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# Deformed Mirror Symmetry for Punctured Surfaces 

Jasper van de Kreeke


2023

# Deformed Mirror Symmetry for Punctured Surfaces 

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aan de Universiteit van Amsterdam
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## Introduction

Mirror symmetry is a phenomenon observed in string theory and has been translated to mathematics by Maxim Kontsevich in 1994. Homological mirror symmetry is broadly speaking the quest for equivalences between Fukaya categories and categories of coherent sheaves. Typically, a symplectic manifold is tied to a complex variety.

## A-side

Symplectic manifold $X$
Fukaya category Fuk $X$
Symplectic form $\omega$
Deformations of $\omega$

$\stackrel{\text { Mirror }}{\longleftrightarrow}$
$\stackrel{\text { Mirror }}{\longleftrightarrow}$

## B-side

Complex variety $\check{X}$
Derived category Coh $\check{X}$
Complex structure $I$

Deformations of $I$

Noncommutative homological mirror symmetry specifically includes the option to make the complex variety noncommutative. In this thesis, we consider the specific case of noncommutative mirror symmetry for punctured surfaces. We deform the A-side and find the matching deformation on the B-side.

## Results

As a starting point, we take mirror symmetry for punctured surfaces according to Bocklandt 18. It asserts an equivalence of $A_{\infty}$-categories:

$$
\operatorname{Gtl} Q \cong \operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)
$$

$A_{\infty}$-categories are algebraic structures which can be deformed in a way similar to associative algebras or commutative varieties. Equivalent $A_{\infty}$-categories have equal deformation theory. The main question in this field is:

Given a deformation on one side, what is the corresponding deformation on the other side?
The Hochschild DGLA is a tool to capture the entire deformation theory of an $A_{\infty}$-category. If $\mathcal{C}$ is an $A_{\infty}$-category, we denote its Hochschild DGLA by $\mathrm{HC}(\mathcal{C})$. An equivalence of $A_{\infty}$-categories $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$ induces a noncanonical $L_{\infty}$-equivalence of their Hochschild DGLAs. For instance, Bocklandt's mirror symmetry for punctured surfaces induces an $L_{\infty}$-equivalence

$$
\mathrm{HC}(\operatorname{Gtl} Q) \xrightarrow{\sim} \mathrm{HC}(\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)) .
$$

It is of utmost importance to compute this map. In this thesis, we have succeeded in computing the left-hand side Paper I), picking a relatively general representative of the left-hand side and investigating it in detail Paper II), and pushing it to the right-hand side Paper IIT).

Main result Theorem 21.28 In this thesis, we prove deformed mirror symmetry for punctured surfaces. As starting point, pick a dimer $Q$ and its gentle algebra Gtl $Q$. We define a specific $A_{\infty^{-}}$ deformation $\operatorname{Gtl}_{q} Q$ of $\operatorname{Gtl} Q$. Under technical conditions on the dimer $Q$ and its dual dimer $\check{Q}$, we construct a deformation $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ together with an equivalence of deformed $A_{\infty^{-}}$ categories

$$
F_{q}: \operatorname{Gtl}_{q} Q \xrightarrow{\sim} \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right) .
$$

We provide an explicit description of the deformed Landau-Ginzburg model $\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ and the image objects $F_{q}(X)$ of the functor. During the course of this thesis, we also derive several results which are of independent interest:

Hochschild cohomology Theorem 3.22) We compute the $A_{\infty}$-Hochschild cohomology of the $A_{\infty^{-}}$ gentle algebra Gtl $Q$ for an arbitrary punctured surface $Q$. We conduct this investigation in order to get a first grasp on deformations of Gtl $Q$. The strategy is to guess cocycles based on the idea of Seidel 63 that deformations of the Fukaya category come from working relative to a divisor. The result is an explicit description of the $A_{\infty}$-Hochschild cohomology including cocycle representatives. It agrees with similar calculations done for more general Fukaya categories by Ganatra 32 and in the mirror model by Wong 71.

Classification of deformations Theorem 3.27) We classify formal deformations of gentle algebras $\mathrm{Gtl} Q$. We conduct this investigation in order to get an overview of which deformations are to be pushed from A-side to B-side. The strategy is to write down a large class of explicit deformations over arbitrary Artin rings and use the computation of Hochschild cohomology to prove this class exhausts all deformations. The result is an explicit enumeration of all formal deformations of $\mathrm{Gtl} Q$ up to gauge equivalence.

Formality of Hochschild DGLA Corollary 3.26 We prove that the Hochschild DGLA of a gentle algebra $\mathrm{Gtl} Q$ is formal. It is part of our effort to understand the deformation theory of $\mathrm{Gtl} Q$. The strategy is to inspect the minimal model of $\mathrm{HC}(\mathrm{Gtl} Q)$ through Kadeishvili's theorem. Combining our explicit knowledge of the Hochschild complex with a grading argument specifically tailored to punctured surfaces, we conclude that the DGLA is formal.

Deformed Kadeishvili theorem Theorem 8.34 We show how to compute minimal models of arbitrary deformations of $A_{\infty}$-categories, including those with (infinitesimal) curvature. From the perspective of formal deformations, it is no surprise that these minimal models exist. From the perspective of curved $A_{\infty}$-categories, the result is a rather stark surprise since (non-infinitesimally) curved $A_{\infty}$-categories admit no minimal models. The strategy is to maximally uncurve the category and then apply a classical Kadeishvili construction by trees. The result is an explicit algorithm to compute minimal models for arbitrary (curved, formal) $A_{\infty}$-deformations.

Uncurving of band objects Theorem 9.20 We show that band objects in almost every deformation of $\mathrm{Gtl} Q$ are generally uncurvable. This is part of our effort to compare the deformations of Gtl $Q$ with relative Fukaya categories. The strategy is to infinitesimally gauge the band objects so that they lose their curvature. Geometrically this makes the band an object of the relative Fukaya category and confirms Seidel's prediction 64.

Discrete relative Fukaya category Theorem 13.31) We show that (part of) the relative Fukaya category is equal to (part of) the derived category of a deformed gentle algebra. More precisely, we compute the $A_{\infty}$-structure on the derived category $\mathrm{HTw}_{\mathrm{Gtl}}^{q}$ $Q$ for a specific deformation $\mathrm{Gtl}_{q} Q$ of $\mathrm{Gtl} Q$. The computation essentially consists of a minimal model calculation, during which we build a large web of data structures that capture output terms of deformed Kadeishvili trees. For every output term appearing in this web, we offer a geometric interpretation. This shows that the $A_{\infty}$-structure agrees with the $A_{\infty}$-structure of the relative Fukaya category.

Deformed Cho-Hong-Lau construction (Theorem 20.50) We build a deformed version of the mirror functor construction of Cho, Hong and Lau 26]. The codomain of our functor $F_{q}: \mathcal{C}_{q} \rightarrow$ $\operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right), \ell_{q}\right)$ consists of a category of deformed matrix factorizations, in which the product of the factors is allowed to differ from the potential $\ell_{q}$ by an infinitesimal term. In contrast to the Cho-HongLau construction, this infinitesimal term serves as curvature and makes the typical image object $F_{q}(X)$ curved. Application of the deformed Cho-Hong-Lau construction to the case of $\mathrm{Gtl}_{q} Q$ gives the desired deformed mirror symmetry $\mathrm{Gtl}_{q} Q \xrightarrow{\sim} \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$.

Flatness of CY3 deformations (Theorem 19.76) We show that a deformation of a CY3 superpotential induces an algebra deformation under a mild boundedness condition. The reason for this investigation is that $\operatorname{mf}\left(\mathrm{Jac}_{q} \check{Q}, \ell_{q}\right)$ only becomes a deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ if $\mathrm{Jac}_{q} \check{Q}$ is a (flat) algebra deformation of $\mathrm{Jac}_{q} \check{Q}$. Our flatness result is a culmination of a long sequence of improvements in the literature. Our starting point is the work of Berger and Ginzburg 11 which requires, like all previous results, that the superpotential $W$ is homogeneous. We show that this condition is superfluous and can be replaced by a mild boundedness condition. The result is a flatness result for formal deformations of CY3 algebras with nonhomogeneous potential. In particular, it follows from this result that $\mathrm{Jac}_{q} \check{Q}$ is a flat deformation of $\operatorname{Jac} \check{Q}$ for almost all dimers $\check{Q}$.

## Context

This thesis can be placed in the context of mirror symmetry, with ties to representation theory and deformation theory. In what follows, we explain several lines of mathematical development in these three areas. For each, we explain basic questions in the field and name results which have inspired this thesis. It is possible to spell out a few results of this thesis on a more philosophical level. Each of the following claims is explained below:

- "Deformed mirror symmetry is deformed Koszul duality" Paper III,
- "Gradedness requirements can be replaced by boundedness requirements" Paper III,
- "Hamiltonian deformations arise naturally from representation theory" Paper II,
- "The curvature problem is not a problem in case of formal deformations" Paper II.

Derived invariants Derived categories capture the true homological nature of categories. Whenever one builds data from a category and the data only depends on the derived category, one speaks of a derived invariant. By definition, derived equivalent categories have the same derived invariants. Given an invariant and two derived equivalent categories $\mathcal{C}, \mathcal{D}$, it is an interesting task to compare how the invariant plays out in $\mathcal{C}$ and in $\mathcal{D}$.

The primary derived invariants we work with in this thesis concern the deformation theory of $A_{\infty^{-}}$ categories. The two invariants we associate with an $A_{\infty}$-category are its Hochschild cohomology and the set of its $A_{\infty}$-deformations up to gauge equivalence. When $\mathcal{C}$ and $\mathcal{D}$ are derived equivalent $A_{\infty^{-}}$ categories, they have the same Hochschild cohomology and $A_{\infty}$-deformations up to gauge equivalence. The interesting task is then to examine how a given Hochschild class or deformation of $\mathcal{C}$ plays out in the category $\mathcal{D}$.

Mirror symmetry Mirror symmetry originally envisions a correspondence between Calabi-Yau manifolds 38 . A Calabi-Yau manifold is a manifold which has both a symplectic and a complex structure. The original mirror symmetry observation entails that for some Calabi-Yau manifolds $X$ there are Calabi-Yau manifolds $\check{X}$ such that symplectic invariants of $X$ are related to complex invariants of $\check{X}$. The manifold $X$ with its symplectic invariants is also called the A-side and the manifold $\check{X}$ with its complex invariants is called the B -side. Mirror symmetry envisions that the Hodge diamonds of $X$ and $\check{X}$ are related by a flip over the diagonal and the Gromov-Witten invariants of $X$ are related to periods of $\check{X} 6$. Since the numbers in the Hodge diamond characterize the dimensions of symplectic and complex deformations, the gist is that the space of symplectic deformations of $X$ should equal the space of symplectic deformations of $\check{X}$.

Kontsevich proposed a categorification of mirror symmetry in 1994 43. In his vision, the Fukaya category of the Calabi-Yau manifold $X$ is derived equivalent to the category of coherent sheaves of $\check{X}$. Kontsevich also expected the equivalence to hold on the level of $A_{\infty}$-categories. The Fukaya-category Fuk $X$ is already an $A_{\infty}$-category by nature. Its objects are the Lagrangian submanifolds of $X$, the hom spaces are spanned by intersection points of Lagrangian submanifolds and the products are given by counting pseudoholomorphic disks between intersection points. The category of coherent sheaves Coh $\tilde{X}$ can also be turned into an $A_{\infty}$-category. The procedure consists of replacing every coherent sheaf by a projective resolution, defining a natural structure of dg category and taking the minimal $A_{\infty}$-model. Kontsevich's homological mirror symmetry asserts that the derived categories $\mathrm{D}^{b}$ Fuk $X$ and $\mathrm{D}^{b} \operatorname{Coh} \check{X}$ are equivalent as $A_{\infty}$-categories.

Elliptic curves are Calabi-Yau manifolds of complex dimension 1 and form a simple instance of homological mirror symmetry. The A-side is the real torus $X=\mathbb{R}^{2} /(\mathbb{Z} \oplus \mathbb{Z})$ with complexified Kähler form
$\omega=b+i \omega^{\prime}$ and the B-side is the elliptic curve $\check{X}=\mathbb{C} /(\mathbb{Z}+\tau \mathbb{Z})$ with $\tau=\int_{X} \omega$. The mirror equivalence $\mathrm{D}^{b}$ Fuk $X \cong \mathrm{D}^{b} \operatorname{Coh} \check{X}$ was proven by Polishchuk and Zaslow 61 .

Mirror symmetry asserts that the construction $X \mapsto \bar{X}$ of $\dot{X}$ from $X$ is an involution in the sense that $\check{\tilde{X}}=X$. Therefore one typically has the following pair of equivalences:

$$
\begin{aligned}
& \mathrm{D}^{b} \text { Fuk } X \cong \mathrm{D}^{b} \operatorname{Coh} \check{X}, \\
& \mathrm{D}^{b} \operatorname{Fuk} \check{X} \cong \mathrm{D}^{b} \operatorname{Coh} X
\end{aligned}
$$

Deformations versus stability conditions Hochschild cohomology and deformations of the A-side and B-side of mirror symmetry have a long tradition in the literature. For instance, work of Ganatra 32 interprets Hochschild cohomology of Fukaya categories in terms of symplectic cohomology of the given symplectic manifold. Work of Seidel 64 shows how to deform Fukaya categories explicitly by working relative to a divisor. Work of Wong 71] determines the compactly supported Hochschild cohomology of the category of matrix factorizations $\operatorname{MF}(\operatorname{Jac} \check{Q}, \ell)$ when $\check{Q}$ is a torus dimer.

Stability conditions are another popular derived invariant, introduced by Bridgeland 22 . The idea is to associate with a triangulated category a space of possible ways to declare certain objects "semistable" and to assign "phases" to all objects in a way such that every object can be built from a sequence of semistable objects with increasing phase and morphisms only exist between objects of increasing phase. Possible intuition is that stability conditions capture all possible ways to immerse a category into the real plane. For gentle algebras $\operatorname{Gtl} Q$, the spaces of stability conditions have been determined by Haiden, Katzarkov and Kontsevich 35. For matrix factorizations, there is work of Bocklandt 19. The following diagram is a non-exhaustive summary:

|  | A-side |  | B-side |
| :---: | :---: | :---: | :---: |
| Deformations | Ganatra | 32 | , Seidel 64 |
| Stability conditions | Joyce $\boxed{37}$, | Wong | 71 |

The relation between stability conditions on $\mathrm{D}^{b}$ Fuk $X$ and structures on $X$ is mysterious. The understanding of the relation between deformations of $\mathrm{D}^{b} \operatorname{Coh} X$ and complex structures on $X$ is far from complete as well. The holy grail is to identify the space of stability conditions on $\mathrm{D}^{b} \operatorname{Coh} \check{X}=\mathrm{D}^{b}$ Fuk $X$ as "stringy Kähler moduli space" of $\check{X}$ and the space of deformations of $\mathrm{D}^{b} \operatorname{Coh} X=\mathrm{D}^{b}$ Fuk $\check{X}$ as "extended moduli space of complex structures" of $X$.

It is a folklore conjecture that deformations and stability conditions are connected via mirror symmetry 21. The conjecture entails that stability conditions on $\mathrm{D}^{b}$ Fuk $X$ are deformations of $\mathrm{D}^{b}$ Coh $X$. Alternatively, stability conditions on $\mathrm{D}^{b} \operatorname{Coh} X$ are deformations of $\mathrm{D}^{b}$ Fuk $X$. The two spaces are of course not directly equal, instead one conjectures that suitable enlargements of both spaces agree.

Elliptic curves provide a simple illustration of the folklore conjecture. Let $X=\mathbb{C} /(\mathbb{Z}+\tau \mathbb{Z})$ be an elliptic curve. The space of stability conditions on $\mathrm{D}^{b} \operatorname{Coh} X$ has been identified as the universal cover of $\mathrm{GL}^{+}(2, \mathbb{R})$ by Bridgeland 22 . It has two complex dimensions. The space of deformations of $\mathrm{D}^{b}$ Fuk $X$ can be guessed from letting the symplectic structure of $X$ run to the large volume limit, where $X$ turns into a 1-punctured torus. The space of deformations for the Fukaya category of the 1-punctured torus has been determined to be two-dimensional as well by Lekili and Perutz 46. The two dimension numbers seemingly match, although this constitutes by no means an actual correspondence between individual stability conditions and deformations.

Punctured surfaces provide a further illustration of the conjecture. Stability conditions on the Fukaya category of a punctured surface $X$ have been explicitly matched with flat structures on $X$ by Haiden et al. 35. A flat structure on $X$ is the datum of a complex structure together with a quadratic holomorphic differential. An expectation is that the space of complex structures is again part of the space of deformations of $\mathrm{D}^{b} \operatorname{Coh} X$. This provides evidence for the folklore conjecture that the space of stability conditions on $\mathrm{D}^{b}$ Fuk $X$ agrees with the space of deformations of $\mathrm{D}^{b} \operatorname{Coh} X$ once the spaces are suitably enlarged.

Any correct solution of the folklore conjecture will include constructing candidate maps between the different moduli spaces. The candidate maps might be constructed directly on the level of moduli spaces of stability conditions and deformations, or alternatively via the moduli spaces of symplectic or complex deformations of $X$ or $\check{X}$. Whichever way the candidate maps are constructed, a solution needs to match deformations of $\mathrm{D}^{b}$ Fuk $X$ and $\mathrm{D}^{b} \operatorname{Coh} \ddot{X}$. This is what we undertake in the present thesis in case $X$ is a punctured surface.

Calabi-Yau algebras Calabi-Yau manifolds play a central role in mirror symmetry and Calabi-Yau algebras have been introduced as their noncommutative analog by Ginzburg 34. Modules over CalabiYau algebras come with particularly simple Serre duality and the $A_{\infty}$-category of modules comes with a cyclic structure. The cyclic structure is nowadays considered a "Calabi-Yau category" structure by itself. There are generalizations available, notably the so-called pre-Calabi-Yau category structures 45 .

Mirror symmetry has originally been a duality between Calabi-Yau manifolds. This is also reflected in noncommutative mirror symmetry. In Bocklandt's noncommutative mirror symmetry of punctured surfaces, the B-side is built upon the Calabi-Yau $\operatorname{algebra} \operatorname{Jac} \check{Q}$ of dimension 3. The A-side is the gentle algebra $\mathrm{Gtl} Q$, which is presumably a pre-Calabi-Yau category as well. The systematic construction of mirror functors due to Cho, Hong and Lau 26 explains this connection on an abstract level.

A core feature of associative Calabi-Yau algebras of dimension 3 is that they can be typically captured as a Jacobi algebra of a quiver with a superpotential. When we deform mirror symmetry, we have to deal with deformations of this superpotential. It is a priori unclear that a deformation of the superpotential provides a (flat) deformation of the Jacobi algebra.

Work of Berger, Ginzburg and Taillefer 11,12 concerns this flatness problem in the context of PBW deformations. Their result confirms that if the original superpotential is homogeneous, then a deformation of the superpotential leads to a PBW deformation of the algebra. In the context of mirror symmetry, the superpotential of $\operatorname{Jac} \check{Q}$ is however not homogeneous and the result fails to apply.

In the present thesis, we show that the original argument of Berger and Ginzburg can be extended beyond the homogeneous case by assuming a simple boundedness condition. The core observation is that the homogeneity requirement only serves to finish an induction argument. Without the homogeneity assumption, one is confronted with the task of getting an inductively defined sequence of paths in $Q$ under control. The boundedness assumption limits the possible growth of this sequence and therefore allows us to finish the proof of flatness without homogeneity requirement. It seems that this boundedness argument is rather versatile and should be applicable to other problems of algebra wherever homogeneity is only required to finish induction arguments and limit term growth.

Frobenius manifolds The original working title of this thesis project concerned "Frobenius manifolds and their relation to homological mirror symmetry". In a very implicit way, we have made this proposal true. The bridge between this thesis and Frobenius manifolds is best explained on the level of Hochschild cohomology and moduli spaces of deformations. The starting point consists of the two $A_{\infty}$-categories $\operatorname{Gtl} Q$ and $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ both of which enjoy Calabi-Yau-type structures. The philosophy is that the Calabi-Yau structures turn their Hochschild cohomology into Frobenius algebras. More generally, one is interested in forming actual moduli spaces of $A_{\infty}$-deformations, whose tangent spaces are Hochschild cohomology. The philosophy is that the Calabi-Yau structures turn the moduli spaces of deformations into Frobenius manifolds.

|  | A-side | B-side |
| :---: | :---: | :---: |
| Category | $\mathrm{Gtl} Q$ | $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ |
| Frobenius manifold | $\mathcal{M}$ | $\mathcal{M}^{\prime}$ |
| Frobenius algebra | $\mathrm{HH}(\operatorname{Gtl} Q)$ | $\mathrm{HH}(\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell))$ |

Koszul duality Mirror symmetry conjectures the existence of $A_{\infty}$-functors between seemingly unrelated geometric objects. One looks for an interpretation of how $X$ and $\check{X}$ are related or can be constructed from each other. Mirror symmetry also requires us to understand why two seemingly unrelated categories can be equivalent. The quest of finding mirror pairs $(X, \check{X})$ has sparked the development of functor constructions. Popular slogans nowadays include that "Mirror symmetry is T-duality" 68 and "Mirror symmetry is dimer duality" 18 .

Koszul duality is a phenomenon which connects for instance $A_{\infty}$-categories and dg algebras. To an $A_{\infty}$-algebra $A$ with certain properties Koszul duality associates a dg algebra $A^{!}$. This duality typically satisfies $\left(A^{!}\right)^{!} \cong A$ and comes with a functor $F: \operatorname{Mod}^{\mathrm{fd}} A \rightarrow \mathrm{Tw} A^{!}$between the category of finitedimensional right $A$-modules and the twisted completion of $A^{!}$. The work of Cho, Hong and Lau 26 shows how to tweak Koszul duality in order to obtain the mirror symmetry for punctured surfaces $\operatorname{Gtl} Q \rightarrow \operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$. We can phrase the slogan of Cho, Hong and Lau as "Mirror symmetry is Koszul duality".

The construction of Cho, Hong and Lau explicitly builds a functor $F: \mathcal{C} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right)$ from a given $A_{\infty}$-category $\mathcal{C}$ and a cyclic subcategory $\mathbb{L} \subseteq \mathcal{C}$. In this thesis, we provide a deformed Cho-Hong-Lau construction which gives rise to functors $F_{q}: \mathcal{C}_{q} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right), \ell_{q}\right)$. Behind the scenes,
this construction is really a deformation of Koszul duality. We can phrase our interpretation of mirror symmetry as "Deformed mirror symmetry is deformed Koszul duality".

The curvature problem In $A_{\infty}$-deformation theory, one regards infinitesimal deformations of a given $A_{\infty}$-structure such that the $A_{\infty}$-relations are preserved. An issue is that only deforming the $A_{\infty}$-structure does not give a notion of deformations that is invariant under quasi-equivalence. In order to obtain a notion invariant under quasi-equivalence, one needs to permit the deformation to have curvature. Dealing with curvature is however regarded as tedious, because the curvature prevents the differential from squaring to zero. The presence of curvature is often referred to as the "curvature problem". A main question is how to gauge away the curvature or otherwise how to deal with the remaining curvature. An instance of the uncurving problem has been studied by Lowen and Van den Bergh 50, on which we comment in section F.1.3.

In the present thesis, we consider the derived category $\mathrm{HTw} \operatorname{Gtl} Q$ of a gentle algebra $\mathrm{Gtl} Q$. The objects of the derived category $\mathrm{HTw} \operatorname{Gtl} Q$ have been classified up to isomorphism by Haiden, Katzarkov and Kontsevich 35. They fall into two classes, known as string objects and band objects. Geometrically, a string corresponds to a curve $\gamma:[0,1] \rightarrow|Q|$ in the surface which starts and ends at punctures. A band object corresponds to a closed curve $\gamma: S^{1} \rightarrow|Q|$ in the surface which does not hit any punctures.

Both string objects and band objects can be interpreted as twisted complexes lying in Tw Gtl $Q$. They can also be interpreted as objects in the deformed twisted completion $\mathrm{Tw}_{\mathrm{Gtl}}^{q}$ $Q$. The strings and bands in $\mathrm{Tw}_{\mathrm{Gtl}_{q} Q}$ are however curved objects. The curvature on $\mathrm{Tw} \mathrm{Gtl}_{q} Q$ is in principle regarded as problematic and for instance an obstruction to forming the derived category. The "curvature problem" in this situation asks how to deal with this situation.

The philosophy of Seidel 63 envisions that this curvature is non-essential for most band objects and that it can be removed by means of a gauge functor. In the present thesis, we provide a trick aimed at gauging the band objects in $\mathrm{Tw}_{\mathrm{Gtl}}^{q}$ $Q$ slightly such that the objects become curvature-free. Geometrically this trick entails adding an infinitesimal copy of the curve which runs on the opposite side of all the punctures that the band object lies close to. We check that the trick succeeds in removing curvature from almost all band objects, verifying Seidel's vision from the perspective of gentle algebras and solving the "curvature problem" for $\mathrm{Tw} \mathrm{Gtl}_{q} Q$.

Derived categories The derived category of an $A_{\infty}$-category is defined as the minimal model of its twisted completion. If $\mathcal{C}$ is an $A_{\infty}$-category, then its twisted completion is denoted $\operatorname{Tw} \mathcal{C}$ and the minimal model is denoted HC . The derived category is then $\mathrm{HTw} \mathcal{C}$. Its degree zero part $\mathrm{H}^{0} \mathrm{Tw} \mathcal{C}$ is the analog of the classical derived category in the abelian setting.

Part of the "curvature problem" is that curved $A_{\infty}$-categories do not have derived categories because their differential already fails to square to zero. A particular instance appears in the present thesis where we build a mirror functor by deforming the construction of Cho, Hong and Lau 26. Since the classical construction already works with the derived category $\mathrm{HTw} \mathrm{Gtl} Q$, our deformed construction will need to work with $\mathrm{H} \mathrm{Tw} \mathrm{Gtl}_{q} Q$. In particular, we are required to define and construct part of the derived category $\mathrm{H} \mathrm{Tw} \mathrm{Gtl}_{q} Q$. We therefore set out to build a good theory of derived categories of $A_{\infty}$-deformations.

As a first cue towards derived categories for $A_{\infty}$-deformations, we take the observation that $A_{\infty}$ deformation theory is a derived invariant. When $\mathcal{C}$ and $\mathcal{D}$ are $A_{\infty}$-categories and $F: \mathcal{C} \rightarrow \mathcal{D}$ is a quasiequivalence or derived equivalence, one can in principle push any deformation $\mathcal{C}_{q}$ of $\mathcal{C}$ to a deformation $\mathcal{D}_{q}$ of $\mathcal{D}$. This phenomenon can be explained on the abstract level of Hochschild DGLAs. The idea is that the deformation $\mathcal{C}_{q}$ gives rise to a Maurer-Cartan element of the Hochschild DGLA HC(C) and the quasiequivalence $F$ gives rise to a non-canonical pushforward $L_{\infty}$-quasi-isomorphism $F_{*}: \operatorname{HC}(\mathcal{C}) \rightarrow \operatorname{HC}(\mathcal{D})$. Correspondingly, the deformation $\mathcal{C}_{q}$ can be pushed to a deformation $\mathcal{D}_{q}$ of $\mathcal{D}$. This pushforward has the property that there is still a functor $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ of $A_{\infty}$-deformations which is a deformation of the functor $F$.

In the present thesis, we define twisted completions, minimal models and derived categories for $A_{\infty^{-}}$ deformations. The idea is to take the twisted completion, minimal model or derived category without deformation and induce the deformation only afterwards. Under the context of an $A_{\infty}$-category $\mathcal{C}$, we contend that the correct way to define the twisted completion $\mathrm{Tw}_{\mathrm{w}} \mathcal{C}_{q}$ is by taking $\mathrm{Tw} \mathcal{C}$ and inducing the deformation $\mathcal{C}_{q}$ onto $\operatorname{Tw} \mathcal{C}$ via the embedding $\mathcal{C} \rightarrow \operatorname{Tw} \mathcal{C}$. Similarly, we contend that the correct way to define the minimal model $\mathrm{H} \mathcal{C}_{q}$ is by taking HC and inducing the deformation $\mathcal{C}_{q}$ onto HC via the quasi-isomorphism $\pi: \mathcal{C} \rightarrow \mathrm{HC}$. We define the derived category $\mathrm{HTw} \mathcal{C}_{q}$ as the composition of these two procedures. This way, every $A_{\infty}$-deformation has a derived category.

The definition of derived categories for $A_{\infty}$-deformations completely avoids the problems coming from curvature. The downside is that inducing deformations afterwards makes actual computation of
the derived category very difficult. Forming the twisted completion $\operatorname{Tw} \mathcal{C}_{q}$ is not very problematic, but inducing a deformation to HC is close to impossible.

We therefore offer a deformed Kadeishvili theorem which produces the minimal model $\mathrm{H} \mathcal{C}_{q}$ directly. In the world of $A_{\infty}$-categories without deformation, the Kadeishvili theorem is the classical way to build the minimal model HC . It provides an explicit mains of computing the $A_{\infty}$-structure of HC , although it depends on a lot of choices and actual calculations are often hard. In the world of $A_{\infty}$-deformations, we devise a deformed Kadeishvili theorem which builds the minimal model $\mathrm{H} \mathcal{C}_{q}$ explicitly. It comes with the same downside of computational hardness, but has the feature that it applies to any $A_{\infty}$-deformation, including those with curvature.

Minimal model calculations are scarce in the literature. The reason is that they are hard to conduct and one can sometimes guess the result without performing the calculation until the end. In the present thesis, we build deformed mirror symmetry by deforming the construction of Cho, Hong and Lau 26 . An essential requirement for this deformed construction is the knowledge of the $A_{\infty}$-structure on part of the derived category $\mathrm{H} \mathrm{Tw}_{\mathrm{Gtl}}^{q}$ $Q$. We apply our deformed Kadeishvili theorem to this situation and perform the minimal model calculation until the end. The difficulty lies in getting the large amount of data under control which keep spinning out of the gentle algebra and its Kadeishvili construction.

Smooth versus discrete Smooth mathematical structures originate in an observation of the physical world. Whether it concerns the classification of closed surfaces, existence of solutions to PDEs or detecting geometry through curvature, one starts by studying differentiable properties and differential equations. Discretization of mathematical structures aims at getting grip on the smooth structures when they seem to slip away upon investigation. From Conway's zip proof for the classification of closed surfaces to finite elements serving as Galerkin schemes in the solution of PDEs to discrete differential geometry capturing curvature, discretization helps to extract the core of a smooth mathematical concept and prove properties which would otherwise have remained inaccessible.

Mirror symmetry is also subject to discretization. The problem in the field is that Fukaya categories and categories of coherent sheaves are generally too large to describe them in one breath. From the perspective of mirror symmetry, one aims at finding small models for Fukaya categories or categories of coherent sheaves which make actual calculations more tractable. Still, one tries to retain essential properties of the large geometric categories in the small discrete models in order to understand which properties actually cause a given mirror equivalence.

Discretization of Fukaya categories aims at representing Fukaya categories through small models. In contrast to the large geometric categories, the small models may also exhibit gluability properties which means that one can build up a Fukaya category by gluing it from pieces. Unfortunately such procedure is known not to work for Fukaya categories in general. Instances in which gluing via a discrete interface is possible include gluing Fukaya categories from cosheaves 35 and Nadler's arborealization program 4.

In the case of punctured surfaces, the gentle algebras Gtl $Q$ play the role of a discretized Fukaya category thanks to 18 . The $A_{\infty}$-structure on these gentle algebras is rather easy to define, in contrast to the extremely hard $A_{\infty}$-structure on smooth Fukaya categories. Gentle algebras do not contain information on all objects of the Fukaya category directly, instead this information has to be obtained by passing to the derived category $\mathrm{HTw} \mathrm{Gtl} Q$ of the gentle algebra. The drawback of gentle algebras is then that the $A_{\infty}$-structure on the derived category is again hard to compute explicitly.

In the present thesis, we extend the discretization of Fukaya categories to the deformed setting. The idea is to construct a deformation of the gentle algebra $\mathrm{Gtl} Q$ that behaves in a way analogous to Seidel's relative Fukaya categories. We define the candidate deformation $\operatorname{Gtl}_{q} Q$ directly in the hope that it is the correct implementation of Seidel's idea on the discrete side. We put our expectations on a test and compute part of the derived category $\mathrm{HTw}_{\mathrm{Gtl}}^{q} \boldsymbol{Q}$. Limited to the subcategory of zigzag paths, our calculation shows that $\mathrm{H} \mathrm{Tw} \mathrm{Gtl}_{q} Q$ agrees with the relative Fukaya category. We therefore consider $\mathrm{Gtl}_{q} Q$ the correct transport of Seidel's vision to gentle algebras and may think of it as a "discrete relative Fukaya category".

## Deformed smooth

Relative Fukaya category relFuk $Q$
$\square$

## Deformed discrete

Deformed gentle algebra $\mathrm{Gtl}_{q} Q$

Hamiltonian deformations Implementing Hamiltonian deformations is one of the difficulties one encounters when defining smooth Fukaya categories. In the discrete world, one circumvents this problem by choosing such a small set of generators that the Hamiltonian deformations can be chosen canonically and disappear completely from the picture. When passing to the derived category $\mathrm{H} \mathrm{Tw} \mathrm{Gtl} Q$, we however
expect the full generality of the smooth Fukaya category to reappear. In particular, we expect to find $A_{\infty}$-products on some non-transversal sequences and expect that we can explain these products as an incarnation of Hamiltonian deformation.

In the present thesis, we compute the precise $A_{\infty}$-products on $\mathrm{HL} \subseteq \mathrm{HTw} \mathrm{Gtl} Q$ and on the deformed category $\mathrm{HL}_{q} \subseteq \mathrm{HTw}_{\mathrm{Gtl}}^{q} \boldsymbol{Q}$. As a starting point, we take the Kadeishvili theorem whose essential ingredient is a choice of a so-called homological splitting for $\mathbb{L}$. The Kadeishvili theorem then describes the minimal model $\mathrm{H} \mathbb{L}_{q}$ in terms of sums over trees. One enters the calculation with the expectation that $\mathrm{H} \mathbb{L}_{q}$ has the same $A_{\infty}$-products as the relative Fukaya category. We verify this expectation by matching every individual "result component" of a Kadeishvili tree with an immersed disk.


As expected, we can interpret even the products on non-transversal sequences geometrically by means of Hamiltonian deformation. While Hamiltonian deformations are an ingredient which has to be incorporated into the definition of smooth Fukaya categories from the beginning, they appear naturally through the Kadeishvili construction of the minimal model $\mathrm{H}_{\mathbb{L}_{q}}$.

The precise shape of the products of $\mathrm{H}_{q}$ depends on the choice of homological splitting for $\mathbb{L}$. Nevertheless, different homological splittings give quasi-equivalent minimal models $\mathrm{H} \mathbb{L}_{q}$. We have selected one specific splitting which makes it particularly easy to identify the minimal model as the relative Fukaya category. When choosing a slightly different splitting, we still expect to obtain the same products on transversal sequences, but the products on non-transversal sequences will typically change. These changed products can be interpreted geometrically as products in the relative Fukaya category under application of a different Hamiltonian deformation. While homological splittings for $\mathbb{L}$ are a discrete and representation-theoretic notion, Hamiltonian deformations are a smooth and geometric notion. Highly simplified, choices of homological splittings correspond to choices of Hamiltonian deformations:

## Hamiltonian deformations for zigzag Lagrangians

## Homological splittings <br> for zigzag paths

Deformed mirror symmetry Mirror symmetry has been deformed in several cases before. As a starting point, one takes a deformation of the Fukaya category Fuk $X$, for instance the relative Fukaya category introduced by Seidel 64 . Deformed mirror symmetry then seeks to find a corresponding deformation of $\check{X}$ such that the deformed Fukaya category of $X$ is still equivalent to the deformed category of coherent sheaves of $\check{X}$.

Lekili and Perutz 46 find a commutative mirror for the relative Fukaya category of the 1-punctured torus, apparently the first use of a relative Fukaya category in mirror symmetry. Lekili and Polishchuk 47 generalize this result to the case of the $n$-punctured torus. The strategy is to depart from a finite collection of split-generators of the Fukaya category. Then one computes part of their deformed products in the relative Fukaya category and guesses the corresponding deformation of the mirror.

Mirror symmetry in the original context of Calabi-Yau manifolds asserts that the construction $X \mapsto \check{X}$ is an involution. The dimer duality $Q \mapsto \check{Q}$ of Bocklandt is indeed an involution and therefore

$$
\begin{aligned}
& \operatorname{Gtl} Q \cong \operatorname{mf}\left(\operatorname{Jac} \check{Q}, \ell_{\check{Q}}\right), \\
& \operatorname{Gtl} \check{Q} \cong \operatorname{mf}\left(\operatorname{Jac} Q, \ell_{Q}\right) .
\end{aligned}
$$

The clue is that the dual dimer $\check{Q}$ serves as starting point for a gentle algebra Gtl $\check{Q}$ again. In contrast, in deformed mirror symmetry it is hard to continue the involution property when $X$ gets deformed. Specifically, in our deformed mirror $\operatorname{symmetry}^{\operatorname{Gtl}}{ }_{q} Q \cong \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ it is unclear how to interpret the deformed Jacobi algebra $\mathrm{Jac}_{q} \check{Q}$ as a starting point for a gentle algebra construction. A conjectural way out consists of interpreting the deformation $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ as a stability condition on $\mathrm{Gtl} \check{Q}$ instead.

## Assembly of the materials

We explain here the assembly of the main result and the distribution of the material into the four constituent parts of this thesis. Our aim is to explain how one idea leads to the next and how the main result evolves.


Paper 1 This text deals with gentle algebras and their deformations. The starting point is an arc system $\mathcal{A}$ on a punctured surface. The gentle algebra $\mathrm{Gtl} \mathcal{A}$ itself is defined by counting immersions of polygons into the surface. The deformed gentle algebras we construct are defined by additionally counting immersions of "orbigons", weighted by the punctures that the orbigon covers. We also devote ourselves to studying Hochschild cohomology and deformation theory of gentle algebras. We show that under certain assumptions on $\mathcal{A}$, the Hochschild DGLA is formal and that our explicit construction of deformed gentle algebras captures all formal deformations of $\operatorname{Gtl} \mathcal{A}$.

Paper II This text prepares us for applying the Cho-Hong-Lau construction to deformations of gentle algebras. We focus on the case where the arc system is a geometrically consistent dimer $Q$. Out of the whole range of deformations constructed in Paper I. we select one specific deformation $\mathrm{Gtl}_{q} Q$. As preparation for applying the Cho-Hong-Lau construction to $\mathrm{Gtl}_{q} Q$, we construct the deformed category of zigzag paths $\mathbb{L}_{q} \subseteq \mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} Q$. The rest of the text is concerned with computing the minimal model $\mathrm{H}_{q}$. We achieve this by introducing a deformed Kadeishvili theorem and working through all the Kadeishvili trees that appear. The result is an explicit description of the minimal model by means of what we call CR, ID, DS and DW disks.

Paper III This text introduces a deformed Cho-Hong-Lau construction and applies it to $\operatorname{Gtl}_{q} Q$. The essential input data for applying the construction are the $A_{\infty}$-structure on the minimal model $\mathrm{H}_{q}$. Fortunately, we have already described this structure in terms of CR, ID, DS and DW disks in Paper II We obtain a candidate functor $F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$. The rest of the text is concerned with making the description of $\mathrm{Jac}_{q} \check{Q}$ as explicit as possible and checking that $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ is indeed a deformation of $\operatorname{mf}(\operatorname{Jac} \breve{Q}, \ell)$. This is the logical end of the thesis.

Note IV This text is a stand-alone appendix to Paper I An issue observed in Paper I is that the implicit construction of the even Hochschild classes from the odd Hochschild classes fails if the arc system has only a single puncture. We solve this here by constructing the even Hochschild cocycles explicitly.

Fast track It is possible to understand this thesis without reading it in entirety. The fastest track consists of reading only section 17, 18, 20 and 21.

A more thorough understanding of the technicalities can be achieved as follows: Start with the definition of deformed gentle algebras in Paper I or section 6 Try to get an understanding of zigzag paths in dimers via section 10.1 Digest the thought that zigzag paths become a curved subcategory $\mathbb{L}_{q} \subseteq \mathrm{Tw}_{\mathrm{Gtl}}^{q}$ $Q$ in section 11.1. Get a grasp of the deformed Kadeishvili theorem in section 8. Quickly browse through section 11 till 13 to get an impression of the sheer amount of technicalities that go into computation of the minimal model $\mathrm{H} \mathbb{L}_{q}$. Let the pictures of section A $\operatorname{sink}$ in. Take as a milestone the comparison of $\mathrm{H}_{L_{q}}$ with the relative Fukaya category in Theorem 13.26 or 13.31

Then skip to Paper III and get an impression of how the Cho-Hong-Lau construction gives mirror equivalences in section 20.2. Take for granted that this construction can be deformed easily and in principle gives rise to a deformed mirror functor $\mathrm{Gtl}_{q} Q \rightarrow \operatorname{mf}\left(\mathrm{Jac}_{q} \check{Q}, \ell_{q}\right)$. Get a taste of the boundedness condition in section 19.4 which we use to show that the algebra $\mathrm{Jac}_{a} \check{Q}=\mathbb{C} \mathscr{Q} \llbracket Q_{0} \rrbracket / \overline{\left(\partial_{a} W_{q}\right)}$ is indeed a deformation of $\operatorname{Jac} Q=\mathbb{C} \check{Q} /\left(\partial_{a} W\right)$. Then skip to the main result section 21.7

History of this thesis My PhD project was supervised by Raf Bocklandt. From the beginning, we put a focus on mirror symmetry for punctured surfaces and the associated derived invariants. Raf Bocklandt made me acquainted with the popular conjecture that deformations and stability conditions are related through mirror symmetry. In an effort to turn this conjecture into a rigorous statement, I organized a summer camp for PhD students and postdocs which became a unique event during the Covid summer of 2020.

Severin Barmeier participated in this event and subsequently introduced me to formal deformation theory through the lens of $L_{\infty}$-algebras. I realized that formal deformations are the right vehicle for transferring deformations. It turned out Raf Bocklandt had many beautiful guesses about deformation theory of gentle algebras. We decided I would work on transferring deformations via mirror symmetry for a while. I then spent several months on combining theory learnt from Severin Barmeier with mirror symmetric insight brought to me by Raf Bocklandt.

After developing a concrete strategy, one cornerstone of my implementation failed. I had to abandon the approach and we decided I would follow Raf Bocklandt's earlier suggestion of deforming mirror symmetry by means of the Cho-Hong-Lau construction. This approach is rather straightforward, but it has led to an enormous complexity of unavoidable calculations all of which are documented in the present thesis.

Investigations into the relationship between deformations and stability conditions, classifications of matrix factorizations, gluing of mirror symmetry and classifications of cohomological field theories were started, but did not find their way into this thesis.

## Release

The thesis will be released in four pieces, some of which coauthored by Raf Bocklandt. The following list documents the contributions of the individual authors:

- Deformations of Gentle $A_{\infty}$-Algebras (Paper I), by Raf Bocklandt and Jasper van de Kreeke. This project was conducted in cooperation of both authors and the text was written jointly. The construction of deformed gentle algebras falls rather under Raf Bocklandt's regime. The Hochschild cohomology computation falls rather under Jasper van de Kreeke's regime. The formality and classification result are the culmination of a long series of improvements from both authors.
- Relative Fukaya Categories via Gentle Algebras Paper II, by Jasper van de Kreeke. This project was proposed and supervised by Raf Bocklandt. The research was conducted and the text written by Jasper van de Kreeke.
- Deformed Mirror Symmetry for Punctured Surfaces Paper III, by Raf Bocklandt and Jasper van de Kreeke.
This project was proposed and supervised by Raf Bocklandt. The research was conducted and the text written by Jasper van de Kreeke.
- Explicit Hochschild classes for Gentle Algebras Note IV, by Jasper van de Kreeke. This note is a stand-alone appendix to Paper I It was supervised by Raf Bocklandt and written by Jasper van de Kreeke.


## Paper I

## Deformations of Gentle $A_{\infty}$-Algebras

## 1 Introduction

To a collection of oriented arcs on a closed surface with marked points, one can associate an $A_{\infty}$-algebra called the gentle $A_{\infty}$-algebra. These were introduced in 18,35 to study wrapped Fukaya categories of punctured surfaces 3, where the punctured surface is obtained by removing the marked points. These algebras are also generalizations of well-known algebras that are studied in representation theory 5,57 , 48.

In this paper we study the deformation theory of these $A_{\infty}$-algebras in detail. The intuition from mirror symmetry tells us that deforming these algebras as curved $A_{\infty}$-algebras should correspond to filling in these punctures with normal points or orbifold points 65 . 30 .

We will work out this idea as follows. Starting from an arc collection on a closed surface with marked points, we introduce the combinatorial notion of an orbigon and use it to define a family of higher products that count these orbigons. We show that these structures satisfy the curved $A_{\infty}$-axioms and indeed deform the gentle $A_{\infty}$-algebra coming from the arc collection.

To show that these deformations solve the deformation problem we look at the Hochschild cohomology. We show that each element in the Hochschild cohomology is determined by its zeroth and first component; in other words its nullary and unary product. In the case that the arc collection has no loops or two-cycles we obtain an explicit basis for the Hochschild cohomology. This enables us to give a description of the Gerstenhaber algebra structure on the Hochschild cohomology. These computations match work done by Wong 71 on the Borel-Moore cohomology for matrix factorizations of dimer models (which is the B-side analogon of our setting, from the perspective of mirror symmetry [18). Furthermore we can show that the bracket from the Gerstenhaber structure is formal. Finally, using this description we conclude that each solution of the Maurer-Cartan equation for the Hochschild cohomology is gauge equivalent to one of the curved $A_{\infty}$-structures we introduced.

## 2 Curved gentle algebras

In this section we will introduce curved gentle algebras. These are curved deformations of the gentle $A_{\infty}$-algebras that were defined in 18 and 35 .

### 2.1 Arc collections and gentle algebras.

Definition 2.1. A marked surface $(S, M)$ is a pair consisting of a compact oriented surface without boundary $S$ and a finite subset of marked points $M \subset S$. We will denote the genus of the surface by $g$ and the number of marked points by $n$. Furthermore we will assume that $2-2 g-n<0$, which means that the surface with the marked points removed has a negative Euler characteristic.

An arc collection $\mathcal{A}$ is a set of oriented curves $a:[0,1] \rightarrow S$ that meet $M$ only at the end points $\left(a^{-1}(M)=\{0,1\}\right)$ and do not (self)-intersect internally. We say that an arc collection splits the surface if the complement of the arcs is the disjoint union of discs. These disks are called the faces and we denote the set of faces by $F$. Each face can be seen as a polygon bounded by arcs.

An arc collection that splits the surface satisfies

- the no monogons or digons condition [NMD], if no single arc or pair of arcs bounds a face. In this case the surface is split in $n$-gons with $n \geq 3$.
- the no loops or two cycles condition [NL2] if every arc has two different end points and no two arcs share more than one end point.
- the dimer condition, if the arcs around each disk form an oriented cycle. In case of a dimer, we will call a face positive or negative depending on whether the cycle around it is anticlockwise or clockwise. We can partition $F$ accordingly: $F=F^{+} \cup F^{-}$.
From now on we will assume that [NMD] holds, but in later sections we will sometimes have to assume the stronger [NL2] condition.

Recall that a quiver is a four-tuple $Q=\left(Q_{0}, Q_{1}, h, t\right)$ representing an oriented graph with vertices $Q_{0}$, arrows $Q_{1}$ and maps $h, t: Q_{1} \rightarrow Q_{0}$ that assign to each arrow its head and tail. A quiver is graded by a group $G$ if it comes with a map $|\cdot|: Q_{1} \rightarrow G$.

The path algebra $\mathbb{C} Q$ of a quiver $Q$ is the complex vector space spanned by the paths with as product concatenation of paths if possible and zero otherwise. We write the arrows as going from right to left so $\alpha \beta$ is a genuine path if $t(\alpha)=h(\beta)$. Every vertex of the quiver $v \in Q_{0}$ gives rise to a path of length zero, which is called the vertex idempotent $\mathbb{1}_{v}$. These span a semisimple subalgebra which we will denote by $\mathbb{k} \cong \mathbb{C}^{Q_{0}}$. If $Q$ is $G$-graded then $\mathbb{C} Q$ is a graded $\mathbb{k}$-algebra by assigning to each arrow its $G$-degree and to each vertex idempotent degree 0 .

It is tempting to consider the arc collection itself as a quiver, but we will not do this. Instead we will consider a different quiver, which is obtained by putting a vertex in the middle of each arc.

Definition 2.2. Given an arc collection $\mathcal{A}$ that splits $(S, M)$, we define a $\mathbb{Z}_{2}$-graded quiver $Q_{\mathcal{A}}$ as follows.

- The vertices of the quiver are the arcs: $\left(Q_{\mathcal{A}}\right)_{0}=\mathcal{A}$.
- For each angle of a face we define an arrow that corresponds to the internal anticlockwise angle between consecutive arcs that meet in that corner.
- An arrow has degree zero if both arcs have the same direction at the marked point (both outwards or both inwards), and degree one otherwise. Note that $\mathcal{A}$ is a dimer if and only if all arrows have $\mathbb{Z}_{2}$-degree 1 .

$|\alpha|=0$

$|\alpha|=0$

$|\alpha|=1$

$|\alpha|=1$

We will denote these 'angle arrows' by Greek letters and use $h(\alpha)$ and $t(\alpha)$ to denote the arcs (vertices) that correspond to the head and tail of the arrows. Each angle arrow also turns around a unique marked point and is contained in a unique face. We will denote these by $m(\alpha)$ and $f(\alpha)$. In this quiver two consecutive arrows either are angles that turn around a common marked point or they correspond to consecutive angles in a face.

$$
t(\alpha)=h(\alpha) \Longrightarrow m(\alpha)=m(\beta) \text { or } f(\alpha)=f(\beta)
$$

Definition 2.3. The gentle algebra $\mathrm{Gtl}_{\mathcal{A}}$ of an $\operatorname{arc}$ collection $\mathcal{A}$ is the path algebra of $Q_{\mathcal{A}}$ modulo the ideal of relations spanned by the products of arrows that are consecutive angles in a face.

$$
\left.\operatorname{Gtl}_{\mathcal{A}}=\mathbb{C} Q_{\mathcal{A}} /\langle\alpha \beta| t(\alpha)=h(\beta) \text { and } f(\alpha)=f(\beta)\right\rangle
$$

The product rule in this algebra can be illustrated pictorially as follows.

$\alpha \beta \neq 0$


$$
\alpha \beta=0
$$

Remark 2.4. The number of arrows is $2|\mathcal{A}|$ because in each vertex precisely two arrows arrive and two arrows leave. More precisely for every angle arrow $\alpha$ there are precisely two angle arrows $\beta_{1}$ and $\beta_{2}$ with $h(\alpha)=t\left(\beta_{i}\right)$. One will satisfy $\beta_{1} \alpha=0$ and for the other one we have $\beta_{2} \alpha \neq 0$. Likewise there are two $\gamma_{i}$ with $h\left(\gamma_{i}\right)=t(\alpha)$, one with $\alpha \gamma_{1}=0$ and one with $\alpha \gamma_{2} \neq 0$. This notion of a gentle algebra refers to this property and derives from representation theory 5. The usual definition of a gentle algebra also entails that the algebra is finite-dimensional but this is not the case for our algebras. It is possible to obtain finite dimesnional algebras by looking at surfaces with marked points on the boundary 48,57 .

Example 2.5. In the picture below on the left you see an arc collection on a torus with two marked points. It has 4 arcs and two square faces. The quiver $Q_{\mathcal{A}}$ has 4 vertices and 8 angle arrows. The angle arrows all have degree 1. The relations are generated by all paths $\alpha_{i} \beta_{j}$ and $\beta_{i} \alpha_{j}$, whenever the arrows are composable. This implies that every nonzero path in the gentle algebra is either a sequence of only $\alpha^{\prime} s$ or only $\beta^{\prime} s$. The former turn arrow the first marked points, while the latter turn around the second marked point.


In general, the nonzero paths of a gentle algebra correspond to positive angles between arcs that share a marked point. Every nonzero angle paths is uniquely determined by it sequence of angle arrows and products of angles paths turning around different marked points are zero.

For each $m \in M$ we define

$$
\ell_{m}=\alpha_{1} \ldots \alpha_{k}+\cdots+\alpha_{k} \ldots \alpha_{1}
$$

where $\alpha_{1}, \ldots, \alpha_{k}$ are the angle arrows around $m$ ordered in anticlockwise direction. These elements represent single loops around the marked points and the generate the center of the algebra.

## Lemma 2.6.

$$
Z\left(\operatorname{Gtl}_{\mathcal{A}}\right)=\mathbb{C}\left[\ell_{m} \mid m \in M\right] /\left(\ell_{i} \ell_{j} \mid i \neq j\right)
$$

Proof. One can easily check that the $\alpha \ell_{m}=\ell_{m} \alpha$ if $m(\alpha)=m$ and $\alpha \ell_{m}=\ell_{m} \alpha=0$ if $m(\alpha) \neq m$. This also implies that $\ell_{u} \ell_{v}=0$ if $u \neq v$.

Suppose $z=c \beta+\ldots$ is a nonzero central element containing the angle path $\beta$ and let $\alpha$ be the angle arrow that follows $\beta$ in the cycle around $m$. Then we have that $\alpha \beta \neq 0$, so if $\alpha z=z \alpha$ then $\alpha \beta$ must end in $\alpha$ and hence $\beta$ must be a cycle that winds a number of full turns around $m$. Therefore $z$ will contain $\ell_{m}^{r}$ and the $\ell_{m}^{r}$ form a basis for the part of the center with path length $>0$. The length 0 part of the center is $\mathbb{C}$ because the quiver is connected.

Lemma 2.7. As a $Z\left(\mathrm{Gtl}_{\mathcal{A}}\right)$-module the outer $\left(\mathbb{Z}_{2}\right.$-graded) $\mathbb{k}$-derivations are generated by the Euler derivations

$$
E_{\alpha}:=\alpha \partial_{\alpha}
$$

Proof. Let $\alpha$ be an angle arrow. If $d$ is a $\mathbb{k}$-derivation then $h(d \alpha)=h(\alpha)$ and $t(d \alpha)=t(\alpha)$. If $d \alpha$ contains a term $\gamma$ that does not turn around the same marked point as $\alpha$, let $\beta$ be the angle such that $\gamma \beta \neq 0$ is cyclic. We then have that $\alpha \beta=0$ because they turn around different marked points. Therefore

$$
0=d(\alpha \beta)=\gamma \beta+\cdots \pm \alpha d \beta
$$

but this is impossible because these two components turn around different marked points and hence cannot cancel each other.

So suppose $d \alpha$ turns around the same marked point $m$ as $\alpha$ In that case there are the following possibilities.

1. If $a=h(\alpha)=t(\alpha)$ then either $a$ forms a monogon (which is excluded by condition [NMD]) or $\alpha=\ell_{m}$ is a full turn around a marked point $m$ with one arc $a$ arriving.


In the latter case $d \alpha=g(\alpha) \mathbb{1}_{a}$ for some polynomial $g$. This polynomial cannot have a constant term: let $\beta$ be the arrow that follows $\alpha$ in the face $f(\alpha)$. Then $\alpha$ and $\beta$ turn around different marked points so

$$
0=d(\beta \alpha)=(d \beta) \alpha \pm \beta f(\alpha)= \pm g_{0} \beta
$$

So $d \alpha=\left(g(\alpha) \alpha^{-1}\right) \alpha=z E_{\alpha}(\alpha)$ for $z=g\left(\ell_{m}\right) \ell_{m}^{-1}$.
2. If $a=h(\alpha) \neq t(\alpha)$ and $a$ is a loop then there is an angle path $\kappa: a \rightarrow a$ following $\alpha$ that does not form a full turn around $m$.


In that case $d \alpha$ can contain a term of the form $\kappa \alpha$. Let $\beta$ be the angle arrow that directly follows $\kappa$. Note that $\beta \neq \alpha$ otherwise the arc $a$ forms a monogon. Because $\beta$ and $\alpha$ are on different ends of the arc $a$, we have that

$$
d(\beta \alpha)=\beta(c \kappa \alpha+\ldots) \pm(d \beta) \alpha=0
$$

Therefore $d(\beta)$ contains $c \beta \kappa$. The commutator [ $c \kappa,-]$ is only nonzero for the angle arrows $\alpha$ and $\beta$. By substracting this from $d$ we can make these terms disappear.
3. If $a=t(\alpha) \neq h(\alpha)$ and $a$ is a loop we can do a similar reasoning as above.
4. If $a=t(\alpha) \neq h(\alpha)=b$ and $a, b$ are both loops then $d \alpha$ can contain a term of the form $\kappa_{1} \alpha \kappa_{2}$. If $\kappa_{1}$ is not a full turn then let $\beta$ be the angle arrow such that $\beta \kappa_{1} \alpha \kappa_{2} \neq 0$. Because $\beta \alpha \neq 0$ we get

$$
d(\beta \alpha)=\beta\left(c \kappa_{1} \alpha \kappa_{2}+\ldots\right) \pm(d \beta) \alpha=0
$$

So $\kappa_{2}$ must end in $\alpha$ but then $\kappa_{2}$ is a full turn, so $\kappa_{1} \alpha \kappa_{2}=\ell_{m}^{r} \kappa_{1} \alpha$, and we can make this term disappear with $\left[\ell_{m}^{r} \kappa_{1},-\right]$. If $\kappa_{1}$ is a full turn we can do the same.
5. If neither $h(\alpha)$ or $t(\alpha)$ are loops then $d(\alpha)$ then every path in $d(\alpha)$ is of the form $\ell_{m}^{r} \alpha$.

From the discussion above we see that we can remove all terms that are not of the form $\ell_{m}^{r} \alpha$ by subtracting commutators.

Remark 2.8. Lemma 2.6 holds more generally for marked surfaces with bouundary, but lemma 2.7 does not hold in this generality (think of the cylinder with one marked point on each boundary circle, this has an arc collection whose gentle algebra is the Kronecker quiver $\circ \Longrightarrow \circ$. However if we impose the [NL2] condition the lemma still holds.

We end this section with a nice interpretation of Koszul duality in this setting. If $\mathcal{A}$ is an arc collection on $(S, M)$, choose a set of points $F \subset S$ in the centers of the faces. For each arc $a$ draw a perpendicular $\operatorname{arc} a^{\perp}$ that connects the centers of the faces adjacent to $a$ and points inside the left face. The dual arc collection $\mathcal{A}^{\perp}=\left\{a^{\perp} \mid a \in \mathcal{A}\right\}$ forms an arc collection for $(S, F)$.

Theorem 2.9. The Koszul dual of the gentle algebra is a gentle algebra of the dual arc collection:

$$
\left(\mathrm{Gtl}_{\mathcal{A}}\right)^{!} \cong \mathrm{Gtl}_{\mathcal{A}^{\perp}}
$$

Proof. From Bardzell 7 and 18 we know that $A=\mathrm{Gtl}_{\mathcal{A}}$ has a bimodule resolution $B^{\bullet}$ spanned by $A \otimes_{\mathfrak{k}} b \otimes_{\mathfrak{k}} A$ where $b=\beta_{l} \beta_{l-1} \cdots \beta_{1} \in \mathbb{C} Q_{\mathcal{A}}$ is a path of angle arrows such that all products $\beta_{i} \beta_{i-1}$ are zero. In other words paths that turn around faces. The maps between the terms have the following form

$$
1 \otimes b_{k} \ldots b_{1} \otimes 1 \mapsto b_{k} \otimes b_{k-1} \ldots b_{1} \otimes 1-(-1)^{k} \otimes b_{k} \ldots b_{2} \otimes b_{1}
$$

Therefore the Koszul dual $\operatorname{Ext}{ }_{A}^{\bullet}(\mathbb{k}, \mathbb{k})=\operatorname{Hom}\left(B^{\bullet},{ }_{L} \mathbb{k} \otimes \mathbb{k}_{R}\right)$ is spanned by the dual basis $b^{\vee}$. The element $b^{\vee}$ corresponds to the (right) module extension

$$
0 \leftarrow \mathbb{C}_{t\left(\beta_{l}\right)} \stackrel{\beta_{l}}{\leftarrow} \mathbb{C}_{h\left(\beta_{l}\right)} \leftarrow \cdots \leftarrow \mathbb{C} \mathbb{1}_{t\left(\beta_{1}\right)} \stackrel{\beta}{1}^{\leftarrow} \mathbb{C} \mathbb{1}_{h\left(\beta_{1}\right)} \leftarrow 0 .
$$

Note we use right modules because in that way $\alpha$ and $\alpha^{\vee}$ run in opposite directions. Stitching together two of these module extensions shows that the ext product matches the (reverse) concatenation of paths if the paths turn around the same face. All other products are zero. In particular the ext algebra is spanned by dual angle arrows and $\alpha^{\vee} \beta^{\vee}=0$ if $m(\alpha)=m(\beta)$. Finally if $\alpha$ is an angle arrow then the dual $\alpha^{\vee}$ in the Koszul dual has degree $1-|\alpha|$. All this can be realised geometrically by considering the dual angles as angles between perpendicular arcs.


### 2.2 Orbigons

Now let $R$ be a complete local commutative ring over $\mathbb{C}$ with maximal ideal $R^{+}$and residue field $R / R^{+}=$ $\mathbb{C}$. For each element $r \in Z\left(\mathrm{Gtl}_{\mathcal{A}}\right) \widehat{\otimes} R^{+}$we will define a curved $R$-linear $A_{\infty}$-structure over $\mathrm{Gtl}_{\mathcal{A}} \widehat{\otimes} R$. When tensored with $R / R^{+}$the result will be the gentle $A_{\infty}$-algebra as defined in 18, 35, so our construction gives a family of deformations of the latter.

