In the next section, we will show that the SD equation (3.2.46) is equivalent to the topological recursion relation for JT gravity. The recursion relation is supplemented with initial input, given by the free one- and two-point functions (3.2.23) derived in the previous section:

$$\langle\mathcal{J}(z)\rangle_0^{\mathsf{c}} = \omega(z)\ ,\quad \langle\mathcal{J}(z)\mathcal{J}(w)\rangle_0^{\mathsf{c}} = \mathsf{B}(z,w)\ . \tag{3.2.47}$$

We will show that these input data also agree with those of JT gravity, namely the disk and annulus contributions.

3.3 Connection to JT gravity

The observables in the KS theory that are relevant for JT gravity are defined as the inverse Laplace transform of $\mathcal{J}$:

$$Z(\beta) \equiv \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz\, \mathcal{J}(z)\, e^{\beta z^2} \ . \tag{3.3.1}$$

The integration contour is along the interval $(-i\infty, i\infty)$ which is shifted slightly to the right by a small parameter $c > 0$ to avoid possible poles of $\mathcal{J}$ at the imaginary axis[2]. From the gravity perspective the observables in (3.3.1) should be thought of as creating an asymptotic boundary in spacetime (or string world-sheet) of renormalized length β. In this section, we will argue that the spacetime wormhole contributions to the JT gravity path integral will be given by *connected* correlation functions of these observables in the KS theory:

$$\mathcal{Z}^{\mathsf{c}}_{\mathrm{JT}}(\beta_1, \ldots, \beta_n) = \langle Z(\beta_1) \cdots Z(\beta_n) \rangle^{\mathsf{c}}_{\mathsf{KS}} \ . \tag{3.3.2}$$

These correlation functions can be expanded in the coupling constant λ of the KS theory by matching $\lambda = e^{-S_0}$. The full genus expansion of the JT partition function now follows from the perturbative expansion of the KS path integral. In analogy with string field theory, each term in this expansion corresponds to a world-sheet with a fixed topology.

3.3.1 Matching KS theory with JT gravity

First, we show that the disk and annulus partition functions are obtained using the inverse Laplace transform (3.3.1) of the free one- and two-point functions (3.2.47). After that, we match the higher genus contributions for an arbitrary number of boundaries.

The disk. To obtain the one-point function of $Z(\beta)$ in the KS theory we need to compute the following inverse Laplace transform:

$$\langle Z(\beta) \rangle^{\mathsf{c}}_{\mathsf{KS},0} = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} \, \langle \mathcal{J}(z) \rangle_0 \; e^{\beta z^2} = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\, \omega(z)\, e^{\beta z^2} \ . \tag{3.3.3}$$

We see that the KS theory determines the leading term in the genus expansion from the holomorphic one-form, which encodes the complex structure of the spectral

[2] In the case that $\mathcal{J}$ is regular at the origin, it need not be the Laplace transform of some function. The definition (3.3.1) should then be understood in a distributional sense.

curve:

$$\omega(z)dz = y(z)dx(z) = \frac{1}{2\pi} z \sin(2\pi z) dz \ . \tag{3.3.4}$$

Since $\omega(z)$ is regular at $z = 0$, we can set $c = 0$ and substitute $z \to -iw$, giving:

$$\langle Z(\beta) \rangle^{\mathsf{c}}_{\mathsf{KS},0} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dw\, w \sinh(2\pi w)\, e^{-\beta w^2} = \frac{1}{4\pi^{1/2}\beta^{3/2}} e^{\pi^2/\beta} \ . \tag{3.3.5}$$

This expression agrees with the path integral of the disk in JT gravity.

The annulus. Next, we compute the free two-point function of $Z(\beta)$ in KS theory and match it to the annulus amplitude. For that we need to compute the inverse Laplace transform of the free bosonic two-point function:

$$\langle Z(\beta_1) Z(\beta_2) \rangle^{\mathsf{c}}_{\mathsf{KS},0} = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} \frac{dw}{2\pi i} \langle \mathcal{J}(z) \mathcal{J}(w) \rangle^{\mathsf{c}}_0 \, e^{\beta_1 z^2 + \beta_2 w^2} \ , \tag{3.3.6}$$

where

$$\langle \mathcal{J}(z) \mathcal{J}(w) \rangle^{\mathsf{c}}_0 = \mathsf{B}(z, w) = \frac{1}{(z-w)^2} + \frac{1}{(z+w)^2} \ . \tag{3.3.7}$$

Again, we may rotate the contours to the real axis, and then note that both terms in (3.3.7) give the same contribution upon sending $w \to -w$ in the second integral. Expanding $(z-w)^{-2}$ as a power series, and using gamma functions to compute the resulting Gaussian moments, it can be shown that the double integral gives:

$$\langle Z(\beta_1) Z(\beta_2) \rangle^{\mathsf{c}}_{\mathsf{KS},0} = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} dz dw\, \frac{e^{-\beta_1 z^2 - \beta_2 w^2}}{(z-w)^2} = \frac{1}{2\pi} \frac{\sqrt{\beta_1 \beta_2}}{\beta_1 + \beta_2} \ . \tag{3.3.8}$$

This matches the Euclidean wormhole contribution in JT gravity.

Higher genus amplitudes. The higher genus corrections in JT gravity are computed recursively, either using Mirzakhani's recursion for the Weil-Petersson volumes $V_{g,n}(\boldsymbol{\ell})$, or, after a Laplace transform, using Eynard and Orantin's topological recursion. In the KS theory, we can also compute higher-order corrections to connected correlation functions of $\mathcal{J}(z)$ using the SD equation (3.2.46). We show that the topological recursion is retrieved as the perturbative expansion of the SD equation.

Let us denote the connected correlation functions by

$$\mathcal{W}_n(z_1, \ldots, z_n) \equiv \langle \mathcal{J}(z_1) \cdots \mathcal{J}(z_n) \rangle^{\mathsf{c}}_{\mathsf{KS}} = \frac{\delta^n W_{\mathsf{KS}}}{\delta \mu_{\mathcal{J}}(z_1) \cdots \delta \mu_{\mathcal{J}}(z_n)} \Big|_{\mu_{\mathcal{J}}=0} \ . \tag{3.3.9}$$

We can expand the free energy $W_{\mathsf{KS}}[\mu_{\mathcal{J}}]$ of the KS theory in terms of connected correlation functions as:

$$W_{\mathsf{KS}}[\mu_{\mathcal{J}}] = \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} dz_i \frac{\mu_{\mathcal{J}}(z_1)\cdots\mu_{\mathcal{J}}(z_1)}{n!} \mathcal{W}_n(z_1,\ldots,z_n) \ . \tag{3.3.10}$$

Plugging this expansion in to the SD equation (3.2.46) and comparing powers of $\mu_{\mathcal{J}}$, the SD equation takes the form:

$$\begin{aligned}\mathcal{W}_{n+1}(z_0,z_I) = \frac{\lambda}{4}\oint_{\gamma}\frac{dz}{2\pi i}\frac{\langle\mathcal{J}(z_0)\Phi(z)\rangle_0}{\omega(z)}\Big[&\mathcal{W}_{n+2}(z,z,z_I)\\ &+\sum_{J_1\sqcup J_2=I}\mathcal{W}_{1+|J_1|}(z,z_{J_1})\mathcal{W}_{1+|J_2|}(z,z_{J_2})\Big] \ .\end{aligned} \tag{3.3.11}$$

The sum in (3.3.11) is over subsets $J_1 \sqcup J_2 = I = \{1,\ldots,n\}$, and the multi-index notation is given by $z_J \equiv (z_j)_{j\in J}$. Next, consider the perturbative expansion in powers of the KS coupling constant λ:

$$\mathcal{W}_n(z_1,\ldots,z_n) = \sum_{g=0}^{\infty} \lambda^{2g-2+n}\mathcal{W}_{g,n}(z_1,\ldots,z_n) \ . \tag{3.3.12}$$

Substituting this into (3.3.11) and matching the terms with the same powers of λ we obtain a system of recursive equations:

$$\begin{aligned}\mathcal{W}_{g,n+1}(z_0,z_I) = \operatorname*{Res}_{z\to 0}\mathcal{K}(z_0,z)\Big[&\mathcal{W}_{g-1,n+2}(z,z,z_I)\\ &+\sum_{h=0}^{g}\sum_{J_1\sqcup J_2=I}^{\prime}\mathcal{W}_{h,1+|J_1|}(z,z_{J_1})\mathcal{W}_{g-h,1+|J_2|}(z,z_{J_2})\Big] \ .\end{aligned} \tag{3.3.13}$$

Here, we have defined the recursion kernel $\mathcal{K}(z_0,z)$ in terms of the twisted propagator as:

$$\mathcal{K}(z_0,z) \equiv \frac{\langle\mathcal{J}(z_0)\Phi(z)\rangle_0}{4\,\omega(z)} = \frac{1}{2}\left(\frac{1}{z_0-z}-\frac{1}{z_0+z}\right)\frac{1}{2\omega(z)} \ , \tag{3.3.14}$$

where $\omega(z) = \frac{1}{2\pi}z\sin(2\pi z)$. The prime indicates that terms involving $(g,n) = (0,1)$ should be excluded from the summation. We have replaced the contour integral around the branch point by a residue at $z = 0$. The recursion relation is therefore determined by the pole structure of the correlation functions in the complex plane.

Importantly, the recursion in (3.3.13) corresponds precisely to the topological recursion relations applied to the spectral curve $\mathscr{S}_{\rm JT}$, with input data[3]:

$$\mathcal{W}_{0,1}(z) = \omega(z) \ , \quad \mathcal{W}_{0,2}(z_1, z_2) = \mathsf{B}(z_1, z_2) \ . \tag{3.3.15}$$

The relevant background for the formalism of topological recursion is summarized in appendices 2.2.3 and A.2. We can combine this result with Eynard and Orantin's observation [80] that the Weil-Petersson volumes $V_{g,n}$ are related to the 'symplectic invariants' $\mathcal{W}_{g,n}$ by a Laplace transform:

$$\mathcal{W}_{g,n}(z_1, \ldots, z_n) = \int_0^\infty \prod_{i=1}^n d\ell_i \, \ell_i \, e^{-z_i \ell_i} V_{g,n}(\ell_1, \ldots, \ell_n) \ , \quad (\chi < 0) \ . \tag{3.3.16}$$

Using our proposal (3.3.1) for relating JT to KS, we go to the β-variable by applying the inverse Laplace transform for each z_i:

$$\langle Z(\beta_1) \cdots Z(\beta_n) \rangle_{\mathsf{KS}}^{\mathsf{c},(g)} = \int_{c-i\infty}^{c+i\infty} \prod_{i=1}^n \frac{dz_i}{2\pi i} \, e^{\beta_i z_i^2} \mathcal{W}_{g,n}(z_1, \ldots, z_n) \tag{3.3.17}$$

$$= \int_0^\infty \prod_{i=1}^n d\ell_i \, \ell_i \, V_{g,n}(\ell_1, \ldots, \ell_n) \int_{-\infty}^\infty \prod_{i=1}^n \frac{dw_i}{2\pi} e^{-\beta_i w_i^2 - i\ell_i w_i} \tag{3.3.18}$$

$$= \int_0^\infty \prod_{i=1}^n d\ell_i \, \ell_i \, V_{g,n}(\ell_1, \ldots, \ell_n) Z_{\mathsf{trumpet}}(\beta_i, \ell_i) \ . \tag{3.3.19}$$

This indeed agrees with the Euclidean path integral $Z^{\mathsf{c}}_{g,n}(\beta_1, \ldots, \beta_n)$ in (2.1.7) for the stable surfaces with $\chi < 0$. Multiplying by λ^{2g-2+n} and summing over the genus we conclude that the perturbative expansion of the universe field theory matches with the genus expansion of the gravitational path integral. Since all correlation functions can be expressed in terms of connected correlations functions, we see that the full JT gravity n-boundary path integral is the n-point function of the boundary creation operators $Z(\beta)$:

$$\mathcal{Z}_{\rm JT}(\beta_1, \ldots, \beta_n) = \langle Z(\beta_1) \cdots Z(\beta_n) \rangle_{\mathsf{KS}} \ . \tag{3.3.20}$$

Let us emphasize that the right-hand side is a *non-gravitational* Euclidean path integral with n operator insertions, whereas the left-hand side is the *gravitational* path integral of JT gravity. We have thus expressed JT gravity as a Euclidean 'universe field theory' on the spectral curve. This provides a non-perturbative completion (at least on a formal level) of the topological expansion of the Euclidean JT path integral.

[3]This can either be verified using direct computation, or the generic argument presented in (A.2.93).

Given the geometric interpretation of the chiral boson $\mathcal{J}(z)$ as describing the quantum fluctuations of the target space geometry around the classical value $\omega(z)$, we can think about the 'ensemble average' $\langle\cdots\rangle_{\mathsf{KS}}$ associated to JT gravity, roughly speaking, as describing an average over background geometries in which the JT string propagates.

3.4 Non-perturbative effects

Given the KS theory description for JT gravity and its embedding in the B-model topological string theory, we can invoke intuition and tools from string theory to study non-perturbative effects of order $\mathcal{O}(e^{1/\lambda})$ in the universe field theory. We will study the insertion of certain topological D-branes [38–40, 126] in the target space geometry. Their effect will be doubly non-perturbative in G_N, as can be seen from the identification $\lambda = e^{-S_0}$. These contributions are very interesting from the point of view of gravity, as they form an indirect probe of the discreteness of the spectrum in a candidate microscopic theory.

In Section 3.4.1, we will study non-perturbative effects due to the D-branes. These correspond to fermionic objects in the KS theory. In Section 3.4.2, we will give these D-branes an interpretation in JT gravity as hypersurfaces on which fixed energy boundaries can end. These boundaries are described by a boundary term in the JT gravity action that can be obtained from the standard Dirichlet-type boundary action by a Legendre transform, which on the level of the path integral becomes a Laplace transform. As an application of the D-brane formalism, we will show in Section 3.4.3 how to obtain non-perturbative corrections to the density-density correlator, giving the 'plateau' feature of the spectral form factor [127].

3.4.1 Branes in KS theory

Since the basic KS field is a 2-dimensional chiral boson, we can use the familiar boson-fermion correspondence and introduce the following fermionic fields[4]:

$$\psi(z) = e^{\Phi(z)} \ , \qquad \psi^\dagger(z) = e^{-\Phi(z)} \ . \tag{3.4.1}$$

The exponentials are normal-ordered by subtracting the OPE singularities of $\Phi(z)\Phi(w) \sim \log(z-w)$ in the expansion of the exponential. These fermionic fields have an interpretation as D-branes in the KS theory. Namely, recall that

[4] By $\Phi(z)$ we mean the full chiral boson with both positive and negative modes. We treat it as the indefinite integral of $\mathcal{J}(z)$. For an account of the boson-fermion correspondence for twisted fields, see [128, 129].

the spectral curve is embedded in the following non-compact Calabi-Yau:

$$\mathsf{CY}: \quad uv - y^2 + \frac{1}{(4\pi)^2}\sin^2(2\pi\sqrt{x}) = 0 \ . \tag{3.4.2}$$

The base of this Calabi-Yau is the spectral curve $\mathscr{S}_{\mathrm{JT}}$. This is where the bosonic fields $\mathcal{J}$ and Φ live. The fibers over the spectral curve are defined by $u = 0$ and $v = 0$: If we specify a base point on $\mathscr{S}_{\mathrm{JT}}$ these are complex one-dimensional manifolds in the geometry, which can be wrapped by topological D2-branes. In the topological string terminology, $u = 0$ is wrapped by a brane, while a brane that wraps the transverse fiber $v = 0$ has opposite flux and can be thought of as an 'anti-brane' [130]. The fibers are parametrized by a point on the spectral curve, so we can talk about a brane 'inserted' at a point $\zeta \in \mathscr{S}_{\mathrm{JT}}$.

In the topological string B-model, integrating out open strings ending on the brane deforms the geometry in which the closed strings propagate [107]. This can be thought of as the backreaction of a brane on the geometry, which deforms the complex structure of CY. As before, the change in complex structure is encoded in the period integral of the holomorphic $(3,0)$-form Ω_{CY} around a 3-cycle $\widetilde{C}$ surrounding the D-brane. The change in complex structure due to a single D-brane is found to be [123]:

$$\delta \int_{\widetilde{C}} \Omega_{\mathsf{CY}} = \lambda \ . \tag{3.4.3}$$

We can follow the same steps as in (3.2.7) to reduce the period integral on the complex 3-cycle surrounding the 2-dimensional brane to a period integral of ω on a contour $C(\zeta)$ surrounding the point ζ. The result is simply:

$$\delta \oint_{C(\zeta)} \omega = \lambda \ . \tag{3.4.4}$$

What this equation is saying is that the insertion of a brane above the point ζ deforms the complex structure of the spectral curve $\mathscr{S}_{\mathrm{JT}}$ by a small amount λ. This is implemented in the quantum theory by a field $\psi(\zeta)$. Rescaling $\omega \to \omega/\lambda$, the property (3.4.4) can be written as an operator product, to be read inside correlation functions:

$$\oint_{C(\zeta)} dz\, \partial\Phi(z)\, \psi(\zeta) = \psi(\zeta) \ . \tag{3.4.5}$$

We used the defining relation $\delta\omega = d\Phi$ for the chiral boson Φ. We now recognize the operator product expansion of a complex fermion of conformal weight $h = \frac{1}{2}$ with the holomorphic bosonic current $\mathcal{J}(z) = \partial\Phi(z)$:

$$\partial\Phi(z)\psi(\zeta) \sim \frac{1}{z-\zeta}\psi(\zeta) \ . \tag{3.4.6}$$

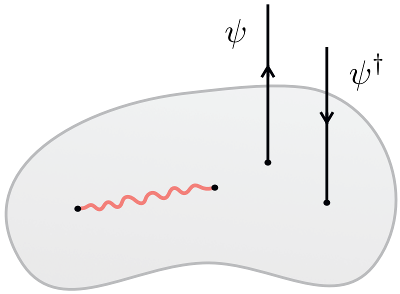


Figure 3.4: *A pictorial representation of non-compact D-brane insertions on the spectral curve $\mathscr{S}$. The straight lines correspond to the non-compact fiber directions $u = 0$ or $v = 0$ which are wrapped by branes ψ and anti-branes $\psi^\dagger$ respectively, having opposite flux as indicated by the direction of the arrow. The branch cut is denoted by a red wiggly line.*

Conversely, we can obtain $\mathcal{J}(z)$ by taking the coincident limit of a brane and an anti-brane:

$$\partial\Phi(z) = \lim_{z' \to z} \left\{ \psi(z')\psi^\dagger(z) \right\} . \tag{3.4.7}$$

The accolades signify normal ordering, by subtracting the OPE divergence $\sim \frac{1}{z'-z}$. We conclude that the non-compact D-branes described above are indeed nothing but complex fermions on the spectral curve[5].

So far, we have defined the fermions on the double cover, which is also how we have presented the KS theory. However, ultimately we will be interested in extracting non-perturbative information from the fermions to correlation functions of the density of states $\rho(E)$, where E is related to the base space coordinate $x = -E$. In particular, these quantities will be sensitive, at least semiclassically [132], to the branched structure of the spectral curve. Therefore, we also want to define fermionic fields $\psi(x)$ on the spectral plane $x = z^2$. However, this requires us to choose a branch of $z = \sqrt{x}$. We therefore use the formalism developed in [133] to describe $\mathbb{Z}_2$-twisted fermions on sheeted Riemann surfaces. To connect to the $\mathbb{Z}_2$-twisted boson formalism outlined in Appendix A.2, we will use the spectral plane variable x, and obtain physical quantities like $\rho(E)$ by evaluating at $x = -E$ in the end.

[5]This is true locally. As explained in [40], for an arbitrary spectral curve $\psi(z)$ is only defined patch by patch and transforms as a wavefunction, instead of as a fermionic 'half-differential' $\psi = \psi(z)\sqrt{dz}$. The wavefunction interpretation of $\psi(z)$ appears naturally from a Schrödinger equation satisfied by $\langle\psi(z)\rangle_{\mathsf{KS}}$, see, e.g., [131].

$\mathbb{Z}_2$-twisted fermions

We think of the spectral curve as two copies of the spectral x-plane, glued together along the branch cut on the negative real axis. On each sheet, labelled by indices $0, 1$, we define a bosonic field, such that after a $2\pi i$ rotation the fields are rotated into each other:

$$\Phi_0(e^{2\pi i}x) = \Phi_1(x) \ , \quad \Phi_1(e^{2\pi i}x) = \Phi_0(x) \ . \tag{3.4.8}$$

In Appendix A.2.6, we introduce explicit mode expansions for Φ_0 and Φ_1 and show that the KS field $\Phi(x)$ is the combination that diagonalizes the monodromy:

$$\Phi(x) = \frac{1}{\sqrt{2}}(\Phi_0(x) - \Phi_1(x)) \ , \quad x \in \mathbb{C} \setminus \mathbb{R}_{\leq 0} \ . \tag{3.4.9}$$

Then, $\Phi(e^{2\pi i}x) = -\Phi(x)$, so Φ is indeed an odd function of z. In terms of x, it has an expansion in only half-integer powers of x. So in particular it has a discontinuity across the branch cut, which we will call $\Omega(x)$. The discontinuity can be expressed alternatively in terms of the fields on opposite sheets as they approach each other on the negative real axis:

$$\Omega(x) \equiv \lim_{x' \to x} (\Phi_0(x) - \Phi_1(x')) \ , \quad x \in \mathbb{R}_{\leq 0} \ . \tag{3.4.10}$$

Next, on each sheet we introduce the following bosonized fermions:

$$\psi_a(x) = \mathsf{c}_a e^{\Phi_a(x)} \ , \quad \psi_a^\dagger(x) = \mathsf{c}_a e^{-\Phi_a(x)} \ , \quad a = 0, 1 \ . \tag{3.4.11}$$

Again, the exponentials are implicitly normal ordered by subtracting the divergences. Furthermore, we have used what is known as the Jordan-Wigner trick to multiply the vertex operators by a cocycle c_a that ensures the correct anti-commutation between fermions on opposite sheets [134]. A consistent choice of cocycles in this case is simply:

$$\mathsf{c}_0 = 1 \ , \quad \mathsf{c}_1 = (-1)^{N_f + 1} \ , \tag{3.4.12}$$

where N_f is the fermion number operator[6]. This ensures that fermions on opposite sheets anti-commute, for example:

$$\psi_0^\dagger(x)\psi_1(x') = e^{-\Phi_0(x)}(-1)^{N_f+1}e^{\Phi_1(x')} = -\psi_1(x')\psi_0^\dagger(x) \ . \tag{3.4.13}$$

[6] It can be bosonized as $N_f = \oint \frac{dz}{2\pi i}\partial\Phi_a(z) = \alpha_0$, which, being the coefficient of $\frac{1}{z}$, can be seen as the momentum of $\partial\Phi_a$.

For fields on the same sheet, the cocycles square to one and the anti-commutation is ensured by the OPE:

$$\psi_a(x)\psi_b^\dagger(x') \sim \frac{\delta_{ab}}{x-x'} + \text{reg.} \tag{3.4.14}$$

This is of course the expected OPE for fermion fields. As before, we have a boson-fermion correspondence for fermions on the same sheet:

$$\partial\Phi_a(x) = \lim_{x'\to x}\left\{\psi_a(x')\psi_a^\dagger(x)\right\} \equiv \lim_{x'\to x}\left(\psi_a^\dagger(x')\psi_a(x) - \frac{1}{x'-x}\right) . \tag{3.4.15}$$

On the other hand, for two fermions on opposite sheets, we do not have to normal order since $\Phi_0(x')\Phi_1(x)$ is regular, and we can simply add the exponentials in a single normal-ordered exponential:

$$\psi_0(x)\psi_1^\dagger(x) \equiv \lim_{x'\to x}\psi_0(x')\psi_1^\dagger(x) = \mathsf{c}_0\mathsf{c}_1\, e^{\Phi_0(x)-\Phi_1(x)} . \tag{3.4.16}$$

Usually for OPE's we implicitly demand the radial ordering $|x'| > |x|$. But here we should be careful about the ordering of the x-arguments when we take the coincident limit, since the points are on different sheets. We choose the convention that $\psi_0(x')$ is always left of $\psi_1(x)$ when we take the coincident limit. With this convention, the product of fields when they approach each other from opposite sheets gives the following weight one vertex operators:

$$e^{\Omega(x)} = \lim_{x'\to x}\psi_0(x')\psi_1^\dagger(x) , \quad e^{-\Omega(x)} = \lim_{x'\to x}\psi_0^\dagger(x')\psi_1(x) . \tag{3.4.17}$$

These operators will play an important role in the next section. The fermions have the following monodromies when going around the branch point:

$$\langle\sigma|\,\psi_0(e^{2\pi i}x) = -\langle\sigma|\,\psi_1(x) , \quad \langle\sigma|\,\psi_1(e^{2\pi i}x) = -\langle\sigma|\,\psi_0(x) . \tag{3.4.18}$$

We have multiplied from the left by the free bosonic twisted vacuum $\langle\sigma|$, so that the cocycles $\mathsf{c}_0, \mathsf{c}_1$ become ± 1, respectively. However, from now on we will leave the left-vacuum implicit. This is justified because, as we will see, to extract the non-perturbative physics we will not need the higher genus corrections from the interacting $|\mathsf{KS}\rangle$ vacuum; we will only need the free vacuum $|\sigma\rangle$ correlation functions.

3.4.2 Interpretation in JT gravity

To connect the discussion of branes on the spectral curve to JT gravity, we should introduce a type of boundary directly in the gravitational theory which can end on branes in the target space geometry. We will thus introduce a set of boundary conditions for the JT universes which do not have a fixed length, but are 'hovering' in the bulk at some finite distance, and which have a fixed energy E. The observables in the matrix model can now be obtained from the JT path integral with this choice of modified boundary conditions, which on the level of the action amounts to a Legendre transform[7].

Fixed energy boundary conditions

Instead of fixing the dilaton and the boundary metric as in (2.1.4), one can also impose Dirichlet-Neumann (DN) boundary conditions, in which one fixes both the dilaton and its normal derivative at the boundary, but leave the metric free [121]:

$$\phi_{\partial M} = \frac{\phi_r}{\varepsilon} \ , \qquad \partial_n \phi_{\partial M} = \frac{\phi_r'}{\varepsilon} \ . \tag{3.4.19}$$

Here, we have normalized the normal vector n, so that $\partial_n \phi$ has the same dimensions as ϕ. In this case, the following boundary action must be added to the bulk JT gravity action:

$$S^{\partial}_{DN} = -\int_{\partial M} du \sqrt{\gamma_{uu}} (\partial_n \phi - \phi \mathcal{K}) \ . \tag{3.4.20}$$

The DN boundary conditions are related to the standard DD boundary conditions by a Legendre transform. To see this, we rewrite (3.4.20) in the following form:

$$S^{\partial}_{DN} = \frac{1}{\varepsilon} \int_{\partial M} du \sqrt{\gamma_{uu}} (\phi_r - \phi_r') + \int_{\partial M} du \sqrt{\gamma_{uu}}\, \phi(\mathcal{K} - 1) \tag{3.4.21}$$

$$= \int_0^\beta du \sqrt{\gamma_{uu}}\, E + S^{\partial}_{DD}[\gamma] \ . \tag{3.4.22}$$

We have written explicitly a dependence on the boundary metric γ_{uu} in the last term, because in the DN action γ is kept free. We recognize the Legendre transform[8], with conjugate variables β and

$$E \equiv \frac{\phi_r - \phi_r'}{\varepsilon} \ . \tag{3.4.23}$$

One can show that (3.4.23) corresponds to a fixed energy $E = \phi_r \mathrm{Sch}(x, u)$ in the boundary Schwarzian theory, when $\epsilon \to 0$. Therefore, the input of a particular Dirichlet type boundary in the JT path integral is some temperature β describing a *canonical* ensemble, whereas the input of a Neumann type boundary is some

[7]This can be viewed as a particular instance of the more general result in the AdS/CFT correspondence [135], namely that a Legendre transformation in the boundary field theory at large N leads to a change in the boundary conditions for the fields on the gravity side.

[8]Usually, the Legendre transform has a relative minus sign. The plus sign here means that on the level of the path integral, we will get an inverse Laplace transform.

fixed energy E describing a *microcanonical* ensemble in the boundary theory.

On the level of the path integral, the Legendre transform becomes an inverse Laplace transform [121]:

$$\mathcal{Z}_{\rm JT}(E) = \int_{c-i\infty}^{c+i\infty} \frac{\mathcal{D}\gamma}{\text{Diff } S^1} e^{\int_0^\beta du \sqrt{\gamma_{uu}}\, E} \mathcal{Z}_{\rm JT}[\gamma] \ . \tag{3.4.24}$$

We will often omit the superscript DN, as it should be clear from using the variable E that we mean the path integral with DN boundary conditions. We can go to a gauge where $\sqrt{\gamma_{uu}}$ is constant, and then we have to divide by β to account for the time reparametrization symmetry:

$$\mathcal{Z}_{\rm JT}(E) = \int_{c-i\infty}^{c+i\infty} \frac{d\beta}{\beta} e^{\beta E} \mathcal{Z}_{\rm JT}(\beta) \ . \tag{3.4.25}$$

For example, we can compute the DN partition function of the disk (with $\phi_r = \frac{1}{2}$) to be

$$Z_{\textsf{disk}}(E) = \frac{e^{S_0}}{8\pi^4} \Big(2\pi\sqrt{E}\cosh(2\pi\sqrt{E}) - \sinh(2\pi\sqrt{E}) \Big) \ . \tag{3.4.26}$$

Furthermore, the trumpet partition function becomes

$$Z_{\textsf{trumpet}}(E,\ell) = \int_{c-i\infty}^{c+i\infty} \frac{d\beta}{\beta} e^{\beta E} \frac{1}{\sqrt{4\pi\beta}} e^{-\frac{\ell^2}{4\beta}} = \frac{1}{\pi\ell}\sin(\ell\sqrt{E}) \ . \tag{3.4.27}$$

Looking at the form of the higher-genus partition functions (2.1.7), we see that the β-dependence only comes in via the trumpets. So, the only modification to the perturbative formula of $\mathcal{Z}_{\rm JT}(E)$ will be to change the integration kernel of the trumpet to its DN counterpart:

$$Z_{g,n}(E_1,\dots,E_n) = \frac{1}{\pi} \int_0^\infty \prod_{i=1}^n d\ell_i \sin(\ell_i\sqrt{E_i}) V_{g,n}(\ell_1,\dots,\ell_n) \ . \tag{3.4.28}$$

The ℓ_i from the gluing measure has cancelled with the ℓ_i in the denominator of (3.4.27). So $Z_{g,n}(E_1,\dots,E_n)$ is simply multiple Fourier-type transform of the Weil-Petersson volumes. We must be careful in evaluating these integrals, as the volumes $V_{g,n}$ are polynomials in ℓ_i^2 and so the above integral in general is divergent. However, this divergence can be easily regularized, for example by introducing a small exponential regulator.

Relation to matrix integrals

The fixed energy JT partition function $\mathcal{Z}_{\rm JT}(E)$ has a direct interpretation in the dual matrix model. To see this, one can write $\mathcal{Z}_{\rm JT}(\beta)$ as the difference of two integrals in spectral plane coordinate $x = z^2$ just above and below the negative real axis:

$$\mathcal{Z}_{\mathrm{JT}}(\beta) = -\lim_{\epsilon\to 0}\int_{-\infty}^{0}\frac{dx}{2\pi i}e^{\beta x}\Big[\langle\partial\Phi(x+i\epsilon)\rangle_{\mathsf{KS}} - \langle\partial\Phi(x-i\epsilon)\rangle_{\mathsf{KS}}\Big]\ . \tag{3.4.29}$$

To obtain $\mathcal{Z}_{\mathrm{JT}}(E)$ from this expression, we use the following integral representation of the delta function:

$$\delta(E+x) = \int_{-i\infty}^{i\infty} d\beta\, e^{\beta(E+x)}\ . \tag{3.4.30}$$

Sending $x \to -x$, the DN path integral can be expressed in terms of KS field insertions as:

$$\mathcal{Z}_{\mathrm{JT}}(E) = \lim_{\epsilon\to 0}\int^{E} dE'\int_{0}^{\infty}\frac{dx}{2\pi i}\delta(E'-x)\Big[\langle\partial\Phi(-x+i\epsilon)\rangle_{\mathsf{KS}} - \langle\partial\Phi(-x-i\epsilon)\rangle_{\mathsf{KS}}\Big]\ . \tag{3.4.31}$$

Since $E' \in \mathbb{R}_{\geq 0}$, the delta function sets $x = E'$. From now on we use the short-hand notation $\partial\Phi(E) \equiv \partial\Phi(x)|_{x=-E}$, when $\partial\Phi$ is viewed as a function of E. This amounts to moving the branch cut to the positive real axis. On the right-hand side we then recognize the discontinuity of $\partial\Phi$ across the branch cut:

$$\operatorname{disc}\partial\Phi(E) \equiv \frac{1}{2\pi i}\lim_{\epsilon\to 0}\Big(\langle\partial\Phi(E+i\epsilon)\rangle_{\mathsf{KS}} - \langle\partial\Phi(E-i\epsilon)\rangle_{\mathsf{KS}}\Big)\ . \tag{3.4.32}$$

We find that $\partial\Phi(E)$ plays the role of the *resolvent* and its discontinuity across the branch cut is the *density of states*. Therefore, we will match our notation with that from double-scaled matrix models and write:

$$\rho(E) \equiv \operatorname{disc}\partial\Phi(E)\ , \qquad E \in \mathbb{R}_{\geq 0}\ , \tag{3.4.33}$$

$$R(E) \equiv \partial\Phi(E)\ , \qquad E \in \mathbb{C}\setminus\mathbb{R}_{\geq 0}\ . \tag{3.4.34}$$

The DN path integral can now be expressed as an insertion of the integrated density of states:

$$Z(E) \equiv \int^{E} dE'\,\rho(E')\ . \tag{3.4.35}$$

These are the analogues of the boundary creation operators (3.3.1) in the case of DN boundary conditions. To be precise, we can obtain the JT path integral with DN boundary conditions by inserting these observables in the KS theory:

$$\mathcal{Z}_{\mathrm{JT}}(E_1,\ldots,E_n) = \langle Z(E_1)\cdots Z(E_n)\rangle_{\mathsf{KS}}\ . \tag{3.4.36}$$

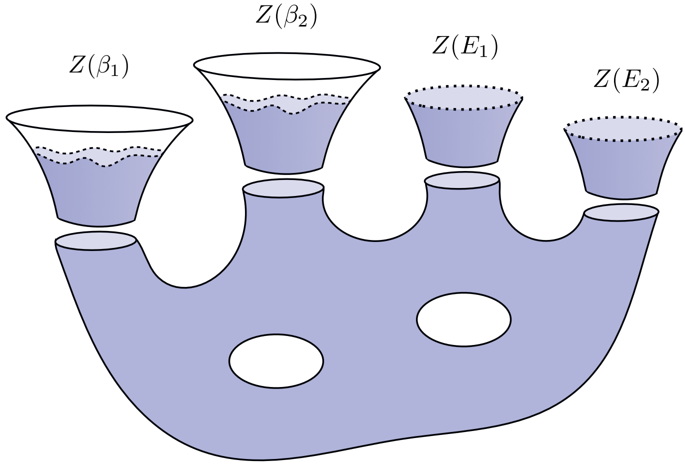


Figure 3.5: *The creation of DD and DN boundary trumpets $Z(\beta)$ and $Z(E)$ in JT gravity, indicated by blue and yellow boundaries respectively.*

This completes the dictionary between JT gravity with DN boundary conditions and the KS theory.

For example, the one-point function $\langle \partial\Phi(x)\rangle_{\mathsf{KS},0} = \omega(x)$ becomes the leading order density of states:

$$\rho_0(E) \equiv \operatorname{disc}\omega(E) = \frac{1}{4\pi^2}\sinh(2\pi\sqrt{E}) \ . \tag{3.4.37}$$

Note that it has the correct universal $\sqrt{E}$-behaviour for low energy, typical of double-scaled matrix models. Integrating, we obtain the disk contribution to the DN path integral, which agrees with our previous answer (3.4.26):

$$\begin{aligned} Z_{\mathsf{disk}}(E) &= \int^E dE' e^{S_0}\, \langle \operatorname{disc}\partial\Phi(E')\rangle_{\mathsf{KS},0} && (3.4.38)\\ &= \frac{e^{S_0}}{8\pi^4}\Big(2\pi\sqrt{E}\cosh(2\pi\sqrt{E}) - \sinh(2\pi\sqrt{E})\Big) \ . && (3.4.39)\end{aligned}$$

Note that the E-integral came from the factor of $1/\beta$ present in the definition of the microcanonical path integral. It arose from gauge fixing the $U(1)$ symmetry of trumpet boundary. If we had assumed some marked point on the DN boundary, there would be no such factor of $1/\beta$, and $\mathcal{Z}_{\mathrm{JT}}(E)$ would be computed from insertions of $\rho(E)$. Such operators were considered from the matrix model point of view in [136], where they were called '*energy eigenbranes*', because they fix a particular energy eigenvalue in the matrix integral.

As another example, we can easily evaluate the contribution from two fixed energy boundaries connected by a wormhole. We do this by glueing two DN trumpets along their common geodesic boundary:

$$\int_0^\infty d\ell\, \ell\, Z_{\mathsf{trumpet}}(E_1,\ell) Z_{\mathsf{trumpet}}(E_2,\ell) = \frac{1}{2\pi^2} \log\left(\frac{\sqrt{E_1}+\sqrt{E_2}}{\sqrt{E_1}-\sqrt{E_2}}\right) . \tag{3.4.40}$$

As expected, the right-hand side of (3.4.40) is the discontinuity of the free two-point function $\langle \Phi(E_1)\Phi(E_2)\rangle_0$. Indeed, one can easily verify that taking derivatives with respect to E_1 and E_2, followed by an inverse Laplace transform, precisely returns the universal wormhole contribution (3.3.8).

We can now borrow the matrix model intuition to understand why we found a twisted bosonic[9] field $\partial\Phi$ to describe JT gravity. We have just seen that the energy E that was fixed as a DN boundary condition, and which is dual to the temperature β, is related to the spectral plane coordinate as $x = -E$. The branch cut in the spectral x-plane is therefore mapped to the positive real axis in the E-plane. We can view the branch cut, and therefore the fact that $\partial\Phi$ had to be $\mathbb{Z}_2$-twisted, as a direct consequence of a continuous eigenvalue density created by $\rho_0(E)$. This is the reason we called x the 'spectral plane' in the first place: a path in the x-plane represents the *spectrum* of a double-scaled matrix model. A given DD boundary, for which the temperature is fixed, can be thought of as having an energy that is randomly drawn from a continuous statistical ensemble.

The double-scaled matrix model that gives rise to density of states $\rho_0(E)$ can be identified explicitly in the topological string theory setup. In fact, there is a precise way in which a stack of topological D2-branes wrapped around compact[10] cycles in the target space geometry (3.2.4) give rise to a large N matrix integral, which is dual to the closed string theory [39]. In that sense, the random matrix H that leads to $\rho_0(E)$ is describing the open string degrees of freedom associated to this brane configuration. The localization of the open string field theory, which in this case is a 6-dimensional holomorphic Chern-Simons theory associated to the space-filling D6-brane, to a matrix integral can be done explicitly [38] (see also [138]). The notation that was used heuristically in (3.4.33) and (3.4.34) can then be understood more formally as a statement of the open/closed duality in topological string theory. Hence, from the perspective of the 'JT string' it follows

[9] The reason that we obtained a *bosonic* theory, is that the collective excitations of eigenvalues in the double-scaled matrix model behave like bosons, even though single eigenvalues behave fermionically. For more on the relation between conformal field theory and double-scaled matrix models, see for example [137].

[10] Importantly these D-branes are *compact* and distinct from the *non-compact* D-branes introduced before.

that the double-scaled matrix integral is actually dual to the KS field theory, rather than to JT gravity itself. This gives a new perspective on the role of the matrix ensemble associated to the gravitational path integral.

3.4.3 Application: spectral correlation functions

Now we apply the formalism introduced in the previous section to extract non-perturbative corrections to the leading-order result for density and density-density correlation functions. First, recall that the density of states $\rho(E)$ is the discontinuity of the resolvent operator:

$$\rho(E) = \frac{1}{2\pi i}(\partial\Phi_0(E) - \partial\Phi_1(E)) \ . \tag{3.4.41}$$

The (first order correction to the) collision of two branes on the same sheet $\psi_0(E)\psi_0^\dagger(E)$ leads to the insertion of a closed string state $\partial\Phi$ as in (3.4.7). The contribution coming from the interaction of branes on opposite sheets is given by

$$\langle\psi_0(E)\psi_1^\dagger(E)\rangle_{\mathsf{KS}} = \langle e^{\Omega(E)}\rangle_{\mathsf{KS}} \approx e^{\frac{2\pi i}{\lambda}\int^E dE'\rho_0(E')} \ . \tag{3.4.42}$$

Here, we have used that the operator $\Omega(E)$ can be written as

$$\Omega(E) = 2\pi i\int^E dE'\rho(E') \ , \tag{3.4.43}$$

and kept only the genus zero contribution to the expectation value. The result is entirely localized at the branch cut, and non-perturbative in the coupling constant. Intuitively, one may think about this quantity as a 'geometric phase' that a brane picks up when it is transported around the branch point, see Figure 3.6.

It turns out that this result captures non-perturbative contributions to the density of states if we add the following corrections to the perturbative expansion:

$$\partial\Phi_0(E)_{\mathrm{np}} \sim \partial\Phi_0(E) + ie^{\Omega(E)} \ , \tag{3.4.44}$$

$$\partial\Phi_1(E)_{\mathrm{np}} \sim \partial\Phi_1(E) - ie^{-\Omega(E)} \ . \tag{3.4.45}$$

Crucially, the symbol $\sim$ indicates that the above operator identifications should be read inside *perturbative* expectation values $\langle\cdots\rangle_{\mathsf{KS}}$ of the KS theory. This is the whole point of the construction: we are trying to extract non-perturbative physics using perturbative computations. The precise mechanism that underlies the above identifications is still rather mysterious, for now one should view it as an observation. We expect that the answer can be found in the open string theory dual to KS, and we hope to address this in future work.

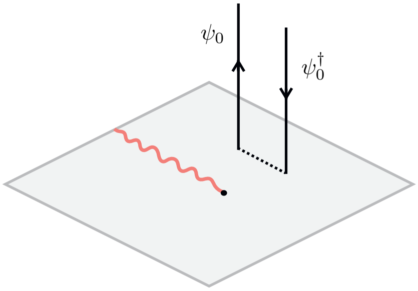


Figure 3.6: *A pictorial representation of (Left) the collision of two fermions ψ_0 and $\psi_0^\dagger$ on the same sheet, (Right) the collision of two fermions ψ_0 and $\psi_1^\dagger$ on opposite sheets. Effectively, the latter is obtained from moving the anti-brane around the branch point along the striped line, and then bringing the fermions together on the branch cut (indicated by a red wiggly line).*

In terms of the matrix model $\partial\Phi(E) = R(E)$ is the resolvent, so the fermions $\psi(E) = \det(H - E)$ and $\psi^\dagger(E) = 1/\det(H - E)$ correspond to (inverse) determinant operators. In particular, $\psi^\dagger$ has singularities at the real axis and should be regularized. This leads to two fermions $\psi_0^\dagger$ or $\psi_1^\dagger$ depending on the sign in the $\pm i\epsilon$ prescription. Taking a single eigenvalue E using the probe brane $\psi_0^\dagger$ and have it circle the branch point once results in the operator $\psi_1^\dagger$. In the process, the eigenvalue 'feels' the force of the other eigenvalues, which is proportional to the number of eigenvalues given by $\int^E \rho_0(E')dE'$, and this effect is captured by the non-perturbative phase in $\langle\psi_0(E)\psi_1^\dagger(E)\rangle_{\mathsf{KS}}$. We now make the following proposal for an observable in the universe field theory that captures non-perturbative corrections to the density of states, by analogy with (3.4.41):

$$\rho_{\text{np}}(E) \sim \frac{1}{2\pi i}(\partial\Phi_0(E)_{\text{np}} - \partial\Phi_1(E)_{\text{np}}) \ . \tag{3.4.46}$$

The superscript 'np' indicates that we have defined an observable which takes the non-perturbative corrections into account. With this proposal for a non-perturbative density of states, we can compute perturbative correlation functions in the KS theory.

Density correlator. The one-point function of $\rho_{\text{np}}(E)$ is computed straightforwardly:

$$\langle\rho_{\text{np}}(E)\rangle_{\mathsf{KS}} \sim \langle\rho(E)\rangle_{\mathsf{KS}} + \frac{1}{2\pi}\left(\langle e^{\Omega(E)}\rangle_{\mathsf{KS}} + \langle e^{-\Omega(E)}\rangle_{\mathsf{KS}}\right) \ . \tag{3.4.47}$$

In principle, we can compute this using the full interacting theory, including all genus corrections, but it turns out that the leading-order result already has the features that we are interested in, so we keep only the 'disk' and 'annulus' contributions. Clearly, we have

$$\langle \rho(E) \rangle_{\mathsf{KS}} \approx e^{S_0} \rho_0(E) \ . \tag{3.4.48}$$

For the other contribution we use the following identity:

$$\langle e^{\pm\Omega(E)} \rangle_{\mathsf{KS}} \approx \exp\left[\pm \langle \Omega(E) \rangle_0^{\mathsf{c}} + \frac{1}{2} \langle \Omega(E)^2 \rangle_0^{\mathsf{c}} \right] \ . \tag{3.4.49}$$

The annulus contribution, which should be appropriately normal ordered, can be computed using the twisted two-point functions $\langle \Phi_a(x) \Phi_b(x') \rangle_{\mathsf{KS}}$:

$$\frac{1}{2} \langle \Omega(x)^2 \rangle_0^{\mathsf{c}} \equiv \frac{1}{2} \lim_{x' \to x} \left\langle \{ \Omega(x) \Omega(x') \} \right\rangle_0^{\mathsf{c}} \tag{3.4.50}$$

$$= \lim_{x' \to x} \left[\log(\sqrt{x} - \sqrt{x'}) - \log(\sqrt{x} + \sqrt{x'}) - \log(x - x') \right] \tag{3.4.51}$$

$$= \log \frac{1}{4x} \ . \tag{3.4.52}$$

Setting $x = -E$, and putting everything together, we thus find the leading-order result:

$$\langle \rho_{\rm np}(E) \rangle_{\mathsf{KS}} \approx e^{S_0} \rho_0(E) - \frac{1}{4\pi E} \cos\left(2\pi \, e^{S_0} \int^E dE' \rho_0(E') \right) . \tag{3.4.53}$$

This is the same result was found in [36] using matrix model techniques. As one can see, the non-perturbative effects give small oscillations on top of the perturbative leading order density of states $\rho_0(E) = \frac{1}{4\pi^2} \sinh(2\pi\sqrt{E})$, where the size of the oscillations is controlled by $\lambda = e^{-S_0}$.

Density-density correlator. Next, we want to compute the density-density correlator:

$$\langle \rho_{\rm np}(E_1) \rho_{\rm np}(E_2) \rangle_{\mathsf{KS}} \ . \tag{3.4.54}$$

Of course, there will be the factorized contribution, but we will see that the interesting result comes from the connected contributions, which assemble into the so-called *sine-kernel*, well-known in the matrix model literature. Expanding the expression, there are many products that we need to compute:

$$\rho_{\rm np}(E_1) \rho_{\rm np}(E_2) = \rho(E_1) \rho(E_2) + \frac{1}{2\pi} \rho(E_1) (e^{\Omega(E_2)} + e^{-\Omega(E_2)}) \tag{3.4.55}$$

$$+\frac{1}{2\pi}(e^{\Omega(E_1)}+e^{-\Omega(E_1)})\rho(E_2)+\frac{1}{4\pi^2}(e^{\Omega(E_1)}+e^{-\Omega(E_1)})(e^{\Omega(E_2)}+e^{-\Omega(E_2)})\ .$$

The singularities in the cross-terms of $\rho(E)$ with $e^{\Omega(E)}+e^{-\Omega(E)}$ cancel, while the OPE's of $e^{\Omega(E_1)}e^{\Omega(E_2)}$ and $e^{-\Omega(E_1)}e^{-\Omega(E_2)}$ are also regular as $E'\to E$. The only singular contributions come from the products $\rho(E_1)\rho(E_2)$ and $e^{\pm\Omega(E_1)}e^{\mp\Omega(E_2)}$. The first gives the perturbative contribution, keeping only the genus zero terms:

$$\langle\rho(E_1)\rho(E_2)\rangle_{\mathsf{KS}}\approx-\frac{1}{2\pi^2(E_1-E_2)^2}+\text{reg.}\tag{3.4.56}$$

where we have neglected terms which are regular as $E'\to E$. This perturbative contribution to the density-density correlator is called the 'ramp', because after a double Fourier transform it gives rise to the linear growth of the spectral form factor. To obtain the second term, we compute the product:

$$e^{\Omega(E_1)}e^{-\Omega(E_2)}=\frac{1}{(E_1-E_2)^2}\left\{e^{\Omega(E_1)-\Omega(E_2)}\right\}\ .\tag{3.4.57}$$

Here, we combined the product of normal-ordered exponentials into a single normal-ordered exponential, with the normal ordering $\{\dots\}$ given by subtracting $\log(E_1-E_2)$ from the singular products $\Phi_0\Phi_0$ and $\Phi_1\Phi_1$, leading to the multiplicative factor. Now we can take the expectation value and keep only the genus zero contributions:

$$\langle e^{\Omega(E_1)}e^{-\Omega(E_2)}\rangle_{\mathsf{KS}}\approx\frac{1}{(E_1-E_2)^2}e^{\langle\Omega(E_1)-\Omega(E_2)\rangle_0+\frac{1}{2}\left\langle(\Omega(E_1)-\Omega(E_2))^2\right\rangle_0^{\mathsf{c}}}\ .\tag{3.4.58}$$

The square in the last term of the exponent should be appropriately normal-ordered by subtracting the singular pieces, as was the case for the density correlator. It can be easily evaluated using the free two-point functions to be:

$$\frac{1}{2}\left\langle(\Omega(x)-\Omega(y))^2\right\rangle_0^{\mathsf{c}}=\frac{1}{2}\Big[\langle\Omega(x)^2\rangle_0^{\mathsf{c}}-2\left\langle\{\Omega(x)\Omega(y)\}\right\rangle_0^{\mathsf{c}}+\langle\Omega(y)^2\rangle_0^{\mathsf{c}}\Big]\tag{3.4.59}$$

$$=\log\frac{1}{4x}+\log\frac{1}{4y}-2\left[\log\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)-\log(x-y)\right]\tag{3.4.60}$$

$$=\log\frac{(\sqrt{x}+\sqrt{y})^4}{16xy}\ .\tag{3.4.61}$$

Sending $x=-E_1$ and $y=-E_2$ and plugging this into (3.4.58), we find:

$$\langle e^{\Omega(E_1)}e^{-\Omega(E_2)}\rangle_{\mathsf{KS}}\approx\frac{1}{(E_1-E_2)^2}\frac{(\sqrt{E_1}+\sqrt{E_2})^4}{16E_2E_1}\,e^{\langle\Omega(E_1)-\Omega(E_2)\rangle_0}\tag{3.4.62}$$

$$= \left[\frac{1}{(E_1-E_2)^2} + \text{reg.}\right] e^{\langle \Omega(E_1)-\Omega(E_2)\rangle_0} \, . \tag{3.4.63}$$

In the last line, we again expanded E_1 around E_2 and kept only the singular piece. Repeating the calculation above we find the other OPE with signs flipped:

$$\langle e^{-\Omega(E_1)} e^{\Omega(E_2)} \rangle_{\mathsf{KS}} \approx \left[\frac{1}{(E_1-E_2)^2} + \text{reg.}\right] e^{-\langle \Omega(E_1)-\Omega(E_2)\rangle_0} \, . \tag{3.4.64}$$

Putting everything together gives the connected contribution to the non-perturbative density-density correlator:

$$\begin{aligned} &\langle \rho_{\mathrm{np}}(E_1)\rho_{\mathrm{np}}(E_2)\rangle^{\mathsf{c}}_{\mathsf{KS}} \sim \\ &\qquad \langle \rho(E_1)\rho(E_2)\rangle_{\mathsf{KS}} + \frac{1}{4\pi^2}\left(\langle e^{\Omega(E_1)} e^{-\Omega(E_2)}\rangle_{\mathsf{KS}} + \langle e^{-\Omega(E_1)} e^{\Omega(E_2)}\rangle_{\mathsf{KS}}\right) &(3.4.65)\\ &\qquad \approx -\frac{1}{2\pi^2 (E_1-E_2)^2}\Big[1-\cosh\left(\langle \Omega(E_1)-\Omega(E_2)\rangle_0\right)\Big] \, . &(3.4.66) \end{aligned}$$

Using that $\langle \Omega(E)\rangle_0 = \frac{2\pi i}{\lambda}\int^E \rho_0(E')dE'$ we conclude:

$$\langle \rho_{\mathrm{np}}(E_1)\rho_{\mathrm{np}}(E_2)\rangle^{\mathsf{c}}_{\mathsf{KS}} \approx -\frac{1}{\pi^2(E_1-E_2)^2}\sin^2\left(\pi e^{S_0}\int_{E_2}^{E_1} \rho_0(E')dE'\right) \, . \tag{3.4.67}$$

For $E_2 \to E_1$, the integral can be approximated by $(E_1-E_2)\rho(E_2)$. Adding the disconnected piece, we arrive at the main result of this section:

$$\begin{aligned} &\langle \rho_{\mathrm{np}}(E_1)\rho_{\mathrm{np}}(E_2)\rangle_{\mathsf{KS}} \approx \\ &\qquad \langle \rho_{\mathrm{np}}(E_1)\rangle_0 \langle \rho_{\mathrm{np}}(E_2)\rangle_0 - \frac{\sin^2\!\left(\pi e^{S_0}(E_1-E_2)\rho_0(E_2)\right)}{\pi^2(E_1-E_2)^2} \, . &(3.4.68) \end{aligned}$$

This is only an approximate answer, in the sense that we only considered genus zero contributions and were interested in the singular part of the products. The above computation was merely meant to show the universal behaviour of $\langle \rho(E_1)\rho(E_2)\rangle_{\mathsf{KS}}$ for $|E_1 - E_2| \ll 1$. Here, we have shown that the non-perturbative contributions in the universal form of the sine-kernel [90] can be understood as arising from branes in the KS theory. From the universe field theory side, these are described by (bilinears of) fermion fields, while on the JT gravity side, they describe D-branes where fixed-energy boundaries can end on.

3.5 Discussion

Let us now discuss some subtleties, open questions and directions for future research.

Open/closed duality. We have proposed that the modified holographic dictionary which relates JT gravity to a matrix integral is in fact a consequence of a more standard open/closed duality in topological string theory. Formulating JT gravity in terms of the KS theory allows for a direct interpretation in string theory and the 'ensemble average' should then correspond to the path integral in the open string field theory dual of the KS theory, as outlined in Figure 3.2. Although we have identified the relevant quantities on both sides, a detailed account of the duality is still an open question. This requires a more careful study of both the compact and non-compact D-branes that we have introduced.

An interesting step in this direction might be found in [139], where a Kontsevich matrix model arises from the localization of a cubic open string field theory. It would be interesting to make the connection between this perspective and the dynamics of the non-compact branes more precise. More speculatively, we expect that the open string field, which can be represented by a Hermitian matrix H, should be viewed as the Hamiltonian of some quantum mechanical system associated to the branes in the theory. It would be interesting to see if the underlying fermionic theory of the open string degrees of freedom can be in some way related to the SYK model.

Baby universes and α-states. Moreover, we expect that the KS theory gives a well-defined construction of the baby universe Hilbert space (as defined in [100]) for JT gravity. One can formally represent the path integral of KS theory in terms of an operator formalism. This would naturally lead to a notion of boundary operators $\widehat{Z}(\beta)$. In fact, we expect a slight modification of the construction by Marolf and Maxfield [100] in the sense that we need to consider a larger algebra of observables by adding the 'canonical momentum' of $\widehat{Z}(\beta)$. We would then have a non-commutative algebra of observables with not only boundary creation operators but also boundary annihilation operators. This is close to the original approach taken in [101].

In particular, this construction would lead to a precise definition of the Hartle-Hawking state $|\mathrm{HH}\rangle$ with non-trivial topology, in terms of the interacting vacuum of the KS theory. We can represent this state geometrically by an integration over half the spectral curve. That is, we cut open the path integral on the slice $\mathrm{Re}(z) = 0$. The resulting baby universe Hilbert space is infinite-dimensional and distinct from the baby universe Hilbert space that will be discussed in Appendix

A.2, since the in- and out-states are treated *symmetrically*. In principle, this construction would give us a definition of the microscopic α-states $|\alpha\rangle$ in JT gravity, and an understanding of their role in the factorization problem [140].

Non-perturbative effects. It would be interesting to study further the non-perturbative effects that were touched upon in Section 3.4. Although it seems that the brane/anti-brane perspective leads to the correct results (3.4.53) and (3.4.68), the geometrical interpretation of the precise mechanism is still rather mysterious. For example, the non-perturbative correction in (3.4.44) and (3.4.45) decouples the the two fields $\partial\Phi_0$, and $\partial\Phi_1$ in the sense that they are not anymore related to each other by a 2π rotation. This seems to agree with the perspective that non-perturbative effects have a dramatic effect on the target space decoupling the two sheets of the branched geometry [132]. In particular, it leads to a branch cut extending over the whole real axis $[-\infty, \infty]$.

It would also be interesting to understand the connection to [49], where an effective field theory for the late-time behaviour of quantum chaotic systems is presented (see also [141]). For example, one could try to find an interpretation of the Altshuler-Andreev saddle and notion of causal symmetry breaking, that are important in the computation of the 'plateau' feature of the spectral form factor, in the topological string theory setup. We expect these effects to become visible in the open string field theory description dual to KS theory. These results are presented in Chapter 4.

Super JT gravity. There are some generalizations of the construction which are worth studying. It would be interesting to carry out a similar analysis in the case of JT supergravity [92, 142]. It is defined on super Riemann surfaces for which a recursion relation similar to Mirzakhani's is derived. The topological recursion for the matrix model associated to super JT is related to the Brezin-Gross-Witten and the Bessel model [143]. Introducing both fermions and bosons, the super-Virasoro algebra generated by the combined stress tensor leads to super-Virasoro constraints [144]. A natural question is if these are equivalent to the 'super-Mirzakhani recursion' and if we can extend the KS theory to a supersymmetric model, whose SD equations impose the super-Virasoro constraints.

Pure 3d gravity as an ensemble. The precise form of the KS action and its relation to JT gravity relied heavily on the interpretation of the spacetime in terms of a string world-sheet. In that sense, the construction seems to be very specific to models of 2-dimensional gravity. There is some evidence that wormholes in pure 3-dimensional gravity can also be understood in terms of some averaging prescription (research in this direction includes [95, 145–151]). Our derivation was fundamentally based on the universal recursive structure expressed in terms of the SD equation. If one could unearth a similar recursive structure in higher-dimensional theories of quantum gravity, this would open up a way for finding a similar field theory description.

We hope to address some of these questions in future work.

4 Chaos in 2D gravity

4.1 Introduction

In recent years, there has been a renewed interest in two-dimensional quantum gravity, most notably JT gravity [33, 36, 56, 64, 82], its supersymmetric cousins [83, 92, 143], and more general dilaton gravity theories [98, 152, 153]. In particular, the focus has been on understanding these toy models of quantum gravity at a fully non-perturbative level. Much research has concentrated on the proposed completions as double-scaled matrix models, whose universal features capture hallmarks of an underlying chaotic microscopic theory, such as the plateau in the spectral form factor [43, 93].

An alternative approach proposed [1] treats the two-dimensional JT universe as the worldsheet of a closed topological string, whose splitting and joining is described by a field theory in target space. The diagrammatic expansion of this simple interacting 2d CFT – dubbed a 'universe field theory', after [105] – corresponds to the genus expansion of the gravitational path integral, while non-perturbative information can be accessed by allowing JT strings to end on D-branes. Besides conceptually clarifying the matrix model origins of JT gravity, it also offers the technical advantage of working directly in the double-scaling limit.

In this chapter, we use our universe field theory to connect JT gravity to a hallmark in the field of quantum chaos, namely the 'supersymmetry method', as pioneered by Efetov [154] and recently applied to holography in [49, 141]. Given some ensemble that models a quantum chaotic system, denoted by $\langle \ldots \rangle_H$, the supersymmetry method extracts moments of the ensemble from ratios of determinants

$$D_n(X) = \left\langle \frac{\det(x_1 + H)\det(x_2 + H)\ldots\det(x_n + H)}{\det(\mathsf{x}_1 + H)\det(\mathsf{x}_2 + H)\ldots\det(\mathsf{x}_n + H)} \right\rangle_H . \tag{4.1.1}$$

The complex variables $x_i = -E_i \pm i\eta$ and $\mathsf{x}_i = -\mathsf{E}_i \pm i\eta$ contain real energy arguments as well as infinitesimal imaginary offsets which mark the causal structure of the correlator[1] [155, 156]. By taking derivatives of $D_n(X)$ and setting the energy arguments equal, one obtains nth moments[2] of the spectral density $\rho(E) = \operatorname{tr}\delta(E-H)$. The method is called 'supersymmetric' because the determinants and inverse determinants can be represented by integrals over fermionic and bosonic vector degrees of freedom, manifesting an underlying $\mathrm{U}(n|n)$ supergroup structure. In the limit where the energy differences $\Delta E = |E_i - E_j|$ become small and approach the mean level spacing Δ, the global $\mathrm{U}(n|n)$ is spontaneously broken to $\mathrm{U}(\frac{n}{2}|\frac{n}{2}) \times \mathrm{U}(\frac{n}{2}|\frac{n}{2})$ and the correlator (4.1.1) reduces to a non-linear σ-model [157]:

$$D_n(X) \simeq \int_{\mathrm{AIII}_{n|n}} dQ \, \exp\left[i\frac{\pi}{\Delta}\mathrm{str}(XQ)\right] \,. \tag{4.1.2}$$

on the coset manifold [158]

$$\mathrm{AIII}_{n|n} \equiv \frac{\mathrm{U}(n|n)}{\mathrm{U}(\frac{n}{2}|\frac{n}{2}) \times \mathrm{U}(\frac{n}{2}|\frac{n}{2})} \,. \tag{4.1.3}$$

This σ-model universally captures the late time physics of any quantum chaotic system: it only depends on the mean level spacing Δ and the symmetry class of the ensemble[3]. For JT gravity, the mean level spacing at energy E is determined to first order in e^{-S_0} by the Schwarzian density of states [34]

$$\Delta^{-1} = \frac{e^{S_0}}{4\pi} \sinh\sqrt{2\pi E} \,. \tag{4.1.4}$$

This is a perturbative input derived from semi-classical gravity. Crucially, we will show that JT gravity, *non-perturbatively* completed by its universe field theory, is capable of reproducing the full σ-model (4.1.2), thus demonstrating its quantum chaotic nature.

From KS theory to the σ-model of quantum chaos. To derive the σ-model directly from JT universe field theory, we study correlation functions of vertex operators $e^{\Phi(x)}$ and $e^{-\Phi(\mathsf{x})}$, which can be seen as the double-scaled analogs of the determinant and inverse determinant operators in (4.1.1). The field $\Phi(x)$ is a $\mathbb{Z}_2$-twisted chiral boson living on the JT spectral curve, as we showed in Chapter 3. Previously, we showed that correlation functions of the current $\partial\Phi$ compute all-

[1] We will always use serif variables for arguments of determinants, and sans serif for inverse determinants.

[2] For example, from $\mathcal{D}_2(X)$ one extracts the density-density correlator, which after Laplace transform to $\langle Z(\beta_1)Z(\beta_2)\rangle$ and analytic continuation of the inverse temperatures yields the spectral form factor.