To describe these deformations, we need the notion of an orbigon. Morally this is a branched cover from a disk to the surface such that the boundary is mapped to arcs and the branch points are mapped to marked points. We can construct such branched covers by joining faces together. We will define this concept combinatorially and inductively in two steps.

Definition 2.10. Tree-gons are certain sequences of angles up to cyclic permutation. The basic treegon come from faces: $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ is a tree-gon if they form the consecutive angles of a face such that $h\left(\alpha_{i}\right)=t\left(\alpha_{i+1}\right)$.

If $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ and $\left(\beta_{l}, \ldots, \beta_{1}\right)$ are tree-gons such that $\alpha_{1} \beta_{l} \neq 0$ and $\alpha_{k} \beta_{1} \neq 0$ in $\operatorname{Gtl}_{\mathcal{A}}$ then we define a new tree-gon

$$
\left(\beta_{1} \alpha_{k}, \alpha_{k-1}, \ldots, \alpha_{2}, \alpha_{1} \beta_{l}, \ldots, \beta_{2}\right)
$$

Geometrically this operation stitches the two tree-gons together over the common arc $h\left(\beta_{l}\right)=t\left(\alpha_{1}\right)$.


Remark 2.11. From this definition one can deduce that tree-gons are sequences of internal angles of faces stitched together in a tree-like way. This tree, whose nodes are the faces and whose edges are the stitched arcs, can be reconstructed solely from the angle sequence.

Let $\left(\gamma_{u}, \ldots, \gamma_{1}\right)$ be the sequence of all the indecomposable angle arrows in the tree-gon. Define a permutation $\sigma$ on $\{1, \ldots, u\}$ as follows: for every $i$ there will be a unique shortest nontrivial path $\gamma_{j} \ldots \gamma_{i+1}$ with $i+u \leq j<i$ that lifts to a contractible loop in the universal cover of $S \backslash M$. Set $\sigma(i)=j$ $\bmod u$. One can easily show that for a tree-gon we have that $\sigma^{2}=\mathrm{Id}$ and $\sigma(i)=i$ if and only if $h\left(\gamma_{i}\right)$ is an arc on the boundary of the tree-gon. This means that the internal arcs are in 1-1 correspondence with the 2 -cycles in $\sigma$. The nodes of the tree on the other hand correspond to the orbits of the permutation $\sigma \circ(u \ldots 1)$. Indeed this permutation follows the angles and crosses over if the arc is internal, and therefore it cycles around the faces. An edge and a node are incident if their orbits intersect.

Now we will allow to fold together two consecutive arcs on the boundary of a tree-gon that are identical. The result is a disk-like shape with internal marked points, so some of the angles sit in the interior of the disk. We will use square brackets to denote what is interior. In general we get sequences of angles separated by square brackets and commas such as $(\alpha, \beta[\gamma[\delta] \epsilon] \zeta[\eta])$. The reduced sequence cuts out anything that is between square brackets: e.g. $(\alpha, \beta \gamma)$. Again we work inductively.
Definition 2.12. An orbigon is a bracketed cyclic sequence of angles. The basic orbigons are tree-gons without any brackets and if $(\ldots, U, \ldots)$ is an orbigon for which the reduced sequence of $U$ is an angle that turns $r$ full turns around a marked point $m$ then $(\ldots[U] \ldots)$ is also an orbigon. The type of the orbigon is a multiset containing all pairs $p=(m, r)$ that were needed to introduce the brackets. Such a pair is also called an orbifold point.

Different tree-gons can be folded together to form the same disk, therefore we also need to impose an equivalence relation on the orbigons. Suppose $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ and $\left(\beta_{l}, \ldots, \beta_{1}\right)$ are tree-gons and

$$
(\beta_{1} \alpha_{k}|\ldots| \alpha_{j}[\underbrace{\alpha_{j-1}|\ldots| \alpha_{1}}_{U_{1}} \underbrace{\beta_{l}|\ldots| \beta_{i+1}}_{U_{2}}] \beta_{i}|\ldots| \beta_{2})
$$

is an orbigon, where the |-separators can be commas or brackets. Then the shift rule shifts the position of a piece between brackets and reverses the order of the two parts $U_{1}, U_{2}$ that are in thee different tree-gons as follows

$$
(\beta_{1}[\underbrace{\beta_{l}|\ldots| \beta_{i+1}}_{U_{2}} \underbrace{\alpha_{j-1}|\ldots| \alpha_{1}}_{U_{1}}] \alpha_{k}|\ldots| \alpha_{j} \beta_{i}|\ldots| \beta_{2})
$$

where brackets in $U_{1}$ that are paired up with a bracket $U_{2}$ and vice versa are flipped, for the formula to make sense.


The shift rule generates an equivalence relation on the orbigons that preserves the type and the reduced sequence.

Remark 2.13. Every tree-gon corresponds to a tree with nodes labeled by faces and edges labeled by arcs. Each folding operation adds another edge to the graph corresponding to the arc $a=h(U)$. In this way we end up with a graph of which this tree is a spanning tree. The graph, which we will sometimes refer to as the face graph, is planar. It divides the plane into regions which can be labeled by a pair ( $m, r$ ) from the type multiset. Clearly the equivalence relation changes the underlying spanning tree while keeping the face graph the same. For the example above the face graph is a grid with $3 \times 3$ nodes and the spanning trees are indicated in bold.


The spanning tree of the face graph completely determines the unreduced sequences of the orbigon because it is the sequence of angles that runs around the tree. It also determines the brackets: each time you cross an edge of the face graph that is not in the spanning tree you open or close a bracket depending on whether you crossed it the first or the second time.

Lemma 2.14. Two orbigons are equivalent if and only if their face graphs are equivalent as labeled planar graphs.

Proof. By construction the shift rule does not change the underlying face graph. Now suppose that the two orbigons have isomorphic face graphs, then we have to show that we can move from one spanning tree $T_{1}$ to another spanning tree $T_{2}$ via the shift rules. If $a$ is an edge (or dually an arc) in $T_{1}$ not contained in $T_{2}$ then if we remove $a, T_{1}$ will split in two parts (or dually tree-gons) denote the angles in the first tree-gon by $\alpha_{k}, \ldots, \alpha_{1}$ and those in the second tree-gon by $\beta_{l}, \ldots, \beta_{1}$ such that $a=t\left(\beta_{1}\right)=h\left(\alpha_{k}\right)$.

Because $T_{2}$ is connected it must contain an arc $b$ that connects the two parts of $T_{1} \backslash\{a\}$. Now find the indices $i, j$ such that $b=h\left(\beta_{i}\right)=t\left(\alpha_{j}\right)$ and perform the corresponding shift rule. The result will be an orbigon with a spanning tree equal to $T_{1} \backslash\{a\} \cup\{b\}$, which is one edge closer to $T_{2}$. Keep repeating this procedure until the spanning tree is $T_{2}$.

Remark 2.15. Note that we can also stitch orbigons together over a common arc in their reduced sequences by stitching the underlying tree-gons and transferring the brackets to the stiched tree-gon. The face graph of the new orbigon consists of the two graphs of the smaller orbigons joined together by one edge labeled by the common arc. Moreover because the common arc must be contained in all spanning trees, this implies that the small orbigons are uniquely determined by the big one.

Lemma 2.16. If $\left(\alpha_{k}, \ldots, \xi \eta, \ldots \alpha_{1}\right)$ is the reduced sequence of an orbigon and $\xi, \eta$ are nontrivial angle paths then there are two possibilities:

A the orbigon is stitched together of two smaller orbigons with reduced sequences $\left(\beta_{r}, \ldots, \xi\right)$ and $\left(\gamma_{r}, \ldots, \eta\right)$ such that $\gamma_{r} \beta_{r}=\alpha_{r}$ for some $r$.
B the orbigon is folded together from an orbigon with reduced sequence $\left(\alpha_{k}, \ldots, \xi, \ell_{m}^{r} h(\eta), \eta, \ldots, \alpha_{1}\right)$ in both cases these orbigons are unique.

Proof. Look at the face graph of the orbigon. The arc $t(\xi)=h(\eta)$ will correspond to an edge in this graph. If the graph remains connected after removing this edge then this new graph will be the graph of an orbigon that can be folded to the old orbigon and we are in situation B. Otherwise the old orbigon is stitched together from two smaller orbigons and we are in situation A.

Lemma 2.17. For a given type and reduced sequence there are at most a finite number of orbigons (up to equivalence).
Proof. Each bracketing changes two commas in brackets and introduces an extra pair in the multiset of the type. Therefore the length of the unreduced sequence (the number of commas and brackets) is 2 times the size of the multiset longer than the unreduced sequence (the number of commas).

Lemma 2.18. If the arc collection satisfies

- the no monogon or digon condition [NMD] then all tree-gons have length at least 3 .
- the no loops or two-cycles condition [NL2] then all orbigons have reduced sequences of length at least 3 .
Proof. If [NMD] holds then all faces are at least 3 -gons. Gluing an $n$-gon to an $m$-gon results in an $n+m-2$-gon, and $n+m-2>2$ if $n, m>2$. Furthermore, an orbigon with a reduced sequence of length one or two would give rise to a loop or two-cycle of arcs.


## $2.3 \quad A_{\infty}$-structures on the gentle algebra.

Fix a commutative ring $R$. Recall that if $A$ is a $\mathbb{Z}$ or $\mathbb{Z} / 2 \mathbb{Z}$-graded projective $R$-module then a curved $A_{\infty}$-structure is a collection of $R$-linear maps

$$
\mu^{k}: A^{\otimes_{R} i} \rightarrow A
$$

of degree $2-k$, satisfying the curved $A_{\infty}$-axioms:

$$
\sum_{k \geq l \geq m \geq 0}(-1)^{\left\|x_{m}\right\|+\ldots+\left\|x_{1}\right\|} \mu\left(x_{k}, \ldots, \mu\left(x_{l}, \ldots, x_{m-1}\right), x_{m}, \ldots, x_{1}\right)=0
$$

Here $\|x\|$ is shorthand for the shifted degree: $\|x\|=|x|-1$. If $l=m$ then we interpret the middle $\mu()$ as the element $\mu^{0}:=\mu^{0}(1) \in A$. This is called the curvature and if it is zero the structure is called uncurved.

If $A$ is an algebra, we say that $\mu$ is an extension of the product if $a \cdot b=(-1)^{\operatorname{deg} b} \mu^{2}(a, b)$. If $A=\mathbb{C} Q / I \otimes R$ comes from a path algebra of a quiver we will take tensor products over $\mathbb{k} \otimes R$ instead of over $R$ and we ask that the vertex idempotents $\mathbb{1}_{v}$ are strict: all products $\mu^{\neq 2}$ for which one of the entries is such an idempotent are zero.

Now let $R$ be a local nilpotent or complete ring over $\mathbb{C}$ with maximal ideal $R^{+}$. Fix an element $r \in Z\left(\mathrm{Gtl}_{\mathcal{A}}\right) \widehat{\otimes} R^{+}$. We will write this element as

$$
r_{0}+\sum_{m} r_{m}\left(\ell_{m}\right)
$$

where $r_{0} \in R^{+}$, the $r_{m}(t) \in R^{+}[t]$ are polynomials without a constant term. We will write the $j^{t h}$ coefficient of $r_{m}(t)$ as $r_{p}$ where $p=(m, j)$ is viewed as an orbifold point. We will also write $\ell_{p}$ as shorthand for $\ell_{m}^{j}$. We are now ready to define a family of extensions of the gentle algebra.
Definition 2.19. For any $r \in Z\left(\mathrm{Gtl}_{\mathcal{A}}\right) \widehat{\otimes} R^{+}$we define ${ }^{r} \mu_{\bullet}$ on $\mathrm{Gtl}_{\mathcal{A}} \widehat{\otimes} R$ as follows

- For each orbifold point $p=(m, j)$ we define a nullary product

$$
{ }^{r} \mu_{p}^{0}: \mathbb{k} \rightarrow \operatorname{Gtl}_{\mathcal{A}} \widehat{\otimes} R^{+}: 1 \mapsto r_{p} \ell_{m}^{j}
$$

- For each orbigon $\psi$ with reduced sequence $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ of length $k$ we define a $k$-ary product ${ }^{r} \mu_{\psi}$. If $\beta$ and $\gamma$ are angles such that $\beta \alpha_{k+i} \neq 0$ and $\alpha_{i} \gamma \neq 0$ then we set

$$
{ }^{r} \mu_{\psi}\left(\beta \alpha_{k+i}, \ldots, \alpha_{i}\right)=r_{\psi} \beta \text { and }{ }^{r} \mu_{\psi}\left(\alpha_{k+i}, \ldots, \alpha_{i} \gamma\right)=(-1)^{|\gamma|} r_{\psi} \gamma
$$

where $r_{\psi}$ is the product of all $r_{p}$ with $p$ running over the orbifold points in the type of $\psi$. If the type is empty (i.e. $\psi$ is a tree-gon) then we put $r_{\psi}=1$. All other ${ }^{r}{ }_{\mu} \mu_{\psi}\left(\beta_{1}, \ldots, \beta_{n}\right)$ where the $\beta_{i}$ are angles are zero.
The total product is the sum of all these products together with an overall curvature coming from $r_{0}$ :

$$
\begin{aligned}
{ }^{r} \mu^{0} & =r_{0} \mu_{\mathbb{1}}^{0}+\sum_{p}{ }^{r} \mu_{p} \\
{ }^{r} \mu^{1} & =\sum_{|\psi|=1}{ }^{r} \mu_{\psi} \\
{ }^{r} \mu^{2} & ={ }^{r} \mu_{\text {ord }}^{2}+\sum_{|\psi|=2}{ }^{r} \mu_{\psi} \\
{ }^{r} \mu^{k} & =\sum_{|\psi|=k}{ }^{r} \mu_{\psi} .
\end{aligned}
$$

where $\mu_{\mathbb{1}}^{0}$ is the nullary product with $\mu_{\mathbb{1}}^{0}(1)=1$ and ${ }^{r} \mu_{\text {ord }}^{2}$ the $R$-linear extension of the ordinary product in $\mathrm{Gtl}_{\mathcal{A}}$. This is well defined because by lemma 2.17 there are only a finite number of $k$-orbigons for each type and $R$ is nilpotent or complete.

Remark 2.20. The condition [NMD] implies that there are no tree-gons with length 1 or 2 . This means that ${ }^{r} \mu^{1}=0 \bmod R^{+}$and ${ }^{r} \mu^{2}=\mu_{\mathrm{Gtl}_{\mathcal{A}}}^{2} \bmod R^{+}$. The condition [NL2] implies that there are no orbigons with length 1 or 2 . This means that ${ }^{r} \mu^{1}=0$ and ${ }^{r} \mu^{2}={ }^{r} \mu_{\text {ord }}^{2}$.

Remark 2.21. If $r=0$, the only ${ }^{r} \mu_{\psi}$ that contribute are those coming from tree-gons and each contributes with a factor $r_{\psi}=1$. These are precisely those that correspond to immersed disks that do not cover marked points with their interior. Therefore the definition is equivalent to the definition in 35 In 18 the definition of the higher product is given inductively, with an induction step that is analogous to the induction step we used to define a tree-gon. Hence the definition is also equivalent to the definition in 18 . From both papers we can conclude that ${ }^{r} \mu$ is an uncurved $A_{\infty}$-structure if $r=0$. If $r \neq 0$ then ${ }^{r} \mu$ will be a curved deformation of this uncurved $A_{\infty}$-structure.

Remark 2.22. Each higher product removes a sequence of angles that forms an orbigon. If the angle that remains comes from the first entry, we will call that the front of the product, if it comes from the last we call it the back of the product.
Proposition 2.23. If [NL2] holds then ${ }^{r} \mu$ is a well defined curved $A_{\infty}$-structure on $\mathrm{Gtl}_{\mathcal{A}} \widehat{\otimes} R$, which is strict over $\mathbb{k}=\mathbb{C}^{\mathcal{A}}$.

Proof. We need to check the curved $A_{\infty}$-axioms. The first two axioms

$$
{ }^{r} \mu^{1}\left({ }^{r} \mu^{0}(1)\right)=0, \quad{ }^{r} \mu^{1}\left({ }^{r} \mu^{1}(\alpha)+{ }^{r} \mu^{2}\left({ }^{r} \mu^{0}(1), \alpha\right)-{ }^{r} \mu^{2}\left(\alpha,{ }^{r} \mu^{0}(1)\right)=0\right.
$$

follow easily from the facts that ${ }^{r} \mu^{0}(1)$ is a central element and ${ }^{r} \mu^{1}=0$. The third axiom holds because ${ }^{r} \mu_{\text {ord }}$ is associative and there are no ${ }^{r} \mu_{3}\left(\ldots,{ }^{r} \mu_{0}(1), \ldots\right)$ terms as this would imply an orbigon with reduced sequence $\left(\ldots\left[\ell_{m}^{k}\right] \ldots\right)$ of length 1 . This contradicts lemma 2.18

To show the higher axioms, first note that by construction all the ${ }^{r} \mu$ are strict over $\mathbb{k}$, so we only need to check the axioms when all entries are nontrivial paths. Fix an angle sequence ( $\gamma_{r}, \ldots, \gamma_{1}$ ) of length $\geq 4$ and assume that there is a nonzero double product

$$
{ }^{r} \mu_{u}\left(\gamma_{s+t-1}, \ldots, \gamma_{i+t},{ }^{r} \mu_{v}\left(\gamma_{i+t-1}, \ldots, \gamma_{i}\right), \gamma_{i-1}, \ldots, \gamma_{1}\right)
$$

where ${ }^{r} \mu_{u}$ can be a ${ }^{r} \mu_{\text {ord }}($ if $s=2)$ or a ${ }^{r} \mu_{\psi}$ if $s>2$, and ${ }^{r} \mu_{v}$ can be a ${ }^{r} \mu_{p}$ (if $\left.t=0\right)$, a ${ }^{r} \mu_{\text {ord }}($ if $t=2)$ or a ${ }^{r} \mu_{\psi}($ if $t>2)$. Such a product is uniquely characterized by the triple $(u, v, i)$.

Each nonzero double product is a path multiplied with a factor $\pm r_{u} r_{v}$ (where we use the convention that $r_{\text {ord }}=1$ ). We will show there is a unique other triple with a nonzero double product cancel that cancels it.

To do this, we will go through all possible triples $(u, v, i)$ systematically, making a distinction between whether $i$ is $1, s$ or somewhere in the middle. For each case we draw a diagram of the possible situations and every situation will occur exactly twice (see figure 2.1. Note that most diagrams have two versions depending on whether the (only/outer) higher product has a front or a back. We will only consider the latter cases. The diagrams are named after the operations of type $A, B$ from lemma 2.16 that can be performed on the orbigons to stitch or fold them together.

1. $(\psi, p, i)$

- If $i=1$ then $\ell_{p}$ contains the back of ${ }^{r} \mu_{\psi}$ and we are in situation $O 1$. We can compensate this term with a term of a similar type but where $\ell_{p}$ contains the front:

$$
{ }^{r} \mu_{\psi}\left(\gamma_{s}, \ldots, \gamma_{1},{ }^{r} \mu_{p}\right) \text { cancels with }{ }^{r} \mu_{\psi}\left({ }^{r} \mu_{p}, \gamma_{s}, \ldots, \gamma_{1}\right)
$$

- If $1<i<s$ then $\psi$ has $\ell_{p}$ as one of its angles and we are in situation $B 1$. We can fold the orbigon together to form a new orbigon $\psi^{\prime}$ with $r_{\psi^{\prime}}=r_{\psi} r_{p}$. The other term that cancels it is of the form ( $\psi^{\prime}$, ord, $i-1$ ).
- If $i=s$ and $\ell_{p}$ comes before $\gamma_{s}$ then either $\gamma_{s}$ is shorter than the back of ${ }^{r} \mu_{\psi}$ or longer. If it is shorter, we are in situation $B 2$. If it is longer than the back, the arc $t\left(\gamma_{1}\right)$ will lie inside $\psi$. In that case we distinguish type $B A$ if $t\left(\gamma_{1}\right)$ cuts $\psi$ in two pieces, or $B B$ if it opens a second fold at a marked point $q$. The former is compensated by a terms of type ( $\left.\psi_{1}, \psi_{2}, i\right)$, while the latter is compensated by a term $\left(\psi_{1}, q, i\right)$. In both cases $\psi_{1}$ includes the orbifold point $p$ and has a front)

2. $(\psi$, ord,$i)$

- If $1<i \leq s$ then the inner product ${ }^{r} \mu^{2}\left(\gamma_{i+1}, \gamma_{i}\right)$ is an angle from the orbigon $\psi$ then $h\left(\gamma_{i}\right)$ either opens a fold of $\psi(B 1)$ or it cuts the orbigon in two $(A 1 L, A 1 R, A 1 M)$. The first is compensated by a term of type ( $\psi^{\prime}, p, i+1$ ), while the latter three by a term of type $\left(\psi_{1}, \psi_{2}, j\right)$.
- If $i=1$ and $\gamma_{1}$ is part of the back of ${ }^{r} \mu_{\psi}$ then $h\left(\gamma_{1}\right)$ will cut the back in two. This is situation $O 2$ and it is compensated by a term of the form (ord, $\psi, 2$ ).
If the back of ${ }^{r} \mu_{\psi}$ is part of $\gamma_{1}$ then $h\left(\gamma_{1}\right)$ either opens a fold of $\psi$ or it cuts the orbigon in two. Just like when $i \neq 1$ above, we are in cases $(B 1, A 1 L, A 1 R, A 1 M)$ but now with $\gamma_{1}$ equal to the top of the hexagon instead of the bottom. They are also compensated by terms of type $\left(\psi^{\prime}, p, i+1\right)$ and $\left(\psi_{1}, \psi_{2}, j\right)$.

3. (ord, $\psi, i)$

- If $i=1$ and ${ }^{r} \mu^{2}\left(\gamma_{t+1},{ }^{r} \mu_{\psi}\left(\ldots, \gamma_{1}\right)\right)$ puts something in front of the back of ${ }^{r} \mu_{\psi}$, three things can happen. If $\gamma_{t+1}$ is shorter than the first angle of $\psi$ then $h\left(\gamma_{t+1}\right)$ either cuts $\psi$ in two (A2) or opens a fold of $\psi(B 2)$. These cases are compensated by terms of the form $\left(\psi_{1}, \psi_{2}, j\right)$ and $\left(\psi^{\prime}, p, t\right)$.
If $\gamma_{t+1}$ is longer than the first angle we are in situation $O 3$ and can compensate it with a term of the same type but with a head

$$
{ }^{r} \mu^{2}\left({ }^{r} \mu_{\psi}\left(\gamma_{t+1}, \ldots, \gamma_{2}\right), \gamma_{1}\right) .
$$

- If $i=2$ then ${ }^{r} \mu^{2}\left({ }^{r} \mu_{\psi}\left(\ldots, \gamma_{2}\right), \gamma_{1}\right)$ adds something to the back of ${ }^{r} \mu_{\psi}$ and we are in situation O2.

4. $\left(\psi_{1}, \psi_{2}, i\right)$

- If $i=1$ and ${ }^{r} \mu_{\psi_{2}}$ has a back then we are in $A 1 M$. If ${ }^{r} \mu_{\psi_{2}}$ has a front we are in situation $O 4$. Then we can commute the order of $\psi_{1}$ and $\psi_{2}$, this gives two terms that cancel:

$$
{ }^{r} \mu_{\psi_{1}}\left(\gamma_{s+t-1}, \ldots,{ }^{r} \mu_{\psi_{2}}\left(\ldots, \gamma_{1}\right)\right) \text { and }{ }^{r} \mu_{\psi_{2}}\left({ }^{r} \mu_{\psi_{1}}\left(\gamma_{s+t-1}, \ldots\right), \ldots, \gamma_{1}\right)
$$

- If $1<i<s$ and ${ }^{r} \mu_{\psi_{2}}$ has a back equal to an angle of $\psi_{1}$ we are in situation $A 1 L$, while if it has a front equal to an angle, we are in situation $A 1 R$.
- If $i=s$ and ${ }^{r} \mu_{\psi_{2}}$ has a front we are in $A 1 R$. If it has a back then it depends on the back of ${ }^{r} \mu_{\psi_{1}}$. If it is at least as long as the last angle of $\psi_{2}$ we are in situation $A 2$. Otherwise $t\left(\gamma_{1}\right)$ will





Figure 2.1: The possible diagrams that can occur in as double products
cut $\psi_{2}$ in two pieces (situation $A A$ ) or it will open a fold $(A B)$. The former is compensated by a term of type $\left(\psi_{1}^{\prime}, \psi_{2}^{\prime}, j\right)$ while the latter is compensated by a term $\left(\psi_{1}^{\prime}, p, j\right)$. In both cases the outer product now has a front. Note that type $A B$ and $B A$ only seem to occur once in the list but this is because they are canceled by terms with a front and these are not in the list.

Remark 2.24. The proof is analogous to the one in 35, but because of the internal orbifold points extra cases needed to be considered. These are all the cases with a $B$ in their label. The proof can also be extended to the case where [NL2] does not hold but this will include a lot more extra cases.

Remark 2.25. It is also possible to weight the faces in the orbigons using elements in $R^{+}$. Choose an element $s_{(f, i)} \in R^{+}$for each face $f$ and $i \in \mathbb{N}$. The weight of a tree-gon is then defined inductively by giving $\left(\alpha_{i k}, \ldots, \alpha_{1}\right)$ weight $s_{(f, i)}$ if the angle sequence turns $i$ times around $f$ (note that this definition allows for orbifold faces as well). If you stitch two tree-gons together their weight is defined as the product. The folding operation adds an additional factor $r_{(m, j)}$ in the same way as before.

This gives a family of products ${ }^{r, s} \mu^{\bullet}$ on $\mathrm{Gtl}_{\mathcal{A}} \widehat{\otimes} R$ that depends on two parameters sets $r: M \times \mathbb{N} \rightarrow R^{+}$ and $s: M \times \mathbb{N} \rightarrow R^{+}$. Note that the curvature only depends on $r$ not on $s$.

This same parameter set can be used for the Koszul dual $\mathrm{Gtl}_{\mathcal{A}^{\perp}}$ but now the role of $r$ and $s$ are reversed because for the Koszul dual the role of the marked points and the faces are swapped. In this case $s$ will give rise to curvature, while $r$ only contributes to the higher products. Note however that if $s_{(f, j)}$ is nonzero for an infinite number of $(f, j)$ we should go to the completed version $\widehat{\mathrm{Gtl}}_{\mathcal{A}^{\perp}}$ for the curvature to make sense.

## 3 Hochschild Cohomology of Gentle $A_{\infty}$-algebras

### 3.1 Definitions

Definition 3.1. If $(A, \mu)$ is a $\mathbb{Z}$ - or $\mathbb{Z}_{2}$-graded $A_{\infty}$-algebra over a semisimple algebra $\mathbb{k}$, we define the $A_{\infty}$-Hochschild complex as

$$
\mathrm{HC}^{\bullet}(A)=\operatorname{Hom}_{\mathrm{k}}\left(\bigoplus_{i \geq 0} A[1]^{\otimes_{\mathrm{k}} i}, A[1]\right)
$$

Every element $\nu \in \operatorname{HC}^{j}(A)$ can be seen as a collection of $n$-ary products $\nu^{n}$ of degree $1+j-n$, one for each $n \in \mathbb{Z}_{\geq 0}$. We will call $\nu^{n}$ the $n$th component of $\nu$.
Remark 3.2. Note that the grading on this space is not the classical grading on Hochschild cohomology coming from the number of entries in the products, but the one coming from the degrees of the maps $A[1]^{\otimes_{k} i} \rightarrow A[1]$. For this grading the product $\mu$ can be seen as an element of $\mathrm{HC}^{1}(A)$. To avoid confusion we will call this degree the $\infty$-degree $\|\bullet\|$. If we work $\mathbb{Z}_{2}$-graded we will also refer to this grading as the parity and denote the homogeneous parts as $\mathrm{HC}^{\text {even }}(A)$ and $\mathrm{HC}^{\text {odd }}(A)$.

- On $\mathrm{HC}^{\bullet}(A)$ we have a bracket of degree 0 :

$$
\begin{aligned}
{[\kappa, \nu]\left(a_{r}, \ldots, a_{1}\right):=} & \sum_{0 \leq i \leq j \leq r}(-1)^{\left(\left\|a_{1}\right\|+\ldots+\left\|a_{i}\right\|\right)\|\nu\|} \kappa\left(a_{r}, \ldots, a_{j+1}, \nu\left(a_{j}, \ldots, a_{i+1}\right), a_{i}, \ldots, a_{1}\right) \\
& -(-1)^{\|\nu\|\|\kappa\|+\left(\left\|a_{1}\right\|+\ldots+\left\|a_{i}\right\|\right)\|\kappa\|} \sum_{0 \leq i \leq j \leq r} \nu\left(a_{r}, \ldots, a_{j+1}, \kappa\left(a_{j}, \ldots, a_{i+1}\right), a_{i}, \ldots, a_{1}\right) .
\end{aligned}
$$

The $A_{\infty}$-axioms for the product $\mu$, can be rephrased as $[\mu, \mu]=0$, which also means that $d=[\mu,-]$ is a differential of degree 1 and the triplet

$$
\left(\mathrm{HC}^{\bullet}(A), d,[,]\right)
$$

is a differential graded Lie algebra (DGLA). The solutions to the Maurer-Cartan equation

$$
d \nu+\frac{1}{2}[\nu, \nu]=0
$$

describe the deformations of $\mu$ as a curved $A_{\infty}$-structure.

- There is also a second product on $\mathrm{HC}^{\bullet}(A)$ : the cup product.

$$
(\kappa \smile \nu)\left(a_{r}, \ldots a_{1}\right):=\sum_{0 \leq i \leq j \leq u \leq v \leq r}(-1)^{2} \mu\left(a_{r}, \ldots, \kappa\left(a_{v}, \ldots, a_{u+1}\right), \ldots, \nu\left(a_{j}, \ldots, a_{i+1}\right), \ldots, a_{1}\right)
$$

with $=\left(\left\|a_{1}\right\|+\ldots+\left\|a_{u}\right\|\right)\|\kappa\|+\left(\left\|a_{1}\right\|+\ldots+\left\|a_{i}\right\|\right)\|\nu\|+\|\nu\|+1$. This product has degree 1 and together with $d$ satisfies the it satisfies the graded Leibniz rule. Therefore

$$
\left(\mathrm{HC}^{\bullet}(A)[-1], d, \smile\right)
$$

is a differential graded algebra DGA and if we go to homology the triplet

$$
\left(\mathrm{HH}^{\bullet}(A)[-1], \smile,[,]\right)
$$

is a Gerstenhaber algebra.
Remark 3.3. These two classical structures on Hochschild cohomology of ordinary ungraded algebras were analyzed first by Gerstenhaber [33. Both definitions have been extended to $A_{\infty}$-algebras and are well-known in the literature, see e.g. 56 .

Mescher 56 uses a different sign convention for $A_{\infty}$-categories. This means his definition of the cup product also has different signs. We have adapted the signs in order to suit the signs in our definition of $A_{\infty}$-categories. In particular, the cup product as defined is graded symmetric, with respect to a degree shift of 1 on the Hochschild cohomology:

$$
\begin{equation*}
\nu \smile \eta=(-1)^{(\|\nu\|-1)(\|\eta\|-1)} \eta \smile \nu . \tag{3.1}
\end{equation*}
$$

The unexpected shift by 1 in this sign rule is desired. In fact, this "shifted graded symmetry" renders the cup product truly graded symmetric with respect to the "traditional" grading of the Hochschild complex, which differs from the $A_{\infty}$-grading precisely by one. In other words, if one regards an ordinary ungraded algebra and starts grading the Hochschild cohomology at zero (instead of minus one), the cup product becomes graded symmetric.

We have tried to arrange the signs such that the cup product together with the Gerstenhaber bracket turns $\mathrm{HH}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$ into a Gerstenhaber algebra with correct signs. In order to be a Gerstenhaber algebra, sign conventions of cup product and bracket need to be tuned to each other. The relevant compatiblity condition is the signed Leibniz rule which reads

$$
[\nu, \eta \smile \omega]=[\nu, \eta] \smile \omega+(-1)^{\|\nu\|(\|\eta\|-1)} \eta[\nu, \omega] .
$$

We have tried to arrange the signs with the Leibniz rule in mind, but we have not conducted the tedious checks. We have however chosen the signs in analogy with 56 . In particular, the signs are such that the cup product descends to Hochschild cohomology and is graded symmetric, the latter we did check by hand.

Remark 3.4. If $(A, \mu)$ is a strictly unital $A_{\infty}$-algebra over $\mathbb{k}$, i.e. $\mu\left(x_{1}, \ldots, x_{n}\right)=0$ if $n \neq 2$ and at least one of the $x_{i} \in \mathbb{k}$, one can construct the normalized Hochschild cochain complex $\underline{H C}^{\bullet}(A)$. This is the subspace of all cochains that evaluate to zero if at least one of the entries is in $\mathbb{k}$. This subspace is closed under all the operations above the embedding is a quasi-isomorphism 39. The Maurer-Cartan equation for $\underline{\mathrm{HC}}^{\bullet}(A)$ classifies deformations of $\mu$ for which $\mathbb{k}$ remains strict.

We will now specialize to the case where $A=\mathrm{Gtl}_{\mathcal{A}}$ is the gentle algebra of an arc collection $\mathcal{A}$ equipped with $\mu={ }^{r} \mu$ for $r=0$. We have seen that the $\mathrm{Gtl}_{\mathcal{A}}$ has a natural $\mathbb{Z}_{2}$-grading, so the $\infty$-degree is a $\mathbb{Z}_{2}$-degree, which we refer to as the parity. It is possible to lift this $\mathbb{Z}_{2}$-grading to a $\mathbb{Z}$-grading. To do this we assign to each angle $\alpha$ a degree $\operatorname{deg} \alpha \in \mathbb{Z}$ with the same parity as $|\alpha|$. In order to make this degree compatible with $\mu$ we have to ensure that $\mu$ has $\infty$-degree 1 . This happens only when the total degree of the angles in a $k$-gon is $k-2$. As every angle occurs only in one $k$-gon this can always be done but not canonically.

The gentle $A_{\infty}$-algebra also has an extra grading coming from the relative first homology

$$
G=H_{1}(S \backslash M, \mathcal{A}, \mathbb{Z})
$$

This group can be described in terms of the angles and faces

$$
G=\frac{\bigoplus_{\alpha \in\left(Q_{\mathcal{A}}\right)_{1}} \mathbb{Z} \alpha}{\left.\left\langle\alpha_{1}+\cdots+\alpha_{k}\right|\left(\alpha_{1}, \ldots, \alpha_{k}\right) \text { is a face }\right\rangle}
$$

The group $G$ comes with two natural maps

- $\pi: G \rightarrow \mathbb{Z}^{\mathcal{A}}: \alpha \mapsto h(\alpha)-t(\alpha)$
- $\iota: \mathbb{Z}^{M} \rightarrow G: m \mapsto \sum_{\alpha \in \ell_{m}} \alpha$

We have $\pi \circ \iota=0$. The image of the $\pi$ has corank 1 , while the kernel of $\iota$ is $(1, \ldots, 1)$. Therefore if $\# M \geq 2$ the map $\iota$ is nonzero.

If we grade $\mathrm{Gtl}_{\mathcal{A}}$ by giving each angle its corresponding degree in $G$, it is clear that all products $\mu$ have degree $0 \in G$. We can transfer the $G$-grading to the Hochschild complex and it is easy to see that it contains only nonzero elements for degrees in $\operatorname{Ker} \pi$. Furthermore the natural operations [,], $\smile$ and $d$ all have $G$-degree 0 .

Remark 3.5. It is also clear from this construction that if deg and $\mathrm{deg}^{\prime}$ are two different $\mathbb{Z}$-lifts of the $\mathbb{Z}_{2}$-grading then their difference factors through $G$ because deg and deg' assign the same degree to a face. More precisely, the possible $\mathbb{Z}$-gradings form a $\operatorname{Hom}(G, 2 \mathbb{Z})$-torsor.

Lemma 3.6. If $\operatorname{deg}$ is a $\mathbb{Z}$-lift of the $\mathbb{Z}_{2}$-degree on $\operatorname{Gtl}_{\mathcal{A}}$ then

$$
\sum_{m \in M} \operatorname{deg} \ell_{m}=4-4 g-2 \# M=2 \chi(S, M)<0
$$

Here $g$ is the genus of the marked surface $(S, M)$ and $\chi(S, M)$ its Euler characteristic (which is negative by assumption).

Proof.

$$
\begin{aligned}
\sum_{m \in M} \operatorname{deg} \ell_{m} & =\sum_{\alpha \in\left(Q_{\mathcal{A}}\right)_{1}} \operatorname{deg} \alpha=\sum_{f \in F} \operatorname{deg} f \\
& =\sum_{f \in F}(2-\#\{\alpha \in f\})=2 \# F-2 \# \mathcal{A}=2(2-2 g)-2 \# M
\end{aligned}
$$

### 3.2 Reduction to the zeroth and first component

In this section we will show that the Hochschild cohomology class of a cocycle can be read off from the 0 th and 1st component. In other words, if $\nu^{0}$ and $\nu^{1}$ are both zero then $\nu$ is a coboundary. Remark 3.4 allows us to restrict to cochains which belong to the normalized Hochschild cohomology. For such classes we will construct an $\epsilon \in \underline{\mathrm{HC}^{\bullet}}(A)$ such that $\nu-d \epsilon$ evaluates zero on all sequences $\left(\beta_{k}, \ldots, \beta_{1}\right)$ where the $\beta_{i}$ are nontrivial angle paths and $t\left(\beta_{i}\right)=h\left(\beta_{i+1}\right)$. We will often refer to the procedure of adding a coboundary as gauging.

Definition 3.7. Let $\beta=\left(\beta_{k}, \ldots, \beta_{1}\right)$ be a sequence of nontrivial angles with $h\left(\beta_{i}\right)=t\left(\beta_{i+1}\right)$ for $i<k$ (we do not impose that it cycles so $h\left(\beta_{i}\right), t\left(\beta_{1}\right.$ can be different).

- We call $\beta$ elementary if all $\beta_{i}$ are indecomposable angles (angle arrows).
- An index $i$ is called a contractible index if $\beta_{i+1} \beta_{i} \neq 0$.
- Denote the set of contractible indices by $\operatorname{Contr}(\beta)$ and if $S \subset \operatorname{Contr}(\beta)$ then the contracted sequence $\beta^{S}$ will be the sequence where consecutive angles separated by a contractible index are multiplied together.
- Every angle sequence is of the form $\beta^{S}$ where $\beta$ is elementary and $S$ a set of contractible indices. Moreover $\beta$ and $S$ are uniquely determined. Two sequences are of the same type if they are contractions of the same elementary sequence.
- The total length of a sequence will be the length of its underlying indecomposable sequence.
- For an elementary sequence we have that $\operatorname{Contr}(\beta)=\{ \}$ if and only if it consists of consecutive angles of a polygonal face. Therefore we will call such sequences polygon sequences.

Lemma 3.8. Let $\nu \in \operatorname{Ker}(d) \subset \underline{\operatorname{HC}^{\bullet}}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$ with $(\mathrm{A}) \nu^{0,1}=0$. Then $\nu$ can be gauged to satisfy additionally that (B) $\nu^{2}(\alpha, \beta)=0$ for all pairs of angle paths with $\alpha \beta \neq 0$. During the gauging procedure, the value of $\nu$ does not change on polygon sequences.

Proof. We will construct an $\varepsilon=\varepsilon^{1}$ such that $\nu^{2}(\alpha, \beta)=d \varepsilon^{2}(\alpha, \beta)$ if $\alpha \beta \neq 0$. Set $\varepsilon^{1}(\alpha)=0$ for every indecomposable angle $\alpha$. Now define inductively $\varepsilon^{1}$ for decomposable angles (angle paths) by the rule

$$
\begin{equation*}
\varepsilon^{1}(\alpha \beta)=\varepsilon^{1}(\alpha) \beta+\alpha \varepsilon^{1}(\beta)-(-1)^{|\beta|} \nu^{2}(\alpha, \beta) \tag{3.2}
\end{equation*}
$$

Let us check that extending $\varepsilon^{1}$ according to this rule is well-defined, that is, $\varepsilon^{1}((\alpha \beta) \gamma)=\varepsilon^{1}(\alpha(\beta \gamma))$. Indeed,

$$
\begin{aligned}
& \left(\varepsilon^{1}(\alpha \beta) \gamma+\alpha \beta \varepsilon^{1}(\gamma)-(-1)^{|\gamma|} \nu^{2}(\alpha \beta, \gamma)\right)-\left(\varepsilon^{1}(\alpha) \beta \gamma+\alpha \varepsilon^{1}(\beta \gamma)-(-1)^{|\beta \gamma|} \nu^{2}(\alpha, \beta \gamma)\right) \\
& =\varepsilon^{1}(\alpha) \beta \gamma+\alpha \varepsilon^{1}(\beta) \gamma-(-1)^{|\beta|} \nu^{2}(\alpha, \beta) \gamma+\alpha \beta \varepsilon^{1}(\gamma)-(-1)^{|\gamma|} \nu^{2}(\alpha \beta, \gamma) \\
& -\varepsilon^{1}(\alpha) \beta \gamma-\alpha \varepsilon^{1}(\beta) \gamma-\alpha \beta \varepsilon^{1}(\gamma)+(-1)^{|\gamma|} \alpha \nu^{2}(\beta, \gamma)+(-1)^{|\beta \gamma|} \nu^{2}(\alpha, \beta \gamma) \\
& =-\left(-\left.1\right|^{|\beta|} \nu^{2}(\alpha, \beta) \gamma-(-1)^{|\gamma|} \nu^{2}(\alpha \beta, \gamma)+(-1)^{|\gamma|} \alpha \nu^{2}(\beta, \gamma)+(-1)^{|\beta \gamma|} \nu^{2}(\alpha, \beta \gamma)\right. \\
& =(-1)^{|\beta|}(d \nu)(\alpha, \beta, \gamma)=0 .
\end{aligned}
$$

In the final row we have used that $\nu^{0}=\nu^{1}=0$. We conclude that for angles $\alpha, \beta$ with $\alpha \beta \neq 0$ we have

$$
(d \varepsilon)(\alpha, \beta)=(-1)^{|\beta|} \varepsilon^{1}(\alpha) \beta+(-1)^{|\beta|} \alpha \varepsilon^{1}(\beta)-(-1)^{|\beta|} \varepsilon^{1}(\alpha \beta)=\nu^{2}(\alpha, \beta)
$$

as desired. Furthermore $(d \varepsilon)^{0}=(d \varepsilon)^{1}=0$ and if $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ is a polygon sequence, then

$$
(d \varepsilon)\left(\alpha_{k}, \ldots, \alpha_{1}\right)=\varepsilon\left(\mu\left(\alpha_{k}, \ldots, \alpha_{1}\right)\right)+\mu\left(\ldots, \varepsilon\left(\alpha_{i}\right), \ldots\right)=0 .
$$

because all $\alpha_{i}$ are indecomposable so $\varepsilon\left(\alpha_{i}\right)=0$.
Lemma 3.9. Let $\nu \in \operatorname{Ker}(d) \subset \underline{\operatorname{HC}^{\bullet}}\left(\operatorname{Gtl}_{\mathcal{A}}\right)$ with $(\mathrm{A}) \nu^{0,1}=0$ and (B) $\nu^{2}(\alpha, \beta)=0$ if $\alpha \beta \neq 0$. Then $\nu$ can be gauged to (C) evaluate zero on polygonal sequences without affecting the conditions (A), (B).

Proof. We start off with a little remark that is important to follow the argument. Note that when going around a polygon $\left(\alpha_{N}, \ldots, \alpha_{1}\right)$ the boundary arcs can be oriented clockwise or anticlockwise. If two consecutive $\operatorname{arcs} a_{i}=t\left(\alpha_{i}\right)$ and $a_{i+1}=h\left(\alpha_{i}\right)$ have the same orientation then $\left|\alpha_{i}\right|=1$ and $\left\|\alpha_{i}\right\|=0$.


Therefore if we have a polygonal sequence $\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)$ then the orientation of $h\left(\alpha_{i+k}\right)$ and $t\left(\alpha_{i+1}\right)$ will be the same (different) if the total shifted degree $\left\|\alpha_{i+k}\right\|+\ldots+\left\|\alpha_{i+1}\right\|$ is zero (one).

Working with the shifted degree is useful because the shifted degree of $\left\|\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)\right\|$ is the total shifted degree $\left\|\alpha_{i+k}\right\|+\ldots+\left\|\alpha_{i+1}\right\|$ plus the parity of $\nu$. With this in mind we will distinguish two cases depending on the parity of $\nu$.

- The parity of $\nu$ is odd. We proceed by induction. Regard an elementary polygon $\left(\alpha_{N}, \ldots, \alpha_{1}\right)$ of length $N$ and assume $\nu$ already vanishes on its polygon sequences of length $\leq k-1$. We will simultaneously gauge away $\nu$ on the polygon sequences of length $k$ of this polygon.
First note that an arc can occur at most twice on the boundary of a face and if it does it must be oriented once clockwise and once anticlockwise around the face because the surface is orientable. This implies that if $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)$ contains an identity, then $k$ must be a multiple of $N$. Indeed, if $k<N$ and there is an identity $\mathbb{1}_{a}$ in the result then $t\left(\alpha_{i+1}\right)=h\left(\alpha_{i+k}\right)=a$. Because the two orientations of $a$ around the face are different the shifted degree of $\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)$ is odd. As $\nu$ is odd, $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)$ will be even and hence it cannot contain $\mathbb{1}_{a}$ (whose shifted degree is odd).
Let us now assume $k$ is not a multiple of $N$. Then we can write $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)=\alpha_{i+k} \delta^{(i+k)}+$ $\gamma^{(i+1)} \alpha_{i+1}$. Note that

$$
\begin{aligned}
0= & (d \nu)\left(\alpha_{i+k+1}, \ldots, \alpha_{i+1}\right) \\
= & (-1)^{\left\|\alpha_{i+k}\right\|+\ldots+\left\|\alpha_{i+1}\right\|} \alpha_{i+k+1}\left(\alpha_{i+k} \delta^{(i+k)}+\gamma^{(i+1)} \alpha_{i+1}\right) \\
& +(-1)^{\left\|\alpha_{i+1}\right\|+\left|\alpha_{i+1}\right|}\left(\alpha_{i+k+1} \delta^{(i+k+1)}-\gamma^{(i+2)} \alpha_{i+2}\right) \alpha_{i+1} \\
= & \alpha_{i+k+1}\left((-1)^{\left\|\alpha_{i+k}\right\|+\ldots+\left\|\alpha_{i+1}\right\|} \gamma^{(i+1)}-\delta^{(i+k+1)}\right) \alpha_{i+1} .
\end{aligned}
$$

In evaluating the first row we have used the induction hypothesis. Independent of arc directions, the angles $\alpha_{i+k+1}$ and $\gamma^{(i+1)}$ are composable and $\delta^{(i+k+1)}$ and $\alpha_{i+1}$ are composable. We deduce $\delta^{(i+k+1)}=(-1)^{\left\|\alpha_{i+k}\right\|+\ldots+\left\|\alpha_{i+1}\right\|} \gamma^{(i+1)}$ for all $i$. Set $\varepsilon\left(\alpha_{i+k-1}, \ldots, \alpha_{i+1}\right)=(-1)^{\left\|\alpha_{i+k-1}\right\|+\ldots+\left\|\alpha_{i+1}\right\|+1} \delta^{(i+k)}$ for all $i$. Then

$$
\begin{aligned}
(d \varepsilon)\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right) & =\alpha_{i+k} \delta^{(i+k)}+(-1)^{\left\|\alpha_{i+k}\right\|+\ldots+\left\|\alpha_{i+2}\right\|+1+\left|\alpha_{i+1}\right|} \delta^{(i+k+1)} \alpha_{i+1} \\
& =\alpha_{i+k} \delta^{(i+k)}+\gamma^{(i+1)} \alpha_{i+1}=\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right) .
\end{aligned}
$$

We have $(d \varepsilon)^{0}=(d \varepsilon)^{1}=0$. Let us now check that $d \varepsilon$ does not affect any other elementary polygon sequences or the sequences in the same polygon with length $l \leq k-1$. Regarding shorter sequences in the same polygon, we have

$$
(d \varepsilon)\left(\alpha_{i+l}, \ldots, \alpha_{i+1}\right)=\mu(\ldots, \varepsilon, \ldots)+\varepsilon\left(\ldots, \mu\left(\alpha_{i+s+t N}, \ldots, \alpha_{i+s+1}\right), \ldots\right)
$$

where $N$ is the length of the polygon. The first summand vanishes, since $l \leq k-1$. In the second summand, the inner $\mu$ may only yield an identity and $\varepsilon$ vanishes. For sequences in other polygons, similar arguments apply. We have safeguarded that $\nu^{0,1}=0$ is preserved when gauging by $\varepsilon$. For $k>2$, also $(d \varepsilon)^{2}$ vanishes. In case $k=2$ the $(d \epsilon)^{2}(\alpha, \beta)$ may be nonzero in rare cases, but we fix this by applying Lemma 3.8, without changing $\nu$ on polygon sequences.
Finally, let us treat the case where $k$ is a multiple of the length $N$ of the polygon. Apart from the identities, we can gauge everything in $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)$ away as above. It remains to check out the identities. Write $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)=c_{i} \mathbb{1}_{a_{i}}$. Note we have

$$
\begin{aligned}
0 & =(d \nu)\left(\alpha_{i+k+1}, \ldots, \alpha_{i+1}\right) \\
& =\alpha_{i+k+1} \nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)-\nu\left(\alpha_{i+k+1}, \ldots, \alpha_{i+2}\right) \alpha_{i+1} \\
& =c_{i} \alpha_{i+k+1}-c_{i+1} \alpha_{i+1}=\left(c_{i}-c_{i+1}\right) \alpha_{i+1} .
\end{aligned}
$$

Here we have used that $\nu$ already vanishes on sequences of the same polygon of length $\leq k-1$. We obtain that all $c_{i}$ around the polygon are equal. Denote this value by $c$. Choose an angle $\alpha_{1}$ in the polygon and define $\varepsilon\left(\alpha_{k-N+1}, \ldots, \alpha_{1}\right):=c \alpha_{1}$. As desired,

$$
\begin{aligned}
(d \varepsilon)\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right) & =\mu\left(\ldots, \varepsilon\left(\alpha_{k+1}, \ldots, \alpha_{1}\right), \ldots\right) \\
& =c \mu\left(\alpha_{i+N}, \ldots, \alpha_{i+1}\right)=c \mathbb{1}_{a_{i}}
\end{aligned}
$$

In the first row we have used that the sequence $\alpha_{1}, \ldots, \alpha_{k-N+1}$ of length $k-N+1$ appears precisely once in the sequence $\alpha_{i+k}, \ldots, \alpha_{i+1}$ and hence an inner $\varepsilon$ can be applied precisely once. We see
that $(d \varepsilon)^{0}=(d \varepsilon)^{1}=0$. Let us check that $d \varepsilon$ vanishes on any polygon sequence $\beta_{1}, \ldots, \beta_{l}$ of length $l \leq k-1$ in the same polygon. Indeed,

$$
(d \varepsilon)\left(\beta_{l}, \ldots, \beta_{1}\right)=\varepsilon(\ldots, \mu(\ldots), \ldots)+\mu^{2}(\ldots, \varepsilon(\ldots), \ldots)+\mu^{\geq 3}(\ldots, \varepsilon(\ldots), \ldots)
$$

The first summand vanishes, because the inner $\mu$ only gives identities. The second summand vanishes, because the result of the inner $\varepsilon$ is a multiple of the first, equivalently last input of its input sequence, hence not composable with the input angle of the outer $\mu^{2}$. The third summand vanishes, because $\varepsilon$ consumes $k-N+1$ inputs and the outer $\mu^{\geq 3}$ needs $N-1$ more inputs, while the sequence $\beta_{1}, \ldots, \beta_{l}$ has length only $l \leq k-1$. Similarly, one checks that $d \varepsilon$ vanishes entirely on polygon sequences in other polygons. For $k>N$, we have $(d \varepsilon)^{2}=0$. For $k=N$, it may happen that $(d \varepsilon)^{2} \neq 0$, but we fix this by applying Lemma 3.8 .
In total, we have gauged $\nu$ infinitely many times during this proof. However, the $\varepsilon$ gauges have higher and higher input length. Moreover we only invoke Lemma 3.8 finitely many times, this means that the total sum of the gauges is defined in $\operatorname{Hom}_{\mathbf{k}}\left(\bigoplus_{i \geq 0} A[1]^{\otimes_{\mathbf{k}} i}, A[1]\right)$ and we conclude it is a Hochschild cochain.

- The parity of $\nu$ is even. As in the odd case, we proceed again by induction over the length $k$ of the polygon sequence.
Let us first check for possible identities in $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)$. In case $k$ is a multiple of $N$, the source arc of $\alpha_{i+1}$ is equal to the target arc of $\alpha_{i+k}$, in particular $\left\|\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)\right\|$ is even and does not contain identities. In case $k$ is not a multiple of $N$, we observe

$$
(d \nu)\left(\alpha_{i+k+1}, \alpha_{i+k}, \ldots, \alpha_{i+1}\right)=\alpha_{i+k+1} \nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)+\nu\left(\alpha_{i+k+1}, \ldots, \alpha_{i+2}\right) \alpha_{i+1}
$$

Since $k$ is not a multiple of $N$, we have $\alpha_{i+k+1} \neq \alpha_{i+1}$ and conclude that $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)$ contains no identities.
We now proceed with gauging $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)$ to zero. By abuse of wording, let us say "head" and "tail" of a polygon's arc $a$ to mean the vertex at the clockwise end and at the counterclockwise end of $a$. For every $i$ we write

$$
\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)=\kappa_{i}+\lambda_{i}
$$

where $\kappa_{i}$ contains the angle paths that starts at the "head" of $a_{i+1}$ and $\lambda_{i}$ starts at the "tail" of $a_{i+1}=t\left(\alpha_{i+1}\right)$.


We show that $\lambda_{i}$ necessarily vanishes. Regard the source arc $a_{i+1}$ of $\alpha_{i+1}$ and target arc $a_{i+k+1}$ of $\alpha_{i+k}$. We have

$$
0=(d \nu)\left(\alpha_{i+k+1}, \ldots, \alpha_{i+1}\right)=\alpha_{i+k+1}\left(\kappa_{i}+\lambda_{i}\right)+\left(\kappa_{i+1}+\lambda_{i+1}\right) \alpha_{i+1}
$$

We know $\kappa_{i}$ is odd or even, depending on whether $a_{i+1}$ and $a_{i+k+1}$ are equally oriented with respect to the polygon or not. Since $\kappa_{i}$ always starts at the "head" of $a_{i+1}$, we deduce from its degree that it always ends at the "tail" of $a_{i+k+1}$. In particular $\alpha_{i+k+1} \kappa_{i}=0$. Similarly, $\lambda_{i}$ starts at the "tail" of $a_{i+1}$, hence ends at the "head" of $a_{i+k+1}$, whether $a_{i+1}$ and $a_{i+k+1}$ are oriented equally or not. In particular $\alpha_{i+k+1}$ and $\lambda_{i}$ are composable. But $\alpha_{i+1}$ starts at the "head" of $a_{i+1}$, while $\lambda_{i}$ starts at the "tail" of $a_{i+1}$, hence the summands $\alpha_{i+k+1} \lambda_{i}$ and $\left(\kappa_{i+1}+\lambda_{i+1}\right) \alpha_{i+1}$ consist of disjoint sets of angles. Since the whole sum is supposed to vanish, we conclude $\lambda_{i}=0$.
Since $\kappa_{i}$ starts at the side of $\alpha_{i+1}$ and $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)$ contains no identities, we can write $\kappa_{i}=$ $\gamma_{i} \alpha_{i+1}$. We aim at gauging $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)$ to zero. Let us first gauge away all terms except the possible scalar multiple of $\alpha_{i+1}$, which comes from a possible identity in $\gamma_{i}$. Put $\varepsilon\left(\alpha_{i+k}, \ldots, \alpha_{i+2}\right):=$ $\gamma_{i}$. Then

$$
(d \varepsilon)\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)=\gamma_{i} \alpha_{i+1}+\alpha_{i+k} \gamma_{i-1}
$$

Apart from identities in $\gamma_{i-1}$, the angles $\alpha_{i+k}$ and $\gamma_{i-1}$ are not composable. To see this, consider two cases. If $a_{i}$ and $a_{i+k}$ are oriented opposite, then $\nu\left(\alpha_{i+k-1}, \ldots, \alpha_{i}\right)=\kappa_{i-1}$ is even. Hence $\gamma_{i-1}$ enters $a_{i+k}$ at the opposite side of where $\alpha_{i+k}$ leaves. If $a_{i}$ and $a_{i+k}$ are oriented equally, then $\kappa_{i-1}$ is odd and $\gamma_{i-1}$ enters $a_{i+k}$ still at the opposite side of where $\alpha_{i+k}$ leaves. Either way, we have $\alpha_{i+k} \gamma_{i-1}=0$ and $\gamma_{i} \alpha_{i+1}=\kappa_{i}$ remains as desired.

Let us check that $d \varepsilon$ vanishes on polygon sequences in other polygons and on polygon sequences shorter than $k$ in the same polygon. Indeed,

$$
(d \varepsilon)\left(\beta_{l}, \ldots, \beta_{1}\right)=\varepsilon(\ldots, \mu(\ldots), \ldots)+\mu(\ldots, \varepsilon(\ldots), \ldots) .
$$

In the first summand, the inner $\mu$ can only give identities, on which $\varepsilon$ vanishes. In the second summand, the input sequence of the inner $\varepsilon$ must be from the same polygon as $\alpha_{1}, \ldots, \alpha_{N}$ and of length precisely $k-1$. Since the outer $\mu$ is $\mu^{\geq 2}$, this means that $\beta_{1}, \ldots, \beta_{l}$ has length at least $k$, which was not to be assumed.
Finally, let us gauge away the remaining scalar multiples of $\alpha_{i+1}$ in $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)$. For $i \in \mathbb{Z} / N \mathbb{Z}$, write $\nu\left(\alpha_{i+k}, \ldots, \alpha_{i+1}\right)=c_{i} \alpha_{i+1}$. If $c_{i} \neq 0$, we deduce that the target arc of $\alpha_{i+k}$ is equal to the target arc of $\alpha_{i+1}$, that is, $a_{i+k+1}=a_{i+2}$. Moreover, $\nu$ is even and hence $a_{i+k+1}$ and $a_{i+2}$ are oriented equally. In other words, $a_{i+k+1}$ and $a_{i+2}$ are equal as arcs and are oriented equally in the polygon. As we remarked in the odd case, this is only possible if $i+k+1=i+2$ in $\mathbb{Z} / N \mathbb{Z}$. We conclude that if some $c_{i}$ does not vanish, then $k=l N+1$ for some $l \geq 1$.
Assuming $k=l N+1$, let us show that the different $c_{i}$ along the polygon are related. Indeed,

$$
0=(d \nu)\left(\alpha_{(l+1) N}, \ldots, \alpha_{1}\right)=\sum_{i=1}^{N} c_{i} \mathbb{1}_{a_{1}}+\mu^{2}\left(\alpha_{(l+1) N}, \nu(\ldots)\right)+\mu^{2}\left(\nu(\ldots), \alpha_{1}\right)
$$

We have used that an inner $\nu$ can be placed in precisely $N$ ways, replicating the corresponding angle $\alpha_{i}$ and forming a disk together with the remaining angles. Moreover, an inner $\mu$ is impossible, since it only yields identities. An outer $\mu^{2}$ is possible, but gives non-empty angles only which can be distinguished from the identities. We deduce that the sum of all $c_{i}$ vanishes. Put

$$
\varepsilon\left(\alpha_{i+l N}, \ldots, \alpha_{i+1}\right):=-\left(\sum_{j=1}^{i-1} c_{j}\right) \mathbb{1}_{a_{i+1}}
$$

Then

$$
(d \varepsilon)\left(\alpha_{i+l N+1}, \ldots, \alpha_{i+1}\right)=-\sum_{j=1}^{i-1} c_{j} \alpha_{i+l N+1}+\sum_{j=1}^{i} c_{j} \alpha_{i+1}
$$

This definition makes sense for $i \in \mathbb{Z} / N \mathbb{Z}$, since the sum over $c_{i}$ vanishes. The sums cancel each other because $\alpha_{i+l N+1}=\alpha_{i+1}$, and $c_{i} \alpha_{i+1}$ remains as desired. It is standard to check that $d \varepsilon$ does not have values on shorter polygon sequences or other polygons.

Lemma 3.10. Let $\nu \in \operatorname{Ker}(d) \subset \underline{\operatorname{HC}}^{\bullet}\left(\mathrm{Gtl}_{\mathcal{A}}\right.$ with (A) $\nu^{0,1}=0$, (B) $\nu(\alpha, \beta)=0$ if $\alpha \beta \neq 0$ and suppose (C) $\nu$ vanishes on polygonal sequences. Then $\nu$ can be gauged to zero on all angle sequences.

Proof. We prove the statement by gauging $\nu$ to zero inductively on the total length of the angle sequences. After each step of gauging, we prove that all additional assumptions (A-C) still hold.

We already know $\nu^{1}=0$ and if $(\alpha, \beta)$ is a sequence of length two then either $\alpha \beta \neq 0$ of $(\alpha, \beta)$ is a polygon sequence, so by $(\mathrm{B})$ and $(\mathrm{C})$ we are done for total length at most 2 .

Now let $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ be any sequence of $k \geq 3$ angles $\alpha_{i}: a_{i} \rightarrow a_{i+1}$. By induction, we assume that $\nu$ already vanishes on all shorter sequences. Since $\nu$ already vanishes on polygon sequences, we can assume ( $\alpha_{k}, \ldots, \alpha_{1}$ ) is not a polygon sequence.

We will gauge $\nu$ on $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ simultaneously with all other sequences of the same type and therefore we will assume $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ is elementary. Now $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ consists of indecomposable angles of which at least one pair is composable (otherwise it would be a polygon sequence). Orientations of arcs will play no role in this proof, apart from determining the degrees of the angles. For simplicity, let us assume all $\alpha_{i}$ have odd degree. (This happens in case of a dimer.) Similarly, $\nu$ is either of odd or even parity, and the signs we write are for the odd case.

Let $\operatorname{Contr}(\beta)$ be the set of all contractible indices and $s=\max (\operatorname{Contr}(\beta))$. We will now gauge all $\beta^{S}$ for $S \subseteq \operatorname{Contr}(\beta)$ simultaneously. Put

$$
\varepsilon\left(\beta^{\{s\} \cup T}\right)=(-1)^{|T|} \nu\left(\beta^{T}\right), \quad T \subseteq \operatorname{Contr}(\beta) \backslash\{s\}
$$

We show that $(d \varepsilon)\left(\beta^{S}\right)=\nu\left(\beta^{S}\right)$ for all $S \subseteq \operatorname{Contr}(\beta)$.

- In case $s \notin S$, we have

$$
(d \varepsilon)\left(\beta^{S}\right)=\varepsilon\left(\beta^{\{s\} \cup S}\right)=(-1)^{|S|} \nu\left(\beta^{S}\right)
$$

by definition. Note that any $\varepsilon\left(\ldots, \mu^{\geq 3}, \ldots\right)$ and $\mu(\ldots, \varepsilon, \ldots)$ vanish since their input sequence of $\varepsilon$ is shorter than $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$.

- In case $s \in S$, first observe that for $K \subseteq \operatorname{Contr}(\beta)$ we have

$$
\sum_{t \notin K}(-1)^{1+\left|K_{<t}\right|} \nu\left(\beta^{K \cup\{t\}}\right)=(d \nu)\left(\beta^{K}\right) \mp \nu\left(\ldots, \mu^{\geq 3}, \ldots\right) \mp \mu(\ldots, \nu, \ldots)=0 .
$$

Here $\left|K_{<t}\right|$ denotes the number of indices in $K$ smaller than $t$. Indeed, the second and third summands vanish because the input sequence of $\nu$ is shorter than $\beta$. Now using this observation for $K=S \backslash\{s\}$ we get

$$
\begin{aligned}
(d \varepsilon)\left(\beta^{S}\right) & =-\sum_{t \notin S}(-1)^{1+\left|S_{<t}\right|} \varepsilon\left(\beta^{S \cup\{t\}}\right) \\
& =\sum_{t \notin S}(-1)^{|S \backslash\{s\} \cup\{t\}|+\left|S_{<t}\right|} \nu\left(\beta^{S \backslash\{s\} \cup\{t\}}\right) \\
& =\sum_{t \notin S \backslash\{s\}}(-1)^{|S|+1}(-1)^{\left|(S \backslash\{s\})_{<t}\right|+1} \nu\left(\beta^{S \backslash\{s\} \cup\{t\}}\right)-(-1)^{|S|+1}(-1)^{\left|S_{<s}\right|+1} \nu\left(\beta^{S \backslash\{s\} \cup\{s\}}\right) \\
& =0+\nu\left(\beta^{S}\right)
\end{aligned}
$$

as desired. In the first row, we have used that all $\varepsilon\left(\ldots, \mu^{\geq 3}, \ldots\right)$ and $\mu(\ldots, \varepsilon, \ldots)$ vanish since their input sequence of $\varepsilon$ is shorter than $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$.
Finally, $(d \varepsilon)^{0}=(d \varepsilon)^{1}=0$ and $d \varepsilon$ vanishes on polygon sequences $\gamma_{l}, \ldots, \gamma_{1}$. Indeed,

$$
\begin{equation*}
(d \varepsilon)\left(\gamma_{l}, \ldots, \gamma_{1}\right)=\varepsilon(\ldots, \mu, \ldots)+\mu(\ldots, \varepsilon, \ldots) \tag{3.3}
\end{equation*}
$$

The first summand vanishes because the inner $\mu$ yields only vertex idempotents. The second summand vanishes, since the input sequence of $\varepsilon$ consists only of indecomposable angles, while a sequence $\beta^{\{s\} \cup T}$ on which $\varepsilon$ is defined has at least one decomposable angle.

Next, note that $d \varepsilon$ vanishes if an input is a vertex idempotent. Now assume $\gamma_{l}, \ldots, \gamma_{1}$ is any sequence of non-empty angles of total length less than or equal to that of $\alpha_{k}, \ldots, \alpha_{1}$. Consider the sum 3.3) again. The first summand vanishes in case of $\mu^{\geq 3}$ because then the input sequence of $\varepsilon$ has less total length than $\alpha_{1}, \ldots, \alpha_{k}$. If the first summand does not vanish for some $\mu^{2}$, we conclude that the indecomposable constituents of the sequence are equal to $\alpha_{1}, \ldots, \alpha_{k}$. In other words, it is one of those sequences we have just gauged correctly already. The second summand vanishes, since the input sequence of $\varepsilon$ is shorter than $\alpha_{k}, \ldots, \alpha_{1}$.

Theorem 3.11. Let $\nu \in \operatorname{Ker}(d)$ with $\nu^{0,1}=0$ then $\nu$ is zero in $H^{\bullet}\left(\operatorname{Gtl}_{\mathcal{A}}\right)$.
Proof. This is an immediate consequence of the previous lemmas.
This theorem implies that the cohomology class of the cocycles can be read off from its zeroth and first components.

Lemma 3.12. If $\nu$ is a cocycle then

- $\nu^{0}$ is a central element,
- if $\nu^{0}=0$ then $\nu^{1}$ is a derivation.

Proof. We have that $\nu^{0}(1)$ is a central element because

$$
0=(d \nu)^{1}(\alpha)=\mu^{2}\left(\alpha, \nu^{0}(1)\right)+(-1)^{\|\alpha\|} \mu^{2}\left(\nu^{0}(1), \alpha\right)
$$

If $\nu^{0}=0$ then $\nu^{1}$ is a derivation because

$$
\begin{aligned}
& 0=(-1)^{|\beta|}(d \nu)(\alpha, \beta)=\nu^{1}(\alpha) \beta+\alpha \nu^{1}(\beta)-\nu(\alpha \beta) \\
& 0=\nu^{1}\left(\mathbb{1}_{a}\right) \mathbb{1}_{a}+\mathbb{1}_{a} \nu^{1}\left(\mathbb{1}_{a}\right)-\nu\left(\mathbb{1}_{a}\right)
\end{aligned}
$$

Depending on the parity of the cocycle we can be even more precise: odd cocycles are determined by $\nu^{0}$, which is a central element, while even cocycles are determined by $\nu^{1}$, viewed as an outer derivation of $\mathrm{Gtl}_{\mathcal{A}}$.

Theorem 3.13. Let $\mathcal{A}$ be an arc collection.

1. The map

$$
\operatorname{HH}^{\text {even }}\left(\operatorname{Gtl}_{\mathcal{A}}\right) \rightarrow \operatorname{OutDer}\left(\mathrm{Gtl}_{\mathcal{A}}\right): \nu \mapsto \nu^{1}
$$

is a well-defined embedding.
2. The map

$$
\operatorname{HH}^{\mathrm{odd}}\left(\mathrm{Gtl}_{\mathcal{A}}\right) \rightarrow Z\left(\mathrm{Gtl}_{\mathcal{A}}\right): \nu \mapsto \nu^{0}(1)
$$

is a well-defined embedding.
Proof. For the first statement note that the center only has cycles of even degree, which represent odd maps $A[1]^{\otimes 0} \rightarrow A[1]$. Therefore $\nu^{0}=0$ if the parity of $\nu$ is even and hence by the previous lemma $\nu^{1}$ is indeed a derivation. The map is also well defined because

$$
(d \kappa)^{1}(\alpha)=\mu^{1}\left(\kappa^{1}(\alpha)\right)+\mu^{2}\left(\kappa^{0}, \alpha\right)-\mu^{2}\left(\alpha, \kappa^{0}\right)=\left[\kappa^{0}, \alpha\right]
$$

Furthermore if $\nu^{1}$ is inner and $\nu^{0}=0$ then we can find a $\kappa$ such that $(\nu-d \kappa)^{1}=0$ zero. Because $(\nu-d \kappa)^{0}$ is zero for degree reasons we have by theorem 3.11 that $\nu=0$ in $H^{\text {even }}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$. Therefore the map is an embedding.

For the second statement, first note that this map is well-defined because $(d \kappa)^{0}=\mu^{1} \circ \kappa^{0}=0$. Now suppose that $\nu^{0}(1)=0$, then we will construct an even $\varepsilon$ such that $(d \varepsilon)^{1}(\alpha)=\nu^{1}(\alpha)$ for all indecomposable angles. Because $\nu^{0}(1)=(d \varepsilon)^{0}=0$ both $(d \varepsilon)^{1}$ and $\nu^{1}$ are derivations and therefore $(d \varepsilon)^{1}=\nu^{1}$. This means that $(\nu-d \epsilon)^{\leq 1}=0$ and hence $\nu=0$ in $H^{\text {odd }}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$. Therefore the map is an embedding.

To construct $\epsilon$, let us first make sure that $\nu^{1}(\alpha)$ does not include the identity. Indeed let $\alpha: a \rightarrow a$ be an indecomposable angle. Since $a$ is not contractible by assumption, $\alpha$ is not the only angle in the polygon. In particular, $\alpha$ does not go from head to tail of $a$ or from tail to head of $a$. We conclude $\alpha$ is even and $\nu^{1}(\alpha)$ is odd, hence does not include an identity.

Regard the polygon that $\alpha$ sits in. By abuse of wording, let us say "head" and "tail" of a polygon's arc $a$ to mean the vertex at the clockwise end and at the counterclockwise end of $a$. Independent of arrow directions, $\nu^{1}(\alpha)$ can be decomposed into a part running from the tail of $a$ to the tail of $b$ and a part running from the head of $a$ to the head of $b$. Write this decomposition as $\nu^{1}(\alpha)=\alpha \delta^{(\alpha)}+\gamma^{(\alpha)} \alpha$. Whether $\alpha$ is even or odd, $\delta^{(\alpha)}$ and $\gamma^{(\alpha)}$ are always odd since $\nu^{1}$ is.

Now let $a$ be an arc. Denote by $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}$ the angles incident at $a$ as in Figure 3.1a. We have

$$
\begin{aligned}
0 & =(d \nu)\left(\alpha_{2}, \alpha_{4}\right)=(-1)^{\left\|\alpha_{4}\right\|} \mu^{2}\left(\nu^{1}\left(\alpha_{2}\right), \alpha_{4}\right)+\mu^{2}\left(\alpha_{2}, \nu^{1}\left(\alpha_{4}\right)\right)+\nu^{1}\left(\mu^{2}\left(\alpha_{2}, \alpha_{4}\right)\right) \\
& =-\gamma^{\left(\alpha_{2}\right)} \alpha_{2} \alpha_{4}-\alpha_{2} \delta^{\left(\alpha_{2}\right)} \alpha_{4}+(-1)^{\left\|\alpha_{4}\right\|} \alpha_{2} \gamma^{\left(\alpha_{4}\right)} \alpha_{4}+(-1)^{\left\|\alpha_{4}\right\|} \alpha_{2} \alpha_{4} \delta^{\left(\alpha_{4}\right)}+0 \\
& =\alpha_{2}\left((-1)^{\left\|\alpha_{4}\right\|} \gamma^{\left(\alpha_{4}\right)}-\delta^{\left(\alpha_{2}\right)}\right) \alpha_{4} .
\end{aligned}
$$

We conclude $\delta^{\left(\alpha_{2}\right)}=(-1)^{\left\|\alpha_{4}\right\|} \gamma^{\left(\alpha_{4}\right)}$. Note that we have used that $\alpha_{2} \alpha_{4}=0$ and that $\gamma^{\left(\alpha_{4}\right)}$ and $\delta^{\left(\alpha_{2}\right)}$ both run from the tail of $a$ to the head of $a$, and that $\alpha_{4}$ ends at the tail of $a$ and $\alpha_{2}$ starts at the head of $a$. Similarly,

$$
\begin{aligned}
0 & =(d \nu)\left(\alpha_{3}, \alpha_{1}\right)=(-1)^{\left\|\alpha_{1}\right\|} \mu^{2}\left(\nu^{1}\left(\alpha_{3}\right), \alpha_{1}\right)+\mu^{2}\left(\alpha_{3}, \nu^{1}\left(\alpha_{1}\right)\right)+\nu^{1}\left(\mu^{2}\left(\alpha_{3}, \alpha_{1}\right)\right) \\
& =-\alpha_{3} \delta^{\left(\alpha_{3}\right)} \alpha_{1}-\gamma^{\left(\alpha_{3}\right)} \alpha_{3} \alpha_{1}+(-1)^{\left\|\alpha_{1}\right\|} \alpha_{3} \alpha_{1} \delta^{\left(\alpha_{1}\right)}+(-1)^{\left\|\alpha_{1}\right\|} \alpha_{3} \gamma^{\left(\alpha_{1}\right)} \alpha_{1}+0 \\
& =\alpha_{3}\left(-\delta^{\left(\alpha_{3}\right)}+(-1)^{\left\|\alpha_{1}\right\|} \gamma^{\left(\alpha_{1}\right)}\right) \alpha_{1} .
\end{aligned}
$$

We conclude $\delta^{\left(\alpha_{3}\right)}=(-1)^{\left\|\alpha_{1}\right\|} \gamma^{\left(\alpha_{1}\right)}$. Let us now put

$$
\varepsilon_{a}^{0}:=(-1)^{\left\|\alpha_{4}\right\|+1} \gamma^{\left(\alpha_{4}\right)}-\delta^{\left(\alpha_{3}\right)}=(-1)^{\left\|\alpha_{1}\right\|+1} \gamma^{\left(\alpha_{1}\right)}-\delta^{\left(\alpha_{2}\right)}
$$

The expression can be read independent on the arrow direction of $a$ : It stays invariant under rotating the labels $\alpha_{1}, \alpha_{2}, \alpha_{3}$ and $\alpha_{4}$ by 180 degrees. In other words, $\varepsilon_{a}^{0}$ can be seen either as the (signed) difference of $\gamma$ and $\delta$ of its incident angles at its head or at its tail. This makes it easy to check $(d \varepsilon)(\alpha)=\nu^{1}(\alpha)$ for

(a) Surrounding angles

(b) Surrounding angles

Figure 3.1
any indecomposable angle $\alpha: a \rightarrow b$. Denote by $\beta_{1}$ and $\beta_{2}$ its predecessor and successor indecomposable angles around the same puncture as in Figure 3.1b. Then

$$
\begin{aligned}
(d \varepsilon)(\alpha) & =(-1)^{|\alpha|} \varepsilon_{b}^{0} \alpha-\alpha \varepsilon_{a}^{0} \\
& =(-1)^{|\alpha|}\left((-1)^{\|\alpha\|+1} \gamma^{(\alpha)}-\delta^{\left(\beta_{2}\right)}\right) \alpha-\alpha\left((-1)^{\left\|\beta_{1}\right\|+1} \gamma^{\left(\beta_{1}\right)}-\delta^{(\alpha)}\right) \\
& =\gamma^{(\alpha)} \alpha+(-1)^{\|\alpha\|} \delta^{\left(\beta_{2}\right)} \alpha+(-1)^{\left\|\beta_{1}\right\|} \alpha \gamma^{\left(\beta_{1}\right)}+\alpha \delta^{(\alpha)} \\
& =\gamma^{(\alpha)} \alpha+\alpha \delta^{(\alpha)}=\nu^{1}(\alpha) .
\end{aligned}
$$

In the penultimate equality, we have used that $\delta^{\left(\beta_{2}\right)}$ ends where $\alpha$ starts and is odd, therefore not composable with $\alpha$. Similarly $\alpha \gamma^{\left(\beta_{1}\right)}=0$. We conclude that $(d \varepsilon)^{1}=\nu^{1}$ on indecomposable angles.

Remark 3.14. If $\nu$ is an even cocycle then $\nu^{0}$ is zero, but if $\nu$ has odd parity then $\nu^{1}$ need not to be zero or not even a derivation. An example of this occurs when $m$ is a marked point surrounded by a loop arc. This loop gives rise to an orbigon of length 1 , so the corresponding product $A_{\infty}$-deformation has a nontrivial ${ }^{r} \mu^{1}$ if $r_{(m, 1)} \neq 0$ and therefore $\nu_{(m, 1), o}$, as defined in definition 3.15 will be an odd cocycle with nontrivial $\nu^{1}$.

### 3.3 A set of generators

In this section we will describe a special set of Hochschild classes that span the Hochschild cohomology.
The first class we need is the unit class $\nu_{\mathbb{1}}$, which corresponds to a single nullary product: $\nu_{\mathbb{1}}^{0}: \mathbb{k} \rightarrow$ $A[1]: \mathbb{1}_{a} \mapsto \mathbb{1}_{a}$. Clearly this is a Hochschild cycle because the $\mathbb{1}_{a}$ are strict idempotents. Its parity is odd.

Definition 3.15. For each orbifold point $p=(m, r) \in M \times \mathbb{N}$ we define its orbifold point homology class as

$$
\nu_{p, o}=\frac{1}{\hbar}\left({ }^{r} \mu-\mu\right) \text { with } r=\ell_{m}^{r} \hbar \in Z(A) \otimes \mathbb{C}[\hbar] /\left(\hbar^{2}\right)
$$

Lemma 3.16. $\nu_{p, o}$ is a cocycle of odd parity.
Proof. These are clearly cocycles because ${ }^{r} \mu$ satisfies the Maurer-Cartan equation, which reduces to $d\left({ }^{r} \mu\right)=0$ in the first order.

A second set of cocycles are characterized as follows. Fix a map $\lambda:\left(Q_{\mathcal{A}}\right)_{1} \rightarrow \mathbb{C}$. Define the first component of $\nu_{\lambda}$ to be the derivation such that $\nu_{\lambda}(\alpha)=\lambda_{\alpha} \alpha$ and set $\nu_{\lambda}^{\neq 1}=0$. The derivation condition implies that for a path $\beta=\beta_{k} \ldots \beta_{1}$ we have $\nu_{\lambda}(\beta)=\sum_{i} \lambda_{\beta_{i}} \beta$, so it makes sense to define $\lambda_{\beta}:=\sum_{i} \lambda_{\beta_{i}}$.
Lemma 3.17. $\nu_{\lambda}$ is a cocycle if and only if for every polygon $\left(\alpha_{N}, \ldots, \alpha_{1}\right)$ we have $\sum_{i} \lambda_{\alpha_{i}}=0$.
Proof. The condition that $\sum_{i} \lambda_{\alpha_{i}}=0$ for every polygon implies that $\sum_{\alpha_{i}} \lambda_{\alpha_{i}}=0$ for every tree-gon $\left(\alpha_{k}, \ldots, \alpha_{1}\right)$. Therefore

$$
\left(d \nu_{\lambda}\right)\left(\alpha_{1}, \ldots, \alpha_{k} \beta\right)=\left(\sum \lambda_{\alpha_{i}}+\lambda_{\beta}\right) \mu\left(\alpha_{k}, \ldots, \alpha_{1} \beta\right)-\nu_{\lambda}(\beta)=0
$$

On the other hand if $\nu^{\neq 1}=0$ then we know from lemma 3.12 that $\nu$ must be a derivation, while for every polygon $d\left(\nu_{\lambda}\right)\left(\alpha_{N}, \ldots, \alpha_{1}\right)=0$ implies that $\sum_{i} \lambda_{\alpha_{i}}=0$.

The set of all $\nu_{\lambda}$ with these properties form a vector space $S$. If two elements in $S$ differ by a commutator with an element in $\mathbb{k}$ they will represent the same homology class.

Lemma 3.18. $\operatorname{dim} S /[\mathbb{k},-]=2 g-1+\left|Q_{0}\right|$.
Proof. The dimension of $S$ is $2 \# \mathcal{A}-\#\{$ faces $\}$ because there are 2 angles arriving in each arc and every face gives one linear condition. All these conditions are linearly independent because every angle occurs only in one face. The kernel of the map $\mathbb{k} \rightarrow[\mathbb{k},-]$ is $\mathbb{C}$, so the image of $[\mathbb{k},-]$ in $S$ is of dimension $\# \mathcal{A}-1$ and therefore

$$
\operatorname{dim} S /[\mathbb{k},-]=2 \# \mathcal{A}-\#\{\text { faces }\}-\# \mathcal{A}+1=2 g-2+\left|Q_{0}\right|+1
$$

It is easy to find a basis for $S /[\mathfrak{k},-]$. Look again at Figure 3.1a For each arc $a$ there are 4 angles: two $\alpha_{2}, \alpha_{3}$ leaving $a$ and two $\alpha_{1}, \alpha_{2}$ arriving in $a$. The angles $\alpha_{2}$ and $\alpha_{4}$ sit in the face on the right of $a$ and $\alpha_{1}, \alpha_{3}$ in the face on the left.

We define $\nu_{a, L}$ to be the derivation with

$$
\lambda\left(\alpha_{1}\right)=1, \lambda\left(\alpha_{3}\right)=-1
$$

while all other entries of $\lambda$ are zero. Similarly we define $\nu_{a, R}$ with

$$
\lambda\left(\alpha_{2}\right)=1, \lambda\left(\alpha_{4}\right)=-1
$$

Note that because $\left[\mathbb{1}_{a},-\right]=\nu_{a, L}-\nu_{a, R}$ we have that up to homology $\nu_{a, L}=\nu_{a, R}$, so we will drop the subscript $R, L$ and set $\nu_{a}:=\nu_{a, L}$.
Lemma 3.19. If $\mathcal{T}$ is a set of arcs that forms a spanning tree in the face graph then

$$
\left\{\nu_{a} \mid a \in \mathcal{A} \backslash \mathcal{T}\right\}
$$

is a basis for $S /[\mathbb{k},-]$.
Proof. The sum of the $\nu_{a}$ where $a$ runs over the arcs going around a given face is zero. Because $\mathcal{T}$ is a spanning tree, the $\left\{\nu_{a} \mid a \in \mathcal{A} \backslash T\right\}$ are independent. Moreover because a spanning tree has $\#\{$ faces $\}-1$ arcs, the cardinality of this set is the dimension of $S /[\mathfrak{k},-]$.

Remark 3.20. We will call the $\nu_{a}$ and more general the $\nu_{\lambda}$, which are linear combinations of the $\nu_{a}$, arc classes.

Let $p=(m, j)$ be an orbifold point and $a$ be any arc arriving (or leaving) the marked point. We now define

$$
\nu_{p, e}:=\nu_{p, o} \smile \nu_{a} \text { or }-\nu_{a} \text { if } a \text { leaves } m .
$$

Note that if $a$ is not a loop we have that $\nu_{p, e}$ is zero for every angle arrow except for $\alpha_{1}$ (or $\alpha_{3}$ in the case $a$ leaves $m$ ).

Lemma 3.21. If the arc collection $\mathcal{A}$ satisfies [NL2] then the homology class of $\nu_{p, e}$ does not depend on the choice of $a$.

Proof. Let $a$ and $b$ be neighboring arcs incident at the same puncture, such that $b$ comes after $a$ in clockwise order. Let $\varepsilon=\varepsilon^{0}$ be the odd cochain given by $\varepsilon^{0}=\ell_{m}^{j} \mathbb{1}_{a}$. For any angle $\alpha$ winding around puncture $\mathrm{m}(\alpha)$ we have

$$
d(\varepsilon)(\alpha)= \begin{cases}+\ell_{m}^{j} \alpha & t(\alpha)=a \text { and } \mathrm{m}(\alpha)=m \\ -\ell_{m}^{j} \alpha & h(\alpha)=a \text { and } \mathrm{m}(\alpha)=m \\ 0 & \text { otherwise }\end{cases}
$$

Because $a$ is not a loop, the two nonzero cases each happen for just one $\alpha_{1}$ and $\alpha_{3}$ angle. We conclude

$$
[d(\varepsilon)]^{1}=\left[\nu_{p, e} \smile \nu_{a}-\nu_{p, e} \smile \nu_{b}\right]^{1} .
$$

In other words the difference between $\nu_{p, e} \smile \nu_{a}$ and $\nu_{p, e} \smile \nu_{b}$ is homotopic to a cocycle $\kappa$ with $\kappa^{0}, \kappa^{1}=0$. In combination with theorem 3.11 this means that two consecutive arcs around $m$ define the same $\nu_{p, e^{-}}$ class, and hence all arcs around $m$ do.
Theorem 3.22. Let $\mathcal{A}$ be an arc collection and $\mathcal{T}$ a spanning tree in the face graph.

1. The cocycles $\nu_{\mathbb{1}}, \nu_{p, o}$ where $p$ runs over all orbifold points form a basis for $H^{\text {odd }}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$.
2. If $\mathcal{A}$ satisfies [NL2] then the cocycles $\nu_{p, e}, \nu_{a}$ where $p$ runs over all orbifold points and $a \in \mathcal{A} \backslash \mathcal{T}$ form a basis for $H^{\text {even }}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$.
Proof. The zeroth components of $\nu_{\mathbb{1}}, \nu_{p, o}$ form a basis for $Z\left(\mathrm{Gtl}_{\mathcal{A}}\right)$, so the embedding $\operatorname{HH}^{\text {odd }}\left(\mathrm{Gtl}_{\mathcal{A}}\right) \rightarrow$ $Z\left(\mathrm{Gtl}_{\mathcal{A}}\right)$ in lemma 3.13 is a bijection.

To show the second part, note that the fact that there are no loops or two-cycles implies that all nonzero angle paths with the same head and tail as $\alpha$ turn around the same marked point as $\alpha$ and are of the form $\ell_{m}^{r} \alpha$.

If $\nu$ is $G$-homogeneous this means that there are two possibilities

- If the degree is 0 then $\nu$ must be a linear combination of the $\nu_{a}$.
- $\nu(\alpha)=\lambda_{\alpha} \ell_{m}^{j} \alpha$ with $j>0$ then the only other $\beta$ for which $\nu(\beta) \neq 0$ must turn around $m$. Up to an inner derivation this $\nu$ is determined by the sum $\sum \lambda_{\alpha}$ where $\alpha$ runs over the indecomposables turning around $m$.


### 3.4 Gerstenhaber structure product on Hochschild cohomology

In this section, we compute the Gerstenhaber algebra structure on $\mathrm{HH}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$. In other words, we determine the bracket and the cup product on Hochschild cohomology. The bracket agrees with the computations by Wong for the Borel-Moore cohomology of matrix factorizations for dimer models [71, which is conjecturally equivalent to $\mathrm{HH}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$.

To describe the bracket and cup product, we use the basis for odd and even Hochschild cohomology constructed earlier: the odd classes $\nu_{(m, j), o}$ for a puncture $m \in M$ and $j \geq 0$, the even classes $\nu_{(m, j), e}$, and the arc classes $\nu_{\lambda}$ with $\lambda:\left(Q_{\mathcal{A}}\right)_{1} \rightarrow \mathbb{C}$.

We start with the cup-product
Proposition 3.23. Let $m, n \in M$ be two marked points, let $i, j \geq 1$ be two indices, and $\nu_{\lambda}, \nu_{\kappa}$ two arc classes. Then the cup product in cohomology reads as follows:

$$
\begin{aligned}
\nu_{(m, i), o} \smile \nu_{(n, j), o} & =\delta_{m n} \nu_{(m, i+j), o}, \\
\nu_{(m, i), o} \smile \nu_{(n, j), e} & =\delta_{m n} \nu_{(m, i+j), e}, \\
\nu_{(m, i), o} \smile \nu_{\lambda} & =\lambda_{\ell_{m}} \nu_{(m, i), e}, \\
\nu_{\kappa} \smile \nu_{\lambda} & =0, \\
\nu_{(p, i), e} \smile \nu_{\lambda} & =0 .
\end{aligned}
$$

Proof. We compute the cup products of the given Hochschild cocycles first on chain level. Then we compute their projection to cohomology. In fact, for the odd products it suffices to compute the curvature component $(\nu \smile \eta)^{0}$ and for the even products it suffices to compute the first component $(\nu \smile \eta)^{1}$. Also recall that the parity of $\smile$ itself is odd, so the product of two classes with the same (different) parity is odd (even). We are now ready to start the calculations. For the first identity, regard

$$
\left(\nu_{(m, i), o} \smile \nu_{(n, j), o}\right)^{0}=\mu^{2}\left(\nu_{(m, i), o}^{0}(1), \nu_{(n, j), o}^{0}(1)\right)=\delta_{m n} \mu^{2}\left(\ell_{m}^{i}, \ell_{n}^{j}\right)=\ell_{m}^{i+j}
$$

This is precisely the curvature of the Hochschild cohomology class $\nu_{(m, i+j), o}$ and hence projects to it. For the second identity and third identity one can do a similar calculation but now one has to look a the first component. Alternatively, we can use the definition of $\nu_{(p, i), e}$ in combination with the associativity of $\smile$ on the Hochschild cohomology.

For the last two identities we again have to look at the zeroth component and these are trivially zero because both factors have trivial curvature.

Proposition 3.24. Let $m, n \in M$ be two marked points, let $i, j \geq 1$ be two indices, and $\nu_{\lambda}, \nu_{\kappa}$ two arc classes. Then the Gerstenhaber bracket in cohomology reads as follows:

$$
\begin{aligned}
{\left[\nu_{(m, i), o}, \nu_{(n, j), o}\right] } & =0, \\
{\left[\nu_{(m, i), e}, \nu_{(n, j), o}\right] } & =\delta_{m n} \cdot j \cdot \nu_{(m, i+j), o}, \\
{\left[\nu_{(m, i), e}, \nu_{(n, j), e}\right] } & =\delta_{m n} \cdot(j-i) \cdot \nu_{(m, i+j), e}, \\
{\left[\nu_{\lambda}, \nu_{(m, i), o}\right] } & =i \lambda_{\ell_{m}} \cdot \nu_{(m, i), o}, \\
{\left[\nu_{\lambda}, \nu_{(m, i), e}\right] } & =i \lambda_{\ell_{m}} \cdot \nu_{(m, i), e}, \\
{\left[\nu_{\kappa}, \nu_{\lambda}\right] } & =0 .
\end{aligned}
$$

Proof. The strategy for this calculation is the same as for the cup product. We calculate the zeroth or first component of the bracket.

$$
\begin{aligned}
{[\nu, \eta]^{0} } & =\nu^{1}\left(\eta^{0}\right)-(-1)^{\|\nu\|\|\eta\|} \eta^{1}\left(\nu^{0}\right) \\
{[\nu, \eta]^{1}(\alpha) } & =\nu^{1}\left(\eta^{1}(\alpha)\right)+(-1)^{\|\eta\|\| \| \alpha \|} \nu^{2}\left(\eta^{0}, \alpha\right)+\nu^{2}\left(\alpha, \eta^{0}\right) \\
& -(-1)^{\|\nu\|\|\eta\|}\left(\eta^{1}\left(\nu^{1}(\alpha)\right)+(-1)^{\|\nu\|\|\alpha\|} \eta^{2}\left(\nu^{0}, \alpha\right)+\eta^{2}\left(\alpha, \nu^{0}\right)\right) .
\end{aligned}
$$

The main difference is now that the bracket has even parity, so we need to check the zeroth component whenever the two entries have different parity. In that case the result is the outer derivation of the even class applied to the central element of the odd class. This takes care of the second and fourth identity.

The first component of the bracket of two even classes is the commutator of their outer derivations because they have no $\nu^{0}$. This takes care of the third and the last two identities.

Finally the bracket of two odd classes is zero because they have no $\nu^{1}, \nu^{2}$ if [NL2] holds. If [NL2] fails and $\alpha$ is an indecomposable angle then $\nu^{1}(\alpha)$ can only be nonzero if $a=h(\alpha)$ is a loop around a puncture but then the arc collection does not split the surface. Secondly for our deformed products a term of the form ${ }^{r} \mu^{2}\left(\ell_{p}, \alpha\right)$ is always cancelled by ${ }^{r} \mu^{2}\left(\alpha, \ell_{p}\right)$ and therefore the same holds for the odd classes. This implies that $[\nu, \eta]^{1}(\alpha)$ is zero for all indecomposable $\alpha$ and because $[\nu, \eta]^{1}$ is a derivation it is identically zero.

The Lie-bracket on $\mathrm{HH}^{\bullet}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$ is part of an $L_{\infty}$-structure. This is a graded vector space $L$ together with brackets [,..., $]^{k}: L^{\otimes k} \rightarrow L$ of degree $2-k$ which satisfy the $L_{\infty}$-axioms 67 .

If $K^{\bullet}, d,[]$ is a DGLA then we can construct an $L_{\infty}$-structure [,] on the homology such that $H K^{\bullet}, 0,[,]^{\bullet}$ is quasi-isomorphic to $C^{\bullet}, d,[]$.

These products can be constructed using the homotopy transfer lemma (see e.g. [54). The construction can be summarized as follows:

- Split $K^{\bullet}$ as a direct sum of graded vector spaces

$$
K^{\bullet}=H \oplus I \oplus R
$$

such that $\operatorname{Im} d=I$ and $\operatorname{Ker} d=H \oplus I$. In this way $H \cong H K^{\bullet}$ and $d$ restricts to an isomorphism $d_{I R}: R \rightarrow I$.

- Let $h$ be the map

$$
h: H \oplus I \oplus R \rightarrow H \oplus I \oplus R:(u, v, w) \mapsto\left(0,0, d_{R \rightarrow I}^{-1}(v)\right) .
$$

In this way the projection onto $H$ becomes $\pi=\mathbb{1}-d h-h d$.

- for $x_{1}, \ldots, x_{k} \in H$ we define

$$
\left[x_{1}, \ldots, x_{k}\right]^{k}=\sum_{t} c_{t} \pi\left[\ldots, h\left[x_{i}, x_{j}\right], \ldots,\right]
$$

as a linear combination of all possible ways of fully bracketed expressions where each internal bracket is composed with an $h$. (The precise coefficients $c_{t}$ will not matter in our discussion.)
In our case we will take $K=\operatorname{HC}\left(\mathrm{Gtl}_{\mathcal{A}}\right), H \cong \operatorname{HH}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$ to be the span of the classes we constructed, and we take $I$ and $R$ to be compatible with the $G$-grading.
Theorem 3.25. If $\mathcal{A}$ satisfies [NL2] then we can choose a split such that all higher brackets are zero.
Proof. First note that $\left[\nu_{\mathbb{1}},-\right]=0$ in $\underline{\mathrm{HC}}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$, so $\left[\nu_{\mathbb{1}},-, \ldots,-\right]^{k}$ will also be zero.
To show that the other brackets are zero, pick any $\mathbb{Z}$-grading of the gentle algebra. Then we can assume that that the bracket $[,]^{k}$ has degree $2-k$ for this grading and degree 0 for the $G=\mathrm{H}(S \backslash M, \mathcal{A}, \mathbb{Z})$ grading. As we assumed that $\mathcal{A}$ has no loops, there are at least two punctures so the $G$-degree of $\ell_{m}$ is nontrivial for each marked point.

If all the entries of the bracket have $G$-degree 0 (which means that they are all arc classes, which have $\mathbb{Z}$-degree 0 ), then the homotopy transfer lemma tells us that the product must be zero because all pairs [ $\left.\nu_{\lambda}, \nu_{\kappa}\right]$ are already zero in $\mathrm{HC}\left(\mathrm{Gtl}_{\mathcal{A}}\right)$.

Suppose that there is at least one $\nu_{p, e}$ in one entry. We make the following distinctions:

- All the entries have $G$-degree a multiple of $\operatorname{deg}_{G} \ell_{m}$. In that case the $G$-degree of the result will be $r \operatorname{deg}_{G} \ell_{m}$ for some $r \in \mathbb{N}$ and therefore the $\mathbb{Z}$-degree must be either $r \operatorname{deg}_{\mathbb{Z}} \ell_{m}$ if the result is even or $r \operatorname{deg}_{\mathbb{Z}} \ell_{m}-1$ if the result is odd. The total $\mathbb{Z}$-degree of the entries is at most $r \operatorname{deg}_{\mathbb{Z}} \ell_{m}$ and this only happens if all entries are all even. Because the product has degree $2-k$ it can only be nonzero for degree reasons if $2-k \geq-1$, or in other words if $k=2,3$.
- Some entries have degrees that turn around different punctures. If the result has $G$-degree $\operatorname{deg}_{G} \ell_{m}^{j}$ then the $G$-degree of the entries must also be $\operatorname{deg}_{G} \ell_{m}^{j}$ but if there are degrees for more than one marked point all marked points must appear because the only relation between the degrees of the $\ell_{m}$ is that $\sum_{m \in M} \operatorname{deg}_{G} \ell_{m}=0$. On the other hand by lemma 3.6 we have that $\sum_{m} \operatorname{deg}_{\mathbb{Z}} \ell_{m} \leq-1$, so the $\mathbb{Z}$-degree of the result must be $\leq \operatorname{deg}_{\mathbb{Z}} \ell_{m}^{j}+2-k-1$. This is impossible if $k \geq 3$.

From the discussions above we see that because of degree reasons the only nontrivial higher brackets are of the form

$$
\left[\nu_{\left(m, i_{1}\right), e}, \nu_{\left(m, i_{2}\right), e}, \nu_{\left(m, i_{3}\right), e}\right]=\lambda \nu_{\left(m, i_{1}+i_{2}+i_{3}\right), o}
$$

or cases where one or more $\nu_{\left(m, i_{j}\right), e}$ are $\nu_{\lambda} \mathrm{S}$. To show that these products are all zero, we have to look at the homotopy transfer construction in detail. The triple product can be written down in terms of expressions of the form

$$
\pi\left[h\left[\nu_{\left(m, i_{1}\right), e}, \nu_{\left(m, i_{2}\right), e}\right], \nu_{\left(m, i_{3}\right), e}\right]
$$

where $\pi$ is the projection onto homology. We will now argue why such terms are zero.
From our computation in 3.24 we know that

$$
\begin{aligned}
& {\left[\nu_{\left(m, i_{1}\right), e}, \nu_{\left(m, i_{2}\right), e}\right]^{0}=0} \\
& {\left[\nu_{\left(m, i_{1}\right), e}, \nu_{\left(m, i_{2}\right), e}\right]^{1}=\left(\left(i_{2}-i_{1}\right) \nu_{\left(m, i_{1}+i_{1}\right), e}\right)^{1}}
\end{aligned}
$$

So $\left[\nu_{\left(m, i_{1}\right), e}, \nu_{\left(m, i_{2}\right), e}\right]=\left(i_{2}-i_{1}\right) \nu_{\left(m, i_{1}+i_{2}\right), e}+d \kappa$ with $(d \kappa) \leq 1=0$. From the construction in section 3.2. we can choose $\kappa$ such that $\kappa^{0}=0$. For what follows, choose $R^{\text {odd }}=R_{1} \oplus R_{2}$ (in a $G$-graded way) such that $R_{1}^{\neq 0}=R_{2}^{0}=0$ and $R_{1} \cap Z\left(\mathrm{Gtl}_{\mathcal{A}}\right)=0$. With respect to the direct sum $\mathrm{HC}^{\text {odd }}=H^{\text {odd }} \oplus I^{\text {odd }} \oplus R_{1} \oplus R_{2}$ write $\kappa=h+i+r_{1}+r_{2}$. Then $0=\kappa^{0}=h^{0}+r_{1}^{0}$, hence $r_{1}=r_{1}^{0}=-h^{0} \in R_{1} \cap Z\left(\mathrm{Gtl}_{\mathcal{A}}\right)=0$ and we conclude $\kappa=i+r_{2}$. Simply subtracting $i$ from $\kappa$ keeps $\kappa^{0}=0$ and brings $\kappa$ into $R_{2} \subseteq R$. Finally

$$
\left[h\left[\nu_{\left(m, i_{1}\right), e}, \nu_{\left(m, i_{2}\right), e}\right], \nu_{\left(m, i_{3}\right), e}\right]^{0}=\left[\kappa, \nu_{\left(m, i_{3}\right), e}\right]^{0}=\nu_{\left(m, i_{3}\right), e}\left(\kappa^{0}\right) \pm \kappa^{1}\left(\nu_{\left(m, i_{3}\right), e}^{0}\right)=0
$$

Therefore all contributions are zero.
Corollary 3.26. If $\mathcal{A}$ has no loops or two-cycles then $\operatorname{HC}\left(\mathrm{Gtl}_{\mathcal{A}}\right), d,[$,$] is formal. In other words there is$ an $L_{\infty}$-quasi-isomorphism between $\operatorname{HC}\left(\mathrm{Gtl}_{\mathcal{A}}\right), d,[$,$] and \operatorname{HH}\left(\mathrm{Gtl}_{\mathcal{A}}\right), 0,[$,$] .$

### 3.5 Classifying curved deformations

As an application of our computations we will show that every curved deformation of $\mathrm{Gtl}_{\mathcal{A}}$ is equivalent to one of the ${ }^{r} \mu$. Remember that if $(A, \mu)$ is an $A_{\infty}$-algebra over $\mathbb{k}$ and $R$ is a complete local Noetherian unital commutative $\mathbb{C}$-algebra with maximal ideal $R^{+}$and residue field $R / R^{+}=\mathbb{C}$ then a curved deformation is odd an element $\nu \in \operatorname{Hom}\left(\bigoplus_{i} A^{\otimes_{\mathbf{k}}^{i}}, A\right) \widehat{\otimes} R^{+}$such that $\mu+\nu$ satisfies the curved $A_{\infty}$-axioms. As we indicated before this equation is equivalent to the Maurer-Cartan equation for the Hochschild cochain complex together with the Gerstenhaber bracket 44 .

Theorem 3.27. If [NL2] holds, then every curved $A_{\infty}$-deformation of $\mathrm{Gtl}_{\mathcal{A}}$ over $R$ is equivalent to one of the ${ }^{r} \mu$.
Proof. We give a proof in terms of deformation functors, in the language of 52 . In short, we interpret our explicit class of deformations ${ }^{r} \mu$ as a functor of Artin rings. Gauging by even elements acts on the values of this functor and lands in Maurer-Cartan elements. The first part of the proof deals with the case of $R$ being Artinian. In the second part of the proof, we pass to the non-Artinian case.

Let Art denote the category of Artinian local Noetherian unital commutative rings over $\mathbb{C}$ with residue field $\mathbb{C}$, with morphisms being local $\left(\varphi\left(R^{+}\right) \subseteq S^{+}\right)$and unital $\left(\varphi\left(1_{R}\right)=1_{S}\right)$. We build three functors $G, F, \mathrm{MC}:$ Art $\rightarrow$ Set. The functor $G$ is the gauge group functor, $F$ is the functor of our deformation parameters $r$, and MC is the standard Maurer-Cartan functor. More precisely, define

$$
\begin{aligned}
G(R) & :=\exp \left(\mathrm{HC}^{\text {even }}\left(\mathrm{Gtl}_{\mathcal{A}}\right) \widehat{\otimes} R^{+}\right), \\
F(R) & :=Z\left(\mathrm{Gtl}_{\mathcal{A}}\right) \widehat{\otimes} R^{+} \\
\operatorname{MC}(R) & :=\operatorname{MC}\left(\mathrm{HC}\left(\mathrm{Gtl}_{\mathcal{A}}\right), R\right) .
\end{aligned}
$$

All three assignments come with natural restriction maps $G(R) \rightarrow G(S), F(R) \rightarrow F(S)$ and MC(R) $\rightarrow$ $\mathrm{MC}(S)$ for every morphism $\varphi: R \rightarrow S$ in Art. It is standard to check that all three are deformation functors in the sense of 52, Definition 2.5]. In fact, $G$ and $F$ as well as their product functor $G \times F$ are
smooth (unobstructed) in the sense of 52, Definition 2.8]. Regard now the morphism of functors given by

$$
\Phi(R): G(R) \times F(R) \rightarrow \mathrm{MC}(R),(g, r) \mapsto g .{ }^{r} \mu
$$

Let us define obstruction theories $O_{G \times F}$ and $O_{\mathrm{MC}}$ for $G \times F$ and MC by defining both $O_{G \times F}:=0$ and $O_{\mathrm{MC}}:=0$ to be trivial. We will now show that $\Phi$ is smooth by applying [52, Proposition 2.17]. In the terminology of that paper, we have to check three items: (1) $\Phi(\mathbb{C}[\varepsilon])$ is surjective where $\mathbb{C}[\varepsilon]=\mathbb{C}[X] /\left(X^{2}\right)$ is the ring of dual numbers, (2) the obstruction theory $O_{G \times F}=0$ is complete, (3) the morphism between obstruction theories $O_{G \times F} \rightarrow O_{\mathrm{MC}}$ is injective and compatible with $\Phi$.

We check the three conditions. (1) Let $\varepsilon c \in \operatorname{MC}(\mathbb{C}[\varepsilon])$. Then $c$ is a Hochschild cocycle and can be gauged by some $g \in G(\mathbb{C}[\varepsilon])$ to be equal to an ${ }^{r} \mu$ for some $r \in \varepsilon Z\left(\mathrm{Gtl}_{\mathcal{A}}\right)$. In other words, we have $\Phi(\mathbb{C}[\varepsilon])(g, r)=\varepsilon c$. This proves $\Phi(\mathbb{C}[\varepsilon])$ surjective. Item (2) and item (3) are trivial because $G \times F$ is smooth.

The standard smoothness criterion [52, Proposition 2.17] implies that $\Phi$ is smooth. Putting $S=\mathbb{C}$ in the definition of smoothness implies that $\Phi(R): G(R) \times F(R) \rightarrow \mathrm{MC}(R)$ is surjective for every $R \in$ Art. In other words, every deformation of $\mathrm{Gtl}_{\mathcal{A}}$ over $R \in$ Art is equivalent to an ${ }^{r}{ }_{\mu}$ by gauge equivalence.

In the second part of the proof, we generalize to the non-Artinian case. Let $R$ be a complete local Noetherian unital commutative $\mathbb{C}$-algebra with residue field $\mathbb{C}$. Let $\mu \in \operatorname{MC}\left(\mathrm{HC}\left(\mathrm{Gtl}_{\mathcal{A}}\right), R\right)$ be a deformation over $R$, meaning a Maurer-Cartan element in the completed tensor product $\mu \in \operatorname{HC}\left(\mathrm{Gtl}_{\mathcal{A}}\right) \widehat{\otimes} R^{+}$. Our aim is to show that $\mu$ is gauge equivalent to some ${ }^{r} \mu$ with $r \in Z\left(\mathrm{Gtl}_{\mathcal{A}}\right) \widehat{\otimes} R^{+}$.

Our strategy is to truncate $\mu$ to $R /\left(R^{+}\right)^{i}$ for every $i$ and use the first part of the proof to construct an element $r_{i}$ and a gauge $g_{i}$. We use smoothness of $\Phi$ to force both sequences $\left(r_{i}\right)$ and $\left(g_{i}\right)$ to converge.

Put $\mu_{i}:=\pi_{i}(\mu) \in \operatorname{MC}\left(R /\left(R^{+}\right)^{i}\right)$. We shall construct sequences $r_{i} \in F\left(R /\left(R^{+}\right)^{i}\right)$ and $g_{i} \in G\left(R /\left(R^{+}\right)^{i}\right)$ such that (1) $\Phi\left(R /\left(R^{+}\right)^{i}\right)\left(g_{i}, r_{i}\right)=\mu_{i}$, (2) $\pi_{i}\left(r_{i+1}\right)=r_{i}$ and (3) $\pi_{i}\left(g_{i+1}\right)=g_{i}$ for all $i \in \mathbb{N}$. For the induction base $i=1$, let $g_{1}:=1 \in G(\mathbb{C})$ and $r_{1}:=0 \in F(\mathbb{C})$. Since $\Phi(\mathbb{C})\left(g_{1}, r_{1}\right)=0=\mu_{1} \in \operatorname{MC}(\mathbb{C})$, the three conditions are satisfied at $i=1$.

For induction hypothesis, assume the sequences have already been constructed until index $i$. Since $\Phi$ is smooth, we have a surjection

$$
G\left(R /\left(R^{+}\right)^{i+1}\right) \times F\left(R /\left(R^{+}\right)^{i+1}\right) \rightarrow\left(G\left(R /\left(R^{+}\right)^{i}\right) \times F\left(R /\left(R^{+}\right)^{i}\right)\right) \times_{\mathrm{MC}\left(R /\left(R^{+}\right)^{i}\right)} \mathrm{MC}\left(R /\left(R^{+}\right)^{i+1}\right)
$$

Pick $\left(g_{i}, r_{i}, \mu_{i+1}\right)$ on the right hand side. Indeed, $\Phi\left(R /\left(R^{+}\right)^{i}\right)\left(g_{i}, r_{i}\right)=\mu_{i}=\pi_{i}\left(\mu_{i+1}\right)$ by assumption and construction. By surjectivity there is a lift $\left(g_{i+1}, r_{i+1}\right)$ such that (1) $\Phi\left(R /\left(R^{+}\right)^{i+1}\right)\left(g_{i+1}, r_{i+1}\right)=\mu^{i+1}$ and (2) $\pi_{i}\left(r_{i+1}\right)=r_{i}$ and (3) $\pi_{i}\left(g_{i+1}\right)=g_{i}$. This finishes the induction step.

Finally, we have constructed the desired sequences $\left(r_{i}\right)$ and $\left(g_{i}\right)$. Since $\pi_{i}\left(r_{i+1}\right)=r_{i}$, the sequence $r_{i}$ converges to some $r \in Z\left(\mathrm{Gtl}_{\mathcal{A}}\right) \widehat{\otimes} R^{+}$and $g_{i}$ converges to some $g \in \exp \left(\mathrm{HC}^{\text {even }}\left(\mathrm{Gtl}_{\mathcal{A}}\right) \widehat{\otimes} R^{+}\right)$. Within $\operatorname{MC}\left(R /\left(R^{+}\right)^{i}\right)$ we have

$$
\pi_{i}\left(g \cdot{ }^{r} \mu\right)=\Phi\left(R /\left(R^{+}\right)^{i}\right)\left(g_{i}, r_{i}\right)=\mu_{i}=\pi_{i}(\mu), \quad \forall i \in \mathbb{N} .
$$

Passing to the limit gives that $g \cdot{ }^{r} \mu=\mu$ within $\mathrm{MC}(R)$. In other words, $\mu$ is gauge equivalent to ${ }^{r} \mu$.
Remark 3.28. In remark 2.25 we extended the notion of orbigons to allow weights on the faces and the marked points. This allows us to construct curved deformations ${ }^{r, s} \mu$ of the (completed) gentle algebra without its $A_{\infty}$-structure coming from the Wrapped Fukaya category. It is also possible to show that every deformation of the completed gentle algebra is equivalent to one of these forms. This nicely fits into the framework of Koszul duality because Koszul dual $A_{\infty}$-algebras have the same deformation theory.

## Paper II

## Relative Fukaya Categories via Gentle Algebras

## 4 Introduction

Fukuya categories capture the global geometry of manifolds. They are complicated even to define and hard to study. In the case of punctured surfaces, gentle algebras were introduced as a remedy by Bocklandt 18. They have become a successful discrete version of the Fukaya category, having already served as A-side in mirror symmetry 18 and as standard model to study homological properties and stability conditions 35 .

Smooth<br>Fukaya category Fuk $Q$



Discrete Gentle algebra Gtl $Q$

On the smooth side, Seidel introduced in 2002 a procedure to deform Fukaya categories 63. The result is now known as the "relative Fukaya category" and has already been used e.g. as A-side in deformed mirror symmetry for the $n$-punctured torus by Lekili-Perutz-Polishchuk 46, 47. Surprisingly, a rigorous construction of the relative Fukaya category was only finished in 2022 by Perutz and Sheridan 59.

On the discrete side, Bocklandt and the author recently proposed an analog of Seidel's procedure in Paper I Our analog consists of an explicit deformation of the gentle algebra. Ultimately, these "deformed gentle algebras" will serve as A-side in our proof of deformed mirror symmetry for arbitrary punctured surfaces.

In this paper, we prove that the smooth and discrete deformations are equivalent. More precisely, we show that on the subcategories of zigzag curves, the smooth and discrete deformations have the same $A_{\infty}$-structure:

Deformed smooth
Relative Fukaya category relFuk $Q$


Deformed Discrete
Deformed gentle algebra $\mathrm{Gtl}_{q} Q$

## Assembling deformed mirror symmetry

This paper is the second part in a series of three papers aimed at proving deformed noncommutative mirror symmetry for punctured surfaces. The results of this paper seem to be interesting enough to stand on their own, but their full value becomes visible when viewed in the context of the series. Here we recall the overall aim of the series and the special relevance that this paper has to the workings of the series.

Mirror symmetry for punctured surfaces Original mirror symmetry of punctured surfaces due to Bocklandt 18 considers as A-side the gentle algebra $\mathrm{Gtl} Q$ of a dimer $Q$ and as B-side a category of matrix factorizations $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ of the dual dimer $\check{Q}$. Under the assumption that $\check{Q}$ is zigzag consistent, Bocklandt proves the existence of an $A_{\infty}$-quasi-isomorphism

$$
\operatorname{Gtl} Q \cong \operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)
$$

This is known as noncommutative mirror symmetry for punctured surfaces. A natural question is which deformation $\operatorname{Gtl} Q$ corresponds to which deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q} \ell)$. We shall focus on the specific deformation $\operatorname{Gtl}_{q} Q$ we defined in Paper I and ask which deformation $\operatorname{mf}_{q}(\operatorname{Jac} \check{Q}, \ell)$ of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ corresponds to $\operatorname{Gtl}_{q} Q$ such that there is still a quasi-isomorphism of deformed $A_{\infty}$-categories $\operatorname{Gtl}_{q} Q \cong \operatorname{mf}_{q}(\operatorname{Jac} \check{Q}, \ell)$.

The Cho-Hong-Lau construction Proving mirror symmetry is inherently difficult because two categories only vaguely resembling each other need to be matched. More precisely, there exists typically no strict $A_{\infty}$-isomorphism between the categories involved. To construct a non-strict functor as in 18 requires one to recognize that two given $A_{\infty}$-structures are equal up to a kind of homotopy. The analogous question in case of deformations of $A_{\infty}$-categories is how to decide whether two given $A_{\infty}$-deformations are gauge-equivalent. There are apparently very few tools available to decide this question.

The game changes as soon as we take the work of Cho, Hong and Lau 26 into account. They explain how to construct a mirror equivalence for punctured surfaces by a version of Koszul duality. Their paper shows how to systematically obtain both the dual dimer $\check{Q}$ and Bocklandt's mirror equivalence from a systematic construction:

$$
\begin{gathered}
\mathbb{L} \subseteq \mathcal{C} \\
A_{\infty} \text {-category with subcategory }
\end{gathered}
$$

$$
\begin{aligned}
F: \mathcal{C} & \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right) \\
& \text { mirror functor }
\end{aligned}
$$

The mirror category $\operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right)$ is a category of matrix factorizations. The Jacobi algebra $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)=\mathbb{C} Q^{\mathbb{L}} /\left(\partial_{a} W\right)$ and the potential $\ell \in \operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$ are determined by a kind of Koszul transform of the $A_{\infty}$-structure on the subcategory $\mathbb{L} \subseteq \mathcal{C}$. Mirror symmetry of punctured surfaces is a special case of the Cho-Hong-Lau construction: We set $\mathcal{C}=\mathrm{HTw} \operatorname{Gtl} Q$ and let $\mathrm{H} \mathbb{L} \subseteq \mathrm{H} \mathrm{Tw} \mathrm{Gtl} Q$ be the subcategory of zigzag paths $(26$, Chapter 10] $)$. This way, the Jacobi algebra $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$ becomes the Jacobi algebra $\operatorname{Jac} \check{Q}$ of the dual dimer and $\ell$ becomes the standard central element $\ell \in \operatorname{Jac} \check{Q}$. The mirror functor $F: \operatorname{Gtl} Q \rightarrow \operatorname{MF}(\operatorname{Jac} \check{Q}, \ell)$ one obtains this way is an explicit incarnation of Bocklandt's mirror symmetry for punctured surfaces.

A deformed Cho-Hong-Lau construction The aim of this series of three papers is to prove a broad range of deformed mirror equivalences for punctured surfaces. We achieve this by constructing a deformed Cho-Hong-Lau construction and applying it to $\mathrm{H}_{q} \subseteq \mathrm{HTw} \mathrm{Gtl}_{q} Q$ instead of $\mathrm{H} \mathbb{L} \subseteq \mathrm{HTw} \operatorname{Gtl} Q$. The result is a deformed mirror functor $F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$. Here $\mathrm{Jac}_{q} \check{Q}$ is a deformation of the algebra $\mathrm{Jac} \check{Q}$ and the central element $\ell_{q}$ is a deformation of $\ell$.

$$
\begin{aligned}
& \left.\mathrm{H}_{L_{q}} \subseteq \mathrm{HTw}_{\mathrm{Ttl}}^{q} \boldsymbol{Q} \sim_{\sim} \quad F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, W_{q}\right), \ell_{q}\right) \\
& \text { Deformed category of zigzag paths }
\end{aligned}
$$

The assembly of deformed mirror symmetry is divided into the three papers as follows: In Paper I we classify all deformations of $\operatorname{Gtl} Q$ up to gauge equivalence. In the present second paper, we select one certain broad deformation $\mathrm{Gtl}_{q} Q$ of $\mathrm{Gtl} Q$. This deformation induces a deformation $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ of the derived category $\mathrm{HTw} \operatorname{Gtl} Q$. We calculate the deformed $A_{\infty}$-structure on the subcategory $\mathrm{H}_{q} \subseteq$ $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ given by zigzag paths. In the third paper, we prove a deformed version of the Cho-HongLau construction. Simply plugging in the description of $H \mathbb{L}_{q}$ from the present paper gives the desired mirror functor $F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$. This result amounts to a wide range of deformed mirror equivalences for punctured surfaces.

## Results

We present here the results of this paper in a non-technical manner. The precise statements can be found in Theorem $8.349 .20,13.26$ and 13.31 .

Deformed Kadeishvili theorem The classical Kadeishvili theorem states that every $A_{\infty}$-category has a minimal model. By definition, a minimal model of an $A_{\infty}$-category $\mathcal{C}$ is any $A_{\infty}$-category $\mathcal{D}$ with vanishing differential $\mu_{\mathcal{D}}^{1}$ such that $\mathcal{C}$ and $\mathcal{D}$ are quasi-isomorphic. When $\mathcal{C}_{q}$ is an (infinitesimally curved) deformation of $\mathcal{C}$, it is not clear a priori what a minimal model should be and whether it exists. In the present paper, we fix a definition of minimal models for deformed $A_{\infty}$-categories and show that all deformed $A_{\infty}$-categories have minimal models. In Theorem 8.34 we show that a minimal model for any $A_{\infty}$-deformation $\mathcal{C}_{q}$ can be explicitly computed by the following Kadeishvili construction:

1. Choose a homological splitting for $\mathcal{C}$.
2. Perform an infinitesimal base change on the homological splitting in order to adapt it to $\mu_{q}^{1}$.
3. Gauge away part of the deformation's curvature.
4. Repeat steps 2 and 3 indefinitely. Take the limit of this process.
5. Calculate the deformed codifferential $h_{q}$ and projection $\pi_{q}$.
6. Define the structure of $\mathrm{H} \mathcal{C}_{q}$ by sums over deformed Kadeishvili trees.

Uncurving of band objects The objects of the derived category H Tw Gtl $Q$ have been classified up to isomorphism in 35. They fall into two classes: the string objects and the band objects. Geometrically, a string corresponds to a curve $\gamma:[0,1] \rightarrow|Q|$ which starts and ends at punctures. A band object corresponds to a closed curve $\gamma: S^{1} \rightarrow|Q|$ which does not hit any punctures. Both string objects and band objects can also be interpreted as objects in the deformed twisted completion $\mathrm{Tw}_{\mathrm{Gtl}}^{q}$ $Q$. In Theorem 9.20, we show that for the typical band object this curvature can be gauged away.

Minimal model of the deformed category of zigzag paths We explicitly describe the minimal model $\mathrm{H} \mathbb{L}_{q} \subseteq \mathrm{HTw} \mathrm{Gtl}_{q} Q$ in terms of immersed disks. We find four types of immersed disks, the CR, ID, DS and DW disks. The precise description is stated in Theorem 13.26 Once we restrict to the transversal part of $\mathrm{H}_{q}$, the description reduces to the smooth immersed disks used for the definition of the relative Fukaya category. Explicitly, the transversal part of $H \mathbb{L}_{q}$ agrees with the subcategory of relFuk ${ }^{\text {pre }} Q$ given by zigzag curves.

## The minimal model calculation

The main storyline of this paper is the calculation of the minimal model $\mathrm{H} \mathbb{L}_{q}$ by means of our deformed Kadeishvili theorem. We describe here how every of the six steps in the calculation of $\mathrm{H} \mathbb{L}_{q}$ play out in practice. This description covers the materials contained in section 10 till 13

Step 1 For the first step of the Kadeishvili construction, we are supposed to choose a homological splitting $H \oplus I \oplus R$ for $\mathbb{L}$. There are many possible homological splittings, but not all make sense from a geometric point of view. In section 10, we choose one specific homological splitting. To define our splitting, we have to choose an explicit basis for the cohomology of every hom spaces in $\mathbb{L}$. Since we expect to obtain the relative Fukaya category as minimal model, we choose cohomology basis elements which are geometrically located as close as possible to the intersection points of the zigzag curves

Step 1A Deviating slightly from the general procedure of the deformed Kadeishvili theorem, we already here gauge away curvature from $\mathbb{L}_{q}$. The gauge consists of applying our "complementary angle trick" which we define and treat in the larger generality of band objects. The idea is that the twisted complexes contained in $\mathbb{L}_{q}$ consist of sums of arcs, with twisted differential given by angles between those arcs. Every angle comes with a certain complementary angle. Our "complementary angle trick" consists of adding the complements of these angles, weighted by deformation parameters, to the twisted differential. This trick succeeds at removing the curvature of $\mathbb{L}_{q}$.