[3] In the above discussion we have assumed the ensemble to be unitary invariant, but there are in total 10 distinct symmetry classes, following the classification of [159].

genus multi-boundary wormhole amplitudes in JT gravity, after inverse Laplace transform [1]. This time we take an inverse Laplace (or Fourier) transform of the vertex operators $e^{\pm\Phi(x)}$ to demonstrate, in Section 4.3, that their correlation function can be rewritten as a *flavor matrix integral* over the space of Hermitian supermatrices in $\mathrm{GL}(n|n)$:

$$\left\langle \left\{ e^{\Phi(x_1)} e^{-\Phi(\mathsf{x}_1)} \cdots e^{\Phi(x_n)} e^{-\Phi(\mathsf{x}_n)} \right\} \right\rangle_{\mathsf{KS}} = \int_{(n|n)} dA\, e^{-e^{S_0}\Gamma(A) + e^{S_0}\mathrm{str}(XA)} \, . \quad (4.1.5)$$

Here the curly brackets denote a normal ordering prescription for the product of vertex operators, and the angular brackets denote the expectation value in Kodaira-Spencer (KS) universe field theory. On the right-hand side, the matrix potential $\Gamma(A)$ can be computed perturbatively, to arbitrary order in e^{-S_0}, from the topological recursion relations satisfied by $\langle \partial\Phi(x_1) \dots \partial\Phi(x_n) \rangle_{\mathsf{KS}}$. To leading order, it is given by the single supertrace $\Gamma(A) = \mathrm{str}\,\Gamma_0(A)$, where the functional form of $\Gamma_0(y)$ is determined by solving the spectral curve equation $H(x,y) = 0$ and integrating $-xdy$, the one-form dual to the canonical holomorphic one-form $\omega = ydx$, giving

$$\Gamma_0(y) = -\int^{y} x(y')dy' \, . \quad (4.1.6)$$

In the case of the 'Airy' spectral curve, $H(x,y) = y^2 - x$, which governs the low energy behavior of all topological gravity models [76], the potential is cubic $\Gamma(A) = \frac{1}{3}\mathrm{str}(A^3)$, and the flavor matrix integral becomes a graded version of the celebrated Kontsevich matrix model [88]. For JT gravity, the spectral curve is derived from the Schwarzian density of states (4.1.4), and is given by $H(x,y) = y^2 - \frac{1}{(4\pi)^2}\sin^2(2\pi\sqrt{x})$. Like the Kontsevich model, one has to select appropriate integration contours for the eigenvalues of A, which are analyzed in Appendix B.

After establishing the duality (4.1.5), we perform a stationary phase analysis of the flavor matrix integral in the limit that the probe energies approach each other, $\Delta E \to 0$, and the mean level spacing Δ (and hence e^{-S_0}) goes to zero, while keeping their ratio fixed to

$$s = \Delta E / \Delta \, . \quad (4.1.7)$$

Since $\Delta E \to 0$, we are probing the very late time behavior of the system. In this 'late time limit' there is a whole saddle-point manifold over which one should integrate, which turns out to be precisely the coset manifold $\mathrm{AIII}_{n|n}$. In fact, we show that the flavor matrix theory reduces to the non-linear σ-model of quantum chaos in the late time limit

$$\int_{(n|n)} dA\, e^{-e^{S_0}\Gamma(A) + e^{S_0}\mathrm{str}(XA)} \xrightarrow[\text{phase}]{\text{stationary}} \int_{\mathrm{AIII}_{n|n}} dQ\, e^{i\frac{\pi}{\Delta}\mathrm{str}(XQ)} \, , \quad (4.1.8)$$

where Δ^{-1} is given by the disk JT density of states (4.1.4). This result demonstrates that the completion of JT gravity in terms of its universe field theory knows both about the perturbative (in e^{-S_0}) sum over topologies, as well as the fully non-perturbative late time ergodic physics. For example, when $s \gg 1$, the main contribution to the σ-model comes from a perturbative expansion around the standard saddle. For $s \gtrsim 1$, a new class of supersymmetry breaking saddles becomes important, the so-called Andreev-Altshuler saddle points [160], as described in [49]. When $s < 1$, one needs to integrate Q over the full Goldstone manifold, corresponding to a phase where causal symmetry is restored. As one can see, each of these phases is captured by universe field theory, and so the result (4.1.8) improves on [1] where similar vertex operator calculus was used to derive the sine kernel in JT gravity.

The open string perspective. It may seem like a miracle that a theory of semi-classical gravity should be sensitive to the late time quantum chaotic properties of the underlying microscopics. Our aim is to give a gravitational interpretation of this result, using intuition from (topological) string theory.[4] The main idea is to access non-perturbative information (in e^{-S_0}) by allowing JT strings to end on D-branes embedded in a higher dimensional target space CY. This six-dimensional Calabi-Yau is a fibration of the spectral curve, defined by

$$uv - H(x,y) = 0 \ , \tag{4.1.9}$$

where $u, v, x, y \in \mathbb{C}$. To the Calabi-Yau one associates a holomorphic $(3,0)$-form

$$\Omega = \frac{du}{u} \wedge dx \wedge dy \ . \tag{4.1.10}$$

We can wrap a one-complex-dimensional brane on the non-compact submanifold $u = 0$, which is parametrized by v and a point on the spectral curve. Similarly, we define anti-branes by wrapping them on $v = 0$. From $uv = 0$ we see that $\frac{du}{u} = -\frac{dv}{v}$, so that exchanging u and v amounts to a change of sign for Ω. This shows that anti-branes can be viewed as branes with the opposite orientation. Inserting a vertex operator $e^{\Phi(x)}$ creates a brane, while $e^{-\Phi(\mathsf{x})}$ creates an anti-brane, above a point x on the spectral curve. The real part of x is interpreted as an energy E in the Schwarzian theory, which is kept fixed by imposing fixed energy boundary conditions on the JT gravity action [121, 136]. On the level of JT gravity, the fixed energy boundary term is the Legendre transform of the usual Dirichlet boundary term for fixed inverse temperature β. So, in the universe field theory language, inserting a vertex operator $e^{\Phi(x)}$ creates a brane in target space

[4]These ideas trace back to the seminal work of Dijkgraaf and Vafa on topological strings and matrix models [37–40], as well as work by Maldacena, Moore, Seiberg and Shih on minimal string theory [81, 132].

on which JT worldsheets with boundary energy E can end.

This D-brane point of view gives a simple explanation for the appearance of the flavor matrix theory (4.1.5). Namely, inserting n pairs of vertex operators $e^{\Phi(x_i)}e^{-\Phi(\mathsf{x}_i)}$ creates n brane/anti-brane pairs. When we consider the limit of coincident probe energies, $x_i \to \mathsf{x}_i$, we get a stack of branes and anti-branes and the gauge group enhances to $\mathrm{U}(n|n)$ [130]. This is precisely the 'causal symmetry' alluded to in the context of quantum chaos. Indeed, we show in Section 4.4.1 that the effective brane worldvolume theory is to leading order in e^{-S_0} equal to the flavor matrix integral. The small imaginary offsets of the brane positions x_i spontaneously break the $\mathrm{U}(n|n)$ causal symmetry and finite energy differences ΔE give rise to massive modes – this is the well-known Higgs effect for D-branes.

The open string perspective also provides an explanation for the color-flavor duality which connects the double-scaled matrix model of Saad, Shenker and Stanford to the Kontsevich-like flavor matrix integral presented in this chapter. This duality has been established using the supersymmetry method and a generalized Hubbard-Stratonovich transformation in [161, 162]. In Section 4.4, we give this duality an interpretation using the open string field theory in the target space CY. Namely, the degrees of freedom of the flavor matrix theory are open JT strings ending on the non-compact branes introduces above, whereas those of the color matrix theory are open JT strings ending on *compact* branes, described in detail in Section 4.4.2, which wrap the blown-up singularities of the spectral curve.

Connection to SYK. Finally, our results establish an interesting new connection to the SYK model. As is well known, the SYK model [57, 58] reduces at low energies to the Schwarzian theory [60], which is the same as JT gravity on the disk. This relates JT and SYK on the most coarse-grained level, or at early times. Interestingly, we have found that also the very late time description of (non-perturbatively completed) JT quantum gravity, namely the non-linear σ-model (4.2.5), is precisely the same as the late time ergodic phase of the SYK model derived in [50]. Of course, this does not prove a full duality between JT and SYK, but it shows that they are in the same universality class: they have the same mean level spacing (4.1.4) and the same pattern of causal symmetry breaking.

Outline: This chapter is meant to bridge a gap between the communities of quantum chaos and holography. We have therefore summarized some necessary background in Section 4.2: flavor matrix theory (fMT) is introduced as an organizing principle for quantum chaos. A more comprehensive treatment of can be found in [49]. After setting the scene, we derive the non-linear σ-model of quantum chaos directly from JT universe field theory in Section 4.3. This is accomplished

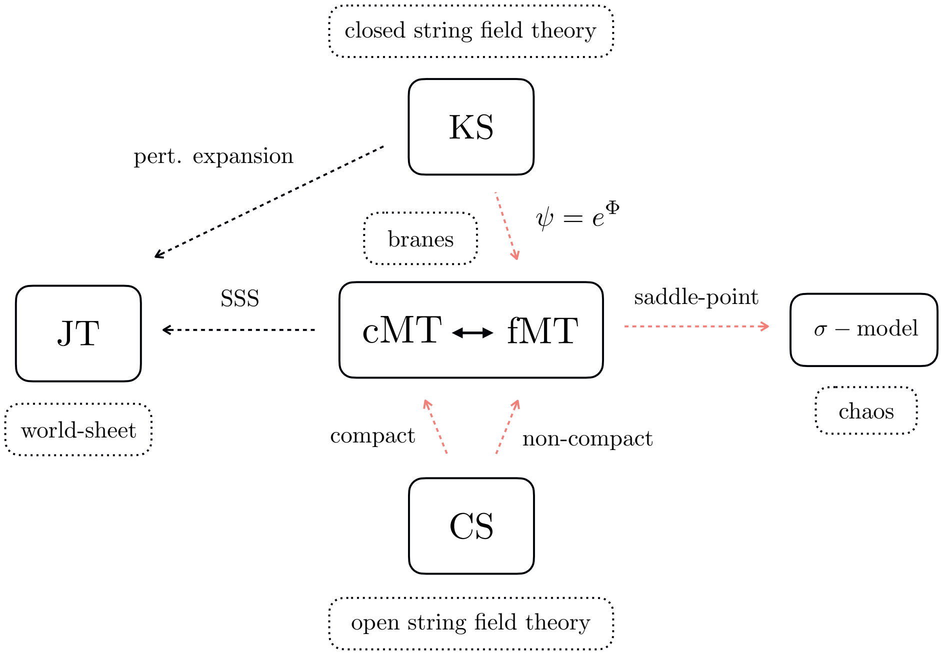


Figure 4.1: *A comprehensive diagram of the relevant theories in this chapter. Euclidean JT gravity (left) is defined perturbatively as a sum over topologies. It may be completed non-perturbatively – via the Saad, Shenker, Stanford (SSS) duality – by a large L color matrix theory (cMT), double-scaled to the spectral edge, or by a flavor matrix integral (fMT) using the color-flavor duality (middle). Both descriptions can be derived exactly from the KS theory (top) or from the brane world-volume Chern-Simons (CS) theory (bottom), by the insertion of suitable D-branes. A saddle point approximation of fMT then leads to the σ-model of quantum chaos (right). The arrows that will be covered in detail are highlighted in red.*

by inserting brane/anti-brane operators and Fourier transforming them to a fMT, whose saddle-point approximation in the late time limit is the sought-after σ-model. In Section 4.4, we interpret this result by studying the brane worldvolume theory, whose effective description when the branes coincide is given by a dimensional reduction of $\mathrm{U}(n|n)$ holomorphic Chern-Simons theory. By identifying both compact and non-compact branes in the target space geometry, we give an open string interpretation of the color-flavor map. In Appendix B we perform a stationary phase analysis of the flavor matrix integral and select the defining integration contours. It is shown how Stokes' phenomena lead to causal symmetry breaking, which allows us to identify which saddle points contribute. A schematic overview of the relevant concepts and their interrelations is presented in Figure 4.1.

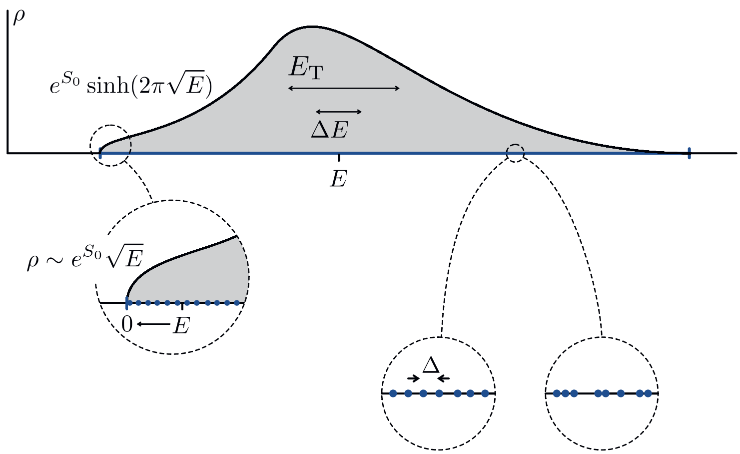


Figure 4.2: *Spectral density of a chaotic quantum system. A total number of L microstates is contained in a compact spectral support defined by a non-vanishing average spectral density $\rho(E)$. In the gravitational context, one is often interested in energies 'double scaled' to the ground state, $E = 0$, inset left. Different from the energy levels of a generic system (right inset), levels of chaotic systems are almost uniformly spaced (middle) and cannot 'touch'.*

4.2 Setting the scene

4.2.1 The nonlinear σ-model of quantum chaos

One of the beautiful aspects of ergodic chaotic quantum systems is that they essentially all behave the same way. This phenomenon is often paraphrased as *random matrix universality*: "Quantum systems that are classically chaotic are equivalently described by random matrix theory at large time scales" [44]. Here, 'large time scales' refers to scales longer than the time it takes to establish ergodic equilibration of the dynamics, called the Thouless time t_{T}. This scale is non-universal, and needs to be determined on a system-to-system basis. The longest characteristic time scale is the Heisenberg time (aka *plateau time*), $t_{\mathrm{H}} = \Delta^{-1}$, where $\Delta = \langle\rho(E)\rangle^{-1}$ is the average microstate spacing at the chosen probe energy E. For generic chaotic quantum systems, $\langle\rho(E)\rangle$ of a system with a total number of L microstates exhibits smooth dependence on E, see Figure 4.2.

One might object that the statement of random matrix universality is somewhat of a tautology. Conceptually, a random matrix Hamiltonian is just another representative in the class of chaotic quantum systems. So the statement is but repeating that they all behave identically. A more substantial characterization is as follows: chaotic quantum systems spontaneously break a continuous symmetry related to the causal structure of time evolution. In ergodic quantum dynamics, this symmetry gets restored after Heisenberg time t_H. This symmetry breaking principle finds its quantitative formulation in a simple mean field theory, which takes the form of a non-linear σ-model. Much as mean field theory for, say, a magnetization order parameter captures the universal features of of ferromagnetism, the σ-model describes the universality class of ergodic quantum chaos.

In order to explain the symmetry breaking principle in general terms, let us write the determinant ratio from the Introduction, (4.1.1), as a superdeterminant

$$D_n(X) = \left\langle \mathrm{Sdet}(X \otimes \mathbb{1}_\mathrm{c} + \mathbb{1}_\mathrm{f} \otimes H) \right\rangle_H \equiv \langle \mathrm{Sdet}(\Xi) \rangle_H \,. \tag{4.2.1}$$

We think of Ξ as an operator acting in a product Hilbert space $\mathcal{H} = \mathcal{H}_\mathrm{f} \otimes \mathcal{H}_\mathrm{c}$ of a $2n$-dimensional graded 'flavor-space' $\mathcal{H}_\mathrm{f} = \mathbb{C}^{n|n}$ and L-dimensional 'color space' $\mathcal{H}_\mathrm{c} = \mathbb{C}^L$. As such, Ξ carries an adjoint representation under the group $\mathrm{U}(nL|nL)$. Transformations under this group change $\Xi \to U \Xi U^{-1}$ but naturally leave our determinant correlation functions invariant. This huge symmetry group possesses two interesting subgroups, the color group $\mathrm{U}_\mathrm{c} = \mathbb{1}_\mathrm{f} \otimes \mathrm{U}(L)$ which acts on $H \to UHU^{-1}$, leaving X invariant, and the $2n$ dimensional flavor group $\mathrm{U}_\mathrm{f} = \mathrm{U}(n|n) \otimes \mathbb{1}_\mathrm{c}$, changing $X \to TXT^{-1}$, but leaving H invariant. The interplay of the dual pair defined by the color and the flavor symmetry group is key to the characterization of universality in quantum chaos.

The idea behind the construction of effective field theories of quantum chaos is to turn the complexity of the theory in color space into an advantage: Averaging over realizations of H will eradicate contributions to the spectral determinant that fluctuate strongly under the action of the color group[5]. Eventually, in ergodic limits, only contributions in the color singlet representation survive. In this projection onto the color singlet sector, the measure associated to the integration over the H-measure gets converted into an integration over flavor degrees of freedom. This *color-flavor* duality assumes its purest form in the case of invariant matrix ensembles, where $\langle \ldots \rangle_H = \int dH \, \exp(-L\mathrm{tr}\, V(H))$, with a potential function $V(H)$. It can be shown that [49]

$$\left\langle \mathrm{Sdet}(X \otimes \mathbb{1}_\mathrm{c} + \mathbb{1}_\mathrm{f} \otimes H) \right\rangle_H = \left\langle \mathrm{Sdet}(X \otimes \mathbb{1}_\mathrm{c} + A \otimes \mathbb{1}_\mathrm{c}) \right\rangle_A \,, \tag{4.2.2}$$

where $A \in \mathrm{GL}(n|n)$ is a flavor matrix, with n 'bosonic'and n 'fermionic'eigenvalues,

[5] The minimal averaging measures required to block color space fluctuations must be determined on a case to case basis. However, a rule of thumb is that for systems with ergodic chaotic dynamics, averages over parameter windows corresponding to just a few microstate spacings can already be sufficient.

and the flavor matrix integral is $\langle \ldots \rangle_A = \int dA \exp(-L \operatorname{str} W(A))$. In the Gaussian case, $V(H) = H^2$, the flavor matrix potential $W(A) = A^2$ is also quadratic in A [48]. For general potentials, the color and flavor ensembles agree on the level of the generating function [49]. The representation on the right then defines the starting point for the construction of an effective flavor matrix theory. Besides A being a low dimensional matrix, the advantage of this representation is that $\operatorname{Sdet}(X \otimes \mathbb{1}_\mathrm{c} + A \otimes \mathbb{1}_\mathrm{c}) = \operatorname{sdet}(X + A)^L = \exp(-L \operatorname{str} \ln(X + A))$, due to color isotropy: the flavor theory is amenable to a stationary phase analysis stabilized by the large parameter L.

To anticipate what is awaiting us in this large-L analysis, let us go one step back to the theory before having taken any averages. For small differences between the probe arguments, (x_i, x_i), flavor symmetry is an approximate symmetry not just of the correlation functions but of the operator Ξ itself: $T\Xi T^{-1} \approx \Xi$. In view of what has been said above, we expect these transformations to become soft degrees of freedom of the putative flavor matrix theory (fMT). To understand the structure of these theories, it is key to realize that they all (irrespective of the detailed realization of H) live under the spell of a symmetry breaking principle. In physically meaningful correlation functions, flavor symmetry is never realized in a strict sense: we need our probe arguments infinitesimally shifted into the complex plane as, say, $\operatorname{Im}(x_j) = \pm i\eta$, and the same with the probes in the denominator, x_i. Even in the limit $\operatorname{Re}(x_j) = \operatorname{Re}(\mathsf{x}_j) = -E$, flavor symmetry remains infinitesimally broken by these increments. Referring to their physical meaning as indicators of causality, we refer to the approximate symmetry under flavor transformations as *causal symmetry* of the theory.

This is as much as can be said in the most general terms; no reference to chaos or concrete realizations of H is made so far. However, let us now turn back to the role played by the ensemble average $\langle \ldots \rangle_H$ over microscopically different realizations of any model. The eigenvalues of individual H's define a set of finely spaced, yet isolated singularities of Ξ in the complex $x-$plane. Averaging over an ensemble will blur these point singularities into a cut structure, see Figure 4.2. On the level of a crudest 'mean field' approximation applied to the averaged theory, we expect the correlation function to assume the form

$$\langle \operatorname{Sdet}(\Xi) \rangle_H \longrightarrow \operatorname{Sdet}\left((X + i\gamma\tau_3) \otimes \mathbb{1}_\mathrm{c} + \mathbb{1}_\mathrm{f} \otimes H_0\right) , \tag{4.2.3}$$

where H_0 is a possible un-averaged contribution to H, $\gamma = \gamma(E)$ a finite imaginary offset non-vanishing along the support set of the theory (the interval(s) of E for which $\rho(E) > 0$), and $\hat{\tau}_3 = \tau_3 \otimes \mathbb{1}_\mathrm{f}$ is a flavor Pauli matrix with sign structure set by the η-parameters.[6] The key point here is that the infinitesimal η sets the

[6] Without loss of generality, we assume as many positive as negative increments $i\eta$.

sign of the finite γ: causal symmetry is spontaneously broken at the mean field level inside the spectral curve. Still, no reference to chaos is made: averaging over realizations of an integrable theory defines a cut structure too.

It is natural to expect that in a theory with large symmetry group, the mean field configuration will be subject to fluctuations. To get an idea of their influence, consider the full symmetry group action of $U(nL|nL)$ on (4.2.3). Such transformations preserve eigenvalues, and hence are compatible with the analytic structure of the theory. However, at this point, differences between integrable and chaotic parent theories begin to show. In the former case, fluctuations with non-trivial color space structure may be physically significant even at large time scales (provided they commute with H_0.) However, in a chaotic ergodic phase the above mentioned projection onto the color singlet sector becomes effective: Only $T \in \mathrm{U_f}$ remains in the fluctuation spectrum, i.e. we are reduced to the degrees of freedom of fMT. In fact, a stronger statement can be made. For n even, the effective degrees of freedom assume the form

$$Q \equiv T\hat{\tau}_3 T^{-1} \in \frac{\mathrm{U}(n|n)}{\mathrm{U}(\frac{n}{2}|\frac{n}{2}) \times \mathrm{U}(\frac{n}{2}|\frac{n}{2})}\ , \tag{4.2.4}$$

where the divisor represents the unbroken symmetry group (transformations commuting with $\hat{\tau}_3$). As in the introduction, we denote the coset supersymmetric target space with Cartan's notation $\mathrm{AIII}_{n|n}$. Referring for a more detailed discussion to Section 4.3.3, these coset degrees of freedom arise as the reduction of fMT to a non-linear σ-model. For finite differences between the probe arguments contained in X, the fluctuations Q acquire a mass. To lowest order in this explicit symmetry breaking, we will end up with a fluctuation integral

$$D_n(X) \simeq \int_{\mathrm{AIII}_{n|n}} dQ\, e^{iS[Q]}\ , \qquad S[Q] \equiv \frac{\pi}{\Delta}\mathrm{str}(XQ)\,, \tag{4.2.5}$$

where $\Delta = \langle \rho(E) \rangle^{-1}$ is the averaged spectral density at the center value E (the only characteristic energy scale in the problem). This is the non-linear σ-model mentioned in the introduction. It provides a complete description of the ergodic phase of quantum chaos.

To see how, let us discuss the role of the flavor coset space fluctuations described by (4.2.5). In the limit of small energy differences $|E_i - E_j| \equiv \Delta E \sim \Delta$, large fluctuations signal that the 'true configurations' of the theory are far detached from the naive cut-saddle points. These fluctuations act to restore the previously broken causal symmetry. Remembering that causal symmetry breaking was equivalent to the emergence of a cut structure along the spectral curve, the restoration of this symmetry in the limit of small energy difference, or large times, must amount to the re-emergence of information on the discrete pole structure of the chaotic spectrum. Indeed, one can do the Q integral in closed form to verify that it produces the exact correlation functions ('ramp+plateau') of ergodic quantum chaos.

En route to the deep limit $\Delta E \lesssim \Delta$, one encounters various physically interesting intermediate structures: for $\Delta E \gg \Delta$ the perturbative expansion around the 'standard saddle point' $Q_{st} = \hat{\tau}_3$ defines an asymptotic series equal to the perturbative expansion of other effective theories of quantum chaos. Specifically, it can be shown to be identical to the topological expansion of conventional color matrix theory (more precisely to the limit of that expansion for small differences $\gamma \gg \Delta E \gtrsim \Delta$), or to the mini-universe expansion of JT gravity in the same limit. For $\Delta E \sim \Delta$, and $n = 2$, a second, supersymmetry breaking saddle point $Q_{AA} = \tau_3 \otimes \tau_3$ begins to play a rôle. This saddle point is known as the Altshuler-Andreev saddle, and related to a standard saddle by a discrete (Weyl group) transformation in $\mathrm{U_f}$ [49].

Summarizing, a combination of phenomenological arguments and symmetry considerations identifies the σ-model (4.2.5) as the effective theory of the quantum ergodic phase. An implicit assumption in this construction was that our parent theory, H, contains no anti-linear symmetries besides hermiticity. More generally, one needs to distinguish between ten different classes of anti-linear symmetries, and in the consequence ten different incarnations of fMT's [159]. All have in common that they assume the form of integrals over low dimensional 'classical' supergroups or -coset spaces. In view of the generality of the construction, one expects *any* theory describing an ergodic quantum phase must collapse to one of these variants in the long time limit. In this chapter, we demonstrate this reduction principle for a family of theories of two-dimensional gravity that includes JT gravity. To understand how this happens, we first introduce universe field theory, which will be our main tool connecting JT to quantum chaos.

4.3 fMT from universe field theory

In this section we derive a flavor matrix theory from brane creation operator insertions in KS universe field theory. Our construction of the flavor matrix theory in this section proceeds in three steps. In Section 4.3.1 we introduce vertex operators in KS theory. Seen through the lens of the flavor matrix model, they probe eigenvalue correlations along the spectral curve. From the target space point of view, they create branes and anti-branes. Either way, they play the same the role as the determinant operators in (4.1.1), but instead of averaging over large color matrices, we are computing a Euclidean correlation function in KS field theory. We show in Section 4.3.2 that the correlator of brane/anti-brane vertex operators leads to an eigenvalue representation of a flavor matrix integral. The crucial ingredient is to use the transformation properties of $e^{\pm\Phi(x)}$ under symplectic transformations. Having identified the fMT of JT gravity, the stationary phase analysis of this integral (Section 4.3.3) naturally gives rise to the nonlinear σ-model discussed in Section 4.2.1.

4.3.1 (Anti-)brane creation operators

In Chapter 3 we have argued that semiclassical JT gravity is captured by the perturbative expansion of the KS field theory on the spectral curve $\mathscr{S}_{\mathsf{JT}}$. However, our goal is to show that the fully *non-perturbative* physics of the theory is described by a fMT, and upon further reduction the universal non-linear σ-model (4.2.5) presented in Section 4.2.1.

As a first step towards realizing the fMT theory in the KS framework, we introduce D-brane-like objects in the target space geometry assuming the role of the the probe determinants in (4.1.1). In fact, the B-model topological string theory allows for certain non-compact branes that do precisely that (see [40, 123], where they are referred to as 'B-branes'): they probe a particular 'eigenvalue' in the spectral $x-$plane. Topological (anti-)branes wrapped around the submanifold

$$\mathcal{B} : u = 0\,, \qquad (x, y) \in \mathscr{S}_{\mathrm{JT}}\,, \tag{4.3.1}$$

in the Calabi-Yau (3.4.2) give rise to vertex operators[7]

$$\psi(x) = e^{\Phi(x)}\,, \qquad \psi^{\dagger}(x) = e^{-\Phi(x)}\,, \tag{4.3.2}$$

on the spectral curve. Given the identification $\mathrm{Re}(x) = -E$, we think about the insertion of a fermionic field in (4.3.2) as defining a topological (anti-)brane, on which JT universes with 'fixed energy boundaries' can end. The precise boundary condition for the JT 'worldsheet' theory is Dirichlet-Neumann (DN), where one fixes the dilaton and its normal derivative at the boundary. (Various choices of boundary conditions in JT gravity, including DN, are nicely summarized in [121].) This is to be contrasted with the Dirichlet-Dirichlet (DD) boundary condition, that leads to the canonical partition function where one fixes the dilaton as well as boundary lengths $\beta_1, \beta_2, \ldots$. These correspond to temperatures in the Schwarzian boundary theory. On the level of the action, one can move between both choices of boundary condition by a suitable Legendre transform, so we can think about the DN boundary conditions as fixing the energies $E_1, E_2, \ldots$ in the boundary theory: it defines a microcanonical partition function. The path integral associated to

[7] These are normal ordered exponentials, meaning that the OPE divergences from $\Phi\Phi$ contractions have been subtracted. Technically, Φ should be understood here as the indefinite integral of $\mathcal{J}$.

the fixed energy boundaries is related to the fixed temperature boundaries by yet another inverse Laplace transform

$$\mathcal{Z}(E_1,\ldots,E_n) = \int_{c-i\infty}^{c+i\infty} \prod_{i=1}^{n} \frac{d\beta_i}{\beta_i} e^{\beta_i E_i} \mathcal{Z}(\beta_1,\ldots,\beta_n)\,. \tag{4.3.3}$$

For example, we can compute the DN partition function of the disk to be

$$Z_{\mathsf{disk}}(E) = \int_{c-i\infty}^{c+i\infty} \frac{d\beta}{\beta} e^{\beta E} Z_{\mathsf{disk}}(\beta) = e^{S_0} \int^{E} dE'\, \rho_0(E')\,, \tag{4.3.4}$$

which corresponds to the insertion of an integrated density of states. The DN boundaries conditions relate to the presence of so-called *energy-eigenbranes* (as defined in [136]) which fix a particular eigenvalue in the dual (color) matrix model.

The vertex operators in KS theory satisfy some useful properties. Firstly, they obey the boson-fermion correspondence [123], which states that

$$\lim_{z'\to z} \left\{\psi(z)\psi^\dagger(z')\right\} = \partial\Phi(z)\,. \tag{4.3.5}$$

Here the accolades signify subtracting the OPE singularity $\sim (z-z')^{-1}$. This property is the analog of the random matrix identity in which the derivative of a ratio of determinants gives the resolvent, upon taking the energy arguments equal. The role of the derivative, in the case at hand, is played by Wick's theorem in combining ψ and $\psi^\dagger$ into a single normal ordered exponential. In [1], these brane/anti-brane insertions (on the two-sheeted spectral curve $\mathscr{S}_{\mathrm{JT}}$) were used to study non-perturbative corrections to resolvent and spectral density correlation functions.

Secondly, the vertex operators transform under coordinate transformations of x and y that leave the symplectic form $dx\wedge dy$ invariant. Their transformation properties are inherited from the higher-dimensional closed string field theory. In short, the full six-dimensional KS theory on the Calabi-Yau (3.4.2) has a large symmetry group, namely diffeomorphisms that leave the holomorphic $(3,0)$-form invariant. Upon reduction to the spectral curve, the symmetry is broken to diffeomorphisms that preserve the symplectic form $dx\wedge dy$. The chiral boson Φ can be seen as the Goldstone boson for this broken symmetry. The broken symmetry generates Ward identities in the quantum theory, which coincide with the SD-equations discussed above [40]. Diffeomorphisms that leave the symplectic form invariant are called *symplectomorphisms*, and the symplectic group of $\mathbb{C}^2$ is just $Sp(2,\mathbb{C}) \cong SL(2,\mathbb{C})$.

So a symplectic transformation acts simply as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.3.6}$$

with $ad-bc = 1$. This leaves invariant $dx' \wedge dy' = dx \wedge dy$, whereas the holomorphic $(1,0)$-form $\omega = ydx$ changes up to a total derivative

$$y'dx' - ydx = dS\,. \tag{4.3.7}$$

The vertex operator $\psi(x) = e^{\Phi(x)}$ transforms under the symplectomorphism with a weight determined by the function $S(x, x')$ as

$$\widehat{\psi}(x') = \int \frac{dx}{\sqrt{\lambda}} e^{S(x,x')/\lambda} \psi(x)\,. \tag{4.3.8}$$

Similarly, the anti-brane operator $\psi^\dagger(x)$ transforms with the opposite sign of $S(x, x')$. Here $\lambda = e^{-S_0}$ is the KS coupling constant. (4.3.8) can be interpreted by saying that the open string partition function transforms like a wave function [163–166]. We will be most interested in the 'S-transform', sometimes called 'x-y symmetry', which exchanges the coordinates $x' = y$ and $y' = -x$. In [167,168] it is explicitly shown, using topological recursion, that this transformation is a symmetry of the KS partition function, to all orders in the genus expansion.

The observables $\psi, \psi^\dagger$ transform in a natural way under the S-transform. Namely, the classical action $S(x, x')$ corresponding to this transformation can be found using the definition (4.3.7), $x = -\partial_y S$ and $y = -\partial_x S$. This is solved by $S = -yx$, and so $\psi(x)$ and $\psi^\dagger(x)$ transform by Fourier (or inverse Laplace) transforms under the symplectic S-transformation

$$\widehat{\psi}(y) = \int \frac{dx}{\sqrt{\lambda}} e^{-xy/\lambda} \psi(x)\,, \quad \widehat{\psi}^\dagger(y) = \int \frac{dx}{\sqrt{\lambda}} e^{xy/\lambda} \psi^\dagger(x)\,. \tag{4.3.9}$$

Here the integration contours should be chosen such that the integrals converge — we will come back to this point momentarily. By inverting the above Fourier transform, we can similarly express $\psi(x)$ and $\psi^\dagger(x)$ in terms of the Fourier transformed (anti-)brane operators.

We can study the brane operators (4.3.9) by inserting them in KS correlation functions. Their perturbative expansion admits an 'open string' expansion

$$\langle \widehat{\psi}(y) \rangle_{\mathsf{KS}} = \exp\left[-\frac{1}{\lambda} \sum_{n=0}^{\infty} \lambda^n \Gamma_n(y) \right] \tag{4.3.10}$$

To leading order in λ, the disk contribution to the 1-point function is simply $\exp \frac{1}{\lambda} \langle \widehat{\Phi}(y) \rangle_0$, where

$$\langle \widehat{\Phi}(y) \rangle_0 = - \int^y x(y')dy' \tag{4.3.11}$$

is the integral of the dual one-form $-xdy$, and $x(y)$ is determined by the spectral curve equation $H(x,y) = 0$. Therefore, in the limit $\lambda \to 0$ we see that the Fourier transforms in (4.3.9) implement a Legendre transform of the disk potential

$$\langle \Phi(x) \rangle_0 = \int^x y(x')dx' \,. \tag{4.3.12}$$

In the case that the spectral curve is given by $y^2 - x = 0$, the potential in (4.3.11) becomes the cubic

$$\Gamma_0(y) = - \langle \widehat{\Phi}(y) \rangle_0 = \frac{1}{3} y^3 \tag{4.3.13}$$

characteristic of the Airy integral. For the case of JT gravity, we can solve the spectral curve $\mathscr{S}_{\mathrm{JT}}$ for $x = \arcsin^2(y)$ (absorbing the factors of 2π for convenience), and the leading order potential becomes[8]

$$\Gamma_0(y) = -2y + 2\sqrt{1-y^2}\arcsin y + y \arcsin^2 y \,. \tag{4.3.14}$$

One can check that around $y = 0$, the JT potential can be expanded as $\frac{1}{3}y^3 + \mathcal{O}(y^5)$. However, its behaviour away from the origin is quite different from the Airy potential, as will be discussed in Appendix B.

The reason to introduce the canonically conjugate coordinate y may seem a bit mysterious at this stage. However, from the point of view of JT gravity the Fourier transformed vertex operators are rather natural objects. If we interpret the real part of x as parametrizing the energy space of the boundary Schwarzian theory, it makes sense to interpret the conjugate variable y's real part as a temperature, β, or boundary length. So we think of the insertion of $\widehat{\psi}(y)$ as creating a D-brane on which arbitrarily many open JT worldsheets can end. Notice the similarity to our identification of the boundary creation operators $Z(\beta)$ as the inverse Laplace transform of $\mathcal{J} = \mathcal{J}(x)dx = \mathcal{J}(z)dz$:

$$Z(\beta) = \int_{c-i\infty}^{c+i\infty} dx\, e^{\beta x} \mathcal{J}(x) \,, \quad \widehat{\psi}(y) = \int_{c-i\infty}^{c+i\infty} dx\, e^{-yx} e^{\Phi(x)} \,. \tag{4.3.15}$$

The above intuition is strengthened by the observation in [112] that the Fourier transform maps the Virasoro constraints for correlation functions of vertex operators to the open topological recursion of [169, 170]. Moreover, we will see in the

[8]This potential has appeared in the literature once before (as far as we know), cf. [109].

next section that the y-variable naturally arises as a matrix *eigenvalue* in the dual flavor matrix theory.

4.3.2 fMT representation of the brane correlator

Having introduced the vertex operators $\psi(x)$, $\psi^\dagger(\mathsf{x})$ and their transformation properties under symplectomorphisms, we can go on to study correlation functions of brane/anti-brane pairs,

$$D_n(X) = \left\langle \psi(x_1)\psi^\dagger(\mathsf{x}_1)\cdots\psi(x_n)\psi^\dagger(\mathsf{x}_n)\right\rangle_{\mathsf{KS}}. \tag{4.3.16}$$

These are the analogs of determinant/inverse determinant insertions in a random (color) matrix theory, so we expect to reduce their correlation function to a suitable flavor matrix integral whose dimension is set by the number of vertex operator insertions. To demonstrate this, let us invert the Fourier transforms in (4.3.9):

$$\psi(x) = \int_{\mathcal{C}} \frac{dy}{\sqrt{\lambda}}\, e^{xy/\lambda}\widehat{\psi}(y)\,, \qquad \psi^\dagger(\mathsf{x}) = \int_{\mathcal{C}'} \frac{d\mathsf{y}}{\sqrt{\lambda}}\, e^{-\mathsf{x}\mathsf{y}/\lambda}\widehat{\psi}^\dagger(\mathsf{y})\,. \tag{4.3.17}$$

Substituting these symplectic transformations into the correlation function $D_n(X)$ gives

$$\begin{aligned}&\left\langle \psi(x_1)\psi^\dagger(\mathsf{x}_1)\cdots\psi(x_n)\psi^\dagger(\mathsf{x}_n)\right\rangle_{\mathsf{KS}} = \\ &\qquad\lambda^{-n}\int dY\, e^{\mathrm{str}(XY)/\lambda}\left\langle \widehat{\psi}(y_1)\widehat{\psi}^\dagger(\mathsf{y}_1)\cdots\widehat{\psi}(y_n)\widehat{\psi}^\dagger(\mathsf{y}_n)\right\rangle_{\mathsf{KS}}.\end{aligned} \tag{4.3.18}$$

Here we have defined $dY = \prod_i dy_i d\mathsf{y}_i$ and collected the exponentials into a supertrace over graded diagonal matrices $X = \mathrm{diag}(x_1,\ldots,x_n|\,\mathsf{x}_1,\ldots,\mathsf{x}_n)$ and $Y = \mathrm{diag}(y_1,\ldots,y_n|\,\mathsf{y}_1,\ldots,\mathsf{y}_n)$. As a next step, we use Wick's theorem to write the operator product of the vertex operators $\widehat{\psi}(y) = e^{\widehat{\Phi}(y_i)}$ and $\widehat{\psi}^\dagger(\mathsf{y}) = e^{-\widehat{\Phi}(\mathsf{y}_i)}$ as a single normal-ordered exponential of chiral bosons. As before, we define normal ordering $\{\cdots\}$ by subtracting all singular terms coming from the OPE of the chiral boson $\widehat{\Phi}(y)\widehat{\Phi}(y') \sim \log(y-y') + \mathrm{reg}$. After doing all the Wick contractions, this procedure gives rise to a super-Vandermonde determinant

$$\mathsf{s}\Delta(Y) \equiv \frac{\prod_{i<j}(y_i-y_j)\prod_{k<l}(\mathsf{y}_k-\mathsf{y}_l)}{\prod_{i,k}(y_i-\mathsf{y}_k)}\,. \tag{4.3.19}$$

Using Cauchy's determinant formula, the super-Vandermonde determinant can be written more elegantly as

$$\mathsf{s}\Delta(Y) = \det_{ij}\frac{1}{y_i-\mathsf{y}_j}. \tag{4.3.20}$$

Hence, the brane/anti-brane correlator (4.3.18) can be brought in the following form

$$\lambda^{-n} \int dY \, e^{\mathrm{str}(XY)/\lambda} \, \mathsf{s}\Delta(Y) \Big\langle \big\{ e^{\widehat{\Phi}(y_1) - \widehat{\Phi}(\mathsf{y}_1) + \cdots + \widehat{\Phi}(y_n) - \widehat{\Phi}(\mathsf{y}_n)} \big\} \Big\rangle_{\mathsf{KS}} . \tag{4.3.21}$$

Using the general formula $\langle \exp \mathcal{O} \rangle = \exp \sum_{k=1}^{\infty} \frac{1}{k!} \left\langle \mathcal{O}^k \right\rangle^{\mathsf{c}}$ for going between correlation functions and connected correlation functions, we represent the brane/anti-brane correlator (4.3.21) in a form that closely resembles a flavor matrix integral

$$D_n(X) = \lambda^{-n} \int dY \, \mathsf{s}\Delta(Y) e^{-\Gamma(Y)/\lambda + \mathrm{str}(XY)/\lambda} , \tag{4.3.22}$$

where the potential $\Gamma(Y)$ is defined as a sum of connected correlation functions in KS theory

$$\Gamma(Y) = -\sum_{k=1}^{\infty} \frac{\lambda}{k!} \Big\langle \big\{ \big(\widehat{\Phi}(y_1) - \widehat{\Phi}(\mathsf{y}_1) + \cdots + \widehat{\Phi}(y_n) - \widehat{\Phi}(\mathsf{y}_n) \big)^k \big\} \Big\rangle^{\mathsf{c}}_{\mathsf{KS}} . \tag{4.3.23}$$

As a last step, we normal order the brane correlator once more, but this time in the (x_i, x_i)-coordinates, which gives another super-Vandermonde determinant. We find

$$\Big\langle \Big\{ \psi(x_1) \psi^\dagger(\mathsf{x}_1) \cdots \psi(x_n) \psi^\dagger(\mathsf{x}_n) \Big\} \Big\rangle_{\mathsf{KS}} \tag{4.3.24}$$

$$= \frac{1}{\mathsf{s}\Delta(X)} \Big\langle \psi(x_1) \psi^\dagger(\mathsf{x}_1) \cdots \psi(x_n) \psi^\dagger(\mathsf{x}_n) \Big\rangle_{\mathsf{KS}} \tag{4.3.25}$$

$$= \frac{\lambda^{-n}}{\mathsf{s}\Delta(X)} \int dY \, \mathsf{s}\Delta(Y) \, e^{-\Gamma(Y)/\lambda + \mathrm{str}(XY)/\lambda} . \tag{4.3.26}$$

This is precisely the eigenvalue representation of a $\mathrm{GL}(n|n)$ graded flavor matrix integral with invariant potential $\Gamma(A)$ and 'external source' X. To recognize this, consider a Hermitian supermatrix A in $\mathrm{GL}(n|n)$ with eigenvalues $\{y_i, \mathsf{y}_i\}$, diagonalized by a unitary supermatrix $T \in \mathrm{U}_f = \mathrm{U}(n|n)$:

$$A = TYT^{-1} , \qquad Y = \mathrm{diag}(y_1, \ldots, y_n | \, \mathsf{y}_1, \ldots, \mathsf{y}_n) . \tag{4.3.27}$$

In terms of this decomposition the integration measure dA decomposes as

$$dA = dT dY \, \mathsf{s}\Delta(Y)^2 , \tag{4.3.28}$$

where we have defined $dY = \prod_{a=1}^{n} dy_a d\mathsf{y}_a$, and dT is the Haar measure on $\mathrm{U}(n|n)$. Similar to ordinary matrix integrals, the super-Vandermonde determinant arises as the Jacobian of the change of variables from A to T and Y. It can be easily derived as the volume form corresponding to the metric on the space of Hermitian supermatrices

$$ds^2 = \mathrm{str}(dA^2) = \mathrm{str}(dY^2 + [d\Omega, Y]^2) \ , \tag{4.3.29}$$

where $d\Omega = T^{-1}dT$. Now consider a general flavor matrix integral with invariant potential $\Gamma(A) = \Gamma(TYT^{-1}) = \Gamma(Y)$ and external source term $\mathrm{str}(XA)$. One can see this as the supersymmetric generalization of the Kontsevich matrix integral, whose potential is $\Gamma(A) = \frac{1}{3}\mathrm{str}(A^3)$. However, we will keep $\Gamma(A)$ arbitrary for now, and match it to JT gravity later. We also include a coupling constant λ. Using the eigenvalue decomposition (4.3.27) the flavor matrix integral decomposes as

$$\int_{(n|n)} dA\, e^{-\Gamma(A)/\lambda + \mathrm{str}(XA)/\lambda} = \int dY e^{-\Gamma(Y)/\lambda}\, \mathsf{s}\Delta(Y)^2 \int_{U(n|n)} dT\, e^{\mathrm{str}(XTYT^{-1})/\lambda} \, . \tag{4.3.30}$$

The super-unitary integral appearing in (4.3.30) can be evaluated in closed form [171, 172] and is a supersymmetric generalization of the famous Harish-Chandra-Itzykson-Zuber integral. The integral turns out to be one-loop exact and evaluates to[9]

$$\int_{U(n|n)} dT\, e^{\frac{1}{\lambda}\mathrm{str}(XTYT^{-1})} = C_n\, \lambda^{-n} \frac{\det_{i,j}\left(e^{x_i y_j/\lambda}\right) \det_{k,l}\left(e^{-\mathsf{x}_k \mathsf{y}_l/\lambda}\right)}{\mathsf{s}\Delta(X)\mathsf{s}\Delta(Y)} \, . \tag{4.3.31}$$

Plugging this into (4.3.30), we can evaluate the determinants by exploiting the antisymmetry of the Vandermonde determinants $\Delta(y) = \prod_{j<i}(y_i - y_j)$ and $\Delta(\mathsf{y}) = \prod_{j<i}(\mathsf{y}_i - \mathsf{y}_j)$ and the freedom to relabel dummy variables in the y and y integration. For example, when $n = 2$, there is a factor $-\Delta(y)e^{(x_1 y_2 + x_2 y_1)/\lambda}$, which upon relabeling $y_2 \leftrightarrow y_1$ becomes $+\Delta(y)e^{(x_1 y_1 + x_2 y_2)/\lambda}$. So we are left with only the diagonal contributions of $e^{x_i y_i/\lambda}$ and $e^{-\mathsf{x}_k \mathsf{y}_k/\lambda}$, which assemble into the supertrace of XY. This argument is easily extended to general n, and we arrive at the eigenvalue representation of the flavor matrix integral

$$\int_{(n|n)} dA\, e^{-\Gamma(A)/\lambda + \mathrm{str}(XA)/\lambda} = \tilde{C}_n \frac{\lambda^{-n}}{\mathsf{s}\Delta(X)} \int dY\, \mathsf{s}\Delta(Y)\, e^{-\Gamma(Y)/\lambda + \mathrm{str}(XY)/\lambda} \, . \tag{4.3.32}$$

[9] This is proven in [171] using a heat kernel method for the super-Laplacian operator on the space of Hermitian supermatrices, analogous to Itzykson and Zuber's original proof [173] in the non-supersymmetric case.

The prefactor $\tilde{C}_n$ (which now includes the symmetry factors from the above permutation argument) can be absorbed in an overall normalization of the flavor matrix integral. If we now identify the fMT coupling constant λ with the KS coupling constant, and take as our invariant potential the KS potential (4.3.23), then we see that the flavor matrix integral coincides with the brane/anti-brane correlator (4.3.26). In conclusion, we have shown that the normal ordered expectation value of n brane and n anti-brane creation operators in KS theory, inserted at positions x_i, x_i, is exactly equal to a $(n|n)$ graded flavor matrix integral with external source $X = \mathrm{diag}(x_1, \ldots, x_n | \mathsf{x}_1, \ldots, \mathsf{x}_n)$:

$$\Big\langle \big\{ e^{\Phi(x_1)} e^{-\Phi(\mathsf{x}_1)} \cdots e^{\Phi(x_n)} e^{-\Phi(\mathsf{x}_n)} \big\} \Big\rangle_{\mathsf{KS}} = \int_{(n|n)} dA \, e^{-e^{S_0}\Gamma(A) + e^{S_0}\mathrm{str}(XA)} \,. \quad (4.3.33)$$

This is the main result of this section. In principle, one can compute the potential $\Gamma(Y)$ to any order in the KS perturbation theory in powers of $\lambda = e^{-S_0}$. In the case that the spectral curve is the Airy curve $y^2 - x = 0$, the higher genus contributions vanish and the only non-zero contributions to $\Gamma(Y)$ come from the disk and cylinder amplitudes[10]. For the JT spectral curve, there are non-trivial corrections suppressed in powers of λ (for a single brane insertion these were computed in [109]). In the matrix theory context, such refinement would describe small corrections to the *average* spectral density. However, we are interested in the limit that e^{S_0} is very large, ΔE very small, and the ratio $s \sim e^{S_0}\Delta E$ kept fixed, as explained in Section 4.2.1. In this limit, it suffices to keep only the leading order potential function

$$\Gamma(Y) \approx -\sum_{i=1}^{n} \Big(\langle \widehat{\Phi}(y_i) \rangle_0 - \langle \widehat{\Phi}(\mathsf{y}_i) \rangle_0 \Big) \quad (4.3.34)$$

$$= \sum_{i=1}^{n} \Big(\int^{y_i} x(y) dy - \int^{\mathsf{y}_i} x(y) dy \Big) \,. \quad (4.3.35)$$

As explained before, the function $x(y)$ follows from the spectral curve equation $H(x, y) = 0$, and is given by $x(y) = \arcsin^2(y)$ for the JT spectral curve. And, as advertised, the right-hand side of (4.3.35) can be written as a supertrace, $\Gamma(Y) \approx \mathrm{str}\, \Gamma_0(Y)$, where the function $\Gamma_0(y)$ for the JT spectral curve was given in (4.3.14). Note that for small values of the argument, $x(y) \sim y^2$, and so $\Gamma_0(Y) \sim \mathrm{str}(Y^3)$. Hence, near the spectral edge $E \to 0$ our flavor matrix model is governed by

[10] One way to see this is to note that under the x-y symmetry the dual of the Airy spectral curve has no branch point, since $dy(z) = dz$. So the topological recursion of [167] is identically zero. The claim can also be checked order by order, by computing the 'WKB' form of $\langle \psi(x) \rangle = e^{\frac{1}{\lambda}\sum_n \lambda^n S_n(x)}$ using topological recursion [77], and then doing a stationary phase analysis of the Fourier transform in (4.3.9).

a cubic potential, and thus reduces to graded variant of a Kontsevich matrix model[11]. However, the potential behaves differently at infinity. This will influence the choice of integration contours for the eigenvalues (y_i, y_i), as will be discussed in more depth in Appendix B.

4.3.3 Reduction to the nonlinear σ-model

Having derived a flavor matrix integral from (anti-)brane insertions in KS theory, we go on to show that the nonlinear σ-model introduced in (4.2.5) arises from a stationary phase analysis of the flavor matrix integral (4.3.33) in the limit of large e^{S_0}. More precisely, we study the fMT in the limit discussed in the introduction, where we take $\lambda \to 0, \Delta E \to 0$ and $s \sim \Delta E/\lambda$ held fixed. We will find that which saddles dominate depends on the causal symmetry breaking parameters $\pm i\eta$, which are the infinitesimal imaginary offsets in the (anti-)brane positions $\mathrm{Im}(x_i) = \pm i\eta$ on the spectral curve.

To find the stationary points, we decompose $A = TYT^{-1}$ as before and vary T and Y:

$$\begin{aligned} \mathrm{str}\left[\left(T^{-1}XT + \Gamma_0'(Y)\right)\delta Y\right] = 0\,, \\ \mathrm{str}\left[(YTX - XTY)\delta T\right] = 0\,. \end{aligned} \tag{4.3.36}$$

Let us first discuss the diagonal solutions, for which $T = \mathbb{1}$. In that case, the entries y_i, y_i of Y separately have to satisfy the equations

$$\Gamma_0'(y_i) = x_i\,, \quad \Gamma_0'(\mathsf{y}_i) = \mathsf{x}_i\,. \tag{4.3.37}$$

Looking at the form of $\Gamma_0(y)$ in (4.3.35), these equations are solved by inverting $x(y_i) = x_i$, where $x(y)$ is determined by the spectral curve $\mathscr{S}_{\mathrm{JT}}$. Here the branched structure of the spectral curve rears its head: for each diagonal element y_i, y_i, there are two choices of branch when taking the square root, for example

$$y_1^{\pm} = \pm\sin(\sqrt{x_1}) = \pm i\rho_0(E_1)\,. \tag{4.3.38}$$

This naively gives a total of 2^n diagonal saddles in the fMT. However, precisely which saddle points contribute depends on the integration contour that is part of the definition of the flavor matrix integral. In Appendix B we perform a steepest descent analysis of the fMT with the potential $\Gamma_0(Y)$. We find that the dominant saddle is selected by the $\pm i\eta$ prescription of the external energy arguments. For example, if we take the imaginary part of X to be $i\eta\,\hat{\tau}_3$, where $\hat{\tau}_3 = \tau_3 \otimes \mathbb{1}$ is the

[11] This is similar to how the insertion of N FZZT branes in Liouville theory were shown to give rise to a Kontsevich matrix integral by Gaiotto and Rastelli in [139]. Including anti-FZZT branes [111], one expects to find the graded variant of the Kontsevich matrix integral.

(flavor) Pauli z matrix, then the dominant saddle point will be

$$Y_{st} = i\rho_0(E)\,\tau_3 \otimes \mathbb{1}\,. \tag{4.3.39}$$

This is called the *standard saddle*. We have set the external energies equal to E in Y_{st}, because upon evaluating the potential at the saddle point the linear term $e^{S_0}\mathrm{str}XY_{st}$ should be expanded to leading order in $s \sim e^{S_0}\Delta E$. Besides the standard saddle, there are also subleading *Andreev-Altshuler* (AA) saddles [160], which arise from $i\eta$-prescriptions such that a brane and an anti-brane are on opposite sheets before taking their OPE limit. For example, in computing the density-density correlator ($n = 2$), there is one such AA saddle, given by

$$Y_{AA} = i\rho_0(E)\,\tau_3 \otimes \tau_3\,. \tag{4.3.40}$$

It can easily be verified that evaluating the fMT on the standard saddle gives a vanishing action, while the action for the AA saddle is non-zero and purely imaginary. This gives rise to the well-known oscillatory behavior of the spectral density two-point function (the 'sine kernel') [90].

Having found the diagonal saddle points Y_*, we make the following simple observation: if $[X, T] = 0$, then the saddle point equations (4.3.36) are also solved by TY_*T^{-1}. For generic values of the external energies, X does not commute with T. However, recall that we are considering the limit that ΔE is very small, and so we can approximate $[X, T] \approx 0$. In other words, to leading order in s, TY_*T^{-1} is an *approximate* solution to the saddle point equations. So instead of a sum over distinct saddles, we have to integrate over a whole saddle point manifold.

To determine the saddle point manifold, we need to account for the redundancies corresponding to transformations that commute with the diagonal saddle Y_*. Without loss of generality, consider for Y_* the standard saddle, Y_{st}, which is proportional to $\hat{\tau}_3$. The stabilizer subgroup of Y_{st} is $\mathrm{U}(\frac{n}{2}|\frac{n}{2}) \times \mathrm{U}(\frac{n}{2}|\frac{n}{2})$, which acts on $\mathrm{U}(n|n)$ in an obvious way. So the full saddle point manifold will be the coset manifold

$$\mathrm{AIII}_{n|n} = \frac{U(n|n)}{U(\frac{n}{2}|\frac{n}{2}) \times U(\frac{n}{2}|\frac{n}{2})} \tag{4.3.41}$$

as advertised in Section 4.2.1. The saddle point manifold continuously connects the standard saddle to the AA saddles [49]. Parametrizing the coset by $Q = T\hat{\tau}_3T^{-1}$, and evaluating the fMT action on its solution $A = i\rho_0(E)Q$, we obtain the non-linear σ-model on the Goldstone manifold

$$\int_{(n|n)} dA\, e^{-e^{S_0}\Gamma(A)+e^{S_0}\mathrm{str}(XA)} \simeq \mathcal{N}\int_{\mathrm{AIII}_{n|n}} dQ\, e^{i\frac{\rho_0(E)}{\lambda}\mathrm{str}(XQ)}\,, \tag{4.3.42}$$

to first order in s, in the late time limit $\lambda \to 0$, $\Delta E \to 0$. In the above expression we have absorbed the potential term $\exp\left[-e^{S_0}\mathrm{str}\,\Gamma_0(Y)\right]$ into a normalization $\mathcal{N}$, because it is independent of Q using the cyclicity of the supertrace. Moreover, since Q is supertraceless, we can freely replace X by its symmetry breaking part $X \to X - E\mathbb{1}$ containing the energy differences ΔE only. Concluding, we see that the brane/anti-brane correlator in KS theory captures the universal late time ergodic physics described by the NLSM.

Having derived the non-linear σ-model of quantum chaos from KS theory, one can systematically study perturbative corrections in the parameter s. As shown in [49], the s^{-1}-expansion of spectral correlations is a topological expansion. This expansion should be viewed as a limit of the JT topological expansion for which the probe arguments are sent to small differences, ΔE, and only the relevant diagrams are kept. This can checked by explicit computation: Individual diagrams contributing to the expansion of the NLSM can be represented in a 't Hooft double line syntax [49, 141] to verify that their perturbative s^{-1}-degree maps to the topological order of the KS/JT expansion. Conversely, one may take the limit $E_i \to E_j$ in contributions of given genus order in the expansion of the JT path integral to verify that the topological order determines the order of the highest singularities in $s^{-1} \propto |E_i - E_j|^{-1}$, with matching coefficients.

However, we already mentioned that the correspondence between the asymptotic expansion of the JT path integral in e^{-S_0} to the NLSM is limited to values $s^{-1} < 1$. In the opposite case, corresponding to post-Heisenberg or plateau times, the NLSM leaves the regime of perturbation theory. Instead, the now small coupling constant requires non-perturbative integration over the full graded coset manifold. This integration, which describes the restoration of the causal symmetry previously broken by large fluctuations, has no analog in semi-classical JT gravity. We conclude that the latter knows about perturbative signatures of level correlations, but not about their fine-grained microscopics. In this sense, JT gravity remains 'UV incomplete'. Our discussion has shown that the closure of the theory is provided by KS field theory, which for pre-Heisenberg time scales is perturbatively equivalent to JT, and to the NLSM beyond.

In this context, it is also worth mentioning connections to the SYK model. Early work identified a bridge between SYK and JT at time scales which from the perspective of our present discussion are 'super-short', $t \sim \log(e^{S_0})$. In this regime, reflecting the approximate realization of a conformal symmetry, both reduce to Liouville quantum mechanics [174] as a common effective theory. At larger scales beyond the Thouless time (here identified with the dip time of the SYK form factor), the SYK model is described by fMT [50], which close to the lower spectral edge again takes the form of a graded Kontsevich model [49]. In this way a second link between the SYK model (at the spectral edge) and a gravitational theory is drawn, via universe field theory[12]. This 'late time bridge' relies on conceptually independent insights to that at early times. Its underlying symmetry principle is the causal symmetry breaking/restoration of ergodic quantum systems. Understanding how the respective symmetry principles connect at intermediate time scales remains an open question.

4.4 D-branes and the color-flavor map

In the previous section we have shown how fMT arises directly from universe field theory. This has been derived solely using a closed string field theory framework. However, since a crucial role is played by D-branes in the target space geometry, it is natural to expect that there is also an open string field theory which leads to fMT.

In this section we will show that this is indeed the case, owing to an open-closed duality in string field theory defined on a particular 6d Calabi-Yau. This open-closed duality manifested itself in the previous section – in the 2d target space theory on the spectral curve – as a type of boson-fermion correspondence between 'fermionic' operators $\psi(x)$ and vertex operators $e^{\Phi(x)}$. In the six-dimensional setting, the open-closed duality relates Kodaira-Spencer theory (closed) to holomorphic Chern-Simons theory (open) [107]. Reducing this holomorphic Chern-Simons theory to the worldvolume of a stack of non-compact branes and anti-branes gives rise to the fMT, as will be explained in Section 4.4.1.

Moreover, the open string perspective naturally explains the color-flavor duality described in Section 4.2.1. Recall from that section that the long time limit of ergodic dynamics realized in Hilbert spaces of high color dimension, L, is equivalently described by a matrix theory of low flavor dimension. Here we discuss a gravitational interpretation of the color and flavor degrees of freedom, and describe their duality from a D-brane worldvolume perspective. An idea of the relevant constructions is given in Figure 4.3.

Before describing these D-branes explicitly, let us start with the general picture to have in mind. As always, let us begin from the determinant ratio (4.1.1). We can rewrite it in terms of graded integrals, keeping in mind the interpretation of branes for the determinants in the numerator and anti-branes for those in the

[12] It would be interesting to see how our work relates to another recent connection between SYK and string theory established in [175].

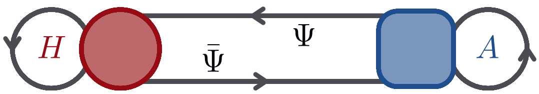


Figure 4.3: *A schematic picture of the different types of D-branes in KS theory. Color branes (red) and flavor branes (blue) each have open string degrees of freedom associated to the branes themselves (indicated by the color matrix H and flavor matrix A resp.) and open string degrees of freedom $\Psi, \overline{\Psi}$ connecting both types of branes.*

denominator. We thus have

$$\mathrm{Sdet}(X \otimes \mathbb{1}_{\mathrm{c}} + \mathbb{1}_{\mathrm{f}} \otimes H) = \int D\bar{\Psi} D\Psi \exp\left[-\bar{\Psi}\left(X \otimes \mathbb{1}_{\mathrm{c}} + \mathbb{1}_{\mathrm{f}} \otimes H\right)\Psi\right] , \tag{4.4.1}$$

where Ψ is a graded vector of dimension $(nL|nL)$. The adjoint flavor group representation carried by the operator $\Xi = X \otimes \mathbb{1}_{\mathrm{c}} + \mathbb{1}_{\mathrm{f}} \otimes H$ in (4.2.1) implies the following transformation in the fundamental representation for the vectors $\Psi, \overline{\Psi}$:

$$\Psi \to T\Psi, \qquad \overline{\Psi} \to \overline{\Psi} T^{-1} , \qquad\qquad T \in \mathrm{U}\,(n|n)\,. \tag{4.4.2}$$

The idea is now to take the integral representation in (4.4.1) seriously, and identify the vector Ψ as describing the open string degrees of freedom that stretch between two types of branes in KS theory: non-compact branes (which we already touched upon in Chapter 3) and compact branes. The causal symmetry given in (4.4.2) can now be identified with the gauge symmetry of the gauge field on the configuration of n coincident branes and anti-branes. Crucially, the Hermitian color matrix H, which describes the microscopic degrees of freedom, is taken to large size $L \to \infty$ where L counts the number of compact branes. However, the flavor matrix A has finite (possibly small) size $2n$, where the index n counts the number of non-compact (anti-)branes.

As explained in Chapter 3, the target space geometry is the non-compact Calabi-Yau

$$\mathsf{CY}: \quad uv - H(x,y) = 0\,, \tag{4.4.3}$$

which is a fibration over the spectral curve defined by $\mathscr{S} : H(x,y) = 0$. One recovers JT gravity for the choice $\mathscr{S} = \mathscr{S}_{\mathrm{JT}}$. There are now two distinct ways of introducing D-branes in the geometry [40]: we can either introduce non-compact 'flavor' branes, which can be related to a Kontsevich-like matrix model (see Section 4.4.1) or compact 'color' branes, which lead to the usual Hermitian matrix model (see Section 4.4.2). The color branes are wrapped over a compact two-cycle in the

CY geometry and their presence introduces a flux for the holomorphic $(3,0)$-form

$$\Omega = \frac{du}{u} \wedge dx \wedge dy \,, \tag{4.4.4}$$

over the three-sphere linking the two-cycle. One can also deform the complex structure at infinity by inserting flavor branes which wrap the non-compact fibers $u = 0, v = 0$ in the CY. The open string sector associated to the branes can be interpreted as an effective change of the geometry in which the closed strings propagate. In both cases the world-volume theory associated to the branes can be derived from the dimensional reduction of a holomorphic CS theory on the space-filling D6 brane wrapping the entire CY [116]

$$S_{\text{open}} = \frac{1}{g_s} \int_{\mathsf{CY}} \Omega \wedge \operatorname{str} \left[\mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right] \,, \tag{4.4.5}$$

where g_s is the (open-)string coupling[13]. The supertrace str indicates that we are considering a slightly unconventional version of the open string field theory, that includes both branes and anti-branes (which have opposite flux). It was argued in [130] that the inclusion of anti-branes can be implemented by upgrading the $(0,1)$-form gauge field $\mathcal{A}$ to be supermatrix-valued. In order to describe, say, a stack of n branes and n anti-branes wrapped on $\mathcal{B}$, we take the gauge group in (4.4.5) to be the supergroup $\mathrm{GL}(n|n)$. This can be argued for topological branes by examining their Chan-Paton factors [130], which carry opposite signs for string world-sheets with an odd number of boundaries on an anti-brane. By examining the four different kinds of annulus diagrams that correspond to the string having endpoints on either a brane or an anti-brane, and assigning a minus sign to each anti-brane boundary, one sees that the physical states arrange themselves into a $\mathrm{U}(n|n)$ superconnection, or rather its complexification $\mathrm{GL}(n|n)$. We therefore end up with a $\mathrm{GL}(n|n)$-valued version of the holomorphic Chern-Simons theory describing the B-model on the CY (4.4.5).