Step 2 For the second step of the Kadeishvili construction, we are supposed to calculate the infinitesimal base change. It requires from us that we evaluate the deformed differential $\mu_{\mathbb{L}_{q}}^{1}$. In section 11 , we execute this by investigating all possible contributions to products $\mu_{\mathbb{L}_{q}}^{1}(\varepsilon)=\mu_{q}^{1}(\delta, \ldots, \varepsilon, \ldots, \delta)$. We introduce the notions of "E, F, G, H disks" and "tails" as bookkeeping tool to systematically construct the required infinitesimal base change.

Step 3, 4 The third and fourth step are vacuous for $\mathbb{L}_{q}$, since we have already gauged away all the curvature in the beginning. Already at the present stage after the fourth step, we have a strong indication that we will obtain the relative Fukaya category as result of the calculation. Take for granted that the basis elements of $H$ can be identified with intersection points of zigzag curves. The infinitesimal base change of the second step adds infinitesimal amounts of $R$ to the cohomology basis elements in $H$. Visually, the interpretation is that the intersection points "grow tails" in all directions where they could possibly bound disks. This is a strong indication that we will obtain the relative Fukaya category as a result.

Step 5 For the fifth step, we are supposed to calculate the deformed codifferential $h_{q}$ and deformed projection $\pi_{q}$. In section 11.5. we calculate the deformed codifferential $h_{q}(\varepsilon)$ and $\pi_{q}(\varepsilon)$ for the most important morphisms $\varepsilon$ between zigzag paths. This requires a detailed analysis of the surroundings of $\varepsilon$, which we capture in terms of what we call situations of type A, B, C and D. It turns out also $h_{q}(\varepsilon)$ comes with an infinitesimal "tail" pointing in all possible directions that can bound disks with $\varepsilon$. We end up with expressions for the deformed codifferential of any morphism.

Very specifically, the reader will see recurrent use of the codifferential expression $h_{q}(\beta \alpha)$ throughout the paper. In this context, the morphism $\beta$ always denotes a $\beta$-angle associated with a "type A situation". The analysis shows that $\beta$-angles act as extending link between multiple portions of a relative Fukaya disk, which renders them the most powerful angles in this paper.

Step 6 For the sixth step, we have to evaluate deformed Kadeishvili trees. In section 12, we start with a careful characterization of all results that can possibly come out of the Kadeishvili trees. The simplest Kadeishvili trees can of course be translated into relative Fukaya disks directly. Results of all other Kadeishvili trees are instead results of iterated applications of the deformed product $\mu_{q}$ of $\mathrm{Gtl}_{q}$. We introduce "result components" as a bookkeeping tool for evaluating Kadeishvili trees and provide a full characterization of how result components are derived from each other within a Kadeishvili tree.

Interpretation of the result In section 13 we show how to turn result components of Kadeishvili trees into disks by an explicit method. Due to our characterization, every result component comes with a history, a way in which it was derived from simpler result components. We devise an inductive procedure to draw a disk from a result component. The type of the outcome is not exactly the same as a relative Fukaya disk, but is what we call an SL disk (shapeless disk). The drawing procedure works as follows: Departing from the leaves of the Kadeishvili tree, we start drawing a small portion of the SL disk. As multiple result components are merged into one at any node in the tree, we glue together their small portions. When we reach the root of the tree, we conclude the drawing by closing the SL disk with an output mark. All in all, we have assigned this way an SL disk to a result component.


As a final step, we classify all SL disks we have obtained this way. It turns out that the SL disks obtained are of four types, which we call CR, ID, DS and DW disks. In other words, the higher products on $\mathrm{H} \mathbb{L}_{q}$ are precisely computed by SL disks in the surface that belong to one of those four types. It is useful to know in advance that many, but not all of these disks are transversal. In fact, the transversal ones among them are all of CR type and match exactly the (transversal) relative Fukaya disks. This finishes the computation of the minimal model $H \mathbb{L}_{q}$.

## Context and philosophical highlights

The results of our paper are very specific. To get a sense of their general meaning, we put the results into context. We comment on the following philosophical highlights:

- Derived categories of $A_{\infty}$-deformations exist.
- Constructions with infinitesimal curvature are performed by inducing the deformation afterwards.
- Computational techniques continue to apply with infinitesimal curvature.
- Minimal model calculations are possible if one is sensitive to the result.
- The deformed gentle algebra $\operatorname{Gtl}_{q} Q$ is a relative wrapped Fukaya category.
- Hamiltonian deformations arise naturally from representation theory.

The curvature problem In formal $A_{\infty}$-deformation theory, one regards infinitesimal deformations of a given $A_{\infty}$-structure such that the $A_{\infty}$-relations are preserved. An issue is that only deforming the $A_{\infty}$-structure does not give a notion of deformations that is invariant under quasi-equivalence of $A_{\infty}$-categories. In order to obtain a notion invariant under quasi-equivalence, one needs to permit the deformation to have curvature. More precisely, the curvature must be infinitesimal in the sense that it lies in a multiple of the maximal ideal of the local ring. Infinitesimal curvature is inevitable for a good notion of $A_{\infty}$-deformations.

Dealing with curvature is however regarded as tedious, because the curvature prevents the differential from squaring to zero. The presence of curvature is often referred to as the "curvature problem". A main question is how to gauge away the curvature or otherwise how to deal with the remaining curvature. An instance of the uncurving problem has been studied by Lowen and Van den Bergh 50, on which we comment in section F.1.3

Derived categories of $A_{\infty}$-deformations Part of the "curvature problem" is that curved $A_{\infty^{-}}$ categories do not have derived categories. In fact, their differential need not square to zero because of the curvature. The game changes when the curvature is only infinitesimal. In the present paper, we highlight that any construction available for $A_{\infty}$-categories can be performed with $A_{\infty}$-deformations as well. The idea is to apply the construction to the non-deformed $A_{\infty}$-category and to induce the deformation on the result afterwards. Regard for example the twisted completion construction or the minimal model construction. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ an $A_{\infty}$-deformation, possibly with infinitesimal curvature. While the category $\mathcal{C}$ has a twisted completion $\operatorname{Tw} \mathcal{C}$ and a minimal model $\mathrm{H} \mathcal{C}$, what should the twisted completion $\operatorname{Tw} \mathcal{C}_{q}$ or the minimal model $\mathrm{H} \mathcal{C}_{q}$ be? The mathematically correct answer is that these categories should be deformations of $\operatorname{Tw} \mathcal{C}$ and HC , with the deformation induced from $\mathcal{C}_{q}$.

The correct way to define $\operatorname{Tw} \mathcal{C}_{q}$ is by gathering the same objects as $\operatorname{Tw} \mathcal{C}$, namely those twisted complexes satisfying the non-deformed Maurer-Cartan equation $\mu^{1}(\delta)+\mu^{2}(\delta, \delta)+\ldots=0$. This is not the same as gathering twisted complexes with twisted differential $\delta$ satisfying the deformed MaurerCartan equation $\mu_{q}^{0}+\mu_{q}^{1}(\delta)+\ldots=0$. In contrast to $\operatorname{Tw} \mathcal{C}$, the category $\mathrm{Tw} \mathcal{C}_{q}$ inherits infinitesimal curvature, stemming precisely from the infinitesimal failure of the twisted differentials to satisfy the deformed Maurer-Cartan equation.

The correct way to define $\mathrm{H} \mathcal{C}_{q}$ consists of taking any minimal model HC and inducing the deformation $\mathcal{C}_{q}$ onto HC via any quasi-isomorphism $\pi: \mathcal{C} \rightarrow \mathrm{HC}$. This abstract approach means that the "minimal model" $\mathrm{H} \mathcal{C}_{q}$ may have infinitesimal curvature as well as a residual infinitesimal differential. This is not the same as taking cohomology of the hom complexes $\left(\operatorname{Hom}_{\mathcal{C}_{q}}(X, Y), \mu_{q}^{1}\right)$. In fact, the deformed differential of an $A_{\infty}$-deformation need not even square to zero because of the curvature.

Minimal models of $A_{\infty}$-categories can classically be computed by means of homological splittings and Kadeishvili trees. In our deformed Kadeishvili theorem, we show that this method carries over to the deformed case. The starting point is an $A_{\infty}$-category together with an deformation $\mathcal{C}_{q}$. The difficulties encountered in constructing the minimal model are the presence of curvature $\mu_{q}^{0}$, the fact that the deformed differential $\mu_{q}^{1}$ does not square to zero and the fact that $\mu_{q}^{1}$ is not compatible with the homological splitting chosen for $\mathcal{C}$. In section 8 , we show how to adapt the Kadeishvili construction to these special circumstances. We view our deformed Kadeishvili theorem as evidence that computational techniques which apply to $A_{\infty}$-deformations can be tweaked in order to apply to $A_{\infty}$-deformations as well.

Discrete relative Fukaya category Many different constructions of Fukaya categories are available in the literature. The most general approach is the reference work of Seidel 64. For relative Fukaya categories, a new reference is the construction of Sheridan and Perutz 59. For the case of punctured surfaces, there are many further specific models available. One can distinguish whether they depart from the discrete side of gentle algebras or from the smooth side of actual Fukaya categories, and whether they consider the punctured surface alone or whether they consider Seidel's deformation. The following is a non-exhaustive overview:

| starting point | non-deformed | deformed |
| :---: | :---: | :---: |
| geometric | 18, Appendix B] | $63,46,47$ |
| discrete | $18,46,35$ | this paper |

The most important reference for us is the construction of the gentle algebras of 18 . In Paper I we proposed a candidate deformation $\operatorname{Gtl}_{q} Q$ with the intention to provide a "relative wrapped Fukaya category" for punctured surfaces. Verifying that the deformed gentle algebra $\mathrm{Gtl}_{q} Q$ deserves this name
would at least entail proving that the transversal part of its derived category is equivalent to the relative Fukaya category, see section F.2.2. It is however quite difficult to actually compute the derived category, as we witness in the present paper. If a "relative wrapped Fukaya category" existed already, this would be greatly eased, see section F.2.3

In the present paper, we succeed in showing that at least on the subcategory of zigzag paths, the derived category $\mathrm{HTw}_{\mathrm{Gt}}^{q} \boldsymbol{} Q$ and the relative Fukaya category relFuk $Q$ agree. Although our calculation is limited to zigzag paths, we consider our calculation strong evidence that $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ indeed contains relFuk $Q$. It is a crude verification that we have correctly transported Seidel's vision to gentle algebras and that $\mathrm{Gtl}_{q} Q$ can be considered a relative wrapped Fukaya category.

Minimal model calculations A highlight in this paper is our explicit computation of an entire minimal model. Such computations are scarce in the $A_{\infty}$-literature and often considered tedious. Indeed, minimal model calculations are hard because of the large amount of Kadeishvili trees involved. For some calculations in the liturature, it is not necessary to perfom the calculation until the end. In the case of 18 it suffices to calculate only part of the minimal model because the rest is determined up to homotopy. In the case of Paper I we also cut short the calculation of an $L_{\infty}$-minimal model by means of grading arguments. In the present paper, we perform the minimal model calculation of $\mathbb{L}_{q}$ until the end.

Minimal model calculations are the core connecting bridge between the discrete and the smooth world. They are regarded as tedious, but we contend that minimal model calculation need not hurt if one has a clue regarding the outcome. The minimal model calculation in the present paper succeeds precisely because we recognize in every step the inherent geometric meaning of the terms that appear. This concerns both the choice of the homological splitting for $\mathbb{L}$ and the evaluation of the Kadeishvili trees. In section F.3 and F.4.4, we offer further explanation on why our method of "result components" works and how to apply it in other situations.

Hamiltonian deformations Implementing Hamiltonian deformations is one of the difficulties one encounters when defining smooth Fukaya categories. In the discrete world, one circumvents this problem by choosing such a small set of generators that the Hamiltonian deformations can be chosen canonically and disappear completely from the picture. When passing to the derived category $\mathrm{HTw} \mathrm{Gtl} Q$, we however expect the full generality of the smooth Fukaya category to reappear. In particular, we expect to find $A_{\infty}$-products on some non-transversal and expect that we can explain these products as an incarnation of Hamiltonian deformation.

In the present paper, we compute the precise $A_{\infty}$-products on the deformed category $\mathrm{H} \mathbb{L}_{q}$. The starting point is our deformed Kadeishvili theorem, whose essential ingredient is a choice of homological splitting for $\mathbb{L}$. As expected, the products of $\mathrm{H}_{\mathbb{L}_{q}}$ agree with the products of the relative Fukaya category on transversal sequences. We however also obtain an explicit description of the products on non-transversal sequences. We show how to interpret even the products on non-transversal sequences geometrically as disks being bounded by zigzag curves and their Hamiltonian deformations. While Hamiltonian deformations have to be incorporated as an ingredient into the definition of smooth Fukaya categories from the beginning, they appear naturally through the Kadeishvili construction of the minimal model $\mathrm{H} \mathbb{L}_{q}$.

The precise shape of the products of $H \mathbb{L}_{q}$ depends on the choice of homological splitting for $\mathbb{L}$. Nevertheless, different homological splittings give quasi-equivalent minimal models $\mathrm{H} \mathbb{L}_{q}$. We have selected one specific splitting which makes it particularly easy to identify the minimal model as the relative Fukaya category. When choosing a slightly different splitting, we still expect to obtain the same products on transversal sequences, but the products on non-transversal sequences will typically change. These changed products can be interpreted geometrically as products in the relative Fukaya category under application of a different Hamiltonian deformation. While homological splittings for $\mathbb{L}$ are a discrete and representationtheoretic notion, Hamiltonian deformations are a smooth and geometric notion. Highly simplified, we may say that choices of homological splittings correspond to choices of Hamiltonian deformations. See also section F.3.2

Strings and bands Gentle algebras $\mathrm{Gtl} Q$ were originally introduced in 18 to provide a combinatorial description of the wrapped Fukaya category of the punctured surface $|Q| \backslash Q_{0}$. In contrast to the Fukaya category of $|Q| \backslash Q_{0}$, the wrapped Fukaya category also includes curves which start and end at punctures. Haiden, Katzarkov and Kontsevich 35 classified the objects of H Tw Gtl $Q$ under the additional datum of a $\mathbb{Z}$-grading. Their classification indeed finds those types of curves expected from the wrapped Fukaya category. Explictly, their classification divides the objects into two classes, known as string objects and band objects. Roughly speaking, a string object is a non-closed curve running between two punctures of $Q$ and a band object is a closed curve that avoids the punctures of $Q$.

A string object or band object given by a curve in $|Q|$ can be explicitly realized as a twisted complex in $\operatorname{Tw} \operatorname{Gtl} Q$. The procedure entails approximating the curve by arcs $a_{1}, \ldots, a_{k}$ of $Q$ together with angles $\alpha_{i}$ between the arcs. One then forms a twisted complex $\left(\bigoplus_{i} a_{i}\left[s_{i}\right], \delta=\sum_{i} \alpha_{i}\right)$ by summing up the arcs and using the angles as twisted differential.

Seidel 63 describes which objects in the relative Fukaya category should have curvature according to his vision. For the case of punctured surfaces, his criterion states that curves which bound a so-called teardrop should have curvature. Also those curves which are contractible in the surface $|Q|$ should have curvature. All other objects in the relative Fukaya category should be curvature-free according to Seidel.

In the present paper, we approach Seidel's vision from the starting point of the deformed gentle algebras $\operatorname{Gtl}_{q} Q$ instead of the relative Fukaya category. Translated to our setting, a band object $\left(\bigoplus_{i} a_{i}\left[s_{i}\right], \delta=\sum_{i} \alpha_{i}\right)$ should be uncurvable if its underlying curve in the closed surface $|Q|$ is not contractible and does not bound a teardrop. In order to make his vision true, we devise a trick to gauge away the curvature of these band objects. Our "complementary angle trick" consists essentially of adding infinitesimal multiples of the complementary angle of $\alpha_{i}$ to $\delta$ for all $i$. In section 9 , we verify that our trick successfully uncurves all band objects whose underlying curve in the closed surface $|Q|$ is not contractible and does not bound a teardrop, making true Seidel's vision.

## Data structures

Most of our calculation does not go beyond simple inspection of arc systems and linear algebra. However, organizing result components and matching them with disks requires us to devise a large amount of data structures and fill them with data. For an overview, we depict in Figure 4.1 the essential data structures. We shall here explain the purpose and development of these datastructures and which data flows from which structure into which one.

The starting point is a dimer $Q$, which is a specific type of quiver embedded in a surface. It gives rise to the discrete notion of zigzag paths and the smooth notion of zigzag curves. On the smooth side, the zigzag curves give rise to the notion of intersection points and smooth immersed disks, the foundations of Fukaya categories.

On the discrete side, we regard the category $\mathbb{L}$ of zigzag paths. A morphism $\varepsilon: L_{1} \rightarrow L_{2}$ between two zigzag paths consists of an angle between arcs of $L_{1}$ and $L_{2}$. We determine a basis of cohomology elements for $\mathbb{L}$. We also define the category $\mathbb{L}_{q}$ of deformed zigzag paths. Examining the deformed differential $\mu_{\mathbb{L}_{q}}^{1}$ of this category gives rise to four types of disks which we call $\mathrm{E}, \mathrm{F}, \mathrm{G}, \mathrm{H}$ disks. We introduce the auxiliary notion of "tails". The tail of an angle $\varepsilon: L_{1} \rightarrow L_{2}$ is a tree whose nodes are decorated with E, F, G, H disks.

The deformed Kadeishvili theorem gives rise to notions of deformed cohomology basis elements, a deformed projection $\pi_{q}$ and a deformed codifferential $h_{q}$. We can describe them explicitly by means of tails. According to the deformed Kadeishvili theorem, the product structure of the minimal model $\mathrm{H} \mathbb{L}_{q}$ is described in terms of sums over trees. We define a notion of "result components" which serves to systematically track the results of evaluations of trees. From a result component we build a "subdisk" by drawing zigzag curve segments and intersection points. Subdisks of result components are immersed disks and fall into four classes which we call the CR, DS, ID and DW disks.

From the perspective of data structures, this finishes the construction of the minimal model $\mathrm{H}_{\mathbb{L}_{q}}$. Both $\mathrm{H} \mathbb{L}_{q}$ and the relative Fukaya category are described by immersed disks. Therefore the category $\mathbb{L}_{q}$ of deformed zigzag paths is quasi-isomorphic to the subcategory of zigzag curves of the relative Fukaya category. On the level of data structures, this finishes the main theorem.

## Structure of the paper

In section 5 we recall $A_{\infty}$-categories and their deformations. In section 6 we recall gentle algebras and deformed gentle algebras. In section 7 we recall basics of Fukaya categories and explain their subcatgories of zigzag curves. In section 8, we build our deformed Kadeishvili theorem. In section 9 we present the uncurving procedure for band objects. In section 10, we exhibit the category of zigzag paths $\mathbb{L}$ together with a homological splitting. In section 11, we present the deformed version of this homological splitting, together with reference material for the rest of the paper. In section 12 we introduce the tool of result components to enumerate products in the minimal model $\mathrm{H} \mathbb{L}_{q}$. In section 13 we devise a simple drawing method to transform these result components into immersed disks. InTheorem 13.26, we describe explicitly the structure of the minimal model $\mathrm{H} \mathbb{L}_{q}$ in terms of immersed disks. Our main result Theorem 13.31 states that $H \mathbb{L}_{q}$ has the same products on transversal sequences as the relative Fukaya category.


Figure 4.1: This graph depicts the essential data structures used in the paper. The left part of the graph depicts data structures used for the computation of the minimal model $H \mathbb{L}_{q}$. The right part depicts the construction of the Fukaya category. At the end of the paper, the minimal model $\mathrm{H} \mathbb{L}_{q}$ is described by means of immersed disks. The relative Fukaya category is defined in terms of immersed disks as well. Ultimately, we conclude that $\mathrm{H}_{\mathbb{L}_{q}}$ is equivalent to the subcategory of the relative Fukaya category given by zigzag curves.

This paper contains several appendices which are devoted to technical proofs and additional explanation. In section A we provide examples of immersed disks together with their corresponding result components. The aim is to facilitate understanding of how disks arise from the minimal model $\mathrm{H} \mathbb{L}_{q}$. In section B, we complete the proof of uncurvability of band objects. In section C, we finish the proof of the main result by providing an explicit inverse construction which maps CR, DS, ID and DS disks to their corresponding result components. In section D, we study the case of specific sphere dimers, among which the pair of pants. These dimers are not geometrically consistent and fall outside of the scope of the rest of the paper, but we have included their calculation due to their relevance in mirror symmetry. In section E, we compute a small class of products in the category H Tw $\mathrm{Gtl}_{q} Q$ which go beyond zigzag paths. Specifically, this concerns products of morphisms between arcs and zigzag paths from which we determine the mirror objects $F_{q}(a) \in \operatorname{MF}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ in the third paper. In section F. 1 we discuss the relation with the literature in more detail. In section F.2 we explain why one is led to believe from an a priori perspective that $\mathrm{H} \mathbb{L}_{q}$ agrees with the relative Fukaya category. In section F.3 we summarize from an a posteriori perspective why the very technical calculation of $H \mathbb{L}_{q}$ contained in this paper succeeds. In section F.4, we share insight on how to reuse the constructions in this paper for other purposes. In section G we collect notation specific to this paper.

## Conventions

During the course of the paper, we play in different contexts. The following are the overarching conventions that are significant to the validity of the results:

- In section 9, we assume that $\mathcal{A}$ is an arc system which has no monogons or digons in the closed surface. We summarize this in the [NMDC] condition.
- Insection 10 till 13 we assume that $Q$ is a geometrically consistent dimer. Every zigzag path is supposed to come with a chosen spin structure and locations of identity and co-identity endomorphism. We summarize this setup in Convention 10.10
- For the purposes of section 5, 8.3 and 9 we have to assume axioms on the invariance of the Hochschild DGLA under $A_{\infty}$-quasi-equivalences. We summarize these axioms in Convention 5.55


## 5 Preliminaries on $A_{\infty}$-categories

In this section, we recollect material on $A_{\infty}$-categories and build theory for $A_{\infty}$-deformations. Theory on $A_{\infty}$-deformations is scarce and implicit in the literature, so we provide a rigorous definition here that includes the infinitesimally curved case. We explain why curvature is inevitable if one wants to obtain a deformation theory invariant under quasi-equivalences.

In section 5.1, we recall $A_{\infty}$-categories. In section 5.2 we recall the completed tensor products of the form $B \overparen{\otimes} X$, where $B$ is a local algebra. In section 5.3 we recall (curved) deformations of $A_{\infty}$ categories. In section 5.4 we recall functors between $A_{\infty}$-categories and between deformations of $A_{\infty^{-}}$ categories. In section 5.5 we recall the twisted completion construction for $A_{\infty}$-categories and define an analogous construction for $A_{\infty}$-deformations. In section 5.6, we recall how to view $A_{\infty}$-deformations as Maurer-Cartan elements via the so-called Hochschild DGLA. In section 5.7, we recall the notion of gauge equivalence of Maurer-Cartan elements and explain its application in the specific case of $A_{\infty}$-deformations. In section 5.8 we recall the generalization of DGLAs known as $L_{\infty}$-algebras. In section 5.9 we explain how to push forward deformations between quasi-equivalent categories.

The interpretation of gauge equivalences and pushforwards of deformations in terms of $L_{\infty}$-theory is very helpful, but we will not prove it here. Rather, we state in section 5.9 a collection of axioms on the Hochschild DGLA which we will just assume in this paper. The construction of twisted completions of $A_{\infty}$-deformations might differ slightly from what the reader expects.

The material in the entire section is not original: Seidel already mentioned curved $A_{\infty}$-deformations in 63. Lowen and Van den Bergh considered the "curvature problem" in 50. The Hochschild DGLA is classical at least in the case of ordinary algebras. Keller proved its invariance under quasi-equivalence in the dg case in 40.

This section serves to support a very specific viewpoint on $A_{\infty}$-deformations. While there is widespread belief that curvature is a hindrance, we want to demonstrate here that it is at least possible to live peacefully with infinitesimal curvature. Let us present which section supports which claim: From section 5.5 we learn that the twisted completion of an $A_{\infty}$-deformation can be formed even under the presence of infinitesimal curvature. This construction is entirely natural in the sense that the deformed twisted completion is a deformation of the twisted completion of the non-deformed $A_{\infty}$-category. From section 5.6
we learn that infinitesimal curvature is inevitable for a good notion of $A_{\infty}$-deformations. The core summary is that curvature does not hurt for constructions with $A_{\infty}$-categories because it enters the picture automatically when a deformation is induced from one category to another.

In this section, we will be comparing different deformations with another. We should set some terminology right before we get started. A "deformation base" $B$ will be a complete local Noetherian unital $\mathbb{C}$-algebra with residue field $B / \mathfrak{m}=\mathbb{C}$. Every deformation $\mathcal{C}_{q}$ over $B$ can be interpreted as object of two different universes: either as deformation of $\mathcal{C}$, or as a deformation of any $A_{\infty}$-category. Let us depict this pictorially:


This perspective makes a difference: The correct notion for two deformations $\mathcal{C}_{q}, \mathcal{C}_{q}^{\prime}$ to be similar on the left side is to be gauge equivalent, while the correct notion for two deformations $\mathcal{C}_{q}$ and $\mathcal{D}_{q}$ to be similar on the right side is to be quasi-equivalent. In the former case, the leading term of the functor connecting the two is supposed to be the identity, in the latter case the leading term can be any quasi-equivalence of $A_{\infty}$-categories.

To distinguish the two perspectives, we will often use the terminology $A_{\infty}$-deformation to refer to the left side, and deformed $A_{\infty}$-category to refer to the right side.

## 5.1 $A_{\infty}$-categories

The notion of $A_{\infty}$-category is now widely used as a homological-algebraic tool to study symplectic and algebraic geometry. Its relation to dg categories, triangulated categories and $\infty$-categories can be described as follows:

- DG categories are more rigid than $A_{\infty}$-categories. DG quasi-isomorphisms cannot be quasi-inverted on the dg level, while they always have a quasi-inverse if one interprets the $\operatorname{dg}$ structures as $A_{\infty^{-}}$ categories. In particular, zigzags of dg quasi-isomorphisms can be resolved into a single $A_{\infty}$-quasiisomorphism. DG structures are very easy to work with, for instance homotopy colimits are easy to calculate due to the well-known Tabuada model structure 69, 41.
- Triangulated categories are weaker than $A_{\infty}$-categories. They do not remember any of the "higher structure" that $A_{\infty}$-categories contain. The higher structure on an $A_{\infty}$-category makes the entire category reconstructible from a subcategory of generators. There is an $A_{\infty}$ notion of derived category $\mathrm{DC}:=\mathrm{HTw} \mathcal{C}$, whose "lower structure" $(\mathrm{DC})^{0}$ is a triangulated category 43.
- Stable $\infty$-categories also remember the homotopy information, similar to $A_{\infty}$-categories. A priori they lack the linearity over a base field, which can be added afterwards. In fact, there is a direct correspondence between $k$-linear stable $\infty$-categories and $A_{\infty}$-categories over $k$, given in one direction by taking the derived category 27,31 .
We work over an algebraically closed field of characteristic zero and always write $\mathbb{C}$. Let us now recall the definition of $A_{\infty}$-categories:

Definition 5.1. A ( $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded, strictly unital) $A_{\infty}$-category $\mathcal{C}$ consists of a collection of objects together with $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded how spaces $\operatorname{Hom}(X, Y)$, distinguished identity morphisms $\operatorname{id}_{X} \in \operatorname{Hom}^{0}(X, X)$ for all $X \in \mathcal{C}$, together with multilinear higher products

$$
\mu^{k}: \operatorname{Hom}\left(X_{k}, X_{k+1}\right) \times \ldots \times \operatorname{Hom}\left(X_{1}, X_{2}\right) \rightarrow \operatorname{Hom}\left(X_{1}, X_{k+1}\right), \quad k \geq 1
$$

of degree $2-k$ such that the $A_{\infty}$-relations and strict unitality axioms hold: For all compatible morphisms $a_{1}, \ldots, a_{k}$ we have

$$
\begin{aligned}
& \sum_{0 \leq n<m \leq k}(-1)^{\left\|a_{n}\right\|+\ldots+\left\|a_{1}\right\|} \mu\left(a_{k}, \ldots, \mu\left(a_{m}, \ldots, a_{n+1}\right), a_{n}, \ldots, a_{1}\right)=0, \\
& \mu^{2}\left(a, \operatorname{id}_{X}\right)=a, \mu^{2}\left(\operatorname{id}_{Y}, a\right)=(-1)^{|a|} a, \mu^{\geq 3}\left(\ldots, \operatorname{id}_{X}, \ldots\right)=0 .
\end{aligned}
$$

Remark 5.2. The first few $A_{\infty}$-relations read

$$
\begin{array}{r}
\mu^{1}\left(\mu^{1}(a)\right)=0, \\
(-1)^{\|b\|} \mu^{2}\left(\mu^{1}(a), b\right)+\mu^{2}\left(a, \mu^{1}(b)\right)+\mu^{1}\left(\mu^{2}(a, b)\right)=0, \\
(-1)^{\|c\|} \mu^{2}\left(\mu^{2}(a, b), c\right)+\mu^{2}\left(a, \mu^{2}(b, c)\right)+\mu^{1}\left(\mu^{3}(a, b, c)\right) \\
+(-1)^{\|b\|+\|c\|} \mu^{3}\left(\mu^{1}(a), b, c\right)+(-1)^{\|c\|} \mu^{3}\left(a, \mu^{1}(b), c\right)+\mu^{3}\left(a, b, \mu^{1}(c)\right)=0 .
\end{array}
$$

In particular, if $\mu^{1}$ or $\mu^{3}$ vanishes, then $\mu^{2}$ is graded associative.
An $A_{\infty}$-category $\mathcal{C}$ is minimal if $\mu_{\mathcal{C}}^{1}=0$. Let $\mathcal{C}$ be a minimal $A_{\infty}$-category. Regard its degree-zero part $\mathcal{C}^{0}$, given by the same objects as $\mathcal{C}$ but only including the degree-zero part of the hom spaces. Then the product composition $a \circ b:=(-1)^{|b|} \mu^{2}(a, b)$ is associative on $\mathcal{C}^{0}$, rendering $\mathcal{C}^{0}$ an ordinary $\mathbb{C}$-linear category.

We recall minimal models in detail in section 8 A minimal model of an $A_{\infty}$-category $\mathcal{C}$ is any minimal $A_{\infty}$-category HC together with a quasi-isomorphism $F: \mathrm{H} \mathcal{C} \rightarrow \mathcal{C}$. We recall twisted completions $\mathrm{Tw} \mathcal{C}$ in detail in section 5.5

There is a good notion of quasi-isomorphism for objects in an $A_{\infty}$-category:
Definition 5.3. Let $\mathcal{C}$ be an $A_{\infty}$-category and $X, Y \in \mathcal{C}$. A morphism $f \in \operatorname{Hom}^{0}(X, Y)$ is a quasiisomorphism if there exist morphisms $g \in \operatorname{Hom}^{0}(Y, X), h_{X} \in \operatorname{Hom}^{-1}(X, X)$ and $h_{Y} \in \operatorname{Hom}^{-1}(Y, Y)$ such that $\mu^{1}(f)=\mu^{1}(g)=0$ and $\mu^{2}(f, g)=\operatorname{id}_{Y}+\mu^{1}\left(h_{Y}\right)$ and $\mu^{2}(g, f)=\operatorname{id}_{X}+\mu^{1}\left(h_{X}\right)$. Two objects $X$ and $Y$ are quasi-isomorphic if there exists a quasi-isomorphism between them.

In other words, a morphism in $\mathcal{C}$ is a quasi-isomorphism if it is closed and descends to an isomorphism in the minimal model HC . Two objects $X$ and $Y$ are quasi-isomorphic if they are isomorphic as objects of the ordinary category $\mathrm{H}^{0} \mathcal{C}$. The notions of quasi-isomorphism and quasi-isomorphic objects are both flexible and sufficient for studying isomorphisms within $A_{\infty}$-categories.

The notion of a curved $A_{\infty}$-category is a generalization of ordinary $A_{\infty}$-category. In contrast to an ordinary $A_{\infty}$-category, a curved $A_{\infty}$-category also has an element $\mu_{X}^{0}$ of degree 2 associated with every object $X \in \mathcal{C}$. The required curved $A_{\infty}$-relations are the same as those of an $A_{\infty}$-category, in particular allowing $\mu_{X}^{0}$ to appear as inner $\mu$. The first two curved $A_{\infty}$-relations read

$$
\mu^{1}\left(\mu^{0}\right)=0, \quad \mu^{1}\left(\mu^{1}(a)\right)+(-1)^{\|a\|} \mu^{2}\left(\mu^{0}, a\right)+\mu^{2}\left(a, \mu^{0}\right)=0 .
$$

Curved $A_{\infty}$-categories are however ill-behaved: They do not have a notion of derived category. One may form the twisted completion, but if one enforces the Maurer-Cartan equation $\mu^{0}+\mu^{1}(\delta)+\ldots=0$, very few objects remain and the construction is probably not functorial. If one instead discards the MaurerCartan requirement for twisted complexes, one obtains a category $\operatorname{Tw} \mathcal{C}$ with curvature, the curvature given by the Maurer-Cartan formula $\mu^{0}+\mu^{1}(\delta)+\ldots$. This category however has no minimal model $\mathrm{H} \operatorname{Tw} \mathcal{C}$, because curvature prevents us from bringing $\mu^{1}$ to zero. In short, curved $A_{\infty}$-categories are not useful, except for the purpose of matrix factorizations.

We would like to stress that infinitesimal curvature however does not hurt. In section 5.3 we recall what infinitesimal curvature entails and we explain in section 5.5 and 8.3 that infinitesimally curved $A_{\infty}$-categories do have derived categories.

### 5.2 The completed tensor product

In this section, we recall the completed tensor products of the form $B \widehat{\otimes} X$, where $B$ is a local algebra and $X$ is a vector space. This serves as a preparation for section 5.3 where we define $A_{\infty}$-deformations.

Throughout this paper, we deform over local rings like $\mathbb{C} \llbracket q \rrbracket$. There are a few more conditions we put on the local ring: In order to work with the $A_{\infty}$-formalism we need the ring to be a $\mathbb{C}$-algebra. In order to speak of a special fiber, or algebraically of an $\mathfrak{m}$-adic leading term, we need to require that its residue field is $\mathbb{C}$ itself. As is customary in deformation theory, we shall also require that the ring be complete and Noetherian. We have decided to give the type of local rings a name in this paper:

Definition 5.4. A deformation base is a complete local Noetherian unital $\mathbb{C}$-algebra $B$ with residue field $B / \mathfrak{m}=\mathbb{C}$. The maximal ideal is always denoted $\mathfrak{m}$.

Remark 5.5. By the Cohen structure theorem, every deformation base is of the form $\mathbb{C} \llbracket x_{1}, \ldots, x_{n} \rrbracket / I$ with $I$ denoting some ideal.

The idea behind formal $A_{\infty}$-deformations is to tensor the hom spaces with a deformation base. One then looks at $A_{\infty}$-structures on the enlarged collection of hom spaces which reduce to the original $A_{\infty^{-}}$ structure once the maximal ideal is divided out. The construction of these tensored hom spaces makes use of the completed tensor product, which we now recall.

Definition 5.6. Let $B$ be a deformation base and $X$ a vector space. Then the completed tensor product $B \widehat{\otimes} X$ is the $B$-module limit

$$
B \widehat{\otimes} X=\lim \left(\ldots \rightarrow B / \mathfrak{m}^{1} \otimes X \rightarrow B / \mathfrak{m}^{0} \otimes X\right)
$$

For simplicity, we write $\mathfrak{m}^{k} X$ to denote the infinitesimal part $\mathfrak{m}^{k} X=\mathfrak{m}^{k} \widehat{\otimes} X \subseteq B \widehat{\otimes} X$.
Remark 5.7. In case $B=\mathbb{C} \llbracket q \rrbracket$, the completed tensor product $B \widehat{\otimes} X$ equals the even more well-known space $X \llbracket q \rrbracket$. The space $X \llbracket q \rrbracket$ consists of formal $X$-valued power series in one variable. This way $X \llbracket q \rrbracket$ becomes naturally a $\mathbb{C} \llbracket q \rrbracket$-module. The space is different from $\mathbb{C} \llbracket q \rrbracket \otimes X$. In fact, elements of $\mathbb{C} \llbracket q \rrbracket \otimes X$ are only those power series which can be written as a finite sum of pure tensors $a \otimes x$. Simply speaking, in $\mathbb{C} \llbracket q \rrbracket \otimes X$, the power series entries are divided into finitely many partitions in which all coefficients are interrelated. Meanwhile in $X \llbracket q \rrbracket$ any entries can be chosen at random. However if $X$ is finite-dimensional, then $X \llbracket q \rrbracket=\mathbb{C} \llbracket q \rrbracket \otimes X$.

There are two popular ways of defining formal deformations of an associative algebra $A$. The first definition asks for a product $\mu_{q}: A \otimes A \rightarrow B \widehat{\otimes} A$ and the second definition asks for a product $\mu_{q}$ : $(B \widehat{\otimes} A) \otimes(B \widehat{\otimes} A) \rightarrow B \widehat{\otimes} A$. The first definition immediately gives rise to a Maurer-Cartan element, while in the second definition associativity is formulated more naturally. In what follows, we shall explain briefly why both are equivalent.

Definition 5.8. Let $B$ be a deformation base. Then the $\mathfrak{m}$-adic topology on $B$ is the topology on $B$ generated by open neighborhoods $x+\mathfrak{m}^{k} \subseteq B$ for $x \in B$ and $k \in \mathbb{N}$. The $\mathfrak{m}$-adic topology on $B \widehat{\otimes} X$ is generated by the open neighborhoods $x+\mathfrak{m}^{k} \widehat{\otimes} X \subseteq B \widehat{\otimes} X$ for every $x \in B \widehat{\otimes} X$ and $k \in \mathbb{N}$. A map $\varphi: B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$ is continuous if it is continuous with respect to the $\mathfrak{m}$-adic topologies. A map $\varphi:\left(B \widehat{\otimes} X_{k}\right) \otimes \ldots \otimes\left(B \widehat{\otimes} X_{1}\right) \rightarrow B \widehat{\otimes} Y$ is continuous if for every $1 \leq i \leq k$ and every sequence of elements $x_{1}, \ldots, \hat{x}_{i}, \ldots, x_{k}$ the map

$$
\mu\left(x_{k}, \ldots,-, \ldots, x_{1}\right): B \widehat{\otimes} X_{i} \rightarrow B \widehat{\otimes} Y
$$

is continuous.
Remark 5.9. It is well-known that the $\mathfrak{m}$-adic topology turns $B \widehat{\otimes} X$ into a sequential Hausdorff space. It is also well-known that $B \widehat{\otimes} X$ can simultaneously be interpreted as limit and completion. More precisely, $B \widehat{\otimes} X$ is the completion of $B \otimes X$ with respect to the so-called $\mathfrak{m}$-adic metric on $B \otimes X$. For convenience, we may from time to time use expressions like $x=\mathcal{O}\left(\mathfrak{m}^{k}\right)$ to indicate $x \in \mathfrak{m}^{k} X$.
Remark 5.10. Every element in $B \widehat{\otimes} X$ can be written as a series $\sum_{i=0}^{\infty} m_{i} x_{i}$. Here $m_{i}$ is a sequence of elements $m_{i} \in \mathfrak{m}^{\rightarrow \infty}$ and $x_{i}$ is a sequence of elements $x_{i} \in X$. We have used the notation $m_{i} \in \mathfrak{m}^{\rightarrow \infty}$ to indicate that $m_{i} \in \mathfrak{m}^{k_{i}}$ for some sequence $\left(k_{i}\right) \subseteq \mathbb{N}$ with $k_{i} \rightarrow \infty$.
Remark 5.11. Let $\varphi: B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$ be a $B$-linear map. If $\varphi\left(\mathfrak{m}^{k} X\right) \subseteq \mathfrak{m}^{k} Y$, then $\varphi$ is continuous.
Whenever $X \rightarrow B \widehat{\otimes} Y$ is a linear map, we can extend it uniquely to a $B$-linear continuous map $B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$. Conversely, we can restrict any $B$-linear map $B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$ to a linear map $X \rightarrow B \widehat{\otimes} Y$. Restriction and extension are in fact inverse to each other, providing a one-to-one correspondence:

Lemma 5.12. Let $B$ be a deformation base and $X, Y$ be vector spaces. Then a $B$-linear map $B \widehat{\otimes} X \rightarrow$ $B \widehat{\otimes} Y$ is automatically continuous. There is a one-to-one correspondence between:

- linear maps $X \rightarrow B \widehat{\otimes} Y$,
- $B$-linear maps $B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$,
- $B$-linear continuous maps $B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$.

Proof. The proof consists of three parts: First, we show that every $B$-linear map $B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$ is continuous. Second, we recall how to extend a map $X \rightarrow B \widehat{\otimes} Y$ to $B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$. Third, we comment on the one-to-one aspect of the claim.

For the first step, we show that any $B$-linear map $\varphi: B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$ is continuous. Let $k \in \mathbb{N}$. We claim that $\varphi\left(\mathfrak{m}^{k} X\right) \subseteq \mathfrak{m}^{k} Y$. The idea is to exploit the Cohen structure theorem. Write $B=$
$\mathbb{C} \llbracket q_{1}, \ldots, q_{n} \rrbracket / I$, and regard the maximal ideal $\mathfrak{m}=\left(q_{1}, \ldots, q_{n}\right)$. With this in mind, we can write any element $x \in \mathfrak{m}^{k} X$ as a series

$$
x=\sum_{i=0}^{\infty} m_{i} \tilde{m}_{i} y_{i}
$$

Here $m_{i}$ is a monomial of degree $k$ in the variables $q_{1}, \ldots, q_{n}$, the letter $\tilde{m}_{i}$ denotes a sequence $\tilde{m}_{i} \in \mathfrak{m} \rightarrow \infty$, and $y_{i} \in Y$. We conclude

$$
x=\sum_{\substack{\text { monomials } M \\ \text { of degree } k}} M \sum_{\substack{i \geq 0 \\ m_{i}=M}} \tilde{m}_{i} y_{i}
$$

The outer sum is finite. For every monomial $M$ of degree $k$, the inner sum is an element $x_{M} \in B \widehat{\otimes} X$. We get that

$$
\varphi(x)=\sum_{\substack{\text { monomials } M \\ \text { of degree } k}} M \varphi\left(x_{M}\right)
$$

We conclude that $\varphi(x) \in \mathfrak{m}^{k} Y$. This shows $\varphi\left(\mathfrak{m}^{k} X\right) \subseteq \mathfrak{m}^{k} Y$. In particular, $\varphi$ is continuous.
For the second step, denote by $\pi_{i}$ the projection maps $\pi_{i}: B \widehat{\otimes} X \rightarrow B / \mathfrak{m}^{i} \otimes X$ or $B \widehat{\otimes} Y \rightarrow B / \mathfrak{m}^{i} \otimes Y$ and by $\pi_{i j}$ the projection maps $\pi_{i j}: B / \mathfrak{m}^{i} \otimes Y \rightarrow B / \mathfrak{m}^{j} \otimes Y$ for $i>j$. Note that $\left(\pi_{i j} \otimes \operatorname{Id}_{Y}\right) \circ \pi_{i}=\pi_{j}$ for $i>j$.

Let now $F: X \rightarrow B \widehat{\otimes} Y$ be a linear map. In order to define a map $\hat{F}: B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$, we shall build maps $\hat{F}_{i}: B \widehat{\otimes} X \rightarrow B / \mathfrak{m}^{i} \otimes Y$ and then use the universal property to combine them into $\hat{F}$.

Let us now construct auxiliary maps $F_{i}$ and the maps $\hat{F}_{i}$. Define $F_{i}: B / \mathfrak{m}^{i} \otimes X \rightarrow B / \mathfrak{m}^{i} \otimes Y$ as the $B / \mathfrak{m}^{i}$-linear extension of $\pi_{i} F: X \rightarrow B / \mathfrak{m}^{i} \otimes Y$. Then let $\hat{F}_{i}: B \widehat{\otimes} X \rightarrow B / \mathfrak{m}^{i} \otimes Y$ be the composition $F_{i} \pi_{i}$. More directly, we could write $\hat{F}_{i}(z)=F\left(\pi_{i}(z)\right)$, where on the right-hand side $F$ is interpreted $B / \mathfrak{m}^{i}$-linearly.

We claim that $\hat{F}_{j}=\left(\pi_{i j} \otimes \operatorname{Id}_{Y}\right) \circ \hat{F}_{i}$ for $i>j$. Indeed,

$$
\pi_{i j}\left(\hat{F}_{i}(z)\right)=\left(\pi_{i j} \otimes \operatorname{Id}_{Y}\right)\left(F\left(\pi_{i}(z)\right)=F\left(\left(\pi_{i j} \otimes \operatorname{Id}_{X}\right)\left(\pi_{i}(z)\right)\right)=F\left(\pi_{j}(z)\right)=\hat{F}_{j}(z)\right.
$$

This proves that the family of maps $\left\{\hat{F}_{i}\right\}_{i \in \mathbb{N}}$ is compatible with the projections $\pi_{i j} \otimes \operatorname{Id}_{Y}$. By the universal property of $B \widehat{\otimes} Y$, this family of maps factors through the limit $B \widehat{\otimes} Y$, yielding a $B$-linear map

$$
\hat{F}: B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y, \text { with } \pi_{i} \hat{F}=\hat{F}_{i} .
$$

The map $\hat{F}$ can also be written explicitly as follows: Let $x=\sum_{i=0}^{\infty} m_{i} x_{i}$ be an element of $B \widehat{\otimes} X$, with $m_{i} \in \mathfrak{m}^{\rightarrow \infty}$ and $x_{i} \in X$. Then $\hat{F}\left(\sum m_{i} x_{i}\right)=\sum m_{i} F\left(x_{i}\right)$.

As third step of the proof, we comment on the one-to-one aspect in the claim. The only remaining statement to explain is that for a given map $F: X \rightarrow B \widehat{\otimes} Y$, there is only one single $B$-linear continuous extension to a map $B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$. But this is obvious, since $B$-linearity already determines the value on $B \otimes X$ and continuity then determines the value on all of $B \widehat{\otimes} X$. This finishes the proof.

We shall provide a few more standard utilities. Recall that a map $\varphi: X \rightarrow Y$ of topological spaces is an embedding if it is a homeomorphism onto its image, the image being equipped with subspace topology. The leading term of a $B$-linear map $\varphi: B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$ is the map $\varphi_{0}: X \rightarrow Y$ given by the composition $\varphi_{0}=\left.\pi \varphi\right|_{X}$, where $\pi: B \widehat{\otimes} Y \rightarrow Y$ denotes the standard projection.

Lemma 5.13. Let $X, Y$ be vector spaces and $\varphi: B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$ be $B$-linear with injective leading term. Then $\varphi$ is an embedding and its image is closed. If the leading term is surjective, then $\varphi$ is an isomorphism.

Proof. Before we dive into the proof, we start with an observation regarding the leading term. Let $\varphi_{0}: X \rightarrow Y$ be the leading term of $\varphi$ and define for every $i \in \mathbb{N}$ the map

$$
\begin{equation*}
\varphi_{i}: \mathfrak{m}^{i} / \mathfrak{m}^{i+1} \otimes X \rightarrow \mathfrak{m}^{i} / \mathfrak{m}^{i+1} \otimes Y \tag{5.1}
\end{equation*}
$$

induced from $\varphi$. Our observation is that $\varphi_{i}$ is in fact equal to $\mathrm{id}_{\mathfrak{m}^{i} / \mathfrak{m}^{i+1}} \otimes \varphi_{0}$.
We are now ready to start the proof. First we show that $\varphi$ is injective. Second we show that $\varphi^{-1}: \operatorname{Im}(\varphi) \rightarrow B \widehat{\otimes} X$ is continuous. Third we show that $\operatorname{Im}(\varphi)$ is closed. Fourth we prove $\varphi$ surjective if $\varphi_{0}$ is surjective.

For the first part of the proof, we show that $\varphi$ injective. Let $x \in B \widehat{\otimes} X$ with $\varphi(x)=0$. By induction, we show that $x \in \mathfrak{m}^{i} X$ for every $i \in \mathbb{N}$. For $i=0$ this is clear. As induction hypothesis, assume that
$x \in \mathfrak{m}^{i} X$. Then $\varphi_{i}([x])=[\varphi(x)]=0$. Since $\varphi_{i}$ is injective, we obtain $x \in \mathfrak{m}^{i+1} X$. This finishes the induction. In consequence we have $x \in \mathfrak{m}^{i} X$ for every $i \in \mathbb{N}$ and therefore $x=0$. This proves $\varphi$ injective.

For the second part of the proof, we show that $\varphi^{-1}$ preserves the filtration by $\mathfrak{m}$. In other words, we show that $\varphi(x) \in \mathfrak{m}^{i} Y$ implies $x \in \mathfrak{m}^{i} X$. We do this by induction over $i \in \mathbb{N}$. For $i=0$ this is clear. Assume this statements holds for a certain $i \in \mathbb{N}$. Now let $\varphi(x) \in \mathfrak{m}^{i+1} Y$. Then $x \in \mathfrak{m}^{i} X$ by induction hypothesis. We get $\varphi_{i}([x])=0$, since $\varphi(x) \in \mathfrak{m}^{i+1} Y$. Since $\varphi_{i}$ is injective, we get $x \in \mathfrak{m}^{i+1} X$. This finishes the induction. Finally, we have shown $\varphi^{-1}\left(\mathfrak{m}^{i} Y\right) \subseteq \mathfrak{m}^{i} X$. This renders $\varphi^{-1}: \varphi(B \widehat{\otimes} X) \rightarrow B \widehat{\otimes} X$ continuous.

For the third part of the proof, we check that $\operatorname{Im}(\varphi)$ is closed. Since $B \widehat{\otimes} Y$ is a sequential space, it suffices to check that any sequence $\varphi\left(x_{n}\right) \subseteq \operatorname{Im}(\varphi)$ converging in $B \widehat{\otimes} Y$ converges in $\operatorname{Im}(\varphi)$. Pick a sequence $\left(x_{n}\right) \subseteq B \widehat{\otimes} X$ and assume $\varphi\left(x_{n}\right) \rightarrow y \in B \widehat{\otimes} Y$. This makes $\varphi\left(x_{n}\right)-y \in \mathfrak{m}^{\rightarrow \infty} Y$. We have $\varphi\left(x_{n}\right)-\varphi\left(x_{m}\right) \in \mathfrak{m}^{\rightarrow \infty} Y$ and by continuity of $\varphi^{-1}$ we get $x_{n}-x_{m} \in \mathfrak{m}^{\rightarrow \infty}$. In particuar, $x_{n}$ is a Cauchy sequence and converges to some $x \in B \widehat{\otimes} X$. We get $\varphi(x)=\lim \varphi\left(x_{n}\right)=y$ and therefore $y \in \operatorname{Im}(\varphi)$. This proves $\operatorname{Im}(\varphi)$ closed.

For the fourth part of the proof, consider $y \in B \widehat{\otimes} Y$. We construct inductively a sequence $\left(x_{n}\right)$ with $x_{n} \in \mathfrak{m}^{n} \widehat{\otimes} X$ such that $\varphi\left(\sum_{i=0}^{n} x_{i}\right)=\pi_{n}(y)$. For $i=0$, let $x_{0} \in X$ be defined as $x_{0}=\varphi_{0}^{-1}\left(\pi_{0}(y)\right)$. Now assume the sequence has been constructed for indices until $n \in \mathbb{N}$. Put $z:=y-\varphi\left(\sum^{n} x_{i}\right) \in \mathfrak{m}^{n} Y$. Since $\varphi_{0}$ is an isomorphism, the map $\varphi_{n}$ from (5.1) is an isomorphism as well. Put

$$
x_{n+1}^{\prime}:=\varphi_{n+1}^{-1}(z) \in \mathfrak{m}^{n+1} / \mathfrak{m}^{n+2} X
$$

Let $x_{n+1} \in \mathfrak{m}^{n+1} X$ be any lift of $x_{n+1}^{\prime}$. Then we have

$$
\varphi\left(\sum_{i=0}^{n+1} x_{i}\right)=y-z+\varphi\left(x_{n+1}\right)=y+\mathcal{O}\left(\mathfrak{m}^{n+2}\right)
$$

This finishes the inductive construction of the sequence $\left(x_{i}\right)$. Finally, the series $x=\sum x_{i}$ converges and $\varphi(x)=y$. This proves $\varphi$ surjective.

The pathway to defining $A_{\infty}$-deformations is now clear: When $\mathcal{C}$ is an $A_{\infty}$-category, a deformation of $\mathcal{C}$ will always be modeled on the collection of enlarged hom spaces $\left\{B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}(X, Y)\right\}_{X, Y \in \mathcal{C}}$. Any $B$-multilinear product on these hom spaces is automatically continuous. Similarly, functors of $A_{\infty^{-}}$ deformations will be defined as maps between tensor products of the enlarged hom spaces and will be automatically continuous as well.

### 5.3 Deformations of $A_{\infty}$-categories

In this section, we give a quick definition of curved deformations of $A_{\infty}$-categories. We have already used this notion in Paper I. The notion is not surprising and known to experts. Despite being uncomfortable to work with, curvature comes naturally via the Hochschild complex and is necessary if one aims at deformation theory invariant under quasi-equivalences. The curvature in $A_{\infty}$-deformations will always be infinitesimal, which we consider harmless in contrast with the different notion of curved $A_{\infty}$-categories 24.

Let us recall the setup of Hochschild cohomology in the classical case. Let $A$ be an associative algebra. As the reader probably knows, Hochschild cohomology $\operatorname{HH}^{2}(A)$ captures the associative deformations of the algebra over $\mathbb{C}[\varepsilon] /\left(\varepsilon^{2}\right)$. More precisely, a 2-cochain $\nu \in \operatorname{HC}^{2}(A)$ is a cocycle if and only if $\mu+\varepsilon \nu$ is an associative product on $A \oplus A \varepsilon$ (note $\varepsilon^{2}=0$ ). Classical Hochschild cohomology in other degrees than 2 helps characterize the deformation problem, but provides no actual deformations.

Since Hochschild's original definition in 1946, Hochschild cohomology has been generalized to $A_{\infty^{-}}$ categories. The trick is to use the same formula for the differential, with two adaptations: include also higher products $\mu^{k}$ instead of only $\mu^{2}$, and use grading induced from the shift $\mathcal{C}[1]$. We will make this precise in section 5.6

What should an $A_{\infty}$-deformation be then? The naive answer would be to allow deformations of all products $\mu^{k}$. This is however too shortsighted: We contend that for a notion of $A_{\infty}$-deformations whose infinitesimal deformations are classified by a cochain complex (or better a DGLA), the $A_{\infty}$-Hochschild complex is the natural choice. Its Maurer-Cartan elements consist however not only of deformations to the products $\mu^{k}$ with $k \geq 1$, but also introduce curvature $\mu^{0}$.

We are now ready to define $A_{\infty}$-deformations precisely.
Definition 5.14. Let $\mathcal{C}$ be an $A_{\infty}$ category with products $\mu$ and $B$ a deformation base. An (infinitesimally curved) deformation $\mathcal{C}_{q}$ of $\mathcal{C}$ consists of

- The same objects as $\mathcal{C}$,
- Hom spaces $\operatorname{Hom}_{\mathcal{C}_{q}}(X, Y)=B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}(X, Y)$ for $X, Y \in \mathcal{C}$,
- $B$-multilinear products of degree $2-k$

$$
\mu_{q}^{k}: \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{k}, X_{k+1}\right) \otimes \ldots \otimes \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{1}, X_{2}\right) \rightarrow \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{1}, X_{k+1}\right), k \geq 1
$$

- Curvature of degree 2 for every object $X \in \mathcal{C}$

$$
\mu_{q, X}^{0} \in \mathfrak{m} \operatorname{Hom}_{\mathcal{C}_{q}}(X, X)
$$

such that $\mu_{q}$ reduces to $\mu$ once the maximal ideal $\mathfrak{m} \subseteq R$ is divided out, and $\mu_{q}$ satisfies the curved $A_{\infty}$ $\left(c A_{\infty}\right)$ relations

$$
\sum_{k \geq l \geq m \geq 0}(-1)^{\left\|a_{m}\right\|+\ldots+\left\|a_{1}\right\|} \mu_{q}\left(a_{k}, \ldots, \mu_{q}\left(a_{l}, \ldots\right), a_{m}, \ldots, a_{1}\right)=0
$$

The deformation is unital if the deformed higher products still satisfy the unitality axioms

$$
\mu_{q}^{2}\left(a, \mathrm{id}_{X}\right)=a, \mu_{q}^{2}\left(\operatorname{id}_{Y}, a\right)=(-1)^{|a|} a, \mu_{q}^{\geq 3}\left(\ldots, \mathrm{id}_{X}, \ldots\right)=0
$$

We use the terms $A_{\infty}$-deformation and deformation of an $A_{\infty}$-category interchangeably. Whenever we speak of deformations of $A_{\infty}$-categories in this paper, they are allowed to be (infinitesimally) curved.

Remark 5.15. As explained in section 5.2 , the product $\mu_{q}^{k}$ is automatically m-adically continuous. Spelling out this continuity requirement, the datum of the map $\mu^{k}$ is equivalent to the datum of merely multilinear maps

$$
\mu_{q}: \operatorname{Hom}_{\mathcal{C}}\left(X_{k}, X_{k+1}\right) \otimes \ldots \otimes \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, X_{2}\right) \rightarrow B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, X_{k+1}\right)
$$

There is a notion for two objects to be quasi-isomorphic in an $A_{\infty}$-deformation. We provide here an ad-hoc definition which seems odd at first, but we will encounter evidence in section 8 and section 9 which supports correctness of the definition.

Definition 5.16. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Let $X, Y \in \mathcal{C}_{q}$ be two objects. Then $X$ and $Y$ are quasi-isomorphic if they are quasi-isomorphic in $\mathcal{C}$.

### 5.4 Functors between $A_{\infty}$-deformations

In this section, we define the notion of (infinitesimally curved) $A_{\infty}$-functors. These functors serve as a framework for gauge equivalences, quasi-equivalences and pushforwards of $A_{\infty}$-deformations. This class of functors is presumably known to experts. Our sign conventions are those of 35 and 16 , Section 3.1.4/symplectic].