4.4.1 Non-compact branes: flavor

Let us first define the relevant probe flavor branes in the geometry (4.4.3). For a fixed point (x_0, y_0) in the (x,y)-plane the equation

$$uv = H(x_0, y_0) \tag{4.4.6}$$

[13]For the reader who may want to compare with the matrix-model convention frequently employed in the literature, comparing the Dijkgraaf-Vafa matrix potential (4.4.22) with equation (3.9) in [49], reveals the relation $g^2 = g_s L$.

defines a subspace of complex dimension one. When $(x_0, y_0) \in \mathscr{S}$, the above curve develops a node $uv = 0$ and splits in two complex planes: $u = 0$ or $v = 0$. These planes are wrapped by the flavor branes. Let us assume that the branes are at some fixed position

$$\mathcal{B}: \quad u = 0 \ , \quad (x, y) = (x_0, y_0) \ , \tag{4.4.7}$$

where $(x_0, y_0) \in \mathscr{S}$ lies on the spectral curve. The world-volume of the brane – or as will be the case relevant to us, a stack of branes – is parametrized by the complex coordinate $(v, \overline{v})$. We may also wrap anti-branes along these fibres, the only difference being their orientation which is opposite to that of the branes[14]. In the following, we will consider normal deformations of the brane (4.4.7) with suitable boundary conditions at infinity $|v| \to \infty$.

We now want to describe the effective theory associated to the open strings ending on $\mathcal{B}$. We will show that the result takes the form of a flavor matrix model. This derivation closely follows [40, 176] (with the exception that we accommodate both branes and anti-branes). The dimensional reduction of the holomorphic CS action (4.4.5) involves the following decomposition: the gauge field $\mathcal{A}$ splits as a gauge field $\widetilde{\mathcal{A}}$ on the world-volume of the brane and two Higgs fields A and B, which describe the movement of the brane in the transverse direction. We will assume that all fields depend only on the variables $(v, \overline{v})$ along the brane $\mathcal{B}$, so that we can apply a dimensional reduction. Explicitly, the normal deformations of the brane (4.4.7) are given by two scalar fields $B = B(v, \overline{v})$ and $A = A(v, \overline{v})$ in the adjoint representation of $\mathrm{GL}(n|n)$. They represent the deformations of $\mathcal{B}$ in the directions of the (x, y)-plane, according to the identification

$$(x, y) \quad \mapsto \quad (B(v, \overline{v}), A(v, \overline{v})) \, . \tag{4.4.8}$$

Moreover, the holormorphic $(3, 0)$-form (4.4.4) can be written as

$$\Omega = -\frac{dv}{v} \wedge dx \wedge dy \ , \tag{4.4.9}$$

in terms of the variable v. This shows that only non-zero contributions in (4.4.5) come from the $d\overline{v}, d\overline{x}$ and $d\overline{y}$ components of the gauge field $\mathcal{A}$, which we have identified with the fields $\widetilde{\mathcal{A}}(v, \overline{v}), B(v, \overline{v})$ and $A(v, \overline{v})$ respectively. The dimensional reduction now leads to the action [176, 177]

$$S_{\text{flavor}} = -\frac{1}{\lambda} \int_{\mathcal{B}} \frac{i}{2} \frac{dv d\bar{v}}{v} \operatorname{str}\left[B\overline{D}A\right] \ , \tag{4.4.10}$$

[14]Equivalently, one can obtain an anti-brane insertion on the spectral curve by putting a brane along the $v = 0$ direction (instead of $u = 0$). This follows from (4.4.7), which implies $\frac{du}{u} = -\frac{dv}{v}$, and so in terms of v the holomorphic $(3, 0)$-form (4.4.4) acquires a minus sign.

where $\overline{D} = \overline{\partial} + [\tilde{\mathcal{A}}, \cdot\,]$ is the antiholomorphic covariant derivative associated to $\tilde{\mathcal{A}}$ and $\overline{\partial}$ is the derivative with respect to $\overline{v}$. The volume of the transverse directions that have been integrated out have been absorbed into the coupling constant[15], which we identify with the expansion parameter λ, since we are considering a stack of branes in the (double-scaled) closed string background. Integrating the holomorphic three-form Ω yields a factor of Lg_s, measuring the flux of the L compact branes that have been dissolved in the geometry (see Section 4.4.3 below for more detail), and we needed to rescale $A \to A/g_s^{1/4}$, $B \to B/g_s^{1/2}$ to get a finite answer in the double-scaling limit, $L \to \infty, g_s \to \infty$ with $1/\lambda = e^{S_0} = \left(L/g_s^3\right)^{1/4}$ held finite.

Note that this theory inherits a $\mathrm{GL}(n|n)$ gauge symmetry, which acts as

$$B \mapsto gBg^\dagger \qquad A \mapsto gAg^\dagger \qquad \tilde{\mathcal{A}} \mapsto g\tilde{\mathcal{A}}g^\dagger + (\overline{\partial}g)g^\dagger\,, \tag{4.4.11}$$

where $g(v, \bar{v})$ is a $\mathrm{GL}(n|n)$ transformation depending on the fibre coordinate on $\mathcal{B}$. Gauge fixing to $\tilde{\mathcal{A}} = 0$ leaves only the constant $g \in \mathrm{GL}(n|n)$, which for $n = 2$, we recognise as the causal symmetry transformations appropriate for a four-determinant ratio. In fact, we are interested in configurations of the branes such that their classical position is fixed to the diagonal matrix X of external energies introduced in (4.1.1). We see here that this corresponds to fixing the asymptotic positions of the flavor branes on the spectral curve. These boundary conditions can be implemented by a adding a boundary term (which we can think of as a Legendre transform)

$$S_{\text{flavor}} = \frac{1}{\lambda}\int_{\mathcal{B}} \frac{i}{2}\frac{dv d\bar{v}}{v}\,\mathrm{str}\,(B - X)\overline{D}A, \tag{4.4.12}$$

implying that the field B at infinity is now fixed to the constant matrix $B_\infty = X$, while A_∞ at infinity is free, and must be integrated over in the quantum theory. As we described before, if the boundary condition parametrized by X is not proportional to the identity matrix in flavor space, i.e. contains unequal energy arguments on the diagonal, the small differences break the causal symmetry explicitly. This action will give rise to a Kontsevich-like matrix model for the flavor degrees of freedom in terms of the matrix A_∞.

To see this, let us study the action (4.4.10) in a bit more detail. First, the gauge field $\tilde{\mathcal{A}}$ can be set to zero by a suitable gauge transformation. The equation of motion for $\tilde{\mathcal{A}}$, given by $[A, B] = 0$, has to be imposed as a constraint, implying that the matrices associated to the Higgs fields can be simultaneously diagonalized. To simplify our analysis, we assume that the functions B and A are rotationally

[15] A compactification of the transverse directions is necessary to make the volume finite.

symmetric and only depend on the radial direction $r \equiv |v|$ of the brane. One can now perform the integral over the angular coordinate $\theta \equiv \arg v$. This gives an extra factor of 2π and leaves an integral over the radial direction:

$$S_{\text{flavor}} = \frac{2\pi}{\lambda}\text{str}\left[\int_0^\infty dr\ (B-X)\,\partial_r A\right] = \frac{2\pi}{\lambda}\text{str}\left[\int_{A_0}^{A_\infty} (B-X)\ dA\right]\ . \quad (4.4.13)$$

Recall that we have assumed that the brane is at some fixed position $B_\infty = X$ at infinity $v \to \infty$, and takes some possibly different value on the spectral curve at the origin $v = 0$. This leads to the final form of the effective action (up to an irrelevant constant):

$$S_{\text{flavor}} = e^{S_0}\text{str}\left[\int^{A_\infty} BdA - XA_\infty\right], \quad (4.4.14)$$

where we have used that $\lambda = e^{-S_0}$. Note that the equation of motion for this action is given by $B = X$, so we can indeed interpret the matrix X as describing the classical position of the branes in the $x-$plane.

In the case that the spectral curve $\mathscr{S}$ is given by $y^2 - x = 0$ (i.e., pure topological gravity), the action takes a familiar form. The equation $B = B(A)$ can be solved directly as a function of A, and by integrating (4.4.14) we obtain:

$$S_{\text{flavor}} = e^{S_0}\text{str}\left[\frac{1}{3}A^3 - XA\right], \quad (4.4.15)$$

where we have renamed – by a slight abuse of notation – $A \equiv A_\infty$ to conform with the notation that was used before. This is a graded version of the Kontsevich model action with cubic interaction. Since we have Legendre transformed to an open boundary condition on $A(v,\bar{v})$, in the quantum theory we still need to integrate over the value A_∞, which has been identified with the matrix field A. The partition function associated to the flavor branes is therefore given by

$$Z_{\text{flavor}}(X) = \int dA\, \exp\left[-e^{S_0}\text{str}\left(\frac{1}{3}A^3 - XA\right)\right], \quad (4.4.16)$$

up to some overall normalization. For this reason, the action in (4.4.14) gives rise to a graded Kontsevich model. In fact, we can easily generalize this to a general spectral curve $H(x,y) = 0$. In this case the resulting fMT takes the form

$$Z_{\text{flavor}}(X) = \exp\left[-e^{S_0}\text{str}\left(\Gamma_0(A) - XA\right)\right], \quad \text{with} \quad \frac{\delta\Gamma_0(A)}{\delta A} = B(A), \quad (4.4.17)$$

where $B(A)$ is determined by the equation $H(B,A)=0$. It will not have escaped the reader's attention that this is effectively the flavor matrix integral (4.3.33), in the large e^{S_0} limit. This establishes a representation of the fMT (as found in Section 4.3.2) in a gravitational setting, as a theory of non-compact flavor (anti-)branes probing the backreacted closed-string background.

The integration over A in the quantum theory represents an integration over the position of the brane at infinity. In the semiclassical limit $e^{S_0} \gg 1$, where we also take the energy arguments in X close together, we may evaluate this matrix integral by the method of steepest-descent, parallel to the analysis of Section 4.3.3. Among the different possible choices of saddle point configurations of the matrix A two are distinguished by being attainable by a contour deformation from the original contour. We remind the reader that this is explained in detail for both the Airy case and the JT spectral curve in Appendix B. In the present case these correspond to semi-classical configurations of the gauge field $\mathcal{A}_*$, such that the original Chern-Simons $\mathrm{GL}(n|n)$ gauge symmetry, already dimensionally reduced to (4.4.11) is broken to those $g \in \mathrm{GL}(n|n)$ which preserve $\mathcal{A}_*$. In both cases, this corresponds to the breaking

$$\mathrm{GL}(n|n) \to \mathrm{GL}\left(\tfrac{n}{2}|\tfrac{n}{2}\right) \times \mathrm{GL}\left(\tfrac{n}{2}|\tfrac{n}{2}\right)$$

by the classical configuration of the flavor branes.

By considering the non-compact flavor branes in open string field theory, we identified classical saddle point configurations of the branes which encode the standard and Altshuler-Andreev saddle points. One may morally compare this situation with that of adding flavor degrees of freedom in AdS/QCD [178]: there one places flavor D8-branes into the backreacted geometry of backreacted D4-branes, which have been replaced by the closed string geometry, containing an AdS_5 factor. The action of the flavor-branes is the familiar (non-abelian) DBI action. Here we think of the spectral curve $H(x,y)$ as the closed-string geometry, and the flavor branes we place into this background are described by the holomorphic Chern-Simons theory. The important difference to standard AdS/CFT is that in our picture, the flavor branes are objects that arise as boundary conditions on JT (or topological gravity) universes, while in the AdS/QCD context they are boundary conditions on fundamental strings embdedded *inside* the AdS background. In both cases the relevant symmetry (causal or chiral symmetry) is broken by specific semiclassical brane configurations. It may be enlightening to pursue this analogy further.

4.4.2 Compact branes: color

We will now address the question of how to realize the color matrix H in the KS theory. This will involve the introduction of a set of compact branes in a slightly different geometry, that is related to the closed-string background (4.4.3) by a geometric transition. In particular, we will find that the large L Hermitian matrix integral will give rise to the JT target space geometry, which provides an interesting perspective on how the microscopic degrees of freedom are converted into geometry.

The CY geometry that gives rise to this type of matrix integral takes the form of (4.4.3) with [38, 39]:

$$H(x,y) = y^2 - V'(x)^2 \,. \tag{4.4.18}$$

The function V is directly related to the potential of the Hermitian matrix integral, $V(H)$. Note that the CY geometry defined by (4.4.18) is singular along the slice $u = v = y = 0$: there are singularities at the critical points x_{c} satisfying $V'(x_{\mathrm{c}}) = 0$. It is useful to keep in mind the example $V'(x) = x$, which corresponds to a Gaussian matrix integral and has a single critical point at $x_{\mathrm{c}} = 0$. To make the singularity manifest, we can redefine $u = u' - iv'$, $v = u' + iv'$, $y = iy'$ so that the geometry takes the form $u'^2 + v'^2 + x^2 + y'^2 = 0$, which is the defining equation of a conifold. For a more general choice of potential, the geometry in (4.4.18) looks locally like a conifold near each one of the critical points, as long as $V''(x_{\mathrm{c}}) = 0$.

A well-known procedure for removing such conifold singularities is by 'blowing up' the relevant singular points into finite two-spheres. The resulting geometry is referred to as the resolved CY, for which we write $\mathsf{CY}_{\mathrm{res}}$, and is, locally near the singular point, a fiber bundle over the blown-up $\mathbb{P}^1$. The coordinate on the $\mathbb{P}^1$ is denoted by z, while the two fiber directions are represented by sections χ, φ with corresponding transition functions

$$z' = 1/z \,, \quad \chi' = \chi \,, \quad \varphi' = z^2\varphi + V'(\chi)z \,, \tag{4.4.19}$$

in going from the patch associated to the north pole (indicated by the coordinate z) to the south pole (indicated by the coordinate z'). We can rewrite the above transition map in a slightly different way by defining the coordinates

$$x \equiv \chi \,, \quad u \equiv 2\varphi' \,, \quad v \equiv 2\varphi \,, \quad y \equiv 2z'\varphi' - V'(x) \,. \tag{4.4.20}$$

Then, (4.4.19) becomes

$$\mathsf{CY}: \quad uv - y^2 + V'(x)^2 = 0 \,. \tag{4.4.21}$$

Note that this is precisely the geometry given by (4.4.18). The blown-up two-spheres are located at the zeros of V', because this is the only way to have $\varphi = \varphi' = 0$.

The relation of the resolved geometry CY_{res} to the $L \times L$ Hermitian matrix integral is through a stack of L topological branes that wrap the blown-up singularity[16]. The effective theory associated to the compact branes can again be extracted from (4.4.5), now defined on the target space (4.4.21). The dimensional reduction involves the movement of the brane in the transverse fiber directions χ, φ. Since we are dealing with a configuration of multiple branes, these fields are upgraded to matrices in the adjoint reprentation of the gauge group $\mathrm{U}(L)$. Following a similar procedure as for the non-compact branes in Section 4.4.1, one finds that the partition function of the color branes localizes to a Dijkgraaf-Vafa matrix integral with potential V (the details of this derivation are nicely worked out in [38, 138]):

$$\mathcal{Z}_{\text{color}} = \int dH e^{-\frac{1}{g_s} \operatorname{tr} V(H)} \ . \tag{4.4.22}$$

The constant $L \times L$ matrix H is identified with the scalar χ. We have also introduced the coupling constant g_s associated to the open string background (4.4.21) (pre-double-scaling). Going from the open-string to the closed-string background involves a double-scaling procedure, while simultaneously zooming in on the edge of the eigenvalue spectrum by taking the 't Hooft parameter $g^2 \equiv g_s L$ large. The definition of the matrix integral in (4.4.22) depends on a choice of contour integral. In particular, one can choose a contour that leads to real eigenvalues, and (4.4.22) then takes the form of an $L \times L$ Hermitian matrix integral. The eigenvalues of H corresponds (using the definition (4.4.20)) to the $x-$position of the branes in the geometry (4.4.21). The classic equation of motion is given by $V'(H) = 0$ and therefore the eigenvalues of H are classically located at the critical points of the potential. This is precisely the configuration of topological branes that we are considering. We conclude that, in our construction, the cMT is realized as the effective theory associated to the compact branes.

4.4.3 Color-flavor map and the geometric transition

Having described the color and the flavor perspective, let us now discuss the geometric interpretation of their *duality*, and its relation to the gravitational theory. To describe the multi-determinant operators in (4.4.1) we consider the situation where we have both color and flavor branes in the geometry (4.4.21). The flavor branes introduce a new sector of open strings that stretch between both types

[16] In the case of more than one singularity, one can divide the branes over the different two-spheres.

of branes. These are described by fields living on the intersection locus (which can be taken to be a single point on the blown-up two-sphere, say at $z = 0$): a bi-fundamental field $\Psi = \Psi^a_\mu$, whose Chan-Paton index μ labels the L-dimensional color Hilbert space, while a labels the flavor Hilbert space. The flavor indices have a $(n|n)$ grading so that Ψ^a contains both fermionic and bosonic components, indicating the presence of both branes and anti-branes. See the left panel of Figure 4.4 for an overview of the relevant D-brane configuration.

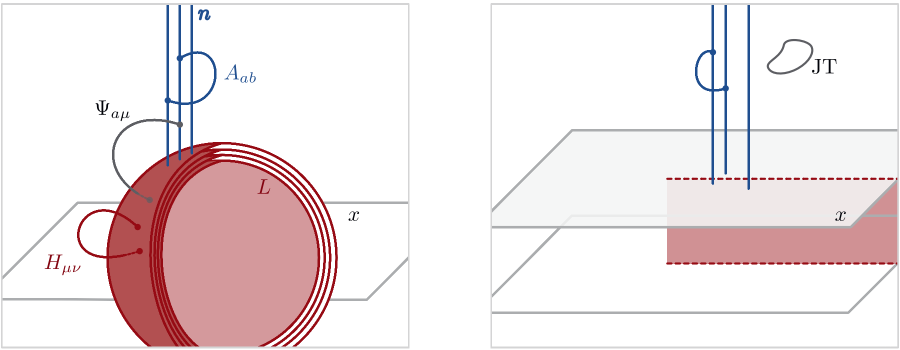


Figure 4.4: *Left: The setup of a stack of L color branes and n flavor (anti-)branes. The open string degrees of freedom that stretch between the two types of branes can be described by a field Ψ^a_μ in the bi-fundamental representation of $U(L) \otimes \mathrm{U}(n|n)$. Right: After taking $L \to \infty$ the color branes dissolve into a flux for the holomorphic three-form, which is represented by a branch cut (dashed line) in the x-plane. The closed string description is that of JT gravity on the world-sheet. The flavor branes (blue) in the closed string background, on which JT universes with fixed energy boundaries can end, correspond to probes for the color degrees of freedom.*

The idea is now to start from the expression in (4.4.1) involving the open string stretching Ψ^a_μ between the color and flavor branes, and integrate out the color degrees of freedom. We will now highlight some of the important features of this computation. As was shown in Section 4.4.2, the theory associated to the compact branes is a Hermitian matrix integral with potential $\operatorname{tr} V(H)$. Therefore, this step involves taking the average with respect to $\langle \ldots \rangle_H$. This effectively introduces bound states of open strings (which are the equivalent of 'mesons' in QCD):

$$\Pi^{ab} = \Psi^a_\mu \overline{\Psi}^b_\mu \, , \tag{4.4.23}$$

where the color index is summed over, so that they are indeed color singlets. To be precise, the terms involving the color matrix H are replaced by a potential term $\operatorname{str} V(\Pi)$ for the Ψ-fields. One can now integrate in a graded matrix A^{ab} that couples to Π^{ab} through $\operatorname{str} A\Pi = \overline{\Psi}(A \otimes \mathbb{1}_c)\Psi$ and effectively replaces the potential by $\operatorname{str} V(A)$. In the target space geometry, the field A^{ab} describes open strings stretching between two flavor branes (as indicated in Figure 4.4). The integral over Ψ-fields leaves a determinant operator $\operatorname{Sdet}(X \otimes \mathbb{1}_{\mathrm{c}} + A \otimes \mathbb{1}_{\mathrm{c}})^L$, which realizes the color-flavor duality (4.2.2). After applying a double-scaling limit (where both the size of the color matrix size L and the open string coupling constant g are taken to be large, while their ratio is kept fixed) to the flavor matrix integral, one precisely lands on the graded Kontsevich model that we found as an effective theory of the flavor branes in Section 4.4.1.

From the gravitational perspective, we can think about the above discussion in terms of an open/closed duality. By integrating out the color degrees of freedom, and taking the double-scaling limit $L \to \infty$, we are, in fact, replacing the compact branes by a backreacted target space geometry for the closed string. To be precise, the color branes dissolve into a branch cut, which leads to a non-trivial spectral curve $\mathscr{S}$. The theory of closed string propagation in this modified target space geometry is precisely the KS theory. Its mini-universe expansion in different world-sheet topologies gives rise to the universes on which the gravitational theory lives. In the case of $\mathscr{S} = \mathscr{S}_{\mathrm{JT}}$ this description is JT gravity. The open-string geometry in Figure 4.4 left is, therefore, to be contrasted with the closed-string geometry in Figure 4.4 right. We have effectively replaced the H/color-description with a geometric description in terms of JT gravity.

Given that the probe flavor branes are still there after the large L transition (while the color branes have disappeared), we obtain an additional sector in the theory. On the closed string side of the duality, the probe branes introduce another set of open string degrees of freedom, namely of open JT universes with fixed energy boundaries, that stretch between two flavor branes (these are precisely the A-fields). The KS theory allows for the inclusion of such degrees of freedom in terms of vertex operators $\psi = e^{\Phi}, \psi^\dagger = e^{-\Phi}$. This provides a geometric understanding of the relation (4.3.33) between vertex operator insertions and the fMT that was derived by an explicit computation in Section 4.3.

Let us end this section with some more details on the relation between the open- and closed-string background geometries. We have seen that there is a way to resolve the singularities in (4.4.21) by inserting a $\mathbb{P}^1$ at each of the singular points, leading to $\mathsf{CY}_{\mathrm{res}}$. There is actually another way of smoothing out the singularities by deforming the complex structure. For the conifold geometry this can be done by turning on a parameter μ on the right-hand side of the equation: $u'^2 + v'^2 + x^2 + y'^2 = \mu^2$. Having $\mu > 0$ corresponds to inflating a three-sphere of radius μ: the S^3 appears as a real section of the conifold. For a general singularity of the form (4.4.21) one needs a polynomial $\mu(x)$ of degree $n-1$ to deform all the singularities

$$\mathsf{CY}_{\text{def}}: \quad uv - y^2 + V'(x)^2 = \mu(x) \ . \tag{4.4.24}$$

Near each of the singular points the geometry looks like the deformed conifold. Viewing the inflated S^3 as a two-sphere fibered over an interval in the complex $x-$plane, which appears as a branch cut. Indeed, taking again the conifold as example the relevant interval is $-\sqrt{\mu} \leq x \leq \sqrt{\mu}$. For fixed x the deformed conifold describes a two-sphere of radius $\sqrt{\mu^2 - x^2}$, which disappears at the endpoints of the interval: the total space is therefore a three-sphere. Effectively, in going from CY_{res} to CY_{def}, we thus replace each critical point (or 'blown-up' $\mathbb{P}^1$) by a branch cut (or 'inflated' S^3). This is precisely what happens in the double-scaling limit of the cMT (4.4.22), where the eigenvalues cluster together around each critical point leading to a branch cut in the spectral $x-$plane.

The above transition between the resolved and deformed conifold is known as the '*conifold transition*' [118,179], and it is the archetypical example of an open/closed duality. The physical interpretation of the open/closed duality (or 'geometric transition') is that the open string theory on CY_{res} with L branes wrapping the two-spheres is equivalent in the double-scaling limit $L \to \infty$ to the closed topological string theory on CY_{def}, without the D-branes. The D-branes have been replaced by fluxes for the holomorphic $(3,0)$-form. The effective theory associated to the open strings is a matrix integral (4.4.22) with potential V. The closed string theory, on the other hand, is the KS theory of complex structure deformations of the spectral curve $\mathscr{S}$. In that sense, the KS on $\mathscr{S}$ is dual to the large L matrix integral [39], with $e^{S_0} \sim L^{1/4}/g_s^{3/4}$. As we have mentioned before, the precise ratio of $L^{1/4} g_s^{-4/3}$ comes from an additional technical step in going from the background (4.4.24) to, for example, the spectral curve $\mathscr{S}$ of pure topological gravity or JT gravity, where we zoom in on the edge of the eigenvalue spectrum while taking $L \to \infty$.

4.5 Discussion

In this chapter we have laid out an arc spanning all the way from the theory of quantum chaos to that of 2d quantum gravity. The guiding principle, from the gravity perspective is the (doubly) non-perturbative completion of the semi-classical path integral of JT gravity. From the perspective of quantum chaos, the relevant contributions determine the hyper-fine structure of the spectrum of energy eigenstates, which are quasi regularly ordered with an average spacing of e^{-S_0} (see Figure 4.2). In quantum chaotic theories this spectral structure is universal and determined by a symmetry principle: the breaking and restoration of causal symmetry, [49], as described by a remarkably simple low-dimensional matrix theory, the fMT introduced in Section 4.2.1.

The one line summary of our story is that all relevant players in the two-dimensional gravitational framework – Kodaira-Spencer field theory, a system of D-branes introduced into a six-dimensional Calabi-Yau manifold, and the SYK model (at the spectral edge) as a putative boundary theory – map onto that fMT. This proves that they all faithfully describe the universal ergodic phase of quantum chaos – a finding not easily established otherwise. It also establishes their quantitative equivalence in the long time limit. The setting is minimal in that further reduction, such as the perturbative representation of Kodaira-Spencer field theory in terms of the JT gravitational path integral, looses information on the hyperfine structure of the spectrum.

The mechanism behind the reduction to fMT is the fast ergodization of systems with quantum chaotic dynamics. It implies the efficient entangling of all states in a Hilbert space of high 'color' dimension, to the effect that long time correlations are described by a universal effective theory whose low 'flavor' dimension is determined by the number of probes (or the order of correlation functions) into the chaotic phase. This color-flavor duality is universally observed in ergodic quantum chaos and, as we show in this chapter, the effective theories of two-dimensional gravity are no exception. The relationship between large-L 'color' matrix model and finite-size 'flavor' matrix models, in particular of the Kontsevich had been appreciated before in the topological- and minimal-string context [40, 132], but it is very illuminating to see that it is in fact an expression of the universality of quantum chaotic correlations even in the gravitational context, as we have established in this work.

In Section 4.3 we described the above reduction to flavor theory for Kodaira-Spencer (KS) field theory, with brane and anti-brane insertions assuming the role of spectral probes. This setting led to a beautiful geometric view of the analytic structure of the spectrum of 2D quantum gravity: KS theory is defined on the multi-sheeted spectral curve of JT gravity with a branch cut describing the coarse grained spectral density. Its perturbative expansion in e^{-S_0} equals the 'mini-universe expansion' of JT gravity, and at the same time reveals long ranged (on scales of the microstate spacing) correlations in the spectrum. However, it takes a computation non-perturbative in e^{S_0}, in the presence of probe branes, to reveal the micro structure of states as poles, rather than elements of a continuous cut. Within the fMT framework, the discontinuity across the cut, and its resolution into individual poles at hyperfine scales are associated to the breaking and restoration of causal symmetry, respectively.

It would be interesting to connect this to recent work on the discrete spectrum of a putative non-perturbative completion of JT gravity [180][17]. In Section 4.4 we

[17]That work also noted the analogy to (classical) D-brane positions being 'smeared' into a con-

looked at these principles from an even further expanded geometric perspective. Starting from the parent geometry of a six-dimensional CY manifold, we showed how it emerges from the world-volume theory of non-compact branes in a closed-string background obtained by dissolving a stack of compact branes into the geometry. In this setting, causal symmetry arises as the world-volume symmetry of the flavor branes. The breaking and restoration of causal symmetry indicates a change in the analytical structure of the target space perceived by the flavor branes: At perturbative level in e^{S_0} the target space presents branch cuts, which are resolved into poles at the fully non-perturbative level. The semi-classical brane partition function expanded around the saddle points presents branch points, while the full quantum result is an entire function of the energy, that is target-space coordinate (e.g. the Airy function at the topological point).

In theories of quantum chaos, the restoration of causal symmetry in the long time limit reflects the hyperfine structure of the chaotic spectrum: Levels repel and ultimately 'crystallize' into an approximately evenly spaced structure. Our work shows how the same phenomenon occurs in the gravitational framework, but now can be given a geometric interpretation, as outlined above. We note that story is reminiscent of what happens in the study of the target space geometry of minimal string theory [132] The fact that different effective descriptions of two-dimensional gravity 'want' to contract to fMT at large time scales demonstrates the prevalence of the ergodic quantum chaotic phase in this context.

tinuous spectral density. It would be interesting to find the analog of the Fredholm determinant used by [180] directly in topological string / KS theory.

5 Virasoro entanglement Berry phases

5.1 Introduction

A particular goal of holography is to understand the emergence of geometry from the boundary conformal field theory. Recent applications of quantum information theory in holography have given a means of directly probing geometry of the bulk, and thus have provided a promising avenue for addressing this question.

One geometrical application of entanglement is an auxiliary space for holography known as kinematic space, which can be defined as the space of pairs of spacelike points in a CFT_d [181, 182]. Perturbations of entanglement entropy are seen to propagate as fields on this space [183]. For CFT_2, kinematic space can additionally be obtained from the set of entanglement entropies associated to intervals [184]. While fixed by the asymptotic conformal symmetry, kinematic space provides a tool for the reconstruction of bulk geometry in certain sufficiently symmetrical and controlled settings. For instance, it reconstructs geometry for locally AdS_3 spacetimes [185]. It also probes the geometry only outside of entanglement shadow regions that are inaccessible to spacelike geodesics [186,187]. This auxiliary space is a symplectic manifold, specifically it is a particular coadjoint orbit of the conformal group [188].

The drawback here is of course the reliance on symmetries and special geometries. Is it possible to use entanglement to probe more general geometries? To this end, transport for 2d kinematic space was generalized to a parallel transport process for the modular Hamiltonian [53, 54]. In this setup, there is an associated Berry connection on kinematic space that computes lengths of curves in the bulk. More generally, a modular Berry connection can be shown to relate frames for CFT algebras associated to different states and subregions. Entanglement provides a connection that sews together nearby entanglement wedges and probes the geometry near the extremal surface. This connection builds spacetime from entanglement, reminiscent of the ER=EPR proposal [52]. While the modular

Hamiltonian admits a particularly simple, local description only in special cases, the parallel transport of modular Hamiltonians is true more generally, and its bulk description relies only on leading order in $1/N$ and sufficient smoothness of the extremal surface.

The parallel transport of modular Hamiltonians has been studied in the setting where the interval shape is varied, which connects to kinematic space [181]. Shape-changing parallel transport has also been applied to study cases in holography where the modular chaos bound is saturated, which is governed by a certain algebra of modular scrambling modes that generate null deformations close to the extremal surface [189].

We are interested in generalizing beyond the case where the shape or interval location is varied, to consider modular parallel transport governed by a change of global *state* (see also [190] for a similar approach). For instance, one could imagine acting on a CFT on the cylinder by a large diffeomorphism contained in the Virasoro algebra. This would modify the algebra of operators on the interval. The redundancy by certain symmetries known as *modular zero modes* which change the algebra but leave physical observables fixed results in a connection and non-trivial parallel transport, even in the case where the interval remains fixed. A general modular transport problem would consist of an amalgamation of these two kinds of parallel transport, with a simultaneous modification of both the state and interval shape.

Ultimately, we consider special transformations which do not lie in the Virasoro algebra as typically defined since they are not analytic, rather they vanish at the interval endpoints and are non-differentiable at these points. The reason for this is technical: to uniquely isolate the zero mode contribution it is necessary to have a decomposition into kernel and image of the adjoint action of the modular Hamiltonian. As we explain in Appendix C.3, this is not possible for the Virasoro algebra. This is a subtlety that, to our knowledge, has not been previously studied. For a large class of transformations which obey certain properties, we derive a general expression for the Berry curvature in Appendix C.2. We also explain how these non-standard vector fields have a simple interpretation as plane waves in the hyperbolic black hole geometry using the map of Casini, Huerta and Myers [191].

We define a suitable algebra of vector fields on the circle constructed from wave packets of these eigenstates. Much as similar group-theoretic parallel transport problems are governed by the geometry of symplectic manifolds known as coadjoint orbits, here that is the case as well. We show that the Berry curvature for state-changing parallel transport is equal to the Kirillov-Kostant symplectic form on an orbit associated to this algebra of vector fields.

State-changing parallel transport can also be related to bulk geometry. This has the advantage of accessing different geometrical data in the bulk, compared to the setting where only the interval shape is varied. We find that the Berry curvature for a fixed interval and changing state computes the symplectic form for a Euclidean conical singularity geometry obtained from the backreaction of a cosmic brane, subject to a suitable principal value prescription for regulating divergences near the interval endpoint. To match the curvature, we must impose Dirichlet boundary conditions at the location of the extremal surface. We interpret this as describing (and defining) a symplectic form associated to the entanglement wedge. In the discussion, we connect to earlier work on the holographic interpretation of the Berry curvature, and comment on the relation to the entanglement wedge symplectic form in the case of operator-based parallel transport.

Modular parallel transport, either in the case of a changing shape or a changing state, is a parallel transport of *operators* and density matrices. It is distinct from existing algebraic applications of parallel transport of *states*, which for instance transform under unitary representations of a symmetry group. As part of this work we hope to clarify some of the differences, as well as various applications of each. In particular, we both review how kinematic space for CFT_2 can be understood in the language of operator-based parallel transport in Section 5.2.1, while also providing a new derivation of this same kinematic space using state-based parallel transport in Appendix C.1. This gives two different ways of viewing the same problem, both utilizing group theory, reminiscent of the 'Heisenberg' versus 'Schrödinger' pictures for quantum mechanics.

Outline: We begin in Section 5.2 by giving a summary of both state and operator-based parallel transport, and providing a few examples of each. In Section 5.3, we derive the boundary parallel transport process for transformations that diagonalize the adjoint action and compute the curvature in an example. We go into further detail in Section 5.4 about the algebraic structure and the connection to coadjoint orbits. In Section 5.5, we present our proposal for the bulk dual using the symplectic form for Euclidean conical singularity solutions created from the backreaction of a cosmic brane. We end with a discussion about some subtleties and suggest future research directions. In Appendix C.1, we provide a derivation of kinematic space using state-based parallel transport, and in Appendix C.2 we derive a general expression for the curvature for operator-based parallel transport, which applies for any algebra. Finally, in Appendix C.3 we discuss some subtleties about diagonalization of the adjoint action for the Virasoro algebra.

5.2 Geometric Berry phases

Geometric phases can arise in quantum mechanics when a Hamiltonian depends continuously on certain parameters, such as an external magnetic field. This results in a state that differs from the starting state by a phase under a closed path in parameter space. Several generalizations of this notion have recently arisen in studies of conformal field theory and holography, relying for instance on the fact that entanglement can act as a connection that relates the Hilbert spaces of different subsystems.

The applications to holography utilize group-based generalizations of the familiar geometric phases of quantum mechanics. In this section, we will review the tools that are relevant, making a distinction between two different approaches for group-based parallel transport depending on whether it is applied to states (a Schrödinger-type picture) or density matrices (a Heisenberg approach). Before moving on to new results, we give some examples of how these different approaches have so far been applied to holography.

5.2.1 States

We begin by describing the parallel transport of states that transform under a unitary representation of a group (see [192] for applications to the Virasoro group). The basic idea is to generalize beyond a path in a space of parameters, as in quantum mechanics, to a path in a group representation. A gauge connection can be defined relating different tangent spaces along the path. If some unitaries in the representation act trivially on a starting state, this constitutes a redundancy by which the state may not return to itself under a closed path through the group manifold.

Specifically, consider a group G with Lie algebra $\mathfrak{g}$, and a unitary representation $\mathcal{D}$ which acts on a Hilbert space $\mathcal{H}$. Take a state $|\phi\rangle \in \mathcal{H}$ that is an eigenstate of all elements in a 'stabilizer' subalgebra $\mathfrak{h} \subset \mathfrak{g}$, or equivalently it is left invariant up to a phase under the action of the corresponding subgroup $H \subset G$. Let $U(\gamma(t)) \in \mathcal{D}$ with $\gamma(t) \in G$, $t \in [0, T]$ be a continuous path through this representation, which corresponds to a continuous path of states $|\phi(t)\rangle = U(\gamma(t))\,|\phi\rangle$. The states $|\phi(t)\rangle$ for all $\gamma(t)$ are often called generalized coherent states, and they parametrize the coset space G/H [193, 194].

The Berry connection is defined as

$$A = i\,\langle\phi(t)|\,d\,|\phi(t)\rangle = i\,\langle\phi|\,U^{-1}dU\,|\phi\rangle \ , \tag{5.2.1}$$

where d is the exterior derivative on the group manifold, and we have used

$U^\dagger = U^{-1}$ since the representation is unitary. The connection is just $A = i \langle\phi| \mathcal{D}(\Theta) |\phi\rangle$ with Θ the Maurer-Cartan form associated to the group, $\Theta(\dot{\gamma}(t)) = \frac{d}{d\tau}\big|_{\tau=t} [\gamma(t)^{-1}\gamma(\tau)]$. Under action by an element of the stabilizer subgroup, the state changes by a phase $|\phi(t)\rangle \to e^{i\alpha} |\phi(t)\rangle$. The connection then transforms as a gauge field, $A \to A - d\alpha$.

The associated Berry curvature is

$$F = dA \;, \tag{5.2.2}$$

and the geometric phase is defined as

$$\theta(\gamma) = \int_\gamma A \;. \tag{5.2.3}$$

This phase is in general gauge dependent, but is gauge invariant when the path γ is closed. In this case, we can write

$$\theta(\gamma) = \oint_\gamma A \; = \int_{B|\partial B = \gamma} F \;, \tag{5.2.4}$$

where in the last line we have used Stokes' theorem to convert this to the flux of the Berry curvature over any surface B with boundary γ. This measures the phase picked up by the state $|\phi\rangle$ under a closed trajectory through the group representation.

Similar techniques are relevant in the study of Nielsen complexity, which describes the geometry of the space of states related by unitaries, starting from a given reference state. A specific path through unitaries is known as a 'circuit.' In conformal field theory, one can choose a reference state such as a primary that is invariant under a subset of the conformal symmetry. Defining the complexity further requires a notion of distance between states. Certain choices have relations to the Berry connection or curvature of state-based parallel transport [195–201].

Another application arises in a subfield of holography known as 'kinematic space,' which studies the geometric properties of the space of spacelike pairs of points in a CFT$_d$ and their role in probing the geometry of the bulk anti-de Sitter (AdS) spacetime [181–184]. It was demonstrated that certain bilocal operators in a CFT pick up phases under a parallel transport that displaces the location of the spacelike points where they are evaluated. In the bulk AdS spacetime this was shown to compute the length of a curve traced out by geodesics limiting to these point pairs on the boundary (see Figure 5.1) [53]. As we show in Appendix C.1, these results for kinematic space can be understood using the language of state-based parallel

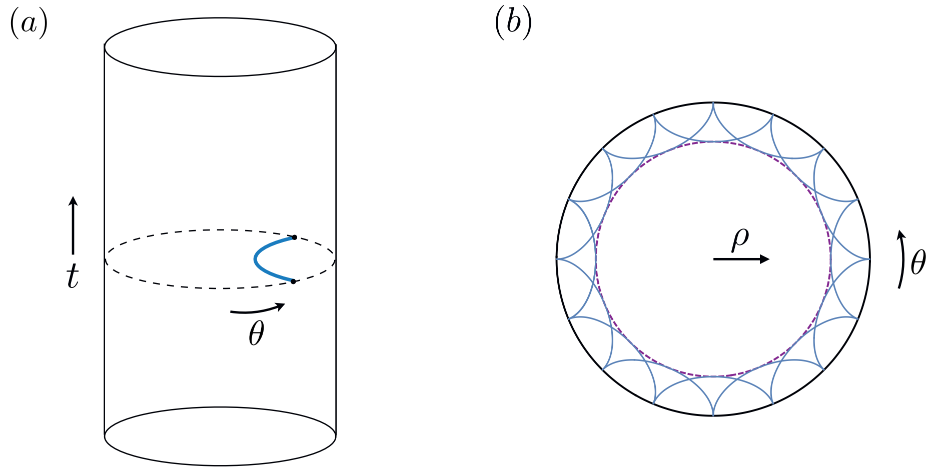


Figure 5.1: *(a) Kinematic space can be defined as the space of pairs of spacelike separated points in a CFT, which are in correspondence with bulk minimal area spacelike geodesics ending on these points. The blue curve is one such geodesic, in the special case that the endpoints lie on the same time slice. (b) The parallel transport of operators in kinematic space can be related to lengths in the bulk AdS spacetime. Depicted here is a constant time slice of anti-de Sitter spacetime. Pairs of points on the boundary define bulk geodesics (blue, solid curves). As the interval position is varied, these trace out an envelope in the bulk (dashed purple circle). The length of this envelope is directly related to the Berry phase associated to the boundary parallel transport of bilocal operators evaluated at the endpoints [53].*

transport.

5.2.2 Density matrices

Consider a subregion A on a time slice of a CFT. Associated to this region is an algebra of operators $\mathcal{A}_A$. Assuming some short distance cutoff, the state is described by a reduced density matrix ρ_A, obtained from tracing the full state over the complement $\bar{A}$ of A. From this we can define the modular Hamiltonian H_{mod} through $\rho_A = e^{-H_{\text{mod}}}/(\text{tr } e^{-H_{\text{mod}}})$. The modular Hamiltonian encodes information about the entanglement properties of the state. It will be formally useful to refer to the 'complete' modular Hamiltonian $H_{\text{mod,A}} - H_{\text{mod,}\bar{\text{A}}}$. We will often drop the subscript A, and additionally allow the modular Hamiltonian to depend on some parameter $H_{\text{mod}}(\lambda)$. This could for instance encode changes in the size of region A as was studied in [53, 54], or a change of state as we describe in the next section.

The physical data associated to A is not the set of operators in $\mathcal{A}$, but rather their expectation values. As such, there can be symmetries, i.e, transformations which act on the algebra while leaving no imprint on measurable quantities. We define a *modular zero mode* Q_i as a Hermitian operator that commutes with the modular Hamiltonian,

$$[Q_i, H_{\mathrm{mod}}] = 0 \ . \tag{5.2.5}$$

The modular zero mode can be exponentiated to the unitary

$$V = e^{-i\sum_i s_i Q_i} \ . \tag{5.2.6}$$

Under the flow $\mathcal{O} \to V^\dagger \mathcal{O} V$, the expectation values of algebra elements are left unchanged while taking the algebra to itself. The transformation by modular zero modes therefore constitutes a kind of gauge redundancy.

Given an operator, it is often useful to separate the zero mode part out from a contribution that is non-ambiguous. In the finite-dimensional case, we can compute the zero mode contribution by using the projection operator

$$P_0[\mathcal{O}] = \sum_{E, q_i, q_i'} |E, q_i\rangle \langle E, q_i| \, \mathcal{O} \, |E, q_i'\rangle \langle E, q_i'| \ , \tag{5.2.7}$$

where $|E, q_i\rangle$ are simultaneous eigenstates of H_{mod} and Q_i. Note that later we will be working with an infinite-dimensional algebra, where this formula no longer applies. We will show how to define an appropriate projection relevant for that situation in Section 5.3.

The zero mode frame redundancy leads to a Berry transport problem for operators. Imagine a process that modifies the algebra $\mathcal{A}_A$ depending on a parameter λ, for instance by changing the interval A or the state. We start by diagonalizing the modular Hamiltonian,

$$H_{\mathrm{mod}} = U^\dagger \Delta U \ , \tag{5.2.8}$$

where Δ is a diagonal matrix of eigenvalues. H_{mod}, U and Δ are functions of λ that vary along the path. Taking the derivative gives the 'parallel transport equation,'

$$\dot{H}_{\mathrm{mod}} = [\dot{U}^\dagger U, H_{\mathrm{mod}}] + U^\dagger \dot{\Delta} U \ , \tag{5.2.9}$$

where $\cdot = \partial_\lambda$. The first term on the right-hand side lies in the image of the adjoint action, $[\cdot, H_{\mathrm{mod}}]$. The second term encodes the change of spectrum under the parallel transport. It is a zero mode since it commutes with the modular Hamiltonian, in other words, it lies in the kernel of the adjoint action. We will assume that there is a unique decomposition into the image and kernel of the adjoint action, so that the entire zero mode contribution can be isolated from the second term: $P_0[\dot{H}_{\rm mod}] = U^\dagger \dot{\Delta} U$. For a discussion of subtleties associated with this assumption for the Virasoro algebra, see Appendix C.3.

This equation exhibits a redundancy due to the presence of modular zero modes. For instance, the modular Hamiltonian together with (5.2.9) could be equally well expressed in terms of $U \to \tilde{U} = UV$ where V given by (5.2.6) is generated by a modular zero mode. Instead of (5.2.8) this gauge choice leads to

$$H_{\rm mod} = V^\dagger U^\dagger \Delta U V \ . \tag{5.2.10}$$

A reasonable choice for fixing this ambiguity is to impose that

$$P_0[\partial_\lambda \tilde{U}^\dagger \tilde{U}] = 0 \ . \tag{5.2.11}$$

Since V preserves the zero mode space, $P_0[V^\dagger \dot{U}^\dagger U V] = V^\dagger P_0[\dot{U}^\dagger U] V$ from (5.2.7). Likewise, $\dot{V}^\dagger V$ is a modular zero mode from (5.2.6), so it projects to itself. Thus, this condition reduces to

$$-V^\dagger \dot{V} + V^\dagger P_0[\dot{U}^\dagger U] V = 0 \ , \tag{5.2.12}$$

where we have used $\dot{V}^\dagger V = -V^\dagger \dot{V}$ since V is unitary. We therefore obtain a more familiar expression for parallel transport of the operator V,

$$\left(\partial_\lambda - \Gamma\right) V = 0 \ , \tag{5.2.13}$$

where

$$\Gamma = P_0[\dot{U}^\dagger U] \tag{5.2.14}$$

is a Berry connection that encodes information about how the zero mode frame changes as we vary the modular Hamiltonian. It transforms as $\Gamma \to V^\dagger \Gamma V - V^\dagger \dot{V}$ under $U \to UV$. After performing the parallel transport around a closed loop, $\dot{U}^\dagger U$ has a definite value by (5.2.11). However, U itself may differ by a modular zero mode,

$$U(\lambda_f) = U(\lambda_i) e^{-i \sum_i \kappa_i Q_i} \ . \tag{5.2.15}$$

Here, $\lambda_f = \lambda_i$ are the endpoints of a closed path. The coefficients κ_i contain information about the loop.

There is also a curvature, F, associated to this parallel transport process. We can evaluate the curvature by performing parallel transport around a small loop. Here, 'small' means that we replace the derivatives with infinitesimal transformations. We can think of the operator $S_{\delta\lambda} = \tilde{U}^\dagger \delta_\lambda \tilde{U}$ as a generator of parallel transport.

It transforms as a gauge field

$$S_{\delta\lambda} \to V^\dagger S_{\delta\lambda} V + V^\dagger \delta_\lambda V \tag{5.2.16}$$

under a change of modular frame $\tilde{U} \to \tilde{U}V$ and satisfies $P_0[S_{\delta\lambda}] = 0$ by (5.2.11). The curvature F associated to this gauge field is what we call the *modular Berry curvature*. It can be represented in the usual way by performing two consecutive infinitesimal transformations $\lambda_i \to \lambda_i + \delta_1\lambda$, followed by $\lambda_i + \delta_1\lambda \to \lambda_i + \delta_1\lambda + \delta_2\lambda$. Doing the same with $(1 \leftrightarrow 2)$ and taking the difference gives a closed loop with

$$F = (1 + S_{\delta_2\lambda}(\lambda_i + \delta_1\lambda))(1 + S_{\delta_1\lambda}(\lambda_i)) - (1 \leftrightarrow 2)\ . \tag{5.2.17}$$

Here, we use that the holonomy operator along the line $[\lambda_i, \lambda_i + \delta\lambda]$ is given by

$$\exp\left(\int_{\lambda_i}^{\lambda_i+\delta\lambda} \tilde{U}^\dagger \delta_\lambda \tilde{U}\right) = 1 + S_{\delta\lambda}(\lambda_i)\ . \tag{5.2.18}$$

In Appendix C.2, we will derive a general expression for the curvature, (5.2.17), and we apply it in Section 5.3 to the case of state-changing parallel transport.

Example: Shape-changing parallel transport

As an example, we will review how this framework for parallel transport of operators can be applied to a parallel transport process of the modular Hamiltonian intervals whose location is varied in the CFT vacuum. This reduces to the study of kinematic space, which we see can also be described using state-based parallel transport in Appendix C.1.

We consider our subregion A to be an interval on a fixed time slice of the CFT with endpoints located at θ_L and θ_R. Generalizing to subregions with endpoints which are not in the same time slice is straightforward. The modular Hamiltonian associated to A can be written in terms of $\mathfrak{sl}(2,\mathbb{R})$ generators as

$$H_{\text{mod}} = s_1 L_1 + s_0 L_0 + s_{-1} L_{-1}\ . \tag{5.2.19}$$

Here, we have omitted the $\bar{L}$ operators for simplicity. The coefficients in (5.2.19) depend on θ_L, θ_R and can be determined by requiring that the generator keeps the interval fixed. Explicitly, they are given by

$$s_0 = -2\pi \cot\left(\frac{\theta_R - \theta_L}{2}\right)\ , \quad s_{\pm 1} = \frac{2\pi \cot\left(\frac{\theta_R - \theta_L}{2}\right)}{e^{\pm i\theta_R} + e^{\pm i\theta_L}}\ ,. \tag{5.2.20}$$

In case of A extending along half the interval, taking for example $\theta_R = -\theta_L = \pi/2$, the modular Hamiltonian can be found from (5.2.20) to be $H_{\text{mod}} = \pi(L_1 + L_{-1})$.

We now construct a one-parameter family of modular Hamiltonians by changing the shape of the interval. The simplest trajectory is given by just changing one of the endpoints, e.g., taking the parameter $\lambda = \theta_L$. The change in modular Hamiltonian is now captured by the parallel transport equation (5.2.9), which in this case reads

$$\delta_{\theta_L} H_{\text{mod}} = [S_{\delta\theta_L}, H_{\text{mod}}] \ . \tag{5.2.21}$$

We can solve (5.2.21) for the shape-changing parallel transport operator $S_{\delta\theta_L}$ by first diagonalizing the action of the modular Hamiltonian

$$[H_{\text{mod}}, V_\mu] = i\mu V_\mu \ , \tag{5.2.22}$$

with $\mu \in \mathbb{R}$. It is not difficult to see that the following operators are solutions

$$V_{-2\pi} = \partial_{\theta_L} H_{\text{mod}} \ , \quad V_0 = H_{\text{mod}} \ , \quad V_{2\pi} = \partial_{\theta_R} H_{\text{mod}} \ , \tag{5.2.23}$$

with $\mu = -2\pi, 0, 2\pi$ respectively. The operators $V_{2\pi}$ and $V_{-2\pi}$ saturate the modular chaos bound [189]. Importantly, notice that this class of deformations is characterized by imaginary eigenvalues in (5.2.22). The generator of modular parallel transport therefore takes the form

$$S_{\delta\theta_L} = -\frac{i}{2\pi}\partial_{\theta_L} H_{\text{mod}} \ . \tag{5.2.24}$$

For this particular operator (5.2.11) is automatically satisfied, since it can be written as the commutator of H_{mod}. Similarly, one can show that $S_{\delta\theta_R} = \frac{i}{2\pi}\partial_{\theta_R} H_{\text{mod}}$. Then, using (5.2.17) one can compute the modular Berry curvature for this shape-changing transport to be

$$F = [S_{\delta\theta_L}, S_{\delta\theta_R}] = -\frac{i}{4\pi}\frac{H_{\text{mod}}}{\sin^2\left(\frac{\theta_R - \theta_L}{2}\right)} \ . \tag{5.2.25}$$

In particular, applying the projection P_0 to this expression does not change it, as the curvature is proportional to a zero mode. In Appendix C.1, we rederive the result in (5.2.25) from the point of view of kinematic space. The curvature, (5.2.25), is simply the volume form on kinematic space.

5.3 State-changing parallel transport

Let us apply the formalism above to a parallel transport process that modifies not the location of the entangling interval, but rather the state of the system. For definiteness, we work on the AdS$_3$ cylinder with a choice of time slice in the boundary CFT$_2$.

Consider a change of state by acting by an element $\xi(z)$ of Diff(S^1), starting from the vacuum of AdS_3. The operator that implements this is

$$X_\xi = \frac{1}{2\pi i} \oint \xi(z) T(z)\, dz \ , \tag{5.3.1}$$

where $T(z)$ is the stress tensor of the boundary CFT. In particular, the diffeomorphism $\xi(z) = z^n$ is implemented by the usual Virasoro mode operator $X_{z^n} = L_{n-1}$.

Under such a general transformation, the modular Hamiltonian H_{mod} associated to some interval on the boundary transforms as

$$\delta_\xi H_{\mathrm{mod}} = [X_\xi, H_{\mathrm{mod}}] \ . \tag{5.3.2}$$

Notice that this is just the parallel transport equation, (5.2.9), minus the zero mode piece.

Now imagine computing the curvature, (5.2.17), by taking the parallel transport along a small square, i.e., first performing a transformation ξ_1 followed by a transformation ξ_2, then subtracting the opposite order. The result for the curvature is derived in Appendix C.2 and is given by

$$F = P_0([X_{\xi_1}, X_{\xi_2}]) \ , \tag{5.3.3}$$

where P_0 projects to the zero mode of its argument, and the operators X_{ξ_i} are assumed to have no zero modes themselves. We note that while we focus here on CFT_2, this is a quite general result that applies to any parallel transport process of the form (5.3.2). (5.3.3) together with its application in an explicit example constitute the main results of this section.

The projection operator in (5.3.3) is defined by the property that it gives a nonzero answer when evaluated on the modular Hamiltonian (and in general, any other operators that commute with it). Meanwhile, it evaluates to zero on any other operators, which we have assumed take the form $[\cdot, H_{\mathrm{mod}}]$ in the decomposition (5.2.9). It is possible to construct the projection explicitly in cases where the modular Hamiltonian is known, for instance in our case of CFT_2. Let θ be the spatial boundary coordinate on a constant time slice. The modular Hamiltonian for an interval of angular radius α centered around $\theta = 0$ on the cylinder is [202,203]

$$H_{\mathrm{mod}} = \int_{-\alpha}^{\alpha} d\theta \, \frac{\cos\theta - \cos\alpha}{\sin\alpha} \, T_{00}(\theta) \ . \tag{5.3.4}$$

Here, the units are chosen so that the stress energy tensor is dimensionless, $T_{00} \sim -c/12$ in the vacuum on the cylinder, with $T_{00}(\theta) \equiv -(T(\theta) + \overline{T}(\theta))$.

It will be useful to work in planar coordinates. We consider the conformal transformation

$$z = e^{i\theta} \tag{5.3.5}$$

to map the cylinder to the plane (with radial ordering). In particular, the interval $[-\alpha, \alpha]$ in the θ-coordinate is mapped to the circle arc with opening angle 2α in the z-plane. The stress tensor transforms as

$$T(\theta) = \left(\frac{\partial z}{\partial \theta}\right)^2 T(z) + \frac{c}{12}\{z, \theta\} \ , \tag{5.3.6}$$

where the Schwarzian derivative is defined by

$$\{z, \theta\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2 \ . \tag{5.3.7}$$

Applying the transformation (5.3.5), we find that the modular Hamiltonian on the plane is given by

$$H_{\text{mod}} = \frac{1}{i} \oint_{|z|=1} \frac{\frac{1}{2}(1+z^2) - z\cos\alpha}{\sin\alpha} T(z)\, dz \ . \tag{5.3.8}$$

Notice that in (5.3.8) we have converted to the *complete* modular Hamiltonian by integrating over the full range of coordinates instead of $[-\alpha, \alpha]$. The reason is that an integration over the full circle allows for an expansion of quantities in terms of Virasoro modes. Moreover, we have conveniently subtracted the vacuum energy of the cylinder in going from Eq. (5.3.4) to Eq. (5.3.8) and only kept the holomorphic part of the stress tensor.

For simplicity, we will take $\alpha = \pi/2$ so that the interval extends along half of the cylinder (from $z = -i$ to $z = i$ in the Euclidean plane). The generalization to intervals with arbitrary α is straightforward. With this convention the modular Hamiltonian simplifies to

$$H_{\text{mod}} = \frac{1}{2i} \oint (1+z^2) T(z)\, dz \ . \tag{5.3.9}$$

We can also express this in terms of the Virasoro modes on the plane,

$$L_n = \frac{1}{2\pi i} \oint z^{n+1} T(z) dz \quad , \tag{5.3.10}$$

which satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \ . \tag{5.3.11}$$

Then, (5.3.9) can be re-expressed as

$$H_{\text{mod}} = \pi(L_{-1} + L_1) \ . \tag{5.3.12}$$

In the following, it will be useful to write formulae in terms of the diffeomorphism ξ directly, rather than in terms of the corresponding operator X_ξ. In particular, we identify the modular Hamiltonian H_{mod} with the vector field $\xi(z) = \pi(1 + z^2)$, as follows from (5.3.9). Moreover, if we take an operator of the form

$$X_\xi = \frac{1}{2\pi i} \oint \xi(z) T(z) \, dz \ , \tag{5.3.13}$$

the commutator with H_{mod} can also be expressed in ξ directly. Using Eqs. (5.3.9) and (5.3.13), applying the OPE

$$T(w)T(z) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + ... \tag{5.3.14}$$

and integrating by parts we find

$$[H_{\text{mod}}, X_\xi] = \frac{1}{2i} \oint \left[2z\xi(z) - (1+z^2)\xi'(z) \right] T(z) \, dz \ . \tag{5.3.15}$$

Applying several integration by parts directly onto (5.3.9), the term proportional to the central charge identically vanishes in this case.

To implement Eq. (5.3.3) for the modular Berry curvature one needs to define the operator P_0 which projects onto the zero mode. Following the general prescription in Section 5.2.2, one would like to decompose an arbitrary operator X into the image and the kernel of the adjoint action of H_{mod},

$$X = \kappa H_{\text{mod}} + [H_{\text{mod}}, Y] \ , \tag{5.3.16}$$

where κ is the zero mode that needs to be extracted. However, it turns out that there is a subtlety associated with the above decomposition in the case of the Virasoro algebra. In general, there are operators which are neither in the kernel, nor in the image of the adjoint action[1], which leads to an ambiguity in the definition of the zero mode projection P_0. We refer to Appendix C.3 for a discussion of these issues in the case of the Virasoro algebra. For this reason, we will consider a different class of transformations, i.e., those which diagonalize the adjoint action of the modular Hamiltonian H_{mod} (see [204] where a similar diagonalization in

[1]For finite-dimensional vector spaces this is not the case if the kernel and image are disjoint, as follows from a simple dimension counting. In the infinite-dimensional set-up the situation is more complicated, e.g., one can write down linear maps which are injective but not surjective.

terms of so-called modular eigenmodes was considered). Therefore, we start from the eigenvalue equation

$$[H_{\text{mod}}, X_\lambda] = \lambda X_\lambda \ , \tag{5.3.17}$$

where we have used the short-hand notation $X_\lambda \equiv X_{\xi_\lambda}$ for the operator associated to the transformation ξ_λ. Using (5.3.15) it is not difficult to see that (5.3.17) is solved by

$$\xi_\lambda(z) = \pi(1+z^2)\left(\frac{1-iz}{z-i}\right)^{-i\lambda/2\pi} \ . \tag{5.3.18}$$

In particular, we see that the operator with eigenvalue zero, $\lambda = 0$, is the modular Hamiltonian itself, as one would expect from (5.3.17). Notice that the solutions in (5.3.18) go to zero at the endpoints of the interval:

$$\xi_\lambda(z) \to 0 \quad \text{as} \quad z \to \pm i \ . \tag{5.3.19}$$

The eigenfunctions of H_{mod} therefore correspond to the transformations which change the state, but not the location of the boundary interval. They are not analytic at $z = \pm i$,[2] so strictly speaking they are not part of the Virasoro algebra (defined in the usual way as the space of smooth vector fields on the circle). However, they seem to be the natural transformations to consider in this context. We will refer to them as *state-changing* transformations as opposed to the shape-changing transformations in Section 5.2.2.

From (5.3.17) combined with the Jacobi identity, these eigenfunctions form an algebra with commutation relations

$$[X_\lambda, X_\mu] = (\lambda - \mu) X_{\lambda+\mu} \ , \tag{5.3.20}$$

which defines a continuous version of the Virasoro algebra[3] with generators X_λ labeled by a continuous parameter $\lambda \in \mathbb{R}$. Note that in the following we are leaving out the central extension (so strictly speaking we are working with a continuous version of the Witt algebra). We will return to discuss how to include the central extension in Section 5.3.3.

It is natural to define the transformations in (5.3.18) to have support only on the subregion A. In the case at hand, this makes all the contour integrals collapse to integrals over the semicircle from $-i$ to i, e.g., the $\lambda = 0$ eigenfunction does not correspond to the *complete* modular Hamiltonian, but simply to the half-sided

[2] Note that due to (5.3.19), it is valid to apply a single integration by parts. Thus, (5.3.15) is maintained.

[3] A Virasoro algebra with continuous index also appears in the context of the so-called dipolar quantization of 2d CFT [205, 206] which is related to the sine-square deformation [207, 208], as well as in the study of non-equilibrium flows in CFT [209].

one. The state-changing vector fields, which might look unfamiliar in terms of the z-coordinate, take a more familiar form when we map the entanglement wedge to a hyperbolic black hole geometry using [191].

This can be seen in the following way. Starting with the boundary CFT_d on the Euclidean cylinder $\mathbb{R} \times S^{d-1}$ with metric

$$ds^2 = dt_E^2 + d\theta^2 + \sin^2\theta \; d\Omega_{d-2}^2 \ , \tag{5.3.21}$$

we consider a fixed sphere at $t_E = 0$, $\theta = \theta_0$. We can apply the following conformal transformation considered in [191]:

$$\begin{aligned} \tanh t_E &= \frac{\sin\theta_0 \sin\tau}{\cosh u + \cos\theta_0 \cos\tau} \ , \\ \tan\theta &= \frac{\sin\theta_0 \sinh u}{\cos\theta_0 \cosh u + \cos\tau} \ , \end{aligned} \tag{5.3.22}$$

which conformally maps the causal development of the sphere to the hyperbolic geometry $\mathbb{R} \times \mathbb{H}^{d-1}$ given by

$$ds^2 = \Omega^2 \left(d\tau^2 + du^2 + \sinh^2 u \; d\Omega_{d-2}^2 \right) \ , \tag{5.3.23}$$

with conformal factor

$$\Omega^2 = \frac{\sin^2\theta_0}{(\cosh u + \cos\theta_0 \cos\tau)^2 - \sin^2\theta_0 \sin^2\tau} \ . \tag{5.3.24}$$

Taking $d = 2$ and $\theta_0 = \pi/2$ for the half interval entangling surface, the transformation (5.3.22) at the $\tau = 0$ (or equivalently $t_E = 0$) time slice reduces simply to

$$\tan\theta = \sinh u \ . \tag{5.3.25}$$

Written in terms of the coordinate $z = e^{i\theta}$ this leads to

$$e^u = \frac{1 - iz}{z - i} \ . \tag{5.3.26}$$

Recall that the boundary region A corresponds to $|z| = 1$ and $-\pi/2 \leq \arg(z) \leq \pi/2$ in the plane, so it is mapped to $u \in \mathbb{R}$. Moreover, the components of the vector field transform according to

$$\xi_\lambda(z) \frac{\partial}{\partial z} = \xi_\lambda(u) \frac{\partial}{\partial u} \tag{5.3.27}$$

with

$$du = -2i\frac{dz}{1+z^2} \ , \tag{5.3.28}$$

so that the transformations take the simple form

$$\xi_\lambda(u) = -2\pi i\, e^{-i\lambda u/2\pi} \ . \tag{5.3.29}$$

Hence, we find that the state-changing transformations, when written in terms of the u-variable, are simply plane wave solutions with frequency $\lambda/2\pi$ in this black hole background. Therefore, they are natural objects to consider in this geometry.