Recall that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ of $A_{\infty}$-categories is a map which intertwines the products of $\mathcal{C}$ and $\mathcal{D}$ :

Definition 5.17. Let $\mathcal{C}$ and $\mathcal{D}$ be $A_{\infty}$-categories. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of a map $F: \mathrm{Ob}(\mathcal{C}) \rightarrow$ $\mathrm{Ob}(\mathcal{D})$ together with for every $k \geq 1$ a degree $1-k$ multilinear map

$$
F^{k}: \operatorname{Hom}_{\mathcal{C}}\left(X_{k}, X_{k+1}\right) \otimes \ldots \otimes \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, X_{2}\right) \rightarrow \operatorname{Hom}_{\mathcal{C}}\left(F X_{1}, F X_{k+1}\right)
$$

such that the $A_{\infty}$-functor relations hold:

$$
\begin{aligned}
& \sum_{0 \leq j<i \leq k}(-1)^{\left\|a_{j}\right\|+\ldots+\left\|a_{1}\right\|} F\left(a_{k}, \ldots, a_{i+1}, \mu\left(a_{i}, \ldots, a_{j+1}\right), a_{j}, \ldots, a_{1}\right) \\
&=\sum_{\substack{l \geq 0 \\
1=j_{1}<\ldots<j_{l} \leq k}} \mu\left(F\left(a_{k}, \ldots, a_{j_{l}}\right), \ldots, F\left(\ldots, a_{j_{2}}\right), F\left(\ldots, a_{j_{1}}\right)\right) .
\end{aligned}
$$

When $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{E}$ are $A_{\infty}$-functors, then their composition $G F$ is given on objects by $G \circ F: \mathrm{Ob}(\mathcal{C}) \rightarrow \mathrm{Ob}(\mathcal{E})$ and on morphisms by

$$
(G F)\left(a_{k}, \ldots, a_{1}\right)=\sum G\left(F\left(a_{k}, \ldots\right), \ldots, F\left(\ldots, a_{1}\right)\right)
$$

We are now ready to explain the natural extension of $A_{\infty}$-functors to the deformed case.
Definition 5.18. Let $\mathcal{C}, \mathcal{D}$ be two $A_{\infty}$-categories and $\mathcal{C}_{q}, \mathcal{D}_{q}$ deformations. A functor of deformed $A_{\infty}$-categories consists of a map $F_{q}: \operatorname{Ob}(\mathcal{C}) \rightarrow \operatorname{Ob}(\mathcal{D})$ together with for every $k \geq 1$ a $B$-multilinear degree $1-k$ map

$$
F_{q}^{k}: \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{k}, X_{k+1}\right) \otimes \ldots \otimes \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{1}, X_{2}\right) \rightarrow \operatorname{Hom}_{\mathcal{D}_{q}}\left(F_{q} X_{1}, F_{q} X_{k+1}\right)
$$

and infinitesimal curvature $F_{q, X}^{0} \in \mathfrak{m} \operatorname{Hom}_{\mathcal{D}}^{1}\left(F_{q} X, F_{q} X\right)$ for every $X \in \mathcal{C}$, such that the curved $A_{\infty}$-functor relations hold:

$$
\begin{aligned}
\sum_{0 \leq j \leq i \leq k}(-1)^{\left\|a_{j}\right\|+\ldots+\left\|a_{1}\right\|} F_{q}\left(a_{k}, \ldots,\right. & \left.a_{i+1}, \mu_{q}\left(a_{i}, \ldots, a_{j+1}\right), a_{j}, \ldots, a_{1}\right) \\
& =\sum_{\substack{l \geq 0 \\
1=j_{1} \leq \cdots \leq j_{l} \leq k}} \mu_{q}\left(F_{q}\left(a_{k}, \ldots, a_{j_{l}}\right), \ldots, F_{q}\left(\ldots, a_{j_{2}}\right), F_{q}\left(\ldots, a_{j_{1}}\right)\right) .
\end{aligned}
$$

Remark 5.19. A functor $F_{q}$ of deformed $A_{\infty}$-categories consists of maps between hom spaces which are allowed to have deformed (nonconstant) terms themselves. As we have seen insection 5.2 the maps $F_{q}^{k}$ are automatically continuous. Apart from the components $F_{q}^{\geq 1}$, the functor is allowed to have infinitesimal curvature $F_{q}^{0}$. This curvature is a feature of the deformed world where infinitesimal curvature is not only welcome, but is necessary. The first two curved $A_{\infty}$-functor relations read

$$
\begin{aligned}
F_{q}^{0}+F_{q}^{1}\left(\mu_{\mathcal{C}_{q}, X}^{0}\right)= & \mu_{\mathcal{D}_{q}}^{1}\left(F_{q, X}^{0}\right), \\
F_{q}^{1}\left(\mu_{\mathcal{C}_{q}}^{1}(a)\right)+(-1)^{\|a\|} F_{q}^{2}\left(\mu_{\mathcal{C}_{q}, Y}^{0}, a\right)+F_{q}^{2}\left(a, \mu_{\mathcal{C}_{q}, X}^{0}\right)= & \mu_{\mathcal{D}_{q}}^{1}\left(F_{q}^{1}(a)\right)+\mu_{\mathcal{D}_{q}}^{2}\left(F_{q, Y}^{0}, F_{q}^{1}(a)\right) \\
& +\mu_{\mathcal{D}_{q}}^{2}\left(F_{q}^{1}(a), F_{q, X}^{0}\right), \quad \forall a: X \rightarrow Y .
\end{aligned}
$$

We can interpret a functor $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ as an extension of an ordinary $A_{\infty}$-functor $\mathcal{C} \rightarrow \mathcal{D}$. Indeed, when forgetting the terms of $F=\left\{F_{q}^{k}\right\}_{k \geq 0}$ in higher $\mathfrak{m}$-adic order, we get a collection of maps $F=\left\{F^{k}\right\}_{k \geq 1}$. Since $F_{q}$ satisfies the curved $A_{\infty}$-functor relations, its reduction $F$ satisfies the ordinary $A_{\infty}$-functor relations. We fix this terminology as follows:

Definition 5.20. Let $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ be a functor of deformed $A_{\infty}$-categories. Then its leading term is the functor $F: \mathcal{C} \rightarrow \mathcal{D}$ obtained by dividing out the maximal ideal $\mathfrak{m}$.

There are several notions for functors between $A_{\infty}$-categories to be equivalences, which we recall as follows:

Definition 5.21. Let $\mathcal{C}$ and $\mathcal{D}$ be two $A_{\infty}$-categories and $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then $F$ is

- an isomorphism if it is an isomorphism on object level and $F^{1}: \operatorname{Hom}(X, Y) \rightarrow \operatorname{Hom}(F X, F Y)$ is an isomorphism for every $X, Y \in \mathcal{C}$,
- a quasi-isomorphism if the induced functor $\mathrm{H} F: \mathrm{H} \mathcal{C} \rightarrow \mathrm{H} \mathcal{D}$ is an isomorphism,
- a quasi-equivalence if $(\mathrm{H} F)^{1}: \operatorname{Hom}_{\mathrm{H} \mathcal{C}}(X, Y) \rightarrow \operatorname{Hom}_{\mathrm{H} \mathcal{D}}(F X, F Y)$ is an isomorphism for every $X, Y \in \mathcal{C}$ and if $\mathrm{H} F$ reaches every object in $\mathcal{D}$ up to quasi-isomorphism,
- a derived equivalence if the induced functor $\mathrm{HTw} F: \mathrm{HTw} \mathcal{C} \rightarrow \mathrm{HTw} \mathcal{D}$ is an equivalence.

There are several notions for $A_{\infty}$-categories to be equivalent, without explicit reference to a functor:
Definition 5.22. Let $\mathcal{C}$ and $\mathcal{D}$ be two $A_{\infty}$-categories. Then $\mathcal{C}$ and $\mathcal{D}$ are

- isomorphic if there exists an isomorphism $F: \mathcal{C} \rightarrow \mathcal{D}$,
- quasi-isomorphic if there exists a quasi-isomorphism $F: \mathcal{C} \rightarrow \mathcal{D}$,
- quasi-equivalent if there exists a quasi-equivalence $F: \mathcal{C} \rightarrow \mathcal{D}$,
- derived equivalent if $\operatorname{Tw} \mathcal{C}$ and $\operatorname{Tw} \mathcal{D}$ are quasi-equivalent.

Remark 5.23. No functor in either direction is required for two $A_{\infty}$-categories to be derived equivalent. The contrast with quasi-isomorphisms is due to the lack of functors $\operatorname{Tw} \mathcal{C} \rightarrow \mathcal{C}$, in contrast to the existence of natural functors $\mathrm{H} \mathcal{C} \rightarrow \mathcal{C}$ and $\mathcal{C} \rightarrow \mathrm{H} \mathcal{C}$. There are many more equivalent and equally esthetic ways to define every of the above notions, so we have only presented a selection.

We now pull the notions of equivalences between $A_{\infty}$-categories over to the world of deformed $A_{\infty^{-}}$ categories. The main idea is to declare a functor $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ an isomorphism if its leading term $F: \mathcal{C} \rightarrow \mathcal{D}$ is an isomorphism. Note that this definition is not vacuous: It still requires the (curved) $A_{\infty}$-relations for $F_{q}$, but displays the isomorphism property as a side issue which is dealt with on the leading term part.

Definition 5.24. Let $\mathcal{C}, \mathcal{D}$ be $A_{\infty}$-categories and $\mathcal{C}_{q}, \mathcal{D}_{q}$ deformations. Let $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ be a functor of deformed $A_{\infty}$-categories and denote by $F: \mathcal{C} \rightarrow \mathcal{D}$ its leading term. Then $F_{q}$ is

- an isomorphism if $F$ is an isomorphism,
- a quasi-isomorphism if $F$ is a quasi-isomorphism,
- a quasi-equivalence if $F$ is a quasi-equivalence,
- a derived equivalence if $F$ is a derived equivalence.

Two deformed $A_{\infty}$-categories $\mathcal{C}_{q}$ and $\mathcal{D}_{q}$ are

- isomorphic if there is an isomorphism $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$,
- quasi-isomorphic if there is a quasi-isomorphism $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$,
- quasi-equivalent if there is a quasi-equivalence $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$,
- derived equivalent if $\operatorname{Tw} \mathcal{C}_{q}$ and $\operatorname{Tw} \mathcal{D}_{q}$ are quasi-equivalent.

We explain in Lemma 5.57 why relations such as quasi-equivalence and derived equivalence are equivalence relations among deformed $A_{\infty}$-categories. Among the $A_{\infty}$-deformations of one single category $\mathcal{C}$, there is one further notion of equivalence, known as gauge equivalence:

Definition 5.25. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}, \mathcal{C}_{q}^{\prime}$ be deformations. Then a gauge equivalence between $\mathcal{C}_{q}$ and $\mathcal{C}_{q}^{\prime}$ is a functor $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{C}_{q}^{\prime}$ of deformed $A_{\infty}$-categories whose leading term $F: \mathcal{C} \rightarrow \mathcal{C}$ is the identity.

We elaborate gauge equivalence further in the context of the Hochschild DGLA in section 5.7

### 5.5 Twisted completion

In this section, we recall the twisted completion construction and extend it to $A_{\infty}$-deformations. The ordinary case is standard, see for example 35, [16. Our definition in the deformed case may differ from what readers expect. We follow the sign convention of 35 . For the purposes of this section, we denote by [1] the right-shift, as opposed to the more common interpretation as left-shift.

To get started, let us recall the definition of additive completion for ordinary $A_{\infty}$-categories. This category consists of formal sums of shifted objects. The hom space between two objects consists of matrices of morphisms between the summands:

Definition 5.26. Let $\mathcal{C}$ be an $A_{\infty}$ category with product $\mu_{\mathcal{C}}$. The additive completion $\operatorname{Add} \mathcal{C}$ of $\mathcal{C}$ is the category of formal sums of shifted objects of $\mathcal{C}$ :

$$
A_{1}\left[k_{1}\right] \oplus \ldots \oplus A_{n}\left[k_{n}\right] .
$$

The hom space between two such objects $X=\bigoplus A_{i}\left[k_{i}\right]$ and $Y=\bigoplus B_{i}\left[m_{i}\right]$ is

$$
\operatorname{Hom}_{\text {Add } \mathcal{C}}(X, Y)=\bigoplus_{i, j} \operatorname{Hom}_{\mathcal{C}}\left(A_{i}, B_{j}\right)\left[m_{j}-k_{i}\right]
$$

Here [-] denotes the right-shift. The products on Add $\mathcal{C}$ are given by multilinear extensions of

$$
\mu_{\mathrm{Add} \mathcal{C}}^{k}\left(a_{k}, \ldots, a_{1}\right)=\sum(-1)^{\sum_{j<i}\left\|a_{i}\right\| l_{j}} \mu_{\mathcal{C}}^{k}\left(a_{k}, \ldots, a_{1}\right) .
$$

Here each $a_{i}$ lies in some $\operatorname{Hom}\left(X_{i}\left[k_{i}\right], X_{i+1}\left[k_{i+1}\right]\right)$. The integer $l_{i}$ denotes the difference $k_{i+1}-k_{i}$ between the shifts and the degree $\left\|a_{i}\right\|$ is the degree of $a_{i}$ as element of $\operatorname{Hom}_{\mathcal{C}}\left(X_{i}, X_{i+1}\right)$.

We now extend the notion of additive completion to the case of $A_{\infty}$-deformations. Let $\mathcal{C}$ be an $A_{\infty^{-}}$ category and $\mathcal{C}_{q}$ a deformation. The aim is to define an additive completion Add $\mathcal{C}_{q}$ in such a way that it is a deformation of $\operatorname{Add} \mathcal{C}$. This is straight-forward:

Definition 5.27. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Then the additive completion $\operatorname{Add} \mathcal{C}_{q}$ is the deformation of $\operatorname{Add} \mathcal{C}$ given by the following deformed product:

$$
\mu_{\operatorname{Add} \mathcal{C}_{q}}^{k}\left(\alpha_{k}, \ldots, \alpha_{1}\right)=\sum(-1)^{\sum_{j<i}\left\|\alpha_{i}\right\| l_{j}} \mu_{\mathcal{C}_{q}}^{k}\left(\alpha_{k}, \ldots, \alpha_{1}\right),
$$

with the same sign convention as in the non-deformed case.
Remark 5.28. The only difference between $\mu_{\mathrm{Add} \mathcal{C}}$ and $\mu_{\mathrm{Add}} \mathcal{C}_{q}$ lies in using the non-deformed product $\mu_{\mathcal{C}}$ for the former and the deformed product $\mu_{\mathcal{C}_{q}}$ for the latter.

Let us now recall twisted completion for ordinary $A_{\infty}$-categories. The idea is to form virtual complexes of objects of $\mathcal{C}$ :

Definition 5.29. Let $\mathcal{C}$ be an $A_{\infty}$-category. A twisted complex in $\mathcal{C}$ is an object $X \in \operatorname{Add} \mathcal{C}$ together with a morphism $\delta \in \operatorname{Hom}^{1}(X, X)$ of degree 1 such that $\delta$ is strictly upper triangular and satisfies the Maurer-Cartan equation:

$$
\operatorname{MC}(\delta):=\mu^{1}(\delta)+\mu^{2}(\delta, \delta)+\ldots=0
$$

We frequently refer to the morphism $\delta$ as the twisted differential or $\delta$-differential. The $A_{\infty}$-category $\operatorname{Tw} \mathcal{C}$ is the category whose objects are twisted complexes. Its hom spaces are the same as for the additive completion:

$$
\operatorname{Hom}_{\mathrm{Tw}} \mathcal{C}(X, Y)=\operatorname{Hom}_{\mathrm{Add} \mathcal{C}}(X, Y)
$$

The products on $\operatorname{Tw} \mathcal{C}$ of $\mathcal{C}$ are given by embracing with $\delta$ 's:

$$
\mu_{\mathrm{Tw} \mathcal{C}}^{k}\left(\alpha_{k}, \ldots, \alpha_{1}\right)=\sum_{n_{0}, \ldots, n_{k} \geq 0} \mu_{\operatorname{Add} \mathcal{C}}(\underbrace{\delta, \ldots, \delta}_{n_{k}}, \alpha_{k}, \ldots, \alpha_{1}, \underbrace{\delta, \ldots, \delta}_{n_{0}}) .
$$

Remark 5.30. Upper triangularity of $\delta$ ensures that the product $\mu_{\mathrm{Tw}} \mathcal{C}$ is well-defined.
We now extend the twisted completion construction to the case of $A_{\infty}$-deformations. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. The aim is to define a twisted completion Tw $\mathcal{C}_{q}$ in such a way that it is a deformation of $\operatorname{Tw} \mathcal{C}$. In particular, the category $\operatorname{Tw} \mathcal{C}_{q}$ should have the same objects as $\operatorname{Tw} \mathcal{C}$. With this in mind, we define:

Definition 5.31. Let $\mathcal{C}$ be an $A_{\infty}$ category with products $\mu_{\mathcal{C}}$ and $\mathcal{C}_{q}$ a deformation with products $\mu_{\mathcal{C}_{q}}$. Then the twisted completion $\operatorname{Tw} \mathcal{C}_{q}$ is the (possibly curved) deformation of $\operatorname{Tw} \mathcal{C}$ given by the deformed products

$$
\mu_{\mathrm{Tw} \mathcal{C}_{q}}^{k}\left(\alpha_{k}, \ldots, \alpha_{1}\right)=\sum_{n_{0}, \ldots, n_{k} \geq 0} \mu_{\operatorname{Add} \mathcal{C}_{q}}(\underbrace{\delta, \ldots, \delta}_{n_{k}}, \alpha_{k}, \ldots, \alpha_{1}, \underbrace{\delta, \ldots, \delta}_{n_{0}}) .
$$

Remark 5.32. As one may have expected, the product $\mu_{\mathrm{Tw} \mathcal{C}_{q}}$ now simply uses $\mu_{\mathcal{C}_{q}}$ instead of $\mu_{\mathcal{C}}$. The set of objects of $\operatorname{Tw} \mathcal{C}_{q}$ may however be surprising: The objects of $\operatorname{Tw} \mathcal{C}_{q}$ are not formed with twisted differentials $\delta \in \operatorname{Hom}_{\mathcal{C}_{q}}^{1}(X, X)$. Instead, the objects of $\operatorname{Tw} \mathcal{C}_{q}$ are twisted complexes $(X, \delta) \in \operatorname{Tw} \mathcal{C}$, in other words, their $\delta$-differential must lie in $\operatorname{Hom}_{\mathcal{C}}^{1}(X, X)$. It is easily checked that $\mu_{\mathrm{Tw}} \mathcal{C}_{q}$ satisfies the (curved) $A_{\infty}$-axioms, rendering $\operatorname{Tw} \mathcal{C}_{q}$ a genuine $A_{\infty}$-deformation of $\mathrm{Tw} \mathcal{C}$.

Remark 5.33. Denoting the twisted completion of $\mathcal{C}_{q}$ by $\operatorname{Tw} \mathcal{C}_{q}$ constitutes a slight abuse of notation: " $\mathrm{Tw} \mathcal{C}_{q}$ " suggests that twisted complexes are to be taken with the $\delta$-differential formed from values in $\mathcal{C}_{q}$, which is not the case. A more proper notation for the twisted completion of $\mathcal{C}_{q}$ would have been $(\mathrm{Tw} \mathcal{C})_{q}$, which we however found too complicated.

Remark 5.34. Typical objects in $\operatorname{Tw} \mathcal{C}_{q}$ have curvature. Indeed, according to Definition 5.31, an object $(X, \delta) \in \operatorname{Tw} \mathcal{C}_{q}$ has curvature

$$
\mu_{(X, \delta)}^{0}=\mu_{X, \operatorname{Add} \mathcal{C}_{q}}^{0}+\mu_{\operatorname{Add} \mathcal{C}_{q}}^{1}(\delta)+\mu_{\operatorname{Add} \mathcal{C}_{q}}^{2}(\delta, \delta)+\ldots
$$

The curvature $\mu_{X, \operatorname{Add} \mathcal{C}_{q}}^{0}$ can be spelled out more concretely as the sum of the curvatures of all constituents $A_{i}$ of $X$. The differential $\mu_{\mathrm{Add}_{\mathcal{C}}}^{1}(\delta)$ is concretely the sum of the deformed differentials $\mu_{\mathcal{C}_{q}}^{1}$ applied to each entry of $\delta$ as a matrix. Whatever the value of $\mu_{(X, \delta)}^{0}$ adds up to, the reader can see that it does typically not vanish because the twisted differential $\delta \in \operatorname{Hom}^{1}(X, X)$ only satisfies the Maurer-Cartan equation with respect to the non-deformed differential $\mu_{\mathcal{C}}^{1}$.

Remark 5.35. A popular way to form twisted completions of curved categories is to pick only curvaturefree twisted complexes. This might be the way to luck in case of curved $A_{\infty}$-categories, because a curved twisted completion cannot be passed to the minimal model. In contrast, for $A_{\infty}$-deformations, curvature on $\operatorname{Tw} \mathcal{C}_{q}$ is not problematic at all. Picking all twisted complexes of $\mathcal{C}$ is in fact necessary in order to render $\operatorname{Tw} \mathcal{C}_{q}$ a deformation of $\operatorname{Tw} \mathcal{C}$.

Twisted completion in the case of deformed $A_{\infty}$-categories offers properties familiar from the twisted completion of ordinary $A_{\infty}$-categories, for instance:
Lemma 5.36. The inclusion $\mathcal{C}_{q} \subseteq \operatorname{Tw} \mathcal{C}_{q}$ is a derived equivalence of deformed $A_{\infty}$-categories.
Proof. Due to our definition of quasi-equivalences of deformed $A_{\infty}$-categories, this statement is trivial. Indeed, regard the inclusion functor $F_{q}: \mathcal{C}_{q} \rightarrow \operatorname{Tw} \mathcal{C}_{q}$. This is by definition a functor of deformed $A_{\infty^{-}}$ categories. Its leading term is the standard inclusion $F: \mathcal{C} \rightarrow \operatorname{Tw} \mathcal{C}$. Ultimately, the functor $F$ is a derived equivalence and we conclude that $F_{q}$ is a derived equivalence according to Definition 5.24.

Remark 5.37. It is possible to define a variant $\mathrm{Tw}^{\prime} \mathcal{C}_{q}$ of the twisted completion of $A_{\infty}$-deformations by allowing additional infinitesimal entries anywhere in the $\delta$-matrix. Let us describe this version in detail: The objects of $\mathrm{Tw}^{\prime} \mathcal{C}_{q}$ shall be pairs

$$
\left(X, \delta=\delta_{0}+\delta^{\prime}\right), \quad X \in \operatorname{Add} \mathcal{C}, \quad \delta_{0} \in \operatorname{Hom}_{\mathcal{C}}^{1}(X, X), \quad \delta^{\prime} \in \mathfrak{m} \operatorname{Hom}_{\mathcal{C}}^{1}(X, X)
$$

Here we require only the leading part $\delta_{0}$ to be upper triangular and satisfy the Maurer-Cartan equation with respect to $\mu_{\mathcal{C}}$. The infinitesimal part $\delta^{\prime}$ can also lie below the diagonal.

The hom spaces and higher products of $\mathrm{Tw}^{\prime} \mathcal{C}_{q}$ shall be defined by embracing with $\delta$ as in Definition 5.31. Let us check that products of this deformed $A_{\infty}$-category $\mathrm{Tw}^{\prime} \mathcal{C}_{q}$ are still well-defined: Regard a sequence of $k$ compatible entries of $\delta$. Then at least $k-d$ of them are infinitesimal, where $d$ is the dimension of the $\delta$-matrix. We conclude that a term

$$
\mu_{q}(\underbrace{\delta, \ldots, \delta}_{k_{n+1} \text { times }}, a_{n}, \underbrace{\delta, \ldots, \delta}_{k_{n} \text { times }}, \ldots, a_{1}, \underbrace{\delta, \ldots, \delta}_{k_{1} \text { times }})
$$

lies in the $K:=\left(k_{1}+\ldots+k_{n+1}-(n+1) d\right)$-th power of the maximal ideal $\mathfrak{m}$. When the number $k_{1}+\ldots+k_{n+1}$ of total $\delta$ insertions goes to infinity, this exponent $K$ tends towards infinity as well. This renders all products in $\mathrm{Tw}^{\prime} \mathcal{C}_{q}$ well-defined.

The category $\mathrm{Tw}^{\prime} \mathcal{C}_{q}$ is clearly a deformed $A_{\infty}$-category. When dividing out the maximal ideal we do not exactly reach $\operatorname{Tw} \mathcal{C}$ though, but a version of $\operatorname{Tw} \mathcal{C}$ with lots of isomorphic objects: one copy for every choice of infinitesimal terms being added in the $\delta$ matrix.
Lemma 5.38. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Then $\operatorname{Tw} \mathcal{C}_{q}$ and $\mathrm{Tw}^{\prime} \mathcal{C}_{q}$ are quasiequivalent as deformed $A_{\infty}$-categories. Moreover, let $S:=\left\{\left(X_{i}, \delta_{i}\right)\right\}_{i=1, \ldots, n}$ be a collection of objects in $\operatorname{Tw} \mathcal{C}_{q}$ and $\delta_{i}^{\prime} \in \mathfrak{m} \operatorname{End}^{1}\left(X_{i}\right)$ be infinitesimal terms. Then the category

$$
S^{\prime}=\left\{\left(X_{i}, \delta_{i}+\delta_{i}^{\prime}\right)\right\}_{i=1, \ldots, n} \subseteq \operatorname{Tw}^{\prime} \mathcal{C}_{q}
$$

is gauge equivalent to $S$.
Proof. For the first part, regard the inclusion $\operatorname{Tw} \mathcal{C}_{q} \subseteq \mathrm{Tw}^{\prime} \mathcal{C}_{q}$. Upon dividing out the maximal ideal $\mathfrak{m}$, this inclusion reduces to the inclusion of $\operatorname{Tw} \mathcal{C}$ into a version of $\operatorname{Tw} \mathcal{C}$ with lots of copies of objects. We conclude that the inclusion $\operatorname{Tw} \mathcal{C}_{q} \subseteq \mathrm{Tw}^{\prime} \mathcal{C}_{q}$ is a quasi-equivalence of deformed $A_{\infty}$-categories.

For the second part, build the functor

$$
F_{q}: S^{\prime} \rightarrow S, \quad F_{q,\left(X_{i}, \delta_{i}+\delta_{i}^{\prime}\right)}^{0}=\delta_{i}^{\prime}, \quad F_{q}^{1}=\mathrm{Id}, \quad F_{q}^{\geq 2}=0
$$

This functor is the identity on objects and indeed satisfies the curved $A_{\infty}$-functor relations:

$$
\begin{aligned}
\sum \mu_{S}(\underbrace{F_{q}^{0}, \ldots, F_{q}^{0}}_{\geq 0 \text { times }}, F_{q}^{1}\left(a_{k}\right) & , \underbrace{F_{q}^{0}, \ldots, F_{q}^{0}}_{\geq 0 \text { times }}, \ldots, F^{1}\left(a_{1}\right), \underbrace{F_{q}^{0}, \ldots, F_{q}^{0}}_{\geq 0 \text { times }}) \\
& =\sum \mu_{\text {Add } \mathcal{C}_{q}}(\underbrace{\delta_{i}, \delta_{i}^{\prime}, \ldots}_{\text {any mix }}, a_{k}, \ldots, a_{1}, \underbrace{\delta_{i}, \delta_{i}^{\prime}, \ldots}_{\text {any mix }}) \\
& =\sum \mu_{\text {Add } \mathcal{C}_{q}}(\underbrace{\delta, \ldots, \delta}_{\geq 0 \text { times }}, a_{k}, \underbrace{\delta, \ldots, \delta}_{\geq 0 \text { times }}, \ldots, a_{1}, \underbrace{\delta, \ldots, \delta}_{\geq 0 \text { times }}) \\
& =F_{q}^{1}\left(\mu_{S^{\prime}}\left(a_{k}, \ldots, a_{1}\right) .\right.
\end{aligned}
$$

This shows that $F_{q}$ is a gauge-equivalence $S^{\prime} \rightarrow S$.

### 5.6 The Hochschild DGLA

In this section, we recall how to view $A_{\infty}$-deformations from the DGLA point of view. The material is all known to experts and nicely shows how curvature enters the picture. Useful references include 53 , Chapter V] and 72 .

Before we recall the concept of DGLA and Maurer-Cartan elements, let us summarize the philosophy: The aim is to capture every deformation problem by a DGLA. The solutions of the deformation problem should then correspond to Maurer-Cartan elements of the DGLA. This empowers the mathematician to use the force of DGLA theory. Standard questions in DGLAs include: to find a quasi-isomorphism $F: L \rightarrow L^{\prime}$ between two DGLAs, or to classify all Maurer-Cartan elements of $L$ up to gauge equivalence.

$$
\text { deformation problem } \underbrace{\text { 1. reformulation }}_{\text {3. harvest }} \text { DGLA }
$$

2. algebraic power
$\qquad$

We are now ready to recall the notion of DGLAs and Maurer-Cartan elements.
Definition 5.39. A DG Lie algebra (DGLA) is a $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded vector space $L$ together with

- a differential $d: L^{i} \rightarrow L^{i+1}$,
- a bracket $[-,-]: L \times L \rightarrow L$ of degree zero,
satisfying skew-symmetry, Leibniz rule and Jacobi identity:

$$
\begin{aligned}
& {[a, b]=(-1)^{|a||b|+1}[b, a]} \\
& d([a, b])=[d a, b]+(-1)^{|a|}[a, d b] \\
& (-1)^{|a||c|}[a,[b, c]]+(-1)^{|b||a|}[b,[c, a]]+(-1)^{|c||b|}[c,[a, b]]=0 .
\end{aligned}
$$

For example, the bracket $[-,-]$ is commutative on odd elements. With this consideration, we recall the definition of Maurer-Cartan elements:
Definition 5.40. Let $L$ be a DGLA and $B$ a deformation base. Regard the tensored DGLA $B \widehat{\otimes} L$ with differential $d$ and bracket $[-,-]$ simply extended continuously. A Maurer-Cartan element of $L$ over $B$ is an element $\nu \in B \widehat{\otimes} L^{1}$ which satisfies the Maurer-Cartan equation

$$
d \nu+\frac{1}{2}[\nu, \nu]=0 .
$$

The set of Maurer-Cartan elements of $L$ over $B$ is denoted $\operatorname{MC}(L, B)$. In case the DGLA $L$ is $\mathbb{Z} / 2 \mathbb{Z}$ graded, a Maurer-Cartan element is supposed to lie in $B \widehat{\otimes} L^{\text {odd }}$.

In the rest of this section, we will make sense of these definitions in the case of the so-called Hochschild DGLA:
Definition 5.41. Let $\mathcal{C}$ be a $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded $A_{\infty}$-category. Then its Hochschild complex $\operatorname{HC}(\mathcal{C})$ is the graded vector space

$$
\operatorname{HC}(\mathcal{C})=\prod_{\substack{X_{1}, \ldots, X_{k+1} \in \mathcal{C} \\ k \geq 0}} \operatorname{Hom}\left(\operatorname{Hom}_{\mathcal{C}}\left(X_{k}, X_{k+1}\right)[1] \otimes \ldots \otimes \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, X_{2}\right)[1], \quad \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, X_{k+1}\right)[1]\right)
$$

Here [1] denotes the left-shift and $\|a\|=|a|-1$ denotes the reduced degree of a morphism $a \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$. The grading $\|\cdot\|$ on $\operatorname{HC}(\mathcal{C})$ is the one induced from the shifted degrees of the hom spaces of $\mathcal{C}$. In other words, we have

$$
\left\|\eta\left(a_{k}, \ldots, a_{1}\right)\right\|=\|\eta\|+\left\|a_{k}\right\|+\ldots+\left\|a_{1}\right\|, \quad \eta \in \mathrm{HC}(\mathcal{C})
$$

For $\eta, \omega \in \operatorname{HC}(\mathcal{C})$, the Gerstenhaber product $\mu \cdot \omega \in \operatorname{HC}(\mathcal{C})$ is defined as

$$
(\eta \cdot \omega)\left(a_{k}, \ldots, a_{1}\right)=\sum(-1)^{\left(\left\|a_{l}\right\|+\ldots+\left\|a_{1}\right\|\right)\|\omega\|} \eta\left(a_{k}, \ldots, \omega(\ldots), a_{l}, \ldots, a_{1}\right) .
$$

The Hochschild DGLA is the following $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded DGLA structure on $\mathrm{HC}(\mathcal{C})$ : The bracket on $\mathrm{HC}(\mathcal{C})$ is the Gerstenhaber bracket

$$
[\eta, \omega]=\eta \cdot \omega-(-1)^{\|\omega\|\|\eta\|} \omega \cdot \eta .
$$

The differential on $\mathrm{HC}(\mathcal{C})$ consists of commuting with the product $\mu_{\mathcal{C}} \in \mathrm{HC}^{1}(\mathcal{C})$ :

$$
d \nu=\left[\mu_{\mathcal{C}}, \nu\right] .
$$

Remark 5.42. It is not hard to check that $\operatorname{HC}(\mathcal{C})$ is indeed a DGLA. The reader who wishes to perform the computation is advised to write all double brackets in terms of Gerstenhaber products and use the associator relation

$$
\begin{aligned}
& (a \cdot b) \cdot c-a \cdot(b \cdot c) \\
& =\sum(-1)^{\|b\|\left(\left\|a_{1}\right\|+\ldots+\left\|a_{i}\right\|\right)+\|c\|\left(\left\|a_{1}\right\|+\ldots+\left\|a_{j}\right\|\right)+\|b\|\|c\|} a\left(\ldots, b(\ldots), a_{i}, \ldots, c(\ldots), a_{j}, \ldots\right) \\
& \left.+\sum(-1)^{\|b\|\left(\left\|a_{1}\right\|+\ldots+\left\|a_{j}\right\|\right)+\|c\|\left(\left\|a_{1}\right\|+\ldots+\left\|a_{i}\right\|\right)} a\left(\ldots, c(\ldots), a_{i}, \ldots, b(\ldots), a_{j}, \ldots\right)\right)
\end{aligned}
$$

The sum in these formulas runs over all ways to insert $b$ and $c$ into $a$. Despite the way the formulas are written, it is not necessary that any elements $a_{i}$ or $a_{j}$ actually lie in between or behind $b$ and $c$. This is merely an artifact needed to define the sign right: the sign shall be the total reduced degree of all elements coming after $b$ or after $c$, respectively.

Remark 5.43. In the terminology of the Hochschild DGLA HC(C), we can interpret $A_{\infty}$-deformations of $\mathcal{C}$ precisely as Maurer-Cartan elements of $\mathrm{HC}(\mathcal{C})$. We start by observing that the $A_{\infty}$-product $\mu_{\mathcal{C}}$ amounts to an element $\mu_{\mathcal{C}} \in \mathrm{HC}^{1}(\mathcal{C})$ and the $A_{\infty}$-relations translate to $\mu_{\mathcal{C}} \cdot \mu_{\mathcal{C}}=0$.

Now let $\mathcal{C}_{q}$ be a deformation of $\mathcal{C}$. Then the curved $A_{\infty}$-relations for $\mu_{\mathcal{C}_{q}}$ translate to $\mu_{\mathcal{C}_{q}} \cdot \mu_{\mathcal{C}_{q}}=0$. Decompose $\mu_{\mathcal{C}_{q}}=\mu_{\mathcal{C}}+\nu$ as non-deformed part $\mu_{\mathcal{C}}$ plus deformation $\nu \in \mathfrak{m} \mathrm{HC}^{1}(\mathcal{C})$. Given that $\mu_{\mathcal{C}}$ already satisfies the $A_{\infty}$-relation $\mu_{\mathcal{C}} \cdot \mu_{\mathcal{C}}=0$, the element $\nu$ itself satisfies the Maurer-Cartan equation

$$
0=\left(\mu_{\mathcal{C}}+\nu\right) \cdot\left(\mu_{\mathcal{C}}+\nu\right)=d \nu+[\nu, \nu] / 2
$$

Conversely, pick a Maurer-Cartan element $\nu \in \operatorname{MC}(\operatorname{HC}(\mathcal{C}), B)$. According to Lemma 5.12, the element $\mu_{\mathcal{C}}+\nu$ extends in a multilinear and $\mathfrak{m}$-adically continuous way to a collection of mappings

$$
\mu_{\mathcal{C}_{q}}^{k \geq 0}:\left(B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}\left(X_{k}, X_{k+1}\right)\right) \otimes \ldots \otimes\left(B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, X_{2}\right)\right) \rightarrow B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, X_{k+1}\right)
$$

The Maurer-Cartan identity for $\nu$ makes that $\mu_{\mathcal{C}_{q}}$ satisfies the curved $A_{\infty}$-relations.
Remark 5.44. In case $L$ is a $\mathbb{Z} / 2 \mathbb{Z}$-graded DGLA, then Maurer-Cartan elements of $L$ are by definition odd elements $\nu \in \mathfrak{m} L^{\text {odd }}$ with $d \nu+[\nu, \nu] / 2=0$. For example, our deformation $\operatorname{Gtl}_{q} Q$ is only a $\mathbb{Z} / 2 \mathbb{Z}$-graded deformation of $\operatorname{Gtl} Q$. In the context of deformations, we have to view both $\operatorname{Gtl} Q$ and its Hochschild DGLA $\mathrm{HC}(\operatorname{Gtl} Q)$ as $\mathbb{Z} / 2 \mathbb{Z}$-graded.

Remark 5.45. For ordinary algebras, which are concentrated in degree zero and have vanishing higher products, the Hochschild cohomology is typically defined without the shifts. This results in a grading difference of 1 from what we present here. For example, the center of the algebra is the classical zeroth Hochschild cohomology. In our $A_{\infty}$-setting, this cohomology will rather be found in degree -1 .

### 5.7 Gauge equivalence

In this section, we recall the notion of gauge equivalence. By virtue of algebraic deformation theory, we have two ways of defining this equivalence: via an explicit definition and via the Hochschild DGLA. Both notions are defined here in parallel. Useful references are 53 and 72 .

> Gauge equivalence
> by functor $F: \mathcal{C}_{q} \rightarrow \mathcal{C}_{q}^{\prime}$$\longleftrightarrow \quad \begin{gathered}\text { Gauge equivalence }\end{gathered}$

Recall from Definition 5.25 that a gauge equivalence between two deformations $\mathcal{C}_{q}, \mathcal{C}_{q}^{\prime}$ of an $A_{\infty^{-}}$ category $\mathcal{C}$ consists of a functor $F: \mathcal{C}_{q} \rightarrow \mathcal{C}_{q}^{\prime}$ whose leading term is the identity.

Remark 5.46. The idea behind Definition 5.25 is that the set of automorphisms $\mathcal{C}_{q} \rightarrow \mathcal{C}_{q}^{\prime}$ as deformed $A_{\infty}$-categories is rather large. The leading term of an automorphism can be any automorphism of $\mathcal{C}$, which is not interesting from the perspective of deformation theory. Therefore one restricts to those functors whose leading term is the identity on $\mathcal{C}$.

After the success of the notion of gauge equivalence throughout mathematics and physics, a definition has also been captured in the abstract DGLA approach. The idea here is that the gauging functor $F_{q}$ can be viewed as an element of a "gauge group". The infinitesimal action of this gauge group can be described purely in terms of the DGLA:

Definition 5.47. Let $L$ be a DGLA and $B$ a deformation base. Then there is a group action by $\exp \left(\mathfrak{m} L^{0}\right)$ on $B \widehat{\otimes} L^{1}$ with infinitesimal generator

$$
\varphi \cdot \nu=d \varphi+[\nu, \varphi] \in B \widehat{\otimes} L^{1}, \quad \nu \in \mathfrak{m} L^{0}
$$

The group action preserves the set of Maurer-Cartan elements $\operatorname{MC}(L ; B)$. Two Maurer-Cartan elements $\nu, \nu^{\prime} \in \mathrm{MC}(L ; B)$ are gauge-equivalent if they lie in the same orbit. The set of Maurer-Cartan elements up to gauge-equivalence is denoted $\overline{\mathrm{MC}}(L ; B)$.
Remark 5.48. In case $L$ is a $\mathbb{Z} / 2 \mathbb{Z}$-graded DGLA, then the gauge group is $\exp \left(\mathfrak{m} L^{\text {even }}\right)$. This group acts on Maurer-Cartan elements of $L$, which are by definition odd elements $\nu \in \mathfrak{m} L^{\text {odd }}$ with $d \nu+[\nu, \nu] / 2=0$.

Let us compare the infinitesimal generators of the action, at the non-deformed product $\mu_{\mathcal{C}}$ : An "infinitesimal functor" Id $+\varepsilon F$ pushes the non-deformed product $\mu_{\mathcal{C}}$ to some $\mu^{\prime}$ such that $(\operatorname{Id}+\varepsilon F) \cdot \mu_{\mathcal{C}}=$ $\mu^{\prime} \circ(\operatorname{Id}+\varepsilon F)$. Here $\circ$ denotes functor composition, in contrast to the Gerstenhaber product ".". Setting $\varepsilon^{2}=0$, we read off

$$
\mu^{\prime}=\mu_{\mathcal{C}}+\varepsilon\left[\mu_{\mathcal{C}}, F\right]
$$

Meanwhile, the trivial deformation $\mu_{\mathcal{C}}$ corresponds to the zero Maurer-Cartan element in $\operatorname{MC}(\operatorname{HC}(\mathcal{C}))$. Gauging it by $\varepsilon F$ under $\varepsilon^{2}=0$ gives the element

$$
(\varepsilon F) \cdot 0=0+d(\varepsilon F)+[0, \varepsilon F]=\varepsilon\left[\mu_{\mathcal{C}}, F\right] .
$$

This element corresponds to the deformation $\mu_{\mathcal{C}}+\varepsilon\left[\mu_{\mathcal{C}}, F\right]$. We see that gauging a deformation by a gauge functor $\operatorname{Id}+\varepsilon F$ is the same as gauging its corresponding Maurer-Cartan element in the Hochschild DGLA:

$$
\begin{array}{c|c}
\text { Infinitesimal functor } \operatorname{Id}+\varepsilon F & \text { Infinitesimal DGLA gauge } \varepsilon F \\
\mu^{\prime}=\mu_{\mathcal{C}}+\varepsilon\left[\mu_{\mathcal{C}}, F\right] . & \mu^{\prime}=\mu_{\mathcal{C}}+(d(\varepsilon F)+[0, \varepsilon F])
\end{array}
$$

More precisely, two deformations are gauge equivalent in the sense of Definition 5.25 if and only if their corresponding Maurer-Cartan elements are gauge equivalent in the sense of Definition 5.47.
Remark 5.49. $A_{\infty}$-Hochschild cohomology of $\mathcal{C}$ is defined as the cohomology of $\mathrm{HC}(\mathcal{C})$, merely regarded as a cochain complex instead of DGLA. This way, Hochschild cohomology is the linear approximation of Maurer-Cartan elements up to gauge equivalence.

## 5.8 $\quad L_{\infty}$-algebras

In this section we recall the formalism of $L_{\infty}$-algebras. In our context of $A_{\infty}$-categories, we namely want to transport deformations from one category to another, so that one needs morphisms between their Hochschild DGLAs. The world of DGLAs and their morphisms is quite restrictive, but there is a more flexible version of DGLAs known as $L_{\infty}$-algebras. We finish this section with a definition and brief discussion of the $L_{\infty}$-theory. We follow version 3 of 8 . The reader be cautioned that version 4 of that paper has changed signs.
Definition 5.50. An $L_{\infty}$-algebra is a graded vector space $L$ together with multilinear maps

$$
l_{k}: \underbrace{L \times \ldots \times L}_{k \text { times }} \rightarrow L
$$

of degree $2-k$ which are graded skew-symmetric and satisfy the higher Jacobi identities:

$$
\begin{aligned}
& l_{k}\left(x_{s(1)}, \ldots, x_{s(k)}\right)=\chi(s) l_{k}\left(x_{1}, \ldots, x_{k}\right), \\
& \sum_{\substack{i+j=k+1 \\
i, j \geq 1}} \sum_{s \in S_{i, k-i}}(-1)^{i(n-i)} \chi(s) l_{j}\left(l_{i}\left(x_{s(1)}, \ldots, x_{s(i)}\right), x_{s(i+1)}, \ldots, x_{s(k)}\right)=0 .
\end{aligned}
$$

Here $S_{i, k-i}$ denotes the set of shuffles, i.e. $s \in S_{k}$ with $s(1)<\ldots<s(i)$ and $s(i+1)<\ldots<s(k)$. The $\operatorname{sign} \chi(s)$ is the product of the signum of $s$ and the Koszul sign of $s$. The Koszul sign of a transposition $(i j)$ is $(-1)^{\left|x_{i}\right|\left|x_{j}\right|}$, and this rule is extended multiplicatively to arbitrary permutations.

With this sign convention, a DGLA as in Definition 5.39 is simply an $L_{\infty}$-algebra without higher products. In particular, the Hochschild DGLA can automatically be regarded as an $L_{\infty}$-algebra.

Morphisms between $L_{\infty}$-algebras are indeed more flexible than between DGLAs: $L_{\infty}$-morphisms are allowed to have higher components, just like $A_{\infty}$-functors allow for higher components.

Definition 5.51. A morphism of $L_{\infty}$-algebras $\varphi: L \rightarrow L^{\prime}$ is given by multilinear maps

$$
\varphi^{k}: L \times \ldots \times L \rightarrow L
$$

of degree $1-k$ for all $k \geq 1$ such that $\varphi\left(x_{s(1)}, \ldots, x_{s(k)}\right)=\chi(s) \varphi\left(x_{1}, \ldots, x_{k}\right)$ for any $s \in S_{k}$ and

$$
\begin{aligned}
& \sum_{\substack{i+j=k+1 \\
i, j \geq 1}} \sum_{s \in S_{i, n-i}}(-1)^{i(k-i)} \chi(s) \varphi\left(l\left(x_{s(1)}, \ldots, x_{s(i)}\right), x_{s(i+1)}, \ldots, x_{s(k)}\right) \\
&=\sum_{\substack{1 \leq r \leq k \\
i_{1}+\ldots+i_{r}=k}} \sum_{t}(-1)^{u} \chi(t) l_{r}^{\prime}\left(\varphi\left(x_{t(1)}, \ldots, x_{t\left(i_{1}\right)}\right), \ldots, \varphi\left(x_{t\left(i_{1}+\ldots+i_{r-1}+1\right)}, \ldots, x_{t(k)}\right)\right),
\end{aligned}
$$

where $t$ runs over all $\left(i_{1}, \ldots, i_{r}\right)$-shuffles for which

$$
t\left(i_{1}+\ldots+i_{l-1}+1\right)<t\left(i_{1}+\ldots+i_{l}+1\right)
$$

and $u=(r-1)\left(i_{1}-1\right)+\ldots+2\left(i_{r-2}-1\right)+\left(i_{r-1}-1\right)$. A morphism $\varphi: L \rightarrow L^{\prime}$ of $L_{\infty}$-algebras is a quasi-isomorphism if $\varphi^{1}$ is a quasi-isomorphism of complexes.

Definition 5.52. Let $L$ be an $L_{\infty}$-algebra and $B$ a deformation base. Then a Maurer-Cartan element is an element $x \in \mathfrak{m} \widehat{\otimes} L^{1}$ satisfying the Maurer-Cartan equation

$$
\sum_{k \geq 1} \frac{l_{k}(x, \ldots, x)}{k!}=0
$$

We write $\operatorname{MC}(L, B)$ for the set of Maurer-Cartan elements of $L$ over $B$.
In contrast to the DGLA case, there is no gauge group acting on the Maurer-Cartan elements. Instead, one should regard an equivalence relation of "homotopy". All we should assume here is that the notion of homotopy exists and gives rise to a quotient set $\overline{\mathrm{MC}}(L, B)$, just as in the DGLA case.

### 5.9 Axioms on $A_{\infty}$-deformations

There is a slight gap in our treatment of $A_{\infty}$-deformations: We cannot prove invariance of the Hochschild DGLA under quasi-equivalences. While derived invariance is known in the dg case due to 40, according to private communication with Keller there is no literature available for the $A_{\infty}$-case. We do not fill the gap here. Instead, we state a small set of axioms in this section which we will simply assume for the purpose of section 9 and section 8.3

The motivation for our axioms is that we need to be able to push deformations from one category to another. Assume two $A_{\infty}$-categories $\mathcal{C}$ and $\mathcal{D}$ are quasi-equivalent by means of a quasi-equivalence $F: \mathcal{C} \rightarrow \mathcal{D}$. Intuition says that a deformation $\mathcal{C}_{q}$ of $\mathcal{C}$ can be "pushed" via $F$ to a deformation $\mathcal{D}_{q}=F_{*}\left(\mathcal{C}_{q}\right)$ of $\mathcal{D}$ such that $\mathcal{C}_{q}$ and $\mathcal{D}_{q}$ still quasi-equivalent to each other by a deformation of the functor $F$. We formalize this notion as follows:

Definition 5.53. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ be a deformation. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a quasiequivalence. Then we call any deformation $\mathcal{D}_{q}$ of $\mathcal{D}$ a naive pushforward of the deformation $\mathcal{C}_{q}$ along $F$ if there exists a functor $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ with leading term $F$.

Remark 5.54. If forming Hochschild DGLAs were functorial, pushforwards would be easy. Indeed, a quasi-equivalence $\mathcal{C} \rightarrow \mathcal{D}$ would ideally induce a quasi-isomorphism of $L_{\infty}$-algebras $\mathrm{HC}(\mathcal{C}) \rightarrow \mathrm{HC}(\mathcal{D})$ and therefore a bijection $\overline{\mathrm{MC}}(\mathcal{C}, B) \rightarrow \overline{\mathrm{MC}}(\mathcal{D}, B)$ of Maurer-Cartan elements. The naive pushforward of $\mathcal{C}_{q}$ would then simply be obtained as the image under this map of the Maurer-Cartan elemement defined by $\mathcal{C}_{q}$. It is an awkward fact of algebra that however neither the Hochschild DGLA nor Hochschild cohomology is functorial. For instance, even the center $\operatorname{HH}^{0}(A)=Z(A)$ of an algebra $A$ is not functorial a property. Two quasi-equivalent $A_{\infty}$-categories however have quasi-isomorphic Hochschild DGLAs. At least, this is a folklore statement, with actual proof only available by Keller 40 in the dg case. See also the discussion in section F.1.1.

This definition of naive pushforwards raises many detail questions. For instance, let $G: \mathcal{D} \rightarrow \mathcal{E}$ be yet another quasi-equivalence. Then we are interested in the question whether the double pushforward of $\mathcal{C}_{q}$ along $F$ and $G$ is gauge-equivalent to the single pushforward of $\mathcal{C}_{q}$ along $G F$. We shall provide answer to this question in the form of axioms:

Convention 5.55. We assume the following axioms regarding the Hochschild DGLA: Let $B$ be a fixed deformation base. Then:
(A1) Two $A_{\infty}$-deformations $\mathcal{C}_{q}, \mathcal{C}_{q}^{\prime}$ over $B$ are gauge equivalent if and only if they are gauge equivalent as Maurer-Cartan elements of $\mathrm{HC}(\mathcal{C})$.
(A2) Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a quasi-equivalence of $A_{\infty}$-categories. Then there exists a quasi-isomorphism of $L_{\infty}$-algebras $F_{*}: \mathrm{HC}(\mathcal{C}) \rightarrow \mathrm{HC}(\mathcal{D})$.
(A3) We call $F_{*}$ the pushforward of $F$. The pushforward is noncanonical. However its induced map $\left(F_{*}\right)^{\mathrm{MC}}: \overline{\mathrm{MC}}(\mathrm{HC}(\mathcal{C}), B) \rightarrow \overline{\mathrm{MC}}(\mathrm{HC}(\mathcal{D}), B)$ has the following property: $\mathcal{D}_{q}$ is a naive pushforward of $\mathcal{C}_{q}$ along $F$ if and only if $\mu_{\mathcal{D}_{q}}=F_{*}^{\mathrm{MC}}\left(\mu_{\mathcal{C}_{q}}\right)$.
(A4) Let $\mathcal{C} \subseteq \mathcal{D}$ be a subcategory such that the inclusion $i: \mathcal{C} \rightarrow \mathcal{D}$ is a quasi-equivalence. Then $i_{*}^{\mathrm{MC}}\left(\left.\mu_{\mathcal{D}_{q}}\right|_{\mathcal{C}}\right)=\mu_{\mathcal{D}_{q}}$.
(A5) Push-forward is functorial on Maurer-Cartan elements: $(G F)_{*}^{\mathrm{MC}}\left(\mu_{\mathcal{C}_{q}}\right)=G_{*}^{\mathrm{MC}}\left(F_{*}^{\mathrm{MC}}\left(\mu_{\mathcal{C}_{q}}\right)\right.$.
Remark 5.56. There is a slight abuse of the notation in Convention 5.55 Where we have written $F_{*}^{\mathrm{MC}}\left(\mu_{\mathcal{C}_{q}}\right)$, we actually mean the Maurer-Cartan element $\mu_{\mathcal{C}_{q}}-\mu_{\mathcal{C}}$ instead of $\mu_{\mathcal{C}_{q}}$. In fact, the element $\mu_{\mathcal{C}_{q}} \in B \widehat{\otimes} \mathrm{HC}^{1}(\mathcal{C})$ is not a Maurer-Cartan element itself. In similar abuse of notation, we may occasionally write $F_{*}^{\mathrm{MC}}\left(\mathcal{C}_{q}\right)$ instead of $F_{*}^{\mathrm{MC}}\left(\mu_{\mathcal{C}_{q}}\right)$.

As an application of Convention 5.55, we show here that quasi-equivalence of deformed $A_{\infty}$-categories is an equivalence relation. By quasi-equivalence of two deformed $A_{\infty}$-categories, we refer to the relation defined in Definition 5.24 A proof without assuming the axioms would either need to deal with complicated explicit constructions, or make use of an $\infty$-categorical level. In other words, assuming the axioms seems to be a healthy midway for the scope of the paper.

Lemma 5.57. Quasi-equivalence of deformed $A_{\infty}$-categories is an equivalence relation. Even stronger, for every quasi-equivalence $F: \mathcal{C} \rightarrow \mathcal{D}$ there exists a quasi-equivalence $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $G_{*} F_{*}=$ id. Derived equivalence is an equivalence relation as well.

Proof. We need to prove reflexivity, transitivity and symmetry of the quasi-equivalence relation. The first two properties are easy: The identity functor obviously provides a quasi-equivalence from any deformation $\mathcal{C}_{q}$ to itself. And if $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ and $G_{q}: \mathcal{D}_{q} \rightarrow \mathcal{E}_{q}$ are quasi-equivalences, then the composition $G_{q} F_{g}: \mathcal{C}_{q} \rightarrow \mathcal{E}_{q}$ is a quasi-equivalence as well. We conclude that only symmetry remains to be proven.

The proof of symmetry consists of five steps: First, we define a set of "good" quasi-equivalences $F$ for which there exists a quasi-equivalence $G$ with $G_{*} F_{*}=\mathrm{id}$. The second, third and fourth step establish basic properties of this "good" set. In the fifth step, we show that those properties already make every quasi-equivalence lie in $\mathcal{F}$.

As step 1 , let us recall our context. We are interested in the set of quasi-equivalences $F: \mathcal{C} \rightarrow \mathcal{D}$ such that there exists a quasi-equivalence $G: \mathcal{D} \rightarrow \mathcal{C}$ with $G_{*} F_{*}=\mathrm{id}$. Denote this set by $\mathcal{F}$ :

$$
\mathcal{F}:=\left\{F: \mathcal{C} \rightarrow \mathcal{D} \text { q.e. } \mid \exists G: \mathcal{D} \rightarrow \mathcal{C} \text { q.e. }: G_{*} F_{*}=\text { id }\right\} .
$$

Our aim is to show that any quasi-equivalence lies in $\mathcal{F}$. Steps $2,3,4$ are devoted to proving several properties of $\mathcal{F}$.

As step 2, we prove the property

$$
F \in \mathcal{F} \text { with } G \text { q.e. such that } G_{*} F_{*}=\mathrm{id} \quad \Longrightarrow \quad F_{*} G_{*}=\text { id and } G \in \mathcal{F} .
$$

Indeed, pick $F$ as on the left-hand side. Since both $F$ and $G$ are quasi-equivalences, both pushforwards $F_{*}$ and $G_{*}$ are bijections. Together with $G_{*} F_{*}=\mathrm{id}$, we conclude that $G_{*}$ and $F_{*}$ are simply inverses to each other. In other words, $F_{*} G_{*}=$ id holds as well. We conclude that $G \in \mathcal{F}$.

As step 3, we prove for composable quasi-equivalences $F, G$ the property

$$
F, G \in \mathcal{F} \quad \Longleftrightarrow \quad G F \in \mathcal{F}
$$

One should think of this as a strong version of the two-out-of-three property. To prove it, pick $F, G \in \mathcal{F}$ with $F_{*}^{\prime} F_{*}=\mathrm{id}$ and $G_{*}^{\prime} G_{*}=\mathrm{id}$. We get $\left(F^{\prime} G^{\prime}\right)_{*}(G F)_{*}=F_{*}^{\prime} F_{*}=$ id, which renders $G F \in \mathcal{F}$. Conversely assume $G F \in \mathcal{F}$ with $H_{*}(G F)_{*}=$ id. Then $(H G)_{*} F_{*}=$ id, hence $F \in \mathcal{F}$. Step 2 implies $F_{*}(H G)_{*}=$ id. In other words $(F H)_{*} G_{*}=$ id, hence $G \in \mathcal{F}$. We conclude that both $F$ and $G$ lie in $\mathcal{F}$, as desired.

As step 4, we show that the following lie in $\mathcal{F}$ :

> quasi-equivalences with a one-sided inverse, inclusions of skeletal subcategories, inclusion $i: \mathrm{HC} \rightarrow \mathcal{C}$ and projection $\pi: \mathcal{C} \rightarrow \mathrm{HC}$

To this end, assume $F$ and $G$ are quasi-equivalences with $F G=\mathrm{Id}$. Combining $\operatorname{Id} \in \mathcal{F}$ with step 3 , we deduce $F, G \in \mathcal{F}$. In particular, isomorphisms fall under this regime. Inclusions of skeletal subcategories also fall under this regime, since a skeletal subcategory $S \subseteq \mathcal{C}$ produces a quasi-equivalence $\mathcal{C} \rightarrow S$ which reduces to the identity on $S$. Now regard a category $\mathcal{C}$ and its minimal model HC. The minimal model is not unique, but every minimal model comes with quasi-isomorphisms $i: \mathrm{H} \mathcal{C} \rightarrow \mathcal{C}$ and $\pi: \mathcal{C} \rightarrow \mathrm{H} \mathcal{C}$. Regard the map $\pi i: \mathrm{HC} \rightarrow \mathrm{HC}$. Since both $i$ and $\pi$ are quasi-isomorphisms, $\pi i$ is a quasi-isomorphism as well. Moreover, HC is already a minimal category, hence $\pi i$ is an isomorphism. We conclude that $\pi i \in \mathcal{F}$. By step 3, we deduce that both $\pi$ and $i$ lie in $\mathcal{F}$. This finishes step 4 .

As final step 5 , we prove that any quasi-equivalence lies in $\mathcal{F}$. To this end, let $F: \mathcal{C} \rightarrow \mathcal{D}$ be any quasi-equivalence. Our strategy is to build a diagram to whose arrows we can apply step 3 and 4 to deduce that $F$ also lies in $\mathcal{F}$. In order to write down the diagram, pick minimal models $\mathrm{H} \mathcal{C}$ and $\mathrm{H} \mathcal{D}$, together with inclusion map $i_{\mathcal{C}}: \mathrm{HC} \rightarrow \mathcal{C}$ and projection $\pi_{\mathcal{D}}: \mathcal{D} \rightarrow \mathrm{H} \mathcal{D}$. Define $F^{\prime}:=\pi_{\mathcal{D}} F i_{\mathcal{C}}$. This gives a diagram, commutative by definition,


Choose a skeletal subcategory $S_{\mathcal{C}} \subseteq \mathrm{H} \mathcal{C}$ and set $S_{\mathcal{D}}:=F^{\prime}\left(S_{\mathcal{C}}\right)$. Then $S_{\mathcal{D}} \subseteq \mathrm{H} \mathcal{D}$ is a skeletal subcategory as well: Any object $X \in \mathrm{H} \mathcal{D}$ is isomorphic to some $F^{\prime}(Y)$, and $Y$ in turn is isomorphic to some $Z \in S_{\mathcal{C}}$, hence $X \cong F^{\prime}(Z) \in S_{\mathcal{D}}$. Moreover if $Y, Z \in S_{\mathcal{C}}$ and $F^{\prime}(Y) \cong F^{\prime}(Z)$, then $Y \cong Z$. This implies $Y=Z$ because $S_{\mathcal{C}}$ is a skeleton. In total, we conclude that $S_{\mathcal{D}} \subseteq \mathrm{H} \mathcal{D}$ is a skeletal subcategory, and we obtain a restricted quasi-equivalence $F^{\prime \prime}: S_{\mathcal{C}} \rightarrow S_{\mathcal{D}}$. Putting everything together, we have the commutative diagram


Here we denoted by $I$ and $J$ the inclusion of the full subcategories $S$ and $S^{\prime}$ into $\mathrm{H} \mathcal{C}$ resp. $\mathrm{H} \mathcal{D}$. The top and bottom square are strictly commutative by definition of $F^{\prime}$ and $F^{\prime \prime}$.

We now count everything together: $F^{\prime \prime}$ is an isomorphism and $J$ is an inclusion of a skeletal subcategory. By step 4 , we get $J, F^{\prime \prime} \in \mathcal{F}$. By step 3 , we get $J F^{\prime \prime} \in \mathcal{F}$. By commutativity of the bottom of the diagram we have $F^{\prime} I=J F^{\prime \prime}$. By step 3, this implies $F^{\prime} \in \mathcal{F}$. By commutativity of the top of the diagram we have $\pi_{\mathcal{D}} F i_{\mathcal{C}}=F^{\prime} \in \mathcal{F}$. A double application of step 3 renders $F \in \mathcal{F}$. Since $F$ was arbitrary, this shows that all quasi-equivalences lie in $\mathcal{F}$. In other words, for every $F: \mathcal{C} \rightarrow \mathcal{D}$ there exists a $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $G_{*} F_{*}=$ id. In particular, this shows quasi-equivalence is an equivalence relation.

Let us now explain why derived equivalence is an equivalence relation as well. Indeed, $\mathcal{C}_{q}$ and $\mathcal{D}_{q}$ derived equivalent according to Definition 5.24 if $\operatorname{Tw} \mathcal{C}_{q}$ and $\operatorname{Tw} \mathcal{D}_{q}$ are quasi-equivalent. Since we have just shown that quasi-equivalence is an equivalence relation, we conclude that derived equivalence is an equivalence relation. This finishes the proof.

Remark 5.58. Pushing deformations from one category to another is not only possible via quasiequivalences. We can also push forward deformations from one category to a derived equivalent category. Namely, let $F: \operatorname{Tw} \mathcal{C} \rightarrow \operatorname{Tw} \mathcal{D}$ be a quasi-equivalence and let $\mathcal{C}_{q}$ be a deformation of $\mathcal{C}$. Then the twisted completion $\operatorname{Tw} \mathcal{C}_{q}$ from Definition 5.31 is canonically a deformation of $\mathrm{Tw} \mathcal{C}$. The pushforward $F_{*}^{\mathrm{MC}}$ on Maurer-Cartan elements now transports this deformation $\operatorname{Tw} \mathcal{C}_{q}$ to a deformation $F_{*}^{\mathrm{MC}}\left(\mu_{\mathrm{Tw} \mathcal{C}_{q}}\right)$ of $\operatorname{Tw} \mathcal{D}$.


Figure 6.1: The three-punctured sphere, the four-punctured sphere and the one-punctured torus

Restricting this deformation to $\mathcal{D} \subseteq \operatorname{Tw} \mathcal{D}$ gives a deformation of $\mathcal{D}$. Via the route of twisted completion, pushforward and restriction, the deformation $\mathcal{D}_{q}$ corresponds to the deformation $\mathcal{C}_{q}$. We may equally call $\mathcal{D}_{q}$ the pushforward of $\mathcal{C}_{q}$ along $F$.

## 6 Preliminaries on gentle algebras

In this section, we concisely recapitulate background on gentle algebras in order to bring the reader into touch with the relevant tools of this paper. We provide definitions of all preliminaries and explain alternative points of view on them. In particular, we will explain how every definition is used in the paper. We follow mostly 18 and Paper I

### 6.1 Punctured surfaces

Punctured surfaces belong to the family of two-dimensional oriented manifolds, while at the same time facilitating singular behavior at the punctures.