We can reintroduce both the right- and the left-movers by replacing $u \to u + i\tau$ in (5.3.26), so that z is allowed to take values in the half plane $\operatorname{Re} z \geq 0$ (the radial direction in the z-plane corresponds to time evolution in τ). (5.3.27) is therefore modified according to

$$\xi_\lambda(z)\frac{\partial}{\partial z} = \xi_\lambda(u+i\tau)\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial \tau}\right) \ , \tag{5.3.30}$$

$$\xi_\lambda(\bar{z})\frac{\partial}{\partial \bar{z}} = -\xi_\lambda(-u+i\tau)\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial \tau}\right) \ . \tag{5.3.31}$$

By setting $\lambda = 0$ and adding the right- and left-moving contributions, we see that the modular Hamiltonian indeed acts by time translation in the black hole geometry:

$$H_{\text{mod}} \sim \frac{\partial}{\partial \tau} \ . \tag{5.3.32}$$

Working in the algebra associated to the eigenfunctions of H_{mod}, we do have a unique decomposition of the form (5.3.16): one simply decomposes an arbitrary operator into eigenoperators, which have either $\lambda = 0$ or $\lambda \neq 0$. Given such a decomposition it is easy to write down an operation which extracts the zero mode κ, namely a linear functional P_0 which satisfies[4]

$$P_0(H_{\text{mod}}) \sim \delta(0) \ , \quad P_0([H_{\text{mod}}, Y]) = 0 \ . \tag{5.3.33}$$

In the u-coordinate such a functional can be written as

$$P_0(X_\xi) = \lim_{\Lambda\to\infty} \frac{i}{2\pi}\int_{-\Lambda}^{\Lambda} \xi(u)\, du \ . \tag{5.3.34}$$

[4]For technical reasons we set $P_0(H_{\text{mod}}) \sim \delta(0)$, instead of $P_0(H_{\text{mod}}) \sim 1$ as one might have naively expected. This results from the plane-wave normalizability of the eigenfunctions, (5.3.29). It ensures the modular Berry curvature is finite when evaluated on wave packets in Section 5.3.1.

Using the coordinate change (5.3.28), we can represent the projection in the z-coordinate as

$$P_0(X_\xi) = \lim_{\Lambda\to\infty} \frac{i}{2\pi} \int_{-\Lambda}^{\Lambda} \xi(u)\, du = \frac{1}{\pi} \int_{-i}^{i} \frac{\xi(z)}{(1+z^2)^2}\, dz \ . \tag{5.3.35}$$

When applied to the eigenfunctions of $H_{\rm mod}$ the projection becomes

$$P_0(X_\lambda) = \lim_{\Lambda\to\infty} 2\pi \int_{-\Lambda}^{\Lambda} e^{i\lambda u}\, du = 4\pi^2 \delta(\lambda) \ , \tag{5.3.36}$$

which is a standard representation of the Dirac delta function. To show that P_0 vanishes on commutators of the form $[H_{\rm mod}, Y]$, it suffices to remark that one can take Y to satisfy $[H_{\rm mod}, Y] = \lambda Y$ with $\lambda \neq 0$ without loss of generality. This shows that (5.3.34) defines a good projection operator in the sense of (5.3.33). Unlike for the case of the ordinary Virasoro algebra treated in Section C.3.3, there is no ambiguity in the resulting projection.

5.3.1 Example

We now have all the ingredients to compute the curvature in an explicit example. We consider a general perturbation of the form

$$z' = z + \epsilon\, \xi(z) + \mathcal{O}(\epsilon^2) \ , \tag{5.3.37}$$

where $\xi(z)$ is a wave packet

$$\xi(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\lambda) \xi_\lambda(z)\, d\lambda \ , \tag{5.3.38}$$

with $\xi_\lambda(z)$ defined in (5.3.18). We start by obtaining the correction to the transformed modular Hamiltonian upon acting with (5.3.37). Let us expand both the modular Hamiltonian and the parallel transport operator to first order in the small parameter ϵ:

$$H'_{\rm mod} = H^{(0)} + \epsilon\, H^{(1)} + \mathcal{O}(\epsilon^2) \ , S = S^{(0)} + \epsilon\, S^{(1)} + \mathcal{O}(\epsilon^2) \ . \tag{5.3.39}$$

Using that $z = z' - \epsilon\xi(z') + \mathcal{O}(\epsilon^2)$, one can expand the transformed $H_{\rm mod}$ to order $\mathcal{O}(\epsilon^2)$. One finds that $H^{(0)} = H_{\rm mod}$ is the original modular Hamiltonian, while the correction is given by

$$H^{(1)} = -\frac{1}{2i} \oint \left[2z\xi(z) - (1+z^2)\xi'(z)\right] T(z)\, dz \ . \tag{5.3.40}$$

Here, we have neglected the Schwarzian contribution for simplicity. It will be treated separately in Section 5.3.3. We now expand the parallel transport equation

$$\delta H_{\text{mod}} = [S, H_{\text{mod}}] \tag{5.3.41}$$

to first order in ϵ. This gives two separate equations:

$$0 = [S^{(0)}, H^{(0)}] \ , \quad H^{(1)} = [S^{(0)}, H^{(1)}] + [S^{(1)}, H^{(0)}] \ . \tag{5.3.42}$$

Solving (5.3.42) for the correction $S^{(1)}$ to the parallel transport operator gives the solution

$$S^{(0)} = 0 \ , S^{(1)} = X_\xi \ . \tag{5.3.43}$$

Both $S^{(0)}$ and $S^{(1)}$ are defined up to a zero mode, meaning that one can add to it an extra operator Q for which $[Q, H_{\text{mod}}] = 0$ (e.g., the modular Hamiltonian itself) and the parallel transport equation would still be satisfied.

To compute the curvature we need to consider two different parallel transport operators S_1 and S_2 which we take to be defined according to the transformations

$$\xi_1(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c_1(\lambda)\xi_\lambda(z)\, d\lambda \ , \quad \xi_2(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c_2(\lambda)\xi_\lambda(z)\, d\lambda \ , \tag{5.3.44}$$

respectively. After projecting out their zero modes, we take the commutator and project to the zero modes again to obtain the value of the curvature component. Therefore, we need to compute

$$[S_1^{(1)} - \kappa_1 H^{(0)}, S_2^{(1)} - \kappa_2 H^{(0)}] \ , \tag{5.3.45}$$

where $\kappa_i = P_0(S_i)$, is the zero mode coefficient of the parallel transport operator S_i. We can split (5.3.45) into terms that we can treat separately. Notice that the term proportional to $[H^{(0)}, H^{(0)}]$ is zero and can be removed. Moreover, the definition of the projection operator immediately implies

$$P_0([S_1^{(1)}, H^{(0)}]) = P_0([S_2^{(1)}, H^{(0)}]) = 0 \ . \tag{5.3.46}$$

To evaluate the last commutator we use the commutation relations in (5.3.20) to obtain

$$[S_1^{(1)}, S_2^{(1)}] = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\lambda_1 - \lambda_2)c_1(\lambda_1)c_2(\lambda_2)X_{\lambda_1+\lambda_2}\, d\lambda_1 d\lambda_2 \ . \tag{5.3.47}$$

Applying the projection operator sets $\lambda_1 = -\lambda_2$, so that we find

$$P_0([S_1^{(1)}, S_2^{(1)}]) = 2\int_{-\infty}^{\infty} \lambda\, c_1(\lambda)c_2(-\lambda)\, d\lambda\ . \tag{5.3.48}$$

Therefore, the final result for the modular Berry curvature associated to the state-changing transport problem is given by

$$F = 2\int_{-\infty}^{\infty} \lambda\, c_1(\lambda)c_2(-\lambda)\, d\lambda\ . \tag{5.3.49}$$

Note that the curvature appropriately vanishes when two perturbations lie along the same direction, $c_1(\lambda) = c_2(\lambda)$. If we take the modes to be peaked at the eigenfunctions $\xi_{\lambda_i}(z)$ themselves, $c_i(\lambda) = \delta(\lambda - \lambda_i)$, the above formula reduces to

$$F = (\lambda_1 - \lambda_2)\delta(\lambda_1 + \lambda_2)\ , \tag{5.3.50}$$

which is a local formula in terms of the parameters λ_i.

5.3.2 Lie algebra

To diagonalize the adjoint action, we saw that we must work with a continuous version of the Virasoro algebra. Viewed in terms of vector fields on the circle, we must consider non-smooth vector fields on the circle, (5.3.18), which have support only along the interval. When mapped to the real line, these are just plane waves, (5.3.29). In the last section, we performed parallel transport using wave packets constructed out of these eigenfunctions. In terms of the coordinates on the real line,

$$\xi(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(\lambda)\xi_\lambda(u) d\lambda\ . \tag{5.3.51}$$

Now we would like to be more precise about the sense in which the corresponding vector fields form a Lie algebra. This amounts to imposing extra conditions on $c(\lambda)$ for these to form a closed algebra, along with any other desirable properties.

The simplest choice would be to demand that the $\xi(u)$ be smooth. Then, since the smoothness of functions is preserved under pointwise multiplication, the corresponding vector fields $\xi(u)\partial_u$ will form a closed algebra. However, an arbitrary $\xi(u)$ will not necessarily have finite zero mode projection, nor will there necessarily exist a natural definition for a dual space. To define sensible wave packets we will impose two additional requirements:

- ◇ There is a notion of Fourier transform that maps the space to itself,
- ◇ The $\xi(u)$ are integrable. This means that the projection, (5.3.34), is finite,

and this property is preserved under commutation of the vector fields $\xi(u)\partial_u$. It also allows us to define a dual space in terms of distributions.

To accomplish this, it is convenient to work with wave packets $\xi(u)$ that are *Schwartz functions*. These are smooth, bounded functions whose derivatives are also all bounded: $|u^\alpha \partial^\beta \xi(u)| < \infty$ for all $\alpha, \beta > 0$. In other words, they rapidly go to zero as $u \to \pm\infty$, faster than any reciprocal power of u. This definition excludes for example polynomials, but includes polynomials weighted by an exponential $e^{-c|u|^2}$ for $c \in \mathbb{R}$. By the Leibniz rule, the Schwartz space $\mathcal{S}$ is closed under pointwise multiplication, thus the corresponding vector fields form a closed Lie algebra. We denote $\mathcal{S}$ for the space of Schwartz functions and $\mathfrak{s}$ for the corresponding algebra of vector fields.

Since these functions are integrable, it is natural to define a dual space $\mathcal{S}'$ consisting of linear functionals $T : \mathcal{S} \to \mathbb{C}$, in terms of distributions:

$$T[\xi(u)] = \int_{-\infty}^{\infty} \xi(u) T(u) du \ . \tag{5.3.52}$$

A pairing between Schwartz functions and dual elements can be defined from this as $\langle T, \xi \rangle \equiv T[\xi(u)]$. Likewise, there is also a dual space $\mathfrak{s}^*$ consisting of linear functionals on $\mathfrak{s}$, the algebra of vector fields. This is inherited from the dual space $\mathcal{S}'$, i.e., it consists of the space of distributions evaluated on Schwartz functions. There is a pairing $\langle \cdot, \cdot \rangle$ between $\mathfrak{s}$ and $\mathfrak{s}^*$ which descends from the pairing on $\mathcal{S}$ and $\mathcal{S}'$.

Notice that, evaluated on the wave packets (5.3.51), the projection operator (5.3.34)

$$P_0 : \xi(u) \mapsto 2\pi c(0) \tag{5.3.53}$$

is a linear functional, and thus it is an element of the dual space. The pairing is given by $\langle P_0, \xi \rangle = P_0(\xi) = 2\pi c(0)$.

In the coordinates on the circle, recall that this dual element can be expressed from (5.3.35) as

$$P_0 : \xi(z) \mapsto \frac{1}{\pi} \int dz \frac{\xi(z)}{(1+z^2)^2} \ . \tag{5.3.54}$$

Notice that this dual element is not a smooth quadratic form on the circle as is typically considered in treatments of the dual space of the Virasoro algebra, but rather a more general distribution that involves singularities at $z = \pm i$[5]. A

[5] In the usual discussion of the Virasoro algebra the dual space is identified with the space of smooth quadratic differentials. Formally, one could argue that distributions such as $\delta(z - z_0)$ and $\delta'(z - z_0)$ are also part of some suitably defined notion of the dual space. Indeed, they define

standard definition of the dual space is an attempt to get a space that is roughly the same size as the algebra itself. For infinite-dimensional spaces the formal dual is much larger and one needs some additional structure, e.g., that of a Hilbert space, to limit it.

We emphasize that there is considerable freedom in these definitions. A different choice would amount to taking a different set-up for varying the state in the parallel transport process. Our definitions allow us to perform parallel transport using wavefunctions that are 'physical' in the sense of being Fourier transformable and integrable. The existence of a natural dual space also allows for contact with a geometrical picture in terms of coadjoint orbits, which we describe in the next section.

5.3.3 Central extension

We have so far only considered changing the state with a transformation of the circle. When the transformations are diffeomorphisms on the circle, the group $\mathrm{Diff}(S^1)$ gets centrally extended to the full Virasoro group, $\mathrm{Diff}(S^1) \times \mathbb{R}$. Here we are considering a continuous version of the Virasoro generated by the transformations, (5.3.18). For the central extension, we proceed in direct analogy with the Virasoro case. In the following, the vector fields $\xi(z)$ should be understood to have non-zero support only between $z = \pm i$, so that this is the only part of the integral over the full circle that contributes.

We consider pairs (ξ, α), where ξ is a vector field of the form (5.3.18), which diagonalizes the adjoint action, and $\alpha \in \mathbb{R}$. The Lie bracket is defined as

$$[(\xi,\alpha),(\chi,\beta)] = \left(-[\xi,\chi], -\frac{1}{48\pi}\oint dz\,(\xi(z)\chi'''(z) - \xi'''(z)\chi(z))\right) , \tag{5.3.56}$$

where $[\xi,\chi] := \xi\chi' - \chi\xi'$ is the commutator of vector fields. This is identical to the commutators for the Virasoro algebra, with the only difference being that we integrate only over half the circle, and also consider transformations ξ which are not smooth at the endpoints. In terms of the operators X_λ, this extends the algebra in (5.3.20) to

$$[\bar{X}_{\bar{\lambda}}, \bar{X}_{\bar{\mu}}] = (\bar{\lambda} - \bar{\mu})\bar{X}_{\bar{\lambda}+\bar{\mu}} + \frac{c}{12}\bar{\lambda}(\bar{\lambda}^2+1)\delta(\bar{\lambda}+\bar{\mu}) . \tag{5.3.57}$$

linear functionals

$$\xi \mapsto \xi(z_0) , \quad \xi \mapsto -\xi'(z_0) , \tag{5.3.55}$$

which evaluate a function (or its derivative) at some point z_0. The projection operator P_0 in Eq. (5.3.54), when integrated over the full circle and properly regularized, can be regarded in this fashion. See Appendix C.3 for more details, for example, Eqs. (C.3.29) - (C.3.31).

where we have defined rescaled barred variables through $X_\lambda = -2\pi\bar{X}_\lambda, \lambda = -2\pi\bar{\lambda}$ to bring this to a form that more closely resembles the usual Virasoro algebra with discrete labels.

One often introduces a new generator, denoted by c, which commutes with all other elements in the algebra, to write

$$(\xi, \alpha) = \xi(z)\partial_z - i\alpha c \ . \tag{5.3.58}$$

By definition, the central element c commutes with H_{mod}, i.e., $[H_{\text{mod}}, c] = 0$. Therefore, we can think about the central element as another zero mode in the parallel transport problem.

Luckily, the situation for the central element is simpler than for the modular Hamiltonian itself. From the form of H_{mod}, Eq. (5.3.12), and the algebra, Eq. (5.3.56), we see that the central element c does not appear in commutators of the form $[H_{\text{mod}}, X]$. Therefore, the projection onto the coefficient of c is simply given by the linear functional

$$(\xi, \alpha) \to \alpha \ . \tag{5.3.59}$$

One way to include the information of the central term is to make the Berry curvature give a $U(1)\times U(1)$-valued number (organized in terms of an extra element which we take to be c). More precisely, we define the zero mode projection operator P_0^c, which depends on c, by

$$P_0^c((X_\xi, \alpha)) = P_0(X_\xi) - \alpha c \ . \tag{5.3.60}$$

The first term is the usual zero mode, while the second term keeps track of the central zero mode. It is easy to see how the result for the Berry curvature gets modified. Using Eq. (5.3.3) with P_0^c instead of P_0, we see that the formula for the Berry curvature is given by

$$F = P_0([X_{\xi_1}, X_{\xi_2}]) + \frac{c}{48\pi}\oint dz \ (\xi_1(z)\xi_2'''(z) - \xi_1'''(z)\xi_2(z)) \ . \tag{5.3.61}$$

As a consistency check, we can go back to our example in Section 5.3.1 and consider the contribution from the Schwarzian term in (5.3.40). Expanding the parallel transport equation, we need to solve

$$H^{(1)} = [S^{(1)}, H^{(0)}] \ , \tag{5.3.62}$$

where the change in the modular Hamiltonian due to the Schwarzian derivative to first order is given by

$$H^{(1)}_{\text{Schw}} = \frac{c}{24i} \oint dz (1+z^2) \xi'''(z) \ , \tag{5.3.63}$$

having used that $\{z', z\} = \epsilon \xi''' + \mathcal{O}(\epsilon^2)$. On the full circle, applying three integration by parts, this is just $H^{(1)}_{\text{Schw}} = 0$ (equivalently, no diffeomorphism has $\xi''' = z^{-1}$ or z^{-3} which would give a pole). The situation is a bit more subtle on the half circle, since due to non-differentiability at the endpoints it is no longer valid to apply integration by parts multiple times. However, it is still the case that none of the eigenfunctions, (5.3.18), have $\xi''' = z^{-1}$ or z^{-3}, and so the Schwarzian contribution vanishes. Thus, in either case the solution to (5.3.62) with the new Lie bracket (5.3.56) is still given by $S^{(1)} = X_\xi$. The extra contribution to the commutator $[S^{(1)}_1, S^{(1)}_2]$ due to the central charge is indeed given by (5.3.61).

Note while it is not possible to apply integration by parts multiple times on (5.3.63) for the half circle, we have *defined* the central extension as the version that obeys integration by parts three times. This is because we have chosen the antisymmetric combination for the central charge part in (5.3.56). As a result, our bracket respects the properties of the commutator, $[X_\xi, X_\chi] = -[X_\chi, X_\xi]$. Likewise, one can check that the Jacobi identity is satisfied. Given elements $(\xi, \alpha), (\chi, \beta), (\rho, \gamma)$ which satisfy the algebra (5.3.56), we have

$$\begin{aligned}
&[(\xi,\alpha),[(\chi,\beta),(\rho,\gamma)]] + [(\chi,\beta),[(\rho,\gamma),(\xi,\alpha)]] + [(\rho,\gamma),[(\xi,\alpha),(\chi,\beta)]] \\
&\quad = \left(0, -\frac{1}{48\pi} \oint \left([\chi,\rho]\, \xi^{(3)} + [\rho,\xi]\, \chi^{(3)} + [\xi,\chi]\, \rho^{(3)} \right) \right) \ .
\end{aligned} \tag{5.3.64}$$

We can see this is identically zero by integrating each term by parts once onto the commutator, which vanishes at the interval endpoints by (5.3.19) so that there is no boundary contribution. These properties are sufficient to ensure the consistency of the central extension.

5.4 Coadjoint orbit interpretation

Various versions of the parallel transport problem we consider exhibit connections to the geometry of symplectic manifolds known as coadjoint orbits. For the state-based parallel transport summarized in Section 5.2.1 applied to the Virasoro algebra, connections to coadjoint orbits were described in [192]. In Appendix C.1, we additionally explain how to use state-based parallel transport to obtain coadjoint orbits of $SO(2,1)$, which describe kinematic space [188]. We will begin by reviewing the notion of coadjoint orbits, and then we explain how our operator-based parallel transport can be related to the geometry of orbits.

Consider a Lie group G with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual space, i.e., the space of linear maps $T : \mathfrak{g} \to \mathbb{C}$. This defines an invariant pairing $\langle T, X\rangle \equiv T(X)$ for $X \in \mathfrak{g}, T \in \mathfrak{g}^*$. The group G acts on the algebra $\mathfrak{g}$ through the adjoint action,

$$\mathrm{Ad}_g(X) = \frac{d}{d\lambda}\left(g e^{\lambda X} g^{-1}\right)\Big|_{\lambda=0} \, , \quad g \in G, \, X \in \mathfrak{g} \, . \tag{5.4.1}$$

For matrix groups such as $SO(2,1)$, which we consider in Appendix C.1, (5.4.1) is just $\mathrm{Ad}_g(X) = gXg^{-1}$.

The adjoint action of the algebra on itself can be defined from this as

$$\mathrm{ad}_X(Y) = \frac{d}{d\rho}\left(\mathrm{Ad}_{e^{\rho X}}(Y)\right)\big|_{\rho=0} = [X, Y] \, , X, Y \in \mathfrak{g} \, . \tag{5.4.2}$$

The adjoint action descends to an action on the dual space. This *coadjoint action* ad^*_X on $\mathfrak{g}^*$ is defined implicitly through

$$\langle \mathrm{ad}^*_X z, Y\rangle = \langle z, \mathrm{ad}_X Y\rangle \, , z \in \mathfrak{g}^*, \, X, Y \in \mathfrak{g} \, . \tag{5.4.3}$$

For a given $T \in \mathfrak{g}^*$, the orbit $\mathcal{O}_T = \{\mathrm{ad}^*_X(T) \,|\, X \in \mathfrak{g}\}$ generated by the coadjoint action is known as a *coadjoint orbit*.

Let x_1, x_2 be coadjoint vectors tangent to the orbit $\mathcal{O}_T$, and let X_1, X_2 be the adjoint vectors that are dual to these through the invariant pairing. Then, the *Kirillov-Kostant symplectic form* associated to this orbit is [65, 210–212]

$$\omega(x_1, x_2) = \langle T, [X_1, X_2]\rangle \, . \tag{5.4.4}$$

This is manifestly anti-symmetric and G-invariant. It is also closed and nondegenerate [210], and hence it defines a symplectic structure on $\mathcal{O}_T$. Thus, coadjoint orbits are naturally symplectic manifolds. For matrix groups, the algebra and dual space are isomorphic through the Cartan-Killing form, which is non-degenerate in this case. It suffices to consider an orbit of the adjoint action, and these generate symplectic manifolds. This is the setting of Appendix C.1. We emphasize that in the general case this is not true and one must work in the dual space.

It will be useful to review the case of the Virasoro group, along with a suitable generalization given by the algebra described in Sections 5.3.2 and 5.3.3 that applies to our case of interest. Recall that the Virasoro group consists of $\mathrm{Diff}(\mathrm{S}^1)$ together with its central extension, $\widehat{\mathrm{Diff}(\mathrm{S}^1)} = \mathrm{Diff}(\mathrm{S}^1) \times \mathbb{R}$. For our problem, we are considering a continuous version of the ordinary Virasoro algebra, with a central extension described in Section 5.3.3. In either case, the formulae will be the same, with the difference that in the second scenario the vector fields ξ should be understood to be non-differentiable at the interval endpoints, with vanishing support outside the interval. Thus, in the latter case all integrals should be understood to cover only the range of the interval rather than the full circle.

For either algebra we consider elements $\xi(z)\partial_z - i\alpha c$ where $\xi(z)\partial_z$ is a vector field on the circle (smooth for Virasoro, and of the form (5.3.18) for its generalization) and $\alpha \in \mathbb{R}$ is a parameter for the central extension, generated by the algebra element c. The only non-trivial commutators are

$$[\xi_1(z)\partial_z, \xi_2(z)\partial_z] = -(\xi_1\xi_2' - \xi_1'\xi_2)\partial_z + \frac{ic}{48\pi}\oint dz\,(\xi_1\xi_2''' - \xi_1'''\xi_2)\ . \tag{5.4.5}$$

In the Virasoro case, using $L_n = z^{n+1}\partial_z$ the bracket (5.3.56) indeed leads to the usual form of the Virasoro algebra, (5.3.11).

For both algebras we can define a pairing between an adjoint vector (ξ, α) and a coadjoint vector (T, β) given by

$$\langle (T,\beta), (\xi,\alpha)\rangle = -\left[\oint dz\, T(z)\xi(z) + \alpha\beta\right]. \tag{5.4.6}$$

Now consider algebra elements $X_{\xi_1} = (\xi_1, \alpha_1)$ and $X_{\xi_2} = (\xi_2, \alpha_2)$, and let x_{ξ_1}, x_{ξ_2} be the corresponding dual elements. The Kirillov-Kostant symplectic form through dual element (T, β) is

$$\begin{aligned}\omega(x_{\xi_1}, x_{\xi_2}) &= \langle (T,\beta), [X_{\xi_1}, X_{\xi_2}]\rangle \\ &= \oint dz\left[T(\xi_1\xi_2' - \xi_1'\xi_2) + \frac{\beta}{48\pi}(\xi_1\xi_2''' - \xi_1'''\xi_2)\right]\ .\end{aligned} \tag{5.4.7}$$

Focusing now on the case of our non-smooth generalization of the Virasoro algebra, we can define the coadjoint orbit $\mathcal{O}_{T_*}$ through the unorthodox element $T_* = (P_0, c)$ of the dual space defined by the projection operator, (5.3.53), together with its central extension c in the full algebra. Again considering elements x_{ξ_1}, x_{ξ_2} in the dual space that correspond to algebra elements X_{ξ_1}, X_{ξ_2} through the pairing, and using (5.3.35), this becomes

$$\begin{aligned}\omega(x_{\xi_1}, x_{\xi_2}) &= \langle T_*, [X_{\xi_1}, X_{\xi_2}]\rangle \\ &= P_0([X_{\xi_1}, X_{\xi_2}]) + \frac{c}{48\pi}\oint dz\,[(\xi_1\xi_2''' - \xi_1'''\xi_2)]\ .\end{aligned} \tag{5.4.8}$$

This is precisely (5.3.61) for the curvature. Thus, the modular Berry curvature for state-changing parallel transport is now related to the symplectic form on this orbit.

What is the holographic bulk interpretation of such a non-standard orbit? We will argue that the corresponding geometry is related to the backreaction of a cosmic brane.

5.5 Bulk phase space interpretation

A Berry curvature for pure states constructed from Euclidean path integrals was shown to be equal to the integral of the bulk symplectic form over a Cauchy slice extending into the bulk in [55, 213] (see also [214]). The notion of Uhlmann holonomy is one particular generalization of Berry phases to mixed states, and it was argued in [190] that its holographic dual is the integral of the bulk symplectic form over the entanglement wedge. However, the arguments for arriving at this result for Uhlmann holonomy are purely formal, and to the best of our knowledge this identification has not been worked out in an explicit example. The derivation also lacks a precise definition for the entanglement wedge symplectic form, which we will provide.

In this section, we will comment on a possible bulk interpretation of the modular Berry curvature for state-changing parallel transport. We will see that the result for the curvature that we obtained in the previous sections is closely related to an integral of a bulk symplectic form on a geometry with a conical singularity. See [215–219] for a related discussion of this geometry.

5.5.1 The conical singularity geometry

We consider a Euclidean geometry obtained through the backreaction of a codimension-2 brane homologous to the boundary interval A. This leads to a family of Euclidean bulk solutions, which we denote by $\mathcal{M}_n$, where n is a function of the tension of the brane [216]:

$$\mathcal{T}_n = \frac{n-1}{4nG} \ . \tag{5.5.1}$$

In the limit $n \to 1$, the cosmic brane becomes tensionless and settles on the location of the the usual RT surface associated to the entangling region, but for non-zero tension the brane backreacts on the geometry. The resulting geometries $\mathcal{M}_n$ are used in the context of the holographic computation of Rényi entropies S_n in the boundary CFT, and we will argue that these are also relevant for a holographic interpretation of the modular Berry curvature.

Let us first examine the boundary dual of the backreaction process. Inserting a

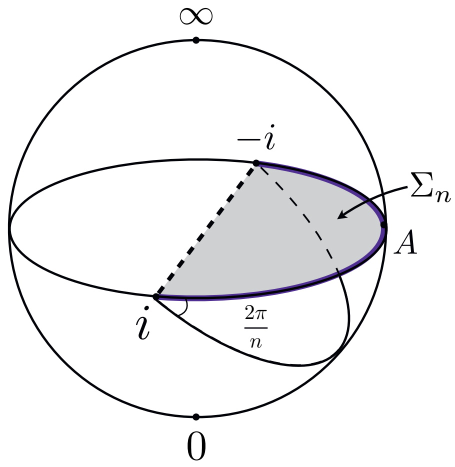


Figure 5.2: *The conical singularity geometry $\mathcal{M}_n$ and entanglement wedge region Σ_n corresponding to the boundary region A. The thick striped line corresponds to the cosmic brane extending from $-i$ to i. The backreaction process creates a conical singularity of opening angle $2\pi/n$.*

cosmic brane which anchors the boundary at z_1 and z_2 corresponds to the insertion of twist fields $\mathcal{O}_n$ in the CFT at z_1 and z_2 [218]. The field $\mathcal{O}_n(z)$ is a (spinless) conformal primary of dimension [220]

$$\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right) \ . \tag{5.5.2}$$

We use the fact that the cosmic brane can be computed as a correlation function of $\mathbb{Z}_n$ twist operators $\mathcal{O}_n, \mathcal{O}_{-n}$ in the boundary theory [216, 218].

Geometrically, we can think about the twist field as creating a conical singularity at the insertion point. Let us denote the two-dimensional geometry obtained from $\mathcal{O}_n(z_1), \mathcal{O}_{-n}(z_2)$ by $\mathcal{B}_n$. We are interested in the stress tensor profile on the boundary of the backreacted geometry, which by this reasoning is given by the stress tensor on the plane in the background of two twist fields:

$$\langle T(z)\rangle_{\mathcal{B}_n} = \frac{\langle T(z)\mathcal{O}_n(z_1)\mathcal{O}_{-n}(z_2)\rangle_{\mathbb{C}}}{\langle \mathcal{O}_n(z_1)\mathcal{O}_{-n}(z_2)\rangle_{\mathbb{C}}} \ . \tag{5.5.3}$$

Using the general form of the three-point function in a CFT in terms of conformal dimensions, it now follows that $T(z)$ has poles of order two at z_1 and z_2 respectively.

To describe the geometry $\mathcal{M}_n$ explicitly, we consider the complex plane with coordinate z which is flat everywhere except for two conical singularities at $z = z_1$ and $z = z_2$. The singular points are assumed to have a conical deficit of magnitude

$$\Delta\varphi = 2\pi\left(1 - \frac{1}{n}\right). \tag{5.5.4}$$

We can use a uniformizing function $f(z)$ to map the z-plane with conical singularities to the smooth covering space, which we denote by $\widetilde{\mathcal{B}}_n$, which is a complex plane with coordinate z' defined by

$$z' = f(z) = \left(\frac{z - z_1}{z - z_2}\right)^{\frac{1}{n}}. \tag{5.5.5}$$

This maps $z_1 \to 0$ and $z_2 \to \infty$ so that the interval between z_1 and z_2 goes to the positive real axis $[0, \infty)$. The power of $\frac{1}{n}$ removes the conical singularity by gluing the n sheets of the z-plane together, each represented by a wedge of opening angle $\frac{2\pi}{n}$.

In terms of the coordinate z' we extend $\widetilde{\mathcal{B}}_n$ into the bulk by introducing a 'radial' coordinate w' with metric of the form

$$ds^2 = \frac{dw'^2 + dz'd\bar{z}'}{w'^2}. \tag{5.5.6}$$

Here, we restrict the range of z' by the identification $z' \sim e^{2\pi i/n}z'$, as this represents a fundamental domain $\widetilde{\mathcal{B}}_n/\mathbb{Z}_n$ in the covering space. The bulk coordinate approaches the boundary in the limit $w' \to 0$. The metric in (5.5.6) is a wedge of three-dimensional hyperbolic space $\mathbb{H}^3$. We now use the following transformation:

$$w' = w\frac{1}{N}\sqrt{f'(z)\bar{f}'(\bar{z})}, \quad z' = f(z) - w^2\frac{1}{N}\frac{f'(z)\bar{f}''(\bar{z})}{2\bar{f}'(\bar{z})}, \tag{5.5.7}$$

where $f(z)$ is defined in (5.5.5) and

$$N = 1 + w^2\frac{f''(z)\bar{f}''(\bar{z})}{4f'(z)\bar{f}'(\bar{z})}. \tag{5.5.8}$$

This transformation reduces to the conformal transformation in (5.5.5) when we go to the boundary $w \to 0$. The metric in the new coordinates reads

$$ds^2 = \frac{dw^2}{w^2} + \frac{1}{w^2}\left(dz - w^2\frac{6}{c}\bar{T}(\bar{z})d\bar{z}\right)\left(d\bar{z} - w^2\frac{6}{c}T(z)dz\right), \tag{5.5.9}$$

where

$$T(z) = \frac{c}{12}\{f(z), z\} = \frac{c}{24}\left(1 - \frac{1}{n^2}\right)\frac{(z_1 - z_2)^2}{(z - z_1)^2(z - z_2)^2}\ , \tag{5.5.10}$$

with a similar expression holding for the anti-holomorphic component of the stress tensor $\bar{T}(\bar{z})$. The metric (5.5.9) falls into the class of Bañados geometries [221], and $T(z)$ has the interpretation of the expectation value of the stress tensor in the boundary CFT on $\mathcal{B}_n$. Therefore, (5.5.10) agrees with the expression, (5.5.3), in terms of twist fields. The formula for $T(z)$ can also be seen more directly from the way the stress tensor in a CFT transforms under a conformal transformation. Starting from the vacuum stress tensor in the z'-coordinate, $T(z') = 0$, and applying (5.5.5), the transformation picks up precisely the Schwarzian contribution in Eq. (5.5.10).

We can also give a description for these geometries in the language of Chern-Simons (CS) theory. It is known that Euclidean AdS_3 can be described by two copies of a Chern-Simons theory with gauge connections $A, \bar{A}$ valued in $\mathfrak{sl}(2,\mathbb{C})$, and where the Chern-Simons coupling is related to Newton's constant by $k = (4G_3)^{-1}$ [222]. We can expand these connections (with complex coefficients) over $\mathfrak{sl}(2,\mathbb{R})$ generators $L_0, L_\pm$ satisfying $[L_0, L_\pm] = \mp L_\pm$, $[L_+, L_-] = 2L_0$. In an explicit two-dimensional representation of the algebra, these are

$$L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,\quad L_+ = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\ ,\quad L_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ . \tag{5.5.11}$$

We can then describe the geometries, (5.5.9), using the connections

$$A = \frac{1}{2w}\begin{pmatrix} dw & -2\,dz \\ w^2\,\frac{12}{c}T(z)\,dz & -dw \end{pmatrix},\quad \bar{A} = -\frac{1}{2w}\begin{pmatrix} dw & w^2\,\frac{12}{c}\bar{T}(\bar{z})\,d\bar{z} \\ -2\,d\bar{z} & -dw \end{pmatrix}\ . \tag{5.5.12}$$

Each metric in this family of solutions corresponds to a choice of gauge connections, (5.5.12), with the same $T(z), \bar{T}(\bar{z})$ through the relation $ds^2 = \frac{1}{2}\mathrm{tr}((A - \bar{A})^2)$.

It will be useful to extract the radial dependence in (5.5.12) by using a suitable gauge transformation

$$A = bab^{-1} + bdb^{-1}\ , \bar{A} = b^{-1}\bar{a}b + b^{-1}db\ , \tag{5.5.13}$$

with gauge parameters

$$a = \begin{pmatrix} 0 & -dz \\ \frac{6}{c}T(z)\,dz & 0 \end{pmatrix}\ ,\ \bar{a} = \begin{pmatrix} 0 & -\frac{6}{c}\bar{T}(\bar{z})\,d\bar{z} \\ d\bar{z} & 0 \end{pmatrix}\ ,\ b = \begin{pmatrix} \frac{1}{\sqrt{w}} & 0 \\ 0 & \sqrt{w} \end{pmatrix}\ . \tag{5.5.14}$$

5.5.2 Symplectic form

We now turn our attention to the bulk symplectic form. It is useful to work in the Chern-Simons formulation of three-dimensional gravity. For a similar discussion of the symplectic structure of 3d gravity in this setting, especially as pertains to the connection to coadjoint orbits, see [223–226].

The CS action with CS coupling k and gauge connection A is given by

$$S_{\mathrm{CS}} = \int \mathcal{L}_{\mathrm{CS}} = \frac{k}{4\pi}\int \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \; . \tag{5.5.15}$$

We would like to evaluate the symplectic form. Taking the variation of the action for a single copy gives

$$\delta \mathcal{L}_{\mathrm{CS}} = \frac{k}{2\pi} \mathrm{tr}\left(\delta A \wedge F\right) + d\Theta \tag{5.5.16}$$

in terms of field strength $F = dA + A \wedge A$, and where $\Theta = \frac{k}{4\pi}\mathrm{tr}(A \wedge \delta A)$. The symplectic form for CS theory on some spatial region Σ is then given by

$$\omega = \int_\Sigma \delta\Theta = \frac{k}{4\pi}\int_\Sigma \mathrm{tr}(\delta_1 A \wedge \delta_2 A) \; . \tag{5.5.17}$$

In the following, we will assume that Σ is topologically a disk, i.e., it has a single boundary but no singularities in the interior. The symplectic form is a two-form on the space of classical solutions satisfying $F = 0$. Because we are working with a disk which admits no nontrivial cycles, a variation δA which leaves this condition invariant is of the form

$$\delta A = d_A \zeta \equiv d\zeta + [A, \zeta] \tag{5.5.18}$$

for some gauge transformation ζ, as follows from $\delta F = d_A \delta A = d_A^2 \zeta = 0$.

We now consider the symplectic form for such a transformation. Using the identity

$$\mathrm{tr}\left([A,\zeta] \wedge \delta A\right) = -\mathrm{tr}\left(\zeta \wedge [A, \delta A]\right) \tag{5.5.19}$$

and integrating by parts we obtain

$$\begin{aligned}\omega &= \frac{k}{4\pi}\int_\Sigma \mathrm{tr}(d_A\zeta \wedge \delta A) = \frac{k}{4\pi}\oint_{\partial\Sigma} \mathrm{tr}(\zeta \wedge \delta A) - \frac{k}{4\pi}\int_\Sigma \mathrm{tr}(\zeta \wedge d_A \delta A) \\ &= \frac{k}{4\pi}\oint_{\partial\Sigma} \mathrm{tr}(\zeta \wedge \delta A) \; . \end{aligned} \tag{5.5.20}$$

From (5.5.20) we see that the symplectic form ω is localized at the boundary of Σ.

Suppose that $\partial\Sigma$ lies in the asymptotic boundary of the geometry, in the $w = 0$ plane, and that we have gauged away the radial dependence. Using the explicit form of the connections, Eqs. (5.5.12) and (5.5.14), we can evaluate the symplectic form in (5.5.20). We see that the field variation can be expressed in terms of the stress tensor as

$$\delta A = \frac{6}{c}\begin{pmatrix} 0 & 0 \\ \delta T & 0 \end{pmatrix} dz \ . \tag{5.5.21}$$

It is also possible to solve (5.5.18) for δT. Decomposing ζ over the $\mathfrak{sl}(2,\mathbb{R})$ generators as $\zeta = \zeta_- L_{-1} + \zeta_0 L_0 + \zeta_+ L_1$ and using the form of the gauge field in (5.5.14), one can compute $d_A\zeta$. Matching with (5.5.21) gives a solution of the form

$$\delta T = \frac{c}{12}\xi''' + 2T\xi' + \partial T\xi \ , \tag{5.5.22}$$

where we have written $\xi \equiv -\zeta_-$ for the component of the gauge transformation associated to the L_{-1} generator. This is the usual stress tensor transformation law. From the form of the gauge transformation and the variation δA in (5.5.18), and using the Brown-Henneaux relation $(4G_3)^{-1} = c/6$ combined with the gravitational value for the CS coupling, we find that

$$\omega = \frac{1}{4\pi}\oint_{\partial\Sigma} dz\, \xi \wedge \delta T \ . \tag{5.5.23}$$

Using (5.5.22) the symplectic form becomes

$$\omega = \frac{1}{4\pi}\oint_{\partial\Sigma} dz\left(\frac{c}{12}\xi \wedge \xi''' + 2T\,\xi \wedge \xi'\right). \tag{5.5.24}$$

Plugging in two diffeomorphisms ξ_1 and ξ_2, the final result for the symplectic form reads:

$$\omega = \frac{1}{2\pi}\oint_{\partial\Sigma} dz\ \left(T\left(\xi_1\xi_2' - \xi_2\xi_1'\right) + \frac{c}{24}\left(\xi_1\xi_2''' - \xi_2\xi_1'''\right)\right) \ . \tag{5.5.25}$$

When the stress tensor $T(z) = T$ is a constant, (5.5.25) is reminiscent of the Kirillov-Kostant symplectic form on the coadjoint orbit

$$\mathcal{O} = \mathrm{Diff}(S^1)/U(1) \ , \tag{5.5.26}$$

(or $\mathcal{O} = \mathrm{Diff}(S^1)/SL(2,\mathbb{R})$ for the vacuum stress tensor) of the Virasoro group $\widehat{\mathrm{Diff}(S^1)}$ with central charge c. However to match onto the Berry curvature,(5.3.3), with the zero mode projection (5.3.35), we must consider a non-constant vacuum stress tensor. In fact the the stress tensor profile that reproduces the correct projection is of the form (5.5.10). In other words, the zero mode projection for the parallel transport process is implemented by integrating against the stress-tensor expectation value in the presence of two twist fields. We will now argue more precisely that in order to match the modular Berry curvature we need to consider a non-standard orbit corresponding to the conical singularity geometry described in Section 5.5.1.

5.5.3 Contour prescription

Let us return to the Euclidean geometry $\mathcal{M}_n$, which is obtained from the backreaction of a cosmic brane with tension $\mathcal{T}_n$. We showed that the stress tensor profile at the boundary is given by (5.5.10). Let us now restrict to transformations which leave the interval at the boundary fixed. This corresponds to Dirichlet boundary conditions $\delta A = 0$ at the cosmic brane.

We consider the symplectic form

$$\omega_n = \frac{k}{4\pi} \int_{\Sigma_n} \mathrm{tr}(\delta_1 A \wedge \delta_2 A) \ , \tag{5.5.27}$$

supported on some region Σ_n which corresponds to the entanglement wedge in the geometry $\mathcal{M}_n$, see Figure 5.2. The subscript in the symplectic form indicates that it depends on n. The entanglement wedge has two boundary components:

$$\partial\Sigma_n = \gamma_n \cup \mathrm{Brane}_n \ , \tag{5.5.28}$$

where γ_n is the entangling region at the asymptotic boundary extending between z_1 and z_2 and Brane_n is the cosmic brane anchored at those points. In Section 5.5.2, we have seen that the bulk symplectic form localizes to the boundary of Σ_n (using that the region is topologically trivial), because $\mathrm{tr}(\delta_1 A \wedge \delta_2 A) = d\eta$ is an exact form with $\eta = \mathrm{tr}(\xi \wedge \delta A)$. The expression for ω_n therefore reduces to a boundary term of the form

$$\omega_n = \frac{k}{4\pi} \left[\int_{\gamma_n} \eta + \int_{\mathrm{Brane}_n} \eta \right] \ . \tag{5.5.29}$$

The contribution at the cosmic brane vanishes due to the boundary conditions we put on the field variations there, i.e., $\delta A = 0$ at Brane_n. We are therefore left with the integral over the entangling region γ_n at the asymptotic boundary. There, η takes the form

$$k\,\eta = \xi \wedge \delta T = \frac{c}{12} \left(\xi_1 \xi_2''' - \xi_2 \xi_1''' \right) + 2\,T[\xi_1, \xi_2] \ , \tag{5.5.30}$$

in terms of the boundary stress tensor profile T of the geometry $\mathcal{M}_n$. Plugging in

(5.5.10) with $z_1 = i$ and $z_2 = -i$, we find that

$$\omega_n = \frac{c}{12\pi}\left(1 - \frac{1}{n^2}\right)\int_{\gamma_n} \frac{[\xi_1, \xi_2]}{(z^2+1)^2}\, dz + \frac{c}{48\pi}\int_{\gamma_n} (\xi_1\xi_2''' - \xi_2\xi_1''')\, dz \ . \qquad (5.5.31)$$

Note that the integrand is singular at the endpoints of the integration region γ_n. Therefore, we should implement some kind of regularization procedure for the integral to avoid the twist field insertion points. A standard choice would be the *principal value* prescription, where we excise a small ball of size ϵ around each of the singularities located at the endpoints of γ_n. After computing the integral, we take $\epsilon \to 0$. The resulting expression for ω_n is UV divergent ($\omega_n \sim \log \epsilon$).

In the limit $n \to 1$ the first term in (5.5.31) vanishes. This is expected, since as the cosmic branes becomes tensionless the geometry reduces to pure AdS$_3$, for which the bulk symplectic form is identically zero (up to the central charge term). To extract a non-zero answer from ω_n, we first take a derivative with respect to n and define

$$\omega \equiv \lim_{n\to 1} \frac{\partial}{\partial n}\frac{\omega_n}{k} \ . \qquad (5.5.32)$$

This corresponds to studying the first order correction of the backreaction process. The appearance of the operator $\lim_{n\to 1}\partial_n$ is not unfamiliar in the context of computing entanglement entropy using Euclidean solutions with conical singularties of the form $\mathcal{M}_n$[6]. (5.5.32) is our proposal for the bulk symplectic form associated to the entanglement wedge, and we will now show that it matches the modular Berry curvature.

To make the connection with the boundary computation, we rewrite the integral over the entangling region in terms of the variable u defined in (5.3.26). Notice that the unit semicircle $-\pi/2 \le \arg(z) \le \pi/2$ is mapped to the line $u \in [-\infty, \infty]$, since $z = 1$ goes to $u = 0$. In particular, the points $u = \pm\Lambda$ correspond to

$$z = \frac{1 + ie^{\pm\Lambda}}{e^{\pm\Lambda} + i} \sim e^{\pm i\left(\frac{\pi}{2} - \epsilon\right)} \ , \qquad (5.5.33)$$

if we identify Λ with the UV regulator by $\Lambda = -\log\frac{\epsilon}{2}$, in the limit $\Lambda \to \infty, \epsilon \to 0$. In the limit $\Lambda \to \infty$, the endpoints go to $z \to \pm i$ along the unit circle, so (5.5.33) is precisely the principal value prescription for γ_n.

Moreover, under the transformation in (5.3.26) the integration measure changes as (5.3.28). Therefore, we can represent the integral over the entangling region γ_n

[6]In fact, the entanglement entropy S associated to the subregion A can be computed by the formula $S = -\lim_{n\to 1}\partial_n \log Z_n$, where $\log Z_n \sim -I[\mathcal{M}_n]$ is the classical action evaluated on the conical singularity geometry $\mathcal{M}_n$.

in terms of the u-variable as

$$\frac{1}{\pi}\int_{-i}^{i}\frac{\xi(z)}{(1+z^2)^2}\,dz=\lim_{\Lambda\to\infty}\frac{i}{2\pi}\int_{-\Lambda}^{\Lambda}\xi(u)\,du\ , \tag{5.5.34}$$

which is precisely the projection operator $P_0(X_\xi)$ in (5.3.34). Thus, we can rewrite the symplectic form ω_n as

$$\omega_n=\frac{c}{12}\left(1-\frac{1}{n^2}\right)P_0([X_{\xi_1},X_{\xi_2}])+\frac{c}{48\pi}\int_{-i}^{i}(\xi_1\xi_2'''-\xi_2\xi_1''')\,dz\ . \tag{5.5.35}$$

Taking the derivative with respect to n and setting $n\to 1$ according to (5.5.32) gives the final result:

$$\omega=P_0([X_{\xi_1},X_{\xi_2}])\ , \tag{5.5.36}$$

which agrees with the curvature F in (5.3.3). Notice that the information about the central zero mode discussed in Section 5.3.3 is also contained in ω_n: it simply corresponds to taking $\lim_{n\to1}\omega_n$ directly.

5.6 Discussion

We have considered the case of boundary parallel transport of a fixed interval under a change in global state, which is in contrast to the situation considered in [54] where the state is held fixed while the interval location is varied. However, a general parallel transport process will change both the state and the location of the interval. In such a situation, the curvature will contain cross-terms between the X_λ's of (5.3.34) and the V_μ's of Section 5.2.2. Both are eigenoperators of the adjoint action of the modular Hamiltonian, $[H_{\text{mod}},X_\lambda]=\lambda X_\lambda$ and $[H_{\text{mod}},V_\mu]=i\mu V_\mu$, but notice that the eigenvalue of the X_λ's is real while that of V_μ is purely imaginary. By the Jacobi identity, the commutator $[X_\lambda,V_\mu]$ will have an eigenvalue that is the sum of the two, thus it has both a real and imaginary part. This is never zero, which means $[X_\lambda,V_\mu]$ does not have a zero mode. The curvature, (5.3.3), is given by the projection onto this zero mode, which means that computed in these directions that mix changes of state and interval location, it must vanish. Thus, it appears to be sufficient to consider state and interval location-based transport separately.

In the bulk, we have demonstrated an abstract connection between state-changing parallel transport of boundary intervals and a certain family of Euclidean bulk solutions. The holographic dual of the modular Berry curvature was argued to be an entanglement wedge symplectic form on this geometry. This is similar in spirit to the results of [55,213], but in the case of mixed states. However, a direct

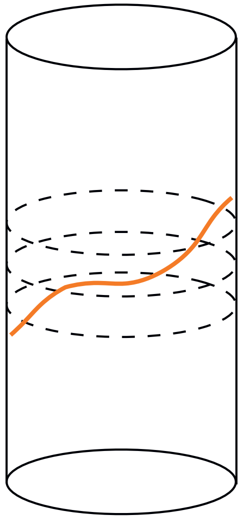


Figure 5.3: *An example of a time-dependent geometry limiting to different boundary states $|\psi_i\rangle$ at each time. Could the Berry phase associated to state-dependent parallel transport compute the length of a curve (such as the thick orange curve) in such a geometry?*

phase space interpretation of this symplectic form in Lorentzian signature is not so obvious. Associating a phase space, i.e., a solution space of a proper initial value problem, to an entanglement wedge involves some subtleties, e.g., the possibility of edge modes [227–229] and boundary ambiguities at the RT surface that must be fixed by a suitable choice of boundary conditions. Possibly, one could exploit the relation to the hyperbolic black hole and identify the relevant phase space with the one associated to the (outside of the) black hole. This would lead to geometric setup for which the Lorentzian continuation is more well-behaved. In particular, this approach requires a further study of the choice of boundary conditions that are natural to put at the horizon.

It would also be interesting to explore a bulk description within a single Lorentzian geometry. For instance, one could imagine constructing a time-dependent geometry by gluing together certain slowly varying time-independent geometries that are each dual to different boundary states. Since this will not in general give an on-shell solution, one could try to turn on suitable sources on the boundary as a function of time, in such a way that time evolution under the modified Hamiltonian (with sources) provides precisely the sequence of states under consideration. In such a situation, one could look for a corresponding on-shell bulk solution with modified asymptotics. It would be interesting to explore whether the Berry phase associated to state-changing parallel transport computes a length within a time-dependent geometry (see Figure 5.3).

Additionally, it would be interesting to explore further the connections to Uhlmann holonomy described in [190]. This is a version of parallel transport constructed from purification of density matrices subject to certain maximization conditions on transition probabilities. Through appropriate insertion of stress tensors at the boundary, this is claimed in [230–233] to describe the shape-changing transport problem considered in Section 5.2.2. In this setting, the Berry curvature associated to a parallel transport process that changes the state was argued to be dual to the symplectic form of the entanglement wedge. While similar in spirit to much of this work, it would be interesting to further study the relation to our work in the context of key differences, such as the need for diagonalizing the adjoint action and the use of non-smooth vector fields.

The problem we study also has relevance for thermalization in 2d CFT. For example, the Krylov complexity contains information about operator growth in quantum chaotic systems. Roughly speaking, this is given by counting the operators that result under nested commutators with respect to a 'Hamiltonian' of the system. In [234], the Krylov complexity was studied for the case where this Hamiltonian takes the form of (5.3.12), using an oscillator representation of the Virasoro algebra. This is similar to the modular Berry transport process we have considered, with the exception again of the use of non-smooth vector fields.

In studying operator-based parallel transport, we uncovered some subtleties regarding the diagonalization of the adjoint action for arbitrary Virasoro generators (an explanation of these issues was given in Appendix C.3). For this reason we considered a set of certain non-smooth vector fields on the circle, (5.3.18), which explicitly diagonalize the adjoint action so that the curvature results of Appendix C.2 may be applied. It would be interesting to further study this issue. For instance, we found that the adjoint action could not be diagonalized over the usual Virasoro algebra, defined as the set of smooth vector fields on the circle.[7] Instead, we saw that the set of generators not expressible as $[H_{\text{mod}}, X]$ was dimension three, larger than the dimension of the kernel (which is in this case one-dimensional and generated by H_{mod}). Furthermore, there was an ambiguity in the non-zero mode piece. One could ask whether it is possible to consider parallel transport generated by elements of the usual Virasoro algebra, and perhaps resolve the ambiguities in the decomposition by taking a suitable choice of norm. Along these lines, one could consider only Virasoro algebra elements that are contained within physical

[7] This is actually not uncommon in the case of infinite-dimensional vector spaces. For example, when one tries to diagonalize the derivative operator on the space of polynomial functions one naturally finds exponential functions, which are not part of the original space. The non-analyticities we found should be regarded in the same way.

correlators. It would be interesting to apply techniques from algebraic quantum field theory to see if this eliminates some of the ambiguities we have encountered.

To properly diagonalize the adjoint action we were led to consider vector fields on the circle that are non-differentiable on the endpoints of the interval. These form a continuous version of the Virasoro algebra. Our Berry curvature can be understood formally as the Kirillov-Kostant symplectic form on an orbit associated to this algebra. It would be interesting to conduct a more rigorous study of this algebra and its central extension. It is also worth noting that we considered a dual space of distributions on the circle, which is larger than the set of smooth quadratic differentials considered in the classification of [210]. For this reason, the orbits we consider differ considerably from known Virasoro orbits since the associated representative, (5.3.54), is not a quadratic form on the circle. To our knowledge, such orbits have not been studied before in the literature. We have identified at least one physical implication of such unconventional orbits, and thus it would be interesting to revisit the classification of Virasoro orbits using more general duals.

6 Modular Berry phases and the bulk symplectic form

6.1 Introduction

The growing interface between quantum information theory and gravity has shed new light on many aspects of quantum gravity; for recent reviews see [235–239]. Within the realm of holography, it has been fruitful to search for bulk duals of quantum information theoretic concepts on the boundary, so as to add new entries to the AdS/CFT dictionary. Contrasted with earlier results in AdS/CFT, the quantum information theory-based part of the dictionary often has a more direct connection to bulk geometry.

In this chapter, we continue the approach of deriving new AdS/CFT dictionary entries from quantum information theoretic quantities on the boundary side. We will investigate a particular new quantum information theoretic boundary quantity, which adapts the Berry parallel transport [240] to trajectories in the space of global states. Unlike Berry transport for pure states in quantum mechanics, this parallel transport transforms operators associated to a spatial subregion. This process has been dubbed *modular* Berry transport because it relies on entanglement properties of subregions, specifically on how the modular Hamiltonian is glued together across different choices of subregion.

Modular Berry transport has been studied in some detail for trajectories defined over kinematic space [184]—ones where boundary subregions vary in shape or location [53,54]. In this case there is a direct bulk geometric dual: The Berry phase reproduces lengths of bulk curves that can be reached by extremal surfaces, and the Berry curvature is related to a bulk curvature. A close cousin of the modular parallel transport generator was recently shown to act in three-dimensional bulk geometries as the generator of ordinary parallel transport, which is described by General Relativity [241].

Our setting here is different from those earlier works. We consider modular parallel transport along trajectories, which visit varying global *states* rather than varying locations or shapes of boundary subregions. This approach was initiated in a more restricted setting in the previous chapter, cf. [3]. There, we showed that the curvature associated to a particular state-changing modular Berry transport could be identified with an appropriately defined symplectic form associated to an entanglement wedge. In that case, state deformations were implemented through the action of a large diffeomorphism, whose form was dictated by the Virasoro symmetry of a CFT_2. (Berry phases on the Virasoro algebra were likewise considered in [192, 197, 204].) This setting further revealed a connection to an auxiliary symplectic geometry derived from the group theory of the Virasoro algebra: a coadjoint orbit. The triality between the Berry curvature, the entanglement wedge symplectic form, and the Kirillov-Kostant symplectic form on an appropriate orbit (see also [201] for a similar triality for bulk duals of complexity) revealed an interplay between group theory and quantum information in this case, giving an additional handle on an important bulk geometric quantity of interest.

Our aim in this chapter is to set up Berry transport for a broad class of state deformations in any dimension. Based on previous results in two dimensions, one might imagine this to be a straightforward task. However, the power of group theory to describe certain state-changing transformations in two dimensions also presents a limitation in its generalization. To generalize state-changing Berry transport to a larger class of state changes including state changes in higher dimensions, one must invoke a very different toolkit. In the present work we make use of the Euclidean path integral to implement state changes (analogously to [55, 213] in the case of pure states, or see [190] for a different version of parallel transport based on the Uhlmann phase[1]). We also use some new (from the perspective of modular Berry transport) techniques such as modular Fourier decompositions, the KMS condition from modular theory, as well as (from the bulk side) the equivalence between bulk and boundary modular flow and the modular extrapolate dictionary. These tools have been useful in proving the ANEC and the quantum null energy condition [231, 244] and in setting the stage for a modular approach to bulk reconstruction [244–246]. Intriguingly, though we employ very different techniques from group theory, coadjoint orbits and Chern-Simons theory as utilized in [3], the end result is similar: The expectation value of the Berry curvature in the global pure state is equal to the symplectic form associated to an entanglement wedge.

Using a similar framework, we can also extract from the full Berry curvature a symmetric quantity. We show that on the boundary, this describes a metric on the space of density matrices, often referred to as the quantum Fisher information

[1] Excellent summaries of the different types of transport in quantum mechanical state spaces, including Berry and Uhlmann transport, are given in [242] and [243]. Note, however, that those works assume that the Hilbert space is finite-dimensional. Infinite-dimensional Hilbert spaces can give rise to subtleties, see for example [3].

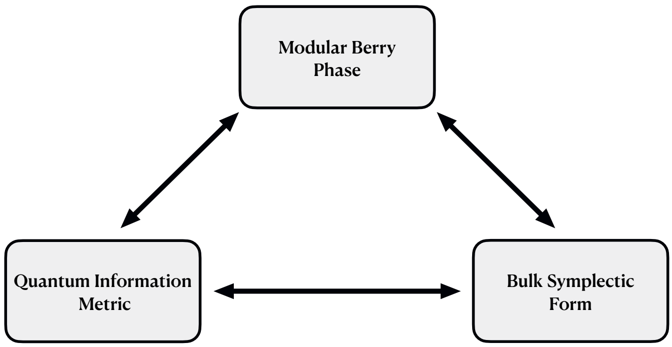


Figure 6.1: *Modular Berry transport provides a framework that encodes information about not only the bulk symplectic form, but also the quantum information metric.*

metric (this also goes by other names). In the bulk, we extract this from the bulk symplectic form by taking a Lie derivative with respect to the generator of modular flow. This describes the canonical energy, which has been used as a tool for deriving the bulk equations of motion from entanglement entropy [247–249]. In the end, we see that the modular Berry phase incorporates more information beyond simply the bulk symplectic form, as is represented in the triangle in Figure 6.1.

Along the way, we can make contact with Berry transport in the shape-changing case, now generalized to higher dimensions. We do so in two ways: first, by considering the specific case of state deformations sourced by the stress tensor, which incorporates shape changes. Next, we act with symmetry generators of the higher dimensional conformal algebra, in a direct generalization of the techniques of [3]. In doing so, we relate the Berry curvature for the higher-dimensional shape-changing case to the Kirillov-Kostant symplectic form on a coadjoint orbit. The full non-abelian Berry curvature lives on the coset space that is relevant for the higher dimensional version of kinematic space, the space of causal diamonds in a CFT [181–184, 250]. The connection to the Kirillov-Kostant symplectic form relies on the fact that in this case, unlike for general state transformations, the deformations which implement parallel transport lie in the symmetry algebra of the boundary.

Outline: We set the stage in Section 6.2 by reviewing modular Berry transport and state preparation using the Euclidean path integral. After introducing some of the language of modular flow and modular Fourier decomposition, we use these tools to derive the modular Berry curvature for general state deformations. We also introduce a symmetric derivative of the Berry curvature (see Appendix C.5 for the quantum information theoretic interpretation). Next, we extend these quantities into the bulk in Section 6.3 using the modular extrapolate dictionary. We show explicitly for operators sourcing bulk scalar fields that this computes the bulk symplectic form (see Section 6.4.1 for a generalization beyond the scalar case). The symmetric offshoot is related to the bulk canonical energy. Section 6.4 presents some explicit examples of the general formalism of the previous sections. Specifically, we consider in Section 6.4.1 the case of a stress tensor source, which in general implements a change of metric but also includes the shape-changing case. Finally, in Section 6.4.2 we explicitly consider the higher-dimensional shape-changing case by acting with symmetry generators, and elucidate the connection to coadjoint orbits. Our conventions for the conformal algebra are presented in Appendix C.4.

6.2 Berry curvature for coherent state deformations

First we consider a parallel transport problem purely defined on the boundary. We will apply the modular Berry formalism to deformations that change the global state on the boundary in arbitrary dimension. In Section 6.2.1, we review how to construct such state deformations using coherent states and the Euclidean path integral, and in Sections 6.2.2 and 6.2.3 we derive new results for the modular Berry curvature and quantum information metric for state deformations. Our results make convenient use of modular eigenstates and a modular Fourier basis.

6.2.1 Coherent state deformations

We would like to consider a modular Berry setup where the variation of η denotes a change of *state* rather than a change of shape or location of the subregion. For a CFT in two dimensions, one particular class of deformations δH_{mod} that implement state changes involve elements of the infinite-dimensional Virasoro symmetry algebra. (Due to certain subtleties, it is necessary to employ a continuous version of the Virasoro algebra, which is described by certain non-smooth vector fields on the circle [3].) For a CFT in $d > 2$ dimensions the story will necessarily be different, since such state-changing transformations no longer lie in the symmetry algebra $\mathfrak{so}(d, 2)$.

To generalize the state-changing Berry construction to accommodate setups in higher dimensions, it will be useful to introduce the language of Euclidean path

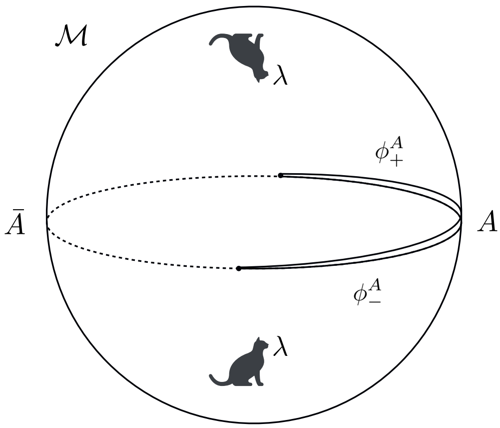


Figure 6.2: *The Euclidean manifold $\mathcal{M}$ that is integrated over to prepare the matrix elements $\langle\phi_+^A|\rho|\phi_-^A\rangle$ of the density matrix. The two hemispheres are glued along the complement region $\bar{A}$, while the boundary conditions at the region A are left open. One can prepare non-trivial coherent states by introducing non-trivial background sources λ (which are represented by cats in the figure).*

integrals.[2] Specifically, we assume that the state $|\Psi\rangle$ is a coherent state, in the sense that it is prepared by the Euclidean path integral with a background source λ. We consider deforming the state through the insertion of some operator $\mathcal{O}$ in the path integral:

$$\delta S = \int d^d x\, \delta\lambda(x)\mathcal{O}(x)\ . \tag{6.2.1}$$

The source $\delta\lambda(x)$ determines the strength of the perturbation. At this point, there is no need to restrict the support of the source, we take it to be anywhere in the Euclidean half-plane.

The perturbation (6.2.1) leads to a change of the density matrix, and hence of the modular Hamiltonian [231]. Denoting the collective field content of the theory by ϕ, one can compute matrix elements of the density matrix by gluing the upper and lower Euclidean half plane along the complement $\bar{A}$ at $t_E = 0$:

$$\langle\phi_+^A|\rho|\phi_-^A\rangle = \frac{1}{Z}\int_{\phi(0^-)=\phi_-^A}^{\phi(0^+)=\phi_+^A} [D\phi]\, e^{-S[\phi]}\ , \quad \text{where} \quad Z \equiv \int [D\phi]\, e^{-S[\phi]}\ . \tag{6.2.2}$$

[2] The Euclidean path integral is also useful for defining a CFT Berry transport process for *pure* states, without restricting to a subregion [55, 213]. One goal of our work is to explicitly adapt this to modular transport for mixed states. For a formal argument involving the mixed state case and a different variety of parallel transport, see [190].

Here, ϕ_+^A and ϕ_-^A denote the value of the field ϕ just above and below the subregion A respectively, and $S[\phi]$ is the Euclidean action of the theory with source λ. One therefore integrates over the full Euclidean manifold $\mathcal{M}$ with a branch cut at the location of the subregion (see Figure 6.2).

We now perturb the state according to (6.2.1). The new density matrix ρ' is given by

$$\langle\phi_+^A|\rho'|\phi_-^A\rangle = \frac{1}{(Z+\delta Z)}\int_{\phi(0^-)=\phi_-^A}^{\phi(0^+)=\phi_+^A}[D\phi]\, e^{-S[\phi]-\int d^d x\,\delta\lambda(x)\mathcal{O}(x)}\ . \tag{6.2.3}$$

Using the geometric series relation

$$\frac{1}{(Z+\delta Z)} = \frac{1}{Z}\left(1-\frac{\delta Z}{Z}+\dots\right)\ , \tag{6.2.4}$$

and expanding the exponential in (6.2.3), we find that the change $\delta\rho \equiv \rho'-\rho$ is given by

$$\langle\phi_+^A|\delta\rho|\phi_-^A\rangle = -\frac{1}{Z}\int_{\phi(0^-)=\phi_-^A}^{\phi(0^+)=\phi_+^A}[D\phi]\, e^{-S[\phi]}\int d^d x\,\delta\lambda(x) : \mathcal{O}(x) : +\dots\ , \tag{6.2.5}$$

where we have introduced the renormalized operator $:\mathcal{O}:\equiv \mathcal{O}-\langle\mathcal{O}\rangle$. From now on we will omit the notation $:\cdot:$, and assume that all operators are background-subtracted. Hence, up to first order in the source the density matrix changes as

$$\delta\rho = -\int d^d x\,\rho\,\delta\lambda(x)\mathcal{O}(x)\ . \tag{6.2.6}$$

Recall that the modular Hamiltonian H_{mod} is related to ρ by

$$\rho = e^{-H_{\text{mod}}}\ . \tag{6.2.7}$$

Using the integral representation of the logarithm,

$$H_{\text{mod}} = -\log\rho = \int_0^\infty d\beta\left(\frac{1}{\rho+\beta}-\frac{1}{1+\beta}\right)\ , \tag{6.2.8}$$

it follows from (6.2.6) that

$$\delta H_{\text{mod}} = \int d^d x\,\delta\lambda(x)\int_0^\infty d\beta\left(\frac{\rho}{\rho+\beta}\mathcal{O}(x)\frac{1}{\rho+\beta}\right)\ . \tag{6.2.9}$$

To proceed, it is useful to use a spectral representation for the density matrix ρ.

We consider modular frequency states $|\omega\rangle$, which are eigenstates of the modular Hamiltonian:[3]

$$H_{\text{mod}}|\omega\rangle = \omega|\omega\rangle \ . \tag{6.2.10}$$

When evaluated in this basis, the change in the modular Hamiltonian takes a relatively simple form. Inserting a resolution of the identity, one finds

$$\langle\omega|\delta H_{\text{mod}}|\omega'\rangle = \int d^d x\, \delta\lambda(x)\langle\omega|\mathcal{O}(x)|\omega'\rangle \int_0^\infty d\beta \left(\frac{e^{-\omega}}{e^{-\omega}+\beta}\frac{1}{e^{-\omega'}+\beta}\right) \ . \tag{6.2.11}$$

The integral over β can be performed easily. Indeed, we find that

$$\int_0^\infty d\beta \left(\frac{e^{-\omega}}{e^{-\omega}+\beta}\frac{1}{e^{-\omega'}+\beta}\right) = \frac{\omega-\omega'}{e^{\omega-\omega'}-1} \ . \tag{6.2.12}$$

Plugging this back into (6.2.11), it follows that

$$\langle\omega|\delta H_{\text{mod}}|\omega'\rangle = \int d^d x\, \delta\lambda(x) n(\omega-\omega')(\omega-\omega')\langle\omega|\mathcal{O}(x)|\omega'\rangle \ , \tag{6.2.13}$$

where here we have introduced the quantity

$$n(\omega) \equiv \frac{1}{e^{\omega}-1} \ , \tag{6.2.14}$$

which will be convenient later. This gives a relatively simple expression for the change in modular Hamiltonian in terms of the matrix elements of the operator $\mathcal{O}$.

Recall that the modular parallel transport problem relies on defining projection P_0 that sends an operator to its zero mode component. There is an ambiguity in how to define this projection. A natural choice is to take the diagonal matrix elements of the operator and multiply by the eigenstate $|\omega\rangle\langle\omega|$:[4]

$$P_0(\mathcal{O}) \equiv \int d\omega\, \langle\omega|\mathcal{O}|\omega\rangle|\omega\rangle\langle\omega| \ . \tag{6.2.15}$$

This procedure defines a diagonal operator, which commutes with the modular

[3] As the existence of such states is only guaranteed in type I von Neumann algebras, our analysis presumes that the more realistic settings of type II algebras (semi-classical gravity) and/or type III algebras (quantum field theory) do not alter the overall picture.

[4] Under general circumstances, the integral in (6.2.15) might involve a non-trivial density of states. The attendant degeneracies among states $|\omega\rangle$ generically arise from additional symmetries, which commute with H_{mod}. If so, one can extend H_{mod} to a complete set of commuting operators and declare ω to denote the corresponding complete set of quantum numbers. In this chapter we assume that any degeneracies in the spectrum have been accounted for in this fashion, and do not include explicit factors of the density of states.

Hamiltonian since $H_{\rm mod}$ is diagonal in its own eigenbasis. It is easy to check that the projection (6.2.15) satisfies

$$P_0(H_{\rm mod}) = H_{\rm mod} \ , \quad P_0([H_{\rm mod}, X]) = 0 \ . \tag{6.2.16}$$

In other words, it is indeed the case that the kernel and image of the adjoint action $[\cdot, H_{\rm mod}]$ with respect to the modular Hamiltonian are disjoint.

We can now use P_0 to define the parallel transport problem. We first subtract off the zero mode part of $\delta H_{\rm mod}$, which is given by the diagonal component of (6.2.13):

$$P_0(\delta H_{\rm mod}) = \int d^d x \, \delta\lambda(x) P_0(\mathcal{O}(x)) \ . \tag{6.2.17}$$

Using the fact that

$$\langle\omega|[X, H_{\rm mod}]|\omega'\rangle = (\omega' - \omega)\langle\omega|X|\omega'\rangle \ , \tag{6.2.18}$$

we recognize that the factor $\omega - \omega'$ in (6.2.13) comes from a commutator. Indeed, we can choose an X with matrix elements

$$\langle\omega|X|\omega'\rangle = -\int d^d x \, \delta\lambda(x) n(\omega - \omega')\langle\omega|\mathcal{O}(x)|\omega'\rangle \ . \tag{6.2.19}$$

We additionally assume that X is zero mode free, $P_0(X) = 0$, which also implies $P_0(\delta H_{\rm mod}) = 0$ by (6.2.17). Then, by comparing (6.2.18) and (6.2.19) with (6.2.13), we see that X satisfies the transport equation

$$\langle\omega|(\delta H_{\rm mod} - P_0(\delta H_{\rm mod}))|\omega'\rangle = \langle\omega|[X, H_{\rm mod}]|\omega'\rangle \ . \tag{6.2.20}$$

Since X is assumed to be zero mode free, we can identify it with the generator of parallel transport whose commutators compute the modular Berry curvature.

6.2.2 Berry curvature

Now that we have computed the matrix elements of $\delta H_{\rm mod}$ in the modular eigenstate basis and derived the generator X of parallel transport, we would like to compute from this the Berry curvature. Recall that given two infinitesimal deformations $\delta_1\lambda, \delta_2\lambda$ and corresponding zero-mode free parallel transport generators X_1, X_2, the Berry curvature is given by

$$F = P_0([X_1, X_2]) \ . \tag{6.2.21}$$

To further evaluate this expression it is useful to decompose the operator $\mathcal{O}$ in a 'modular Fourier basis,' where the action of the modular Hamiltonian is simple. Such a basis was previously used in the context of bulk reconstruction in [245]. Let us first consider the modular flow associated to the algebra $\mathcal{A}$ and state $|\Psi\rangle$, defined by the operation

$$\mathcal{O} \in \mathcal{A} \to \mathcal{O}_s = e^{iH_{\text{mod}}s}\mathcal{O}e^{-iH_{\text{mod}}s} \in \mathcal{A} \ . \tag{6.2.22}$$

One can use the modular flow to make a Fourier decomposition of the form

$$\mathcal{O}_\omega = \int_{-\infty}^{\infty} ds\, e^{-i\omega s}\mathcal{O}_s \ , \tag{6.2.23}$$

where the operators $\mathcal{O}_\omega$ are labeled by some modular frequency ω. We can now decompose an operator $\mathcal{O}$ in terms of the modular Fourier basis as

$$\mathcal{O} = \frac{1}{2\pi}\int d\omega\, \mathcal{O}_\omega \ . \tag{6.2.24}$$

Note that the operators $\mathcal{O}_\omega$ should always be viewed as being integrated against some suitable function of the frequency ω to get finite expectation values. Therefore, using the modular Fourier basis directly will introduce some intermediate δ-functions[5] in the computation, but the final answer for the curvature will be finite.

The action of modular flow (6.2.22) on $\mathcal{O}_\omega$ is particularly simple. By shifting the integration variable in (6.2.23) we find that

$$e^{iH_{\text{mod}}t}\mathcal{O}_\omega e^{-iH_{\text{mod}}t} = e^{i\omega t}\mathcal{O}_\omega \ . \tag{6.2.26}$$

Plugging this into the formula for the commutator

$$[H_{\text{mod}}, \mathcal{O}_\omega] = -i\frac{d}{dt}\Big|_{t=0} e^{iH_{\text{mod}}t}\mathcal{O}_\omega e^{-iH_{\text{mod}}t} \ , \tag{6.2.27}$$

gives the relation

$$[H_{\text{mod}}, \mathcal{O}_\omega] = \omega\mathcal{O}_\omega \ . \tag{6.2.28}$$

We conclude that the operators (6.2.23) constitute a formal spectral decomposition of the adjoint action of H_{mod}.

[5] Note that the Fourier zero mode $\mathcal{O}_0$ commutes with the modular Hamiltonian, but it is not the same as applying the zero mode projection $P_0(\mathcal{O})$. They differ by an infinite normalization factor coming from the extra δ-function:

$$\mathcal{O}_0 = P_0(\mathcal{O}_0) = 2\pi\delta(0)P_0(\mathcal{O}) \ , \tag{6.2.25}$$

which reflects the fact that $\mathcal{O}_0$ by itself is in some sense a singular operator.

The matrix elements of $\mathcal{O}_\omega$ in the modular frequency basis obey

$$\langle\omega'|\mathcal{O}_\omega|\omega''\rangle = \int_{-\infty}^{\infty} ds\, e^{i(\omega'-\omega-\omega'')s}\langle\omega'|\mathcal{O}|\omega''\rangle = 2\pi\delta(\omega'-\omega-\omega'')\langle\omega'|\mathcal{O}|\omega''\rangle \ , \tag{6.2.29}$$

so they are only non-zero when the frequencies satisfy the condition $\omega = \omega' - \omega''$. This can be used to our advantage. In particular, one can use (6.2.29) to show that

$$\int d\omega'' f(\omega'')\langle\omega'|\mathcal{O}_\omega|\omega''\rangle = \int d\omega'' f(\omega'-\omega'')\langle\omega'|\mathcal{O}_{\omega''}|\omega'-\omega\rangle \ , \tag{6.2.30}$$

and similarly

$$\int d\omega'' f(\omega'')\langle\omega''|\mathcal{O}_{-\omega}|\omega'\rangle = \int d\omega'' f(\omega'-\omega'')\langle\omega'-\omega|\mathcal{O}_{-\omega''}|\omega'\rangle \tag{6.2.31}$$

for any function $f = f(\omega)$.

This identity can be used to transform an integral over modular frequency states to an integral over modular frequency operators. Let us first decompose the operator X_1, X_2 into modular Fourier modes $X_{1,\omega_1}, X_{2,\omega_2}$, and compute the commutator

$$\begin{aligned}\langle\omega|[X_{1,\omega_1}, X_{2,\omega_2}]|\omega'\rangle &= \int d^dx \int d^dx'\, \delta_1\lambda(x)\delta_2\lambda(x') \\ &\times \int d\omega''\langle\omega|\mathcal{O}_{\omega_1}(x)|\omega''\rangle\langle\omega''|\mathcal{O}_{\omega_2}(x')|\omega'\rangle n(\omega-\omega'')n(\omega''-\omega') - (1\leftrightarrow 2) \ .\end{aligned} \tag{6.2.32}$$

We have inserted a complete basis of states and used the expression (6.2.19). We will now consider the diagonal part of (6.2.32). First note that from (6.2.29), $\langle\omega|\mathcal{O}_{\omega_1}(x)|\omega''\rangle\langle\omega''|\mathcal{O}_{\omega_2}(x')|\omega\rangle$ is proportional to $\delta(\omega-\omega_1-\omega'')\delta(\omega''-\omega_2-\omega)$. Thus, it is only non-zero when $\omega_1 = -\omega_2$. We are therefore allowed to multiply the equation with a term $\delta(\omega_1+\omega_2)\delta(0)^{-1}$. The extra insertion of $\delta(0)$ will cancel at the end of the computation, when we write the answer in terms of the original operators. Using the identity (6.2.30) we find that

$$\begin{aligned}\int d\omega'' n(\omega-\omega'')n(\omega''-\omega)\langle\omega|\mathcal{O}_{\omega_1}(x)|\omega''\rangle\langle\omega''|\mathcal{O}_{\omega_2}(x')|\omega\rangle &= \delta(\omega_1+\omega_2)\delta(0)^{-1} \\ \times \int d\omega'' n(-\omega'')n(\omega'')\langle\omega|\mathcal{O}_{\omega''}(x)|\omega-\omega_1\rangle\langle\omega-\omega_1|\mathcal{O}_{-\omega''}(x')|\omega\rangle \ .\end{aligned} \tag{6.2.33}$$

By integrating over the modular frequencies ω_1, ω_2 on both sides of the equality using (6.2.24), and then removing a resolution of the identity, one obtains

$$\int d\omega'' n(\omega-\omega'')n(\omega''-\omega)\langle\omega|\mathcal{O}(x)|\omega''\rangle\langle\omega''|\mathcal{O}(x')|\omega\rangle$$
$$= \mathcal{N}^{-1}\int d\omega'' n(-\omega'')n(\omega'')\langle\omega|\mathcal{O}_{\omega''}(x)\mathcal{O}_{-\omega''}(x')|\omega\rangle \ , \quad (6.2.34)$$

where $\mathcal{N} \equiv (2\pi)^2\delta(0)$. Putting this back into the expression for the commutator $[X_1, X_2]$, one finds

$$\langle\omega|[X_1, X_2]|\omega\rangle = \mathcal{N}^{-1}\int d^dx \int d^dx'\, \delta_1\lambda(x)\delta_2\lambda(x')$$
$$\times \int d\omega'' n(-\omega'')n(\omega'')\langle\omega|[\mathcal{O}_{\omega''}(x), \mathcal{O}_{-\omega''}(x')]|\omega\rangle \ . \quad (6.2.35)$$

Since the operator $[\mathcal{O}_{\omega''}(x), \mathcal{O}_{-\omega''}(x')]$ is diagonal already, the projection operator P_0 leaves it invariant. We conclude that the Berry curvature is given by

$$F = \mathcal{N}^{-1}\int d^dx \int d^dx'\, \delta_1\lambda(x)\delta_2\lambda(x')\int d\omega\, n(-\omega)n(\omega)[\mathcal{O}_\omega(x), \mathcal{O}_{-\omega}(x')] \ . \quad (6.2.36)$$

This formula is one of the main results of this section, and it provides a useful representation of the curvature associated to coherent state deformations of the form (6.2.1).

Note that this modular Berry curvature F is operator-valued, due to the fact that our transport problem is suited to density matrices, instead of pure states. In fact it is easy to verify that the curvature is a zero mode, i.e., $F \in \mathcal{A}_0$. By virtue of the Jacobi identity together with (6.2.28),

$$[H_{\mathrm{mod}}, [\mathcal{O}_\omega, \mathcal{O}_{-\omega}]] = [\mathcal{O}_\omega, [H_{\mathrm{mod}}, \mathcal{O}_{-\omega}]] - [\mathcal{O}_{-\omega}, [H_{\mathrm{mod}}, \mathcal{O}_\omega]] = 0 \ , \quad (6.2.37)$$

which shows that the curvature indeed satisfies $[H_{\mathrm{mod}}, F] = 0$. Moreover, the expression (6.2.36) is anti-symmetric under interchanging 1 with 2. This can be most easily seen by substituting ω with $-\omega$ in the integral: While the term $n(-\omega)n(\omega)$ is invariant, the commutator picks up a minus sign.

We would like to extract a number from this operator-valued curvature. Although there is no canonical way to do so[6], a simple and convenient choice is to take the

[6] From a mathematical perspective it corresponds to identifying a suitable dual space of the algebra of zero modes $\mathcal{A}_0$, and corresponding bilinear pairing. In the case of infinite-dimensional algebras this is very subtle (see for example [3] where the case of the Virasoro algebra was discussed).

expectation value of the operator F in the original pure state $|\Psi\rangle$:

$$F_\Psi \equiv \langle\Psi|F|\Psi\rangle = \langle F\rangle \ . \tag{6.2.38}$$

As we will show in Section 6.3, it turns out that (6.2.38) results in the correct identification with the bulk symplectic form. This agreement can be viewed as an argument for why this choice is the most 'physical' one. However, from a mathematical point of view we stress that this choice is by no means unique, and the operator F contains more information.

To proceed in evaluating this expectation value, let us mention a well-known result for two-point functions of operators in the global state $|\Psi\rangle$, the so-called KMS condition. (For a pedagogical exposition of the KMS condition, see for example [251, 252].) Roughly speaking, it says that we can swap operators in a two-point function provided that we evolve one of them in imaginary modular time. To be precise, we introduce the Tomita operator S_Ψ as an anti-linear operator that sends

$$S_\Psi\mathcal{O}|\Psi\rangle = \mathcal{O}^\dagger|\Psi\rangle \ . \tag{6.2.39}$$

The modular operator is now defined by $\Delta = S_\Psi^\dagger S_\Psi$, and satisfies $\Delta|\Psi\rangle = |\Psi\rangle$. Using the definition (6.2.39) together with anti-linearity one can verify that

$$\langle\Psi|\mathcal{O}\mathcal{O}'|\Psi\rangle = \langle\Psi|\mathcal{O}'\Delta\mathcal{O}|\Psi\rangle \ , \tag{6.2.40}$$

for $\mathcal{O}, \mathcal{O}' \in \mathcal{A}$. One can represent the modular operator in terms of the two-sided modular Hamiltonian $\hat{H}_{\text{mod}} \equiv H_{\text{mod}} - \bar{H}_{\text{mod}} = -\log\Delta$ so that the modular flow (6.2.22) is given by $\mathcal{O}_s = \Delta^{-is}\mathcal{O}\Delta^{is}$. Therefore, assuming that the operators $\mathcal{O}_s(x), \mathcal{O}(x')$ are in the algebra $\mathcal{A}$ associated to the subregion[7] one obtains the condition

$$\langle\mathcal{O}_s(x)\mathcal{O}(x')\rangle = \langle\mathcal{O}(x')\mathcal{O}_{s+i}(x)\rangle \ . \tag{6.2.41}$$

The action of modular flow on the Fourier modes $\mathcal{O}_\omega$ is particularly simple, i.e., see (6.2.26), so that the KMS condition (6.2.41) reads

$$\langle\mathcal{O}_\omega(x)\mathcal{O}_{\omega'}(x')\rangle = e^{-\omega}\langle\mathcal{O}_{\omega'}(x')\mathcal{O}_\omega(x)\rangle \ . \tag{6.2.42}$$

By rearranging terms on both sides of the equation one finds the following identity

$$\langle\mathcal{O}_\omega(x)\mathcal{O}_{\omega'}(x')\rangle = n(\omega)\langle[\mathcal{O}_{\omega'}(x'), \mathcal{O}_\omega(x)]\rangle \ , \tag{6.2.43}$$

[7] To ensure this we need to put a restriction on the support of the sources in the perturbation (6.2.6). In the Euclidean picture we assume that the state is perturbed by changing the sources at the branch cut only (using some suitable limiting procedure where we approach it from above and below).

where $n(\omega)$ was defined in (6.2.14). This relation is very useful in practice since we can use it to rewrite the expectation value of a commutator in terms of a two-point function.

We can now use this to evaluate (6.2.38). By recognizing the right-hand side of (6.2.43) in F_Ψ, we obtain

$$F_\Psi = \mathcal{N}^{-1} \int d^d x \int d^d x' \, \delta_1\lambda(x)\delta_2\lambda(x') \int d\omega \, n(\omega) \langle \mathcal{O}_{-\omega}(x)\mathcal{O}_\omega(x')\rangle \ . \tag{6.2.44}$$

One can rewrite the above result by putting one of the two operators in its original form. Using the definition (6.2.23) and the condition $\Delta|\Psi\rangle = |\Psi\rangle$ one can show that the modular Fourier modes satisfy the following relation:

$$\begin{aligned}
\langle \mathcal{O}_\omega(x)\mathcal{O}_{\omega'}(x')\rangle &= \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' e^{-i(\omega s+\omega' s')} \langle\Psi|\mathcal{O}_s(x)\mathcal{O}_{s'}(x')|\Psi\rangle \\
&= \int_{-\infty}^{\infty} ds\, e^{-i\omega s} \int_{-\infty}^{\infty} ds' e^{-i(\omega+\omega')s'} \langle\Psi|e^{iH_{\mathrm{mod}}s'}\mathcal{O}_s(x)\mathcal{O}(x')e^{-iH_{\mathrm{mod}}s'}|\Psi\rangle \\
&= \int_{-\infty}^{\infty} ds\, e^{-i\omega s} 2\pi\delta(\omega+\omega')\langle\Psi|\mathcal{O}_s(x)\mathcal{O}(x')|\Psi\rangle = 2\pi\delta(\omega+\omega')\langle\mathcal{O}_\omega(x)\mathcal{O}(x')\rangle \ .
\end{aligned} \tag{6.2.45}$$

Hence, we conclude that the $\delta(0)$ factor drops out of the final answer, and we obtain

$$F_\Psi = \frac{1}{2\pi} \int d^d x \int d^d x' \, \delta_1\lambda(x)\delta_2\lambda(x') \int d\omega \, n(\omega) \langle \mathcal{O}(x)\mathcal{O}_\omega(x')\rangle \ . \tag{6.2.46}$$

This last equation will be useful in finding a bulk interpretation for the Berry curvature. But first we will show that one can also extract from the Berry curvature a symmetric quantity, which behaves like an information metric on the space of modular Hamiltonians.

6.2.3 Quantum information metric

We can obtain some additional information from F that will be also useful from the bulk perspective. Specifically, it is convenient to construct a symmetric quantity from F by taking one of the perturbations to be of the form $[H_{\mathrm{mod}}, X]$. The quantity

$$G = P_0\left([X_1, [H_{\mathrm{mod}}, X_2]]\right) \tag{6.2.47}$$

is symmetric under exchanging X_1 with X_2. Using an additional commutation $[H_{\mathrm{mod}}, \cdot]$ to turn the antisymmetric object $P_0([X_1, X_2])$ into a symmetric one follows a well-known construction, which applies in finite-dimensional settings [243].

To see how this works, we use the Jacobi identity reorganized in the form

$$[X_1, [H_{\rm mod}, X_2]] = [X_2, [H_{\rm mod}, X_1]] + [H_{\rm mod}, [X_1, X_2]] \ . \tag{6.2.48}$$

Since the last term lies in the image of the adjoint action of $H_{\rm mod}$, it is zero-mode free by (6.2.16), so that

$$P_0([H_{\rm mod}, [X_1, X_2]]) = 0 \ . \tag{6.2.49}$$

Therefore, taking the projection on both sides of (6.2.48) gives the required relation

$$P_0([X_1, [H_{\rm mod}, X_2]]) = P_0([X_2, [H_{\rm mod}, X_1]]) \ . \tag{6.2.50}$$

Using the fact that $\mathcal{O}_\omega$ is an eigenoperator with respect to the adjoint action of $H_{\rm mod}$, (6.2.28), we pick up an extra factor of ω when evaluating $[H_{\rm mod}, X_2]$. Indeed, the formula for F gets modified to

$$G = \mathcal{N}^{-1} \int d^d x \int d^d x' \, \delta_1\lambda(x)\delta_2\lambda(x') \int d\omega \, n(-\omega) n(\omega) \omega [\mathcal{O}_\omega(x), \mathcal{O}_{-\omega}(x')] \ . \tag{6.2.51}$$

Note that this expression is indeed symmetric under the replacement of ω with $-\omega$. As we did for F_Ψ, one can extract from G a number by taking an expectation value, $G_\Psi \equiv \langle \Psi | G | \Psi \rangle$. Going through a similar set of computations one obtains

$$G_\Psi = \frac{1}{2\pi} \int d^d x \int d^d x' \, \delta_1\lambda(x)\delta_2\lambda(x') \int d\omega \, \omega \, n(\omega) \langle \mathcal{O}(x) \mathcal{O}_\omega(x') \rangle \ . \tag{6.2.52}$$

This expression can be rewritten in a form which makes the relation with quantum information theory manifest. Namely, one can undo the Fourier transformation (6.2.23) and write the integral over modular frequencies in terms of an integral over modular time. The extra factor of ω in (6.2.52) comes in handy, since we can replace $\omega n(\omega)$ with the following integral[8]:

$$|\omega| \, n(\omega) = \int_{-\infty - i\epsilon}^{\infty - i\epsilon} ds \frac{\pi}{2\sinh^2(\pi s)} e^{-i\omega s} \ . \tag{6.2.54}$$

[8]This can be derived from an application of the residue formula (by closing the s-contour in the upper/lower half plane depending on the sign of ω) and the geometric series relation. In particular, one uses that the residue at $s = ik$ for $k \in \mathbb{Z}$ is given by

$$\mathrm{Res}_{s=ik} \frac{\pi}{\sinh^2(\pi s)} f(s) = \frac{f'(ik)}{\pi} \ . \tag{6.2.53}$$

Combining (6.2.52) with (6.2.54) and applying the inverse of the Fourier decomposition, (6.2.23), we find that

$$G_\Psi = \int d^dx \int d^dx'\, \delta_1\lambda(x)\delta_2\lambda(x') \int_{-\infty-i\epsilon}^{\infty-i\epsilon} ds \frac{\pi}{2\sinh^2(\pi s)} \langle \mathcal{O}(x)\mathcal{O}_s(x')\rangle \ . \quad (6.2.55)$$

This quantity agrees with a well-known quantum information theoretic 'metric' on the space of mixed states [248, 249], which is obtained from the second variation of the relative entropy. (See Appendix C.5 for more details.) We therefore see that the parallel transport problem for modular Hamiltonians is closely related to a metric on the space of density matrices. This should be reminiscent of the similar situation in the case of pure states, where the Berry phase computes the Fubini-Study metric on the space of pure states [55, 213]. It also provides a natural starting point for investigations of a bulk interpretation.

6.3 Relation to the bulk symplectic form

We would now like to derive a bulk interpretation of the modular Berry curvature and information metric. Let us start out by defining a quantity that generalizes both (6.2.46) and (6.2.52):

$$H_\Psi = \frac{1}{2\pi} \int d^dx \int d^dx'\, \delta_1\lambda(x)\delta_2\lambda(x') \int d\omega\, \mathcal{F}(\omega) \langle \mathcal{O}(x)\mathcal{O}_\omega(x')\rangle \ . \quad (6.3.1)$$

H_Ψ reproduces F_Ψ for the choice $\mathcal{F}(\omega) = n(\omega)$, and G_Ψ for $\mathcal{F}(\omega) = \omega\, n(\omega)$.

We would like to extend H_Ψ into the bulk. Let us for the moment assume that the boundary operator $\mathcal{O}$ used to deform the state in (6.2.1) is a scalar of conformal dimension Δ_+. By general AdS/CFT principles, the dual bulk description is some scalar operator Φ localized in a spacelike slice $\Sigma = \Sigma_A$ of the entanglement wedge associated to the boundary region A. Since the expression (6.3.1) contains operators of the form $\mathcal{O}_\omega$, we first need to describe the bulk analogue of the modular Fourier decomposition. Using the equivalence of bulk and boundary modular flows it is then possible to extend the modular frequency modes into the bulk [245, 253, 254]. We will argue that the two-point function in (6.3.1) behaves like the asymptotic flux of some suitably defined symplectic form. Its bulk extension provides a natural definition for the bulk symplectic form associated to the entanglement wedge. A similar approach was used in [254] to find holographic duals of the α-z relative Rényi divergences, which are certain generalizations of relative entropy.

Note that while we will focus explicitly on the scalar case in this section, it is straightforward to extend to more general field deformations. An important generalization are stress tensor insertions, which in the bulk correspond to perturbations of the geometry, will be treated in Section 6.4.1.

6.3.1 Bulk operator algebra

To make the computation tractable we will make one further assumption, namely that we are working in a free field approximation. We expect this approximation to hold for a generic bulk quantum theory to leading order in $1/N$. Moreover, we are interested in the symplectic form evaluated at a particular point in phase space, so in principle it is possible to find bulk variables Φ and Π that linearize the symplectic form. Interactions can be included perturbatively.

The bulk phase space $\mathcal{A}_\Sigma$ can now be described explicitly in terms of the operators $\Phi(X)$ and canonically conjugate operators $\Pi(X)$ for $X \in \Sigma$. They satisfy the canonical commutation relations

$$[\Phi(Y), \Pi(X)] = i\delta(X - Y) \ . \tag{6.3.2}$$

Let us again now introduce the bulk modular flow associated to Σ, implemented by some bulk density matrix ρ_{bulk}. Following a similar procedure as in the boundary CFT one can now decompose the operators Φ into modular Fourier modes

$$\Phi_\omega = \int_{-\infty}^{\infty} ds\, e^{-i\omega s} \rho_{\text{bulk}}^{-is} \Phi \rho_{\text{bulk}}^{is} \ , \tag{6.3.3}$$

and similarly for Π. Given that the operators $\{\Phi(X), \Pi(X) \,|\, X \in \Sigma\}$ are a formal basis for $\mathcal{A}_\Sigma$, we can express the operator (6.3.3) as the linear combination

$$\Phi_\omega(X) = \int_\Sigma dY \left[\alpha_\omega(X,Y)\Phi(Y) + \beta_\omega(X,Y)\Pi(Y)\right] \ , \tag{6.3.4}$$

with $\alpha_\omega(X,Y), \beta_\omega(X,Y)$ coefficients in the expansion. By acting with the commutator on (6.3.4) and taking the expectation value in the state ρ_{bulk} we find that

$$\alpha_\omega(X,Y) = i\langle[\Pi(Y), \Phi_\omega(X)]\rangle = \frac{i}{n(\omega)} \langle\Phi_\omega(X)\Pi(Y)\rangle \ , \tag{6.3.5}$$

where we have used the KMS condition (6.2.43) adapted to the bulk correlation function. Similarly we have,

$$\beta_\omega(X,Y) = -i\langle[\Phi(Y), \Phi_\omega(X)]\rangle = -\frac{i}{n(\omega)} \langle\Phi_\omega(X)\Phi(Y)\rangle \ . \tag{6.3.6}$$

By plugging this into (6.3.4) one obtains the final result

$$\Phi_\omega(X) = \frac{i}{n(\omega)} \int_\Sigma dY \left[\langle \Phi_\omega(X)\Pi(Y)\rangle \Phi(Y) - \langle \Phi_\omega(X)\Phi(Y)\rangle \Pi(Y) \right] \ . \tag{6.3.7}$$

One can also view (6.3.7) as a Bogoliubov transformation, which changes the operator basis from Φ, Π to modular Fourier modes. As mentioned in Section 6.2.2, the advantage of using the operators Φ_ω is that in this basis the action of the (bulk) modular Hamiltonian is relatively simple. Note that to completely specify the right-hand side of (6.3.7) requires some boundary condition at the finite boundary of Σ, i.e., at the RT surface. We will come back to this issue later, and will argue that the behavior of the integrand near the RT surface is related to the presence of a zero mode in the Berry transport problem.

6.3.2 Modular extrapolate dictionary

Up to this point, we have only used some basic properties of the bulk operator algebra in a free field approximation to write (6.3.7). Let us now invoke the AdS/CFT dictionary to relate the operator Φ_ω to the corresponding boundary operator $\mathcal{O}_\omega$. We denote the holographic direction of the AdS space by z, so that the bulk coordinate is given by $X = (z, x)$. The extrapolate dictionary now states that the properly regularized version of Φ approaches the operator $\mathcal{O}$ near the asymptotic boundary:

$$\lim_{z\to 0} z^{-\Delta_+} \Phi(x, z) = \mathcal{O}(x) \ . \tag{6.3.8}$$

Since we are interested in the modular frequency modes, we will need to use a version of the extrapolate dictionary that is suited to this decomposition. A crucial result was given in [253], where the authors show that the bulk and boundary modular flows agree to first order in $1/N$. This can be used to derive the so-called modular extrapolate dictionary [245]:

$$\lim_{z\to 0} z^{-\Delta_+} \Phi_\omega(x, z) = \mathcal{O}_\omega(x) \ . \tag{6.3.9}$$

We will use (6.3.9) to extend the operator H_Ψ into the bulk. Indeed, we can take the boundary limit on both sides of the equation in (6.3.7):

$$\mathcal{O}_\omega(x) = \frac{i}{n(\omega)} \int_\Sigma dY \left[\langle \mathcal{O}_\omega(x)\Pi(Y)\rangle \Phi(Y) - \langle \mathcal{O}_\omega(x)\Phi(Y)\rangle \Pi(Y) \right] \ . \tag{6.3.10}$$

This formula provides a bulk expression for the boundary operator $\mathcal{O}_\omega(x)$ in terms of some bulk-to-boundary propagators. Note that (6.3.10) is non-local expression in the bulk, which is a reflection of the non-locality of the action of the modular flow. We can now plug (6.3.10) into (6.3.1) to obtain

$$H_\Psi = \frac{1}{2\pi}\int_\Sigma dY \int d^dx \int d^dx'\, \delta_1\lambda(x)\delta_2\lambda(x') \int d\omega\, \mathcal{C}(\omega) \\ \times \left[\langle \mathcal{O}_\omega(x')\Pi(Y)\rangle\langle \mathcal{O}(x)\Phi(Y)\rangle - \langle \mathcal{O}_\omega(x')\Phi(Y)\rangle\langle \mathcal{O}(x)\Pi(Y)\rangle\right] \ . \tag{6.3.11}$$

We have collected the additional dependence on the modular frequency ω in the function $\mathcal{C}(\omega)$. It is given by

$$\mathcal{C}(\omega) \equiv i\mathcal{F}(\omega)n(\omega)^{-1} \ . \tag{6.3.12}$$

The expression (6.3.11) takes a very simple form when written in terms of the bulk fields. Note that the bulk density matrix ρ_{bulk} gets perturbed in a similar way as the boundary density matrix (6.2.6). From the coherent state deformation

$$\delta\rho_{\text{bulk}} = -\int d^dx\, \rho_{\text{bulk}}\delta\lambda(x)\mathcal{O}(x) \ , \tag{6.3.13}$$

we find that the expectation value of the operator Φ in the perturbed density matrix $\delta\rho_{\text{bulk}}$ is given by

$$\delta\phi(Y) \equiv -\int d^dx\, \delta\lambda(x)\langle \mathcal{O}(x)\Phi(Y)\rangle \ . \tag{6.3.14}$$

Note that $\delta\phi$ is a number, while Φ is an operator. Using again that the bulk and boundary modular flows agree we also obtain the relation

$$\delta\phi_\omega(Y) = \int d^dx\, \delta\lambda(x)\langle \mathcal{O}(x)\Phi_\omega(Y)\rangle = \int d^dx\, \delta\lambda(x)\langle \mathcal{O}_{-\omega}(x)\Phi(Y)\rangle \ . \tag{6.3.15}$$

Introducing similar expressions for the canonical conjugate bulk fields $\delta\pi$ and $\delta\pi_\omega$ defined in terms of Π and Π_ω one finds that (6.3.11) simplifies to

$$H_\Psi = \frac{1}{2\pi}\int_\Sigma dY \int d\omega\, \mathcal{C}(\omega)\left[\delta_2\pi_{-\omega}(Y)\delta_1\phi(Y) - \delta_2\phi_{-\omega}(Y)\delta_1\pi(Y)\right] \ . \tag{6.3.16}$$

The variations $\delta_{1,2}$ that we introduced in the above expression correspond to the choice of sources $\delta_{1,2}\lambda$ respectively in (6.2.6).

6.3.3 Entanglement wedge symplectic form

As a final step we will now perform the integral over modular frequencies to obtain the bulk symplectic form. A convenient trick is to first replace the ω-integral by the action of a suitable differential operator on the bulk fields [254]. Specifically, starting with the modular Fourier modes for an arbitrary function f

$$f_{-\omega} = \int_{-\infty}^{\infty} ds\, e^{i\omega s} f_s \ , \tag{6.3.17}$$

we can apply an integration by parts on a wavepacket of such modular Fourier modes to obtain

$$\int d\omega\, \mathcal{C}(\omega) f_{-\omega}(Y) = 2\pi \int_{-\infty}^{\infty} ds\ \left(\mathcal{C}(i\partial_s) f_s(Y)\right) \delta(s) \ . \tag{6.3.18}$$

We thus have

$$\int d\omega\, \mathcal{C}(\omega) f_{-\omega}(Y) = 2\pi \left(\hat{\mathcal{C}}\, f_s(Y)\right)\Big|_{s=0} \ , \tag{6.3.19}$$

where we have defined a differential operator $\hat{C}$ that acts on the modular time s as $\hat{\mathcal{C}} = \mathcal{C}(i\partial_s)$.

Now we have all the ingredients necessary to match our expression for H_Ψ to a bulk symplectic form. Formally, we define the entanglement wedge symplectic form in terms of the field perturbations and corresponding canonical momenta by

$$\Omega(\delta_1\phi, \delta_2\phi) = \int_\Sigma dY \left[\delta_1\phi(Y)\delta_2\pi(Y) - \delta_1\pi(Y)\delta_2\phi(Y)\right] \ . \tag{6.3.20}$$

Note that $\Omega(\delta_1\phi, \delta_2\phi)$ is manifestly anti-symmetric under interchanging 1 with 2. Combining (6.3.16) with (6.3.19) we find that

$$H_\Psi = \Omega(\delta_1\phi, \hat{\mathcal{C}}(\delta_2\phi)_s|_{s=0}) \ . \tag{6.3.21}$$

The modular Berry curvature F_Ψ in (6.2.46) is a particular case of this general relation. Recall that the curvature is described by F_Ψ, which is obtained from H_Ψ by taking the constant function $\mathcal{C}(\omega) = i$. From (6.3.12), this corresponds to the choice $\mathcal{F} = n(\omega)$ in H_Ψ. Then from (6.3.21), the modular Berry curvature is exactly proportional to the bulk symplectic form:

$$F_\Psi = i\, \Omega(\delta_1\phi, \delta_2\phi) \ . \tag{6.3.22}$$

Note that the factor of i comes from the canonical commutation relations, (6.3.2). The above equality constitutes the main result of this section. It provides a bulk dual for the boundary modular Berry curvature.

The modular Berry metric G_Ψ derived in (6.2.52) also arises as an example of H_Ψ with the function $\mathcal{C}(\omega) = i\omega$. In this case, the differential operator acts non-trivially, as $\hat{\mathcal{C}} = -\partial_s$, which corresponds to the infinitesimal action of bulk modular flow. We therefore find that the Berry metric is equal to the bulk symplectic form with an extra action of the modular flow on one of the variations:

$$G_\Psi = \Omega(\delta_1\phi, -\partial_s(\delta_2\phi)_s|_{s=0}) \ . \tag{6.3.23}$$

The presence of bulk modular flow in (6.3.23) can be linked to the extra insertion of the action of the modular Hamiltonian in defining (6.2.47). One can also reverse the logic and argue that the Berry curvature F_Ψ via (6.3.22) provides a natural symplectic form on the space of modular Hamiltonians that agrees with the bulk symplectic form.

Let us now come back to the contribution from the RT surface γ_A in (6.3.10). The contribution that is localized on the RT surface is related to the zero mode of the operator $\mathcal{O}$. From the bulk perspective this is quite easy to see: the action of the bulk modular flow leaves the RT surface fixed so, in particular, operators localized at the RT surface commute with the modular Hamiltonian, i.e., they correspond to modular zero modes. (See Section 4 of [245] for an explicit expression of the zero mode $\mathcal{O}_0$ in terms of an integral over the RT surface γ_A in the case that $\mathcal{O}$ is a scalar.) In our derivation relating the Berry curvature to the bulk symplectic form, we therefore see that the boundary term corresponds to the $\omega = 0$ part of the integral over modular frequencies in (6.2.46). But this term comes from the zero mode in the original transport operator X as computed in (6.2.19). Therefore, imposing $P_0(X) = 0$ by subtracting the zero mode from it, and fixing the zero mode ambiguity in the boundary parallel transport problem, naturally fixes the ambiguity in the boundary condition for the entanglement wedge symplectic form to be Dirichlet.

6.4 Explicit examples

We will now give some explicit examples that illustrate the formalism we have introduced, but restricted to the scenarios where our state transformation are suitable for describing *shape* transformations. In Section 6.4.1, we will consider the case where the perturbing operator $\mathcal{O}$ in the Euclidean path integral (see Section 6.2.1) is given by a stress tensor deformation. Such a deformation will in general cause a change of the boundary metric, so that it lies in the class of *state*-changing transformations. However, for a particular choice of deformation, namely one generated by a conformal Killing vector, this instead implements a change of *shape* of the entangling surface. From the bulk perspective, this example also illustrates how the derivation of the bulk symplectic form given in Section 6.3 straightforwardly generalizes beyond scalar operators.

In Section 6.4.2, we will describe shape deformations in terms of symmetries rather than using the Euclidean path integral, which connects to the language of [3]. We will explain how in the particular case of shape-changing deformations, the Berry curvature is equal to the symplectic form on a special geometry known as a coadjoint orbit. This is reminiscent of the group theoretic structure that was uncovered in two dimensions [3, 188]. However, we emphasize that the connection to coadjoint orbits will not carry over in the more general state-changing case.

6.4.1 Stress tensor insertions

We first consider a specific version of (6.2.6) where we perturb the state by an insertion of the CFT stress tensor. This class of transformations includes the special case of a state transformation that implements a change of shape of the subregion [230–232] but also includes non-trivial changes to the boundary metric. In the bulk dual, this will involve perturbations of the gravitational field.

For concreteness, we consider a d-dimensional CFT on the plane $\mathbb{R}^{1,d-1}$ in some global state $|\Psi\rangle$ with some ball-shaped region A in the $t=0$ slice. We consider the modular Hamiltonian associated to some reduced state $\rho = \rho_A$ that is obtained by tracing out the complement of the ball-shaped region A. In general, the modular Hamiltonian is a complicated non-local operator, but in the case that $|\Psi\rangle = |0\rangle$ is the vacuum state it has an explicit local expression. One can write

$$H_{\text{mod}} = \int_A dS^\nu \xi_A^\mu T_{\mu\nu} \ , \tag{6.4.1}$$

where $T_{\mu\nu}$ is the CFT stress tensor and ξ_A is the vector field that generates modular flow; in particular, it preserves the causal diamond $\mathcal{D}(A)$ of the region A. We would like to deform the modular Hamiltonian via the the action ξ of some coordinate transformation

$$x^\mu \mapsto x'^\mu = x^\mu + \xi^\mu(x) \ . \tag{6.4.2}$$

One can show that the action of ξ is implemented by the action of some operator on the modular Hamiltonian

$$\delta_\xi H_{\text{mod}} - P_0(\delta_\xi H_{\text{mod}}) = [X, H_{\text{mod}}] \ , \tag{6.4.3}$$

where X is defined by

$$X = \int_A dS^\nu \xi^\mu T_{\mu\nu} \ . \tag{6.4.4}$$

The transport problem (6.4.3) is in fact a special example of the coherent state formalism that we discussed in Section 6.2, where we now take $\mathcal{O} = T^{\mu\nu}$ and $\lambda = \partial_\mu \xi_\nu$. Before going into the details, we stress that the equality in (6.4.3) is actually quite subtle. A general coordinate transformation ξ does not leave the metric $h_{\mu\nu}$ of the CFT invariant. Instead, we have

$$\delta h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \ . \tag{6.4.5}$$

If ξ is a Killing vector we have $\delta h_{\mu\nu} = 0$, but in general the variation is non-zero. The idea is, that for a generic transformation ξ, the change in metric can be traded for a change in the state of the CFT implemented by some unitary on the Hilbert space. For this reason, we are able to utilize the formalism of state-changing transformations developed earlier in the chapter to describe *shape* changes by restricting to the particular case where ξ^μ is a conformal Killing vector.

Deforming the boundary metric

We would like to derive the parallel transport equation, (6.4.3), for this special case of stress tensor insertions. The following subsection will review some results derived in [230], while adapting them to the modular Berry setup.

Under a change of the metric the action of the theory picks up a piece of the form

$$\delta S \sim \int d^d x \, \delta h_{\mu\nu}(x) T^{\mu\nu}(x) \ . \tag{6.4.6}$$

Hence, we can think of the deformed state as being obtained from the original state by introducing a source for the stress tensor. We take (6.2.13) as a starting point with the appropriate source and operator. Using a version of the integral formula (6.2.54), one can write this as

$$\langle \omega | \delta H_{\text{mod}} | \omega' \rangle = \int_{-\infty - i\epsilon}^{\infty - i\epsilon} ds \frac{\pi}{2 \sinh^2(\pi s)} \int d^d x \, \delta h_{\mu\nu}(x) \langle \omega | \rho^{is} T^{\mu\nu}(x) \rho^{-is} | \omega' \rangle \ . \tag{6.4.7}$$

The above formula is true for arbitrary metric deformations. Let us now specialize to the case where it is generated by a diffeomorphism:

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \ . \tag{6.4.8}$$

We split the integral over the Euclidean plane into two pieces: a tubular neighborhood R_b of width b around the entangling region ∂A, and its complement $\tilde{R}$. Let us first do the integral over $\tilde{R}$. It can be localized to an integral over the boundary $\partial \tilde{R}$ using an integration by parts:

$$\int_{\tilde{R}} d^d x \, \partial_\mu \xi_\nu T^{\mu\nu} = - \int_{\tilde{R}} d^d x \, \xi_\nu \partial_\mu T^{\mu\nu} + \int_{\partial \tilde{R}} dS_\mu \, \xi_\nu T^{\mu\nu} \ . \tag{6.4.9}$$

By conservation of the stress energy on the support of the diffeomorphism, only the second term in (6.4.9) survives. Let us therefore consider the boundary of $\tilde{R}$ that consist of three parts $\partial\tilde{R} = \partial R_b \cup \partial\tilde{R}_+ \cup \partial\tilde{R}_-$, i.e., the boundary of the tubular neighborhood, and the region just above and below the branch cut at A. We first consider the term coming from the branch cut:

$$\delta H_{\text{mod}}\big|_{\text{cut}} = \int_{-\infty-i\epsilon}^{\infty-i\epsilon} ds \frac{\pi}{2\sinh^2(\pi s)} \left(\int_{\partial\tilde{R}_+} - \int_{\partial\tilde{R}_-} \right) dS_\mu \, \xi_\nu T^{\mu\nu}_{-s} \; . \tag{6.4.10}$$

Here the modular-evolved stress tensor is defined according to (6.2.22) by $T^{\mu\nu}_{-s} = \rho^{is} T^{\mu\nu} \rho^{-is}$. To perform the integral over the branch cut we note that the value of the stress tensor above and below the branch cut are related by modular evolution in Euclidean time. (Recall that Euclidean modular evolution acts geometrically by circular flow around the branch points.) Therefore, we can change the integration region from $\partial\tilde{R}_-$ to $\partial\tilde{R}_+$ by applying a substitution $s \to s + i - 2i\epsilon$ in the integral over s. Hence, it follows that:

$$\delta H_{\text{mod}}\big|_{\text{cut}} = \int_{-\infty}^{\infty} ds \left(\frac{\pi}{2\sinh^2(\pi(s+i\epsilon))} - \frac{\pi}{2\sinh^2(\pi(s-i\epsilon))} \right) \int_{\partial\tilde{R}_+} dS_\mu \, \xi_\nu T^{\mu\nu}_s \; . \tag{6.4.11}$$

Since the contour now only encloses the pole at zero, the integral over s now precisely picks up the double pole at $s = 0$. From (6.2.53) we find that the residue at $s = 0$ is given by

$$\frac{1}{2\pi} \frac{d}{ds}\Big|_{s=0} T^{\mu\nu}_s = \frac{i}{2\pi} [H_{\text{mod}}, T^{\mu\nu}] \; . \tag{6.4.12}$$

In the limit $b \to 0$, the region $\partial\tilde{R}_+$ becomes equal to the subregion A, so we conclude that:

$$\delta H_{\text{mod}}\big|_{\text{cut}} = -\int_A dS_\mu \, \xi_\nu [H_{\text{mod}}, T^{\mu\nu}] \; . \tag{6.4.13}$$

This already reproduces the result (6.4.3). For a detailed derivation of the corner term contribution from ∂R_b, see [230]. For our purposes, we will neglect this term since it is unaffected by a shift in modular time (which is how the modular Hamiltonian acts close to the boundary of the subregion). Thus, it will commute with the modular Hamiltonian and therefore only contributes to the zero mode piece $P_0(\delta H_{\text{mod}})$ in the modular transport problem, and will not affect the Berry curvature.

Gravitational bulk symplectic form

Since the stress tensor perturbations on the boundary are related to perturbations of the bulk geometry, we will compute the gravitational bulk symplectic form explicitly, and compare to the result obtained from the Berry curvature. A standard way to compute the bulk symplectic form is using the covariant phase space formalism [255–258]. This starts from the general action

$$S = \int L \ , \tag{6.4.14}$$

where L is the Lagrangian density which is a $(d+1)$-form on spacetime. We follow standard conventions and denote the exterior derivative on field space by δ, and the exterior derivative on spacetime by d.

We write the variation of the Lagrangian as

$$\delta L = E\, \delta\varphi + d\Theta \ , \tag{6.4.15}$$

where φ denotes the collection of dynamical fields of the theory, and E are the equations of motion (which vanish on-shell). The boundary term Θ is a one-form on field space and a d-form on spacetime. Its variation $\omega \equiv \delta\Theta$ is a two-form on field space that can be integrated to give a symplectic form:

$$\Omega = \int_\Sigma \omega \ . \tag{6.4.16}$$

The d-dimensional surface Σ is usually taken to be a complete Cauchy surface of the bulk spacetime. In our case, we will be interested in the situation where Σ only covers part of the Cauchy slice that corresponds to the entanglement wedge of some boundary subregion. Let us now consider the case of pure Einstein gravity with Lagrangian

$$L = \frac{1}{16\pi G}\,(R - 2\Lambda)\,\epsilon \ , \tag{6.4.17}$$

where ϵ is the $(d+1)$-dimensional volume form. We will take $g_{\alpha\beta}$ to be the bulk metric. It is straightforward to show that Θ takes the form $\Theta = \theta \cdot \epsilon$ with

$$\theta_\alpha = \frac{1}{16\pi G} g^{\beta\gamma}\left(\nabla_\gamma \delta g_{\alpha\beta} - \nabla_\alpha \delta g_{\beta\gamma}\right) \ . \tag{6.4.18}$$

Let us now compute the pullback of θ to the surface Σ. Denoting by n_a the unit normal vector to Σ, and ϵ_Σ the associated volume form, one finds that

$$n^\alpha \theta_\alpha = \frac{1}{16\pi G}\left(n_\rho g^{\beta\sigma} - n^\sigma \delta^\beta_\rho\right)\delta\Gamma^\rho_{\sigma\beta} = \frac{1}{16\pi G}\left(n_\rho \gamma^{\beta\sigma} - n^\sigma \delta^\beta_\rho\right)\delta\Gamma^\rho_{\sigma\beta} \ , \tag{6.4.19}$$

where we have introduced the induced metric $\gamma_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta$ on Σ. Then the pullback of Θ to Σ can be written in terms of the extrinsic curvature as [258, 259]

$$\Theta|_\Sigma = n_\alpha \theta^\alpha \epsilon_\Sigma = \delta\left(-\frac{1}{8\pi G} K \epsilon_\Sigma\right) - \frac{1}{16\pi G}\left(K^{\alpha\beta} - K\gamma^{\alpha\beta}\right)\delta\gamma_{\alpha\beta}\epsilon_\Sigma + dC \ . \tag{6.4.20}$$

The term $C = c \cdot \epsilon_\Sigma$ is given by

$$c^\alpha = -\frac{1}{16\pi G}\gamma^{\alpha\beta} n^\rho \delta g_{\beta\rho} \ , \tag{6.4.21}$$

and vanishes if we impose Dirichlet boundary conditions. In that case, we can express (6.4.20) in terms of the quantities

$$\pi^{\alpha\beta} \equiv -\frac{1}{16\pi G}\left(K^{\alpha\beta} - K\gamma^{\alpha\beta}\right) \ , \qquad \ell \equiv \frac{1}{8\pi G} K\epsilon_\Sigma \ . \tag{6.4.22}$$

Indeed, we have

$$\Theta|_\Sigma = \pi^{\alpha\beta}\delta\gamma_{\alpha\beta}\epsilon_\Sigma - \delta\ell \ . \tag{6.4.23}$$

The fields $\pi^{\alpha\beta}$ will play the role of the canonical momenta associated to the induced metric. Finally, taking another variation of (6.4.23) one finds that

$$\delta\Theta|_\Sigma = \left(\delta\pi^{\alpha\beta} \wedge \delta\gamma_{\alpha\beta}\right)\epsilon_\Sigma \ . \tag{6.4.24}$$

This leads to the final expression for the bulk symplectic form in Darboux form:

$$\Omega(\delta_1 g, \delta_2 g) = \int_\Sigma dX \left[\delta_1\pi^{\alpha\beta}\delta_2\gamma_{\alpha\beta} - \delta_2\pi^{\alpha\beta}\delta_1\gamma_{\alpha\beta}\right] \ . \tag{6.4.25}$$

The boundary quantity (6.3.1) that comes from the Berry transport problem in the case of stress tensor deformations is given by

$$H_\Psi = \frac{1}{2\pi}\int d\omega\,\mathcal{F}(\omega)\int d^d x\int d^d x'\,\delta_1\lambda_{\mu\nu}(x)\delta_2\lambda_{\sigma\tau}(x')\langle T^{\mu\nu}(x)T^{\sigma\tau}_\omega(x')\rangle \ , \tag{6.4.26}$$

where $\delta\lambda_{\mu\nu}(x)$ is generated by a change of boundary metric as in (6.4.6). Let us now compare (6.4.25) to the Berry curvature. The computation is very similar to that of the scalar field, with the difference that some extra indices appear. We denote the bulk operator corresponding to the induced metric $\gamma_{\alpha\beta}$ by $\Gamma_{\alpha\beta}$, and its canonical conjugate operator by $\Pi_{\alpha\beta}$. The commutation relations are analogous to the scalar field case [254]:

$$[\Gamma_{\alpha\beta}(X), \Pi_{\sigma\tau}(Y)] = \frac{i}{2}(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma})\delta(X - Y) \ . \tag{6.4.27}$$

As before, we can define the modular Fourier modes $\Gamma^\omega_{\alpha\beta}$ associated to the operator $\Gamma_{\alpha\beta}$, and expand in terms of $\Gamma_{\alpha\beta}, \Pi_{\alpha\beta}$. Similarly to before, the coefficients can be written in terms of two-point functions using the KMS condition. Applying a version of the modular extrapolate dictionary (6.3.9) that is suited to metric perturbations one finds that

$$T_\omega^{\mu\nu}(x) = \frac{i}{n(\omega)} \int_\Sigma dX \left[\langle T_\omega^{\mu\nu}(x)\Gamma_{\alpha\beta}(X)\rangle \Pi^{\alpha\beta}(X) - \langle T_\omega^{\mu\nu}(x)\Pi_{\alpha\beta}(X)\rangle \Gamma^{\alpha\beta}(X)\right] . \tag{6.4.28}$$

Plugging (6.4.28) into (6.4.26) we find that

$$\begin{aligned} H_\Psi = \frac{1}{2\pi} \int_\Sigma dX \int d\omega\, \mathcal{C}(\omega) \int d^d x \int d^d x'\, \delta_1\lambda_{\mu\nu}(x)\delta_2\lambda_{\sigma\tau}(x') \\ \times \left[\langle T^{\mu\nu}(x)\Pi^{\alpha\beta}(X)\rangle\langle T_\omega^{\sigma\tau}(x')\Gamma_{\alpha\beta}(X)\rangle - \langle T^{\mu\nu}(x)\Gamma^{\alpha\beta}(X)\rangle\langle T_\omega^{\sigma\tau}(x')\Pi_{\alpha\beta}(X)\rangle\right] . \end{aligned} \tag{6.4.29}$$

Similarly to the scalar field case, one can now write the above expression in terms of the metric perturbations and canonical momenta by evaluating the relevant operator in the perturbed state. For example, we have an identity of the form

$$\delta\gamma_{-\omega}^{\alpha\beta} = -\int d^d x\, \delta\lambda_{\mu\nu}(x)\, \langle T_\omega^{\mu\nu}(x)\Gamma^{\alpha\beta}\rangle\ . \tag{6.4.30}$$

Using this together with similar expressions for the perturbations $\delta\gamma_{\alpha\beta}, \delta\pi_{\alpha\beta}, \delta\pi_{-\omega}^{\alpha\beta}$, one can write

$$H_\Psi = \frac{1}{2\pi} \int_\Sigma dX \int d\omega\, \mathcal{C}(\omega) \left[\delta_1\pi_{\alpha\beta}(X)\delta_2\gamma_{-\omega}^{\alpha\beta}(X) - \delta_1\gamma_{\alpha\beta}(X)\delta_2\pi_{-\omega}^{\alpha\beta}(X)\right] . \tag{6.4.31}$$

Applying (6.3.19) to remove the integral over frequencies one finds that

$$H_\Psi = \int_\Sigma dX \left[\delta_1\pi_{\alpha\beta}\,\hat{\mathcal{C}}\left(\delta_2\gamma^{\alpha\beta}\right)_s\Big|_{s=0} - \delta_1\gamma_{\alpha\beta}\hat{\mathcal{C}}\left(\delta_2\pi^{\alpha\beta}\right)_s\Big|_{s=0}\right] , \tag{6.4.32}$$

where the insertion of the operator $\hat{\mathcal{C}}$ is defined in (6.3.19). In the case of the Berry curvature F_Ψ where $\mathcal{C}(\omega) = i$ is the constant function, this explicitly agrees with the gravitational bulk symplectic form (6.4.25).

For the symmetric quantity (6.2.47), which results from taking $\mathcal{C}(\omega) = i\omega$ to no longer be constant, one finds that

$$G_\Psi = \Omega(\delta_1 g, \mathcal{L}_\xi\delta_2 g)\ , \tag{6.4.33}$$

where the bulk modular flow in the vacuum acts geometrically via a Lie derivative $\mathcal{L}_\xi$. This quantity is also known as the canonical energy [248, 254, 260]. From the boundary definition, it is obvious that (6.2.47) defines a symmetric quantity. To see from the bulk perspective that (6.4.33) is symmetric under the interchange of 1 and 2, one can use the product rule and Cartan's magic formula to write

$$\Omega(\delta_1 g, \mathcal{L}_\xi \delta_2 g) - \Omega(\mathcal{L}_\xi \delta_1 g, \delta_2 g) = \int_\Sigma \mathcal{L}_\xi \delta\Theta = \int_\Sigma d(i_\xi \delta\Theta) \ . \tag{6.4.34}$$

At the last equality we have also used that the symplectic potential $\omega = \delta\Theta$ is closed, i.e., $d\omega = 0$. Now we can use Stokes' theorem to localize the integral in (6.4.34) to the boundary $\partial\Sigma$, which consists of the RT surface and the asymptotic boundary. Using the fact that the diffeomorphism ξ is an asymptotic Killing vector which vanishes at the RT surface as well, we find that the boundary terms vanish:

$$\Omega(\delta_1 g, \mathcal{L}_\xi \delta_2 g) - \Omega(\mathcal{L}_\xi \delta_1 g, \delta_2 g) = \int_{\partial\Sigma} i_\xi \delta\Theta = 0 \ . \tag{6.4.35}$$

This confirms that the canonical energy is symmetric, following our derivation of (6.4.33).

6.4.2 Symmetry transformations

We will now study the case where the diffeomorphisms that implement the deformation explicitly lie in the conformal group. This is the direct higher-dimensional generalization of the shape-changing setup that was considered in [53, 54] and reviewed in [3] for the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$. In particular, we will show that the resulting geometric space has the structure of a coadjoint orbit of the conformal group. Notably, the specific state-changing transformations that were considered in [3] are *not* part of the symmetry algebra of CFT_d when $d > 2$, which is finite-dimensional. This is to be contrasted with the situation in $d = 2$, where the symmetry algebra is the infinite-dimensional Virasoro algebra.

Berry curvature

Let us consider a CFT_d in the vacuum state. The modular Hamiltonian associated to a spherical region A of radius R is an element the conformal algebra, $\mathfrak{so}(2, d)$. For example, using planar coordinates $(t, \vec{x})$ for the boundary CFT, and choosing a sphere of midpoint $\vec{x}_0$ and radius R in the $t = 0$ slice, H_{mod} is generated by the conformal Killing vector that preserves a diamond, which is given by [182]

$$H_{\mathrm{mod}} = \frac{\pi}{R}\left[(R^2 - |x - x_0|^2 - t^2)\partial_t - 2t(x^i - x_0^i)\partial_i\right] \ . \tag{6.4.36}$$

Using the conventions of Appendix C.4, we can write this operator in terms of the conformal group generators as

$$H_{\mathrm{mod}} = \frac{\pi}{R}\left[-(R^2 - |\vec{x}_0|^2)P_0 - 2x_0^i M_{0i} - C_0\right] \ . \tag{6.4.37}$$

A crucial ingredient in the computation of the modular Berry curvature is the parallel transportation equation. We will start by changing the modular Hamiltonian by acting with an element in the symmetry group $X \in \mathfrak{so}(2,d)$:

$$\delta H_{\text{mod}} - P_0(\delta H_{\text{mod}}) = [X, H_{\text{mod}}] \ . \tag{6.4.38}$$

These *shape-changing* variations change the spherical region without modifying the global state of the CFT[9]. Recall that (6.4.38) is a special example of (6.4.3) where we take the diffeomorphisms to be conformal Killing vectors.

Clearly, not all generators X in (6.4.38) lead to a change of the modular Hamiltonian. The ones which satisfy $\delta H_{\text{mod}} = 0$, are the modular zero modes, and are formally defined as elements $Q \in \mathfrak{so}(2,d)$ which commute with the modular Hamiltonian:

$$[Q, H_{\text{mod}}] = 0 \ . \tag{6.4.39}$$

This is precisely the definition of the stabilizer subalgebra $\mathfrak{h} \equiv \text{stab}(H_{\text{mod}})$. In the case that H_{mod} is given by (6.4.37) a suitable basis for the space of zero modes can be given by

$$Q_i = \frac{1}{2R}\left[-(R^2 + |\vec{x}_0|^2)P_i - 2x_{0i}D + 2x_0^j M_{ij} + 2x_{0i}x_0^j P_j + C_i\right] \ , \tag{6.4.40}$$

$$Q_{ij} = M_{ij} + x_{0i}P_j - x_{0j}P_i \ , \tag{6.4.41}$$

where $i, j = 1, \dots, d-1$. Indeed, using the conformal algebra one can explicitly check that

$$[Q_i, H_{\text{mod}}] = [Q_{ij}, H_{\text{mod}}] = 0 \ . \tag{6.4.42}$$

The first class of zero modes in (6.4.40) correspond to 'boosts' (directed in the i-th direction of the $\vec{x}$ plane) that preserve the causal diamond associated to the spherical region on the boundary. The second class of zero modes, (6.4.41), rotate the spherical region while leaving the diamond invariant. The algebra of the zero modes is given by:

$$[Q_i, Q_j] = Q_{ij} \ , \qquad [Q_i, Q_{jk}] = Q_k \delta_{ij} - Q_j \delta_{ik} \ . \tag{6.4.43}$$

Together with the modular Hamiltonian itself, the zero mode space $\mathcal{A}_0$ can therefore be identified with the subalgebra

$$\mathfrak{h} = \mathfrak{so}(1,1) \times \mathfrak{so}(1, d-1) \ . \tag{6.4.44}$$

[9] Of course, as explained in Section 6.4.1, one can equivalently think of them in terms of a procedure where we keep the subregion fixed, but change the global state by insertion of a stress tensor operator in the Euclidean path integral.