Definition 6.1. A punctured surface is a closed oriented surface $S$ with a finite set of punctures $M \subseteq S$. We assume that $|M| \geq 1$, or $|M| \geq 3$ if $S$ is a sphere.

A selection of popular punctured surfaces are depicted in Figure 6.1 The condition $|M| \geq 1$ and $|M| \geq 3$ are merely cosmetic and will be explained in section 6.2 .
Remark 6.2. A punctured surface can alternatively be interpreted as a surface with $S^{1}$ boundaries: Let $(S, M)$ be a punctured surface and regard one puncture $q \in M$. The surface around $q$ looks like a punctured disk. Now interpret the punctured disk as an infinitely long cylinder, glued to the rest of the surface. Cut off the cylinder at some distance. We obtain a surface with $S^{1}$ boundaries, which we interpret as markings. In other words, we have a marked surface with only $S^{1}$ boundaries in the sense of 35. For instance, cutting away disks around the punctures in the three-punctured sphere, we obtain the popular pair of pants surface.

### 6.2 Arc systems

In this section, we recall the notion of arcs and arc systems on punctured surfaces. We recall what it means for an arc system to be full, and explain how it cuts the surface into polygons. We fix some terminology regarding polygons, in particular the notion of a polygon's interior angles.

Definition 6.3. Let $(S, M)$ be a punctured surface. An $\operatorname{arc}$ in $S$ is a not necessarily closed curve $\gamma:[0,1] \rightarrow S$ running from one puncture to another. An $\operatorname{arc}$ system $\mathcal{A}$ on a punctured surface is a finite collection of arcs which meet only at the set $M$ of punctures. Intersections and self-intersections are not allowed. The arc system satisfies the no monogons or digons condition [NMD] if

- No arc is a contractible loop in $S \backslash M$.
- No pair of distinct arcs is homotopic in $S \backslash M$.

The arc system satisfies the no monogons or digons in the closed surface condition [NMDC] if

- No arc is a contractible loop in $S$.
- No pair of distinct arcs is homotopic in $S$.

Example 6.4. In Figure 6.2 we have depicted a few arbitrary arc systems on the three- and fourpunctured sphere and one-punctured torus. The drawn three-punctured sphere has the north and south pole marked, as well as a point on the equator lying on the front half of the sphere. The four-punctured sphere has the north and south pole, as well as a point in the far east and far west marked.


Figure 6.2: Arc systems and their properties

Example 6.5. The arc system of Figure 6.2a consists of two half meridians lying in the frontal hemisphere. The arc system of Figure 6.2b consists of the frontal half of the equator and a northern half of a meridian. The one-punctured torus Figure 6.2 c is drawn as a gluing diagram. The arc system consists of the two standard generators of the torus.

Remark 6.6. The configurations banned by [NMD] are depicted in Figure 6.2d and 6.2e The reason to ban these is that the definition of the $A_{\infty}$-structure on the gentle algebras becomes a lot easier, avoiding a so-called monogon or digon rule. This makes checking the $A_{\infty}$-axioms also more tractable. With the [NMDC] condition, we go a step further and ban also monogons and digons in the closed surface $S$. More concretely, we ban loops which become contractible when the punctures are filled. Such a banned configuration is depicted in Figure 6.2f Similarly, we ban pairs of homotopic arcs, the homotopy being allowed to cross punctures. Such a banned configuration is depicted in Figure 6.2g. The purpose of the [NMDC] condition is to avoid the monogon and digon rule also for the deformed gentle algebras.

Definition 6.7. An arc system is full if it cuts the surface into contractible pieces. These pieces are the faces or polygons of the arc system.

In other words, an arc system is full if its complement consists of a disjoint union of topological disks. We usually refer to these pieces as polygons to highlight that they are bounded by arcs of the arc system.

Example 6.8. Of the three arc systems presented in Figure 6.2 , only 6.2 a and 6.2 c are full. Arc system 6.2 b is not full, because the complement of the arcs is a topological disk with a puncture in its interior, the south pole. Removing any arc from 6.2 a or 6.2 c also leads to a non-full arc system. Additional arcs may however be added to make or keep the arc system full. For example, in 6.2 a one may add the full equator as arc and in 6.2 c one may add any diagonal as arc, but not both. All possible types of arc systems with [NMDC] on the three-punctured sphere are depicted in Figure 6.4. In these figures, the directions of the arcs is arbitrary. Only the arc systems in the third and fourth picture are full.

The reason we demand arc systems on spheres to have $|M| \geq 3$ punctures becomes apparent: The [NMD] condition excludes the case of digons bounded by two different arcs, but we also desire to exclude the case where a digon is bounded by twice the same arc, depicted in Figure 6.3b. The only arc system with a digon bounded by twice the same arc is however the two-punctured sphere, depicted in Figure 6.3c. This is the reason we require $|M| \geq 3$.

Definition 6.9. The interior angles of a polygon are the angles in the corners of the polygon. By angle, we refer to the abstract entity (an interval starting at one arc and ending at the other, winding around their common endpoint) instead of the angle value.

Since the punctured surface comes with an orientation, the interior angles of every polygon come with a natural cyclic (clockwise) order, see Figure 6.3a Working with arc systems often requires arguing with properties of the polygons and their angles. Some configurations of arcs and angles are allowed under the [NMDC] condition, others not.

Remark 6.10. In a [NMD] arc system, every polygon is bounded by a sequence of arcs with at least three interior angles in between. Indeed, zero angles bounding a polygon would mean the polygon is


Figure 6.3: Illustrations of arcs and polygons


Figure 6.4: Arc systems on the three-punctured sphere

(a) Sphere

(b) Torus

(c) Not [NMDC]

Figure 6.5: Standard dimer models and the [NMDC] condition
bounded by a single puncture. The punctured surface would necessarily be a one-punctured sphere, which we banned. A single angle bounding a polygon would mean that the polygon is bounded by a loop contractible in $S \backslash M$, which we banned. Two angles bounding a polygon would mean they are equal, or distinct and homotopic in $S \backslash M$. Both options are banned. In summary, every polygon in a [NMD] arc system is bounded by a sequence of arcs with at least three interior angles in between.

### 6.3 Dimers

Dimer models, also referred to as brane tilings, originate in physicists' description of mirror symmetry. The idea is to describe arrangements of branes on the A-side of mirror symmetry in a surface graph. In a dimer model, adjacent nodes have opposite color. Dimer models can be seen as specific instances of punctured surfaces. A comprehensive reference is 15.

Definition 6.11. A dimer $Q$ is a full arc system on a punctured surface such that

- every polygon is bounded by at least three arcs,
- the arcs along the boundary of a polygon are all oriented in the same direction.

The letter $Q$ also denotes the quiver, obtained from the arc system: Its vertex set $Q_{0}$ is the set of punctures and its arrow set $Q_{1}$ is the set of arcs. The underlying closed surface is denoted $|Q|$.

All polygons in a dimer are bounded either entirely clockwise or entirely anticlockwise. Neighboring polygons are bounded opposite: A polygon next to a clockwise polygon is anticlockwise, and a polygon next to an anticlockwise polygon is clockwise. The standard notation for a dimer is the letter $Q$, minding the fact that the punctures together with the arcs can also be interpreted as a quiver embedded in a surface.

Remark 6.12. Every punctured surface has an arc system that is a dimer. Standard dimer models for the $n$-punctured sphere $(n \geq 3)$ and $n$-punctured torus $(n \geq 1)$ are depicted in Figure 6.5a and 6.5b A dimer automatically satisfies the [NMD] condition. There are however dimers which violate the [NMDC] condition, an example is depicted in Figure 6.5c

### 6.4 Gentle algebras

In this section, we recall gentle algebras associated with arc systems. We use a specific definition of gentle algebras, due to Bocklandt 18 . The reason they appear in this paper is that they form discrete models for Fukaya categories of punctured surfaces. In the present section, we describe only the algebra structure. The $A_{\infty}$-structure will be added in section 6.5

As their name suggests, gentle algebras are originally a type of finite-dimensional algebras. In 5, it was shown that so-called "unpunctured marked surface triangulations" naturally give rise to such gentle algebras. For readers familiar with Haiden-Katzarkov-Kontsevich's work 35, these are marked surfaces where all $S^{1}$ boundary components have at least one marking. The construction of gentle algebras from surfaces was subsequently carried over by Bocklandt 18 to the case of marked surfaces with full arc systems, as defined in section 6.2. The definition is essential for this paper:

Definition 6.13. Let $\mathcal{A}$ be a full arc system on a punctured surface. Then the gentle algebra (as ordinary algebra) $\operatorname{Gtl} \mathcal{A}=\mathbb{C R} \mathcal{A} / I$ is the quiver algebra with relations, where:

- The vertices of RA are given by the arc midpoints of the arc system.
- The arrows of $\mathrm{R} \mathcal{A}$ are given by the interior angles of the polygons.
- The relations in $I$ are given by all products of two consecutive interior angles of a polygon.

The quiver $\mathrm{R} \mathcal{A}$ has as many vertices as the arc system has arcs, as many arrows as the arc system has interior angles, and every polygon gives rise to as many relations as it has interior angles. The quiver $\mathrm{R} \mathcal{A}$ is called the rectified quiver in 18 . Figure 6.6 depicts some arc systems together with their rectified quivers. The left part of each graphic is the arc system itself, with arcs drawn thick. The interior angles are drawn as thin arrows; in the three- and four-punctured sphere, the dashed arrows mean the interior angles at the rear, invisible side of the sphere. The right part of each graphic depicts the rectified quiver together with its relations.

A vector space basis for the gentle algebra $\operatorname{Gtl} \mathcal{A}$ consists of all angles around punctures. The basis includes an identity $\operatorname{id}_{a}$ for every arc $a \in \mathcal{A}$, which we may also view as an empty angle. The gentle algebra $\mathrm{Gtl} \mathcal{A}$ of an arc system is not finite-dimensional.

Remark 6.14. By nature, the algebra $\operatorname{Gtl} \mathcal{A}=\mathbb{C R} \mathcal{A} / I$ can be viewed as a $\mathbb{C}$-linear category with objects being the arcs of the arc system. The hom spaces are spanned freely by the angles winding around punctures, starting at one arc and ending at another arc. In 18 , this interpretation of $\mathrm{Gtl} \mathcal{A}$ is also called the "gentle category". We will consistently use the term gentle algebra instead, despite the slight inaccuracy.

Remark 6.15. In the work of Haiden, Katzarkov and Kontsevich 35], gentle algebras were developed further under the name of "topological Fukaya categories". This includes a generalization regarding the type of boundary allowed. The version of gentle algebra we defined above is called a surface with fully marked boundaries of $S^{1}$ type in 35 . If one changes the boundary type to have at least one so-called "boundary arc" on every boundary component, the topological Fukaya category becomes finitedimensional. The gentle algebras $\operatorname{Gtl} \mathcal{A}$ studied in the present paper are however infinite-dimensional by nature: One can keep winding around the punctures as often as one wants, obtaining morphisms of higher and higher length.

It might be worthwhile comparing to the original definition due to 5: A finite-dimensional algebra presented as $\mathbb{C} Q / I$ is gentle if

- At each vertex there start at most two arrows and there end at most two arrows.
- The ideal $I$ is generated by paths of length 2 .
- For every arrow $\beta$, there is at most one arrow $\alpha$ such that $\alpha \beta \in I$, and at most one arrow $\gamma$ such that $\beta \gamma \in I$.
- For every arrow $\beta$, there is at most one arrow $\alpha$ such that $\alpha \beta \notin I$, and at most one arrow $\gamma$ such that $\beta \gamma \notin I$.
The same paper 5 showed how surface triangulations naturally give rise to gentle algebras.


### 6.5 The $A_{\infty}$-structure on $\operatorname{Gtl} \mathcal{A}$

In this section, we recall the $A_{\infty}$-structure on $\operatorname{Gtl} \mathcal{A}$. The starting point is the description of $\mathrm{Gtl} \mathcal{A}$ as ordinary algebra in section 6.4 The idea is to add $A_{\infty}$-structure which captures the topology of the


Figure 6.6: Standard arc systems and their rectified quivers


Figure 6.7: Degree of angle $\alpha$
punctured surface. This $A_{\infty}$-structure was introduced by Bocklandt 18 in order to define a discrete version of the wrapped Fukaya category. In the present section, we recall the $A_{\infty}$-structure briefly and refer to [16. Section 9.1] and 35 and for more insight.

We start with a full arc system $\mathcal{A}$ with [NMD]. The first step towards the $A_{\infty}$-structure is the grading. It is possible to put a $\mathbb{Z}$-grading on $\mathrm{Gtl} \mathcal{A}$ by viewing arcs and angles relative to a vector field on the surface, see section 9.2 The deformations of $\operatorname{Gtl} \mathcal{A}$ that we are interested in exist however only in the $\mathbb{Z} / 2 \mathbb{Z}$-graded world. Consequentially, we define $\operatorname{Gtl} \mathcal{A}$ as a $\mathbb{Z} / 2 \mathbb{Z}$-graded $A_{\infty}$-category from the very beginning. The definition of degrees is depicted in Figure 6.7 and reads as follows:

Definition 6.16. The degree $|\alpha|$ of an angle $\alpha: a \rightarrow b$ is odd if one of the arcs $a, b$ points towards the puncture, and one points away from the puncture. The degree of an angle is even if both arcs point away or both point towards the puncture.

The second step in the definition of the $A_{\infty}$-structure is the definition of the differential $\mu^{1}$ and the product $\mu^{2}$. The differential $\mu^{1}$ is plainly set to zero. We keep the notation $\alpha \beta$ for the concatenation of angles, and define the product $\mu^{2}$ as its signed version:

$$
\mu^{1}:=0, \quad \mu^{2}(\alpha, \beta):=(-1)^{|\beta|} \alpha \beta .
$$

Remark 6.17. The reason we assume the [NMD] condition is that it simplifies the definition of $\mu^{1}$ and $\mu^{2}$. Indeed, can also define the $A_{\infty}$-structure for arc systems without [NMD]. However, the definition of $\mu^{1}$ and $\mu^{2}$ then needs to be tweaked in order to capture the monogons and digons.

The third step is to define the higher products $\mu^{\geq 3}$ of $\operatorname{Gtl} \mathcal{A}$. They capture the topology of the arcs and angles. Roughly speaking, a higher product of a sequence of angles is nonzero if the sequence bounds a disk. Such a disk is given by an immersion of the standard polygon $P_{k}$ into the surface $S$, known as immersed disk. The domain of the immersion mapping is a standard polygon $P_{k}$, depicted in Figure 6.8a


Figure 6.8: Illustration of discrete immersed disks


Figure 6.9: Stitching together polygons yields immersed disks

To distinguish this type of immersed disks from the type used for the Fukaya category, we shall refer to these disks as discrete immersed disks. The precise definition reads as follows:

Definition 6.18. Let $\mathcal{A}$ be a full arc system with [NMD]. A discrete immersed disk in $\mathcal{A}$ consists of an oriented immersion $D: P_{k} \rightarrow S$ of a standard polygon $P_{k}$ into the surface, such that

- The edges of the polygon are mapped to a sequence of arcs.
- The immersion does not cover any punctures.

The immersion mapping $D$ itself is only taken up to reparametrization. The sequence of interior angles of $D$ is the sequence of angles in $\mathcal{A}$ given as images of the interior angles of $P_{k}$ under the map $D$. An angle sequence $\alpha_{1}, \ldots, \alpha_{k}$ is a disk sequence if it is the sequence of interior angles of some discrete immersed disk.

To explain the definition in other words, the image of the interior of the polygon $P_{k}$ consists only of polygon interiors of $\mathcal{A}$ and arcs between punctures, but not punctures themselves. The boundary of $P_{k}$ is mapped to a sequence of arcs, and the corners inside $P_{k}$ are mapped to an angle sequence in the arc system.

Example 6.19. The interior angles of a polygon, when written in clockwise order, are a disk sequence. In particular, if the $\operatorname{arc}$ system $\mathcal{A}$ has a triangle polygon in it, then there exists an disk sequence in $\mathcal{A}$ of just 3 angles. In every case, by the ban on loops and homotopic arcs, an disk sequence $\alpha_{1}, \ldots, \alpha_{k}$ consists of at least three angles, i.e. $k \geq 3$. In Figure 6.8a we have depicted the schematic of a standard polygon. In Figure 6.8 b , we have depicted a discrete immersed disk together with its sequence of interior angles. By definition, this sequence is a disk sequence. In Figure 6.8c we have depicted an angle sequence which is not a disk sequence. The reason it is not a disk sequence is that there is a polygon immersion bounded by the drawn angles, but it covers the puncture at the center of the hexagon. Later on, we will however allow polygon immersions which cover punctures as part of the deformation $\mathrm{Gtl}_{q} \mathcal{A}$.

Disk sequences $\alpha_{1}, \ldots, \alpha_{k}$ can also be described combinatorically: They are either a polygon in $\mathcal{A}$, or stitched together from multiple polygons along arcs. Figure 6.9 depicts two examples of stitching polygons together to form disk sequences. In every example, multiple triangles are stitched together to form a polygon. Thick connectors between two triangles indicate that these triangles are going to be stitched together along their shared edge. The first example is visually easy to grasp, since the three triangles are disjoint. In the second example, seven triangles are stitched together, with one triangle appearing twice. The result is a disk sequence of nine angles $\alpha_{1}, \ldots, \alpha_{9}$, of which one is longer than a full turn. The sketch on the right of the "=" sign provides a visualization of this discrete immersed disk by thinking a third dimension into the picture. In that 3 -dimensional sketch, the angle longer than a full turn is drawn dashed, the other angles are omitted and the outer boundary of the hexagon is depicted as a spiral instead of separate arcs.

Remark 6.20. In section 6.6, we change the terminology. From there on, discrete immersed disks are allowed to cover punctures.

We are now ready to give the definition of the higher products $\mu^{\geq 3}$. Since these products are supposed to be multilinear, it suffices to define them on the basis of $\mathrm{Gtl} \mathcal{A}$ given by angles winding around punctures.

Definition 6.21. Let $\mathcal{A}$ be an full arc system with [NMD]. Then Gtl $\mathcal{A}$ is the $A_{\infty}$-category with objects being the arcs $a \in \mathcal{A}$, hom spaces spanned by angles, and $A_{\infty}$-product $\mu$ defined by $\mu^{1}=0$ and $\mu^{2}(\alpha, \beta)=$ $(-1)^{|\beta|} \alpha \beta$. To define $\mu^{k \geq 3}$, let $\alpha_{1}, \ldots, \alpha_{k}$ be any disk sequence, let $\beta$ be an angle composable with $\alpha_{1}$, i.e. $\beta \alpha_{1} \neq 0$, and let $\gamma$ be an angle post-composable with $\alpha_{k}$, i.e. $\alpha_{k} \gamma \neq 0$. Then

$$
\mu^{k}\left(\beta \alpha_{k}, \ldots, \alpha_{1}\right):=\beta, \quad \mu^{k}\left(\alpha_{k}, \ldots, \alpha_{1} \gamma\right):=(-1)^{|\gamma|} \gamma
$$

The higher products vanish on all angle sequences other than these.
Example 6.22. Let us go through a few examples. Regard the one-punctured torus of Figure 6.6b The angles $\alpha$ and $\gamma$ are odd, and $\beta$ and $\delta$ are even. Angle degrees add up, for instance $\beta \gamma \delta$ is odd and $\gamma \delta \alpha$ is even. The product $\mu^{2}$ has

$$
\mu^{2}(\delta, \gamma)=-\delta \gamma, \quad \mu^{2}(\alpha \beta \gamma \delta \alpha, \beta \gamma \delta)=-\alpha \beta \gamma \delta \alpha \beta \gamma \delta=-(\alpha \beta \gamma \delta)^{2} \text { and } \mu^{2}(\beta, \alpha)=0
$$

The higher product $\mu^{3}$ vanishes because there are no triangles. From $\mu^{4}$ onwards, we have higher products, for instance

$$
\mu^{4}(\delta, \gamma, \beta, \alpha)=\operatorname{id}_{b} \text { and } \mu^{6}(\gamma, \beta \gamma, \beta, \alpha, \delta \alpha, \delta)=\operatorname{id}_{a}
$$

A little less obvious is the $A_{\infty}$-product

$$
\mu^{12}(\alpha, \delta, \gamma \delta, \gamma, \beta \gamma, \beta, \alpha \beta, \alpha, \delta \alpha, \delta, \gamma, \beta \gamma \delta \alpha \beta)=\operatorname{id}_{a}
$$

which is "winds" one and a quarter times around the puncture, without covering the puncture itself though. The second disk sequence in Figure 6.9 is very similar and also yields an identity.

After defining this structure, Bocklandt 18 proved that with this grading and products $\mathrm{Gtl} \mathcal{A}$ is indeed an $A_{\infty}$-category.

Theorem 6.23 ( 18$]$ ). Let $\mathcal{A}$ be a full arc system with [NMD]. Then $\mathrm{Gtl} \mathcal{A}$ is an $A_{\infty}$-category.
Remark 6.24. In 18, the signs in the definition of the higher products $\mu^{k}$ on $\mathrm{Gtl} \mathcal{A}$ differ from the signs presented here. We follow the sign convention of 35.

Remark 6.25. Every angle sequence $\alpha_{1}, \ldots, \alpha_{k}$ either bounds a unique discrete immersed disk or no disk at all. If it bounds a disk, then the products $\mu\left(\alpha_{k}, \ldots, \alpha_{1} \gamma\right)$ and $\mu\left(\beta \alpha_{k}, \ldots, \alpha_{1}\right)$ are nonzero. If it bounds no disk, then the products vanish.

Remark 6.26. Let us explain that that the degrees match. The $A_{\infty}$-product $\mu^{k}$ is required to be of parity $2-k$. If $\alpha_{1}, \ldots, \alpha_{k}$ is any disk sequence, then the total reduced degree

$$
\left\|\alpha_{1}\right\|+\ldots+\left\|\alpha_{k}\right\| \in \mathbb{Z} / 2 \mathbb{Z}
$$

measures how often the boundary of the discrete immersed disk changes orientation when traversing it clockwise. Since a disk sequence traverses the boundary one full time, it ends up with the same orientation as it started. In other words, the total reduced degree of a disk sequence vanishes. This means that $\mu^{k}$ has the right parity.

Remark 6.27. The interior angles of a discrete immersed disk are enumerated clockwise as $\alpha_{1}, \ldots, \alpha_{k}$, while the higher product consumes them only in the order $\alpha_{k}, \ldots, \alpha_{1}$. This seemingly unusual order of the factors $\alpha_{1}, \ldots, \beta \alpha_{k}$ is due to the convention on $A_{\infty}$-categories.

Remark 6.28. It is not possible to write a given angle sequence $\gamma_{k}, \ldots, \gamma_{1}$ as $\beta \alpha_{k}, \ldots, \alpha_{1}$ or $\alpha_{k}, \ldots, \alpha_{1} \gamma$ in two different ways. In fact, the immersion of the polygon is already determined by all angles but one, and an angle sequence of the form $\beta \alpha_{k}, \ldots, \alpha_{1}$ cannot be written as $\alpha_{k}^{\prime}, \ldots, \alpha_{1}^{\prime} \gamma$ with both $\alpha_{1}, \ldots, \alpha_{k}$ and $\alpha_{1}^{\prime}, \ldots, \alpha_{k}^{\prime}$ being disk sequences. This is explained e.g. in 35. We conclude that any angle sequence can be written in at most one way as $\beta \alpha_{k}, \ldots, \alpha_{1}$ or $\alpha_{k}, \ldots, \alpha_{1} \gamma$ with $\alpha_{1}, \ldots, \alpha_{k}$ a disk sequence. This makes the product $\mu^{k}\left(\gamma_{k}, \ldots, \gamma_{1}\right)$ well-defined for every angle sequence $\gamma_{1}, \ldots, \gamma_{k}$.

### 6.6 The deformation $\operatorname{Gtl}_{q} \mathcal{A}$

In this section we define the deformation $\operatorname{Gtl}_{q} \mathcal{A}$ of a gentle algebra $\mathrm{Gtl} \mathcal{A}$. It is a specific instance of the deformations constructed in Paper I and lies at the heart of the present paper. We give an explicit definition in order to provide a feel for this category. We provide a first glance concerning the use of $\operatorname{Gtl}_{q} \mathcal{A}$ in the later sections.

This paper is the second in a series of three, and the first paper Paper I was concerned with classifying the $A_{\infty}$-deformations of $\operatorname{Gtl} \mathcal{A}$. A conclusion from that paper is that all deformations of $\mathrm{Gtl} \mathcal{A}$ up to gauge equivalence can be written down explicitly. In this paper, we select one of these deformations, which we call the deformed gentle algebra and denote by $\operatorname{Gtl}_{q} \mathcal{A}$. Other deformations of $\mathrm{Gtl} \mathcal{A}$ play no role anymore.

The deformation $\operatorname{Gtl}_{q} \mathcal{A}$ is very broad in the sense that it has a lot of deformation parameters, in fact one for each puncture. Any reader who wishes to work with the calculations of this paper can therefore freely set some of these deformation parameters to zero and still have an interesting deformation at hand. Conversely, the deformation $\operatorname{Gtl}_{q} \mathcal{A}$ is so broad that the reader who is interested in deformations not "covered" by $\operatorname{Gtl}_{q} \mathcal{A}$ can still derive qualitative expectations on the behavior of the other deformations.
Remark 6.29. Arguably, one would like to conduct the study of the present paper also for all the other deformations given in Paper I The idea would be to use multiple parameters per puncture, so as to include deformations in $\operatorname{Gtl}_{q} \mathcal{A}$ that measure orbigons around punctures (see Paper I). One reason we restrict to the single deformation $\operatorname{Gtl}_{q} \mathcal{A}$ is that "orbigon deformations" are more difficult to handle than "disk deformations". Another reason is that the relative Fukaya category relFuk $(S, M)$ also has only "disk deformations" as well, so a candidate for a small model of $\operatorname{relFuk}(S, M)$ should only have "disk deformations" at all. This is why we only regard the deformation $\operatorname{Gtl}_{q} \mathcal{A}$, which has one parameter per puncture.

The deformation base of $\operatorname{Gtl}_{q} \mathcal{A}$ is $B=\mathbb{C} \llbracket M \rrbracket$. This is the commutative local ring of power series in $|M|$ variables, one for each puncture. In fact, to capture the punctures covered by an immersion of a standard polygon, every puncture should have one deformation parameter. For this reason we use $B=\mathbb{C} \llbracket M \rrbracket$. Every puncture $q \in M$ gives rise to one deformation parameter, which is also denoted $q$ and lies in the ring $\mathbb{C} \llbracket M \rrbracket$ as one of the generators.
Remark 6.30. We use the letter " $q$ " as in three different meanings in this paper, depending on the context: First, the notation $\operatorname{Gtl}_{q} \mathcal{A}$ is fixed and the letter $q$ does not have any meaning there. Second, whenever a specific puncture is considered, it is typically named $q$. Third, whenever $q$ is used multiplicatively in formulas, then it denotes the infinitesimal parameter $q \in \mathbb{C} \llbracket M \rrbracket$. For example, if $p, q \in M$ are punctures, then $p q$ simply means the product $p q \in \mathbb{C} \llbracket M \rrbracket$.

As a warm-up for the definition of $\operatorname{Gtl}_{q} \mathcal{A}$, recall that the angle sequence $\alpha_{1}, \ldots, \alpha_{6}$ of Figure 6.10 c is not a disk sequence. This means that $\mu^{6}\left(\alpha_{6}, \ldots, \alpha_{1}\right)=0$ in $\operatorname{Gtl} \mathcal{A}$. The deformation $\mathrm{Gtl}_{q} \mathcal{A}$ precisely changes this and similar higher products, while keeping the $A_{\infty}$-relations intact. In short, the deformed higher products of $\operatorname{Gtl}_{q} \mathcal{A}$ precisely capture which and how often punctures are covered by an immersion of a standard polygon. From here on, we drop the requirement that a discrete immersed disk does not cover punctures:
Definition 6.31. Let $\mathcal{A}$ be a full arc system with [NMDC]. A discrete immersed disk in $\mathcal{A}$ is an oriented immersion of a standard polygon $P_{k}$ into $S$ up to reparametrization such that the edges of the polygon are mapped to a sequence of arcs. A disk sequence is an angle sequence together with a choice of discrete immersed disk of which it is the sequence of interior angles. We denote by $M\left(\alpha_{1}, \ldots, \alpha_{k}\right)$ the set of discrete immersed disks $D$ with interior angles $\alpha_{1}, \ldots, \alpha_{k}$. For $D \in M\left(\alpha_{1}, \ldots, \alpha_{k}\right)$, we denote by $q_{D} \in \mathbb{C} \llbracket M \rrbracket$ the product of the punctures covered by $D$.
Remark 6.32. In contrast to section 6.5 for a given angle sequence $\alpha_{1}, \ldots, \alpha_{k}$ there might be multiple discrete immersed disks which have the same interior angle sequence $\alpha_{1}, \ldots, \alpha_{k}$. To see this, regard Bennequin's curve in Figure 6.10a This smooth curve bounds five immersed disks which are not related by reparametrization. If we draw a fine enough grid in the surface and approximate the curve by arcs in the grid, then we obtain an angle sequence which bounds multiple distinct discrete immersed disks.

The deformation $\operatorname{Gtl}_{q} \mathcal{A}$ has infinitesimal curvature, and there are three ways to describe the curvature: Each puncture $q \in M$ contributes curvature $q \ell_{q} \in(M) \widehat{\otimes} \operatorname{Gtl} \mathcal{A}$ to $\operatorname{Gtl}_{q} \mathcal{A}$. Here $(M) \subseteq \mathbb{C} \llbracket M \rrbracket$ denotes the maximal ideal of $\mathbb{C} \llbracket M \rrbracket$ and $\ell_{q}$ denotes the sum of all full turns around $q$, summed over all arc incidences at $q$. In other words, the total curvature $\mu_{q}^{0}$ of $\operatorname{Gtl}_{q} \mathcal{A}$ is the sum over all puncture contributions:

$$
\mu_{q}^{0}:=\sum_{q \in M} q \ell_{q}
$$



Figure 6.10: Illustration of the deformation $\mathrm{Gtl}_{q} \mathcal{A}$

In yet other words, we can describe the individual curvature of an arc $a \in \mathcal{A}$. It carries curvature $\mu_{q, a}^{0}$ given as the sum of the two turns around its endpoints, multiplied by the deformation parameters $h(a), t(a) \in \mathbb{C} \llbracket M \rrbracket$ associated with the arc's endpoints.

In order to define the products on $\mathrm{Gtl}_{q} \mathcal{A}$, it suffices to describe them on basis elements of $\mathrm{Gtl} \mathcal{A}$. The continuous $\mathbb{C} \llbracket M \rrbracket$-multilinear extension is then automatic, according to section 5.2 . We are now ready to state the definition of $\mathrm{Gtl}_{q} \mathcal{A}$ :

Definition 6.33. Let $\mathcal{A}$ be a full arc system with [NMDC]. The deformed gentle algebra $\operatorname{Gtl}_{q} \mathcal{A}$ is the deformation of $\operatorname{Gtl} \mathcal{A}$ over $\mathbb{C} \llbracket M \rrbracket$ given by:

- curvature $\mu_{q}^{0}=\sum_{q \in M} q \ell_{q}$,
- differential $\mu_{q}^{1}=0$ still vanishing,
- product $\mu_{q}^{2}=\mu^{2}$ is not deformed,
- higher products $\mu_{q}^{\geq 3}$ as follows: Let $\alpha_{1}, \ldots, \alpha_{k}$ be an angle sequence and let $\beta, \gamma$ be angles such that $\beta \alpha_{1} \neq 0$ and $\alpha_{k} \gamma \neq 0$. Then set

$$
\begin{aligned}
\mu_{q}^{k}\left(\beta \alpha_{k}, \ldots, \alpha_{1}\right) & =\sum_{D \in M\left(\alpha_{1}, \ldots, \alpha_{k}\right)} q_{D} \beta \\
\mu_{q}^{k}\left(\alpha_{k}, \ldots, \alpha_{1} \gamma\right) & =\sum_{D \in M\left(\alpha_{1}, \ldots, \alpha_{k}\right)}(-1)^{|\gamma|} q_{D} \gamma
\end{aligned}
$$

Example 6.34. For reference, let us go through a few example evaluations: The torus of Figure 6.6b has one puncture, two arcs and four interior angles. The gentle algebra $\mathrm{Gtl} \mathcal{A}$ therefore has two objects and four generators of the morphism spaces. The figure also includes a list of basis elements for $\mathrm{Gtl} \mathcal{A}$. What is the deformation $\operatorname{Gtl}_{q} \mathcal{A}$ here? Since there is just one puncture $q \in M=\{q\}$, the deformation base for $\mathrm{Gtl}_{q} \mathcal{A}$ is $B=\mathbb{C} \llbracket q \rrbracket$. The $\operatorname{arcs} a$ and $b$ get curvature

$$
\mu_{a}^{0}=q \alpha \beta \gamma \delta+q \gamma \delta \alpha \beta \text { and } \mu_{b}^{0}=q \beta \gamma \delta \alpha+q \delta \alpha \beta \gamma .
$$

The product $\mu^{2}$ remains non-deformed, for example we still have $\mu_{q}^{2}(\delta, \gamma)=-\delta \gamma$ and $\mu_{g}^{2}(\beta, \alpha)=0$ as in the non-deformed case, but we can now also insert parameters as in $\mu_{q}^{2}\left(q \delta, q^{2} \gamma\right)=-q^{3} \delta \gamma$. The higher product $\mu^{3}$ remains zero, because there are no triangles. The higher product $\mu^{4}(\delta, \gamma, \beta, \alpha)=\operatorname{id}_{b}$ has still the non-deformed value, and $\mu^{6}(\gamma, \beta \gamma, \beta, \alpha, \delta \alpha, \delta)=\operatorname{id}_{a}$. Deformed products appear for example in

$$
\mu_{q}^{8}(\delta, \gamma \delta, \gamma, \beta \gamma, \beta, \alpha \beta, \alpha, \delta \alpha)=q \operatorname{id}_{b}
$$

the sequence is inscribed in a 2 -by- 2 rectangle covering the puncture once. More generally, we have the $(m-1)$-by- $(n-1)$ rectangles covering the puncture $q$ precisely $(m-1)(n-1)$ times:

$$
\mu_{q}^{2 m+2 n}(\delta, \underbrace{\gamma \delta, \ldots, \gamma \delta}_{m-1}, \gamma, \underbrace{\beta \gamma, \ldots, \beta \gamma}_{n-1}, \beta, \underbrace{\alpha \beta, \ldots, \alpha \beta}_{m-1}, \alpha, \underbrace{\delta \alpha, \ldots, \delta \alpha}_{n-1})=q^{(m-1)(n-1)} .
$$

Remark 6.35. We view $\operatorname{Gtl} \mathcal{A}$ as a $\mathbb{Z} / 2 \mathbb{Z}$-graded $A_{\infty}$-category and $\operatorname{Gtl}_{q} \mathcal{A}$ as a $\mathbb{Z} / 2 \mathbb{Z}$-graded deformation. Let us explain why the deformed products $\mu_{q}^{k}$ satisfy the requirement to be of degree $2-k$. First, the


Figure 6.11: A zigzag path $L$
curvature $\mu_{q, a}^{0}$ on every arc $a \in \mathcal{A}$ consists of a full turn and is automatically even. Second, regard e.g. the deformed higher product

$$
\mu_{q}^{k}\left(\beta \alpha_{k}, \ldots, \alpha_{1}\right)=q_{1} \ldots q_{m} \beta
$$

Here the angles $\alpha_{1}, \ldots, \alpha_{k}$ are a disk sequence possibly covering the punctures $q_{1}, \ldots, q_{m}$. In fact, the disk sequence ends at the opposite side of the arc as it started, so the total reduced degree

$$
\left\|\alpha_{1}\right\|+\ldots+\left\|\alpha_{k}\right\|
$$

is still even, which means that $\beta$ has the same parity as $\alpha_{1}, \ldots, \beta \alpha_{k}$ plus $2-k$. This affirms that the products of $\mathrm{Gtl}_{q} \mathcal{A}$ have the right degree.

Remark 6.36. In the $\mathbb{Z}$-graded world, the deformation $\operatorname{Gtl}_{q} \mathcal{A}$ does not exist. Indeed, the curvature $\mu_{q, a}^{0}$ of an arc $a \in \mathcal{A}$ is a full turn and its degree depends on the vector field used for the $\mathbb{Z}$-grading. It is still possible in the $\mathbb{Z}$-grading to define a deformation $\operatorname{Gtl}_{q} \mathcal{A}$ which only includes those punctures where the full turn has degree 2. This would then also give the deformed higher products the right degree. But such a deformation has far fewer deformation parameters and is less interesting than the $\mathbb{Z} / 2 \mathbb{Z}$-graded version.

In Paper I we have already defined $\operatorname{Gtl}_{q} \mathcal{A}$ from a slightly more general starting point. In fact, the starting point of Paper I is a deformation base $B$ and a deformation parameter $r \in \mathfrak{m} Z(\mathrm{Gtl} \mathcal{A})$, where $Z(\operatorname{Gtl} \mathcal{A})$ denotes the center of $\operatorname{Gtl} \mathcal{A}$ as an ordinary algebra. To the datum of $r$, the construction in Paper I associates a deformed $A_{\infty}$-structure $\mu^{r}$. In order to obtain the specific case of $\mu_{q}$ from this construction, we have to put $B=\mathbb{C} \llbracket M \rrbracket$ and $r=\sum_{q \in M} q \ell_{q}$. We have checked in Paper I that $\mathrm{Gtl}_{r} \mathcal{A}$ satisfies the curved $A_{\infty}$-relations. In particular, this holds for the special case $\mathrm{Gtl}_{q} \mathcal{A}$ :

Theorem 6.37 Paper I]. Let $\mathcal{A}$ be a full arc system with [NMDC]. Then $\operatorname{Gtl}_{q} \mathcal{A}$ is an $A_{\infty}$-deformation of $\operatorname{Gtl} \mathcal{A}$.

### 6.7 Zigzag paths

In this section, we recall the notion of zigzag paths. These are combinatorical tools defined specifically for dimers. The idea is to follow the arrows of a dimer by alternatingly turning left and right. In the presentation of zigzag paths, we mostly follow 18 and 23 .

Zigzag paths appear in this paper for two reasons: First, one uses them to define whether a dimer $Q$ is zigzag consistent or not. Second, zigzag paths themselves can be realized as twisted complexes in $\mathrm{Tw} \mathrm{Gtl}_{q} Q$, and the aim of this paper is to compute the minimal model of this category. Of course, it is not a coincidence that zigzag paths appear twice: The zigzag consistency of $Q$ will help us perform the minimal model calculations of zigzag paths by ruling out lots of difficult cases. We recall zigzag paths as follows:

Definition 6.38. Let $Q$ be a dimer. A zigzag path $L$ is an infinite path $\ldots a_{2} a_{1} a_{0} a_{-1} a_{-2} \ldots$ of arcs in $Q$ together with an alternating choice of "left" or "right" for every $i \in \mathbb{N}$ such that

- $a_{i+1} a_{i}$ lies in a clockwise polygon if $i$ is assigned "right",
- $a_{i+1} a_{i}$ lies in a counterclockwise polygon if $i$ is assigned "left".

We also say that $L$ turns left at $a_{i}$ if $i$ is assigned "left" and turns right if $a_{i}$ is assigned "right". Two zigzag paths are identified if their paths including left/right indications differ only by integer shift.

Since $Q$ is finite, every zigzag path is periodic and has a shortest period $i_{0} \in \mathbb{N}$, which is defined as the smallest integer such that the zigzag path is invariant under shift by $i_{0}$. The period is not necessarily reached when an arc reappears in the zigzag path. The path may namely continue in a different way beyond that arc. In general, the period need not even be reached when a whole sequence of arcs reappears in the zigzag path. An example is depicted in Figure 6.12

Definition 6.39. The length of a zigzag path is the shortest period $i_{0} \in \mathbb{N}$.


Figure 6.12: Despite sharing multiple arcs, the two strands continue differently and do not finish a period.


Figure 6.13: On consistency

Slightly simplified, a zigzag path $L$ is a path in $Q$ that turns alternatingly maximally right and maximally left in $Q$. The typical shape of a zigzag path is drawn in Figure 6.11 If every puncture of $Q$ has valence at least 4 , then a path of length two cannot simultaneously lie in the boundary of a clockwise and a counterclockwise polygon. In this case, the left/right indication for zigzag paths is a superfluous part of the datum of a zigzag path. For other dimers $Q$, the left/right indication is very important. An example is the $n$-punctured sphere $Q_{n}$ of Figure 6.5a If $n$ is odd, then $Q_{n}$ has only one zigzag path, its length is $2 n$. If $n$ is even, then $Q_{n}$ has two zigzag paths each of length $n$. This way, we deviate slightly from the definition of 18 .

### 6.8 Geometric consistency

In this section, we recall what it means for a dimer to be geometrically consistent. This notion is important for the paper, because we will permanently work with a fixed geometrically consistent dimer, see Convention 10.10 Geometric consistency is a specific instance of various consistency conditions which can be imposed on dimers. A summary can be found in 17 , which we also follow here. In this section, we recall universal covers and zigzag rays and then define geometric consistency.

As first step, we recall the universal cover of $Q$. Recall that $Q$ itself consists of a closed surface $|Q|$ together with an arc system that satisfies the dimer property. Regard the universal cover $|\tilde{Q}|$ of the closed surface $|Q|$. We can construct an arc system on $|\tilde{Q}|$ by lifting all punctures and arcs to the universal cover, in all possible ways. This gives an (infinite) arc system on $|\tilde{Q}|$ which also has the dimer property. The universal cover together with its lifted arc system is simply denoted $\tilde{Q}$.

As second step, we recall the notion of zigzag rays. In contrast to zigzag paths, zigzag rays only run in one direction, starting from a given arc. Since we only need zigzag rays in the context of the universal cover, let us directly formulate their definition in $\tilde{Q}$. The four zigzag rays starting at an arc $a \in \tilde{Q}_{1}$ are depicted in Figure 6.13a.
Definition 6.40. Let $Q$ be a dimer, $\tilde{Q}$ its universal cover and $a \in \tilde{Q}_{1}$ an arc. Then the four zigzag rays starting at $a$ are the sequences of $\operatorname{arcs}\left(a_{i}^{1}\right)_{i \geq 0},\left(a_{i}^{2}\right)_{i \geq 0},\left(a_{i}^{3}\right)_{i \geq 0}$ and $\left(a_{i}^{4}\right)_{i \geq 0}$ in $\tilde{Q}$ determined by $a_{0}^{1}=a_{0}^{2}=a_{0}^{3}=a_{0}^{4}=a$ and the following properties:

- The sequences $\left(a_{i}^{1}\right)$ and $\left(a_{i}^{2}\right)$ satisfy $h\left(a_{i}^{1 / 2}\right)=t\left(a_{i+1}^{1 / 2}\right)$.
- The sequences $\left(a_{i}^{3}\right)$ and $\left(a_{i}^{4}\right)$ satisfy $t\left(a_{i}^{3 / 4}\right)=h\left(a_{i+1}^{3 / 4}\right)$.
- The path $a_{i+1}^{1 / 2} a_{i}^{1 / 2}$ lies in the boundary of a counterclockwise polygon when $i$ is odd/even, and clockwise when $i$ is even/odd.
- The path $a_{i}^{3 / 4} a_{i+1}^{3 / 4}$ lies in the boundary of a counterclockwise polygon when $i$ is odd/even, and clockwise when $i$ is even/odd.
A dimer is geometrically consistent if the zigzag rays starting with an arc $a$ in the universal cover intersect nowhere, except at $a$ itself. The precise definition reads as follows:


Figure 6.14: A puncture with four arc incidences

Definition 6.41. Let $Q$ be a dimer. Then $Q$ is geometrically consistent if for every $a \in \tilde{Q}_{1}$ the four zigzag rays $\left(a_{i}^{1}\right),\left(a_{i}^{2}\right),\left(a_{i}^{3}\right)$ and $\left(a_{i}^{4}\right)$ satisfy the following property: Whenever $a_{i}^{k}=a_{j}^{l}$, then $i=j$ and $k=l$, or $i=j=0$.
Example 6.42. A dimer $Q$ on a sphere is never geometrically consistent, because $\tilde{Q}=Q$ and therefore any zigzag rays in $\tilde{Q}$ intersect after a while. There are plenty of geometrically consistent dimers on surfaces of genus $g \geq 1$ though. For example, the $n$-punctured torus dimer of Figure 6.5b is geometrically consistent. Indeed, the universal cover of the torus is the real plane, and the torus dimer lifts to horizontally and vertically repeated copies of Figure 6.5 b . The zigzag rays then run away in different directions in the plane without ever coming closer to each other again. This geometry is a typical example of the toric zigzag fan, see for example 71 .
Remark 6.43. If $Q$ is geometrically consistent, then a zigzag path $L$ on $Q$ may return to an arc twice, however the segment of $L$ between both occurrences is not allowed to be contractible. If it were contractible, then this segment would constitute a zigzag ray cutting itself (the case $i=j$ and $k=l$ ), contradicting geometric consistency.

Remark 6.44. A geometrically consistent dimer automatically satisfies the [NMDC] condition.
Geometric consistency is the strongest consistency condition one can require, apart from R -charge consistency. Indeed, geometric consistency is by definition a stronger version of so-called zigzag consistency, which in turn is known to be stronger than cancellation consistency, see 17:

\[

\]

### 6.9 Terminology for arcs and angles

In this section, we introduce technical terminology that we will use throughout the paper. This terminology is important to describe exactly what happens where in a dimer. It bears no mathematical creativity but is unavoidable for the sake of concise language.

The first notion is that of an arc incidence. This is comparable to half-edges in a ribbon graph. Half-edges are not only a useful tool to describe graphs where one end of some edges is missing, but half-edges are also handy to describe incidences in a graph. Whenever we would like to sum over all edges incident at a given node, letting every loop contribute two (distinct) terms, the right entity to sum over is the set of half-edges incident at the node. Similarly in a dimer $Q$, we would typically like to have a set of all incidences of arcs, where loops contribute both their "head part" and their "tail part". With the terminology of head parts and tail parts, we can also talk about whether an angle starts at the head part or tail part of an arrow. This piece of terminology is depicted in Figure 6.15
Definition 6.45. An arc incidence at a puncture $q \in Q_{0}$ is either an incident head part or an incident tail part of some arc.

For instance, a loop $a \in Q_{1}$ with $h(a)=t(a)=q$ has two arc incidences at the puncture $h(a)=$ $t(a)$. The sample puncture in Figure 6.14 has four arc incidences. Correspondingly, there are four indecomposable angles around the puncture. In that figure, the loop is intended to be topologically nontrivial, indicated by the dots ". ..".

Let us now introduce some terminology for angles in $Q$. Angles always have both an algebraic interpretation as basis morphisms for $\mathrm{Gtl} Q$ and a geometric interpretation as winding around punctures in the surface $Q$. We will therefore use double terminology from time to time: In algebraic contexts, we say an angle is an identity if it is the identity $\mathrm{id}_{a}$ of some arc $a \in Q_{1}$. In geometric context, we call such an angle empty and all other angles non-empty. For instance, a typical usage in a geometric context would be to say that a certain angle $\alpha$ is non-empty and smaller than a full turn.


Figure 6.15: The angle $\alpha$ starts/ends at ...


Figure 6.16: A decomposable and an indecomposable angle

(a) Arc appearing twice in $L$ with different index

Figure 6.17: Terminology in a zigzag path

Definition 6.46. Let $\alpha$ be an angle in $Q$. Then $\alpha$ is an empty angle if it is the identity of some arc. Otherwise $\alpha$ is a non-empty angle.

Given an angle $\alpha$, we would like to distinguish whether it is composed of multiple smaller angles or not. By definition of the angles in $Q$, the smallest units are the interior angles of polygons. This already gives us terminology for a geometric context: We can simply ask whether a given angle $\alpha$ is the interior angle of some polygon or not. We however also need terminology for the algebraic context. Examples are depicted in Figure 6.16. We fix terminology as follows:

Definition 6.47. An angle is decomposable if it is the composition of two non-empty angles. An angle is indecomposable if it is non-empty and not decomposable.

Remark 6.48. An non-empty angle is indecomposable if it is an interior angle of some polygon, and indecomposable otherwise. We regard empty angles as neither decomposable nor indecomposable.

Let us introduce terminology for locations on zigzag paths. Loosely speaking, we want to define an "indexed arc" as an arc $a \in Q_{1}$ lying on $L$, but remember whether $L$ turns left or right after $a$. For example, let $2 k$ be the length of $L$, then $L$ has precisely $2 k$ indexed arcs. Figure 6.17a features a visual explanation: Some arc $a_{3}=a_{42}$ appears twice while traversing $L$, one time at index 3 and one time at index 42. The arc itself is the same in $Q$, but different as indexed arcs of $L$. This amount of precision gives rise to further names for relative positions on $L$. For instance, we can regard indexed segments, depicted in Figure 6.17b We fix terminology as follows:

Definition 6.49. Let $L$ be a zigzag path, given by an infinite path $\ldots a_{1} a_{0} a_{-1} \ldots$ together with left/right indications.

- An indexed arc on $L$ is a tuple $\left(a_{i}, i\right)$ consisting of one of the arcs on $L$ together with its index modulo the period length of the zigzag path.
- The next arc after $\left(a_{i}, i\right)$ is the indexed $\operatorname{arc}\left(a_{i+1}, i+1\right)$.
- The previous arc before $\left(a_{i}, i\right)$ is the indexed $\operatorname{arc}\left(a_{i-1}, i-1\right)$.
- Two consecutive indexed arcs are two indexed arcs on $L$ which can be written in the form $\left(a_{i+1}, i+\right.$ 1) and ( $a_{i}, i$ ) or the other way around.
- An indexed segment of length $k$ on $L$ is the datum of a tuple $\left(a_{i}, \ldots, a_{i+k-1}, i\right)$ of arcs on $L$, remembering the index $i$ modulo the period length of $L$.

Let us introduce terminology for angle sequences. We already have the very fortunate notion of disk sequences available, but in order to analyze products in $\mathrm{Tw} \mathrm{Gtl} Q$ we need flexible terminology to distinguish between the two rules that define higher products in Gtl $Q$. Let $\alpha_{1}, \ldots, \alpha_{k}$ be a disk sequence. Recall that the discrete immersed disk contained in the data of the disk sequence contributes to the product $\mu_{\operatorname{Gtl}_{q} Q}^{k}\left(\alpha_{k}, \ldots, \alpha_{1}\right)$. Now if $\beta$ is an angle such that $\beta \alpha_{k} \neq 0$ and $\gamma$ is an angle such that $\alpha_{1} \gamma \neq 0$, then the discrete immersed disk also contributes to the products $\mu_{\mathrm{Gtl}_{q} Q}^{k}\left(\beta \alpha_{k}, \ldots, \alpha_{1}\right)$ and $\mu_{\mathrm{Gtl}_{q} Q}^{k}\left(\alpha_{k}, \ldots, \alpha_{1} \gamma\right)$. We want to call these contributions final-out and first-out, respectively. This terminology is depicted in Figure 6.18 A more formal definition reads as follows:

Definition 6.50. Let $Q$ be a dimer. Let $\alpha_{1}, \ldots, \alpha_{k}$ be a disk sequence in $Q$ with discrete immersed disk $D$. Let $\beta, \gamma$ be non-empty angles such that $\beta \alpha_{k} \neq 0$ and $\alpha_{1} \gamma \neq 0$. Then:

- The sequence $\alpha_{1}, \ldots, \beta \alpha_{k}$ together with D is a final-out disk. We call $\beta$ the outside morphism and $\alpha_{k}$ the inside morphism. We call $t\left(\alpha_{1}\right)$ the first arc and $t\left(\alpha_{k}\right)$ the final arc.
- The sequence $\alpha_{1} \gamma, \ldots, \alpha_{k}$ together with $D$ is a first-out disk. We call $\gamma$ the outside morphism and $\alpha_{1}$ the inside morphism. We call $h\left(\alpha_{1}\right)$ the first arc and $h\left(\alpha_{k}\right)$ the final arc.
- The sequence $\alpha_{1}, \ldots, \alpha_{k}$ together with $D$ is an all-in disk. We call the $\operatorname{arc} t\left(\alpha_{1}\right)=h\left(\alpha_{k}\right)$ the first, equivalently final arc.

We may call an angle sequence together with a discrete immersed disk a some-out disk if it is first-out or final-out. In the case of a some-out disk, the first and final arc share an endpoint, the concluding puncture of the disk. In the case of an all-in disk, the first and final arc coincide, which is the concluding arc of the disk.

Loosely speaking, all contributions to $\mu{ }_{\overline{\mathrm{G} t 1_{q}} Q}^{\geq 3}$ come from first-out, final-out or all-in disks. Some-out means first-out or final-out. For a some-out disk, the concluding puncture is the one around which the first or final angle winds and it is very important. The first and final arcs are those arcs that neighbor the concluding puncture. For an all-in disk, the first and final arcs are the same and this single arc is very important. Whenever we refer to first-out, final-out or all-in disks, we typicall pass the datum of the discrete immersed disk implicitly. In Figure 6.18 , the first and final arc are drawn thick and the concluding puncture is marked with a dot. We may use wording like "towards the concluding puncture" when referring to the behavior of a sequence of arcs, viewed in the direction of the concluding puncture.

Last but not least, we shall give some means to measure how large the inside angle is by counting the "slots" inside and outside the disk. The terminology is included in Figure 6.18 We formalize this as follows:

Definition 6.51. Let $Q$ be a dimer. Let $\alpha_{1}, \ldots, \alpha_{k}$ be a some-out disk and $\gamma$ be its inside morphism. Write $\ell$ for one full turn around the concluding puncture, starting at the final arc of the disk. Write $\gamma=\gamma^{\prime} \ell^{n}$ for some $n$ such that $\gamma^{\prime}$ is strictly smaller than one full turn. Take the complementary angle $\left(\gamma^{\prime}\right)^{c}$ such that $\left(\gamma^{\prime}\right)^{c} \gamma^{\prime}=\ell$.

- The number of slots inside the disk is the number of indecomposable angles that $\gamma^{\prime}$ consists of.
- The number of slots outside the disk is the number of indecomposable angles that $\left(\gamma^{\prime}\right)^{c}$ consists of.


## 7 Preliminaries of Fukaya categories

In this section, we recall basics of Fukaya categories. One after another, we recall the construction of the Fukaya pre-category, Fukaya category, relative Fukaya pre-category and relative Fukaya category. The core aim of the paper is to define the category $\mathbb{L}_{q}$ and interpret its minimal model $H \mathbb{L}_{q}$ as a part of the relative Fukaya category. The present section aims to facilitate this understanding by preparing the view from the side of Fukaya categories. We have therefore included a dedicated description of the subcategory of the relative Fukaya category given by so-called zigzag curves in section 7.5. Our main references are 1, 26. We comment on results of Efimov, Sheridan and Perutz.

slots inside: 2 slots outside: 4
(a) A final-out disk

(b) An all-in disk

slots inside: 2 slots outside: 4
(c) A first-out disk

Figure 6.18: Illustration of final-out, all-in and first-out disks

### 7.1 The exact Fukaya pre-category

In this section, we review exact Fukaya pre-categories. They are not an immediate necessity for this paper, since we only work with the discrete $\operatorname{model}^{\mathrm{Gtl}_{q}} Q$. The main result however ties $\mathrm{H} \mathrm{Tw} \mathrm{Gtl}_{q} Q$ to the relative Fukaya category, so we will benefit from a review. We follow a combination of the highly recommendable sources 1, 29 and 16. Chapter 6].

In symplectic geometry, one aims at defining a fully-fledged $A_{\infty}$-category Fuk $X$ from a symplectic manifold $X$. A Fukaya category is supposed to have closed Lagrangians as objects and intersection points as basis elements for the hom spaces. The products $\mu^{\geq 2}$ are supposed to be formed from immersed disks between Lagrangians. For Lagrangians lying in general position, this construction works well. It is however not clear what the endomorphism space of a single Lagrangian $L$ should be. We would expect it to be a finite-dimensional vector space, and it should be equal for all small Hamiltonian deformations of $L$. This makes the full set of hom spaces and $A_{\infty}$ products of a Fukaya category very hard to define.

The difficulty in defining a fully-fledged Fukaya category Fuk $X$ has led to the introduction of precategories as partial remedy: Products need not be defined on all sequences of morphisms, only on a choice of transversal sequences.

Definition 7.1. An $A_{\infty}$-pre-category $\mathcal{C}$ consists of the following data:

1. a set of objects $\mathrm{Ob} \mathcal{C}$,
2. for every $N \geq 1$ a set $(\mathcal{C})_{\mathrm{tr}}^{N} \subseteq(\mathrm{Ob} \mathcal{C})^{N}$ of transversal sequences, with $(\mathcal{C})_{\mathrm{tr}}^{1}=\mathrm{Ob} \mathcal{C}$,
3. for every $(X, Y) \in(\mathcal{C})_{\mathrm{tr}}^{2}$ a graded hom space $\operatorname{Hom}(X, Y)$,
4. for every transversal sequence $X_{1}, \ldots, X_{N+1}$ with $N \geq 1$ a degree $2-N$ product map

$$
\mu^{N}: \operatorname{Hom}\left(X_{N}, X_{N+1}\right) \otimes \ldots \otimes \operatorname{Hom}\left(X_{1}, X_{2}\right) \rightarrow \operatorname{Hom}\left(X_{1}, X_{N+1}\right)
$$

such that each subsequence $\left(X_{i_{1}}, \ldots, X_{i_{l}}\right)$ with $1 \leq i_{1}<\ldots<i_{l} \leq n$ of a transversal sequence $X_{1}, \ldots, X_{n}$ is transversal as well, and the $A_{\infty}$-relation holds for $\operatorname{Hom}\left(X_{N}, X_{N+1}\right) \otimes \ldots \otimes \operatorname{Hom}\left(X_{1}, X_{2}\right)$ whenever $X_{1}, \ldots, X_{N+1}$ is a transversal sequence.

Remark 7.2. Staring at the definition seems to imply that the condition on transversal sequences is arbitrarily weak: Setting $(\mathcal{C})_{\mathrm{tr}}^{N}=\emptyset$ for all $N \geq 2$ is possible, and yields a completely vacuous notion of pre-category. The point of 29 is that if one strengthens the conditions suitably, then giving a pre- $A_{\infty^{-}}$ category is the same as giving a full $A_{\infty}$-category. We will comment on this later on. In particular, we will ensure that our definition of transversal sequences is such that it satisfies the condition in 29 .

Abouzaid's exposition 1 exhibits the Fukaya pre-category of a surface with boundary. In particular, we get from his paper a direct construction of the Fukaya pre-category of a punctured surface, by interpreting the punctures as boundary circles. We deviate from Abouzaid's definition by only including exact Lagrangians in the category. This makes it possible to dispose of the Novikov ring and work over $\mathbb{C}$ instead. We are now ready for the first definitions.

Our aim here is to write down the definition of the exact Fukaya pre-category of a punctured surface, such that it is a pre-category in the sense of Definition 7.1 Before we give the definition, we have to recall several concepts from 11: teardrops, spin structures, unobstructed curves, exact curves, transversal sequences of unobstructed curves, degrees of intersection points, immersed disks between unobstructed curves, and the Abouzaid sign rule. We recall these terminologies one by one.

(a) Teardrop

Figure 7.1: Terminology for unobstructed curves


Figure 7.2: Transversality

Definition 7.3. A teardrop of a curve $X: S^{1} \rightarrow S \backslash M$ is an immersion of the monogon $P_{1}$ into $S \backslash M$ which is bounded by a segment of $X$, such that the corner coming from $P_{1}$ is convex.

A teardrop is depicted in Figure 7.1a. In contrast, an interval winding around a puncture does not constitute a teardrop. A curve has a teardrop if and only if it contains an interval that is contractible in $S \backslash M$.

Definition 7.4. A spin structure on a curve consists of putting an arbitrary number of "stars" on distinct points of the curve. We also call these stars the \# signs on a curve.

We regard the stars as \# signs when we think of them as a negative sign -1 . The number of \# signs on a curve is arbitrary, but the resulting isomorphism class of the curve will in fact only depend on the parity of this number. In other words, zero or one \# sign suffice in practice.
Definition 7.5. An unobstructed curve in $S \backslash M$ is a smooth closed immersed curve $X: S^{1} \rightarrow S \backslash M$ with a choice of spin structure, such that $X$ is not contractible and does not bound a teardrop.

Let us now recall the notion of exact curves, a subset of the unobstructed curves. Exact curves serve as objects of the Fukaya (pre-)category. To introduce the notion, put an exact symplectic form $\omega=d \theta$ on $S \backslash M$. The 1-form $\theta$ is then also referred to as the Liouville form.

Definition 7.6. An unobstructed curve $X: S^{1} \rightarrow S \backslash M$ is exact if $X^{*} \theta$ is an exact 1-form, in other words if

$$
\exists f: S^{1} \rightarrow \mathbb{R}: \quad X^{*} \theta=d f, \quad \text { or } \quad \int_{X} \theta=0
$$

These two conditions are equivalent because $\int_{X} \theta=\int_{S^{1}} X^{*} \theta$. The latter integral vanishes if and only if $X^{*} \theta \in \Omega^{1}\left(S^{1}\right)$ has a primitive $f$.

Definition 7.7. A sequence $\left(X_{1}, \ldots, X_{N}\right)$ of unobstructed curves is transversal if

- For $i<j$ the curves $X_{i}$ and $X_{j}$ have only transversal intersection points.
- For $i<j<k$ the curves $X_{i}, X_{j}, X_{k}$ have no triple intersection: $X_{i} \cap X_{j} \cap X_{k}=\emptyset$.