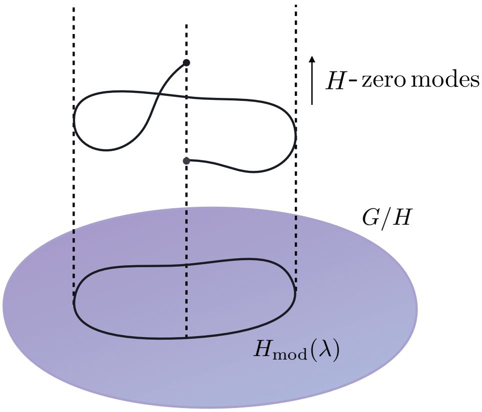


Figure 6.3: *A parallel transport problem. The Berry curvature is associated to the principal H-bundle defined by $G \to G/H$ with fibers that are isomorphic to H. A closed curve of modular Hamiltonians $H_{\mathrm{mod}}(\lambda)$ in the base space G/H is parallel lifted (using the Berry connection) to a non-closed curve in the group G. The endpoints of the curve differ by an element in the zero mode space H.*

Note that the space of zero modes has a non-abelian component $\mathfrak{so}(1, d-1)$.

The general structure of the modular Berry transport can now be described as follows: The space of modular Hamiltonians that we consider is locally given by the variations (6.4.38) and therefore parametrized by $X \in \mathfrak{g}/\mathfrak{h}$. Exponentiating, we conclude that the parameter space is given by the coset space

$$\mathcal{O}_{H_{\mathrm{mod}}} \equiv \frac{SO(2,d)}{SO(1,1) \times SO(1,d-1)} \,. \tag{6.4.45}$$

This is nothing other than the coset space describing the space of causal diamonds in a d-dimensional CFT, known as kinematic space [181–184].

The action of the symmetry group on parameter space is through conjugation and the subgroup of zero modes satisfies

$$V H_{\mathrm{mod}} V^{-1} = H_{\mathrm{mod}} \tag{6.4.46}$$

for $V \in H$. A path in the coset space (6.4.45) can be identified with a one-parameter family of modular Hamiltonians. One can think of this as describing a fiber bundle[10]

$$G \to G/H \ , \tag{6.4.48}$$

which geometrizes the zero mode ambiguity in (6.4.46) by associating to each modular Hamiltonian in the parameter space a fiber of zero modes that projects to the same element (see Figure 6.3).

On an abstract level, the modular Berry connection now corresponds to a one-form on G that takes values in the non-abelian zero mode space. Similarly, the Berry curvature F takes the general form

$$F = F^{H_{\text{mod}}} H_{\text{mod}} + \sum_Q F^Q Q \ , \tag{6.4.49}$$

where the sum over Q indicates a sum over a suitable basis of zero modes (excluding H_{mod} itself). Hence, $F \in \mathfrak{h}$ takes values in a non-abelian zero mode space, and satisfies $[H_{\text{mod}}, F] = 0$. One can compute the Berry curvature associated to two transformations X_1, X_2 from the general formula

$$F = P_0([X_1, X_2]) \ . \tag{6.4.50}$$

The map $P_0 : \mathfrak{g} \to \mathfrak{h}$ denotes the zero mode projector, that extracts the component of the commutator in these directions. Explicitly, decomposing an arbitrary operator X as

$$X = \alpha H_{\text{mod}} + \sum_Q \alpha_Q Q + [H_{\text{mod}}, Y] \ , \tag{6.4.51}$$

the projection operator will extract the parts with coefficients α and α_Q.

Given the non-abelian structure of the zero mode space (6.4.44), one needs to decompose the projector P_0 into subprojectors that extract each of the coefficients in (6.4.51) separately. In general, without introducing more structure, there is no unique procedure for doing this. In fact, one can simply redefine the operators Q that constitute the zero mode basis to get a new set of coefficients α_Q in (6.4.51). However, at this point we can use the fact that we are working with a finite-dimensional Lie algebra and introduce the notion of inner product $\langle \cdot, \cdot \rangle$ on the zero mode space. By choosing an orthonormal basis of zero modes, one can easily distinguish between them. A natural choice of inner product on the Lie algebra

[10] This defines a principal H-bundle in the following sense. There is an action of H on G through left-action:

$$U \to VU \ , \tag{6.4.47}$$

which is compatible with the projection $G \to G/H$, and the isomorphism $\mathfrak{g} \cong \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{h}$ implies that the group G is locally isomorphic to the trivial principal H-bundle.

$\mathfrak{so}(2,d)$ is the Cartan-Killing form given by

$$\langle X, Y\rangle \equiv \frac{1}{2}\mathrm{tr}(XY) \ , \tag{6.4.52}$$

where the trace is taken in the fundamental representation. Let us now choose a linearly independent set of zero mode generators Q_a which are orthonormal with respect to the metric:

$$\langle Q_a, Q_b\rangle = \delta_{ab} \ . \tag{6.4.53}$$

Such an orthonormal basis can, for example, be obtained using the Gram-Schmidt procedure. Moreover, we require that $\langle H_{\mathrm{mod}}, Q_a\rangle = 0$. One can use the metric and corresponding orthonormal basis to extract the coefficients from the operator X. For example, we can define the projection $P_0^{H_{\mathrm{mod}}}$ on the H_{mod}-component of the operator through

$$P_0^{H_{\mathrm{mod}}}(X) \equiv c_{H_{\mathrm{mod}}}^{-1}\langle H_{\mathrm{mod}}, X\rangle = \alpha \ , \tag{6.4.54}$$

where the normalization is such that $c_{H_{\mathrm{mod}}} = \langle H_{\mathrm{mod}}, H_{\mathrm{mod}}\rangle$. One can check that (6.4.54) indeed satisfies the properties that we usually associate with a projection

$$P_0^{H_{\mathrm{mod}}}(H_{\mathrm{mod}}) = 1 \ , \qquad P_0^{H_{\mathrm{mod}}}(Q_a) = 0 \ , \qquad P_0^{H_{\mathrm{mod}}}([H_{\mathrm{mod}}, Y]) = 0 \ , \tag{6.4.55}$$

by using the orthogonality of zero modes. Moreover, the last equality in (6.4.55) can be proved using the cyclicity of the trace

$$\mathrm{tr}(H_{\mathrm{mod}}[H_{\mathrm{mod}}, Y]) = 0 \ . \tag{6.4.56}$$

Using this explicit form of the subprojector, we can compute the curvature component of (6.4.49) in the direction H_{mod} via the formula

$$F^{H_{\mathrm{mod}}} = P_0^{H_{\mathrm{mod}}}([X_1, X_2]) \ . \tag{6.4.57}$$

The non-abelian part of the curvature F can be extracted in a similar fashion. To this end, we construct the subprojection operators onto the other zero modes

$$P_0^{Q_a}(X) \equiv c_{Q_a}^{-1}\langle Q_a, X\rangle = \alpha_{Q_a} \ , \tag{6.4.58}$$

and a different normalization $c_{Q_a} = \langle Q_a, Q_a\rangle$. In particular, the curvature component in the Q_a-direction is given by $F^{Q_a} = P_0^{Q_a}([X_1, X_2])$. This gives a concrete prescription for computing all the components of the modular Berry curvature in the case of shape-changing transformations. We will now show that the numbers that we extract from the operator F can be computed from a symplectic form on certain coadjoint orbits of the conformal group.

Relation to coadjoint orbits

Recall that the parameter space (6.4.45) of the modular Hamiltonian associated to shape-changing transformations is given by

$$\mathcal{O}_{H_{\text{mod}}} = \frac{SO(2,d)}{SO(1,1)\times SO(1,d-1)} \ . \tag{6.4.59}$$

We will now observe that this has the structure of a geometry known as a coadjoint orbit.

Consider our algebra $\mathfrak{g} = \mathfrak{so}(2,d)$. It admits a bilinear pairing $\langle \cdot, \cdot \rangle$ given by (C.4.3) between elements of $\mathfrak{so}(2,d)$. Since the pairing is non-degenerate, the algebra and dual space $\mathfrak{g}^*$ (the space of linear maps on the algebra) are isomorphic. A coadjoint orbit is properly defined in terms of an orbit through the dual space, but due to this isomorphism it suffices to consider orbits of the algebra under a particular action: the adjoint action given by the Lie commutator. Such orbits form symplectic manifolds, and admit a symplectic form known as the Kirillov-Kostant symplectic form [211].

To define the Kirillov-Kostant symplectic form, let us first consider the Maurer-Cartan form

$$\Theta = U^{-1} dU \ , \tag{6.4.60}$$

on the group $U \in SO(2,d)$. Using the dual pairing the Kirillov-Kostant symplectic form is defined as

$$\omega \equiv \langle H_{\text{mod}}, \Theta \wedge \Theta \rangle \ . \tag{6.4.61}$$

To show that ω defines a symplectic form we use the Maurer-Cartan equation:

$$d\Theta + \Theta \wedge \Theta = 0 \ . \tag{6.4.62}$$

Indeed, from (6.4.62) it immediately follows that $d(\Theta \wedge \Theta) = 0$ which shows that $d\omega = 0$. Hence, ω indeed defines a closed form on the group. Moreover, one can check from the definition (6.4.60) that

$$\Theta \wedge \Theta(X_1, X_2) = [X_1, X_2] \ , \tag{6.4.63}$$

so that the Kirillov-Kostant form can also be written as

$$\omega(X_1, X_2) = \langle H_{\text{mod}}, [X_1, X_2] \rangle \ . \tag{6.4.64}$$

Due to the presence of zero modes, (6.4.64) has degeneracies when defined on the full group. The fact that ω descends to a symplectic form on the parameter space $\mathcal{O}_{H_{\mathrm{mod}}}$ follows from the observation:

$$\mathrm{tr}(H_{\mathrm{mod}}[X_1, X_2]) = -\mathrm{tr}([X_1, H_{\mathrm{mod}}]X_2) = 0 \ , \tag{6.4.65}$$

whenever $X_1 \in \mathrm{stab}(H_{\mathrm{mod}})$. Because the stabilizer of the modular Hamiltonian is precisely given by the subgroup $H = SO(1,1) \times SO(1, d-1)$, this shows that ω defines a symplectic form on the coadjoint orbit. Note that (6.4.64) agrees with the formula (6.4.57) for $F^{H_{\mathrm{mod}}}$ up to a normalization constant. Thus, we find that the abelian part of the modular Berry curvature equals the Kirillov-Kostant symplectic form on kinematic space. This result was anticipated for the case $d = 2$ in [3], and we have now established it here in full generality.

For the non-abelian part of the curvature, the situation is slightly different, in the sense that

$$F^{Q_a} = 0 \tag{6.4.66}$$

on the stabilizer $\mathrm{stab}(Q_a)$, which consists of elements that commute with the zero mode Q_a. Of course, H_{mod} is such a stabilizing element (by definition of Q_a), but in general $\mathrm{stab}(H_{\mathrm{mod}}) \neq \mathrm{stab}(Q_i)$. Therefore, the non-abelian components of the curvature do not descend a two-form on $\mathcal{O}_{H_{\mathrm{mod}}}$, but on a different coadjoint orbit $\mathcal{O}_{Q_a}$. Of course, this coadjoint orbit has the same global structure as (6.4.59) (from a mathematical perspective there is nothing special about the zero mode H_{mod} compared to the other Q_a), but the explicit parametrization in terms of conformal group generators will be different. The rest of the arguments that were given above still go through, so that we can identify the Q_a-component of the shape-changing Berry curvature with the Kirillov-Kostant symplectic form on $\mathcal{O}_{Q_a}$.

Low-dimensional examples

Let us now work out some low-dimensional examples, and use the parallel transport formalism to compute the modular Berry curvature by changing the shape of the entangling region. The results will agree with the Crofton formula for computing lengths of geodesics in the bulk [184]. (For the higher-dimensional case, see [261].)

We first restrict to the case of a CFT_2 on the plane. The entangling region on the boundary is an interval (specified by its midpoint x_0 and radius R), with modular Hamiltonian (6.4.37) given by

$$H_{\mathrm{mod}} = \frac{\pi}{R}\left[-(R^2 - x_0^2)P_0 - 2x_0 M_{01} - C_0\right] \ . \tag{6.4.67}$$

For the unit-interval centered at the origin $(x_0, R) = (0, 1)$ this expression reduces to $H_{\mathrm{mod}} = -\pi\left[P_0 + C_0\right]$. A representation of the corresponding vector field is

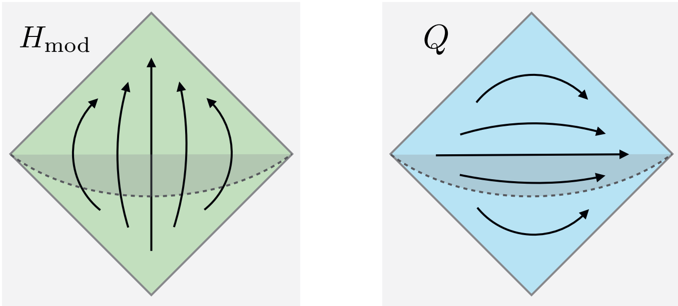


Figure 6.4: *Left: The action of H_{mod} on the causal diamond. Right: The action of Q on the causal diamond.*

provided in Figure 6.4 (left panel). Note that it preserves the causal diamond associated to the interval. The case $d = 2$ allows for one additional zero mode in (6.4.40), which we denote by Q, and is given by

$$Q = \frac{1}{2R}\left[-(R^2 - x_0^2)P_1 - 2x_0 D + C_1\right] \ . \tag{6.4.68}$$

Again, for $(x_0, R) = (0, 1)$ we have $Q = \frac{1}{2}\left[-P_1 + C_1\right]$. It amounts to a spatial 'boost' that fixes the entangling surface at $x = x_0 \pm R$, see Figure 6.4 (right panel).

The Berry transport problem involves a modification of the entangling region by some generators of the conformal group. Given our parametrization of the modular Hamiltonian (6.4.67) in terms of the midpoint x_0 and radius R a natural choice of shape-changing transformations are translations and widenings of the interval. In these cases, the parallel transport equation in (6.4.38) becomes

$$\partial_{x_0} H_{\mathrm{mod}} = [\mathcal{S}_{\delta x_0}, H_{\mathrm{mod}}] \ , \qquad \partial_R H_{\mathrm{mod}} = [\mathcal{S}_{\delta R}, H_{\mathrm{mod}}] \ . \tag{6.4.69}$$

The operators that implement the changes in shape are denote by $\mathcal{S}_{\delta x_0}$ and $\mathcal{S}_{\delta R}$ respectively. Using the commutation relations in Appendix C.4, it is easy to see that the parallel transport operator for translations is

$$\mathcal{S}_{\delta x_0} = P_1 \ . \tag{6.4.70}$$

Similarly, one can show that

$$\mathcal{S}_{\delta R} = -\frac{1}{R}\left(x_0 P_1 - D\right) \ . \tag{6.4.71}$$

Let us now study the relevant subprojection operators $P_0^{H_{\text{mod}}}$ and P_0^Q. The modular Berry curvature associated to this parallel transport problem is given by

$$F^{H_{\text{mod}}} = 0 \ , \qquad F^Q = P_0^Q([\mathcal{S}_{\delta x_0}, \mathcal{S}_{\delta R}]) = \frac{1}{R^2} \ . \tag{6.4.72}$$

Note that it is proportional to the zero mode Q, and for this reason naturally lives on the kinematic space of boundary intervals in CFT_2:

$$\mathcal{O}_Q = \frac{SO(2,2)}{SO(1,1) \times SO(1,1)} \ . \tag{6.4.73}$$

The associated (x_0, R)-component of the Kirillov-Kostant symplectic form is now given by

$$\omega_Q = \frac{1}{R^2} dx_0 \wedge dR \ . \tag{6.4.74}$$

We can rewrite (6.4.74) in a more familiar form by using a cylindrical coordinate θ on the boundary time slice via the identification $x = \tan(\theta/2)$. In particular, identifying

$$R = \tan(\alpha/2) \ , \tag{6.4.75}$$

where the parameter α measures the opening angle of the boundary subregion, the symplectic form (6.4.74) at $x_0 = 0$ becomes

$$\omega_Q = \frac{1}{4\sin^2(\alpha/2)} d\theta \wedge d\alpha \ . \tag{6.4.76}$$

This result agrees with the well-known Crofton formula for RT surfaces on the hyperbolic disk [184], which is identified with the $t = 0$ time slice of AdS_3. In particular, it can be used to compute lengths of curves in the bulk.

Note that the full symplectic form on $\mathcal{O}_Q$ also includes information about shape-changes that, for example, tilt the interval, and take it away from the fixed time slice. To access this information one would need to compute the components of the curvature associated to these deformations as well. For now we will restrict to changes implemented by $\mathcal{S}_{\delta x_0}$ and $\mathcal{S}_{\delta R}$ as in (6.4.72), that act within a single time slice.

Let us also consider the case of CFT_3, where we take the boundary region to be a disk on the (x^1, x^2)-plane with radius R. According to (6.4.37) the modular Hamiltonian associated to this spherical region is given by:

$$H_{\text{mod}} = \frac{\pi}{R}\left[-(R^2 - (x_0^1)^2 - (x_0^2)^2)P_0 - 2(x_0^1 M_{01} + x_0^2 M_{02}) - C_0\right] \ . \tag{6.4.77}$$

For the unit-circle this again reduces to the simple expression $H_{\text{mod}} = -\pi\,[P_0 + C_0]$. There are three distinct zero modes, as can be seen from (6.4.40) and (6.4.41):

$$\begin{aligned} Q_1 &= \frac{1}{2R}\left[-(R^2 + (x_0^2)^2 - (x_0^1)^2)P_1 - 2x_0^1 D + 2x_0^2 M_{12} + 2x_0^1 x_0^2 P_2 + C_1\right] \ , \\ Q_2 &= \frac{1}{2R}\left[-(R^2 + (x_0^1)^2 - (x_0^2)^2)P_2 - 2x_0^2 D - 2x_0^1 M_{12} + 2x_0^2 x_0^1 P_1 + C_2\right] \ , \\ Q_3 &\equiv Q_{12} = M_{12} + x_0^1 P_2 - x_0^2 P_1 \ , \end{aligned} \tag{6.4.78}$$

which constitute a non-abelian $\mathfrak{so}(1,2)$ algebra:

$$[Q_1, Q_2] = Q_3 \ , \qquad [Q_1, Q_3] = Q_2 \ , \qquad [Q_2, Q_3] = -Q_1 \ . \tag{6.4.79}$$

These correspond to two 'spatial' boosts and one rotation that preserve the spherical entangling region $|\vec{x} - \vec{x_0}| = R$. The Berry transport equations (6.4.69) are unchanged, except that we now have two translations indicated by $\mathcal{S}_{\delta x_0^j}$, with $j = 1, 2$. These are given by:

$$\mathcal{S}_{\delta x_0^1} = P_1 \ , \qquad \mathcal{S}_{\delta x_0^2} = P_2 \ , \qquad \mathcal{S}_{\delta R} = \frac{1}{R}\left(D - x_0^1 P_1 - x_0^2 P_2\right) \ . \tag{6.4.80}$$

Now we can compute the commutator associated to a change of center position and a change of radius to be

$$[\mathcal{S}_{\delta x_0^j}, \mathcal{S}_{\delta R}] = -\frac{1}{R} P_j \ . \tag{6.4.81}$$

Hence, we find that the component of the Berry curvature in the H_{mod}-direction vanishes:

$$F^{H_{\text{mod}}} = P_0^{H_{\text{mod}}}\left([\mathcal{S}_{\delta x_0^j}, \mathcal{S}_{\delta R}]\right) = -\frac{1}{R} P_0^{H_{\text{mod}}}(P_j) = 0 \ . \tag{6.4.82}$$

The component in the Q_i-direction will be non-zero. Indeed, the curvature is given by

$$F^{Q_i} = -\frac{1}{R} P_0^{Q_i}(P_j) = \delta_{ij}\frac{1}{R^2} \ , \quad \text{for} \quad i = 1, 2 \ , \tag{6.4.83}$$

and $F^{Q_3} = 0$. Note that the non-zero component of the curvature depends on the direction of the translation: Acting with $\mathcal{S}_{\delta x_0^j}$ leads to $F^{Q_j} \neq 0$. Similarly by setting the center to $\vec{x}_0 = 0$, the relevant component of the symplectic form in the (x_0^i, R)-direction is again given by:

$$\omega_{Q_i} = \frac{1}{R^2} dx_0^i \wedge dR \ . \tag{6.4.84}$$

6.5 Discussion

We have considered modular parallel transport involving a change of *state* in holography in general dimensions. The resulting modular Berry curvature, which is operator-valued, contains information about both the bulk symplectic form as well as the quantum Fisher information metric and its bulk dual, the canonical energy. We additionally treated shape-changing modular transport in higher dimensions, which is a special case of the state-changing transformations, and in this case provided a connection to the geometry of coadjoint orbits.

One could interpret the current work as a continuation of [3], where the modular Berry phase is studied in the specific example of $\text{AdS}_3/\text{CFT}_2$, extended to a larger class of state deformations and to the higher-dimensional setting. Of course, that setting is rather special in the sense that certain properties of AdS_3 gravity and two-dimensional CFTs do not generalize to higher dimensions. For example, the state-changing transformations that were considered in [3] are *not* part of the symmetry algebra of CFT_d when $d > 2$. In higher dimensions, the finite-dimensional conformal group only contains shape-changing transformations. This is to be contrasted with CFT_2, where we have the full infinite-dimensional Virasoro algebra at our disposal. To set up a non-trivial transport problem in higher dimensions we had to introduce a more general formalism of coherent state deformations, that are not restricted to act within the symmetry algebra. Another important difference arises in the bulk computation: While AdS_3 has a topological Chern-Simons theory description that makes the computation of the symplectic form somewhat tractable, no such simplification happens in general Einstein-Hilbert gravity. In the present work, we instead use the covariant phase space formalism directly in the metric formalism to find an expression for the bulk symplectic form. However, as we have shown here, the relation between the Berry phase and symplectic form persists even in this more general setting.

We should also discuss our results in light of previous work on the role of Berry phases in the AdS/CFT correspondence. A notable example involves [55, 213] where an interesting connection between the Berry phase and bulk symplectic form is established. Their computation involves the space of coherent *pure* states that are prepared via the Euclidean path integral by turning on sources. The corresponding Berry phase is shown to agree with the bulk symplectic form associated to the full Cauchy slice. Our approach involves a similar set-up with the important difference that our computations work for deformations of density matrices associated to general subregions in the CFT. The corresponding bulk dual is now the symplectic form supported on the entanglement wedge. In that sense, our work provides a natural extension of these previous results to CFT subregions, and places the Berry phase/bulk symplectic form duality on a more general footing.

To associate a geometric phase to deformations of density matrices we used the construction of the *modular* Berry phase. It is built upon the idea that there is a zero mode ambiguity in the choice of basis frame for the modular Hamiltonian. There is a slightly different version of the parallel transport problem due to Uhlmann that relies on the idea of parallel purifications [262, 263]. The resulting Uhlmann holonomy is closely related to, but not exactly the same as the modular Berry curvature. One difference is that the Uhlmann equations are written in terms of the change of density matrix itself while the modular Berry curvature makes use of the change of the modular Hamiltonian as a starting point. There is a non-trivial transformation, cf. (6.2.9), that relates both perspectives. More importantly, the zero mode projection that is crucial in defining modular Berry transport is absent in the Uhlmann case. While the Uhlmann holonomy is also related to a distance measure on the space of mixed states, i.e., the fidelity, our results indicate that the modular Berry phase is instead related to the quantum Fisher information metric on the space of mixed states. To understand this more deeply would be useful for many reasons. For example, the Uhlmann holonomy was used by [190] to make a claim that is similar in spirit to ours: that there is a direct connection between the geometric phase and some bulk entanglement wedge symplectic form.

7 Conclusion

In this thesis, I have studied certain aspects of black holes in solvable low-dimensional models for holographic quantum gravity. I will conclude by providing an overview of the main results that were obtained – structured according to the three categories: 'wormholes,' 'chaos' and 'holography' – and end with some interesting questions that are left unanswered.

Wormholes

A formal approach towards formulating a theory of quantum gravity involves the gravitational path integral. While generically such a construction, where one integrates over all possible geometries, is beyond what is currently possible, certain simple models of two-dimensional gravity allow for a exact computation of the Euclidean path integral including non-perturbative effects. These non-perturbative effects arise in the form of spacetimes with higher topologies. In particular, Euclidean wormholes, which describe connected spacetimes with disconnected boundaries, seem to play an important role. In this thesis, I have mostly focused on the study of JT gravity: In that case the path integral reduces to a simple expression in terms of the WP volumes associated to the moduli space of Riemann surfaces, and some boundary integral over a coadjoint orbit of the Virasoro group. This computation was explained in Chapter 2.

In Chapter 3, I have presented a field theoretic description of Euclidean JT gravity when different topologies are included. I have shown that the relevant field theory describes the complex structure deformations of the spectral curve. The theory has a cubic interaction supported at the origin of the spectral curve which leads to the correct perturbative expansion. To be precise, the Euclidean wormhole contribution with n boundaries is computed from the n-point function of the field theory. By matching the Schwinger-Dyson equations with the topological recursion relations I have shown that there is agreement up to all order in the genus expansion parameter. Thus, the diagrammatics of the SD equations match one-to-one with the recursion relations between volumes of moduli spaces of Riemann surfaces.

This is reminiscent of a string field theory, albeit in a much simpler context. The name 'universe field theory' alludes to this analogy.

The universe field theory can be obtained from a reduction of the Kodaira-Spencer (KS) theory to the two-dimensional spectral curve $\mathscr{S}_{\mathrm{JT}}$. This shows that JT gravity can be embedded in the well-established topological string theory framework. In the string theory analogy, the spectral curve should be seen as defining the target space in which the JT universes, the equivalent of the strings, propagate. An upshot of the field theory construction is that it is very natural to include certain non-perturbative geometric objects. One such object, a topological D-brane, plays an important role in understanding the quantum chaotic properties of gravity.

Chaos

Black holes represent paradigms of chaotic quantum dynamics. The energy levels of quantum chaotic systems are well-described by random matrix theory, which I have introduced in Chapter 2. An interesting probe for the quantum chaotic spectrum is the spectral form factor. Its characteristic 'dip-ramp-plateau' structure is typical for such systems. While it was argued that including certain non-perturbative effects (i.e., higher topologies) to the gravitational path integral can explain part of this behavior, a full account from the gravity perspective was still lacking. Therefore, we arrive at the following question: Can one find a geometrical interpretation of quantum chaos, also at late plateau times?

In Chapter 4, I have introduced a low-dimensional flavor matrix theory (fMT) as a important concept for studying these effects in the framework of two-dimensional gravity. Crucially, the fMT defines a description of chaotic correlations on all time scales and the duality of a given theory to a particular reduction of fMT, a zero-dimensional flavor nonlinear σ-model (fNLSM), is a sufficient and necessary condition for it to lie in the universality class of ergodic quantum chaos. Generally speaking, the fMT describes a symmetry breaking phenomenon common to all chaotic systems in ergodic phases, 'causal symmetry breaking', and its dynamical restoration at long (plateau) scales.

The fMT description naturally arises in the context of the universe field theory that we described before. This framework naturally incorporates D-branes on which JT universes can end, described by vertex operators. Specifically, I have established a connection between probe D-brane insertions (i.e., correlators of vertex operators) in two-dimensional Kodaira-Spencer (KS) field theory and fMT at long time scales. The advantage of this formulation is that it works directly in the double-scaling limit, as opposed to the the high-rank matrix theories considered in the gravitational setting of e.g., [36, 98, 136], throughout called 'color' matrix theories (cMT). They are dual to fMT via a 'color-flavor duality', that I also interpret geometrically in the universe field theory.

On a more conceptual level, the results of Chapter 4 establish a deep link between the ideas of quantum ergodicity and quantum gravity. The effective field theory of quantum chaos described in [49] can be viewed as a 'third-quantized' field theory of two-dimensional universes: This construction gives a physical picture to the (doubly) non-perturbative contributions needed to restore unitarity at late times (namely during the 'ramp' and the 'plateau' phase) in terms of D-brane dynamics. Finally, in the double-scaled limit there is equivalence to the fMT describing SYK. The emergence of identical fMTs proves that bulk and boundary are in the same universal phase of ergodic quantum chaos.

Holography

Throughout this thesis, we have seen that holography is a powerful tool to study black holes in quantum gravity. In particular, many of the results in this thesis are important for a better understanding of the AdS/CFT correspondence in lower-dimensional models of quantum gravity.

First, in Chapter 3 and 4 I have studied a version of the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence, where JT gravity is dual to a matrix integral. It was shown that this 'modified' holographic dictionary is, in fact, a consequence of a more standard open/closed duality in topological string theory. In KS theory the JT universe plays the role of the closed string world-sheet. It was further shown in [49] that in certain cases, the effective theory of quantum chaos gives rise to a certain matrix model of Kontsevich type, an example of an open string field theory. Thanks to an open-closed type duality between these two descriptions, which is established in Chapter 4, it is possible to understand these two frameworks, the matrix integral and JT gravity, as dual manifestations of the same physics. The ensemble corresponds to the path integral in the open string field theory dual of the KS theory. From a geometric point of view we are 'averaging' over deformations of the background geometry in which the JT universes propagate.

Finally, in Chapter 5 and 6 I have studied certain geometric phases in the AdS/CFT correspondence. The notion of modular Berry phase generalizes the usual quantum mechanical Berry phase to the holographic setting, and can be applied to general mixed states. It incorporates the entanglement spectrum of the CFT in an abstract parallel transport problem for modular Hamiltonians. In chapter 5, I study a class of state-changing transformations in $\mathrm{AdS}_3/\mathrm{CFT}_2$ by acting with Virasoro generators on the vacuum state and keeping the subregion fixed, to obtain a one-parameter family of density matrices. Transporting along an infinitesimal loop of this sort, results in a non-trivial operator-valued Berry curvature. This curvature is explicitly computed in our $\mathrm{AdS}_3/\mathrm{CFT}_2$ setting, and it was shown that it can be computed holographically by some suitably defined entanglement wedge symplectic form. This bulk symplectic form is computed in the background of a Euclidean conical singularity geometry with a backreacted cosmic brane inserted at the RT surface. Using the Chern-Simons formulation of three-dimensional AdS-gravity, the expression for the symplectic form simplifies dramatically and one finds agreement with the modular Berry phase.

In chapter 6, I set up a more general formalism that goes beyond the symmetry-based transport and works in any dimension (so is not specific to CFT_2). The parallel transport problem is now based on perturbations of coherent states that are prepared via the Euclidean path integral. I derive an explicit formula for the modular Berry curvature associated to two such perturbations, and show that this expression agrees with a bulk symplectic form associated to the entanglement wedge. In special cases, where we do act with symmetry generators, this relation recovers the well-known structure of kinematic space and coadjoint orbits. Hence, the results of this chapter provide a new entry in the holograpic dictionary, showing that the modular Berry phase is a nice organizational principle for bulk reconstruction.

Future directions

To end things, I will list some unanswered question and interesting directions for future research. In doing so, I will be a bit more speculative than in the main text.

There are some possible generalizations of the universe field theory construction which are worth studying. It would be interesting to carry out a similar analysis in the case of JT supergravity. The super analogue of the JT/matrix integral correspondence was already discussed in [92]. I expect the relevant universe field theory to come from a supersymmetric generalization of the construction we have presented. Another generalization is provided by the other matrix ensembles (e.g., orthogonal or symplectic), that should be incorporated in a more general framework, called refined topological string. The relevant flavor D-branes that give rise to the specific matrix ensemble should now be of a different type.

Moreover, it might be interesting to study the connection between the KS theory and $c < 1$ minimal string[1]. In this context Liouville gravity is coupled to a $(2, p)$ minimal model with p odd. It is proposed that one can view JT gravity as the $p \to \infty$ of these models by matching the leading order density of states [84]. The precise way JT gravity arises as a limit of such minimal string theories is not yet understood. The KS theory could be useful framework for making this relation precise. At finite p, I expect these p-deformed JT gravity theories to correspond

[1]The relation between minimal string theory and random matrices in the context of two-dimensional gravity goes back to [85–87, 264–266]. For reviews see [267, 268]

to different choices of classical background geometry in the KS theory.

A more ambitious pathway would be to explore the generalization to higher dimensions. It is far from clear if (and how) the above construction generalizes to the higher-dimensional setting. The precise form of the KS action and its relation to JT gravity relied heavily on the interpretation of the spacetime in terms of a string world-sheet. In that sense the construction seems to be very specific to models of two-dimensional gravity. However, the derivation was fundamentally based on the universal recursive structure expressed in terms of the SD equation. If one could find a similar recursive structure in higher-dimensional theories of quantum gravity, this would open up a way for finding such a field theory description.

For example, in three dimensions it is believed that pure quantum gravity does not have a well-defined quantum mechanical dual. There is some evidence that these issues are similar to the ones in JT gravity, and might be resolved if one thinks about the dual in terms of an ensemble of theories. These findings suggest that this ensemble is a generalization of random matrix theory, and is referred to as a 'random CFT' (e.g., [145]). Liouville theory might be the right perspective to study the universal feature of such CFTs. A useful starting point for addressing this question might be to restrict to a class of three-manifolds with topology $\Sigma_{g,n} \times S^1$, as they are closest to the JT gravity geometries. Ideally, one would want to construct a theory for the propagation of branes of this form and find a 'topological recursion' for the correlation functions in this theory. This would give some insight into the higher-dimensional analogue of the duality between JT gravity and a double-scaled matrix integral.

The ideas of Chapter 4 might point to the correct direction. Although the set-up of KS theory seems to be specific to 2D, the chaos story seems to be fairly general: Any quantum chaotic theory in the ergodic phase reduces to a non-linear σ-model. For example, a wide range of ideas about the role of chaos in AdS/CFT, whether it be via random-matrix [145, 269] or ETH-type correlations in the bulk [270] as well as their generalization to statistics of OPE coefficients in $\mathrm{AdS}_3/\mathrm{CFT}_2$ [147, 271, 272], have been considered, and can be studied in the framework of the flavor $\sigma-$model [141, 150]. In addition, it would be very interesting to carry out a computation of entanglement entropies in this set-up, and connect the quantum chaotic structure more directly to the replica wormholes that appear in the 'resolution' to the black hole information problem.

From the boundary perspective, the appearance of a matrix integral (i.e., averaged) dual to JT is quite puzzling: Is the averaging fundamental or more like an effective description of the underlying microscopics? One interesting perspective on the non-factorization of the boundary correlation functions in the presence of

Euclidean wormholes is presented in the work [100] by Marolf and Maxfield on baby universes and α-states. In principle, it should be possible to construct the α-states in JT gravity explicitly. This would shed some light on the microscopic degrees of freedom that are needed to obtain factorization, and I expect them to be related to some D-brane like objects (e.g., [98,121,136]) in the theory. This would show that the averaging is not fundamental, but can be resolved if one introduces D-branes in the theory. However, it turns out that there is an interesting subtlety with diffeomorphism invariance in gravity (that does not arise in the simple topological toy model that is considered in [100]), and prevents a naive implementation of the GNS construction in this setting. A priori, it is not so clear how to implement diffeomorphism invariance as a single null state constraint on the Hilbert space when higher topologies are included. It would be nice to understand these technical obstacles from a physical perspective.

Finally, the theory of modular Berry phases has a very rich structure, and allows for many directions of future research.

The fact that the metric and the symplectic form are related in a simple way through (6.3.23) suggests an underlying geometric structure. In fact, the relation immediately brings to mind the situation for a Kähler manifold where the symplectic form and metric are related by an extra insertion of the (almost) complex structure. This is familiar from the usual Berry phase in finite-dimensional quantum mechanics, where the space of pure states takes the form of a complex projective space, which does indeed exhibit a natural Kähler structure. It is well known that in this case the Berry curvature is closely related to the Fubini-Study metric. However, in the case of mixed states we have found that to go from the modular Berry curvature to the quantum Fisher information metric requires an extra action of the modular Hamiltonian. This procedure does not seem to have a natural interpretation as an almost complex structure: Importantly, it does not square to minus one when acting on general tangent vectors. Only in special cases (for example, when we are acting purely with shape-changing transformations) do we expect that the presence of such underlying geometric structure can be made precise. Nevertheless it would be interesting to understand these observations better.

Likewise, one might ask whether this generalized symplectic structure defines a natural Hilbert space through geometric quantization. In the the shape-changing case, recall that the Berry curvature could be related to the Kirillov-Kostant symplectic form on a special symplectic geometry known as a coadjoint orbit. By the 'orbit method,' which is a version of geometric quantization, such symplectic manifolds can be equated with a particular representation of the group which defines the coadjoint orbit through quotienting [211]. In this more general setting involving state-changes, it would be interesting to learn if similar relations persist, and what one can learn from them about the Hilbert space for quantum gravity.

Finally, the role of operator algebra techniques has gained some renewed interest in the context of holography and black hole physics. In particular, it was argued that the large N limit of the boundary CFT (in the specific setting of the eternal black hole) should be a type III_1 von Neumann algebra [273,274]. Type III von Neumann algebras are rather complicated in the sense that many quantities that we like to use in quantum mechanics (e.g., density matrices, von Neumann entropies) are not well-defined. It is therefore natural to ask how our computations depend on details of the underlying operator algebra. Crucially, entropy differences (e.g., the relative entropy) are well-defined in type III von Neumann algebras. Since the final answer for the Berry curvature is related to the quantum Fisher information metric, which can be written in terms of the relative entropy, it is certainly possible that there exists some suitable continuum limit of our computations. One idea is to define a version of the Berry phase problem in terms of the algebra of observables without any reference to an underlying state deformation. It would be interesting to study further the Berry phase in connection with the emergent type III_1 structure.

A Computations in the universe field theory

A.1 Free two-point functions of the universe field theory

We compute the two-point functions of the free theory including sources (3.2.21) with action (3.2.18). The partition function is just a Gaussian integral in Φ and $\mathcal{J}$, so we can solve it by functional determinants. We will start with the Φ-integral. Integrating by parts, the terms involving Φ are:

$$\frac{1}{2}\Phi\,\partial\overline{\partial}\,\Phi - \Phi\,\overline{\partial}\mathcal{J} - \mu_\Phi\Phi \ . \tag{A.1.1}$$

The Laplacian $\Delta = \partial\overline{\partial}$ on the spectral curve has a Green's function Δ^{-1}, which can be found by first projecting to the spectral x-plane, and then transforming back to the double cover via $x = z^2$. There is a branch cut on the negative real axis and therefore the Green's function on the x-plane can be found using the method of images [275]. Transforming to the double cover, $x = z^2, y = w^2$, we find the result

$$\Delta^{-1}(z,w) = \frac{1}{2}\ln\left|\frac{z-w}{z+w}\right| \ . \tag{A.1.2}$$

We can now solve the Φ-integral by completing the square:

$$\begin{aligned}\int[d\Phi]\exp\left[\frac{1}{2}\Phi\,\partial\overline{\partial}\,\Phi - \Phi\,\overline{\partial}\mathcal{J} - \mu_\Phi\Phi\right]&\\ = N\exp\left[-\frac{1}{2}(\overline{\partial}\mathcal{J}+\mu_\Phi)\Delta^{-1}(\overline{\partial}\mathcal{J}+\mu_\Phi)\right] \ .&\end{aligned} \tag{A.1.3}$$

Here, we have used the condensed notation

$$A\,\Delta^{-1}B \equiv \int d^2z\int d^2w\,A(z)\Delta^{-1}(z,w)B(w) \ . \tag{A.1.4}$$

We have also denoted the functional determinant by N, which is just a (possibly infinite) constant that drops out, because we have normalized the partition function. Explicitly, the functional determinant is

$$N = \int [d\Phi] \exp\left[\frac{1}{2}\Big(\Phi - \Delta^{-1}(\mu_\Phi + \overline{\partial}\mathcal{J})\Big)\Delta\Big(\Phi - \Delta^{-1}(\mu_\Phi + \overline{\partial}\mathcal{J})\Big)\right] \tag{A.1.5}$$

$$= \int [d\Phi'] \exp\left[-\frac{1}{2}\Phi'(-\Delta)\Phi'\right] = \det(-\Delta)^{-1} \; . \tag{A.1.6}$$

In general, the determinant should be regularized, but we do not have to worry about this since we have normalized the partition function with sources. Having done the Φ-integral, we are left with the integration over $\mathcal{J}$:

$$Z_{\mathsf{KS}}^{(0)} = \frac{N}{Z_{\mathsf{KS}}^{(0)}[0]} \int [d\mathcal{J}] \exp\left[-\frac{1}{2}(\overline{\partial}\mathcal{J} + \mu_\Phi)\Delta^{-1}(\overline{\partial}\mathcal{J} + \mu_\Phi) - \mu_{\mathcal{J}}\mathcal{J}\right] \; . \tag{A.1.7}$$

Now we introduce the Green's function for the $\overline{\partial}$-operator on the spectral curve. It is simply the derivative of the Green's function in (A.1.2):

$$\overline{\partial}^{-1} = \partial\Delta^{-1} = \frac{dz}{z-w} - \frac{dz}{z+w} \; . \tag{A.1.8}$$

Writing $\overline{\partial}^{-1} \equiv \overline{\partial}^{-1}(z,w)\, dz$, we perform the shift

$$\mathcal{J} \to \mathcal{J} - \overline{\partial}^{-1} \cdot \mu_\Phi \; , \tag{A.1.9}$$

where the $\cdot$ means that we integrate with respect to the second argument of $\overline{\partial}^{-1}$:

$$\overline{\partial}^{-1} \cdot \mu_\Phi \equiv \int d^2w \, \Big(\overline{\partial}^{-1}(z,w)\mu_\Phi(w)\Big) \, dz \; . \tag{A.1.10}$$

The exponent of (A.1.7) now becomes

$$-\frac{1}{2}\overline{\partial}\mathcal{J}\,\Delta^{-1}\,\overline{\partial}\mathcal{J} - \mu_{\mathcal{J}}\big(\mathcal{J} - \overline{\partial}^{-1} \cdot \mu_\Phi\big) \; . \tag{A.1.11}$$

Integrating the first term by parts, and using that $\overline{\partial}\Delta^{-1} = \partial^{-1}$, we see that the free partition function is

$$Z_{\mathsf{KS}}^{(0)} = \frac{N}{Z_{\mathsf{KS}}^{(0)}[0]} \int [d\mathcal{J}] \exp\left[\frac{1}{2}\mathcal{J}\,\partial^{-1}\overline{\partial}\mathcal{J} - \mu_{\mathcal{J}}\mathcal{J}\right] \exp\left[\mu_{\mathcal{J}}\overline{\partial}^{-1}\mu_\Phi\right] \; . \tag{A.1.12}$$

We can complete the square in the first exponent and do the Gaussian integral, which gives us

$$Z^{(0)}_{\mathsf{KS}}[\mu_\Phi, \mu_{\mathcal{J}}] = \frac{N\,N'}{Z^{(0)}_{\mathsf{KS}}[0]} \exp\left[\frac{1}{2}\mu_{\mathcal{J}}\partial\overline{\partial}^{-1}\mu_{\mathcal{J}} + \mu_{\mathcal{J}}\overline{\partial}^{-1}\mu_\Phi\right] . \tag{A.1.13}$$

We have written the functional determinant N' as

$$N' = \int [d\mathcal{J}] \exp\left[\frac{1}{2}\Big(\mathcal{J} - (\partial^{-1}\overline{\partial})^{-1}\mu_{\mathcal{J}}\Big)\,\partial^{-1}\overline{\partial}\,\Big(\mathcal{J} - (\partial^{-1}\overline{\partial})^{-1}\mu_{\mathcal{J}}\Big)\right] \tag{A.1.14}$$

$$= \int [d\mathcal{J}'] \exp\left[-\frac{1}{2}\mathcal{J}'(-\partial^{-1}\overline{\partial})\mathcal{J}'\right] = \det(-\partial^{-1}\overline{\partial}\,)^{-1} . \tag{A.1.15}$$

In arriving at (A.1.13), we have also used that $(\partial^{-1}\overline{\partial})^{-1} = -\partial\overline{\partial}^{-1}$, as can be verified by acting from the left with $\partial^{-1}\overline{\partial}$ and integrating by parts. The factors N and N' cancel against the normalization, as can be seen by turning off the sources. Therefore, we have shown that:

$$Z^{(0)}_{\mathsf{KS}}[\mu_\Phi, \mu_{\mathcal{J}}] = \exp\left[\frac{1}{2}\mu_{\mathcal{J}}\partial\overline{\partial}^{-1}\mu_{\mathcal{J}} + \mu_{\mathcal{J}}\overline{\partial}^{-1}\mu_\Phi\right] . \tag{A.1.16}$$

In particular, this implies that there are only contractions between $\mathcal{J}$ and $\mathcal{J}$, and between $\mathcal{J}$ and Φ. There is no contraction of Φ with itself. Here, $\overline{\partial}^{-1}$ is given by (A.1.8) and

$$\partial\overline{\partial}^{-1} = \frac{dz\,dw}{(z-w)^2} + \frac{dz\,dw}{(z+w)^2} . \tag{A.1.17}$$

So, if we define the following functions:

$$\mathsf{B}(z,w) = \frac{1}{(z-w)^2} + \frac{1}{(z+w)^2} , \quad \mathsf{G}(z,w) = \frac{1}{z-w} - \frac{1}{z+w} , \tag{A.1.18}$$

we arrive at the result claimed in the main text:

$$\log Z^{(0)}_{\mathsf{KS}} = \int d^2z \int d^2w \left[\frac{1}{2}\mu_{\mathcal{J}}(z)\mathsf{B}(z,w)\mu_{\mathcal{J}}(w) + \mu_{\mathcal{J}}(z)\mathsf{G}(z,w)\mu_\Phi(w)\right] . \tag{A.1.19}$$

A.2 Baby universe Hilbert space and Virasoro constraints

In this appendix, we give another perspective on the duality between KS theory and JT gravity by exploiting the relation to topological gravity, making the underlying Virasoro symmetry manifest. The approach is similar to the SD equation, but makes use of the operator formalism instead of path integrals and should be viewed as complementary. We show that one can reformulate the KS recursion relation in terms of a Virasoro constraint on the partition function, which is also satisfied by the Weil-Petersson volumes (when viewed as intersection numbers of the moduli space). This naturally leads to a definition of a baby universe Hilbert space.

The Weil-Petersson volumes can be expressed in terms of the intersection theory of the moduli space via:

$$V_{g,n}(\boldsymbol{\ell}) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\Omega_{\mathrm{WP}} + \frac{1}{2}\sum_{i=1}^{n}\psi_i \ell_i^2\right) , \tag{A.2.1}$$

where $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n)$, Ω_{WP} is the Weil-Petersson symplectic form on the moduli space and ψ_i are ψ-classes on the moduli space. See, for example, [112] for more details. Attaching the trumpets (we set $\phi_r = 1$ in (2.1.4) for convenience), and using the integral identity $\int_0^\infty x\exp(-\frac{1}{2}ax^2)dx = a^{-1}$, we can write the genus g contribution to the JT path integral with n boundaries of lengths $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ in the following form[1]:

$$\begin{aligned} Z^{\mathsf{c}}_{g,n}(\boldsymbol{\beta}) &= \int_0^\infty \prod_{i=1}^{n} d\ell_i \frac{\ell_i}{\sqrt{2\pi\beta_i}} e^{-\frac{1}{2}\ell_i^2/\beta_i} \int_{\overline{\mathcal{M}}_{g,n}} e^{\Omega_{\mathrm{WP}}+\frac{1}{2}\psi_i\ell_i^2} && \text{(A.2.2)} \\ &= \int_{\overline{\mathcal{M}}_{g,n}} e^{\Omega_{\mathrm{WP}}} \int_0^\infty \prod_{i=1}^{n} d\ell_i \frac{\ell_i}{\sqrt{2\pi\beta_i}} \exp\left(-\frac{1}{2}\left(\beta_i^{-1} - \psi_i\right)\ell_i^2\right) && \text{(A.2.3)} \\ &= \int_{\overline{\mathcal{M}}_{g,n}} e^{\Omega_{\mathrm{WP}}} \prod_{i=1}^{n} \sqrt{\frac{\beta_i}{2\pi}}(1-\beta_i\psi_i)^{-1} . && \text{(A.2.4)} \end{aligned}$$

We can interpret this result in the following way. When we define JT gravity on hyperbolic surfaces without boundaries (allowing only marked points), we do not need to include any boundary terms in the action. So correlation functions in this theory are simply defined by integrating over the moduli space:

$$\langle \cdots \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} e^{\Omega_{\mathrm{WP}}}(\cdots) . \tag{A.2.5}$$

From (A.2.4) we now see that one can create an asymptotically AdS_2 boundary

[1] All integral manipulations with the ψ-classes should be understood formally via the Taylor series of each function in the integrand. This is well-defined because the integral over $\overline{\mathcal{M}}_{g,n}$ only picks out the d-form from the Taylor expansion, where $d = 6g-6+2n$ is the dimension of $\overline{\mathcal{M}}_{g,n}$.

of renormalized length β by inserting the 'observable'

$$Z(\beta) = \lambda\sqrt{\frac{\beta}{2\pi}}(1-\beta\psi)^{-1} \ . \tag{A.2.6}$$

This point of view is familiar in the context of 2d topological gravity [89,113,114]. Note that the observables have an extra factor of λ compared to the main text. Summing over the genus counting parameter $\lambda = e^{-S_0}$, we can thus think of the JT gravity path integral as the correlation function of 'boundary creation operators' $Z(\beta_i)$ in topological gravity:

$$\mathcal{Z}^{\mathsf{c}}_{\chi<0}(\boldsymbol{\beta}) = \langle Z(\beta_1)\cdots Z(\beta_n)\rangle \sim \sum_{\chi<0} \lambda^{2g-2}\langle Z(\beta_1)\cdots Z(\beta_n)\rangle_g \ . \tag{A.2.7}$$

The operators $Z(\beta_i)$ are called *macroscopic loop operators* in the context of matrix models, see, for instance, [264] (and more recently [110]). We will now make the notation of (A.2.7) precise, by rewriting $Z(\beta)$ as a creation operator $Z_+(\beta)$ in a bosonic Fock space.

A.2.1 Virasoro constraints

To turn $Z(\beta)$ into a differential operator, we introduce the generating function F for the intersection numbers of ψ-classes and the Weil-Petersson symplectic form[2]:

$$F(\boldsymbol{t}) = \sum_{g=0}^{\infty} \lambda^{2g-2}\int_{\overline{\mathcal{M}}_{g,n}} \exp(\Omega_{\mathrm{WP}} + \sum_i t_i\sigma_i) \ , \tag{A.2.8}$$

where the t_i are 'sources' for $\sigma_{d_i} \equiv \psi_i^{d_i}$. Expanding $Z(\beta)$ in a geometric series, we have:

$$Z(\beta) = \lambda\sqrt{\frac{\beta}{2\pi}}\sum_{k=0}^{\infty}(\psi\beta)^k = \frac{\lambda}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\sigma_k\beta^{k+\frac{1}{2}} \ . \tag{A.2.9}$$

Therefore, we can write the connected JT path integral over the stable surfaces (with $\chi < 0$) as differential operators acting on the generating function:

$$\mathcal{Z}^{\mathsf{c}}_{\chi<0}(\boldsymbol{\beta}) = Z_+(\beta_1)\cdots Z_+(\beta_n)\, F(\boldsymbol{t})\Big|_{\boldsymbol{t}=0} \ , \tag{A.2.10}$$

[2]Sometimes, the convention is to replace Ω_{WP} by κ_1. The κ-classes are related to the ψ-classes in the following way. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the map that 'forgets' the $(n+1)$-th marked point on a surface. Then we define $\kappa_d = \pi_*(\psi_{n+1}^{d+1})$ as the pushforward of ψ_{n+1}^{d+1} to $\overline{\mathcal{M}}_{g,n}$. Wolpert [68] showed the remarkably simple relation that $\Omega_{\mathrm{WP}} = 2\pi^2\kappa_1$, so the conventions differ only by a numerical factor. Furthermore, one can add a parameter s in front of Ω_{WP} to generate κ-class intersections, but for JT gravity we only need $s = 1$.

where

$$Z_+(\beta_i) = \frac{\lambda}{\sqrt{2\pi}} \sum_{k_i=0}^{\infty} \beta_i^{k_i+\frac{1}{2}} \frac{\partial}{\partial t_{k_i}} \ . \tag{A.2.11}$$

We see that $F(\boldsymbol{t})$ is the generating function for *connected* contributions to the JT path integral, and has the interpretation of a 'free energy'. Therefore, its exponent e^F has the interpretation of the full partition function of no-boundary JT gravity[3]. Acting with suitable trumpet creation operators $Z_+(\beta_i)$ on e^F thus produces both connected and disconnected contributions to the path integral. The free energy F generates just the spacetime wormholes, whereas e^F contains both wormholes and *factorized* contributions. For example, acting with $Z_+(\beta_1)Z_+(\beta_2)$ on e^F produces, graphically, the two contributions in Figure A.1.

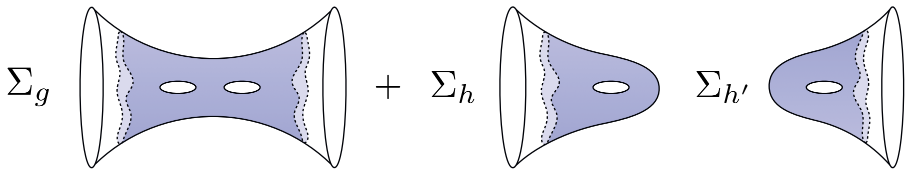


Figure A.1: *Both wormholes and factorized contributions appear in $Z_+(\beta_1)Z_+(\beta_2)\exp F$.*

A natural question is how Mirzakhani's recursion can be rephrased in the operator language that we have just described. Since $F(\boldsymbol{t})$ can be used to generate Weil-Petersson volumes, a natural guess would be that the integral recursion relation of Mirzakhani can be written in a differential version as some combination of trumpet creation operators $Z_+(\beta_i)$ acting on the partition function:

$$\widehat{\mathcal{O}}\big[\{t_i, \partial_{t_i}\}_i\big]\, e^{F(\boldsymbol{t})} = 0 \ . \tag{A.2.12}$$

The precise form of the operator $\widehat{\mathcal{O}}$ can be found in [276]. An intermediate result reads:

$$(2k+3)!!\frac{\partial F}{\partial t_{k+1}} = \sum_{i,j=0}^{\infty} \underbrace{\mathcal{F}_{ijk}\, t_j\, \frac{\partial F}{\partial t_{i+j+k}}}_{a)} \tag{A.2.13}$$

[3] With the exception of the disk and annulus.

$$+\frac{\lambda^2}{4}\sum_{i=0}^{\infty}\sum_{\substack{j_1+j_2=\\ i+k-1}}\widetilde{\mathcal{F}}_{i\,j_1j_2}\Big(\underbrace{\frac{\partial^2 F}{\partial t_{j_1}\partial t_{j_2}}}_{b)}+\underbrace{\frac{\partial F}{\partial t_{j_1}}\frac{\partial F}{\partial t_{j_2}}}_{c)}\Big)\qquad \forall\, k>0\ .$$

The coefficients are given by:

$$\mathcal{F}_{ijk}=\frac{(2(i+j+k)+1)!!}{(2j-1)!!}\widetilde{u}_i\ , \tag{A.2.14}$$

$$\widetilde{\mathcal{F}}_{i\,j_1j_2}=(2j_1+1)!!(2j_2+1)!!\,\widetilde{u}_i\ , \tag{A.2.15}$$

where the 'moduli' $\widetilde{u}_i$ are defined as the Taylor coefficients of the function[4]

$$\frac{4\pi\sqrt{x}}{\sin(2\pi\sqrt{x})}\equiv\sum_{i=0}^{\infty}\widetilde{u}_i x^i\ . \tag{A.2.16}$$

One proceeds by rewriting (A.2.13) in terms of the partition function e^F, and rearranging terms into a single operator acting on e^F. To get rid of the double factorials, we rescale the source parameters t_i as:

$$\mathfrak{t}_{2i+1}\equiv\frac{t_i}{(2i+1)!!}\ . \tag{A.2.17}$$

Including terms for $k=-1,0$ (corresponding to the base cases $V_{0,3}$ and $V_{1,1}$ in Mirzhankani's recursion), equation (A.2.13) is rewritten as:

$$\mathcal{L}_k e^F=0\ ,\qquad \forall k\geq -1\ , \tag{A.2.18}$$

where the differential operators $\mathcal{L}_k$ are given by:

$$\mathcal{L}_k=-\underbrace{\frac{1}{2}\frac{\partial}{\partial \mathfrak{t}_{2k+3}}}_{\text{LHS}}+\underbrace{\left(\frac{\mathfrak{t}_1^2}{\lambda^2}+\frac{\pi^2}{12}\right)\delta_{k,-1}}_{(1,1)}+\underbrace{\frac{\delta_{k,0}}{8}}_{(0,3)} \tag{A.2.19}$$
$$+\frac{1}{2}\sum_{i,j=0}^{\infty}{}'(2j+1)\widetilde{u}_i\,\mathfrak{t}_{2j+1}\frac{\partial}{\partial\mathfrak{t}_{2(i+j+k)+1}}+\frac{\lambda^2}{8}\sum_{i=0}^{\infty}\sum_{\substack{j_1+j_2=\\ i+k-1}}\widetilde{u}_i\frac{\partial^2}{\partial\mathfrak{t}_{2j_1+1}\partial\mathfrak{t}_{2j_2+1}}\ .$$

The first term, labelled LHS, comes from the left-hand side of (A.2.13). The next two terms, denoted by $(1,1)$ and $(0,3)$, arise from the torus with one hole and the pair-of-pants, respectively. The prime indicates that the term with $i=j=0$, $k=-1$ is excluded from that sum. The rest is just a rewriting of the right-hand

[4] A closed form can be given in terms of the Bernoulli number B_{2i} by $\widetilde{u}_i=(-1)^{i-1}(2\pi)^{2i}(2^{2i+1}-4)\frac{B_{2i}}{(2i)!}$.

side of (A.2.13).

We would like to study the algebra associated to the infinite tower of differential equations imposed by $\{\mathcal{L}_k\}_{k\geq -1}$. The commutation relations are given by

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\sum_{i=0}^{\infty} \widetilde{u}_i \mathcal{L}_{m+n+i} \ . \tag{A.2.20}$$

We now apply the simple transformation:

$$\widetilde{L}_k \equiv \sum_{i=0}^{\infty} u_i \mathcal{L}_{k+i} \ , \tag{A.2.21}$$

where u_i are the reciprocal coefficients of $\widetilde{u}_i$, defined by:

$$\frac{\sin(2\pi\sqrt{x})}{4\pi\sqrt{x}} \equiv \sum_{i=0}^{\infty} u_i x^i \ . \tag{A.2.22}$$

One can show that the algebra spanned by the operators $\{\widetilde{L}_k\}$ with $k \geq -1$ is the Virasoro algebra

$$\left[\widetilde{L}_m, \widetilde{L}_n\right] = (m-n)\widetilde{L}_{m+n} \ . \tag{A.2.23}$$

The condition in (A.2.18) is therefore referred to as a *Virasoro constraint*. The structure underlying Mirzakhani's recursion relation is a Virasoro symmetry, which expresses an underlying integrable structure, closely related to the Korteweg-de-Vries (KdV) hierarchy. For more background on its relation to intersection theory on the moduli space of Riemann surfaces, see [89, 114, 276, 277].

A.2.2 Chiral boson with a $\mathbb{Z}_2$ twist

A crucial role in deriving a bosonic theory that describes JT gravity is played by the Laplace transform. Consider the boundary creation operator $Z_+(\beta)$, written in terms of the rescaled parameter $\mathfrak{t}_{2k+1}$:

$$Z_+(\beta) = \frac{\lambda}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{\beta^{k+\frac{1}{2}}}{(2k+1)!!} \frac{\partial}{\partial \mathfrak{t}_{2k+1}} \ . \tag{A.2.24}$$

Using the identity $(2k+1)!! = \frac{2^{k+1}}{\sqrt{\pi}}\Gamma(k+3/2)$, (and setting back $\phi_r = 1/2$, as was used in the main text) we find that the Laplace transform of $Z_+(\beta)$ can be written as:

$$\int_0^{\infty} d\beta\, Z_+(\beta)\, e^{-\beta x} = \frac{\lambda}{2} \sum_{k=0}^{\infty} x^{-k-3/2} \frac{\partial}{\partial \mathfrak{t}_{2k+1}} \ . \tag{A.2.25}$$

Since β has the interpretation of a boundary length it is naturally defined on the positive real axis. The dual variable x can then be taken as a complex 'frequency'. The half-integer powers of x appearing in (A.2.25), show that the coordinate x is only defined on the complex plane with a branch cut, the spectral plane. We will choose the convention that the branch cut lies on the negative real axis. When traversing a rotation of 2π around the branch point, $\sqrt{x}$ picks up a minus sign.

We want to interpret the Laplace transformed operator (A.2.25) as some complex scalar field $\Phi(x)$. By the above argument it should have anti-periodic boundary conditions across the branch cut:

$$\Phi(e^{2\pi i}x) = -\Phi(x) \ . \tag{A.2.26}$$

Therefore, it should be a $\mathbb{Z}_2$-*twisted boson*. To make the correspondence precise, we introduce the following creation and annihilation operators for $k \geq 0$:

$$\alpha_{k+\frac{1}{2}} = \frac{\lambda}{2}\frac{\partial}{\partial \mathsf{t}_{2k+1}} \ , \qquad \alpha_{-k-\frac{1}{2}} = \frac{2}{\lambda}\left(k+\frac{1}{2}\right)\mathsf{t}_{2k+1} \ . \tag{A.2.27}$$

These oscillators generate a representation of a twisted Heisenberg algebra. Namely, evaluating their commutator gives:

$$\left[\alpha_n, \alpha_m\right] = n\delta_{n+m,0} \ , \qquad n, m \in \mathbb{Z} + \frac{1}{2} \ . \tag{A.2.28}$$

The twisted vacuum state $|\sigma\rangle$ is defined by requiring that:

$$\alpha_{k+\frac{1}{2}}\left|\sigma\right\rangle = 0 \ , \quad \left\langle\sigma\right|\alpha_{-k-\frac{1}{2}} = 0 \ , \qquad \forall k \geq 0 \ . \tag{A.2.29}$$

It can be related to the vacuum of the untwisted free boson by the insertion of a twist operator $\sigma(x)$ [278, 279] at the origin and infinity:

$$\left|\sigma\right\rangle = \sigma(0)\left|0\right\rangle \ , \quad \left\langle\sigma\right| = \left\langle 0\right|\sigma(\infty) \ . \tag{A.2.30}$$

The derivative of the field can be expanded in half-integer powers of x as:

$$\partial\Phi(x) = \sum_{k\in\mathbb{Z}} \alpha_{-k-\frac{1}{2}} x^{k-\frac{1}{2}} \ . \tag{A.2.31}$$

We split $\partial\Phi(x)$ into positive and negative modes:

$$\partial\Phi(x) = \partial\Phi_-(x) + \partial\Phi_+(x) = \sum_{k=0}^{\infty} \alpha_{-k-\frac{1}{2}} x^{k-\frac{1}{2}} + \sum_{k=0}^{\infty} \alpha_{k+\frac{1}{2}} x^{-k-\frac{3}{2}} \ . \tag{A.2.32}$$

Then, we recognize the Laplace transform of the trumpet operator $Z_+(\beta)$ as the positive frequency part of $\partial\Phi(x)$:

$$\int_0^\infty d\beta\, Z_+(\beta)\, e^{-\beta x} = \partial\Phi_+(x)\ . \tag{A.2.33}$$

We see that adding a trumpet boundary of length β_i in JT gravity corresponds to inserting $\partial\Phi_+(x_i)$ at a point x_i on the spectral plane, where x_i and β_i are related by the Laplace transform. We now want to relate the Virasoro constraints (A.2.18), which we found to be equivalent to Mirzakhani's recursion, to the stress tensor of the twisted boson.

The stress tensor $T(x)$ of the twisted theory is constructed via a normal ordering prescription:

$$T(x) = \frac{1}{2}\big\{\partial\Phi\partial\Phi\big\}(x) \equiv \frac{1}{2}\lim_{y\to x}\left(\partial\Phi(x)\partial\Phi(y) - \frac{1}{(x-y)^2}\right). \tag{A.2.34}$$

Note that there are two notions of normal ordering. Firstly, there is the normal ordering at the level of modes, which puts all $\alpha_{-n-\frac{1}{2}}$ to the left of the $\alpha_{n+\frac{1}{2}}$, where $n \geq 0$. This respects the twisted vacuum $|\sigma\rangle$, and will be denoted by colons $:\cdots:$. Secondly, there is the normal ordering which subtracts the singular piece from the operator product expansion. This will be denoted by brackets $\big\{\cdots\big\}$.

The two-point function in the twisted vacuum is easily computed to be

$$\begin{aligned}\langle\partial\Phi(x)\partial\Phi(y)\rangle_\sigma &= \sum_{k=0}^\infty\sum_{n=0}^\infty \langle\sigma|\big[\alpha_{k+\frac{1}{2}},\alpha_{-n-\frac{1}{2}}\big]|\sigma\rangle\, x^{-k-3/2}y^{n-1/2} \\ &= \sum_{k=0}^\infty\left(k+\frac{1}{2}\right)x^{-k-3/2}y^{k-1/2} = \frac{1}{2}\frac{\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}}{(x-y)^2}\ .\end{aligned} \tag{A.2.35}$$

This has the correct antiperiodicity in both x and y, and it also exhibits the OPE singularity $\langle\partial\Phi(x)\partial\Phi(y)\rangle \sim (x-y)^{-2}$, as $x \to y$. From the definition of the stress tensor we find that

$$\langle T(x)\rangle_\sigma = \frac{1}{2}\lim_{y\to x}\left\langle\partial\Phi(x)\partial\Phi(y) - \frac{1}{(x-y)^2}\right\rangle_\sigma = \frac{1}{16x^2}\ . \tag{A.2.36}$$

Comparing to the other normal ordering prescription, whose expectation value is zero in the twisted vacuum by construction, we have

$$T(x) = \frac{1}{2}:\partial\Phi(x)\partial\Phi(x): + \frac{1}{16x^2}\ . \tag{A.2.37}$$

The stress tensor has the following mode expansion:

$$T(x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2} \;, \tag{A.2.38}$$

where, for $n \geq -1$, we have

$$L_{-1} = \sum_{k=1}^{\infty} \alpha_{-k-\frac{1}{2}}\alpha_{k-\frac{1}{2}} + \frac{1}{2}(\alpha_{-\frac{1}{2}})^2 \;, \tag{A.2.39}$$

$$L_0 = \sum_{k=0}^{\infty} \alpha_{-k-\frac{1}{2}}\alpha_{k+\frac{1}{2}} + \frac{1}{16} \;, \tag{A.2.40}$$

$$L_{n>0} = \sum_{k=0}^{\infty} \alpha_{-k-\frac{1}{2}}\alpha_{n+k+\frac{1}{2}} + \frac{1}{2}\sum_{k=0}^{n-1} \alpha_{k+\frac{1}{2}}\alpha_{n-k-\frac{1}{2}} \;. \tag{A.2.41}$$

These operators are related to the operators $\widetilde{L}_n$ in (A.2.21) in the following way:

$$\widetilde{L}_n = L_n - \frac{1}{\lambda}\sum_{k=0}^{\infty} u_k \alpha_{k+n+\frac{3}{2}} \;. \tag{A.2.42}$$

To prove this, we rewrite the differential operators $\mathcal{L}_k$ of equation (A.2.19) in terms of the twisted creation and annihilation operators:

$$\begin{aligned}\mathcal{L}_k = &\underbrace{-\frac{1}{\lambda}\alpha_{k+\frac{3}{2}}}_{\text{LHS}} + \underbrace{\left[(\alpha_{-\frac{1}{2}})^2 + \frac{\pi^2}{12}\right]}_{(1,1)}\delta_{k,-1} + \underbrace{\frac{\delta_{k,0}}{8}}_{(0,3)} \\ &+ \sum_{i=0}^{\infty}\widetilde{u}_i\Big[{\sum_{j=0}^{\infty}}' \alpha_{-j-\frac{1}{2}}\alpha_{k+i+j+\frac{1}{2}} + \frac{1}{2}\sum_{\substack{j_1+j_2\\=k+i-1}} \alpha_{j_1+\frac{1}{2}}\alpha_{j_2+\frac{1}{2}}\Big] \;.\end{aligned} \tag{A.2.43}$$

We can incoorporate the terms multiplying $\delta_{k,0}$ and $\delta_{k,-1}$ into the sum over i, by realizing that $\delta_{k+i,-1}$ is only nonzero for $k=-1$ and $i=0$, while $\delta_{k+i,0}$ is nonzero for both $k=i=0$ and $k=-1, i=1$. Using the values for $\widetilde{u}_0 = 2$ and $\widetilde{u}_1 = \frac{4\pi^2}{3}$, we then find for $k \geq -1$:

$$\sum_{i=0}^{\infty}\widetilde{u}_i\left[\frac{1}{2}(\alpha_{-\frac{1}{2}})^2\delta_{k+i,-1} + \frac{\delta_{k+i,0}}{16}\right] = \underbrace{\left[(\alpha_{-\frac{1}{2}})^2 + \frac{\pi^2}{12}\right]}_{(1,1)}\delta_{k,-1} + \underbrace{\frac{\delta_{k,0}}{8}}_{(0,3)} \;. \tag{A.2.44}$$

Therefore, we recognize the operators $\mathcal{L}_k$ to be a simple transformation of the stress tensor modes:

$$\mathcal{L}_k = -\frac{1}{\lambda}\alpha_{k+\frac{3}{2}} + \sum_{i=0}^{\infty} \widetilde{u}_i L_{k+i} \ . \tag{A.2.45}$$

Recalling the definition of the Virasoro operators $\widetilde{L}_n$, we thus prove equation (A.2.42):

$$\widetilde{L}_n = \sum_{k=0}^{\infty} u_k \mathcal{L}_{k+n} = L_n - \frac{1}{\lambda}\sum_{k=0}^{\infty} u_k \alpha_{k+n+\frac{3}{2}} \ , \tag{A.2.46}$$

where we used the fact that u_i and $\widetilde{u}_i$ are reciprocal coefficients, $\sum_{i=0}^{k} u_i \widetilde{u}_{k-i} = \delta_{k,0}$.

We use the moduli u_i to construct the following function:

$$\omega(x) = \frac{1}{\lambda}\sum_{k=0}^{\infty} u_k x^{k+\frac{1}{2}} = \frac{1}{4\pi\lambda}\sin(2\pi\sqrt{x}) \ . \tag{A.2.47}$$

We then define the shifted stress tensor $T_\omega(x)$ by translating $\partial\Phi(x) \to \widetilde{\partial\Phi}(x) = \partial\Phi(x) - \omega(x)$:

$$T_\omega(x) \equiv \frac{1}{2}\{\widetilde{\partial\Phi}\,\widetilde{\partial\Phi}\}(x) = T(x) - \omega(x)\partial\Phi(x) + \frac{1}{2}\omega(x)^2 \ . \tag{A.2.48}$$

It has a mode expansion

$$T_\omega(x) = \sum_{n\in\mathbb{Z}} \widetilde{L}_n x^{-n-2} \ , \tag{A.2.49}$$

where the modes with $n \geq -1$ are precisely the Virasoro operators (A.2.42):

$$\widetilde{L}_n = \oint_0 \frac{dx}{2\pi i} x^{n+1} T_\omega(x) = \oint_0 \frac{dx}{2\pi i} x^{n+1} T(x) - \oint_0 \frac{dx}{2\pi i} x^{n+1}\omega(x)\partial\Phi(x) \tag{A.2.50}$$

$$= L_n - \frac{1}{\lambda}\sum_{k,j=0}^{\infty} u_k\, \alpha_{-j-\frac{1}{2}} \oint_0 \frac{dx}{2\pi i} x^{n+k+j+1} \tag{A.2.51}$$

$$= L_n - \frac{1}{\lambda}\sum_{k=0}^{\infty} u_k \alpha_{k+n+\frac{3}{2}} \ . \tag{A.2.52}$$

We can think of $\omega(x)$ as giving $\widetilde{\partial\Phi}(x)$ a vacuum expectation value. Note that this VEV was chosen in a particular way to match Mirzakhani's recursion, but more generally we could take any set of moduli u_i, and treat $\omega(x)$ as a formal power series. The Virasoro constraints may then still have a geometric interpretation[5], although it will not describe JT gravity. Indeed, various choices of $\omega(x)$ have been related to the generalized Kontsevich-Witten model [282, 283], topological gravity on arbitrary backgrounds [112, 114] and minimal models [81, 133].

We write the generating function in the coherent state basis:

$$e^{F(\boldsymbol{t})} = \langle t|\Sigma\rangle \ , \tag{A.2.53}$$

where the coherent state is expressed as $\langle t| = \langle\sigma|\, e^V$ with

$$V = \frac{2}{\lambda}\sum_{k=0}^{\infty} \mathsf{t}_{2k+1}\alpha_{k+\frac{1}{2}} \ . \tag{A.2.54}$$

The coherent state $\langle t|$ is a left eigenstate of the annihilation operator $\alpha_{-n-\frac{1}{2}}$:

$$\langle t|\,\alpha_{-n-\frac{1}{2}} = \lambda^{-1}(2n+1)\mathsf{t}_{2n+1}\,\langle t| \ . \tag{A.2.55}$$

Therefore, when acting on e^F with an operator $\mathcal{O}(\mathsf{t}_n, \partial_{\mathsf{t}_n})$, we can bring it inside the 'expectation value' $\langle t|\ldots|\Sigma\rangle$ by converting it into oscillator language $\mathcal{O}(\alpha_{-n}, \alpha_n)$. The precise relation is given by (A.2.27). The state $|\Sigma\rangle$ is then fully determined by the Virasoro constraint:

$$\widetilde{L}_n e^F = 0 \qquad \Longleftrightarrow \qquad \widetilde{L}_n\,|\Sigma\rangle = 0, \qquad n \geq -1 \ , \tag{A.2.56}$$

where on the left-hand side it is implied that $\widetilde{L}_n$ is written in terms of t and $\frac{\partial}{\partial \mathsf{t}}$, and on the right-hand side $\widetilde{L}_n$ is expressed in terms of twisted bosonic oscillators.

Looking at the mode expansion (A.2.49) of $T_\omega(x)$, we see that the modes with $n \geq -1$ correspond to the negative powers of x. So instead of the infinite number of equations imposed by (A.2.56), we can write the Virasoro constraint as a single requirement that the expectation value of $T_\omega(x)$ is non-singular as $x \to 0$:

$$\langle t|T_\omega(x)|\Sigma\rangle = \text{analytic} \ . \tag{A.2.57}$$

It is simply the requirement that the theory is conformally invariant at the quantum level. This is a non-trivial requirement, because in terms of the spectral variable x, the twisted boson $\widetilde{\partial\Phi}(x)$ has a branch point at the origin. The branch point breaks the conformal invariance, and the modes of $\partial\Phi$ should be 'dressed' to restore conformal invariance [283]. The dressing is defined through the new

[5] In fact, much recent progress in this direction has been made in [280], where so-called *quantum Airy structures* are introduced as a generalizations of the Virasoro constraints (for an introduction to the subject, see [281]).

$SL(2,\mathbb{C})$-invariant state $|\Sigma\rangle$, which we can formally write as some 'dressing operator' $e^{\lambda\hat{S}}|\sigma\rangle$ acting on the twisted vacuum. We would like to think of $\hat{S}$ as an interaction term in an interacting theory, with coupling constant λ, which perturbs the free theory.

In the case of the so-called *topological point*, for which $\omega(x) \sim \sqrt{x}$, such an operator $\hat{S}$ was explicitly constructed in [284]. It was found to be cubic in the bosonic oscillators α_k. For general $\omega(x)$, one can always use appropriate shift operators V_ω to obtain the solution:

$$|\Sigma\rangle = e^{V_\omega} e^{\lambda\hat{S}}|\sigma\rangle \ . \tag{A.2.58}$$

A.2.3 Back to JT gravity

We have seen that the full JT path integral on connected stable surfaces with n boundaries could be obtained by acting n times with a boundary creation operator $Z_+(\beta_i)$ on the free energy $F(\boldsymbol{t})$. After an n-fold Laplace transform this can be written as

$$\int_0^\infty \prod_{i=0}^n d\beta_i e^{-\beta_i x_i} \mathcal{Z}^{\mathfrak{c}}_{\chi<0}(\boldsymbol{\beta}) = \partial\Phi_+(x_1)\cdots\partial\Phi_+(x_n)F(\mathfrak{t})\Big|_{\mathfrak{t}=0} \ . \tag{A.2.59}$$

We can use the coherent state $\langle t|$ to bring $\partial\Phi_+(x)$ inside the correlation function $\langle t|\ldots|\Sigma\rangle$. This puts all the dependence on the sources into $\langle t|$, and so setting $\mathfrak{t}=0$ boils down to replacing $\langle t|$ by the vacuum $\langle\sigma|$. For the JT gravity path integral, including connected and disconnected spacetimes, this implies:

$$\int_0^\infty \prod_{i=0}^n d\beta_i e^{-\beta_i x_i} \mathcal{Z}_{\chi<0}(\boldsymbol{\beta}) = \frac{\langle\sigma|\partial\Phi_+(x_1)\cdots\partial\Phi_+(x_n)|\Sigma\rangle}{\langle\sigma|\Sigma\rangle} \ . \tag{A.2.60}$$

Here, we divided by the normalization factor $\langle\sigma|\Sigma\rangle$, which has the effect of excluding the contributions from JT universes without boundaries. Note that the dependence on the background $\omega(x)$ is encoded in the state $|\Sigma\rangle$.

Let us interpret this formula from the point of view of the baby universe Hilbert space. The operator insertions of $\partial\Phi(x)$ are radially ordered, and so one may think of equation (A.2.60) as an operator representation of the JT path integral in *radial quantization*. So we should read it from right to left: one in the twisted vacuum $|\sigma\rangle$, which can be thought of as a type of 'Big Bang' for a (possibly disconnected) universe. In particular, the initial state has no boundaries. Then, there is a complicated splitting and joining of baby universes, determined by the interaction $\hat{S}$ in $|\Sigma\rangle = e^{\lambda\hat{S}}|\sigma\rangle$. The requirement that these processes respect modular invariance is imposed by the Virasoro constraint on $|\Sigma\rangle$. Finally, the trumpet boundaries are glued to the spacetime prepared by $|\Sigma\rangle$. The gluing is represented by the overlap

$$\mathcal{Z}_{\chi<0}(\boldsymbol{\beta}) = \langle \beta_1, \ldots, \beta_n | \Sigma \rangle \,, \tag{A.2.61}$$

where the normalized n-trumpet state $\langle \boldsymbol{\beta}|$ is given by:

$$\langle \beta_1, \ldots, \beta_n | = \frac{\langle \sigma | \, Z_+(\beta_1) \cdots Z_+(\beta_n)}{\langle \sigma | \Sigma \rangle} \,. \tag{A.2.62}$$

This intuitive interpretation of the formula (A.2.60) is depicted in Figure A.2. It corresponds to the particular (Euclidean) time slicing in the universe field theory in which all the asymptotically AdS_2 boundaries are in the infinite future. The upshot of this (Euclidean) time-slicing is that it is manifestly invariant under the large diffeomorphisms. However, the downside is that the in and the out-state are treated asymmetrically, contrary to the proposal of [100]: the state $|\Sigma\rangle$ is prepared from one side. Therefore, we could refer to the Hilbert space generated by the states above as the *one-sided* baby universe Hilbert space.

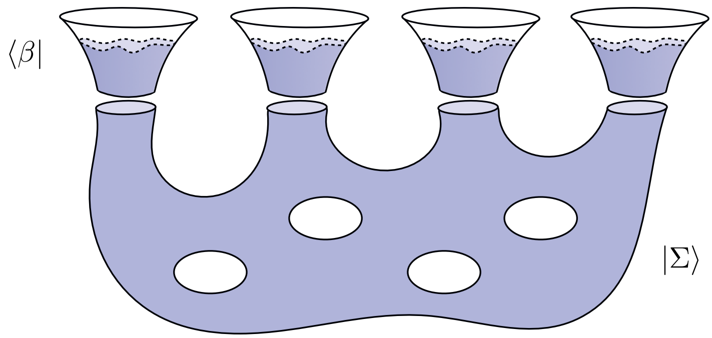


Figure A.2: *An example inner product in the one-sided baby universe Hilbert space: a genus 4 Riemann surface is prepared by $|\Sigma\rangle$, and the overlap is computed with a 4-trumpet state.*

A.2.4 The disk and the annulus

So far, we have mainly focused on the stable ($\chi < 0$) hyperbolic surfaces. However, there are two special contributions to the Euclidean path integral coming from the disk, which represents the classical solution to the JT equations of motion, and the double-trumpet.

The double-trumpet corresponds after Laplace transform to the free two-point function that we computed in (A.2.35), and the disk corresponds to $\omega(x)$, which we interpreted as the classical VEV of the twisted boson. To get the full (disconnected) JT path integral, one should consider expectation values of insertions of the full $\widetilde{\partial\Phi}(x)$ on the spectral plane. To implement the shift by $\omega(x)$ we introduce a shift operator:

$$V_\omega = \oint_0 \frac{dx}{2\pi i}\Phi(x)\omega(x) = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{u_k}{k+\frac{3}{2}}\alpha_{k+\frac{3}{2}} \ . \tag{A.2.63}$$

We see that translation of $\partial\Phi$ by ω is implemented by conjugating with $\exp V_\omega$:

$$e^{-V_\omega}\partial\Phi(y)\, e^{V_\omega} = \partial\Phi(y) - [V_\omega, \partial\Phi(y)] = \partial\Phi(y) - \omega(y) = \widetilde{\partial\Phi}(y) \ . \tag{A.2.64}$$

We can use the shift operator to move around the dependence on $\omega(x)$ inside correlation functions:

$$\langle\sigma|\widetilde{\partial\Phi}(x_1)\cdots\widetilde{\partial\Phi}(x_n)|\Sigma\rangle = \langle\sigma|e^{-V_\omega}\partial\Phi(x_1)\cdots\partial\Phi(x_n)e^{V_\omega}|\Sigma\rangle \ . \tag{A.2.65}$$

Denoting

$$\langle\omega| = \langle\sigma|\, e^{-V_\omega} \quad \text{and} \quad |\Sigma_0\rangle = e^{V_\omega}\,|\Sigma\rangle \ , \tag{A.2.66}$$

the full n-boundary JT gravity partition function, including disk and annulus contributions, can be written as the following integral transform of an n-point correlation function of twisted bosons on the spectral plane:

$$\mathcal{Z}(\boldsymbol{\beta}) = \int_{c-i\infty}^{c+i\infty}\prod_{i=1}^{n}\frac{dx_i}{2\pi i}e^{\beta_i x_i}\frac{\langle\omega|\partial\Phi(x_1)\cdots\partial\Phi(x_n)|\Sigma_0\rangle}{\langle\omega|\Sigma_0\rangle} \ . \tag{A.2.67}$$

For the positive frequencies $\partial\Phi_+$, the formula coincides with equation (A.2.60), because e^{V_ω} commutes with $\partial\Phi_+$. For the negative frequencies, however, we get precisely the disk and annulus contributions that were missing in (A.2.60).

We illustrate this with an instructive example. Consider the two-point function

$$\langle\partial\Phi(x)\partial\Phi(y)\rangle_\Sigma \equiv \frac{\langle\omega|\partial\Phi(x)\partial\Phi(y)|\Sigma_0\rangle}{\langle\omega|\Sigma_0\rangle} \ . \tag{A.2.68}$$

If we split $\partial\Phi$ into positive and negative frequencies, and commute all negative modes to the left, we get four contributions. Using that $\langle\omega|$ is a left eigenstate of

$\partial\Phi_-$ with eigenvalue ω, and computing the commutator

$$\left[\partial\Phi_+(x), \partial\Phi_-(y)\right] = \frac{1}{2}\frac{\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}}{(x-y)^2}\ , \tag{A.2.69}$$

we find for the normalized full two-point function:

$$\begin{aligned}\frac{\langle\omega|\partial\Phi(x)\partial\Phi(y)|\Sigma_0\rangle}{\langle\omega|\Sigma_0\rangle} &= \underbrace{\omega(x)\omega(y)}_{1} + \underbrace{\omega(x)\left\langle\partial\Phi_+(y)\right\rangle_\Sigma + \left\langle\partial\Phi_+(x)\right\rangle_\Sigma \omega(y)}_{2} \\ &\quad + \underbrace{\left\langle\partial\Phi(x)\partial\Phi(y)\right\rangle_\sigma}_{3} + \underbrace{\left\langle\partial\Phi_+(x)\partial\Phi_+(y)\right\rangle_\Sigma}_{4}\ .\end{aligned} \tag{A.2.70}$$

The terms have been labelled by the type of geometry that they represent in the JT path integral, see Figure A.3.

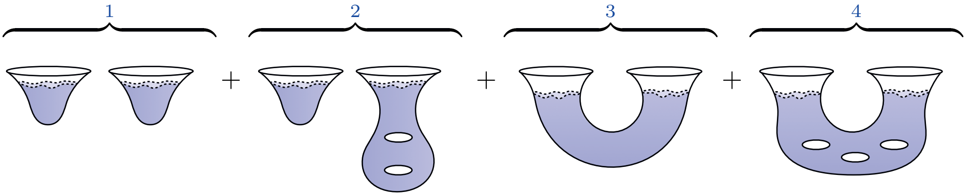


Figure A.3: *The terms in* (A.2.70) *correspond to distinct geometries in the JT path integral. Term* 1 *corresponds to two disks. Term* 2 *is the disconnected contribution of a disk and a sum over genus $g>0$ Riemann surfaces with one trumpet boundary. Term* 3 *is the genus zero wormhole contribution, the double trumpet. And* 4 *is the sum over connected stable spacetime wormholes, $\mathcal{Z}_{\chi<0}(\beta_1,\beta_2)$, with two trumpet boundaries.*

Let us now explicitly check that this correctly reproduces the JT gravity results. For the boundary creation operators, we have already seen that the Laplace transform of $Z_+(\beta)$ gives $\partial\Phi_+(x)$, and so the integral transform

$$\partial\Phi_+ \mapsto \int_{c-i\infty}^{c+i\infty}\frac{dx}{2\pi i}\, e^{\beta x}\partial\Phi_+(x) \tag{A.2.71}$$

is the standard inverse Laplace transform. However, there is no function[6] whose Laplace transform gives the negative modes $\partial\Phi_-(x)$. Nonetheless, the integral transform in (A.2.71) may still exist. In such a case, we will still call it an 'inverse Laplace transform', even though it is not actually the inverse of a convergent Laplace transform.

[6]It can only be defined in a distributional sense, with the help of delta functions.

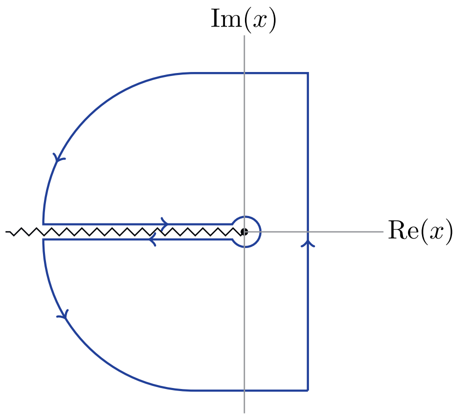


Figure A.4: *The keyhole contour for evaluating the inverse Laplace transform of $\omega(x)$.*

The disk. Indeed, we want to compute the integral

$$\int_{c-i\infty}^{c+i\infty} \frac{dx}{2\pi i}\, e^{\beta x}\omega(x) = \int_{c-i\infty}^{c+i\infty} \frac{dx}{2\pi i}\, e^{\beta x}\frac{\sin(2\pi\sqrt{x})}{4\pi\lambda} \,. \tag{A.2.72}$$

Since we chose the branch cut of $\omega(x)$ to lie on the negative real axis, $e^{\beta x}$ should grow in the right-half plane[7]. So the condition is that $\beta > 0$. The small offset $0 < c \ll 1$ is included to avoid the branch point at $x = 0$. To compute the integral over the offset imaginary axis, we then close the contour in the counter-clockwise direction into a so-called *keyhole contour*, shown in Figure A.4.

Since the integrand is holomorphic inside the keyhole contour, the total contour integral should vanish. As we take the radius of the outer arcs to infinity, the radius of the inner arc to zero, and c to zero, the only non-zero contributions to the integral come from the path along $(-i\infty, i\infty)$ and the two paths just above and below the negative real axis. Therefore, we can express the integral (A.2.72) as

$$\int_{c-i\infty}^{c+i\infty} \frac{dx}{2\pi i}\, e^{\beta x}\omega(x) = \int_0^{\infty} \frac{dx}{2\pi i} e^{-\beta x}\left(\omega(-x+i\epsilon) - \omega(-x-i\epsilon)\right) \,, \tag{A.2.73}$$

where have sent $x \to -x$. Now as we take $\epsilon \to 0$, the second term gets an extra minus sign from the discontinuity of the square root across the branch cut. Using

[7] For the integral to be non-zero, the exponential $e^{\beta x}$ needs to grow in the region of the complex plane where the integrand is holomorphic. If this were not the case, we could close the integration contour in that region and conclude that the integral would be zero.

that $\sin(2\pi i\sqrt{x}) = i\sinh(2\pi\sqrt{x})$, we obtain:

$$\int_{c-i\infty}^{c+i\infty} \frac{dx}{2\pi i}\, e^{\beta x}\omega(x) = 2i\int_0^\infty \frac{dx}{2\pi i} e^{-\beta x}\frac{\sinh(2\pi\sqrt{x})}{4\pi\lambda}\ . \tag{A.2.74}$$

This integral can be evaluated using Gaussian integration. Upon setting $x = z^2$, and writing $\lambda^{-1} = e^{S_0}$, we precisely retrieve the disk partition function:

$$\int_{c-i\infty}^{c+i\infty} \frac{dx}{2\pi i}\, e^{\beta x}\omega(x) = \frac{e^{S_0}}{4\pi^{1/2}\beta^{3/2}} e^{\pi^2/\beta} = Z_{\mathsf{disk}}(\beta)\ . \tag{A.2.75}$$

The annulus. We now show that (A.2.67) reproduces the correct result for the double-trumpet partition function. We want to compute

$$\begin{aligned}\int_{c-i\infty}^{c+i\infty} &\frac{dx}{2\pi i}\frac{dy}{2\pi i} e^{\beta_1 x+\beta_2 y}\,\langle\partial\Phi(x)\partial\Phi(y)\rangle_\sigma \\ &= \int_{c-i\infty}^{c+i\infty} \frac{dx}{2\pi i}\frac{dy}{2\pi i} e^{\beta_1 x+\beta_2 y}\frac{\frac{1}{2}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)}{(x-y)^2}\ .\end{aligned} \tag{A.2.76}$$

Again we can close both the x- and y-contours in the left-half plane via a keyhole contour. For a fixed value of x, the y-contour may enclose a pole at $x = y$, but the residue of this pole is zero:

$$\mathrm{Res}_{x\to y}\langle\partial\Phi(x)\partial\Phi(y)\rangle_\sigma = \lim_{x\to y}\frac{1}{4}\frac{\partial}{\partial x}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right) = 0\ . \tag{A.2.77}$$

So we can express the integral along the imaginary axis in terms of the discontinuity across the negative real axis. Making the substitution $x = -z^2$, $y = -w^2$, we then find:

$$\begin{aligned}\int_{c-i\infty}^{c+i\infty} &\frac{dx}{2\pi i}\frac{dy}{2\pi i} e^{\beta_1 x+\beta_2 y}\,\langle\partial\Phi(x)\partial\Phi(y)\rangle_\sigma \\ &= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} dz dw\, e^{-\beta_1 z^2-\beta_2 w^2}\frac{z^2+w^2}{(z^2-w^2)^2}\ .\end{aligned} \tag{A.2.78}$$

We can split the integrand as a sum of two terms

$$\frac{z^2+w^2}{(z^2-w^2)^2} = \frac{1}{2}\left[\frac{1}{(z-w)^2}+\frac{1}{(z+w)^2}\right]\ , \tag{A.2.79}$$

and then notice that both terms give the same integral, upon sending $w\to -w$ in the second term. Expanding $(z-w)^{-2}$ as a power series, it can be shown that the double integral in (A.2.78) precisely gives the double trumpet partition function

$Z^{\mathsf{c}}_{0,2}(\beta_1, \beta_2)$.

This completes the proof of the dictionary (A.2.67) between the twisted boson and the full n-boundary JT gravity path integral.

A.2.5 Topological recursion in the twisted boson formalism

Using our dictionary between $\mathcal{Z}^{\mathsf{c}}$ and operator insertions $\langle \prod_{i=1}^{n} \partial\Phi(x_i) \rangle_{\Sigma}$, we derive that the symplectic invariants $\omega_{g,n}$ can be expressed as the following correlation functions:

$$\mathcal{W}_{g,n}(\boldsymbol{z}) = \langle \partial\Phi(z_1) \cdots \partial\Phi(z_n) \rangle^{(g)}_{\Sigma,\mathsf{c}} \ , \tag{A.2.80}$$

for $(g, n) \neq (0, 2)$[8]. Here, the subscript c means that we take the connected correlation function, and the superscript g denotes the order λ^{2g-2} contribution. We have summarized the relations between the various quantities in Figure A.5.

$$\begin{tikzcd}
V_{g,n}(\boldsymbol{\ell}) \arrow[r, "\text{Laplace}"] \arrow[d, "\text{Trumpet}"'] & \mathcal{W}_{g,n}(\boldsymbol{z}) \\
Z_{g,n}(\boldsymbol{\beta}) \arrow[r, "\text{Laplace}"] & \left\langle \prod_{i=1}^{n} \partial\Phi(x_i) \right\rangle^{(g)}_{\Sigma} \arrow[u, "\text{Double cover}"']
\end{tikzcd}$$

Figure A.5: *Relations between Weil-Peterson volumes $V_{g,n}$, symplectic invariants $\mathcal{W}_{g,n}$, twisted boson correlators, and Dirichlet-Dirichlet JT partition functions.*

The operators $\partial\Phi(z)$ are related to the twisted boson on the spectral plane $\partial\Phi(x)$ by substituting $x = z^2$ and performing a coordinate transformation as 1-forms:

$$\partial\Phi(z) dz = \partial\Phi(x) dx \ . \tag{A.2.82}$$

Since $\Phi(x)$ is expanded in half-integer powers of x, $\Phi(z)$ is expanded in odd powers of z. Taking into account the extra factor of z from $dx = 2z dz$, we see that $\partial\Phi(z)$ is even.

To make contact with the Virasoro constraints, we will rewrite the topological

[8] In defining $\mathcal{W}_{0,2}$ we subtract the singular part of the OPE

$$\mathcal{W}_{0,2}(z, w) = \langle \partial\Phi(z)\partial\Phi(w) \rangle^{(g=0)}_{\Sigma,\mathsf{c}} - \frac{1}{(z-w)^2} \ . \tag{A.2.81}$$

This finite renormalization is directly related to the normal ordering prescription of the twisted stress tensor.

recursion in a more transparent way. First, we introduce a generating function for insertions of $\partial\Phi(z)$ on the double cover:

$$Z_\Sigma[\mu] = \left\langle \exp \oint_0 \frac{dz}{2\pi i}\mu(z)\partial\Phi(z) \right\rangle_\Sigma \ . \tag{A.2.83}$$

We split $\partial\Phi(z)$ into positive and negative modes $\partial\Phi_+(z)$ and $\partial\Phi_-(z)$. Using that $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ for $[A,B]$ central, we can split the exponential inside the generating function (A.2.83) as:

$$\begin{aligned} Z_\Sigma[\mu] = \langle\omega| e^{\oint \frac{dz}{2\pi i}\mu(z)\partial\Phi_-(z)} e^{\oint \frac{dz}{2\pi i}\mu(z)\partial\Phi_+(z)} \\ e^{\frac{1}{2}\oint \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}\mu(z_1)\mu(z_2)\left[\partial\Phi_+(z_1),\partial\Phi_-(z_2)\right]} |\Sigma_0\rangle \ . \end{aligned} \tag{A.2.84}$$

The commutator between positive and negative modes gives

$$\big[\partial\Phi_+(z), \partial\Phi_-(w)\big] = \frac{1}{(z-w)^2} + \frac{1}{(z+w)^2} = \langle\partial\Phi(z)\partial\Phi(w)\rangle_\sigma \ , \tag{A.2.85}$$

and $\partial\Phi_-(z)$ also pick up a commutator from acting on $\langle\omega| = \langle\sigma|\, e^{V_\omega}$:

$$\big[V_\omega, \partial\Phi_-(z)\big] = \omega(z) \ . \tag{A.2.86}$$

We can therefore write the logarithm of the generating function as:

$$\begin{aligned} \log Z_\Sigma[\mu] = \underbrace{\oint_0 \frac{dz}{2\pi i}\mu(z)\omega(z)}_{\text{Disk}} + \underbrace{\frac{1}{2}\oint_0 \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}\mu(z_1)\mu(z_2)\left\langle\partial\Phi(z_1)\partial\Phi(z_2)\right\rangle_\sigma}_{\text{Annulus}} \\ \underbrace{+ \log\left\langle \exp\oint_0 \frac{dz}{2\pi i}\mu(z)\partial\Phi_+(z)\right\rangle_\Sigma}_{\text{Stable}} \ . \end{aligned} \tag{A.2.87}$$

The first two terms represent the connected contributions from the disk and the annulus, and the last term contains the contributions from all the stable ($\chi < 0$) surfaces. So, we see that the connected correlation functions can be expressed as functional derivatives of $W_\Sigma[\mu] \equiv \log Z_\Sigma[\mu]$ with respect to the sources:

$$\begin{aligned} \langle\partial\Phi(z_1)\cdots\partial\Phi(z_n)\rangle_{\Sigma,\mathsf{c}} &= \frac{\delta}{\delta\mu(z_1)}\cdots\frac{\delta}{\delta\mu(z_n)} W_\Sigma[\mu]\Big|_{\mu=0} \\ &= \omega(z)\delta_{n,1} + \langle\partial\Phi(z_1)\partial\Phi(z_2)\rangle_\sigma \delta_{n,2} + \langle\partial\Phi_+(z_1)\cdots\partial\Phi_+(z_n)\rangle_{\Sigma,\mathsf{c}} \ . \end{aligned} \tag{A.2.88}$$

We can use the generating function $Z_\Sigma[\mu]$ to directly show the equivalence between the Virasoro constraints and the topological recursion. The topological recursion can be written as a functional differential equation for the generating functional of connected correlation functions $W_\Sigma[\mu]$:

$$\frac{\delta W_\Sigma}{\delta\mu(z_0)}\Big|_{\chi<0} = \operatorname*{Res}_{z\to 0} \frac{\langle\partial\Phi(z_0)\Phi(z)\rangle_\sigma}{2\,\omega(z)} \frac{1}{2}\left[\frac{\delta W_\Sigma}{\delta\mu(z)}\frac{\delta W_\Sigma}{\delta\mu(z)} + \frac{\delta^2 W_\Sigma}{\delta\mu(z)\delta\mu(z)}\right] . \tag{A.2.89}$$

Here, it is assumed that the second derivative $\frac{\delta^2}{\delta\mu(z)^2}$ is normal ordered according to (A.2.81). It is also implicit that the $(g,n) = (0,1)$ term is excluded. To show that this equation indeed generates the topological recursion, we expand W_Σ in powers of the coupling constant λ:

$$W_\Sigma[\mu] = \sum_{g=0}^{\infty} \lambda^{2g-2} \sum_{n=0}^{\infty} \frac{\mu(z_1)\cdots\mu(z_n)}{n!} \langle\partial\Phi(z_1)\cdots\partial\Phi(z_n)\rangle_{\Sigma,\mathsf{c}}^{(g)} . \tag{A.2.90}$$

If we insert this expansion into (A.2.89) and compare powers of $\lambda^g\mu^n$, we find the following recursion for $2g-2+n\geq 0$:

$$\begin{aligned}\mathcal{W}_{g,n+1}(z_0,z_I) = \operatorname*{Res}_{z\to 0} \frac{\langle\partial\Phi(z_0)\Phi(z)\rangle_\sigma}{4\,\omega(z)} \Big[&\mathcal{W}_{g-1,n+2}(z,z,z_I) \\ &+ \sideset{}{'}\sum_{\substack{g_1+g_2=g \\ J_1\sqcup J_2=I}} \mathcal{W}_{g_1,1+|J_1|}(z,z_{J_1})\,\mathcal{W}_{g_2,1+|J_2|}(z,z_{J_2})\Big] .\end{aligned} \tag{A.2.91}$$

The normal ordering prescription (A.2.81) has taken care of the term $\mathcal{W}_{0,2}(z,z)$ corresponding to the computation of the one-holed torus amplitude $\mathcal{W}_{1,1}(z)$:

$$\mathcal{W}_{0,2}(z,z) \equiv \lim_{w\to z}\left(\langle\partial\Phi(z)\partial\Phi(w)\rangle_\sigma - \frac{1}{(z-w)^2}\right) = \frac{1}{4z^2} . \tag{A.2.92}$$

At first sight, this recursion looks slightly different from the topological recursion (2.2.29). The $\mathcal{W}_{0,2}$ obtained in the twisted boson formalism after normal ordering is $(z+w)^{-2}$, whereas the Bergmann kernel was $(z-w)^{-2}$. Furthermore, we have dropped the minus signs for $\tilde{z} = -z$. However, the recursion (A.2.91) actually computes the *same* invariants as the topological recursion (2.2.29). This can be checked either by direct computation, or by making the following observations:

- ◇ The recursion kernel (2.2.27) is odd in its first argument, $\mathcal{K}(-z_0,z) = -\mathcal{K}(z_0,z)$. The topological recursion (2.2.29) then implies that all the symplectic invariants except $\omega_{0,2}$ are also odd in their first argument:

$$\omega_{g,n+1}(-z_0,z_I) = -\omega_{g,n+1}(z_0,z_I) . \tag{A.2.93}$$

Therefore, the functions $\mathcal{W}_{g,n}(\boldsymbol{z})$ are all even in their first argument. In [76] it is furthermore shown that the invariants $\omega_{g,n}$ are symmetric multi-differentials, from which it follows that the $\mathcal{W}_{g,n}(\boldsymbol{z})$ are even in each argument. This, of course, agrees with our observation that the fields $\partial\Phi(z)$ are even in z.

◇ The recursion kernel (2.2.27) is even in its second argument, $\mathcal{K}(z_0,-z) = \mathcal{K}(z_0,z)$. Writing $\mathcal{K}(z_0,z) = \kappa(z_0,z)\,dz_0 \otimes \partial_z$, we conclude that $\kappa(z_0,z)$ is an odd function of z. Since all the other $\mathcal{W}_{g,n}(\boldsymbol{z})$ are even, the kernel $\kappa(z_0,z)$ projects to the even part in z of $(z-w)^{-2}$. But the even part in z of $(z-w)^{-2}$ is the same as the even part of $(z+w)^{-2}$. Therefore, we have for all $(g,n) \neq (0,2)$:

$$\operatorname*{Res}_{z\to 0} \kappa(z_0,z)\frac{1}{(z-w)^2}\mathcal{W}_{g,n}(z,z_I) = \operatorname*{Res}_{z\to 0} \kappa(z_0,z)\frac{1}{(z+w)^2}\mathcal{W}_{g,n}(z,z_I)\ . \tag{A.2.94}$$

This allows us to replace the Bergmann kernel by the regularized two-point function.

Lastly, note that the recursion kernel $\mathcal{K}(z_0,z)$ for JT gravity matches the recursion kernel in (A.2.91). Namely, we have:

$$\frac{\langle \partial\Phi(z_0)\Phi(z)\rangle_\sigma}{4\,\omega(z)} = \frac{1}{2}\left(\frac{1}{z_0-z} - \frac{1}{z_0+z}\right)\frac{\pi\lambda}{2z\sin(2\pi z)} = \kappa(z_0,z)\ , \tag{A.2.95}$$

which agrees with (2.2.27).

We would like to show that the topological recursion, rewritten in the form (A.2.91), is equivalent to the Virasoro constraint for the twisted stress tensor. To do so, we write the functional differential equation (A.2.89) as an operator equation (to be read inside correlation functions):

$$\partial\Phi_+(z_0) = \operatorname*{Res}_{z\to 0} \frac{\langle \partial\Phi(z_0)\Phi(z)\rangle_\sigma}{2\,\omega(z)}\frac{1}{2}\big\{\partial\Phi(z)\partial\Phi(z)\big\}\ . \tag{A.2.96}$$

To see that this equation generates (A.2.89), one replaces the operator $\partial\Phi(z)$ by a functional derivative $\frac{\delta}{\delta\mu(z)}$ and act on $Z_\Sigma = e^{W_\Sigma}$. On the right-hand side we now recognize the stress tensor $T(z) = \frac{1}{2}\big\{\partial\Phi\partial\Phi\big\}(z)$. Multiplying by $\omega(z_0)$ and integrating around zero we find:

$$\oint_0 \frac{dz_0}{2\pi i}\frac{\omega(z_0)\partial\Phi(z_0)}{w-z_0} - \oint_0 \frac{dz_0}{2\pi i}\frac{\omega(z_0)}{w-z_0}\oint_0 \frac{dz}{2\pi i}\frac{\langle \partial\Phi(z_0)\Phi(z)\rangle_\sigma}{2\,\omega(z)}T(z) = 0\ . \tag{A.2.97}$$

We have written the residue as a contour integral, and introduced a point w with $|w| > |z_0|$ to project onto the negative powers of z_0. Furthermore, we have replaced $\partial\Phi_+$ by $\partial\Phi$, which can be done because ω is holomorphic.

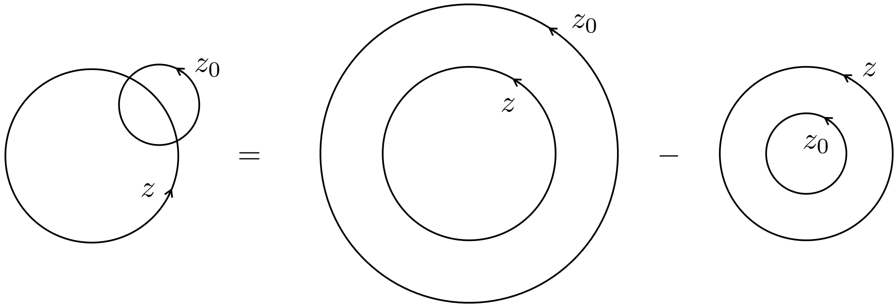


Figure A.6: *The contour deformation argument for* $\oint_0 dz \oint_z dz_0 = \oint_0 dz_0 \oint_0 dz - \oint_0 dz \oint_0 dz_0$.

Next, we deform the z and z_0 contours in the second term according to the contour deformation argument in Figure A.6. Using the fact that $\omega(z_0)$ is holomorphic and $\langle \partial\Phi(z_0)\Phi(z)\rangle_\sigma$ has no poles at $z_0 = 0$, we conclude that the $\oint_0 dz \oint_0 dz_0$ integral vanishes. So, the second term of (A.2.97) can be written as

$$\oint_0 \frac{dz_0}{2\pi i}\frac{\omega(z_0)}{w - z_0}\oint_0 \frac{dz}{2\pi i}\frac{\langle \partial\Phi(z_0)\Phi(z)\rangle_\sigma}{\omega(z)}T(z) \tag{A.2.98}$$

$$= \frac{1}{2}\oint_0 \frac{dz}{2\pi i}\oint_z \frac{dz_0}{2\pi i}\frac{\langle \partial\Phi(z_0)\Phi(z)\rangle_\sigma}{w - z_0}\frac{\omega(z_0)}{\omega(z)}T(z) \tag{A.2.99}$$

$$= \frac{1}{2}\oint_0 \frac{dz}{2\pi i}\left(\frac{1}{w - z} - \frac{1}{w + z}\right)T(z)\ . \tag{A.2.100}$$

The integration kernel $\frac{1}{2}((w - z)^{-1} - (w + z)^{-1})$ is odd in z, so it projects to the even negative powers of $T(z)$. However, since $T(z)$ is already even in z, the kernel simply projects to the negative powers of $T(z)$. Plugging this result into (A.2.97), we obtain

$$\oint_0 \frac{dz}{2\pi i}\frac{1}{w - z}\Big[\omega(z)\partial\Phi(z) - T(z)\Big] = 0\ . \tag{A.2.101}$$

Since $\omega(z)$ is holomorphic, we can freely add $\frac{1}{2}\omega(z)^2$ inside the brackets. Doing so gives

$$\oint_0 \frac{dz}{2\pi i}\frac{1}{w - z}T_\omega(z) = 0\ , \tag{A.2.102}$$

where $T_\omega(z)$ is defined in (A.2.48). In correlation functions this gives the requirement (A.2.57) that the expectation value of T_ω is analytic, which is precisely the Virasoro constraint.

A.2.6 $\mathbb{Z}_2$-twisted fermions

We provide an operator formalism for the twisted fermion fields introduced in Section 3.4.1. Firstly, we introduce $\Phi_0(x)$ and $\Phi_1(x)$ on the two sheets, which we expand in both integer and half-integer powers of x:

$$\partial\Phi_0(x) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \alpha_n x^{-n/2-1} \ , \tag{A.2.103}$$

$$\partial\Phi_1(x) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} (-1)^n \alpha_n x^{-n/2-1} \ . \tag{A.2.104}$$

where the oscillators satisfy the commutation relation $[\alpha_n, \alpha_m] = \frac{n}{2}\delta_{n+m}$. As one can see, the fields rotate into each other due to the square root:

$$\Phi_0(e^{2\pi i}x) = \Phi_1(x) \ , \quad \Phi_1(e^{2\pi i}x) = \Phi_0(x) \ . \tag{A.2.105}$$

Taking the difference leaves only odd $n = 2k+1$ in the sum, and so the following combination diagonalizes the monodromy:

$$\partial\Phi(x) \equiv \frac{1}{\sqrt{2}}(\partial\Phi_0(x) - \partial\Phi_1(x)) = \sum_{k\in\mathbb{Z}} \alpha_{2k+1} x^{-k-\frac{3}{2}} \ . \tag{A.2.106}$$

If we rename $\alpha_{2k+1} \equiv \alpha_{k+\frac{1}{2}}$, we see that $[\alpha_{k+\frac{1}{2}}, \alpha_{l+\frac{1}{2}}] = (k+\frac{1}{2})\delta_{k+l}$, which matches precisely with our definition of the twisted boson in (A.2.28). Now we compute the vacuum two-point functions in the usual way:

$$\langle\partial\Phi_0(x)\partial\Phi_0(y)\rangle_\sigma = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \langle\sigma|\alpha_n\alpha_{-m}|\sigma\rangle\, x^{-n/2-1}y^{m/2-1} \tag{A.2.107}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{2} x^{-n/2-1}y^{n/2-1} = \frac{(\sqrt{x}+\sqrt{y})^2}{4\sqrt{xy}}\frac{1}{(x-y)^2} \ . \tag{A.2.108}$$

This has the correct behaviour of bosonic two-point functions as $x \to y$. Integrating with respect to x and y, we get:

$$\langle\Phi_0(x)\Phi_0(y)\rangle_\sigma = \log(\sqrt{x}-\sqrt{y}) = \log(x-y) + \text{reg.} \tag{A.2.109}$$

The same answer is found for $\langle\partial\Phi_1\partial\Phi_1\rangle_\sigma$ and $\langle\Phi_1\Phi_1\rangle_\sigma$. Next, we compute the free two-point functions for bosons on opposite sheets:

$$\langle\partial\Phi_0(x)\partial\Phi_1(x)\rangle_\sigma = \frac{1}{2}\sum_{n=1}^{\infty}\frac{n}{2}(-1)^n x^{-n/2-1}y^{n/2-1}$$

$$= -\frac{1}{4\sqrt{xy}(\sqrt{x}+\sqrt{y})^2} \ , \qquad \text{(A.2.110)}$$

which is regular as $x \to y$. So in particular, we do not have to normal order combinations of Φ_0 and Φ_1. Integrating, we obtain the two-point function:

$$\langle \Phi_0(x)\Phi_1(y)\rangle_\sigma = \log(\sqrt{x}+\sqrt{y}) \ . \qquad \text{(A.2.111)}$$

The same results hold for $\langle \partial\Phi_1 \partial\Phi_0\rangle_\sigma$ and $\langle \Phi_1\Phi_0\rangle_\sigma$. For consistency, we check that the KS field $\Phi(x)$ has the free two-point function that we derived in (A.2.35):

$$\begin{aligned} &\langle \Phi(x)\Phi(y)\rangle_\sigma \\ &= \frac{1}{2}\Big(\langle \Phi_0(x)\Phi_0(y)\rangle_\sigma + \langle \Phi_1(x)\Phi_1(y)\rangle_\sigma - \langle \Phi_0(x)\Phi_1(y)\rangle_\sigma - \langle \Phi_1(x)\Phi_0(y)\rangle_\sigma \Big) \\ &= \log(\sqrt{x}-\sqrt{y}) - \log(\sqrt{x}+\sqrt{y}) \ . \end{aligned} \qquad \text{(A.2.112)}$$

Having computed the bosonic correlators, we can go on to study the bosonized fermions. First, let us compute the OPE between two fermions on the same sheet:

$$\psi_a(x)\psi_a(y)^\dagger = \{e^{\Phi_a(x)}\}\{e^{-\Phi_a(y)}\} \sim \frac{1}{x-y} e^{\Phi_a(x)-\Phi_a(y)} \sim \frac{1}{x-y} \ . \qquad \text{(A.2.113)}$$

The symbol $\sim$ means that we have only kept singular terms in the limit that $x \to y$. We used the OPE computed above that $\Phi_a(x)\Phi_a(y) \sim \log(x-y)$. As one can see, the cocycles have squared to one. By the boson-fermion correspondence, we have for $a = 0, 1$:

$$\partial\Phi_a(x) = \lim_{y\to x} \{\psi_a(x)\psi_a^\dagger(y)\} \equiv \lim_{y\to x}\left(\psi_a^\dagger(x)\psi_a(y) - \frac{1}{x-y}\right) \ . \qquad \text{(A.2.114)}$$

For two fermions on opposite sheets, we do not have to normal order since $\Phi_0(x')\Phi_1(x)$ is regular, and we can simply add the exponentials in a single normal-ordered exponential:

$$\psi_0(x)\psi_1^\dagger(x) \equiv \lim_{x'\to x} \psi_0(x')\psi_1(x) = \mathsf{c}_0\mathsf{c}_1\{e^{\Phi_0(x)-\Phi_1(x)}\} \ . \qquad \text{(A.2.115)}$$

B Contour analysis for fMT

B.1 Steepest descent contours

Let us study in more detail the steepest descent contours for the eigenvalue integrals of the fMT. Naively, looking at the transformations

$$\psi(x) = \int_{\mathcal{C}} dy\, e^{xy} \widehat{\psi}(y) \ , \qquad \psi^\dagger(x) = \int_{\mathcal{C}'} dy\, e^{-xy} \widehat{\psi}^\dagger(y) \ , \tag{B.1.1}$$

for the brane and anti-brane vertex operators, one might expect that $\mathcal{C}$ will be a contour along the imaginary axis, and $\mathcal{C}'$ along the real axis. This is also what one expects based on the color-flavor duality in the finite L matrix theory. However, in the double-scaling limit we have to make sure that the contours $\mathcal{C}, \mathcal{C}'$ go to infinity in a region where the potential grows, to ensure that the flavor matrix integral converges.

Let us first consider the the case of a single brane insertion

$$\langle \psi(x_i) \rangle_{\mathsf{KS}} = \int_{\mathcal{C}} \frac{dy}{\sqrt{\lambda}}\, e^{\frac{1}{\lambda} x_i y - \frac{1}{\lambda} \Gamma_0(y)(1+\mathcal{O}(\lambda))} \, . \tag{B.1.2}$$

For the purposes of our analysis, we have kept only the leading order term as $\lambda \to 0$. In the case of the Airy curve the potential is given by

$$\Gamma_0(y) = -\langle \widehat{\Phi}(y) \rangle_0 = \int x(y) dy = \frac{y^3}{3} \, , \tag{B.1.3}$$

and so the integral in (B.1.2) becomes the well-known integral representation of the Airy function. The real part of y^3 is positive in three wedges of the complex y-plane, and there are two independent non-trivial choices of contour, defining the 'Airy' and the 'Bairy' function, respectively. The Airy contour $\mathcal{C}$ can be chosen along the imaginary axis, as long as it remains in the left half-plane (see the black striped contour in Fig. B.1.).

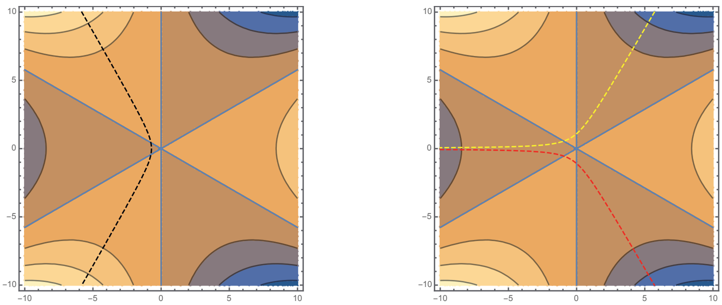


Figure B.1: *Real part of the Airy potential $\frac{y^3}{3}$. The blue shaded areas are regions where the real part is negative. Left, black striped: integration contour $\mathcal{C}$ for the brane insertions. Right, red and yellow striped: two distinct choices of $\mathcal{C}'_\pm$ for the anti-brane insertion.*

Similarly, for a single anti-brane insertion we obtain the integral

$$\langle \psi^\dagger(\mathsf{x}_i) \rangle_{\mathsf{KS}} = \int_{\mathcal{C}'} \frac{d\mathsf{y}}{\sqrt{\lambda}}\, e^{-\frac{1}{\lambda}\mathsf{x}_i\mathsf{y} + \frac{1}{\lambda}\Gamma_0(\mathsf{y})(1+\mathcal{O}(\lambda))}. \tag{B.1.4}$$

In the Airy case, we now see that the naive integral along the real axis diverges, since y^3 blows up as $\mathrm{Re}(y) \to \infty$. The integration contour may start on the negative real axis, but then it should enter into one of the two asymptotic regions $\frac{\pi}{6} < |\mathrm{Arg}(y)| < \frac{\pi}{2}$, such as the red $\mathcal{C}'_-$ or yellow $\mathcal{C}'_+$ striped contours in Fig. B.1. Of course, another valid option would be to integrate parallel to the imaginary axis in the right half-plane, but by Cauchy this contour can always be deformed to a linear combination of the red and yellow contours.

We can repeat the analysis for the JT spectral curve. In this case, the spectral curve equation is solved by $x(y) = \arcsin^2(y)$, and the leading order potential is

$$\Gamma_0(y) = -2y + \sqrt{1-y^2}\arcsin y + y\arcsin^2 y\,. \tag{B.1.5}$$

Its real part has been plotted in Fig. B.2. As one can see, for small y, the real part is very similar to the Airy case, but its large y behavior is different. In particular, for large y, the complex y-plane is divided into only two regions: in the left half-plane, the real part is positive for large enough $\mathrm{Re}(y)$, while in the right half-plane it becomes negative[1]. Moreover, there is a branch cut from the analytic continuation of the $\arcsin(y)$ on two pieces of the real axis $(-\infty,-1] \cup [1,\infty)$. This means that for the brane insertions (B.1.2), we can choose the integration contour $\mathcal{C}$ to be homotopic to the Airy contour, as long as it passes through the real axis in the interval $(-1,1)$ (see also [109]).

[1]The lines where the sign of the potential changes are indicated in light blue.

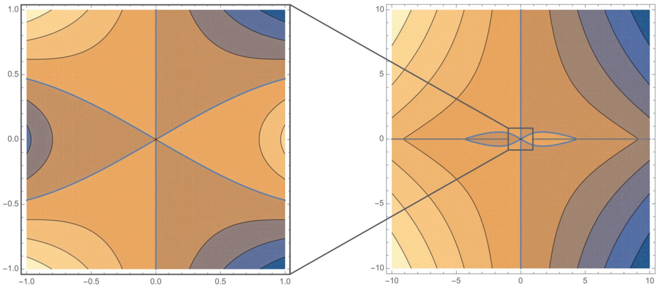


***Figure B.2:** Real part of the JT potential. Close to the origin, the potential looks like Airy potential, while the behavior at infinity is different: the potential is positive and growing for* $\mathrm{Re}(y) \ll 0$*, while it is negative for* $\mathrm{Re}(y) \gg 0$.

However, for the anti-brane (B.1.4), the contours $\mathcal{C}'_{\pm}$ no longer give rise to a convergent integral, because the integral parallel to the negative real axis grows exponentially for sufficiently negative $\mathrm{Re}(\mathsf{y})$. One possible solution is to integrate y parallel to the imaginary axis, shifted into the positive half-plane. However, if we want to make contact with some finite L flavor integral pre-double scaling, this contour choice should be excluded, for the integration contour in the anti-brane sector pre-double scaling is parallel to the real axis (see, for example, the analysis in Appendix A of [141][2]). The resolution of this apparent tension is to take into account the branched structure of the potential $\Gamma_0(y)$. The plot above was generated using the principal branch of $\arcsin(y) = -i\log(iy + \sqrt{1-y^2})$, but there are infinitely many branches, related by $\arcsin(y) + 2\pi k$, for $k \in \mathbb{Z}$. As an example, we have plotted the real part of $\Gamma_0(y)$ taking the branch $k = -1$: we see that a contour which passes through the branch cut onto the next sheet enters a sheet where the potential grows again. We will select precisely such a contour for the anti-brane integrals y_i.

Having discussed the asymptotic behavior of the integration contours, we want to deform them into steepest descent (or stationary phase) contours passing through the saddle points. Let us first briefly describe the Airy case of a single brane insertion. Varying the action in (B.1.2) we find two saddle points, $y^{\pm} = \pm\sqrt{x}$. If we position the brane slightly above the negative x-axis in the spectral x-plane, $x = -E + i\eta$, the saddle points $y^{\pm}$ lie on opposite sides of the imaginary y axis. The steepest descent (and ascent) contours through these saddle points are plotted in figure B.4 (left). The original black dashed Airy contour can be deformed to the sum of the red and blue descent contours, so both saddle points contribute to the

[2] On a technical level, the integration parallel to the real axis ensures that the Hubbard-Stratonovich transform (necessary in the color-flavor duality) is well-defined [162].

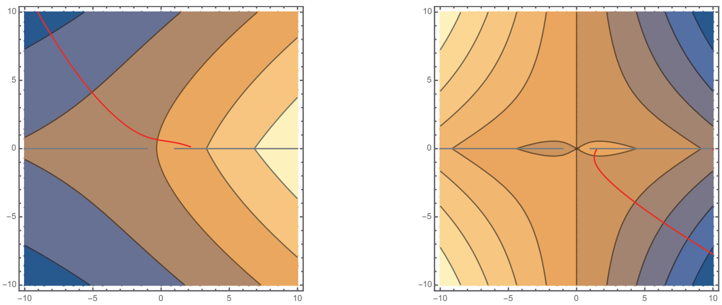


Figure B.3: *Right: principal branch of* $\Gamma_0(y)$*. Left: the* $k = -1$ *sheet of* $\Gamma_0(y)$*. In red a steepest descent contour that goes to infinity in different sheets of the multiple cover.*

integral. However, the contribution of y^+ is exponentially suppressed compared to that of y^-. On the other hand, if we position the brane slightly below the negative x-axis, $x = -E - i\eta$, the dominant contribution will come from y^+ instead of y^-, see Fig. B.4 (right). This is an example of a Stokes' phenomenon: the relative dominance between the saddle points is exchanged when we cross the *anti-Stokes line* on the branch cut of $\sqrt{x}$.

For the anti-brane, the original integration contour can only be deformed to one of the steepest descent contours, passing through only one saddle. The red dashed contour in Fig. B.1 is deformable to the steepest descent contour passing through y^-, while the yellow dashed contour is deformable to the steepest descent path through y^+. This is another example of a Stokes phenomenon: which saddles (cease to) contribute depends on the argument of the external parameter x.

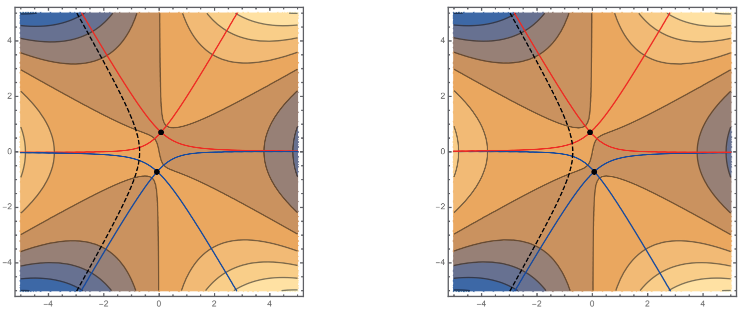


Figure B.4: *Background: real part of* $xy - \frac{y^3}{3}$*. Shaded areas are* < 0*, bright areas* > 0*. Black dots are the saddle points. The steepest descent+ascent contours passing through them are drawn in red and blue. Left:* $x = -E + i\eta$*, right:* $x = -E - i\eta$*.*

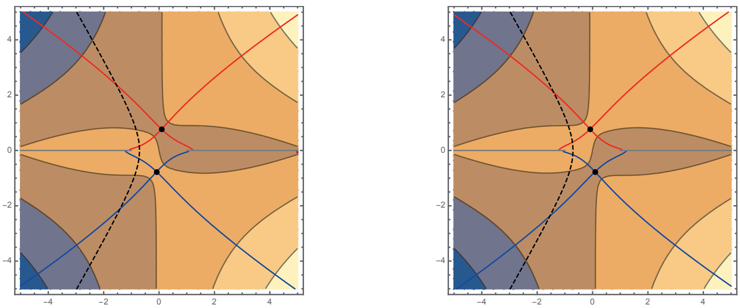


Figure B.5: *Background: real part of $xy - \Gamma_0(y)$. Shaded areas are < 0, bright areas > 0. Black dots are the saddle points. The steepest descent+ascent contours passing through them are drawn in red and blue. Left: $x = -E + i\eta$, right: $x = -E - i\eta$.*

Let us now turn to the case of JT gravity. The saddle points are located at

$$y^{\pm} = \pm \sin \sqrt{x} \ . \tag{B.1.6}$$

The steepest descent (and ascent) contours through the saddles have been plotted in Fig. B.5, for $x = -E + i\eta$ (left) and $x = -E - i\eta$ (right). The black dotted line is the original integration contour for the brane insertions. It can be deformed to the sum of the red and blue steepest descent contours, as these go to the same asymptotic infinity on the next sheet. So both saddle points contribute, and which saddle dominates is determined by the $\pm i\eta$ prescription exactly as in the Airy case. For the anti-brane, one should follow a steepest descent contour that passes through sheet $k = -1$ from an asymptotic infinity, enters the principal branch through the left branch cut and then goes through *only one* of the saddle points. This is again identical to the Stokes' phenomenon of the Airy anti-brane. So we see that the pattern of symmetry breaking precisely follows from the $\pm i\eta$ prescription of the external matrix X, in the same way that this was the case for the finite size matrix integrals of [49].

C Details on Berry phases

C.1 Kinematic space example

We will now describe a version of the state-based parallel transport summarized in Section 5.2.1, which reproduces some of the results from kinematic space for CFT_2 on a time-slice. As we saw in Section 5.2.2, the parallel transport process for kinematic space could also be derived in the operator-based transport language. In this way of formulating the problem, the geometrical description of kinematic space in terms of coadjoint orbits [188] is more readily transparent.

We will start by setting up some geometry that is relevant for this problem. Consider the group $SL(2,\mathbb{R})$. Its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ consists of generators t_μ, $\mu = 0,1,2$ satisfying the commutation relations $[t_\mu, t_\nu] = \epsilon_{\mu\nu}{}^{\rho} t_\rho$, where the indices are raised by a metric η_{ab} with signature $(-,+,+)$. We will make use of an explicit finite-dimensional representation by 2×2 matrices given by

$$t_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;, \quad t_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;, \quad t_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;. \tag{C.1.1}$$

This basis will be most convenient for the calculation of the Berry curvature. It can be easily expressed in terms of the basis used in Section 5.5 as $t_0 = \frac{1}{2}(L_- + L_+), t_1 = \frac{1}{2}(L_- - L_+), t_2 = L_0$. Now consider embedding coordinates (X^0, X^1, X^2) describing 3-dimensional Minkowski spacetime with metric

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 \;. \tag{C.1.2}$$

Recall that $SL(2,\mathbb{R})/\mathbb{Z}_2 \cong SO(2,1)$. A convenient parametrization for the algebra $\mathfrak{sl}(2,\mathbb{R})$ is given through the isomorphism to Mink_3:

$$\frac{1}{2}\begin{pmatrix} X^2 & X^1 + X^0 \\ X^1 - X^0 & -X^2 \end{pmatrix} \leftrightarrow (X^0, X^1, X^2) \;. \tag{C.1.3}$$

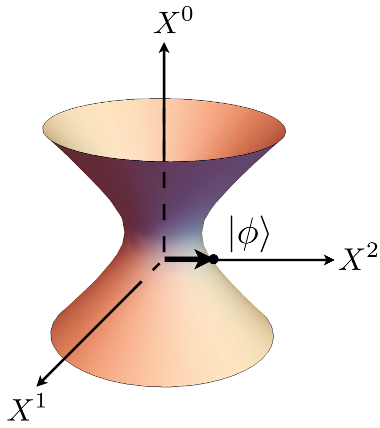


Figure C.1: *The dS_2 hyperboloid describing kinematic space, which is a coadjoint orbit of $SO(2,1)$. The arrow points to a special point that corresponds to the coherent state $|\phi\rangle$.*

The reason to express $\mathfrak{sl}(2,\mathbb{R})$ in this way is that the coadjoint orbits of the Lie group can be realized geometrically in Minkowski space. Any element of $\mathfrak{sl}(2,\mathbb{R})$ lies in one of three conjugacy classes (up to an overall factor ± 1). These can be classified by the value of $\epsilon \equiv |\mathrm{tr}(g)|/2$ where $g \in SL(2,\mathbb{R})$: $\epsilon < 1$ is elliptic, $\epsilon = 1$ is parabolic and $\epsilon > 1$ is hyperbolic. We will assume that our representative is in the diagonal class

$$\Lambda = \mathrm{diag}(\lambda, -\lambda)/2 \tag{C.1.4}$$

with $\lambda \in \mathbb{R}$. Since $|\mathrm{tr}(e^{\Lambda})|/2 > 1$ for all λ, this is a hyperbolic element. Other choices lead to different orbits.