According to the definition, an unobstructed curve is allowed to intersect itself, just as a selfintersection of one unobstructed curve $X_{i}$ is allowed to further intersect with a second unobstructed curve $X_{j}$.

Next, let us recall the degree assigned to an intersection point $p \in X_{1} \cap X_{2}$. The idea is that the intersection $p \in X_{1} \cap X_{2}$ serve as generators of $\operatorname{Hom}\left(X_{1}, X_{2}\right)$, so we have to assign a degree. Since the surface $S$ and the curves $X_{1}, X_{2}$ are oriented, we can distinguish the direction of $X_{1}$ and $X_{2}$ relative to each other at $p$. The degree we assign is depicted in Figure 7.3 In that figure, the shaded area has


Figure 7.3: Intersection degree

(a) Standard polygon $P_{5}$

(b) Immersed disk

(c) Convex corner

(d) Nonconvex corner

Figure 7.4: Immersed disks
no meaning in this definition, but indicates for the convenience of the reader how we are going to use such intersection points as corners of immersed disks. Note that $p$ can be interpreted both as element of $\operatorname{Hom}\left(X_{1}, X_{2}\right)$ and $\operatorname{Hom}\left(X_{2}, X_{1}\right)$. In fact, it has opposite parity in both hom spaces.

Definition 7.8. Let $p$ be a transversal intersection point of $X_{1}$ and $X_{2}$. Then $p$ as morphism from $X_{1}$ to $X_{2}$ is denoted $p: X_{1} \rightarrow X_{2}$. The morphism $p: X_{1} \rightarrow X_{2}$ is odd if a neighborhood of $p \in S$ can be identified in an oriented way with a neighborhood of the origin in $\mathbb{R}^{2}$, mapping $X_{1}$ to the oriented $x$-axis and $X_{2}$ to the oriented $y$-axis. Otherwise $p$ is even.

Let us recall the notion of smooth immersed disks between unobstructed curves. Despite their name, the disks have corners and are therefore actually polygons. We stick to the classical terminology however. Recall that $P_{N+1}$ denotes the standard oriented polygon in $\mathbb{R}^{2}$, with indexed $N+1$ clocNwise indexed edges and $N+1$ indexed corners. The $i$-th corner lies between the $i$-th and $(i+1)$-th edge.

Definition 7.9. Let $X_{1}, \ldots, X_{N+1}$ be a transversal sequence of $N+1 \geq 2$ unobstructed curves. Let $p_{1}, \ldots, p_{N}$ be a sequence of intersection points $p_{i}: X_{i} \rightarrow X_{i+1}$ and let $p \in X_{1} \rightarrow X_{N+1}$ be another intersection point. A smooth immersed disk with inputs $p_{1}, \ldots, p_{N}$ and output $p$ consists of an orientation-preserving polygon immersion $D: P_{N+1} \rightarrow S \backslash M$ up to reparametrization, such that

- the corners of $D$ are all convex,
- the $i$-th edge of $P_{N+1}$ lands on $X_{i}$ for $1 \leq i \leq N+1$,
- the $i$-th corner of $P_{N+1}$ lands on $p_{i}$.

We denote by $M\left(p_{1}, \ldots, p_{N}, p\right)$ the set of smooth immersed disks with inputs $p_{1}, \ldots, p_{N}$ and output $p$.
By convexity of the corners, we mean that the image of any interior angle of the polygon is strictly smaller than half a full turn. Here, an interior angle of the polygon is interpreted as a very small curve near any corner of $P_{N+1}$, and a half turn is the natural half turn given by the two sides of the tangent line of $X_{i}$ at $p_{i} \in S$. All this is depicted in Figure 7.4

Remark 7.10. Regarding the numbering of the disk inputs, we deviate from the Fukaya-theoretic literature. More precisely, the standard convention 11 is to number the disk inputs in counterclockwise order. Instead, we number the disk inputs in clockwise order. The difference is necessary in order to match with the convention for gentle algebras 18].

The orientation of a curve $X_{i}$ involved in a smooth immersed disk $D$ need not agree with the orientation of $\partial P_{N+1}$. We can give the boundary $\partial P_{N+1}$ the orientation pointing in clockwise direction and distinguish whether $X_{i}$ agrees with this orientation or not:
Definition 7.11. Let $D \in M\left(p_{1}, \ldots, p_{N}, p\right)$ be a smooth immersed disk, with $p_{i} \in X_{i} \cap X_{i+1}$. Then:

- $X_{i}$ is oriented clockwise with $D$ if the orientation of $X_{i}$ agrees with clockwise orientation of $\partial P_{N+1}$,


Figure 7.5: Non-exact curves on a torus have endless disks

- $X_{i}$ is oriented counterclockwise with $D$ if the orientation of $X_{i}$ is opposite to the clockwise orientation of $\partial P_{N+1}$.

The differences in orientation give rise to what we call the Abouzaid sign of the disk. This sign is taken from 11 and is the surface world incarnation of a sign rule in higher dimensions.

Definition 7.12. Let $D \in M\left(p_{1}, \ldots, p_{N}, p\right)$ be a smooth immersed disk with inputs $p_{i}: X_{i} \rightarrow X_{i+1}$ and output $p: X_{1} \rightarrow X_{N+1}$. Then the $\operatorname{Abouzaid} \operatorname{sign} \operatorname{Abou}(D) \in \mathbb{Z} / 2 \mathbb{Z}$ is the number of indices $i$ such that $p_{i}$ is odd and $X_{i+1}$ is oriented counterclockwise with $D$, plus one if $p$ is odd and $X_{N+1}$ is oriented counterclockwise with $D$, plus the number of \# signs from the spin structure on the boundary of the disk.

With all devices ready, we can recall the construction of the Fukaya pre-category.
Definition 7.13. Let $(S, M)$ be a punctured surface, with exact symplectic form $\omega=d \theta$. Then the exact Fukaya pre-category $\operatorname{Fuk}^{\text {pre }}(S, M)$ is defined as follows:

- The objects are the exact unobstructed curves in $S \backslash M$ with chosen spin structures.
- The set $\mathcal{C}_{\mathrm{tr}}^{N}$ consists of the tranversal sequences according to Definition 7.7
- For transversal $X, Y$, the hom space $\operatorname{Hom}(X, Y)$ is freely spanned over $\mathbb{C}$ by the intersection points $p \in X \cap Y:$

$$
\operatorname{Hom}(X, Y)=\bigoplus_{p \in X \cap Y} \mathbb{C} p
$$

- For any sequence $p_{1}, \ldots, p_{N}$ of intersection points $p_{i}: X_{i} \rightarrow X_{i+1}$ and $p: X_{1} \rightarrow X_{N+1}$, the higher product is defined as

$$
\mu^{N}\left(p_{N}, \ldots, p_{1}\right)=\sum_{p \in X_{1} \cap X_{N+1}} \sum_{D \in M\left(p_{1}, \ldots, p_{N}, p\right)}(-1)^{\mathrm{Abou}(D)} p
$$

Theorem $7.14(1)$. Fuk $^{\text {pre }}(S, M)$ is an $A_{\infty}$-pre-category.
Remark 7.15. Spin structures determine the signs in the higher products of the Fukaya category. A spin structure on a Lagrangian $X$ can however also be seen as a special case of local system on $X$ : One bakes the spin structure into the local system on $X$. Upon passing to a version of Fukaya category where each Lagrangian comes with a local system of any dimension assigned, the Fukaya category roughly becomes closed under taking cones. In fact, taking a cone amounts to adding up the local systems.

Remark 7.16. There are two reasons we only include exact Lagrangians in the Fukaya pre-category. First, we do not need non-exact curves in this paper at all, since the zigzag curves are already exact. The second reason is due to the Novikov field. Indeed, including non-exact curves allows the set $M\left(p_{1}, \ldots, p_{N}, p\right)$ of immersed disks to be infinite which requires the technical insertion of the Novikov field. The higher product coming from an immersed disk $D$ then gets multiplied by the formal power $t^{\omega(D)}$, where $\omega(D)$ denotes the symplectic area of $D$. This renders the Fukaya pre-category an $A_{\infty}$-precategory over the Novikov field. Since the aim of this paper is to compare the relative Fukaya category to the gentle algebra defined over $\mathbb{C}$, we have decided to avoid the Novikov field early on. It is an interesting question what a discrete model would look like for non-exact parts of the relative Fukaya category that can only be defined with the Novikov field. The discrete model would then need to be defined over the Novikov field instead of $\mathbb{C}$ and its higher products would need to be defined upon a notion of (algebraic) symplectic area.

### 7.2 The exact Fukaya category

In this section, we recall the notion of Fukaya category. We explain how one passes from the pre-category of the previous section to an actual category. In particular, we intend to make the reader acquainted with


Figure 7.6: A Lagrangian and its Hamiltonian deformation
the endomorphism spaces in the Fukaya category and how to extend the $A_{\infty}$-products from transversal sequences to all sequences of morphisms.

The characteristic property of the Fukaya category is that its transversal part is precisely the Fukaya pre-category. By transversal parts we mean the following:

Definition 7.17. Let $\mathcal{C}$ be an $A_{\infty}$-pre-category and $\mathcal{D}$ an $A_{\infty}$-category. Assume $\mathrm{Ob} \mathcal{C}=\mathrm{Ob} \mathcal{D}$. Then the transversal part of $\mathcal{D}$ (with respect to $\mathcal{C}$ ) is the $A_{\infty}$-pre-category ( $\left.\mathcal{D}\right)_{\text {tr }}$ defined by

$$
\begin{aligned}
\left((\mathcal{D})_{\mathrm{tr}}\right)_{\mathrm{tr}}^{N} & =(\mathcal{C})_{\mathrm{tr}}^{N}, \\
\mu_{(\mathcal{D})_{\mathrm{tr}}}^{N} & =\left.\mu_{\mathcal{D}}^{N}\right|_{(\mathcal{C})_{\mathrm{tr}}^{N+1}} .
\end{aligned}
$$

The category $\mathcal{D}$ agrees on the transversal part with $\mathcal{C}$ if $\mathcal{C}=(\mathcal{D})_{\mathrm{tr}}$.
We provide an ad-hoc definition of the Fukaya category as follows:
Definition 7.18. Let $(S, M)$ be a punctured surface. Then the exact Fukaya category $\operatorname{Fuk}(S, M)$ is any $A_{\infty}$-category which agrees with $\operatorname{Fuk}^{\mathrm{pre}}(S, M)$ on the transversal part.

Explicit construction of the Fukaya category exist. The standard reference is Seidel's work 64. The idea is to apply Hamiltonian deformations to make nontransversal pairs of Lagrangians transversal. Most importantly, hom spaces are then also defined for non-transversal pairs. For example, the endomorphism space $\operatorname{End}(L)$ of a Lagrangian $L$ contains an identity and a co-identity element, which we may in the Fukaya category context denote id and id*. The philosophy is that Hamiltonian deformation of $L$ yields a transversal version of $L$, intersecting precisely twice with $L$. This is depicted in Figure 7.6

Remark 7.19. There exist approaches of constructing the Fukaya category from the Fukaya pre-category by purely categorical methods. In 29, Efimov proved the conjecture attributed to Kontsevich-Soibelman that every $A_{\infty}$-pre-category is quasi-equivalent to an $A_{\infty}$-category as $A_{\infty}$-pre-categories. The quasiequivalence relation for $A_{\infty}$-pre-categories is defined in 29, Definition 2.18/2.19]. For ordinary $A_{\infty^{-}}$ categories, this notion coincides with the ordinary notion of quasi-equivalence.

Efimov's theorem implies there is an $A_{\infty}$-category quasi-equivalent to Fuk $^{\text {pre }}(S, M)$. This $A_{\infty}$ category in almost but not quite (a model for) the Fukaya pre-category in the sense of Definition 7.18 , since it may have larger hom spaces than the Fukaya pre-category even on the transversal sequences.

The higher products of non-transversal sequences become very difficult to grasp, since multiple Hamiltonian deformations may need to be performed on the same Lagrangian in order to make all intersections transversal. This results in ambiguities, resolved by providing additional deformation data. In summary, the higher products cannot be determined by simply staring at them. In contrast, our paper provides an explicit realization also of these higher products on non-transversal sequence, at least on zigzag paths. For more on exact Fukaya categories, we refer to 16 and 64 .

### 7.3 The relative exact Fukaya pre-category

In this section, we recall the relative exact Fukaya pre-category for punctured surfaces. The starting point of the relative exact Fukaya pre-category is the exact Fukaya pre-category. The idea is to deform the products by allowing the disk to cover punctures. The resulting object is what we will call an $A_{\infty}$-pre-category deformation.

The history of the subject can be traced back fairly accurately: In 63, Seidel introduced the idea of deforming the Fukaya category by working relative to a divisor. Twenty years later, the relative Fukaya category was finally constructed in 59. Its versality as a deformation of the ordinary Fukaya category was investigated in 66. Lekili, Perutz and Polishchuk 46, 47 proved deformed mirror symmetry for the $n$-punctured torus, apparently the first use of the relative Fukaya category in mirror symmetry.
Definition 7.20. Let $\mathcal{C}$ be an $A_{\infty}$-pre-category and $B$ a deformation base, e.g. $B=\mathbb{C} \llbracket q \rrbracket$. Then an $A_{\infty}$-pre-category deformation $\mathcal{C}_{q}$ of $\mathcal{C}$ is an (infinitesimally curved) and $B$-linear $A_{\infty}$-pre-category structure on $B \widehat{\otimes} \mathcal{C}$. More precisely, this means that $\mathcal{C}_{q}$ has:

- the same objects as $\mathcal{C}$,
- hom spaces $\operatorname{Hom}_{\mathcal{C}_{q}}(X, Y)=B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}(X, Y)$ for $X, Y \in \mathcal{C}_{\mathrm{tr}}^{2}$,
- $B$-multilinear products of degree $2-N$

$$
\mu_{q}^{N}: \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{N}, X_{N+1}\right) \otimes \ldots \otimes \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{1}, X_{2}\right) \rightarrow \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{1}, X_{N+1}\right), N \geq 1
$$

for all transversal sequences $X_{1}, \ldots, X_{N+1}$,

- curvature of degree 2 for every object $X \in \mathcal{C}$ with $(X, X) \in \mathcal{C}_{\mathrm{tr}}^{2}$ :

$$
\mu_{q, X}^{0} \in \mathfrak{m} \operatorname{Hom}_{\mathcal{C}_{q}}(X, X)
$$

such that $\mu_{q}$ reduces to $\mu$ once the maximal ideal $\mathfrak{m} \subseteq B$ is divided out, and $\mu_{q}$ satisfies the curved $A_{\infty}$ relations on transversal sequences.

With this definition in mind, we would like to define the relative version of the Fukaya pre-category. The idea is to define the higher products by counting smooth immersed disks, which are now also allowed to cover punctures. By abuse of terminology, we shall keep using the terminology of "smooth immersed disks" even for those smooth immersed disks which cover punctures:

Definition 7.21. Let $X_{1}, \ldots, X_{N+1}$ be a transversal sequence of $N+1 \geq 2$ unobstructed curves. Let $p_{1}, \ldots, p_{N}$ be a sequence of intersection points $p_{i}: X_{i} \rightarrow X_{i+1}$ and let $p \in X_{1} \rightarrow X_{N+1}$ be another intersection point. A smooth immersed disk with inputs $p_{1}, \ldots, p_{N}$ and output $p$ consists of an orientation-preserving polygon immersion $D: P_{N+1} \rightarrow S$ up to reparametrization, such that

- the corners of $D$ are all convex,
- the $i$-th edge of $P_{N+1}$ lands on $X_{i}$ for $1 \leq i \leq N+1$,
- the $i$-th corner of $P_{N+1}$ lands on $p_{i}$.

We denote by $M_{q}\left(p_{1}, \ldots, p_{N}, p\right)$ the set of smooth immersed disks with inputs $p_{1}, \ldots, p_{N}$ and output $p$.
The deformation base of the relative exact Fukaya pre-category is $B=\mathbb{C} \llbracket M \rrbracket$. This is the power series ring with one variable for each puncture $m \in M$. Correspondingly, every puncture $q \in M$ given an element $q \in \mathbb{C} \llbracket M \rrbracket$. Multiple punctures $q_{1}, \ldots, q_{s} \in M$ can be multiplied to form the element $q_{1} \ldots q_{s} \in \mathbb{C} \llbracket M \rrbracket$.

Definition 7.22. Let $D \in M_{q}\left(p_{1}, \ldots, p_{N}, p\right)$. Then the Abouzaid sign Abou $(D)$ is defined precisely as in Definition 7.12. The $q$-parameter $\operatorname{Punc}(D) \in \mathbb{C} \llbracket M \rrbracket$ is defined as the product of all the punctures reached by the interior of $P_{N+1}$ under $D$, counting multiplicities.

The parameter $\operatorname{Punc}(D)$ is very similar to the deformation parameter in the higher products of $\operatorname{Gtl}_{q} Q$ in section 6.6.

Definition 7.23. The relative exact Fukaya pre-category relFuk ${ }^{\text {pre }}(S, M)$ is the $A_{\infty}$-pre-category deformation of Fuk ${ }^{\text {pre }}(S, M)$ over $B=\mathbb{C} \llbracket M \rrbracket$ given by the deformed $A_{\infty}$-products on transversal sequences

$$
\mu_{q}^{N}\left(p_{N}, \ldots, p_{1}\right)=\sum_{p: X_{1} \rightarrow X_{N+1}} \sum_{D \in M_{q}\left(p_{1}, \ldots, p_{N}, p\right)}(-1)^{\operatorname{Abou}(D)} \operatorname{Punc}(D) p
$$

Checking that relFuk ${ }^{\text {pre }}(S, M)$ is really an $A_{\infty}$-pre-category deformation involves two parts. The first part consists of checking that the higher products $\mu_{q}^{N}$ are well-defined. Indeed, exactness guarantees that the set of disks not covering any puncture is finite, but the case of disks covering some punctures is non-trivial. A likely successful procedure is as follows: For each monomial $q=q_{1} \ldots q_{s} \in \mathbb{C} \llbracket M \rrbracket$, use exactness of the curves to bound the maximum size of disks that cover precisely the punctures $q_{1}, \ldots, q_{s}$. A standard argument is then that the Gromov compactness theorem implies the number of disks is finite. See e.g. 18, Section 6.2.3].

The second check consists of evaluating the $A_{\infty}$-relations on the transversal sequences. This boils down to re-doing the work of Abouzaid 1], now allowing the disks to cover punctures. The procedure should be straightforward and conclude that the $A_{\infty}$-relations still hold. In total, this renders $\operatorname{relFuk}^{\text {pre }}(S, M)$ an $A_{\infty}$-pre-category deformation of $\mathrm{Fuk}^{\mathrm{pre}}(S, M)$.

### 7.4 The relative exact Fukaya category

In this section, we recall the notion of relative exact Fukaya category. The starting point is the relative exact Fukaya pre-category which we sketched in section 7.3 . It is an $A_{\infty}$-pre-category deformation of $\mathrm{Fuk}^{\text {pre }}(S, M)$. In section 7.2 , we have seen that the category $\mathrm{Fuk}^{\text {pre }}(S, M)$ has a lift to an actual $A_{\infty}$-category $\operatorname{Fuk}(S, M)$. The question we discuss in this section is what a lift of the deformation $\operatorname{relFuk}^{\text {pre }}(S, M)$ to a deformation $\operatorname{relFuk}(S, M)$ of $\operatorname{Fuk}(S, M)$ should look like. Our desired lifting procedure is best captured graphically as follows:


Definition 7.24. Let $\mathcal{C}$ be an $A_{\infty}$-pre-category and $\mathcal{C}_{q}$ an $A_{\infty}$-pre-category deformation. Let $\mathcal{D}$ be an $A_{\infty}$-category and $\mathcal{D}_{q}$ an $A_{\infty}$-deformation. Assume $\operatorname{Ob} \mathcal{C}=\operatorname{Ob} \mathcal{D}$. Then the transversal part of $\mathcal{D}_{q}$ (with respect to $\mathcal{C}_{q}$ ) is the $A_{\infty}$-pre-category deformation $\left(\mathcal{D}_{q}\right)_{\mathrm{tr}}$ of $(\mathcal{D})_{\mathrm{tr}}$ defined by

$$
\mu_{\left(\mathcal{D}_{q}\right)_{\mathrm{tr}}}^{N}=\left.\mu_{\mathcal{D}_{q}}^{N}\right|_{(\mathcal{C})_{\mathrm{tr}}^{N+1}} .
$$

We may say that the deformation $\mathcal{D}_{q}$ agrees on the transversal part with $\mathcal{C}_{q}$ if $\mathcal{C}_{q}=\left(\mathcal{D}_{q}\right)_{\mathrm{tr}}$. For sake of explicitness in section 13.6, we provide the following terminology:
Definition 7.25. Let $\mathcal{C}$ and $\mathcal{D}$ be $A_{\infty}$-pre-categories and $\mathcal{C}_{q}$ and $\mathcal{D}_{q}$ be $A_{\infty}$-pre-category deformations. Then a strict isomorphism $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ of $A_{\infty}$-pre-category deformations consists of

- a bijection $F_{q}: \mathrm{Ob} \mathcal{C} \rightarrow \mathrm{Ob} \mathcal{D}$ such that

$$
\forall N \geq 1: \quad(\mathcal{D})_{\mathrm{tr}}^{N}=\left\{\left(F_{q} X_{1}, \ldots, F_{q} X_{N}\right) \mid\left(X_{1}, \ldots, X_{N}\right) \in(\mathcal{C})_{\mathrm{tr}}^{N}\right\}
$$

- for every $X, Y \in(\mathcal{C})_{\operatorname{tr}}^{2}$ a $B$-linear isomorphism $F^{1}: \operatorname{Hom}_{\mathcal{C}_{q}}(X, Y) \rightarrow \operatorname{Hom}_{\mathcal{D}_{q}}\left(F_{q} X, F_{q} Y\right)$ of degree 0 such that

$$
\begin{aligned}
& \forall N \geq 1, \quad\left(X_{1}, \ldots, X_{N+1}\right) \in(\mathcal{C})_{\mathrm{tr}}^{N+1}, \quad a_{i} \in \operatorname{Hom}_{\mathcal{C}}\left(X_{i}, X_{i+1}\right): \\
& \quad F_{q}^{1}\left(\mu_{\mathcal{C}_{q}}\left(a_{N}, \ldots, a_{1}\right)\right)=\mu_{\mathcal{D}_{q}}\left(F_{q}^{1}\left(a_{N}\right), \ldots, F_{q}^{1}\left(a_{1}\right)\right) .
\end{aligned}
$$

We provide an ad-hoc definition of the relative Fukaya category as follows:
Definition 7.26. Let $(S, M)$ be a punctured surface. Then the relative Fukaya category $\operatorname{relFuk}(S, M)$ is any $A_{\infty}$-deformation of $\operatorname{Fuk}(S, M)$ such that $\operatorname{relFuk}(S, M)_{\mathrm{tr}}=\operatorname{relFuk}^{\operatorname{pre}}(S, M)$.
Remark 7.27. Sheridan and Perutz 59 provide explicit constructions of relative Fukaya categories.
Remark 7.28. All concrete models of the Fukaya category give rise to a priori different notions of relative Fukaya category. It should be possible to show that these are in fact all isomorphic.

### 7.5 Zigzag paths as Lagrangians

In the present section, we show how to interpret zigzag paths as objects in Fukaya categories. The starting point is a dimer $Q$. It gives rise to a collection of zigzag paths and we show how to turn them into curves in $|Q|$ which we call "zigzag curves". We recall how to make these curves objects of the relative Fukaya pre-category and look at their hom spaces and higher products. The material can also be found in 26 Chapter 10].

The Fukaya category of a dimer is simply defined as the Fukaya category of its underlying punctured surface. More precisely, let $Q$ be a dimer. Then $Q$ includes the datum of a punctured surface $\left(|Q|, Q_{0}\right)$ and we write

$$
\begin{aligned}
\text { Fuk }^{\text {pre }} Q & =\operatorname{Fuk}^{\text {pre }}\left(|Q|, Q_{0}\right), & & \operatorname{Fuk} Q=\operatorname{Fuk}\left(|Q|, Q_{0}\right), \\
\operatorname{relFuk}^{\text {pre }} Q & =\operatorname{relFuk}^{\text {pre }}\left(|Q|, Q_{0}\right), & & \operatorname{relFuk} Q=\operatorname{Fuk}\left(|Q|, Q_{0}\right) .
\end{aligned}
$$

The first step in this section is to turn zigzag paths into curves. Let $L$ be a zigzag path in $Q$. Then $L$ consists by definition of a path $\ldots a_{1} a_{0} a_{-1} \ldots$ of composable arcs in $Q$, together with left/right indications. The idea to produce a curve $\tilde{L} \subseteq|Q| \backslash Q_{0}$ from $L$ is to stitch together the arcs $a_{i}$ in sequence, minding the left/right indication. The precise definition reads as follows:


Figure 7.7: Zigzag path and zigzag curve

Definition 7.29. Let $Q$ be a dimer and $L$ a zigzag path in $Q$. Then the associated zigzag curve $\tilde{L}: S^{1} \rightarrow S \backslash M$ is defined by connecting the midpoints of the arcs $\ldots, a_{-1}, a_{0}, a_{1}, \ldots$ by means of the angle cutting procedure:

- If $L$ turns left at index $i$, then the midpoint of $a_{i}$ is connected to the midpoint of $a_{i+1}$ by turning clockwise around the puncture $h\left(a_{i}\right)=t\left(a_{i+1}\right)$.
- If $L$ turns right at index $i$, then the midpoint of $a_{i}$ is connected to the midpoint of $a_{i+1}$ by turning counterclockwise around the puncture $h\left(a_{i}\right)=t\left(a_{i+1}\right)$.
The connecting arc between the midpoints of $a_{i}$ and $a_{i+1}$ is to be chosen close enough to the puncture that it does not intersect with the zigzag curves associated with the other zigzag paths.

In Figure 7.7, we have depicted a zigzag path together with its associated zigzag curve. By definition of $\tilde{L}$, we have the freedom to deform $\tilde{L}$ a bit near the punctures. We quote the following lemma from 26 which claims that a small deformation can be chosen in such a way that that $\tilde{L}$ becomes an exact Lagrangian. The condition is that the Liouville form $\alpha$ near the punctures takes the shape $\alpha=d \theta / r$ where $r$ is the distance from the puncture and $\theta$ the polar angle.
Lemma 7.30 ([26| Lemma 10.6]). Let $Q$ be a dimer. Pick a symplectic form $\omega=d \alpha$ on $|Q| \backslash Q_{0}$ such that $\alpha=d \theta / r$ near the punctures. Then the curve $\tilde{L}: S^{1} \rightarrow|Q| \backslash Q_{0}$ can be constructed in such a way that it is exact with respect to $\omega$.

Zigzag curves in a general dimer are not contractible and do not bound teardrops in $|Q| \backslash Q_{0}$. If $Q$ is geometrically consistent, then even in the closed surface $|Q|$ a zigzag curve is not contractible and does not bound a teardrop. Upon specification of spin structures and a symplectic form, the zigzag curves $\tilde{L}$ define object of the exact Fukaya category Fuk $Q$. In analogy to the "category of zigzag paths" constructed combinatorially via $\mathrm{Gtl}_{q} Q$ in section 11, we may give the subcategory of relFuk $Q$ consisting of these objects a name:

Definition 7.31. Let $Q$ be a geometrically consistent dimer. Pick a choice of spin structure for every zigzag curve in $Q$. Then we denote by

$$
\text { Fuk }\left.^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}, \quad \text { Fuk }\left.Q\right|_{\mathrm{Ob} \mathbb{L}},\left.\quad \operatorname{relFuk}^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}, \quad \text { relFuk }\left.Q\right|_{\mathrm{ObL}}
$$

the subcategories of the (relative) Fukaya (pre-)categories given by the zigzag curves with chosen spin structure.

An intersection between two zigzag paths shall be defined as a shared arc of the two zigzag paths. More precisely, we use the following terminology:
Definition 7.32. Let $Q$ be a dimer and $L_{1}, L_{2}$ be two zigzag paths. An (indexed) intersection between $L_{1}$ and $L_{2}$ is a pair $\left(a_{i}, b_{j}\right)$ such that

- $a_{i}$ is an indexed arc of $L_{1}$,
- $b_{j}$ is an indexed arc of $L_{2}$,
- $a_{i}=b_{j}$ as arcs in $Q$,
- $L_{1}$ turns left at $a_{i}$ and $L_{2}$ turns right at $b_{j}$, or the other way around.

Remark 7.33. The cautious wording of Definition 7.32 is necessary in order to make a transversal self-intersection count double.

In Lemma 7.34, we explain that intersections between two distinct zigzag curves $\tilde{L}_{1}$ and $\tilde{L}_{2}$ are precisely the same as (indexed) intersections between $L_{1}$ and $L_{2}$. This is depicted in Figure 7.8 In particular, the hom spaces of Fuk $\left.{ }^{\text {pre }} Q\right|_{\mathrm{ObL}}$ can be identified with spans of (indexed) intersections of zigzag paths. To make this work also in case $L_{1}=L_{2}$, we have to use a model for the Fukaya category as discussed in section 7.2 in which also the endomorphism spaces are spanned by transversal intersections, plus an identity and a co-identity morphism.


Figure 7.8: Shared arcs between $L_{1}$ and $L_{2}$ correspond to intersections $p \in \tilde{L}_{1} \cap \tilde{L}_{2}$

Lemma 7.34. Let $Q$ be a dimer and $L_{1}, L_{2}$ be two zigzag paths. Then transversal intersections points of $\tilde{L}_{1}$ and $\tilde{L}_{2}$, counting transversal self-intersections double if $L_{1}=L_{2}$, are in one-to-one correspondence with indexed intersections between $L_{1}$ and $L_{2}$. Therefore:

$$
\begin{equation*}
\operatorname{Hom}_{\text {Fuk } Q}\left(\tilde{L}_{1}, \tilde{L}_{2}\right)=\operatorname{span}_{\mathbb{C}}\left\{\text { intersections }(a, b) \text { of } L_{1}, L_{2}\right\} \quad\left[\oplus \mathbb{C i d} \oplus \mathbb{C i d}^{*} \text { if } L_{1}=L_{2}\right] . \tag{7.1}
\end{equation*}
$$

Proof. The one-to-one correspondence is a simple inspection. It is worth noting that a transversal selfintersection gives rise to two intersection points between $\tilde{L}_{1}$ and $\tilde{L}_{2}$, according to the double counting, and two indexed intersections between $L_{1}$ and $L_{2}$.

The description of the hom space $\sqrt{7.1}$ for $L_{1} \neq L_{2}$ follows from the definition of Fuk ${ }^{\text {pre }} Q$ and the requirement that $(\operatorname{Fuk} Q)_{\mathrm{tr}}=\operatorname{Fuk}^{\text {pre }} Q$. For $L_{1}=L_{2}, \sqrt{7.1}$ follows from our choice for Fuk $Q$, which describes endomorphism spaces as spans of transversal intersection points plus identity and co-identity. This finishes the proof.

The zigzag curves automatically become objects in the relative Fukaya pre-category and the relative Fukaya category, by virtue of Definition 7.18 and 7.26 . If we choose Fuk $Q$ such that the endomorphism spaces are spanned by transversal intersections plus identity and co-identity, then we know the hom spaces of relFuk $Q$ :

Lemma 7.35. Let $Q$ be a dimer. Then relFuk $Q$ is an $A_{\infty}$-deformation of Fuk $Q$. Let $L_{1}, L_{2}$ be two zigzag paths. Then

$$
\operatorname{Hom}_{\text {relFuk } Q}\left(\tilde{L}_{1}, \tilde{L}_{2}\right)=B \widehat{\otimes} \operatorname{span}\left\{\text { intersections }(a, b) \text { of } L_{1}, L_{2}\right\} \quad\left[\oplus B \text { id } \oplus B \mathrm{id}^{*} \text { if } L_{1}=L_{2}\right]
$$

Proof. By virtue of Definition 7.26, relFuk $Q$ is an $A_{\infty}$-deformation of Fuk $Q$. In particular, its hom spaces are given by

$$
\operatorname{Hom}_{\text {relFuk } Q}\left(\tilde{L}_{1}, \tilde{L}_{2}\right)=B \widehat{\otimes} \operatorname{Hom}_{\text {Fuk } Q}\left(\tilde{L}_{1}, \tilde{L}_{2}\right)
$$

Using the combinatorical description of $\operatorname{Hom}_{\text {Fuk } Q}\left(L_{1}, L_{2}\right)$ from Lemma 7.34 finishes the proof.
Among zigzag curves, it is easy to describe the transversal sequences:
Lemma 7.36. Let $Q$ be a dimer and $L_{1}, \ldots, L_{N+1}$ be a sequence of $N+1 \geq 1$ zigzag paths in $Q$. Then the sequence of zigzag curves $\left(\tilde{L}_{1}, \ldots, \tilde{L}_{N+1}\right)$ is transversal if and only if the zigzag paths $L_{i}$ are pairwise distinct.

Proof. Assume $\left(\tilde{L}_{1}, \ldots, \tilde{L}_{N+1}\right)$ is a transversal sequence. By definition, $\tilde{L}_{i}$ and $\tilde{L}_{j}$ for $i<j$ only have transversal intersection points. Then in particular $\tilde{L}_{i} \neq \tilde{L}_{j}$, hence $L_{i} \neq L_{j}$ as zigzag paths. Conversely, assume all zigzag paths are pairwise distinct. Then for $i<j$ the zigzag curves $\tilde{L}_{i}$ and $\tilde{L}_{j}$ have only transversal intersection points. Moreover, for $i<j<k$ the zigzag curves $\tilde{L}_{i}, \tilde{L}_{j}$ and $\tilde{L}_{k}$ have no common intersection point at all, since an intersection point of zigzag curves is shared between at most two zigzag curves. This shows that $\left(\tilde{L}_{1}, \ldots, \tilde{L}_{N+1}\right)$ is a transversal sequence according to Definition 7.7 and finishes the proof.


Figure 7.9: Illustration of zigzag curves

We can compute some $A_{\infty}$-products in the category of zigzag curves relFuk $\left.Q\right|_{\mathrm{Ob} \mathbb{L}} \subseteq \operatorname{relFuk} Q$. Let $p_{1}, \ldots, p_{N}$ be intersection points with $p_{i}: \tilde{L}_{i} \rightarrow \tilde{L}_{i+1}$. If $\left(\tilde{L}_{1}, \ldots, \tilde{L}_{N+1}\right)$ is a transversal sequence, then the product $\mu_{\text {relFuk } Q}\left(p_{N}, \ldots, p_{1}\right)$ agrees with the product of the relative Fukaya pre-category, which is by definition enumerated by smooth immersed disks with inputs $p_{1}, \ldots, p_{N}$ and arbitrary output $p: \tilde{L}_{1} \rightarrow \tilde{L}_{N+1}$.

Example 7.37. In Figure 7.9a we have depicted a 16 -punctured torus dimer. There are 8 zigzag paths and zigzag curves, depicted in Figure 7.9b A sample smooth immersed disk bounded by four zigzag
 $\tilde{L}_{2}, \tilde{L}_{3}, \tilde{L}_{4}$. The disk has

$$
\text { inputs } \quad p_{1}: \tilde{L}_{1} \rightarrow \tilde{L}_{2}, \quad p_{2}: \tilde{L}_{2} \rightarrow \tilde{L}_{3}, \quad p_{3}: \tilde{L}_{3} \rightarrow \tilde{L}_{4}, \quad \text { output } \quad p: \tilde{L}_{1} \rightarrow \tilde{L}_{4} .
$$

The inputs $p_{1}$ is odd, and the inputs $p_{2}, p_{3}$ are even. The output $p$ is even. It covers six punctures which we denote by $q_{1}, \ldots, q_{6}$. The contribution of this smooth immersed disk to $\mu^{3}\left(p_{3}, p_{2}, p_{1}\right)$ is then

## $\pm q_{1} q_{2} q_{3} q_{4} q_{5} q_{6} p$.

Remark 7.38. When $p_{1}, \ldots, p_{N}$ are morphisms $p_{i} \in \operatorname{Hom}_{\text {Fuk } Q}\left(\tilde{L}_{i}, \tilde{L}_{i+1}\right)$ and the sequence $\left(\tilde{L}_{1}, \ldots, \tilde{L}_{N+1}\right)$ is not transversal, then the product $\mu_{\text {relFuk } Q}\left(p_{N}, \ldots, p_{1}\right)$ is unpredictable. In the present paper, we define a category $\mathrm{H}_{q} \subseteq \mathrm{HTw} \mathrm{Gtl}_{q} Q$ which has the property that its transversal part agrees with relFuk $\left.{ }^{\text {pre }}\right|_{\mathrm{Ob} \mathbb{L}}$. The products of $\mathrm{H} \mathbb{L}_{q}$ are explicitly constructed in section 13 . They provide a candidate for describing the products among non-transversal sequences in relFuk $\left.\right|_{\mathrm{Ob} \mathbb{L}}$.

It seems likely that $\mathrm{H} \mathbb{L}_{q}$ is (gauge equivalent to) relFuk $\left.Q\right|_{\mathrm{Ob} \mathbb{L}}$ (for any model of relFuk $Q$ ). Our main result is however no guarantee for this, since relFuk $Q$ is defined as a lift of the entire relative pre-category and taking subcategories and lifting pre-categories to categories need not commute: Every subcategory of a lift is a lift of the subcategory, but not the other way around.

Combining Definition 7.23 Definition 7.26, Lemma 7.34 and Lemma 7.36 we summarize our findings as follows:

Corollary 7.39. Let $Q$ be a dimer. Then the $A_{\infty}$-pre-category Fuk $\left.{ }^{\text {pre }} Q\right|_{\mathrm{ObL}}$ and its $A_{\infty}$-pre-category deformation relFuk $\left.{ }^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}$ are described as follows:

- The objects are the zigzag curves $\tilde{L}$ for all zigzag paths $L$, with chosen spin structure.
- The set of transversal sequences is

$$
\left(\text { Fuk }\left.^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}\right)_{\mathrm{tr}}^{N}=\left\{\left(\tilde{L}_{1}, \ldots, \tilde{L}_{N}\right) \mid \forall i<j: L_{i} \neq L_{j}\right\} .
$$

- For $\tilde{L}_{1}, \tilde{L}_{2}$ with $L_{1} \neq L_{2}$, the hom space is

$$
\left.\operatorname{Hom}_{\text {Fukpre }} Q\right|_{\text {obL }}\left(\tilde{L}_{1}, \tilde{L}_{2}\right)=\operatorname{span}\left\{p \in \tilde{L}_{1} \cap \tilde{L}_{2}\right\},
$$

- For $\left(\tilde{L}_{1}, \ldots, \tilde{L}_{N+1}\right) \in\left(\text { Fuk }\left.^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}\right)_{\mathrm{tr}}^{N+1}$ and $p_{i} \in \tilde{L}_{i} \cap \tilde{L}_{i+1}$ we have

$$
\left.\mu_{\text {Fuk }}{ }^{\text {pre }} Q \mid \text { ob } \mathbb{L} \text { ( } p_{N}, \ldots, p_{1}\right)=\sum_{p \in \tilde{L}_{1} \cap \tilde{L}_{N+1}} \sum_{D \in M\left(p_{1}, \ldots, p_{N}, p\right)}(-1)^{\operatorname{Abou}(D)} p .
$$

- For $\left(\tilde{L}_{1}, \ldots, \tilde{L}_{N+1}\right) \in\left(\text { Fuk }\left.^{\text {pre }} Q\right|_{\text {ObL }}\right)_{{ }_{\mathrm{tr}}^{N+1}}$ and $p_{i} \in \tilde{L}_{i} \cap \tilde{L}_{i+1}$ we have

$$
\left.\mu_{\text {relFuk }}^{N}{ }^{N} \sum_{p \in \tilde{L}_{1} \cap \tilde{L}_{N+1}} \sum_{D \in M_{q}}\left(p_{N}, \ldots, p_{1}\right)=p_{1}, \ldots, p_{N}, p\right) .
$$

## 8 A deformed Kadeishvili theorem

The aim of this section is to prove a deformed Kadeishvili theorem. The classical Kadeishvili theorem states that every $A_{\infty}$-category has a minimal model, and the minimal model can be computed by a construction with trees. The starting point for the present section is an arbitrary deformed $A_{\infty}$-category. In particular, it may contain curvature and its differential need not square to zero. We nevertheless introduce a notion of minimal model for arbitrary deformed $A_{\infty}$-categories and explain why every deformed $A_{\infty}$-category has a minimal model. In order to find an explicit description, we take the approach of constructing a minimal model via trees. The bottleneck in comparison with the classical case is the curvature and the failure of the differential to square to zero. We are therefore forced to analyze the shape of the differential in detail and build methods that are robust enough to work with less premises than the classical Kadeishvili theorem.

| Classical Kadeishvili: | $A_{\infty}$-category $\mathcal{C}$ | Minimal model $\mathrm{H} \mathcal{C}$ |
| :--- | :--- | :--- |
| Deformed Kadeishvili: | $A_{\infty}$-deformation $\mathcal{C}_{q}$ | $\sim$ |

In section 8.1 we recall homological splittings, a classical basic notion in the construction of minimal models. In section 8.2 we review the classical Kadeishvili theorem and the description of the higher products by trees. In section 8.3 , we define the notion of minimal models for deformed $A_{\infty}$-categories and explain why every deformed $A_{\infty}$-category has a minimal model. In section 8.4 we analyze differentials of $A_{\infty}$-deformations in detail. In section 8.5 we provide a procedure to optimize the curvature of $A_{\infty^{-}}$ deformations. In section 8.6, we provide an auxiliary minimal model construction for deformed $A_{\infty^{-}}$ categories which already have optimal curvature. In section 8.7 we compile all the constructions into a single theorem. Our deformed Kadeishvili theorem Theorem 8.34 states that a minimal model for every deformed $A_{\infty}$-category can be described explicitly, by means of applying the optimization procedure followed by a construction with trees. In section 8.8 we study a special case of the deformed Kadeishvili theorem and relate it back to the classical Kadeishvili theorem.

### 8.1 Homological splittings

In this section we recall the notion of homological splittings. The idea is to split a cochain complex into three direct summands in terms of which the differential becomes easy to describe. This is a classical notion, often just referred to as a "split" 18 . Instead of defining the notion for any cochain complex, we will directly set off in the context of an $A_{\infty}$-category.

Definition 8.1. Let $\mathcal{C}$ be an $A_{\infty}$-category. Then a homological splitting of $\mathcal{C}$ consists of a direct sum decomposition

$$
\operatorname{Hom}_{\mathcal{C}}(X, Y)=H(X, Y) \oplus I(X, Y) \oplus R(X, Y), \quad \forall X, Y \in \mathcal{C}
$$

for all its hom spaces, such that

$$
I(X, Y)=\operatorname{Im}\left(\mu^{1}\right), \quad \operatorname{Ker}\left(\mu^{1}\right)=H(X, Y) \oplus I(X, Y), \quad \forall X, Y \in \mathcal{C}
$$

We frequently denote a homological splitting of $\mathcal{C}$ simply by the letters $H \oplus I \oplus R$, the dependence on $X, Y \in \mathcal{C}$ understood implicitly.

Given a category $\mathcal{C}$, one obtains a homological splitting by choosing $H$ as a space of cocycles that represents the cohomology of the hom complexes. One then chooses $R$ as a complement to $H$ in $\operatorname{Ker}\left(\mu^{1}\right)$. The notation $I$ is simply a shorthand for the image of the differential.

Remark 8.2. Almost everywhere in this section 8 we leave out the letters $X$ and $Y$. All definitions, equations and expressions referring to elements of $H, I$ and $R$ are to be interpreted as being quantified over $X, Y \in \mathcal{C}$. The quantification is understood implicitly. We will even write for instance Hom $\mathcal{C}=H \oplus I \oplus R$, meaning $\operatorname{Hom}_{\mathcal{C}}(X, Y)=H(X, Y) \oplus I(X, Y) \oplus R(X, Y)$ for every $X, Y \in \mathcal{C}$.


Figure 8.1: The differential $\mu^{1}$ in terms of a homological splitting

In the remainder of this section, we record a few consequences of the choice of homological splitting. To start with, in terms of the direct sum decomposition $\operatorname{Hom}_{\mathcal{C}}=H \oplus I \oplus R$, the differential reads

$$
\mu^{1}=\left(\begin{array}{ccc}
0 & 0 & 0  \tag{8.1}\\
0 & 0 & \mu^{1} \\
0 & 0 & 0
\end{array}\right)
$$

Indeed, the spaces $H$ and $I$ are mapped entirely to zero through $\mu^{1}$. The only space not being sent to zero is $R$. This gives the claimed matrix shape 8.1. The action of $\mu^{1}$ on the three summands $H \oplus I \oplus R$ is depicted visually in Figure 8.1

A second observation is that $\left.\mu^{1}\right|_{R}: R \rightarrow I$ is a linear isomorphism and provides an identification between $R$ and $I$. In fact, the map is injective because the kernel of $\mu^{1}$ equals $H \oplus I$ which has vanishing intersection with $R$. Moreover, the map is surjective because $\mu^{1}$ already reaches its entire image on $R$. As a consequence, we can identify $R$ and $I$ by means of $\mu^{1}$. Note that both differ by a shift of 1 . Upon this identification, the remaining $\mu^{1}$ entry in the matrix presentation 8.1 becomes the identity $\operatorname{Id}_{R}$.

Remark 8.3. In the context of homological splittings, we may use tuple notation to indicate an element of the direct sum. Moreover, we may write elements of $\operatorname{Im}\left(\mu^{1}\right)$ explicitly as $\mu^{1}\left(r^{\prime}\right)$ where $r^{\prime} \in R$. In total, we may write an element of $\operatorname{Hom}_{\mathcal{C}}$ as

$$
x=\left(h, \mu^{1}\left(r^{\prime}\right), r\right), \quad \text { with } h \in H, r^{\prime} \in R, r \in R .
$$

Let us set up two more pieces of terminology. The first is the codifferential $h$, which is a zero extension of the inverse of the bijection $\left.\mu^{1}\right|_{R}: R \rightarrow I$. The second is the projection to $H$.

Definition 8.4. Let $H \oplus I \oplus R$ be a homological splitting for $\mathcal{C}$. Then the codifferential is the map

$$
\begin{aligned}
h: \operatorname{Hom}_{\mathcal{C}} & \rightarrow R, \\
\left(h, \mu^{1}\left(r^{\prime}\right), r\right) & \mapsto r^{\prime}, \quad h \in H, r^{\prime} \in R, r \in R .
\end{aligned}
$$

The projection to cohomology is the map

$$
\pi: \operatorname{Hom}_{\mathcal{C}}=H \oplus I \oplus R \rightarrow H
$$

A small abuse of notation consists in the fact that we typically denote elements of the space $H$ by the letter $h$. Typically, there seems to be little chance of confusion.

### 8.2 The classical Kadeishvili theorem

In this section we recall the classical Kadeishvili theorem. This serves as a preparation for our deformed Kadeishvili theorem and fixes pieces of notation. We follow the construction by means of trees, as given in 44 , Chapter 6, 3.3.2]. A good reference is also 16, Section 3.2].

To start with, we recall the standard notion of minimal $A_{\infty}$-categories and minimal models:
Definition 8.5. An $A_{\infty}$-category is minimal if its differential vanishes. Let $\mathcal{C}$ and $\mathcal{D}$ be an $A_{\infty^{-}}$ categories. Then $\mathcal{D}$ is a minimal model for $\mathcal{C}$ if $\mathcal{C}$ and $\mathcal{D}$ are quasi-isomorphic and $\mathcal{D}$ is minimal.

The intention of the Kadeishvili construction is to construct minimal models explicitly. The starting point for the construction is a homological splitting $H \oplus I \oplus R$. The result of the construction


(a) Tree shapes

(b) Decorating tree shapes

Figure 8.2: Illustration of Kadeishvili tree shapes and Kadeishvili $\pi$-trees
is an $A_{\infty}$-structure on $H=\{H(X, Y)\}_{X, Y \in \mathcal{C}}$ which can also be interpreted as an $A_{\infty}$-structure on $\operatorname{HHom}_{\mathcal{C}}=\left\{\operatorname{Hom}_{\mathcal{C}}(X, Y)\right\}_{X, Y \in \mathcal{C}}$, since $H$ and $\operatorname{HHom}_{\mathcal{C}}$ are isomorphic as graded vector spaces through the composition $H \hookrightarrow \operatorname{Ker}\left(\mu^{1}\right) \rightarrow$ H Hom $_{\mathcal{C}}$. Specifically, the $A_{\infty}$-structure on $H$ is defined via trees. We fix terminology as follows:
Definition 8.6. A Kadeishvili tree shape $T$ is a rooted planar tree with $n \geq 2$ leaves whose non-leaf nodes all have at least 2 children. A node in $T$ is internal if it is not a leaf and not the root. The number of internal nodes in $T$ is denoted $N_{T}$. We denote by $\mathcal{T}_{n}$ the set of all Kadeishvili tree shapes with $n$ leaves.

A Kadeishvili $\pi$-tree $\left(T, h_{1}, \ldots, h_{n}\right)$ is a Kadeishvili tree shape $T \in \mathcal{T}_{n}$ with $n \geq 2$ leaves, together with a sequence $h_{1}, \ldots, h_{n}$ of cohomology elements $h_{i} \in H\left(X_{i}, X_{i+1}\right)$. Decorate the leaves by $h_{1}, \ldots, h_{n}$ in sequence. Decorate every non-root node with the operation $h \mu$ and the root with the operation $\pi \mu$. Then the result $\operatorname{Res}\left(T, h_{1}, \ldots, h_{n}\right) \in H\left(X_{1}, X_{n+1}\right)$ of the Kadeishvili $\pi$-tree is the result obtained by evaluating the tree from leaves to the root, according to the decorations.

In other words, to evaluate a $\pi$-tree one inserts the inputs at the leaves, applies $h \mu$ at every internal node and $\pi \mu$ at the root. In every evaluation step, the map $\mu$ is some $k$-ary product of $\mathcal{C}$ and yields an output in some hom space $\operatorname{Hom}_{\mathcal{C}}\left(X_{i}, X_{j}\right)$. The subsequent application of $h$ refers to the codifferential of that hom space $\operatorname{Hom}_{\mathcal{C}}\left(X_{i}, X_{j}\right)$.
Example 8.7. Figure 8.2a depicts all tree shapes with $n=2$ and $n=3$ leaves, as well as a sample tree shape for $n=4$. Figure 8.2b shows how the sample tree shapes of Figure 8.2a together with the input sequences $h_{1}, h_{2}$ or $h_{1}, h_{2}, h_{3}$ or $h_{1}, \ldots, h_{4}$ get decorated. Explicitly, the results of the first three $\pi$-trees read $\pi \mu^{2}\left(h_{2}, h_{1}\right), \pi \mu^{3}\left(h_{3}, h_{2}, h_{1}\right), \pi \mu^{2}\left(h \mu^{2}\left(h_{3}, h_{2}\right), h_{1}\right)$.

Remark 8.8. We have two conventions regarding the order of inputs. Indeed, we sometimes order the elements as $h_{1}, \ldots, h_{n}$ and sometimes as $h_{n}, \ldots, h_{1}$. The convention is that morphisms $h_{1}, \ldots, h_{n}$ indexed by a set of numbers are always compatible in ascending order: $h_{1}$ is a morphism $X_{1} \rightarrow X_{2}$, while $h_{2}$ is a morphism $X_{2} \rightarrow X_{3}$ etc. In particular, due to our "Polish notation" convention of writing $A_{\infty}$-products, the product of the sequence $h_{1}, \ldots, h_{n}$ is written in opposite order as $\mu^{n}\left(h_{n}, \ldots, h_{1}\right)$. The way we draw trees, for example in Figure 8.2b is also in opposite order. In contrast, wherever we refer to the sequence as a whole and evaluation is not immediate, we write the sequence in natural order. An example of natural order is the expression $\operatorname{Res}\left(T, h_{1}, \ldots, h_{n}\right)$.

The construction of the $A_{\infty}$-product $\mu_{H}$ on $H=\mathrm{HC}$ can be summarized as follows: Let $h_{1}, \ldots, h_{n}$ be cohomology elements with $h_{i} \in H\left(X_{i}, X_{i+1}\right)$. Then their higher product is defined as

$$
\mu_{\mathrm{H} \mathcal{C}}\left(h_{n}, \ldots, h_{1}\right)=\sum_{T \in \mathcal{T}_{n}}(-1)^{N_{T}} \operatorname{Res}\left(T, h_{1}, \ldots, h_{n}\right) \in H\left(X_{1}, X_{n+1}\right)
$$

Example 8.9. The first higher products of HC read as follows:

$$
\begin{aligned}
\mu_{\mathrm{H} \mathcal{C}}^{1} & =0 \\
\mu_{\mathrm{H} \mathcal{C}}^{2}\left(h_{2}, h_{1}\right) & =\pi \mu^{2}\left(h_{2}, h_{1}\right) \\
\mu_{\mathrm{H} \mathcal{C}}^{3}\left(h_{3}, h_{2}, h_{1}\right) & =\pi \mu^{3}\left(h_{3}, h_{2}, h_{1}\right)-\pi \mu^{2}\left(h \mu^{2}\left(h_{3}, h_{2}\right), h_{1}\right)-\pi \mu^{2}\left(h_{3}, h \mu^{2}\left(h_{2}, h_{1}\right)\right) .
\end{aligned}
$$

The Kadeishvili theorem claims that $\mathrm{H} \mathcal{C}$ together with the product $\mu_{\mathrm{H} \mathcal{C}}$ is a minimal model for $\mathcal{C}$. The specific version of the theorem is taken from 44, Chapter 6, 3.3.2, 3.3.3]. This source does not provide any sign rule, but we have verified the correctness of the signs $(-1)^{N_{T}}$ in Lemma 8.31.
Theorem 8.10 (Kadeishvili). Let $\mathcal{C}$ be an $A_{\infty^{-}}$-category. Then the products $\mu_{\mathrm{H} \mathcal{C}}^{k}$ form a minimal $A_{\infty^{-}}$ structure on HC . There is a quasi-isomorphism $F: \mathrm{H} \mathcal{C} \rightarrow \mathcal{C}$ which can be constructed by trees as well. Its 1-ary component $F^{1}$ is the standard inclusion $\mathrm{H} \mathcal{C}=H \hookrightarrow \mathcal{C}$.

### 8.3 Existence of minimal models

In this section we define a notion of minimal model for deformed $A_{\infty}$-categories. The definition might differ from the reader's expectation. Indeed, in our notion a minimal model need not have vanishing differential. We finish the section by explaining why every $A_{\infty}$-deformation has a minimal model.

In Definition 8.11, we define the notion of minimal model for $A_{\infty}$-deformations. The aim is to have a definition which is compatible with the classical notion and make minimal models exist for any deformation.

Definition 8.11. Let $\mathcal{C}$ and $\mathcal{D}$ be $A_{\infty}$-categories and $\mathcal{C}_{q}$ and $\mathcal{D}_{q}$ deformations. Then $\mathcal{D}_{q}$ is a minimal model for $\mathcal{C}_{q}$ if $\mathcal{D}$ is a minimal category and there is a functor of deformed $A_{\infty}$-categories

$$
F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}
$$

whose leading term is a quasi-isomorphism $F: \mathcal{C} \rightarrow \mathcal{D}$.
Let us discuss Definition 8.11 Classical minimal models of $A_{\infty}$-categories have vanishing curvature $\mu^{0}$ and differential $\mu^{1}$ by definition. This is not the case anymore for minimal models of deformations. Instead, we require that $\mathcal{D}$ itself is minimal, while $\mathcal{D}_{q}$ is allowed to have both curvature and nonvanishing differential. Of course, the curvature and differential of $\mathcal{D}_{q}$ are both infinitesimal, since $\mathcal{D}_{q}$ is a deformation of the minimal $A_{\infty}$-category $\mathcal{D}$ :

$$
\begin{aligned}
\mu_{\mathcal{D}_{q}, X}^{0} & \in \mathfrak{m} \operatorname{Hom}_{\mathcal{D}}(X, X), \quad \forall X \in \mathcal{D} \\
\mu_{\mathcal{D}_{q}}^{1}(x) & \in \mathfrak{m} \operatorname{Hom}_{\mathcal{D}}(X, Y), \quad \forall X, Y \in \mathcal{D}, x \in \operatorname{Hom}_{\mathcal{D}}(X, Y)
\end{aligned}
$$

Remark 8.12. An alternative definition is obtained by requiring that the map $\mathcal{C} \rightarrow \mathcal{D}$ induced by $F$ be only a quasi-equivalence. The difference in the two notions is merely cosmetic. While in the version of Definition 8.11 the objects of $\mathcal{C}$ and $\mathcal{D}$ are required to match, a definition requiring quasi-equivalence allows for additional bloat: One may add any amount of quasi-isomorphic objects to both $\mathcal{C}$ and $\mathcal{D}$.

There are several equivalent ways of characterizing minimal models. Recall the notion of quasiisomorphism from Definition 5.24
Lemma 8.13. Let $\mathcal{C}, \mathcal{D}$ be $A_{\infty}$-categories and $\mathcal{C}_{q}, \mathcal{D}_{q}$ deformations. The following statements are equivalent:

1. $\mathcal{D}_{q}$ is a minimal model for $\mathcal{C}_{q}$.
2. $\mathcal{C}_{q}$ and $\mathcal{D}_{q}$ are quasi-isomorphic, and $\mathcal{D}$ is minimal.
3. $\mathcal{C}_{q}$ and $\mathcal{D}_{q}$ are quasi-isomorphic, and $\mathcal{D}$ is a minimal model for $\mathcal{C}$.
4. There is a quasi-isomorphism $F: \mathcal{C} \rightarrow \mathcal{D}$ such that $F_{*}^{\mathrm{MC}}\left(\mu_{\mathcal{C}_{q}}\right)=\mu_{\mathcal{D}_{q}}$ and $\mathcal{D}$ is minimal.

Here $F_{*}^{\mathrm{MC}}: \overline{\mathrm{MC}}(\mathrm{HC}(\mathcal{C}), B) \rightarrow \overline{\mathrm{MC}}(\mathrm{HC}(\mathcal{D}), B)$ denotes the push-forward map of Maurer-Cartan elements along $F$.
Proof. This is a simple consequence of the axioms stated in Convention 5.55
A consequence of Lemma 8.13 is that minimal models always exist:
Corollary 8.14. Let $\mathcal{C}$ be an $A_{\infty}$-category. Then any deformation $\mathcal{C}_{q}$ has a minimal model. We may denote the minimal model by $\mathrm{H} \mathcal{C}_{q}$.
Proof. Pick any (classical) minimal model $\mathcal{D}$ for $\mathcal{C}$. Then $\mathcal{D}$ is a minimal category and there is an $A_{\infty}$-quasi-isomorphism $F: \mathcal{C} \rightarrow \mathcal{D}$. Now pick the push-forward

$$
\mu_{\mathcal{D}_{q}}:=F_{*}\left(\mu_{\mathcal{C}_{q}}\right) \in \overline{\mathrm{MC}}(\mathrm{HC}(\mathcal{D}), B) .
$$

This Maurer-Cartan element defines a deformation $\mathcal{D}_{q}$ of $\mathcal{D}$. This satisfies statement 4 of Lemma 8.13

### 8.4 Deformed differentials

In this section, we analyze differentials of deformed $A_{\infty}$-categories in detail. The starting point is a deformed $A_{\infty}$-category $\mathcal{C}_{q}$ together with a homological splitting of $\mathcal{C}$. The homological splitting of $\mathcal{C}$ is naturally not a homological splitting for $\mathcal{C}_{q}$. However, one may try find a decomposition of the hom spaces of $\mathcal{C}_{q}$ with properties that at least resemble those of a homological splitting. The idea is to deform the spaces involved in the homological splitting in order to account for the deformed differential. In the present section, we construct these deformed decompositions and state all properties.

$$
\begin{gathered}
\text { Homological splitting } \\
\operatorname{Hom}_{\mathcal{C}}=H \oplus I \oplus R
\end{gathered} \quad \xrightarrow{\text { upon deformation }} \quad \begin{gathered}
\text { Deformed decomposition } \\
\operatorname{Hom}_{\mathcal{C}_{q}}=H_{q} \oplus \mu_{q}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R)
\end{gathered}
$$

The direct approach of section 8.2 fails in the deformed context. In fact, one of the reasons the construction with trees works well in the classical case is that we have $\mu^{1}(H)=0$. However, it need not be the case that $\mu_{q}^{1}(H)=0$. This makes the direct description of the minimal model by Kadeishvili trees fail in the deformed context.

The present section provides a workaround. A glance at 44, Chapter 6, 3.3] shows that it suffices to require $\mu_{q}^{1}(H) \subseteq H$ instead of $\mu_{q}^{1}(H)=0$. Even better, we may try in the deformed context to find an infinitesimal deformation $H_{q}$ of $H$ with $\mu_{q}^{1}\left(H_{q}\right) \subseteq H_{q}$. Exploiting this observation is the strategy of our deformed Kadeishvili theorem. The present section is devoted to finding this deformation $H_{q}$.

A point of attention is the requirement of a minimal model of $\mathcal{C}_{q}$ to be a deformation of HC . This entails that the hom spaces be identified as $B \widehat{\otimes} \operatorname{Hom}_{H \mathcal{C}}$ and that the leading term of the products is the product $\mu_{\mathcal{C}}$. The present section has been purpose-built to keep track of the identification of $H_{q}$ and $B \widehat{\otimes} H$.