Consider a general group element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \ , \tag{C.1.5}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. The coadjoint orbit is generated by the adjoint action of Λ with arbitrary g,

$$g \cdot \Lambda \cdot g^{-1} = \begin{pmatrix} \frac{\lambda}{2}(bc + ad) & -\lambda ab \\ \lambda cd & -\frac{\lambda}{2}(bc + ad) \end{pmatrix} \ . \tag{C.1.6}$$

The determinant is constant along the orbit, $\det(g \cdot \Lambda \cdot g^{-1}) = -\lambda^2/4$. Applying the map to Minkowski space, (C.1.3), this results in the condition

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 = \lambda^2 \ . \tag{C.1.7}$$

This is the defining equation of a single-sheeted hyperboloid with radius λ. Take the embedding coordinates

$$\begin{aligned} X^0 &= \lambda \cot t \ , \\ X^1 &= \lambda \csc t \cos\theta \ , \\ X^2 &= \lambda \csc t \sin\theta \ . \end{aligned} \tag{C.1.8}$$

These satisfy (C.1.7) and from (C.1.2) result in the induced metric

$$ds^2 = \lambda^2 \csc^2 t(-dt^2 + d\theta^2) \ . \tag{C.1.9}$$

This is just the metric on $\mathrm{dS}_2 \simeq SO(1,2)/SO(1,1)$. We saw that this describes the coadjoint orbit passing through the representative, (C.1.4).

The coadjoint orbit can be thought of as a fiber bundle whose base space is $SO(1,2)/SO(1,1)$ and its fiber is $SO(1,1)$. We want to consider an appropriate section of the fiber bundle. The discussion below follows closely [192]. Using the embedding coordinate (C.1.8) and the map (C.1.3), we obtain the constraints

$$\begin{aligned} 2\mathrm{tr}(g\,\Lambda\, g^{-1}\, t_0) &= -X_0 = -\lambda \cot t \ , \\ 2\mathrm{tr}(g\,\Lambda\, g^{-1}\, t_1) &= X_1 = \lambda \cos\theta \csc t \ , \\ 2\mathrm{tr}(g\,\Lambda\, g^{-1}\, t_2) &= X_2 = \lambda \sin\theta \csc t \ . \end{aligned} \tag{C.1.10}$$

Solving this system of equations, (C.1.10), we obtain

$$b = -\frac{\cot t + \cos\theta \csc t}{2a} \ , \quad c = \frac{a(1 - \sin\theta \csc t)}{\cot t + \cos\theta \csc t} \ , \quad d = \frac{1 + \sin\theta \csc t}{2a} \ . \tag{C.1.11}$$

We have the freedom to impose $a = 1$, in which case the expressions somewhat simplify. Applying this back to (C.1.5), we obtain a section $g : \mathrm{dS}_2 \to SL(2,\mathbb{R})$ for the bundle given by

$$g = \begin{pmatrix} 1 & -\frac{1}{2}(\cos t + \cos\theta)\csc t \\ \tan\left(\frac{t-\theta}{2}\right) & \frac{1}{2}(1 + \csc t \sin\theta) \end{pmatrix} \ . \tag{C.1.12}$$

Notice that g reduces to the identity for $t = \theta = \pi/2$ which corresponds to the point of intersection of the hyperboloid with the axis labeled by the t_2 generator.

Now we will apply some of these tools to the problem of state-based parallel transport for the group $SL(2,\mathbb{R})$, with the aim of describing kinematic space. Recall that to define a state-based Berry phase it is necessary to choose a suitable 'Hamiltonian' with an eigenstate $|\phi\rangle$ that serves as the base state for the parallel transport process. The 'Hamiltonian' is one which generates a specified subgroup of $SL(2,\mathbb{R})$, which we interpret as a flow in time. The state is acted on by group elements in a unitary representation, which we denote by $\mathcal{D}(g)$, $\mathcal{D}(u)$ for $g \in SL(2,\mathbb{R}), u \in \mathfrak{sl}(2,\mathbb{R})$. In the coadjoint orbit language, eigenstates of subalgebras of the symmetry algebra are known as *coherent states*. Specifically, we will choose our Hamiltonian to be t_2, which generates an $\mathfrak{so}(1,1)$ subalgebra. This exponentiates to the hyperbolic group element

$$\mathcal{J} = e^{\eta t_2/2} \tag{C.1.13}$$

with $\eta \in \mathbb{R}$. Taking $X \to \mathcal{J}X\mathcal{J}^{-1}$ using the isomorphism, (C.1.3), we see the adjoint action with respect to $\mathcal{J}$ acts geometrically as

$$X^0 \to X^0 \cosh(\eta/2) + X^1 \sinh(\eta/2) \ , \tag{C.1.14}$$

$$X^1 \to X^0 \sinh(\eta/2) + X^1 \cosh(\eta/2) \ , \tag{C.1.15}$$

$$X^2 \to X^2 \ , \tag{C.1.16}$$

in other words, it acts as a boost with rapidity $-\eta/2$ in the $X^0 - X^1$ direction in embedding space.

We define our coherent state through the condition that the boost leaves it invariant up to a phase,

$$\mathcal{D}(\mathcal{J})\,|\phi\rangle = e^{i\eta\zeta}\,|\phi\rangle \ \ , \mathcal{D}(t_2)\,|\phi\rangle = 2\zeta\,|\phi\rangle \ \ , \tag{C.1.17}$$

with $\zeta \in \mathbb{R}$ since $\mathcal{D}(\mathcal{J})$ is assumed to be unitary and $\mathcal{D}(t_2)$ Hermitian in the representation. By a theorem of Perelomov [193] (see also [194]), coherent states are in one-to-one correspondence with points on an orbit. Our state $|\phi\rangle$ corresponds to the point $(0,0,1)$ on the dS_2 hyperboloid that is left fixed by the action of the boost (see Figure C.1). It is geometrically simple to see that the action of the other generators t_0, t_1 do not leave this point invariant, which corresponds to the statement that $|\phi\rangle$ is not also an eigenstate of these generators.

Recall that the Maurer-Cartan form is given by

$$\Theta = g^{-1}dg \ . \tag{C.1.18}$$

The Berry phase is

$$\theta(\gamma) = \oint_\gamma A \ , \quad A = i \langle\phi| \mathcal{D}(\Theta) |\phi\rangle \ . \tag{C.1.19}$$

We now use (C.1.12) to evaluate the pullback of the Maurer-Cartan form from $SL(2,\mathbb{R})$ to dS_2. Taking the expectation value of the generators in the state $|\phi\rangle$, then applying the commutation relations, the eigenvalue condition (C.1.17) and using $\zeta \in \mathbb{R}$, we see that only t_2 has a nonvanishing expectation value in $|\phi\rangle$. Thus, only this part contributes to the Berry phase. We find

$$A = i \langle\phi| \mathcal{D}(\Theta) |\phi\rangle = i\zeta \csc t \cos\left(\frac{t+\theta}{2}\right) \sec\left(\frac{t-\theta}{2}\right)(dt - d\theta) \ . \tag{C.1.20}$$

From this we can define the Berry curvature

$$F = dA = \frac{i\zeta}{\sin^2 t} dt \wedge d\theta \ . \tag{C.1.21}$$

Using Stokes' theorem one can write the integral of the Berry connection in (C.1.20) as

$$\theta(\gamma) = i\zeta \int_B \frac{1}{\sin^2 t} dt \wedge d\theta \ , \tag{C.1.22}$$

where B is any two-dimensional region with boundary $\partial B = \gamma$.

For a CFT_2 restricted to a time-slice, kinematic space consists of the space of intervals on this time-slice. Given a causal ordering based on containment of intervals, this is just a dS_2 spacetime, (C.1.9), with a time coordinate set by the interval radius, $(\theta_R - \theta_L)/2$ [184]. The curvature, (C.1.21), is a volume form on kinematic space. Recalling the relation between time and interval size, it matches the kinematic space curvature, (5.2.25), derived from the operator-based method in Section 5.2.2 (note that an exact matching of the normalization is unimportant, as the overall normalization for the modular Berry phase will be at any rate affected by the choice of normalization for the modular Hamiltonian). The Berry phase, (C.1.22), computes the volume of region B within this dS_2 spacetime. It also precisely reproduces the Berry phase for kinematic space derived by other means in [53, 54].

C.2 General formulation

We will derive a general formula for the curvature assuming that there is a unique way of separating out the zero mode. As we discuss in the next appendix, this is not generally true when the state-changing transformations are elements of the

Virasoro algebra, however it holds for the transformations that we consider in the main text. The results of Section 5.3 utilize the formula for the curvature presented in this appendix.

Consider a Lie algebra $\mathfrak{g}$ and a trajectory of elements $X(\lambda) \in \mathfrak{g}$ specified by some parameter λ. We write Ad_X for the adjoint action of X on $\mathfrak{g}$, $\mathrm{Ad}_X(Y) = [X, Y]$. We make the assumption that the kernel of Ad_X and the image of Ad_X do not intersect anywhere along the path, which is guaranteed if $[X, Y] \neq 0$ implies $[X, [X, Y]] \neq 0$. Moreover, we will be interested in smooth trajectories $X(\lambda)$ along which the kernel and image of Ad_X vary smoothly. In particular, we will assume their dimensions do not jump.

Crucially, we will make the further assumption[1] that any Y can be uniquely decomposed as $Y = K + I$ with K in the kernel and I in the image of Ad_X. We will call the corresponding projection operators P_K and P_I, with the property that

$$P_I + P_K = 1 \ . \tag{C.2.1}$$

Notice that we are not using an inner product, which means that the projectors are not orthogonal in any sense.

Besides the projectors P_K and P_I, we will denote Ad_X simply by A, and its inverse by A^{-1}. Note that A has a kernel so it does not have an inverse, but since by assumption A defines a non-degenerate map from the image of the image of Ad_X to itself, it does have a well-defined inverse on these subspaces. The map A^{-1} is defined to be the inverse on these subspaces and zero everywhere else. These operators then obey the following set of identities:

$$\begin{aligned} AP_K = P_K A &= 0 \ , && \text{(C.2.2)} \\ A^{-1} P_K = P_K A^{-1} &= 0 \ , && \text{(C.2.3)} \\ AA^{-1} = A^{-1} A &= P_I \ . && \text{(C.2.4)} \end{aligned}$$

We now vary X to $X + \delta X$ by some small change $\delta\lambda$ along the path. In particular, we can use the above identities to express the variations of P_K, P_I and A^{-1} in terms of the variation of A. After some algebra we find that

$$\begin{aligned} \delta P_K &= -\delta P_I = -P_K \delta A A^{-1} P_I - P_I A^{-1} \delta A P_K \ , && \text{(C.2.5)} \\ \delta A^{-1} &= -A^{-1} \delta A A^{-1} + P_I A^{-2} \delta A P_K + P_K \delta A A^{-2} P_I \ . && \text{(C.2.6)} \end{aligned}$$

[1] For finite-dimensional Lie algebras the dimensions of the kernel and the image add up to the total dimension of the Lie algebra. Since they do not intersect, this then implies that the kernel and image of Ad_X together span the full Lie algebra. For infinite-dimensional Lie algebras the situation is more complicated, as we explain in Appendix C.3.

In particular, we used

$$P_K \delta P_I = P_K \delta A^{-1} A P_I \ , \quad P_I \delta P_I = P_I A^{-1} \delta A P_K \ , \tag{C.2.7}$$

for deriving (C.2.5) and

$$P_K \delta A^{-1} = \delta P_I A^{-1} \ , \quad P_I \delta A^{-1} = A^{-1} \delta P_I - A^{-1} \delta A A^{-1} \ , \tag{C.2.8}$$

for (C.2.6). We also used that $P_I A^{-1} = A^{-1} P_I = A^{-1}$ and $P_I A = A P_I = A$.

Given a variation δX, we want to express it as

$$\delta X = [S, X] + P_K \delta X \ , \tag{C.2.9}$$

where $P_K \delta X$ is in the kernel of Ad_X. Moreover, we want to remove the modular zero mode from S, so that S is uniquely defined. We do this by requiring that $P_K S = S P_K = 0$, and with the above equations it is then easy to see that

$$S = -A^{-1}(\delta X) \ . \tag{C.2.10}$$

We are now going to compute the parallel transport along a small square, by first doing the variation $\delta_1 X$ and then $\delta_2 X$, and then subtracting the reverse order. For the difference, we get

$$F = (1 - (A^{-1} + \delta_1 A^{-1})(\delta_2 X))(1 - A^{-1}(\delta_1 X)) - (1 \leftrightarrow 2) \ . \tag{C.2.11}$$

The first order terms vanish, thus it is necessary to expand to second order. One term we get at second order is

$$F_1 = -[A^{-1}(\delta_1 X), A^{-1}(\delta_2 X)] \ . \tag{C.2.12}$$

There is also another term coming from the variations of A^{-1}, which evaluates to

$$F_2 = (A^{-1} \delta_1 A A^{-1} - P_I A^{-2} \delta_1 A P_K - P_K \delta_1 A A^{-2} P_I)(\delta_2 X) - (1 \leftrightarrow 2) \ . \tag{C.2.13}$$

In order to simplify (C.2.13) further, we need several other identities. For example, multiplying

$$A([Y, Z]) = [AY, Z] + [Y, AZ] \tag{C.2.14}$$

by A^{-1} we get the identity

$$A^{-1}([AY, Z] + [Y, AZ]) = P_I([Y, Z]) \ . \tag{C.2.15}$$

From this it follows that

$$A^{-1}[Y, P_K Z] = A^{-1}[P_I Y, P_K Z] = P_I([A^{-1}Y, P_K Z]) \ , \tag{C.2.16}$$

where we used Eqs. (C.2.1), (C.2.3) and (C.2.4).

Next we consider the first term in F_2 minus the same term with 1 and 2 interchanged. It is given by

$$F_2^1 = A^{-1}\delta_1 A A^{-1}(\delta_2 X) - (1 \leftrightarrow 2) \ . \tag{C.2.17}$$

We use $\delta_1 A Y = [\delta_1 X, Y]$ to rewrite it as

$$\begin{aligned} F_2^1 &= A^{-1}([\delta_1 X, A^{-1}(\delta_2 X)] + [A^{-1}(\delta_1 X), \delta_2 X]) \\ &= A^{-1}([(AA^{-1} + P_K)\delta_1 X, A^{-1}(\delta_2 X)] + [A^{-1}(\delta_1 X), (AA^{-1} + P_K)\delta_2 X]) \\ &= P_I([A^{-1}(\delta_1 X), A^{-1}(\delta_2 X)]) + A^{-1}([P_K \delta_1 X, A^{-1}(\delta_2 X)] \\ &\qquad + [A^{-1}(\delta_1 X), P_K \delta_2 X]) \ . \end{aligned} \tag{C.2.18}$$

In the last equality we make use of (C.2.15). Applying (C.2.16) to the last two terms gives

$$\begin{aligned} F_2^1 = P_I([&A^{-1}(\delta_1 X), A^{-1}(\delta_2 X)] \\ &+ [A^{-2}(\delta_1 X), P_K \delta_2 X] - [A^{-2}(\delta_2 X), P_K \delta_1 X]) \ . \end{aligned} \tag{C.2.19}$$

The second term in F_2 reads

$$\begin{aligned} F_2^2 &= -P_I A^{-2}\delta_1 A P_K(\delta_2 X) + P_I A^{-2}\delta_2 A P_K(\delta_1 X) \\ &= -A^{-2}([\delta_1 X, P_K \delta_2 X] - [\delta_2 X, P_K \delta_1 X]) \ . \end{aligned} \tag{C.2.20}$$

Using the identity (C.2.16) twice it follows that

$$F_2^2 = -P_I([A^{-2}(\delta_1 X), P_K \delta_2 X] - [A^{-2}(\delta_2 X), P_K \delta_1 X]) \ . \tag{C.2.21}$$

The last term to consider is

$$\begin{aligned} F_2^3 &= P_K \delta_2 A A^{-2} P_I(\delta_1 X) - P_K \delta_1 A A^{-2} P_I(\delta_2 X) \\ &= P_K([\delta_2 X, A^{-2}(\delta_1 X)] - [\delta_1 X, A^{-2}(\delta_2 X)]) \ . \end{aligned} \tag{C.2.22}$$

This expression does not admit an obvious simplification. Combining all terms we see that the first term in F_2^1 cancels part of F_1, the second and third terms in F_2^1 cancel against F_2^2, so that we are left with a simple and compact expression for the full curvature:

$$F = -P_K([A^{-1}(\delta_1 X), A^{-1}(\delta_2 X)] \\ + [\delta_1 X, A^{-2}(\delta_2 X)] - [\delta_2 X, A^{-2}(\delta_1 X)]) \; . \qquad \text{(C.2.23)}$$

One can easily check that the curvature commutes with X.

Notice that only the P_I components of δX contribute to the curvature due to the observation that

$$P_K([P_I Y, P_K Z]) = P_K([AA^{-1}Y, P_K Z]) = P_K A([A^{-1}Y, P_K Z]) = 0 \; , \qquad \text{(C.2.24)}$$

where we used (C.2.14). Moreover, we find that

$$\begin{aligned} W &= A^2([A^{-2}(\delta_1 X), A^{-2}(\delta_2 X)]) && \text{(C.2.25)} \\ &= 2[A^{-1}(\delta_1 X), A^{-1}(\delta_2 X)] + [P_I \delta_1 X, A^{-2}(\delta_2 X)] + [A^{-2}(\delta_1 X), P_I \delta_2 X] \end{aligned}$$

is almost the same as (C.2.23), except for the factor of two, and the appearance of the projector P_I. It is obvious that $P_K W = 0$ and if we add $P_K W$ to F we can drop the P_I in the resulting expression, as follows from (C.2.24). Therefore, the final expression for the curvature reads

$$F = P_K([A^{-1}(\delta_1 X), A^{-1}(\delta_2 X)]) \; . \qquad \text{(C.2.26)}$$

The simple form of this result suggests that there is a shorter derivation and it would be interesting to further investigate this possibility.

C.3 Non-diagonalization for Virasoro

There are subtleties in expressing a Virasoro generator X as $X = X_0 + [H_{\text{mod}}, Y]$ with X_0 a zero mode of the modular Hamiltonian H_{mod} in the Virasoro algebra. We will give here a summary of why the assumed decomposition, (C.2.9), used to derive the curvature cannot be applied to the full Virasoro algebra, and hence why we have chosen to restrict to a different set of transformations.

We will first be more precise about the notion of 'generator.' A generator of $\text{Diff}(S^1)$ can be expressed as

$$X = \sum_n c_n L_n \; , \qquad \text{(C.3.1)}$$

where the modes L_n satisfy the Virasoro algebra, (5.3.11). We can equivalently represent X as a function on S^1, $f(\theta) = \sum c_n e^{in\theta}$, or as a vector field, $\xi = \sum c_n z^{n+1}\partial_z$ in radial quantization. For the arguments we are interested in the central charge can be considered separately, see Section 5.3.3.

One can ask what values of c_n are allowed in (C.3.1). This leads to different 'definitions' of the Virasoro algebra. Some choices that are preserved under commutation are:

- algebraic: require only a finite number of the c_n to be non-zero ,
- semi-algebraic: require that $c_n = 0$ for n sufficiently negative (alternatively, one could require $c_n = 0$ for n sufficiently positive) ,
- analytic: require the function f or vector field ξ to be smooth .

In the case where the generators are self-adjoint, then semi-algebraic reduces to algebraic.

For each of these choices of infinite-dimensional Lie algebras, we can ask to what extent the statement that any generator X can be written as $X = X_0 + [H_{\text{mod}}, Y]$ with X_0 a zero mode of the modular Hamiltonian H_{mod} holds.

C.3.1 Algebraic and semi-algebraic case

In the algebraic case, one can prove that the only algebra element that commutes with H_{mod} is H_{mod} itself. First, recall that

$$H_{\text{mod}} = \pi(L_1 + L_{-1}) \ . \tag{C.3.2}$$

Now consider elements with only a finite number of non-zero c_n, running from $n = -L, ..., K$, with K and L positive. Then, the commutator

$$[H_{\text{mod}}, \sum_{n=-L}^{K} c_n L_n] = \sum_{n=-L-1}^{K+1} c'_n L_n \tag{C.3.3}$$

maps a vector space of dimension $K+L+1$ into a vector space of dimension $K+L+3$. Its kernel is dimension one so its cokernel must be dimension three. Therefore, the number of generators which can be written as $[H_{\text{mod}}, X]$ is codimension three. In fact, one can write every generator as

$$X = aH_{\text{mod}} + bL_2 + cL_{-2} + [H_{\text{mod}}, Y] \ , \tag{C.3.4}$$

for some a, b, c, which can be seen iteratively by taking a suitable Y with $L = K = 1$ and combining $H_{\text{mod}}, L_2, L_{-2}$ to isolate L_0, then taking a suitable Y with $L = 1, K = 2$ combined with all the previous generators to isolate L_3, and so on and so forth. Crucially, this decomposition is not unique. For instance, we could have equally well written a similar decomposition with L_3, L_{-3} instead of L_2, L_{-2}.

To solve

$$L_{-2} = [H_{\text{mod}}, Y] \ , \tag{C.3.5}$$

it is necessary to express Y as an infinite series $Y = \sum_{k=-3}^{-\infty} c_k L_k$ which is not part of the algebra:

$$Y = \frac{1}{4}L_{-3} - \frac{2}{4\cdot 6}L_{-5} + \frac{2}{6\cdot 8}L_{-7} - \frac{2}{8\cdot 10}L_{-9} + \dots \tag{C.3.6}$$

If we denote by Y_k the sum of the first k terms which truncates at L_{-2k-1}, then we have

$$\frac{1}{\pi}[H_{\text{mod}}, Y_k] = L_{-2} + \frac{(-1)^{k+1}}{k+1}L_{-2k-2} \ , \tag{C.3.7}$$

so that for large k this becomes 'close' to L_{-2}. We can introduce a metric so that this notion of closeness becomes more precise, e.g.,

$$||\sum_n c_n L_n||^2 \equiv \sum_n |c_n|^2 \tag{C.3.8}$$

defines a metric on the Lie algebra. But the Lie algebra is not complete with respect to this metric, i.e., limits of Lie algebra elements which converge in this norm will not in general converge to an element of the Lie algebra.

Even ignoring the fact that the algebra is not complete with respect to (C.3.8), there is the additional issue that this way of interpreting L_{-2} as the commutator of an element of the algebra with Y is too strong. Indeed, we can also find an infinite series Y obeying

$$[H_{\text{mod}}, Y] = H_{\text{mod}} \ , \tag{C.3.9}$$

which looks like

$$Y = \dots + c_6 L_6 + c_4 L_4 + c_2 L_2 + c_{-2} L_{-2} + c_4 L_{-4} + c_{-6} L_{-6} + \dots \tag{C.3.10}$$

This also has the property that if one truncates Y, the Y_k obeys $[H_{\text{mod}}, Y_k] = H_{\text{mod}} + Z_k$, with Z_k small defined with respect to the above norm. This would not allow for a decomposition separating out the zero mode part from the image of the adjoint action without intersection.

Notice that considering the semi-algebraic rather than algebraic case also does not fix the issue. A semi-infinite series in one direction can either remove L_2 or L_{-2} from the expression (C.3.4), but not both.

C.3.2 Analytic case

In the analytic case, the equation $[H_{\text{mod}}, X] = Y$ is the differential equation

$$(1+z^2)X'(z) - 2zX(z) = -\frac{1}{\pi}Y(z) \ , \tag{C.3.11}$$

where we replaced everything by the corresponding smooth function. This differential equation is equivalent to

$$\frac{d}{dz}\left(\frac{X(z)}{1+z^2}\right) = -\frac{1}{\pi}\frac{Y(z)}{(1+z^2)^2} \ . \tag{C.3.12}$$

Therefore,

$$X(z) = -\frac{c_0}{2}(1+z^2) - \frac{1}{\pi}(1+z^2)\int^z \frac{Y(z')}{(1+z'^2)^2}\,dz' \ , \tag{C.3.13}$$

where c_0 is an integration constant, and the integration is over the circle. The differential equation does not have an analytic solution for all $Y(z)$. In fact, we will argue that in order to find an analytic solution we require three conditions on Y, so that once again the space of smooth vector fields which can be written as $[H_{\text{mod}}, X]$ is codimension three.

The first two conditions come from exploring the behavior of the integrand near $z = \pm i$, where we find that there will be logarithmic branch cut singularities unless the residues at $z = \pm i$ vanish. Thus, the first two conditions on $Y(z)$ for (C.3.13) to be analytic are

$$\text{Res}_{z=\pm i}\,\frac{Y(z)}{(1+z^2)^2} = 0 \ . \tag{C.3.14}$$

Note that it is admissible for $Y(z)/(1+z^2)^2$ to have double pole at $z = \pm i$, as these integrate to a single pole, which is then canceled by the $(1+z^2)$ prefactor in (C.3.13). Therefore, the double poles do not give rise to singularities.

There is also another condition, namely that the contour integral of $X'(z)$ around the unit circle vanishes so that we get a periodic function $X(z)$ after integration. Since polynomials in z are automatically periodic, it suffices to consider the behavior of the integrand, $Y(z)/(1+z^2)^2$. Assuming that $Y(z)$ is analytic except possibly at $z = 0$, this amounts to the condition

$$\text{Res}_{z=0}\,\frac{Y(z)}{(1+z^2)^2} = 0 \ . \tag{C.3.15}$$

Note that poles near $z = \pm i$ do not affect the periodicity so we can subtract them before applying this condition if necessary, and we also assume the residues vanish as above, so that we have a well-defined integral.

To see how this works in practice, it is useful to evaluate this for a trial function Y inspired by the algebraic case:

$$Y_0 = a(1+z^2) + bz^{-1} + cz^3 \ , \tag{C.3.16}$$

which contains L_2, L_{-2} and H_{mod}. We notice that

$$\frac{Y_0}{(1+z^2)^2} = \frac{i(b+c)}{4(z-i)^2} + \frac{-b-ia+c}{2(z-i)} + \dots \ , \tag{C.3.17}$$

$$\frac{Y_0}{(1+z^2)^2} = \frac{-i(b+c)}{4(z+i)^2} + \frac{-b+ia+c}{2(z+i)} + \dots \tag{C.3.18}$$

near $z = \pm i$ respectively. The residue of $Y_0/(1+z^2)^2$ at $z = i$ equals $-(iY_0(i) + Y_0'(i))/4$ and the residue at $z = -i$ equals $(iY_0(-i) - Y_0'(-i))/4$, and these are required to vanish by (C.3.14). This translates to $b = c$ and $a = 0$. Recall that the differential equation, (C.3.11), extracts the non-zero mode part, i.e., the vector fields that can be written as $[H_{\text{mod}}, X]$. We could also ask how to extract the zero mode part. In this case it seems the most natural choice to extract a, which is given by the difference of the two residues, as the coefficient of the zero mode.

Even in the case $b = c$ and $a = 0$ with vanishing residues, we see that X will now involve a term $(1+z^2)\log z$ since $Y = z^{-1} + z^3 = (z^2+1)^2 z^{-1} - 2z$. This still has a branch cut singularity, and therefore will not be single-valued. This is where a version of the third condition, (C.3.15), is necessary. To be more precise about this requirement, take a finite polynomial in z, z^{-1} for Y. We first subtract the harmless double poles and the harmful single poles (which we require to vanish independently) so that we get an expression of the type

$$Z(z) \equiv \frac{Y(z) - A - Bz - Cz^2 - Dz^3}{(1+z^2)^2} \ , \tag{C.3.19}$$

where the coefficients A, B, C, D are chosen so as to cancel the single and double poles. To accomplish this, it is necessary for an overall factor $(1+z^2)^2$ to factor out of the numerator. The choice of coefficients can then be determined by the requirement that the numerator of Z and its derivative both vanish at $z = \pm i$. Explicitly, they are given by

$$A = \frac{1}{4}(2Y(-i) + 2Y(i) + iY'(-i) - iY'(i)) \ , \tag{C.3.20}$$

$$B = \frac{1}{4}(3iY(-i) - 3iY(i) - Y'(-i) - Y'(i)) \ , \tag{C.3.21}$$

$$C = \frac{i}{4}(Y'(-i) - Y'(i)) \ , \tag{C.3.22}$$

$$D = \frac{1}{4}(iY(-i) - iY(i) - Y'(-i) - Y'(i)) \ . \tag{C.3.23}$$

With this choice of coefficients the expression, (C.3.19), is now well-behaved everywhere, i.e., the numerator has a factor $(1+z^2)^2$, and the quotient is also a finite polynomial in z and z^{-1}. The only problematic contribution to the integral is coming from the z^{-1} term which does not become a periodic function when integrated. So the remaining number is the coefficient in front of z^{-1} in the polynomial $Z(z)$ in (C.3.19).

We denote by Y_- the terms in Y with a negative power of z. The non-negative powers in Y only give rise to non-negative powers in Z and are never problematic. So we can equivalently consider

$$Z_-(z) \equiv \frac{Y_-(z) - A - Bz - Cz^2 - Dz^3}{(1+z^2)^2} \ , \tag{C.3.24}$$

and we are interested in the coefficient in front of z^{-1} in $Z_-(z)$. We can extract this using a small contour integral. But we might as well extract it using a large contour integral as Z_- is analytic everywhere except at 0 and ∞. Then the integral is dominated by D, so it is necessary that $D = 0$ for the integral to be single-valued. In fact, D is equal to the sum of the residues at $z = i$ and $z = -i$, as can be seen from (C.3.23), so this version of the third condition with the double poles subtracted out reduces to

$$\mathrm{Res}_{z=i} \frac{Y_-}{(1+z^2)^2} + \mathrm{Res}_{z=-i} \frac{Y_-}{(1+z^2)^2} = 0 \ . \tag{C.3.25}$$

Since the residues of the complete $Y/(1+z^2)^2$ have to vanish separately, we could equivalently require the same condition for Y_+.

For more general non-polynomial Y, we can apply the same argument, except that now Y_- is analytic outside the unit disk and Y_+ is analytic inside the unit disk. By the version of the Riemann-Hilbert problem that applies to simple closed curves, a decomposition of analytic functions on the circle of the type $Y_- + Y_+$ exists.

C.3.3 Issues from non-diagonalization

In this subsection, we will show that the ambiguities in the diagonalization of the Virasoro algebra with respect to the adjoint action translate to ambiguities in the projection operator. This leads to different answers for the Berry curvature that are physically inequivalent. As a result, there is no sensible bulk interpretation.

It is because parallel transport acting by elements of the usual Virasoro algebra is plagued with ambiguities that we are forced to extend to a non-standard algebra constructed from vector fields on the half-circle as in Section 5.3, where the construction is unique.

For the ordinary Virasoro case, we want to construct a zero-mode projector P_0 so that it evaluates to zero for the integrand of (5.3.15), while it gives a non-zero value for (5.3.9). In other words we can devise a contour integral prescription in such a way as to satisfy the properties:

◇ The functional is non-zero on the modular Hamiltonian, i.e., $P_0\left(H_{\mathrm{mod}}\right) = 1$,

◇ It projects out the commutator of the modular Hamiltonian with anything else, i.e., $P_0\left([H_{\mathrm{mod}}, X_\xi]\right) = 0$, for any vector field $\xi(z)$.

We emphasize that this is a different projection operator than the one considered in Section 5.3, in particular it is finite rather than a delta function.

There are several different choices that obey both of these properties:

$$P_0^{(1)}(X_\xi) \equiv -\frac{1}{\pi^2}\int_{|z+i\epsilon|=1}\frac{\xi(z)}{(1+z^2)^2}\,dz \ , \tag{C.3.26}$$

$$P_0^{(2)}(X_\xi) \equiv \frac{1}{\pi^2}\int_{|z-i\epsilon|=1}\frac{\xi(z)}{(1+z^2)^2}\,dz \ , \tag{C.3.27}$$

$$P_0^{(3)}(X_\xi) \equiv \frac{1}{2}\left(P_0^{(1)}(X_\xi)+P_0^{(2)}(X_\xi)\right) \ . \tag{C.3.28}$$

By explicitly computing the residues, one can express these in terms of the diffeomorphism ξ and its derivative evaluated at the endpoints of the interval as

$$P_0^{(1)}(X_\xi) = \frac{1}{2\pi}\left[\xi(-i)+i\xi'(-i)\right] \ , \tag{C.3.29}$$

$$P_0^{(2)}(X_\xi) = \frac{1}{2\pi}\left[\xi(i)-i\xi'(i)\right] \ , \tag{C.3.30}$$

$$P_0^{(3)}(X_\xi) = \frac{1}{4\pi}\left[i\,\xi'(-i)-i\,\xi'(i)+\xi(-i)+\xi(i)\right] \ . \tag{C.3.31}$$

Note that the sum of contours $P_0^{(2)} - P_0^{(1)}$ does not satisfy the required properties, as it vanishes on the modular Hamiltonian. The difference of contours, Eqs. (C.3.28) and (C.3.31), is perhaps the most symmetrical choice. It can be seen to result from the decomposition, (C.3.4), by additionally imposing that the linear functional evaluated on the extra terms L_2, L_{-2} in the decomposition give zero. However, recall that this decomposition was not unique. A different choice would have resulted in a different linear functional, and therefore a different P_0.

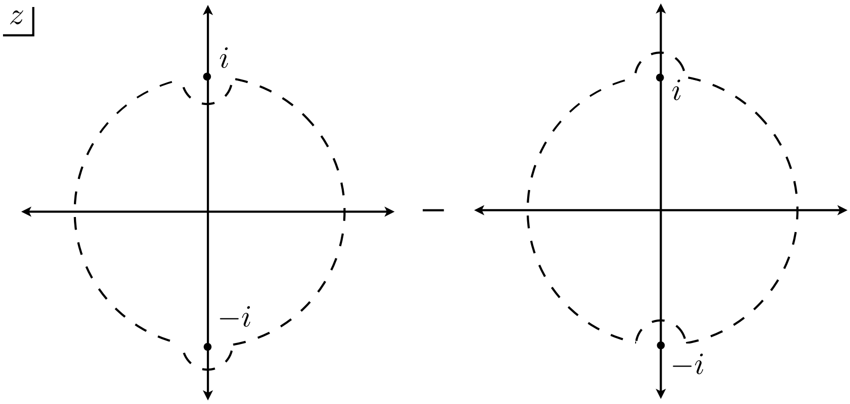


Figure C.2: *One simple choice of linear functional, constructed from the difference of $|z - i\epsilon| = 1$ and $|z + i\epsilon| = 1$ contours. When considering a non-restricted set of generators, there is an ambiguity in the choice of projection. For instance, it is also possible to choose either of these contours separately (but not their sum) and still satisfy the required properties for the linear functional. This ambiguity is tied to the fact that the adjoint action is not diagonalizable over the Virasoro algebra.*

Moreover, we have considered the possibility of defining a zero mode projector P_0 using very early or very late time modular flow. However, we found that this prescription is also ambiguous and depends on whether one considers very early or very late times.

It is also easy to see that this has a direct physical implication by leading to different results for the curvature. For instance, consider an infinitesimal diffeomorphism of the form

$$\theta \to \theta + 2\epsilon \sin(m\theta) \ , \tag{C.3.32}$$

where $m \in \mathbb{Z}$. The parameter ϵ is assumed to be small and dimensionless.

One can consider a parallel transport process consisting of a series of such infinitesimal transformations, where m can vary from step to step. It is described by a function $m(\lambda)$, where λ denotes the point along the path evaluated in the continuum limit. Mapping from the cylinder to the plane using (C.3.32) and expanding to first order in ϵ, this sinusoidal perturbation becomes

$$\xi(z) = z + \epsilon(z^{m+1} - z^{-m+1}) + \mathcal{O}(\epsilon^2) \ . \tag{C.3.33}$$

Up to terms that are higher order in ϵ, (C.3.33) can be inverted to $z = \xi - \epsilon(\xi^{m+1} - \xi^{-m+1}) + \mathcal{O}(\epsilon^2)$. Inserting this in (5.3.9) for H_{mod}, we find the correction to the modular Hamiltonian:

$$H^{(1)} = \pi\left[(m+1)(L_{-m+1}+L_{m-1})+(m-1)(L_{-m-1}+L_{m+1})\right] \ . \tag{C.3.34}$$

Recall that expanding both the parallel transport equation $H_{\text{mod}} = [S, H_{\text{mod}}]$ order by order in ϵ gave (5.3.42). Solving for the correction to the parallel transport operator gives

$$S^{(1)} = L_m - L_{-m} \ , S^{(0)} = 0 \ . \tag{C.3.35}$$

Take two transformations of the form Eq. (C.3.33) with different values for the integer m, say m_1 and m_2. This gives two different parallel transport operators, S_1 and S_2. To compute the curvature, (5.3.3), we are interested in computing the commutator

$$[S_1^{(1)} - \kappa_1 H^{(0)}, S_2^{(1)} - \kappa_2 H^{(0)}] \ , \tag{C.3.36}$$

where $\kappa_i = P_0(S_i)$, is the zero mode coefficient of the parallel transport operator S_i. We can split (C.3.36) into terms that we can treat separately. Notice that the term proportional to $[H^{(0)}, H^{(0)}]$ is zero and can be neglected. By definition, the projection operator vanishes on $[S_i^{(1)}, H^{(0)}]$, so this contribution to the curvature is zero. An explicit computation shows that

$$\begin{aligned}[S_1^{(1)}, S_2^{(1)}] = (m_1 - m_2)&(L_{m_1+m_2} - L_{-m_1-m_2}) \\ &+ (m_1+m_2)(L_{-m_1+m_2} - L_{m_1-m_2}) \ . \end{aligned} \tag{C.3.37}$$

We will now project onto the zero modes of each of the terms. This is where the ambiguity enters since the result depends on the choice of linear functional. We find

$$F^{(1)} = P_0^{(1)}([S_1^{(1)}, S_2^{(1)}]) = \frac{2i}{\pi}(m_2^2 - m_1^2)\sin\left(\frac{m_1\pi}{2}\right)\sin\left(\frac{m_2\pi}{2}\right) \ , \tag{C.3.38}$$

$$F^{(2)} = P_0^{(2)}([S_1^{(1)}, S_2^{(1)}]) = -F^{(1)} \ , \tag{C.3.39}$$

$$F^{(3)} = P_0^{(3)}([S_1^{(1)}, S_2^{(1)}]) = 0 \ . \tag{C.3.40}$$

Notice that in the case where the m_1, m_2 are even, all curvatures agree and in fact identically vanish. Indeed, it is possible to argue that the curvature defined in this way always vanishes for diffeomorphisms that vanish on the interval endpoint. However, in general they do not agree and the result is ambiguous.

C.4 Conformal algebra

We will review here some facts about the d-dimensional conformal algebra, which will set our conventions.

The conformal generators are

$$D = -x^\mu \partial_\mu \ , \ P_\mu = -\partial_\mu \ , \ C_\mu = x^2 \partial_\mu - 2x_\mu x^\rho \partial_\rho \ , \ M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \ . \quad \text{(C.4.1)}$$

The resulting commutation relations are given by

$$\begin{aligned} [D, P_\mu] &= P_\mu \ , \qquad [D, C_\mu] = -C_\mu \ , \\ [C_\mu, P_\nu] &= 2(\eta_{\mu\nu} D - M_{\mu\nu}) \ , \\ [M_{\mu\nu}, P_\rho] &= -\eta_{\mu\rho} P_\nu + \eta_{\nu\rho} P_\mu \ , \\ [M_{\mu\nu}, C_\rho] &= -\eta_{\mu\rho} C_\nu + \eta_{\nu\rho} C_\mu \ , \\ [M_{\mu\nu}, M_{\sigma\rho}] &= -\eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\rho} M_{\nu\sigma} \ . \end{aligned} \quad \text{(C.4.2)}$$

Note that we have written $\mu = (0, i)$, where $i = 1, \ldots, d-1$ is some spatial index.

The bilinear product on the conformal algebra is given by

$$\langle X, Y \rangle \equiv \frac{1}{2} \mathrm{tr}(XY) \ , \quad X, Y \in \mathfrak{so}(2, d) \ , \quad \text{(C.4.3)}$$

where the trace is taken in the fundamental representation. In terms of the above generators the inner product is normalized such that non-zero entries are given by:

$$\langle D, D \rangle = \langle M_{0i}, M_{0i} \rangle = -\langle M_{ij}, M_{ij} \rangle = 1 \ , \quad \langle P_0, C_0 \rangle = -\langle P_i, C_i \rangle = 2 \ . \quad \text{(C.4.4)}$$

C.5 Relative entropy and quantum Fisher information

In this appendix we review the derivation of a metric on the space of density matrices from the second variation of the relative entropy [248, 249]. The relative entropy between two states σ and ρ is given by:

$$S(\sigma || \rho) \equiv \mathrm{tr}(\sigma \log \sigma) - \mathrm{tr}(\sigma \log \rho) \ . \quad \text{(C.5.1)}$$

Let us view σ as obtained from ρ by some small perturbation:

$$\sigma = \rho + \varepsilon \delta\rho + O(\varepsilon^2) \ , \quad \text{(C.5.2)}$$

where ε is some small parameter. Then, the second derivative with respect to this parameter can be expressed as:

$$\frac{d^2}{d\varepsilon^2}S(\sigma||\rho) = \mathrm{tr}\left(\delta\rho\frac{d}{d\varepsilon}\log(\rho+\varepsilon\delta\rho)\right) \ . \tag{C.5.3}$$

To compute the derivative we use the following integral representation for the logarithm of an operator:

$$\log(\rho+\varepsilon\delta\rho) = -\int_0^\infty \frac{ds}{s}\left(e^{-s(\rho+\varepsilon\delta\rho)} - e^{-s}\right) \ . \tag{C.5.4}$$

One can now take the derivative by using the relation

$$\frac{d}{d\varepsilon}e^{A+\varepsilon B} = \int_0^1 dx\, e^{Ax}Be^{(1-x)A} \ , \tag{C.5.5}$$

for two operators A and B. Using (C.5.4) it now follows that

$$\frac{d^2}{d\varepsilon^2}S(\sigma||\rho) = \int_0^1 dx\int_0^\infty ds\,\mathrm{tr}\left(\delta\rho\, e^{-xs\rho}\delta\rho\, e^{-(1-x)s\rho}\right) \ . \tag{C.5.6}$$

We can now evaluate the trace in the eigenbasis of the modular Hamiltonian associated to the state ρ:

$$\rho|\omega\rangle = e^{-\omega}|\omega\rangle \ . \tag{C.5.7}$$

We can write this as:

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}S(\sigma||\rho) &= \int_0^1 dx\int_0^\infty ds\int d\omega\int d\omega'|\langle\omega|\delta\rho|\omega'\rangle|^2 e^{-sx(e^{-\omega}-e^{-\omega'})}e^{-se^{-\omega'}}\\ &= \int d\omega\int d\omega'|\langle\omega|\delta\rho|\omega'\rangle|^2 e^{\omega}(\omega-\omega')n(\omega-\omega') \ .\end{aligned} \tag{C.5.8}$$

Using again the sinh-formula (6.2.54) to replace the integral over frequencies by an integral over modular time, and removing the explicit $|\omega\rangle$ basis we find that this expression is equivalent to

$$\delta^{(2)}S(\sigma||\rho) = \int_{-\infty-i\epsilon}^{\infty-i\epsilon} ds\frac{\pi}{2\sinh^2(\pi s)}\mathrm{tr}(\rho^{-1}\delta\rho\,\rho^{-is}\delta\rho\,\rho^{is}) \ . \tag{C.5.9}$$

This is an expression for the second-order variation of the relative entropy. We will now define a metric on the space of quantum states starting from the above expression. The second derivative of the relative entropy (C.5.9) is a quadratic function in the state perturbations $\delta\rho$, so we can upgrade it to a bilinear form by taking two (possibly) different variations $\delta_1\rho, \delta_2\rho$ on the right-hand side. Plugging in the expressions for $\delta\rho$ in terms of the operators $\mathcal{O}$ using (6.2.6) we find that

$$\delta_1\delta_2 S(\sigma||\rho) = \int d^dx \int d^dx'\, \delta_1\lambda(x)\delta_2\lambda(x') \int_{-\infty-i\epsilon}^{\infty-i\epsilon} ds \frac{\pi}{2\sinh^2(\pi s)} \langle \mathcal{O}(x)\mathcal{O}_s(x')\rangle \ , \tag{C.5.10}$$

where the expectation value is taken in the reference state ρ. This is also known as the *quantum Fisher information metric* [248,249]. This expression agrees with the 'metric' G_Ψ associated to the modular Berry curvature (6.2.55). We have therefore established our identification.

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Samenvatting

Zwarte gaten zijn bijzondere objecten in het heelal. Om ze volledig te begrijpen hebben we een theorie nodig die zowel de effecten van de quantummechanica als van de algemene relativiteitstheorie meeneemt. De eerste is nodig om het universum op de kleinste schaal, de wereld van elementaire deeltjes, te beschrijven. De laatste wordt pas belangrijk op veel grotere schalen, en beschrijft de aantrekkingskracht tussen zware objecten. Het combineren van beide theorieën tot een geheel, ook wel 'een theorie van alles' genoemd, is een van de belangrijkste open problemen in de theoretische natuurkunde.

Wanneer je probeert om te veel massa in een te klein volume te stoppen is het resultaat een object met zo'n sterke aantrekkingskracht dat niets eraan kan ontsnappen. Dit gebeurt bijvoorbeeld als een ster tegen het einde van zijn leven, onder invloed van zijn eigen zwaartekracht, in elkaar stort. Zelfs licht, dat een soort universele snelheidslimiet verschaft, kan niet aan deze aantrekkingskracht ontsnappen. Het resultaat is een waarnemingshorizon: een fictief grensvlak dat alles aan het zicht onttrekt. In een zwart gat vallen is niet bijzonder plezierig. Ervan uitgaande dat je niet uit elkaar wordt getrokken door extreme getijdenkrachten, is de onvermijdelijke bestemming van iedereen die de horizon passeert de singulariteit: een punt in de ruimtetijd met oneindige kromming.

Het kwam als een grote verrassing toen Stephen Hawking aantoonde dat zwarte gaten niet volledig zwart zijn wanneer je quantumeffecten bij de horizon meeneemt. De quantummechanica voorspelt dat zwarte gaten een deel van hun energie verliezen in de vorm van straling. Deze Hawkingstraling geeft aanleiding tot veel van de raadsels rondom zwarte gaten. Aangezien een zwart gat straalt, kun je er een temperatuur aan toekennen. Een object met een temperatuur heeft ook een entropie die volgens de standaardregels van de statistische fysica het aantal microscopische configuraties telt dat je met een bepaalde macroscopische toestand van het systeem kunt associëren.

In gebruikelijke thermodynamische systemen schaalt de entropie met het volume van het systeem. Zwarte gaten daarentegen lijken een ander soort regel te volgen: Ze hebben een entropie die schaalt met het oppervlak van de horizon. Terwijl we voor een heet gas weten wat de statistische entropie telt, namelijk de verschillende configuraties van de individuele moleculen waaruit de gaswolk bestaat, is de situatie voor een zwart gat veel moeilijker. Intuïtief, zouden deze microtoestanden moeten overeenkomen met een soort fundamentele bouwstenen van de ruimtetijd zelf. Maar wat deze bouwstenen precies zijn, en waarom ze een oppervlaktewet volgen, is nog steeds een groot mysterie.

Het feit dat de entropie van een zwart gat een oppervlaktewet volgt gaf aanleiding tot het 'holografisch principe': Een theorie van quantumzwaartekracht is duaal aan een quantumtheorie zonder zwaartekracht die in één dimensie minder leeft. Hierbij gedraagt zwaartekracht zich dus als een soort hologram, waarbij de informatie van een driedimensionaal beeld is gecodeerd in een tweedimensionaal object. In het geval van het zwarte gat wordt de informatie aan de binnenkant in zekere zin op het horizonoppervlak 'opgeslagen'.

Het idee van holografie klinkt op het eerste gezicht misschien wat vreemd. Toch is er een natuurkundig model gevonden waarin het op een prachtige manier wordt gerealiseerd: de AdS/CFT-correspondentie. Deze correspondentie relateert quantumzwaartekracht en quantumveldentheorie op een holografische manier. AdS/CFT vindt zijn oorsprong in de snaartheorie, een kandidaattheorie voor quantumzwaartekracht, waarin elementaire deeltjes worden vervangen door kleine trillende snaren. Helaas gaat de AdS/CFT correspondentie niet over ons eigen universum, maar over zwaartekracht in Anti-de Sitterruimte (AdS), een universum met een negatieve kosmologische constante. Aangenomen wordt dat ons eigen universum een kleine positieve kromming heeft en daarom (bij benadering) wordt beschreven door de Sitterruimte (dS). Vanuit een wiskundig perspectief zijn beide universa echter oplossingen van Einsteins vergelijkingen, en een gedetailleerd begrip van AdS-zwaartekracht kan ons hopelijk veel leren over quantumzwaartekracht in ons eigen universum.

In het kort, zegt AdS/CFT dat quantumzwaartekracht in $(d+1)$-dimensionale AdS-ruimte equivalent kan worden beschreven door een conforme veldentheorie (CFT), een speciale quantumtheorie met veel symmetrie, in d dimensies. Gewoonlijk wordt de CFT gevisualiseerd op de asymptotische rand van de AdS-ruimte. In die zin werkt de dualiteit een beetje als een soepblik: Door de buitenkant te lezen, kun je construeren wat er binnenin gebeurt. De AdS/CFT-correspondentie bestaat uit een uitgebreid 'woordenboek' dat een manier biedt om bepaalde grootheden van de ene theorie naar de andere te 'vertalen,' en vice versa. Zo'n dualiteit is buitengewoon handig: Om een moeilijk probleem in de zwaartekracht op te lossen, kun je het te vertalen naar een gemakkelijker vraagstuk in de CFT, of andersom.

Hoewel we in de loop der jaren veel aspecten van de AdS/CFT-correspondentie hebben begrepen (d.w.z., we hebben een zeer uitgebreid holografisch woordenboek opgebouwd), zijn er nog steeds veel open vragen. Een aantal van deze vragen komen in dit proefschrift aan de orde. Alle ideeën die hier worden gepresenteerd vinden plaats in de context van de AdS/CFT-correspondentie, en de meeste hebben betrekking op een eenvoudig model voor tweedimensionale zwaartekracht, met een ruimte- en een tijdsdimensie. Deze vereenvoudiging maakt dat alle berekeningen exact kunnen worden uitgevoerd

Euclidische wormgaten

Een formele benadering voor het quantiseren van zwaartekracht is de gravitationele padintegraal. In het padintegraalformalisme wordt men geïnstrueerd om bij een berekening alle mogelijke uitkomsten bij elkaar op te tellen, ieder met een bepaald gewicht (meer precies loopt de som over alle mogelijke paden in de ruimte van quantumvelden die een bepaalde begin- en eindtoestand met elkaar verbinden). Hoewel de padintegraal verre van rigoureus is op een wiskundig niveau, is het een essentieel onderdeel in ons begrip van de quantumwereld. Je kunt proberen dezelfde technieken toe te passen op zwaartekracht. Het toevoegen van quantumfluctuaties van de ruimtetijd zelf is echter een zeer ingewikkelde onderneming. In dit proefschrift hebben we ervoor gekozen om een eenvoudig model voor tweedimensionale zwaartekracht te bestuderen waarbij het mogelijk is om een klasse van dergelijke quantumfluctuaties, die overeenkomen met meetkundes met een niet-triviale topologie, precies te definiëren (en te berekenen).

Topologie is de wiskundige term die de globale vorm van een object beschrijft: Het is een van de eigenschappen die een tennisbal en een donut verschillend maakt. In onze setting gebruiken we het concept van topologie om quantumcorrecties op de ruimtetijd te bestuderen. De padintegraalberekening die we uitvoeren omvat meer algemene topologieën, waar de ruimtetijd bijvoorbeeld handvatten (zoals bij een donut) of extra randen kan hebben. Een belangrijke bijdrage wordt geleverd door Euclidische wormgaten die meerdere randen van de ruimtetijd verbindt door een soort tunnel. Een van de voornaamste resultaten van dit proefschrift is de presentatie van een formalisme voor het systematisch bestuderen van deze hogere topologieën, een zogenaamde 'universumveldentheorie.' Je kunt deze theorie gebruiken om de quantumeffecten tot op zeer hoge resolutie te bestuderen. Verrassend genoeg blijken de niet-triviale oppervlakken belangrijke informatie over de onderliggende microscopische theorie te bevatten: Ze wijzen op een diepe connectie met quantumchaos.

Quantumchaos

Chaostheorie gaat over de verborgen structuur in ogenschijnlijk chaotische systemen. In populaire fictie wordt chaos vaak geassocieerd met het 'vlindereffect', waarbij het fladderen van de vleugels van een vlinder de precieze details (bijvoorbeeld de locatie) van een tornado weken later kan beïnvloeden. Met andere woorden: Kleine veranderingen kunnen uit de hand lopen, en dramatische gevolgen hebben. Dit is een kenmerk van chaos in klassieke systemen. Het analogon voor quantumsystemen is veel moeilijker te karakteriseren. Een belangrijk kenmerk dat natuurkundigen hebben ontdekt is dat de energieniveaus van een chaotisch quantumsysteem aan zeer strikte regels voldoen: Twee van deze energieniveaus stoten elkaar af, wat betekent dat ze de neiging hebben om uit elkaars buurt te blijven. Dit gedrag wordt goed gemodelleerd door 'toevalsmatrixtheorie,' een quantummechanisch model waarbij de energieën worden getrokken uit een statistisch ensemble.

Er zijn aanwijzingen dat zwarte gaten voldoen aan dezelfde regels van quantumchaos. Het zijn bijvoorbeeld snelle 'scramblers': Informatie die het zwarte gat binnenkomt wordt heel snel over de horizon verspreid. Bovendien hangt de uitkomst van een deeltjesproces dicht bij de horizon, zeg het verschil tussen naar binnen vallen of ontsnappen, sterk af van de precieze begintoestand. Onze techniek om de chaotische eigenschappen in bredere zin te begrijpen is met behulp van de padintegraal. We laten zien hoe de kleine quantumeffecten die het individuele energieniveau bepalen geometrisch kunnen worden gerealiseerd in termen van de niet-triviale topologieën die we eerder bespraken. Dit toont aan dat zwarte gaten in tweedimensionale AdS-ruimtetijden voldoen aan de regels van quantumchaos, en dat de relevante quantumeffecten geometrisch kunnen worden gerealiseerd in de universumveldentheorie.

Berry-fasen

Het laatste onderwerp dat ik in dit proefschrift heb bestudeerd betreft het concept van Berry-fasen. In de quantummechanica is het mogelijk dat deeltjes veranderen wanneer ze langs een gesloten pad worden getransporteerd: Ze komen anders terug dan dat ze begonnen zijn, en pikken een waarneembare quantumfase op. Het is een beetje zoals een ommetje maken: Bij terugkomst kun je je een heel ander mens voelen dan toen je vertrok. Dergelijke geometrische fasen zijn uitgebreid bestudeerd in de setting van quantummechanica. Een van de doelen van dit proefschrift is om een vergelijkbaar concept te importeren in de holografische context.

Als we aannemen dat AdS-quantumzwaartekracht een duale beschrijving heeft in termen van een CFT, kun je de volgende vraag stellen: Welke zwaartekrachtsberekening komt overeen met de quantummechanische Berry-fase? Met andere woorden: Hoe vertalen we het begrip 'Berry-fase' via het holografisch woordenboek? Allereerst vereist deze vraag een zorgvuldige definitie van wat we bedoelen met Berry-fasen in een holografische CFT. De juiste generalisatie, die ook werkt in het geval van dichtheidsmatrices die behoren bij een deelgebied van de CFT, is de modulaire Berry-fase. Het legt belangrijke informatie vast over het verstrengelingsspectrum van de CFT. Quantumverstrengeling (bekend als 'spookachtige actie op afstand') tussen deeltjes is een vreemde eigenschap die geen klassiek analogon heeft. Dit begrip is van essentieel belang voor manier waarop we tegenwoordig tegen de emergentie van ruimtetijd aankijken.

In dit proefschrift hebben we aangetoond dat de modulaire Berry-fase duaal is aan een geometrische grootheid die geassocieerd is met de 'entanglement wedge,' een bepaald gebied in de AdS-ruimtetijd dat informatie bevat over een deel van de rand. Dit voegt een nieuw 'woord' toe aan het holografische woordenboek.

Summary

Black holes are some of the most mysterious objects in Nature. They provide a unique laboratory, mostly in a theoretical sense, where both the effects of quantum mechanics and general relativity are important. The former is necessary to understand the universe at the smallest scales, those associated with elementary particles. The later becomes important at much larger scales, when we are dealing with the gravitational pull between very massive objects. To combine both theories into a unified framework, sometimes called 'a theory of everything', is one of the most important open problems in theoretical physics.

The laws of gravity dictate that when one tries to put too much mass in too small a volume, the result is an object with a gravitational attraction that is so strong that nothing can escape. Not even light – that provides some sort of universal speed limit – can escape its pull. The result is an event horizon: a fictitious black boundary surface that covers everything from view. Falling into a black hole is not particularly enjoyable: Assuming that you are not ripped apart by extreme tidal forces, the inevitable destination of any in-falling observer is the singularity, a point of zero size and infinite curvature.

It came as a big surprise when Stephen Hawking showed that black holes are not actually black when one includes quantum effects at the horizon. Quantum mechanics predicts that black holes carry away some of their energy in the form of radiation. This so-called Hawking radiation is at the heart of many confusions that arise in the study of black holes. Given that a black hole radiates, one can associate a temperature to it. An object with a temperature also has an entropy which counts, following standard statistical mechanics, the number of microscopic configurations that one can associate to a given macroscopic state of the system.

In usual thermodynamic systems, the entropy scales with the volume of the region. Black holes on the other hand seem to follow a different rule: They have an entropy that scales with the area of the event horizon. While for a hot gas we know that the statistical entropy is counting, namely the different configurations of the individual molecules that make up the cloud, the situation for a black hole is much harder: What are the precise microstates that the black hole entropy is counting? Intuitively, they should correspond to some fundamental building blocks of spacetime itself. What these units are, and why they satisfy an area-scaling law is still an open question, that a full theory of quantum gravity should answer.

An important idea that came out of this observation, and found a concrete realization in string theory, is the holographic principle. It states that a theory of quantum gravity is dual to a quantum theory that lives in one dimension fewer. In that sense, it behaves like a hologram, where the information of a three-dimensional picture is encoded in a two-dimensional object. This explains, in part, why the information inside the black hole should somehow be encoded on its boundary horizon.

The idea of holography might sound a bit strange at first sight. However, we have actually found a physical model where it is beautifully realized: the AdS/CFT correspondence. It relates gravity and quantum field theory in a holographic fashion. Unfortunately, this correspondence does not deal with our own universe, but with gravity in Anti-de Sitter (AdS) space, a universe which has negative cosmological constant. Our own universe is believed to have small positive curvature and is therefore described (in an approximate sense) by de Sitter (dS) space. However, from a mathematical perspective both spacetimes are solutions to Einsteins equations, and a detailed understanding of AdS gravity can hopefully teach us something about quantum gravity in our own universe.

In short, AdS/CFT says that quantum gravity in $(d+1)$-dimensional AdS space can be equivalently described by a conformal field theory (CFT), a special quantum theory with lots of symmetry, in d dimensions. Usually, the CFT is visualized as living on the asymptotic boundary of AdS space. In that sense, the duality works a bit like a soup can: By reading the boundary, one can construct what happens on the inside. Most of the AdS/CFT correspondence consists of an elaborate dictionary that provides a way of translating certain quantities from one theory into the other and vice versa. Such a duality is extremely useful: To solve a difficult problem in gravity, one can try to translate it to an easier question in the CFT, or the other way around.

While over the years, we have understood many aspects of the AdS/CFT correspondence (i.e., we have built up a very extensive holographic dictionary), there are still lots of open questions remaining. A few of those are addressed in this thesis. All the ideas that are presented here take place in the context of the AdS/CFT correspondence, and many involve a simple toy model for two-dimensional gravity, with a single space and a single time dimension. This simplification makes the computations more tractable.

Euclidean wormholes

A formal approach to quantizing gravity is the gravitational path integral. Roughly speaking, in quantum mechanics one is instructed to sum over all possibilities, where each possible path is given some weight. This idea was concretely realized by Richard Feynman in his path integral formalism. Although it is far from rigorous on a mathematical level, the path integral has been an essential tool for our understanding of the quantum world. One could try to naively apply the same techniques to gravity, but there the situation is even worse. To include quantum fluctuations of spacetime itself is a difficult task, and one quickly runs into trouble. In this thesis, we have studied a simple model for two-dimensional gravity where it is possible to exactly define (and compute) a class of such quantum fluctuations, which correspond to geometries with non-trivial topology.

Topology is the mathematical term that describes the global shape of some object: It is one of the differences (besides their respective taste) between a tennis ball and a donut. In our setting, we use the concept of topology to study quantum corrections to spacetime. The path integral computation that we carry out includes more general topologies, where spacetime can exhibit handles or extra boundaries. An important contribution of this sort are given by Euclidean wormholes which connect multiple boundaries by a tube. One of the main results in this thesis is the presentation of a systematic framework for studying these higher topologies, a so-called 'universe field theory.' One can use this framework to carefully study the quantum effects up to very high resolution. Surprisingly, it turns out that the non-trivial shapes carry important information about the underlying microscopic theory: They point towards a deep connection with quantum chaos.

Quantum chaos

Chaos theory deals with the hidden structure in seemingly chaotic systems. In popular fiction, chaos is often associated with the 'butterfly effect,' where the distant flapping of a butterfly's wings can affect the precise details (for example, location or trajectory) of a tornado weeks later. In other words: Small changes, can blow up to have dramatic consequences. This is a characteristic feature of chaos in classical systems. The analogue for quantum systems is much harder to characterize. One important feature that people have found is that the energy levels of a chaotic quantum system satisfy very strict rules: For example, two such energy levels repel each other, meaning that they have the tendency to not be close together. This behavior is well-modeled by random matrix theory, a quantum mechanical model where the energies are drawn from some specific statistical ensemble.

There is evidence that black holes satisfy the same rules of quantum chaos. For example, they are fast scramblers, in the sense that information that enters the black hole is spread around the horizon very rapidly. Moreover, the outcome of a particle process near the horizon, the difference between falling into the interior or escaping to infinity, depends heavily on the precise initial condition. Our technique to exhibit the chaotic properties more broadly is to study quantum gravity via the path integral. We show how the tiny quantum effects that govern the individual energy level can be realized geometrically in terms of the non-trivial topologies that we discussed earlier. This shows that black holes in two-dimensional AdS spacetimes satisfy the rules of quantum chaos, and that the relevant quantum effects can be realized geometrically in the universe field theory.

Berry phases

The last topic that I have studied in this thesis involves the notion of Berry phases. In quantum mechanics, it is possible for particles to change when they get transported along a closed loop: They do not come back the way they started out, and pick up an observable phase. It is a bit like taking a walk: After getting back to your starting position you can feel like a changed person. Such geometric phases have been studied extensively in the setting of quantum mechanics. One of the aims of this thesis is to import a similar concept to the holographic context.

If we assume that AdS quantum gravity has a dual description in terms of a CFT, one can ask the following question: What gravitational quantity corresponds to the quantum mechanical Berry phase? In other words: How do we translate the concept 'Berry phase' via the holographic dictionary? First of all, this question requires a careful definition of what we mean by Berry phases in a holographic CFT. The correct generalization that also work in the case of density matrices associated to some subregion is the modular Berry phase. It captures important information about the entanglement spectrum of the CFT. Quantum entanglement (famously known as 'spooky action' at a distance) between particles is a strange property that does not have a classical analogue. It is one of the most important quantities that we use in our current understanding of what spacetime is built from.

In this thesis, we have argued that the modular Berry phase is dual to some geometrical quantity associated to the entanglement wedge, a certain region in the AdS spacetime that carries information about part of the boundary. This adds a new entry to the holographic dictionary.