In order to find $H_{q}$, we have to analyze the precise shape of the differential $\mu_{q}^{1}$. In section 8.1. we have seen that a differential on an ordinary $A_{\infty}$-category can be written in matrix form. Most matrix entries vanish because of the $A_{\infty}$-relations. For deformed $A_{\infty}$-categories, we can still write down $\mu_{q}^{1}$ in matrix form with respect to $H \oplus I \oplus R$, although no entries vanish by default. A first step is to change $I$ to $\mu_{q}^{1}(B \widehat{\otimes} R)$, which already renders two matrix entries zero:
Lemma 8.15. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Let $H \oplus I \oplus R$ be a homological splitting of $\mathcal{C}$. Then the differential $\mu_{q}^{1}$ restricted to $B \widehat{\otimes} R$ is injective:

$$
\begin{equation*}
\mu_{q}^{1}: B \widehat{\otimes} R \xrightarrow{\sim} \mu_{q}^{1}(B \widehat{\otimes} R), \tag{8.2}
\end{equation*}
$$

and we have a direct sum decomposition of $B$-modules

$$
\begin{equation*}
B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}=(B \widehat{\otimes} H) \oplus \mu_{q}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R) \tag{8.3}
\end{equation*}
$$

With respect to this decomposition, $\mu_{q}^{1}$ takes the shape

$$
\mu_{q}^{1}=\left(\begin{array}{ccc}
D & * & 0  \tag{8.4}\\
\mu_{q}^{1} E & * & * \\
F & * & 0
\end{array}\right)
$$

for some operators

$$
\begin{aligned}
& D: B \widehat{\otimes} H \rightarrow B \widehat{\otimes} H \\
& E: B \widehat{\otimes} H \rightarrow B \widehat{\otimes} R, \\
& F: B \widehat{\otimes} H \rightarrow B \widehat{\otimes} R .
\end{aligned}
$$

Proof. First of all, regard the map

$$
\mu_{q}^{1}: B \widehat{\otimes} R \rightarrow B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}
$$

It is $B$-linear and has leading term the injective map $\left.\mu_{\mathcal{C}}^{1}\right|_{R}$. By Lemma 5.13, it is an embedding. This establishes the first claim.

Second, let us prove the direct sum decomposition. Intuitively, changing the summand $\mu^{1}(B \widehat{\otimes} R)$ to $\mu_{q}^{1}(B \widehat{\otimes} R)$ constitutes only an infinitesimal change and should leave the decomposition intact. Formally, define the map

$$
\begin{aligned}
\psi: \operatorname{Hom}_{\mathcal{C}_{q}} & \rightarrow B \widehat{\otimes} H+\mu_{q}^{1}(B \widehat{\otimes} R)+B \widehat{\otimes} R \hookrightarrow \operatorname{Hom}_{\mathcal{C}_{q}}, \\
\left(h, \mu^{1}(r), r^{\prime}\right) & \mapsto h+\mu_{q}^{1}(r)+r^{\prime}, \quad \text { for } \quad h \in B \widehat{\otimes} H, \quad r \in B \widehat{\otimes} R, \quad r^{\prime} \in B \widehat{\otimes} R .
\end{aligned}
$$

The map $\psi$ has leading term the identity. By Lemma 5.12 it is an isomorphism onto $\operatorname{Hom}_{\mathcal{C}_{q}}$. In particular, this already establishes that 8.3 is a sum decomosition, not necessarily direct. To prove the sum decomposition direct, let $h \in B \widehat{\otimes} H, \mu_{q}^{1}(r) \in \mu_{q}^{1}(B \widehat{\otimes} R)$ and $r^{\prime} \in B \widehat{\otimes} R$ with $h+\mu_{q}^{1}(r)+r^{\prime}=0$ in $\operatorname{Hom}_{\mathcal{C}_{q}}$. This implies $\psi\left(h, \mu_{q}^{1}(r), r^{\prime}\right)=0$ and finally $h=r=r^{\prime}=0$ since $\psi$ is an isomorphism and $\left.\mu_{q}^{1}\right|_{B \widehat{\otimes} R}$ is injective. We conclude that 8.3 is a direct sum decomposition.

To obtain the claimed matrix presentation of $\mu_{q}^{1}$, simply define $D$ and $F$ as $\mu_{B \widehat{\otimes} H}^{1}$ followed by the projections to $B \widehat{\otimes} H$ and $B \widehat{\otimes} R$, respectively. Define $E$ as $\mu_{B \widehat{\otimes} H}^{1}$ followed by projection to $\mu_{q}^{1}(B \widehat{\otimes} R)$ and the inverse of $\mu_{q}^{1}: B \widehat{\otimes} R \xrightarrow{\sim} \mu_{q}^{1}(B \widehat{\otimes} R)$. The vanishing of the two indicated matrix entries is immediate, since $\mu_{q}^{1}$ sends $B \widehat{\otimes} R$ to $\mu_{q}^{1}(B \widehat{\otimes} R)$ by definition. This settles all claims.

We are now ready to define $H_{q}$.
Lemma 8.16. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Assume $H \oplus I \oplus R$ is a homological splitting for $\mathcal{C}$. Let $D, E, F$ denote the operators from Lemma 8.15. Put

$$
H_{q}:=\{h-E h \mid h \in B \widehat{\otimes} H\}
$$

Then we have a direct sum decomposition

$$
\begin{equation*}
B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}=H_{q} \oplus \mu_{q}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R) \tag{8.5}
\end{equation*}
$$

It holds that $\mu_{q}^{1}\left(H_{q}\right) \subseteq H_{q} \oplus B \widehat{\otimes} R$. With respect to this decomposition, $\mu_{q}^{1}$ has the shape

$$
\mu_{q}^{1}=\left(\begin{array}{lll}
* & * & 0 \\
* & * & * \\
0 & * & 0
\end{array}\right)
$$

Proof. The decomposition is achieved easily as in the proof of Lemma 8.15. Namely, $H_{q}$ and $B \widehat{\otimes} H$ only differ by $R$-terms. To show $\mu_{q}^{1}\left(H_{q}\right) \subseteq H_{q} \oplus B \widehat{\otimes} R$, we calculate

$$
\mu_{q}^{1}(h-E h)=D h+\mu_{q}^{1} E(h)+F h-\mu_{q}^{1}(E h)=D h+F h \in B \widehat{\otimes} H \oplus B \widehat{\otimes} R=H_{q} \oplus B \widehat{\otimes} R .
$$

This finishes the proof.
The decomposition 8.5 plays a crucial role throughout this paper. It is not a homological splitting of $\mathcal{C}_{q}$ in any sense, since for example $\mu_{q}^{1}$ need not vanish on $H_{q}$. The decomposition is however an important prerequisite for our deformed Kadeishvili theorem. In particular, whenever computing minimal models of deformed $A_{\infty}$-categories, this decomposition needs to be calculated first.

There is a natural identification between $B \widehat{\otimes} H$ and $H_{q}$. The identification associated an element $h \in B \widehat{\otimes} H$ with $h-E h \in H_{q}$. Since $E h \in B \widehat{\otimes} R$, we can recover $h$ from $h-E h$ by stripping off the $R$ component. This identification plays an important role in this paper. Another important role is played by the map $\mu_{q}^{1}: B \widehat{\otimes} R \rightarrow \mu_{q}^{1}(B \widehat{\otimes} R)$. We call the inverse of this map the deformed codifferential. Let us fix all important notions in the following definition.

Definition 8.17. Let $\mathcal{C}$ be an $A_{\infty}$-category and $H \oplus I \oplus R$ a homological splitting. Let $\mathcal{C}_{q}$ be deformation of $\mathcal{C}$. The deformed decomposition of $\mathcal{C}_{q}$ is the collection of direct sum decompositions 8.5 of all hom spaces in $\mathcal{C}_{q}$. The deformed counterpart of an element $h \in B \widehat{\otimes} H$ is the element $h-E h \in H_{q}$. The correspondence between $H_{q}$ and $B \widehat{\otimes} H$ is denoted

$$
\begin{aligned}
\varphi: H_{q} & \sim \\
h-E h & \longrightarrow \otimes
\end{aligned}, h .
$$

The deformed codifferential of $\mathcal{C}_{q}$ is the $R$-linear map

$$
h_{q}=\left(\left.\mu_{q}^{1}\right|_{B \widehat{\otimes} R}\right)^{-1}: \mu_{q}^{1}(B \widehat{\otimes} R) \longrightarrow B \widehat{\otimes} R
$$

The deformed projection of $\mathcal{C}_{q}$ is the $R$-linear map

$$
\pi_{q}: H_{q} \oplus \mu_{q}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R) \rightarrow H_{q}
$$

### 8.5 Optimizing curvature

In this section, we show how to optimize curvature of an $A_{\infty}$-category. In general, it is not possible to remove curvature from an $A_{\infty}$-deformation entirely. For the purposes of our Kadeishvili theorem, it is however important to tame the curvature as much as possible. In this section, we show that the curvature of any deformation can be reduced sufficiently for our purpose of constructing a deformed Kadeishvili theorem.

$$
\text { Deformation } \mathcal{C}_{q} \xrightarrow[\sim]{\sim} \text { Deformation } \mathcal{C}_{q}^{\text {opt }} \text { with optimal curvature }
$$

Let us start by fixing our terminology:
Definition 8.18. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Let $H \oplus I \oplus R$ be a homological splitting for $\mathcal{C}$ and $H_{q} \oplus \mu_{q}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R)$ be the associated deformed decomposition of $\mathcal{C}_{q}$. Then $\mathcal{C}_{q}$ has optimal curvature if $\mu_{q}^{0} \in H_{q}$.

In the remainder of the section, we show how to gauge an arbitrary deformed $A_{\infty}$-category such that its curvature becomes optimal. We also explain why optimal curvature is the best we can expect. The idea to optimize the curvature is to apply successive gauges. All gauges will be gauge functors $F$ of the form $F^{1}=\mathrm{Id}$ and $F^{0}=r$ and have no higher components. We may also call such functors "uncurving gauges" because they are strong at reducing curvature. The following definition settles our terminology.

Definition 8.19. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Let $r=\left\{r_{X}\right\}_{X \in \mathcal{C}}$ be an element consisting of $r_{X} \in \mathfrak{m} \operatorname{End}_{\mathcal{C}}^{1}(X)$ for every $X \in \mathcal{C}$. Then the uncurving of $\mathcal{C}_{q}$ by $r$ is the category $\mathcal{C}_{q}^{\prime}$ obtained from adding $r_{X}$ as twisted differential to every $X \in \mathcal{C}_{q}$ :

$$
\mathcal{C}_{q}^{\prime}:=\left\{\left(X, r_{X}\right) \mid X \in \mathcal{C}\right\} \subseteq \mathrm{Tw}^{\prime} \mathcal{C}_{q}
$$

The notation $\mathrm{Tw}^{\prime} \mathcal{C}_{q}$ is taken from Remark 5.37 .
Remark 8.20. Let $\mathcal{C}_{q}^{\prime}$ be the uncurving of $\mathcal{C}_{q}$ by $r$. Then the curvature of $\mathcal{C}_{q}^{\prime}$ is

$$
\mu_{\mathcal{C}_{q}^{\prime}}^{0}=\mu_{\mathcal{C}_{q}}^{0}+\mu_{\mathcal{C}_{q}}^{1}(r)+\mu_{\mathcal{C}_{q}}^{2}(r, r)+\ldots
$$

The uncurving $\mathcal{C}_{q}^{\prime}$ is naturally a deformation of $\mathcal{C}$ and comes with a gauge equivalence

$$
F: \mathcal{C}_{q}^{\prime} \xrightarrow{\sim} \mathcal{C}_{q}, \quad \text { given by } \quad F^{0}:=r, \quad F^{1}:=\mathrm{Id}, \quad F^{\geq 2}:=0 .
$$

The category $\mathcal{C}_{q}^{\prime}$ can also be defined by forcing this particular map $F$ to be a functor of deformations. More on uncurving can be found in section 9.1 which focuses on cases where uncurving removes the curvature entirely.

Remark 8.21. The name "uncurving" for the gauge in Definition 8.19 is a slight abuse of terminology: The curvature $\mu_{\mathcal{C}_{q}^{\prime}}^{0}$ will not vanish, but has the chance to be less than $\mu_{\mathcal{C}_{q}}^{0}$. The term uncurving generally refer to any procedure of reducing curvature, while Definition 8.19 restricts usage of the term to a particular class of functors. According to the explanation in section 9.1. this particular class of functors is however the only one that essentially changes curvature, therefore we have adopted the name "uncurving".

Remark 8.22. We can now explain the name "optimal curvature". In fact, any other deformation gauge equivalent to a deformation with optimal curvature will generally have more curvature. To see this, regard a deformation $\mathcal{C}_{q}$ with optimal curvature. Its curvature already lies in $H_{q}$. If we apply uncurving by an element $r$, the new curvature is $\mu_{q}^{0}+\mu_{q}^{1}(r)+\ldots$. If we choose $r \in B \widehat{\otimes} R$, then $\mu_{q}^{1}(r)$ naturally lies in $\mu_{q}^{1}(B \widehat{\otimes} R)$ which already downgrades the curvature. If we choose $r \in \mu_{q}^{1}(B \widehat{\otimes} R)$ or $r \in H_{q}$, then $\mu_{q}^{1}(r)$ typically contains components from $R$ or $\mu_{q}^{1}(B \widehat{\otimes} R)$ as well. The additional summands $\mu_{q}^{2}(r, r)$ even worsen the situation. We see that $\mu_{q}^{0} \in H_{q}$ is generally the best achievable.

In the remainder of this section, we prove that any deformed $A_{\infty}$-category $\mathcal{C}_{q}$ has an uncurving with optimal curvature. The idea is to apply repeated uncurving by elements $s$ which lie in increasingly high order of $\mathfrak{m}$. We take our clue from inspecting the curvature $\mu_{\mathcal{C}_{q}^{\prime}}^{0}=\mu_{\mathcal{C}_{q}}^{0}+\mu_{\mathcal{C}_{q}}^{1}(s)+\ldots$. To get $\mu_{\mathcal{C}_{q}^{\prime}}^{0}$ as close to zero as possible, write $\mu_{\mathcal{C}_{q}}^{0}=h+\mu_{\mathcal{C}_{q}}^{1}(r)+r^{\prime}$ in terms of the deformed decomposition of $\mathcal{C}_{q}$ and choose $s=-r$. The curvature of $\mathcal{C}_{q}^{\prime}$ then reads

$$
\mu_{\mathcal{C}_{q}^{\prime}}^{0}=h+\mu_{\mathcal{C}_{q}}^{1}(r)+r^{\prime}+\mu_{\mathcal{C}_{q}}^{1}(-r)+\mu_{\mathcal{C}_{q}}^{2}(-r,-r)+\ldots=h+r^{\prime}+\mathcal{O}\left(\mathfrak{m}^{2}\right)
$$

This is very productive strategy, since the new curvature has lost its $\mu_{q}^{1}(R)$ component in lowest order. Our idea is to repeat this procedure to eliminate also the higher order terms. A repeated approach is indeed necessary because newly arising curvature terms like $\mu_{\mathcal{C}_{q}}^{2}(r, r)$ may behave unpredictably.
Definition 8.23. Let $\mathcal{C}$ be an $A_{\infty}$-category with deformation $\mathcal{C}_{q}$. Let $H \oplus I \oplus R$ be a homological splitting for $\mathcal{C}$. The curvature optimization procedure is the following inductive procedure, starting with $i=0$ and $\mathcal{C}_{q}^{(0)}:=\mathcal{C}_{q}$.

1. Form the deformed decomposition $H_{q}^{(i)} \oplus \mu_{\mathcal{C}_{q}^{(i)}}^{1}(B \widehat{\otimes} R) \oplus B \widehat{\otimes} R$ of $\mathcal{C}_{q}^{(i)}$.
2. Write the curvature as $\mu_{\mathcal{C}_{q}^{(i)}}^{0}=h^{(i)}+\mu_{\mathcal{C}_{q}^{(i)}}^{1}\left(r^{(i)}\right)+r^{(i) \prime}$ in terms of the decomposition.
3. Define $\mathcal{C}_{q}^{(i+1)}$ to be the uncurving of $\mathcal{C}_{q}^{(i)}$ by $-r^{(i)}$.
4. Repeat.

Remark 8.24. The definition of $r^{(i)}$ in terms of $\mathcal{C}_{q}{ }^{(i)}$ can also be written elegantly as $r^{(i)}=h_{q}{ }^{(i)}\left(\mu^{0} \mathcal{C}_{q^{(i)}}\right)$, where $h_{q}{ }^{(i)}$ is the deformed codifferential of $\mathcal{C}_{q}{ }^{(i)}$. The letters $h^{(i)}, r^{(i)}$ and $r^{(i) \prime}$ are actually families parametrized by objects $X \in \mathcal{C}$. In the statement of Lemma 8.25 we combine this shorthand with the shorthand notation $\operatorname{End}_{\mathcal{C}}=\left\{\operatorname{End}_{\mathcal{C}}(X)\right\}_{X \in \mathcal{C}}$. For instance, $r^{(i)} \in \mathfrak{m}^{2^{i}}$ End $_{\mathcal{C}}$ is to be understood as $r_{X}^{(i)} \in \mathfrak{m}^{2^{i}} \operatorname{End}_{\mathcal{C}}(X)$ for every $X \in \mathcal{C}$.

After running the curvature optimization procedure, we expect the gauges $r^{(i)}$ to combine together to one large gauge. We expect the categories $\mathcal{C}_{q}{ }^{(i)}$ to converge to a limit category $\mathcal{C}_{q}^{\text {opt }}$. We also expect the curvature of $\mathcal{C}_{q}{ }^{(i)}$ to converge to the curvature of $\mathcal{C}_{q}^{\text {opt }}$ and the deformed decompositions of $\mathcal{C}_{q}^{(i)}$ to converge to the deformed decomposition of $\mathcal{C}_{q}^{\text {opt }}$ :

$$
\begin{aligned}
\lim _{i \rightarrow \infty} \mathcal{C}_{q}^{(i)} & =\mathcal{C}_{q}^{\text {opt }}, \\
\lim _{i \rightarrow \infty} \mu_{\mathcal{C}_{q}^{(i)}}^{0} & =\mu_{\mathcal{C}_{q}^{\text {opt }}}^{0}, \\
\lim _{i \rightarrow \infty}\left(H_{q}^{(i)}, \mu_{\mathcal{C}_{q}^{(i)}}^{1}(B \widehat{\otimes} R),(B \widehat{\otimes} R)\right) & =\left(H_{q}^{\text {opt }}, \mu_{\mathcal{C}_{q}^{\text {opt }}}^{1}(B \widehat{\otimes} R),(B \widehat{\otimes} R)\right) .
\end{aligned}
$$

Ultimately, we hope to find $\mu_{\mathcal{C}_{q}}^{0}{ }^{\text {opt }} \in H_{q}^{\text {opt }} \oplus(B \widehat{\otimes} R)$. The next lemma makes this precise, and also shows that we have in fact reached $\mu^{0}{ }_{C_{q}^{\text {opt }}} \in H_{q}^{\text {opt }}$ as desired.
Lemma 8.25. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Let $H \oplus I \oplus R$ be a homological splitting for $\mathcal{C}$. Let $r^{(i)}$ be the sequence obtained from applying the curvature optimization procedure to $\mathcal{C}_{q}$. Then it holds that $r^{(i)} \in \mathfrak{m}^{2^{i}}$ End $_{\mathcal{C}}$. Set $r=\sum_{i \in \mathbb{N}} r^{(i)} \in \mathfrak{m} \operatorname{End}_{\mathcal{C}}$ and define $\mathcal{C}_{q}^{\text {opt }}$ as the uncurving of $\mathcal{C}_{q}$ by $-r$. Then $\mathcal{C}_{q}^{\text {opt }}$ has optimal curvature and comes with an gauge equivalence

$$
F: \mathcal{C}_{q}^{\text {opt }} \rightarrow \mathcal{C}_{q}, \quad \text { given by } \quad F^{0}=-r, \quad F^{1}=\mathrm{Id}, \quad F^{\geq 2}=0
$$

Proof. We divide the proof into three parts. In the first part of the proof we show $r^{(i)} \in \mathcal{O}\left(\mathfrak{m}^{2^{i}}\right)$. Denote by $H_{q}^{\text {opt }} \oplus \mu_{\mathcal{C}_{q}^{\text {opt }}}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R)$ the deformed decomposition of $\mathcal{C}_{q}^{\text {opt }}$. In the second part of the proof, we show that the curvature $\mu_{\mathcal{C}_{q}^{\text {opt }}}^{0}$ lies in $H_{q}^{\text {opt }} \oplus B \widehat{\otimes} R$. In the third part of the proof we conclude that the curvature actually lies in $H_{q}^{\text {opt }}$.

For the first part, let us show $r^{(i)} \in \mathfrak{m}^{2^{i}}$ End $_{\mathcal{C}}$ by induction. For $i=0$, the statement holds. Assume it holds for some $i \in \mathbb{N}$. Recall that $\mathcal{C}_{q}^{(i+1)}$ is the uncurving of $\mathcal{C}_{q}^{(i)}$ by $-r^{(i)}$. Its curvature is

$$
\begin{aligned}
\mu_{\mathcal{C}_{q}^{(i+1)}}^{0} & =\mu_{\mathcal{C}_{q}^{(i)}}^{0}+\mu_{\mathcal{C}_{q}^{(i)}}^{1}\left(-r^{(i)}\right)+\mu_{\mathcal{C}_{q}^{(i)}}^{2}\left(-r^{(i)},-r^{(i)}\right)+\ldots \\
& =h^{(i)}+\mu_{\mathcal{C}_{q}^{(i)}}^{1}\left(r^{(i)}\right)+r^{(i) \prime}-\mu_{\mathcal{C}_{q}^{(i)}}^{1}\left(r^{(i)}\right)+\mathcal{O}\left(\mathfrak{m}^{2^{i+1}}\right) \\
& =h^{(i)}+r^{(i) \prime}+\mathcal{O}\left(\mathfrak{m}^{2^{i+1}}\right)
\end{aligned}
$$

To make statements on $r^{(i+1)}$, write $h_{q}^{(i+1)}$ for the deformed codifferential of $\mathcal{C}_{q}^{(i+1)}$. Then

$$
\begin{aligned}
r^{(i+1)} & =h_{q}^{(i+1)}\left(\mu_{\mathcal{C}_{q}^{(i+1)}}^{0}\right) \\
& =h_{q}^{(i+1)}\left(h^{(i)}+r^{(i) \prime}+\mathcal{O}\left(\mathfrak{m}^{2^{i+1}}\right)\right) \\
& =0+0+\mathcal{O}\left(\mathfrak{m}^{2^{i+1}}\right)
\end{aligned}
$$

In the last row, we have used that

$$
h^{(i)}, r^{(i) \prime} \in H_{q}^{(i)} \oplus(B \widehat{\otimes} R)=(B \widehat{\otimes} H) \oplus(B \widehat{\otimes} R)=H_{q}^{(i+1)} \oplus(B \widehat{\otimes} R)
$$

We have also used that $h_{q}^{(i+1)}: B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}} \rightarrow B \widehat{\otimes} R$ preserves the $\mathfrak{m}$-adic filtration. The reason is that $h_{q}^{(i+1)}$ is $B$-linear and automatically continuous according to Lemma 5.12 . In total, we arrive at $r^{(i+1)} \in \mathfrak{m}^{2^{i+1}} R$ as claimed. This finishes the induction.

For the second part of the proof, we show that $\mu_{\mathcal{C}_{q}^{\text {opt }}}^{0}$ lies in $H_{q}^{\text {opt }} \oplus(B \widehat{\otimes} R)$. For every $i \in \mathbb{N}$, regard

$$
\mu_{\mathcal{C}_{q}^{(i)}}^{0}=h^{(i)}+\mu_{\mathcal{C}_{q}^{(i)}}^{1}\left(r^{(i)}\right)+r^{(i) \prime}
$$

The left-hand side converges to $\mu_{\mathcal{C}_{q}^{\text {opt }}}^{0}$. The third term on the right-hand side converges to zero. Together this means

$$
(B \widehat{\otimes} H) \oplus(B \widehat{\otimes} R) \ni h^{(i)}+\mu_{\mathcal{C}_{q}^{(i)}}^{1}\left(r^{(i)}\right) \xrightarrow{i \rightarrow \infty} \mu_{\mathcal{C}_{q}^{\text {opt }}}^{0} .
$$

We conclude

$$
\mu_{\mathcal{C}_{q}^{\text {opt }}}^{0} \in(B \widehat{\otimes} H) \oplus(B \widehat{\otimes} R)=H_{q}^{\mathrm{opt}} \oplus B \widehat{\otimes} R .
$$

This finishes the second part of the proof.
For the third part of the proof, we show that $\mu_{\mathcal{C}_{q}}^{0}{ }_{q}^{\text {opt }} \in H_{q}^{\text {opt }}$. The idea is to show that the $R$ component of $\mu^{0} \mathcal{C}_{q}^{\text {opt }}$ vanishes. In fact, this is an easy a posteriori observation: Write this curvature as $h+r$ with $h \in H_{q}^{\text {opt }}$ and $r \in B \widehat{\otimes} R$. Then

$$
0=\mu_{\mathcal{C}_{q}^{\text {opt }}}^{1}\left(\mu_{\mathcal{C}_{q}^{\text {opt }}}^{0}\right)=\mu_{\mathcal{C}_{q}^{\text {opt }}}^{1}(h)+\mu_{\mathcal{C}_{q}^{\text {opt }}}^{1}(r) .
$$

On the right hand side, the first summand lies in $H_{q}^{\text {opt }} \oplus B \widehat{\otimes} R$ and the second summand lies in $\mu_{\mathcal{C}_{q}^{\text {opt }}}^{1}(B \widehat{\otimes} R)$. Correspondingly, both summands vanish. While for $h$ this is a weak statement, we immediately derive $r=0$ since $\mu_{\mathcal{C}_{q}}^{1}$ opt injective on $B \widehat{\otimes} R$. This shows $\mu_{\mathcal{C}_{q}^{\text {opt }}}^{0} \in H_{q}^{\text {opt }}$ and finishes the proof.

### 8.6 Auxiliary minimal model procedure

In this section, we construct auxiliary minimal models for deformed $A_{\infty}$-categories with optimal curvature. The idea is to perform a construction with trees as in the classical case. For a given catgeory $\mathcal{C}_{q}$ with optimal curvature, the first step in this section is to provide an explicit description of the auxiliary $A_{\infty^{-}}$ structure on $H_{q}$ and a functor $F_{q}: H_{q} \rightarrow \mathcal{C}_{q}$. We then check that the auxiliary minimal model satisfies the curved $A_{\infty}$-axioms and that $F_{q}$ satisfies the curved $A_{\infty}$-functor axioms.

$$
\text { Deformation } \mathcal{C}_{q} \text { with optimal curvature } \sim \sim \sim \quad \begin{gathered}
\text { Auxiliary deformation } H_{q} \\
\text { Auxiliary functor } F_{q}: H_{q} \rightarrow \mathcal{C}_{q}
\end{gathered}
$$

Remark 8.26. The material in this section is considered auxiliary because we only construct an $A_{\infty^{-}}$ structure on $H_{q}$ and not on $B \widehat{\otimes} H$. The $A_{\infty}$-structure on $B \widehat{\otimes} H$ is obtained in section 8.7 simply by transfer via $\varphi: H_{q} \rightarrow B \widehat{\otimes} H$.

To define the auxiliary $A_{\infty}$-structures, we have to set up some context. Let $\mathcal{C}$ be an $A_{\infty}$-category and $H \oplus I \oplus R$ a homological splitting. Let $\mathcal{C}_{q}$ be a deformation with optimal curvature. Denote by $H_{q} \oplus \mu_{\mathcal{C}_{q}}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R)$ the deformed decomposition of $\mathcal{C}_{q}$. We use the following notation:
Definition 8.27. Consider a sequence $h_{1}, \ldots, h_{n}$ of $n \geq 2$ morphisms with $h_{i} \in H_{q}\left(X_{i}, X_{i+1}\right)$. Let $T \in \mathcal{T}_{n}$ be a Kadeishvili tree shape with $n$ leaves, as in section 8.2. Define

$$
\operatorname{Res}_{q}\left(T, h_{1}, \ldots, h_{n}\right) \in H_{q}\left(X_{1}, X_{n+1}\right)
$$

to be the evaluation of $T$ by decorating the leaves with the inputs $h_{1}, \ldots, h_{n}$, the internal nodes by $h_{q} \mu_{q}$ and the root by $\pi_{q} \mu_{q}$. Define

$$
\operatorname{Res}_{q}^{h}\left(T, h_{1}, \ldots, h_{n}\right) \in B \widehat{\otimes} R\left(X_{1}, X_{n+1}\right)
$$

to be the evaluation of $T$ by decorating the leaves with the inputs $h_{1}, \ldots, h_{n}$ and all other nodes by $h_{q} \mu_{q}$, including the root.


Figure 8.3: Decorating Kadeishvili $\pi$-trees for $\mu_{H_{q}}$

Example 8.28. A few sample decorated trees for the definition of $\operatorname{Res}_{q}\left(T, h_{1}, \ldots, h_{n}\right)$ are depicted in Figure 8.3. For instance, the first three trees give results $\pi_{q} \mu_{q}^{2}\left(h_{2}, h_{1}\right), \pi_{q} \mu_{q}^{3}\left(h_{3}, h_{2}, h_{1}\right), \pi_{q} \mu_{q}^{2}\left(h_{q} \mu_{q}^{2}\left(h_{3}, h_{2}\right), h_{1}\right)$.

We temporarily by $i$ the inclusion map of $H_{q}\left(X_{1}, X_{2}\right)$ into $\operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{1}, X_{2}\right)$. With these preparations, we are ready to define auxiliary product structure on the collection of spaces $H_{q}=\left\{H_{q}(X, Y)\right\}_{X, Y \in \mathcal{C}_{q}}$ and an auxiliary mapping $F_{q}: H_{q} \rightarrow \mathcal{C}_{q}$ :

Definition 8.29. The auxiliary product structure on $H_{q}$ is defined as follows:

$$
\begin{aligned}
\mu_{H_{q}}^{0} & =\mu_{\mathcal{C}_{q}}^{0}, \\
\mu_{H_{q}}^{1} & =\left.\pi_{q} \mu_{q}^{1}\right|_{H_{q}}, \\
\mu_{H_{q}}^{n \geq 2}\left(h_{n}, \ldots, h_{1}\right) & =\sum_{T \in \mathcal{T}_{n}}(-1)^{N_{T}} \operatorname{Res}_{q}\left(T, h_{1}, \ldots, h_{n}\right) .
\end{aligned}
$$

The candidate functor $F_{q}: H_{q} \rightarrow \mathcal{C}_{q}$ is defined by

$$
\begin{aligned}
F_{q}^{0} & =0 \\
F_{q}^{1} & =i, \\
F_{q}^{n \geq 2}\left(h_{n}, \ldots, h_{1}\right) & =\sum_{T \in \mathcal{T}_{n}}(-1)^{N_{T}+1} \operatorname{Res}_{q}^{h}\left(T, h_{1}, \ldots, h_{n}\right) .
\end{aligned}
$$

In words, $\mu_{H_{q}}^{0}$ is defined as $\mu_{\mathcal{C}_{q}}^{0}$ which already lies in $H_{q}$ since $\mathcal{C}_{q}$ has optimal curvature. The differential $\mu_{H_{q}}^{1}$ is defined by projecting $\mu_{\mathcal{C}_{q}}^{1}$ down to $H_{q}$. All higher products are given by trees. The functor component $F_{q}^{0}$ is set to zero, the component $F_{q}^{1}$ is the natural embedding of $H_{q}\left(X_{1}, X_{2}\right)$ into the hom space $\operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{1}, X_{2}\right)$ and the higher components of $F_{q}$ are given by trees.

Checking the functor relations for $F_{q}$ entails switching around projections $\pi_{q}$ and codifferentials $h_{q}$. We need to prepare for this with a simple lemma:
Lemma 8.30. Let $\mathcal{C}_{q}$ be a deformed $A_{\infty}$-category with optimal curvature. Then the projections to $B \widehat{\otimes} R$ and $\mu_{\mathcal{C}_{q}}^{1}(B \widehat{\otimes} R)$ with respect to the deformed decomposition can be written as

$$
\begin{aligned}
\pi_{\mu_{q}^{1}(B \widehat{\otimes} R)} & =\mu_{q}^{1} h_{q}, \\
\pi_{B \widehat{\otimes} R} & =h_{q} \mu_{q}^{1}-h_{q} \mu_{q}^{1} \mu_{q}^{1} h_{q} .
\end{aligned}
$$

Proof. Every hom space in $\mathcal{C}_{q}$ is the direct sum of the three components $H_{q}, \mu_{q}^{1}(B \widehat{\otimes} R)$ and $B \widehat{\otimes} R$. Therefore it suffices to check the identities on these three spaces individually.

On $H_{q}$, both sides of the first formula evaluate to zero by definition. In the second formula, the right hand side evalualates to zero as well, because $\mu_{q}^{1}\left(H_{q}\right) \subseteq H_{q} \oplus \mu_{q}^{1}(B \widehat{\otimes} R)$.

On $B \widehat{\otimes} R$, both sides of the first formula evaluate to zero by definition. In the second formula, the first term evaluates indeed to the identity and the second term vanishes.

On $\mu_{q}^{1}(B \widehat{\otimes} R)$, both sides of the first formula evaluate to the identity. To check the second formula, regard an arbitrary element $\mu_{q}^{1}(r)$ with $r \in B \widehat{\otimes} R$. Then

$$
h_{q} \mu_{q}^{1}\left(\mu_{q}^{1}(r)\right)-h_{q} \mu_{q}^{1} \mu_{q}^{1} h_{q}\left(\mu_{q}^{1}(r)\right)=h_{q} \mu_{q}^{1} \mu_{q}^{1}(r)-h_{q} \mu_{q}^{1} \mu_{q}^{1}(r)=0 .
$$

We conclude that the claimed identities hold on all three direct summand spaces, finishing the proof.
In Lemma 8.31 we prove the desired $A_{\infty}$-relations for $H_{q}$ and $F_{q}: H_{q} \rightarrow \mathcal{C}_{q}$. Strictly speaking, $H_{q}$ itself is not a deformation of any $A_{\infty}$-category. However, it makes perfect sense to check the curved $A_{\infty}$-relations for the structure $\mu_{H_{q}}$ defined on $H_{q}$. The product structure $\mu_{H_{q}}$ is merely an auxiliary tool and will disappear again in section 8.7

Lemma 8.31. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Assume $\mathcal{C}_{q}$ has optimal curvature. Then $\mu_{H_{q}}$ satisfies the curved $A_{\infty}$-relations and $F_{q}: H_{q} \rightarrow \mathcal{C}_{q}$ satisfies the curved $A_{\infty}$-functor relations.
Proof. We prove the statements in reverse order: First we show that $F_{q}$ satisfies the curved $A_{\infty}$-functor relations. Second we conclude that $\mu_{H_{q}}$ satisfies the curved $A_{\infty}$-relations.

For the first part, regard the curved $A_{\infty}$-functor relations for $F_{q}$ :

$$
\begin{equation*}
\sum(-1)^{\left\|a_{1}\right\|+\ldots+\left\|a_{j}\right\|} F_{q}\left(a_{k}, \ldots, \mu_{H_{q}}\left(a_{i}, \ldots, a_{j+1}\right), \ldots, a_{1}\right)=\sum \mu_{\mathcal{C}_{q}}\left(F_{q}\left(a_{k}, \ldots\right), \ldots, F_{q}\left(\ldots, a_{1}\right)\right) . \tag{8.6}
\end{equation*}
$$

We shall first prove these relations separately for $k=0$ and $k=1$ and then for general $k \geq 2$. For $k=0$, the relation reads $F_{q}^{1}\left(\mu_{H_{q}}^{0}\right)=\mu_{H_{q}}^{0}$ since $F_{q}^{0}=0$. This relation holds true by definition of $F_{q}^{1}$ and $\mu_{H_{q}}^{0}$. For $k=1$, we calculate

$$
\begin{aligned}
F_{q}^{1}\left(\mu_{H_{q}}^{1}(a)\right)+F_{q}^{2}\left(a, \mu_{H_{q}}^{0}\right)+(-1)^{\|a\|} F_{q}^{2}\left(\mu_{H_{q}}^{0}, a\right) & =\pi_{q}\left(\mu_{q}(a)\right)-h_{q} \mu_{q}^{2}\left(a, \mu_{q}^{0}\right)-(-1)^{\|a\|} h_{q} \mu_{q}^{2}\left(\mu_{q}^{0}, a\right) \\
& =\pi_{q}\left(\mu_{q}(a)\right)+h_{q}\left(\mu_{q}\left(\mu_{q}(a)\right)\right) \\
& =\mu_{q}(a)=\mu_{\mathcal{C}_{q}}\left(F_{q}^{1}(a)\right) .
\end{aligned}
$$

In the second equality, we have used the curved $A_{\infty}$-relations of $\mathcal{C}_{q}$. In the third equality, we have used the property of the deformed decomposition that $\mu_{q}\left(H_{q}\right) \subseteq H_{q} \oplus(B \widehat{\otimes} R)$. This settles the cases $k=0,1$.

Let us now prove 8.6 in case $k \geq 2$ by projecting both sides onto $H_{q}, \mu_{q}^{1}(B \widehat{\otimes} R)$ and $B \widehat{\otimes} R$ individually. First, regard the projection on $H_{q}$. Since $F_{q}^{\geq 2}$ has image in $B \widehat{\otimes} R$, we have

$$
\pi_{H_{q}}(\mathrm{LHS})=F_{q}^{1}\left(\mu_{H_{q}}\left(a_{k}, \ldots, a_{1}\right)\right)=\mu_{H_{q}}\left(a_{k}, \ldots, a_{1}\right)=\sum \pi_{q} \mu_{q}\left(F_{q}(\ldots), \ldots, F_{q}(\ldots)\right)=\pi_{H_{q}}(\mathrm{RHS})
$$

We have used nothing but the definition of $\mu_{H_{q}}$ and $F_{q}$. Now regard the projection on $\mu_{\mathcal{C}_{q}}^{1}(B \widehat{\otimes} R)$. We have

$$
\begin{aligned}
\pi_{\mu_{\mathcal{C}_{q}}^{1}(B \widehat{\otimes} R)}(\mathrm{RHS}) & =\mu_{q}^{1} h_{q} \mu_{q}^{\geq 2}\left(F_{q}(\ldots), \ldots, F_{q}(\ldots)\right)+\mu_{q}^{1} h_{q} \mu_{q}^{1} F_{q}(\ldots) \\
& =-\mu_{q}^{1} F_{q}(\ldots)+\mu_{q}^{1} F_{q}(\ldots)=0 \\
& =\pi_{\mu_{\mathcal{C}_{q}}^{1}(B \widehat{\otimes} R)}(\mathrm{LHS}) .
\end{aligned}
$$

Lastly, regard the projection to $B \widehat{\otimes} R$. We have

$$
\begin{aligned}
\pi_{B \widehat{\otimes} R}(\mathrm{LHS}) & =(-1)^{\left\|a_{1}\right\|+\ldots+\left\|a_{i}\right\|} F_{q}^{\geq 2}\left(\ldots, \mu_{\bar{H}_{q}}^{\geq 0}(\ldots), a_{i}, \ldots\right) \\
& =(-1)^{\left\|a_{1}\right\|+\ldots+\left\|a_{i}\right\|+1} h_{q} \mu_{q}^{\geq 2}\left(F_{q}, \ldots, F_{q}\left(\ldots, \mu_{\bar{H}_{q}}^{\geq 0}, a_{i}, \ldots\right), F_{q}\left(a_{j}, \ldots\right), \ldots, F_{q}\right) \\
& =(-1)^{\left\|a_{1}\right\|+\ldots+\left\|a_{j}\right\|+1} h_{q} \mu_{q}^{\geq 2}\left(F_{q}, \ldots, \mu_{q}^{\geq 0}\left(F_{q}, \ldots, F_{q}\right), F_{q}\left(a_{j}, \ldots\right), \ldots, F_{q}\right) \\
& =(-1)^{1+1} h_{q} \mu_{q}^{1} \mu_{q}^{\geq 2}\left(F_{q}, \ldots, F_{q}\right)+(-1)^{1+1} h_{q} \mu_{q}^{1} \mu_{q}^{1} F_{q}(\ldots) \\
& =+h_{q} \mu_{q}^{1} \mu_{q}^{\geq 2}\left(F_{q}, \ldots, F_{q}\right)-h_{q} \mu_{q}^{1} \mu_{q}^{1} h_{q} \mu_{q}^{\geq 2}\left(F_{q}, \ldots, F_{q}\right) \\
& =\pi_{B \widehat{\otimes} R}\left(\mu_{q}^{\geq 2}\left(F_{q}, \ldots, F_{q}\right)\right) \\
& =\pi_{B \widehat{\otimes} R}(\operatorname{RHS}) .
\end{aligned}
$$

In the second equality, we have unraveled the definition of $F_{q}$. In the third equality, we have assumed towards induction that 8.6 already holds for a shorter sequence of inputs. In the fourth equality, we have used the (curved) $A_{\infty}$-relation for $\mathcal{C}_{q}$. In the fifth equality, we have unraveled the definition of $F_{q}$ again. In the sixth equality, we have used the expression for the projection according to Lemma 8.30 Finally, we conclude that 8.6 holds on the entire hom spaces of $\mathcal{C}_{q}$. In other words, $F_{q}$ satisfies the (curved) $A_{\infty}$-functor relations.

For the second part of the proof, we show that $\mu_{H_{q}}$ satisfies the curved $A_{\infty}$-relations. The trick is to apply $F_{q}^{1}$ to the $A_{\infty}$-relations for $\mu_{H_{q}}$ and pull terms from inside to outside using the just proven fact that $F_{q}$ satisfies the (curved) $A_{\infty}$-functor relations. We calculate

$$
\begin{aligned}
& F_{q}^{1}\left(\mu_{H_{q}}\left(a_{k}, \ldots, \mu_{\bar{H}_{q}}^{\geq 0}(\ldots), \ldots, a_{1}\right)\right) \\
& =F_{q}^{\geq 2}\left(a_{k}, \ldots, \mu_{\bar{H}_{q}}^{\geq 1}\left(\ldots, \mu_{\bar{H}_{q}}^{\geq 0}(\ldots), \ldots\right), \ldots, a_{1}\right)+F_{q}^{\geq 2}\left(a_{k}, \ldots, \mu_{\bar{H}_{q}}^{\geq 0}(\ldots), \ldots, \mu_{\bar{H}_{q}}^{\geq 0}(\ldots), \ldots\right) \\
& \quad+\mu_{q}^{\geq 1}\left(F_{q}(\ldots), \ldots, F_{q}\left(\ldots, \mu_{\bar{H}_{q}}^{\geq 0}(\ldots), \ldots\right), \ldots, F_{q}(\ldots)\right) \\
& =F_{q}^{\geq 2}\left(a_{k}, \ldots, 0, \ldots, a_{1}\right)+0+\mu_{q}^{\geq 1}\left(F_{q}(\ldots), \ldots, \mu_{q}^{\geq 1}\left(F_{q}(\ldots), \ldots, F_{q}(\ldots)\right), \ldots, F_{q}(\ldots)\right) \\
& =0+0+0 .
\end{aligned}
$$

In the first equality, we have used that $F_{q}$ satisfies the (curved) $A_{\infty}$-functor relation on the sequence $a_{1}, \ldots, \mu_{H_{q}}^{\geq 0}(\ldots), \ldots, a_{k}$. The two terms on the second row come from the $A_{\infty}$-functor relation, and are distinguished by the choice whether the inner $\mu_{H_{q}}^{\geq 0}(\ldots)$ is inserted into the new product $\mu_{H_{q}}^{\geq 0}$ or not. The terms of the type $F_{q}\left(\mu_{H_{q}}, \mu_{H_{q}}\right)$ however appear pairwise and cancel each other. In the second equality, we have used the assumption that $\mu_{H_{q}}$ already satisfies the $A_{\infty}$-relations on shorter sequences.

Finally, we note that $F_{q}^{1}$ is injective on $H_{q}$ and therefore $\mu_{H_{q}}$ satisfies the $A_{\infty}$-relations on the sequence $a_{1}, \ldots, a_{k}$. In total, we conclude that $\mu_{H_{q}}$ satisfies the curved $A_{\infty}$-relations.

### 8.7 The deformed Kadeishvili theorem

In this section, we provide our most general Kadeishvili theorem for deformed $A_{\infty}$-categories. The starting point is an arbitrary deformed $A_{\infty}$-category $\mathcal{C}_{q}$ and the goal is to find a minimal model for $\mathcal{C}_{q}$ in the sense of Definition 8.11. The idea is to apply the curvature optimization procedure to $\mathcal{C}_{q}$, then take the auxiliary minimal model in the sense of section 8.6 and to pull back the structure in order to form a deformation of HC .

Deformed $A_{\infty}$-category $\mathcal{C}_{q} \quad \sim \quad$ Minimal model $\mathrm{H} \mathcal{C}_{q}$
Definition 8.32. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Let $H \oplus I \oplus R$ be a homological splitting for $\mathcal{C}$. Apply the curvature optimization procedure to $\mathcal{C}_{q}$. Let $\mathcal{C}_{q}^{\text {opt }}$ be the result and $r$ be the gauge used. Denote by $H_{q}^{\text {opt }} \oplus \mu_{\mathcal{C}_{q}^{\text {opt }}}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R)$ the deformed decomposition of $\mathcal{C}_{q}^{\text {opt }}$ and by $\varphi: H_{q}^{\text {opt }} \rightarrow B \widehat{\otimes} H$ the associated isomorphism. Apply the auxiliary minimal model procedure to $\mathcal{C}_{q}^{\text {opt }}$. Let $\mu_{H_{q}^{\text {opt }}}$ be the resulting auxiliary $A_{\infty}$-structure and $F_{q}^{\text {opt }}: H_{q}^{\text {opt }} \rightarrow \mathcal{C}_{q}^{\text {opt }}$ be the auxiliary functor.

Then we define the $A_{\infty}$-structure $\mu_{\mathrm{H} \mathcal{C}_{q}}$ on $B \widehat{\otimes} H$ and the functor $F_{q}: \mathrm{H} \mathcal{C}_{q} \rightarrow \mathcal{C}_{q}$ by

$$
\begin{align*}
\mu_{\mathrm{H} \mathcal{C}_{q}} & =\varphi \circ \mu_{H_{q}^{\text {opt }}} \circ \varphi^{-1},  \tag{8.7}\\
F_{q} & =(\mathrm{Id}-r) \circ F_{q}^{\text {opt }} \circ \varphi^{-1} .
\end{align*}
$$

Remark 8.33. In 8.7, the circle symbol denotes composition of curved $A_{\infty}$-functors. By abuse of notation, we have denoted the gauge functor from the curvature optimization procedure by Id $-r$, standing for the functor with 0 -ary component $-r$ and 1 -ary component Id and vanishing higher components. Furthermore, we have interpreted $\varphi$ as an $A_{\infty}$-functor $H_{q}^{\text {opt }} \rightarrow B \widehat{\otimes} H$ with only a 1-ary component. More explicitly, the definition for $\mu_{\mathrm{H} \mathcal{C}_{q}}$ reads

$$
\mu_{\mathrm{H} \mathcal{C}_{q}}^{n \geq 0}\left(h_{n}, \ldots, h_{1}\right):=\varphi \mu_{H_{q}^{\text {opt }}}^{n}\left(\varphi^{-1}\left(h_{n}\right), \ldots, \varphi^{-1}\left(h_{1}\right)\right) .
$$

Theorem 8.34. Let $\mathcal{C}$ be an $A_{\infty}$-category. Let $H \oplus I \oplus R$ be a homological splitting for $\mathcal{C}$ and let HC be the minimal model obtained from this splitting. Then $\mathrm{H} \mathcal{C}_{q}$ is an $A_{\infty}$-deformation of $\mathrm{H} \mathcal{C}$ and $F_{q}: \mathrm{H} \mathcal{C}_{q} \rightarrow \mathcal{C}_{q}$ is a quasi-isomorphism of deformed $A_{\infty}$-categories. In particular, $\mathrm{H} \mathcal{C}_{q}$ is a minimal model for $\mathcal{C}_{q}$.
Proof. It is our task to unwrap all definitions and to apply Lemma 8.31 The application of the curvature optimization procedure has made $\mathcal{C}_{q}^{\text {opt }}$ a category related to $\mathcal{C}_{q}$ by the gauge equivalence Id $-r: \mathcal{C}_{q}^{\text {opt }} \rightarrow \mathcal{C}_{q}$. Subsequent application of the auxiliary minimal model procedure has given $H_{q}^{\text {opt }}$ an $A_{\infty}$-structure with a functor $F_{q}^{\text {opt }}: H_{q}^{\text {opt }} \rightarrow \mathcal{C}_{q}^{\text {opt }}$. Pulling back has given a product structure on $\mathrm{H} \mathcal{C}_{q}$.

A first observation is that $\mathrm{H} \mathcal{C}_{q}$ satisfies the curved $A_{\infty}$-axioms. Indeed, it was merely pulled back form $H_{q}^{\text {opt }}$ and the product structure on $H_{q}^{\text {opt }}$ in turn satisfies the $A_{\infty}$-axioms due to Lemma 8.31 The leading term of $\mu_{\mathrm{H} \mathcal{C}}$ is easily seen to be $\mu_{\mathrm{H} \mathcal{C}}$ and hence $\mathrm{H} \mathcal{C}_{q}$ is a deformation of $\mathrm{H} \mathcal{C}$.

A second observation is that with respect to the three curved $A_{\infty}$-structures on $\mathrm{H} \mathcal{C}_{q}, H_{q}^{\text {opt }}, \mathcal{C}_{q}^{\text {opt }}, \mathcal{C}_{q}$, the following three mappings mappings define curved $A_{\infty}$-functors:

$$
\varphi: H_{q}^{\text {opt }} \rightarrow \mathrm{H} \mathcal{C}_{q}, \quad F_{q}^{\text {opt }}: H_{q}^{\text {opt }} \rightarrow \mathcal{C}_{q}^{\text {opt }}, \quad \mathrm{Id}-r: \mathcal{C}_{q}^{\text {opt }} \rightarrow \mathcal{C}_{q}
$$

For $\varphi$ and Id $-r$, this is the case by definition of pullback/uncurving. For $F_{q}^{\text {opt }}$, this is the statement of Lemma 8.31 In summary, $F_{q}$ is merely a composition of these three functors:

$$
F_{q}: \mathrm{H} \mathcal{C}_{q} \xrightarrow{\varphi^{-1}} H_{q}^{\mathrm{opt}} \xrightarrow{F_{q}^{\mathrm{opt}}} \mathcal{C}_{q}^{\mathrm{opt}} \xrightarrow{\mathrm{Id}-r} \mathcal{C}_{q} .
$$

We conclude that $F_{q}$ itself is a curved $A_{\infty}$-functor. Its leading term is the functor $F: \mathrm{HC} \rightarrow \mathcal{C}$ obtained from the Kadeishvili construction for the non-deformed category $\mathcal{C}$. Since $F$ is a quasi-isomorphism, we conclude that $F_{q}$ is a quasi-isomorphism in the sense of Definition 5.24. This settles all claims and proves that $\mathrm{H} \mathcal{C}_{q}$ is a minimal model for $\mathcal{C}_{q}$ in the sense of Definition 8.11

### 8.8 The $D=0$ case

In this section, we examine the deformed Kadeishvili construction in a special case. The starting point is a curvature-free deformed $A_{\infty}$-category $\mathcal{C}_{q}$ where the deformed differential satisfies $\mu_{q}^{1}(H) \subseteq \mu_{q}^{1}(B \widehat{\otimes} R)$. It turns out that in this case, the differential $\mu_{\mathrm{H} \mathcal{C}_{q}}^{1}$ on the minimal model vanishes. We re-interpret this case as an instance of the Kadeishvili theorem over base rings.

As we explain in Lemma 8.35. there are multiple ways of saying $\mu_{\mathcal{C}_{q}}^{1}(H) \subseteq \mu_{\mathcal{C}_{q}}^{1}(B \widehat{\otimes} R)$. One of them is requiring the operator $D$ appearing in the description Lemma 8.15 to vanish. We may therefore also call the present assumption the " $D=0$ case".

Lemma 8.35. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Let $H \oplus I \oplus R$ be a homological splitting of $\mathcal{C}$. Denote by $D, E, F$ the operators from Lemma 8.15. If $\mathcal{C}_{q}$ is curvature-free, then we have $D^{2}=0$ and $F=-E D$ and the following statements are equivalent:

1. For every $h \in B \widehat{\otimes} H$ there exists an $\varepsilon \in B \widehat{\otimes} R$ such that $\mu_{q}^{1}(h)=\mu_{q}^{1}(\varepsilon)$.
2. We have $\mu_{q}^{1}(H) \subseteq \mu_{q}^{1}(B \widehat{\otimes} R)$.
3. We have $D=F=0$.
4. We have $D=0$.

In the first statement, the element $\varepsilon$ is necessarily infinitesimal: $\varepsilon \in \mathfrak{m} R$. Similarly, the right hand side of the inclusion in $\mu_{q}^{1}(H) \subseteq \mu_{q}^{1}(B \widehat{\otimes} R)$ can be replaced by $\mu_{q}^{1}(\mathfrak{m} R)$.
Proof. Thanks to curvature-freeness, the differential $\mu_{q}^{1}$ squares to zero. The identities $D^{2}=0$ and $F=-E D$ now follow from evaluating $\left(\mu_{q}^{1}\right)^{2}=0$ with respect to the matrix presentation 8.4. The four enumerated statements are all different ways of stating the condition $D=F=0$. The only nontrivial observation is that $D=0$ already implies $F=0$ since $F=-E D$. For the final infinitesimality observations, note that $\mu_{q}^{1}(h)$ is necessarily infinitesimal, since $\mu^{1}(H)=0$ and $\mu_{q}^{1}$ is only an infinitesimal deformation of $\mu^{1}$.

The deformed decomposition of $\mathcal{C}_{q}$ has very favorable properties if $\mathcal{C}_{q}$ is curvature-free and satisfies $D=0$ :

Lemma 8.36. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Let $H \oplus I \oplus R$ be a homological splitting of $\mathcal{C}$. Assume $\mathcal{C}_{q}$ is curvature-free and $D=0$. With respect to the deformed decomposition of $\mathcal{C}_{q}$, the differential $\mu_{q}^{1}$ takes the shape

$$
\mu_{q}^{1}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & * \\
0 & 0 & 0
\end{array}\right) .
$$

Proof. For the third column, note that $\mu_{q}^{1}$ by definition sends $B \widehat{\otimes} R$ to $\mu_{q}^{1}(B \widehat{\otimes} R)$. For the second column, note that $\mu_{q}^{1}$ squares to zero. For the first column, pick an element $h-E h \in H_{q}$ with $h \in B \widehat{\otimes} H$. Then by the first column of 8.4) we have $\mu_{q}^{1}(h)=\mu_{q}^{1} E h$ and hence $\mu_{q}^{1}(h-E h)=0$. This finishes the proof.

According to Lemma 8.36, the deformed decomposition in case $D=0$ has many properties which we expect from a homological splitting. For this reason, we establish the following terminology alias:

Definition 8.37. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. Let $H \oplus I \oplus R$ be a homological splitting of $\mathcal{C}$. Assume $\mathcal{C}_{q}$ is curvature-free and $D=0$. Then the deformed homological splitting of $\mathcal{C}_{q}$ is the deformed decomposition $H_{q} \oplus \mu_{q}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R)$.

The minimal model $\mathrm{H} \mathcal{C}_{q}$ has favorable properties in case $\mathcal{C}_{q}$ is curvature-free and $D=0$. In fact, both curvature $\mu_{\mathrm{H} \mathcal{C}_{q}}^{0}$ and differential $\mu_{\mathrm{H} \mathcal{C}_{q}}^{1}$ vanish by construction. The higher products are computed by Kadeishvili trees, putting $\varphi^{-1}$ on every leaf, $h_{q} \mu^{\mathcal{C}_{q}}$ on every internal node and $\varphi \pi_{q} \mu^{\mathcal{C}_{q}}$ on the root. The entire procedure can be summarized as follows:
Corollary 8.38. Let $\mathcal{C}$ be an $A_{\infty}$-category and $H \oplus I \oplus R$ a homological splitting. Let $\mathcal{C}_{q}$ be a curvaturefree deformation of $\mathcal{C}$ with $D=0$. Then a minimal model $\mathrm{H} \mathcal{C}_{q}$ is determined by the following procedure:

1. For every $h \in H$, let $\varepsilon_{h} \in B \widehat{\otimes} R$ such that $\mu_{\mathcal{C}_{q}}^{1}(h)=\mu_{\mathcal{C}_{q}}^{1}\left(\varepsilon_{h}\right)$.
2. Define $H_{q}=\left\{h-\varepsilon_{h} \mid h \in H\right\}$.
3. Define $\varphi: H_{q} \rightarrow B \widehat{\otimes} H$ by $h-\varepsilon_{h} \mapsto h$.
4. Calculate the deformed codifferential $h_{q}: H_{q} \oplus \mu_{\mathcal{C}_{q}}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R) \rightarrow B \widehat{\otimes} R$.
5. Calculate the deformed projection $\pi_{q}: H_{q} \oplus \mu_{\mathcal{C}_{q}}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R) \rightarrow H_{q}$.
6. Set $\mu_{\mathrm{H} \mathcal{C}_{q}}^{1}=\mu_{\mathrm{H} \mathcal{C}_{q}}^{0}=0$.
7. Regard arbitrary Kadeishvili tree shapes $T$.
8. Decorate $T$ with $\varphi^{-1}$ at the leaves, $h_{q} \mu_{\mathcal{C}_{q}}$ at the internal nodes and $\pi_{q} \mu_{\mathcal{C}_{q}}$ at the root.
9. Define $\mu_{\mathrm{H} \mathcal{C}_{q}}^{\geq 2}\left(h_{k}, \ldots, h_{1}\right)$ as sum over the result of these trees, with $\operatorname{sign}(-1)^{N_{T}}$.

We would like to provide an aftermath to this corollary. More precisely, we will offer an independent explanation of the condition $D=0$. There is namely a classical Kadeishvili theorem that works for $A_{\infty}$-categories defined over rings: Let $S$ be a ring and $\mathcal{C}$ an $S$-linear $A_{\infty}$-algebra. If the cohomology $\mathrm{H}(A)$ is a projective $S$-module, then the projection $\operatorname{Ker}\left(\mu_{A}^{1}\right) \rightarrow \mathrm{H}(A)$ has a lift $\mathrm{H}(A) \rightarrow A$ which is an $S$-linear quasi-isomorphism of complexes. The original construction of Kadeishvili builds noncanonically a minimal $A_{\infty}$-structure on $\mathrm{H}(A)$ together with an $A_{\infty}$-quasi-isomorphism $\mathrm{H}(A) \rightarrow A$. This version of the Kadeishvili theorem can be found for instance in 60 .

It is natural to apply this Kadeishvili theorem to curvature-free deformed $A_{\infty}$-categories. In fact, if $\mathcal{C}_{q}$ is a curvature-free deformed $A_{\infty}$-category, then the classical Kadeishvili theorem gives a minimal model under the condition that $\operatorname{HHom}_{\mathcal{C}_{q}}(X, Y)$ are projective $B$-modules for every $X, Y \in \mathcal{C}$ :

$$
\begin{array}{cc}
\mu_{\mathcal{C}_{q}}^{0}=0 \text { and } & \Longrightarrow \quad \begin{array}{c}
\text { Classical Kadeishvili } \\
\operatorname{HHom}_{\mathcal{C}_{q}}(X, Y) \text { projective } B \text {-modules }
\end{array} \quad \begin{array}{c}
\text { applies to } \mathcal{C}_{q}
\end{array}
\end{array}
$$

In Lemma 8.39 we show that curvature-freeness together with $D=0$ implies the projectivity condition. In particular, we recover Corollary 8.38 as a consequence of the classical Kadeishvili theorem, under the technical assumption that $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is finite-dimensional for all $X, Y \in \mathcal{C}$. We have opted for hiding the quantification by $X, Y \in \mathcal{C}$ in some cases and making it explicit in other cases.

Lemma 8.39. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a curvature-free deformation. Choose a homological splitting $H \oplus I \oplus R$. Denote by $D, E, F$ the operators from Lemma 8.15. Then we have a natural quasi-isomorphism of $B$-modules

$$
\begin{equation*}
\mathrm{H}\left(\operatorname{Hom}_{\mathcal{C}_{q}}, \mu_{q}^{1}\right) \cong \mathrm{H}(B \widehat{\otimes} H, D) \tag{8.8}
\end{equation*}
$$

In particular if $D=0$ and $H(X, Y)$ is finite-dimensional, then $\mathrm{H}\left(\operatorname{Hom}_{\mathcal{C}_{q}}(X, Y), \mu_{\mathcal{C}_{q}}^{1}\right)$ is a projective $B$-module.

Proof. To equate the two cohomology modules, we provide explicit morphisms $\varphi, \psi$ of chain complexes in both directions. Next, we check that the maps actually commute with the differential. We finally show that in cohomology, both compositions $\varphi \psi$ and $\psi \varphi$ descend to the identity.

Our first step is to give explicit morphisms of chain complexes. In terms of the decomposition 8.3 of $\operatorname{Hom}_{\mathcal{C}_{q}}$ into $B \widehat{\otimes} H \oplus \mu_{q}^{1}(B \widehat{\otimes} R) \oplus B \widehat{\otimes} R$, put

$$
\begin{aligned}
\varphi:\left(\operatorname{Hom}_{\mathcal{C}_{q}}, \mu_{q}^{1}\right) & \longrightarrow(B \widehat{\otimes} H, D), \\
\left(h, \mu_{q}^{1}\left(r^{\prime}\right), r\right) & \longmapsto h, \quad \text { for } \quad h \in B \widehat{\otimes} H, \quad r, r^{\prime} \in B \widehat{\otimes} R, \\
\psi:(B \widehat{\otimes} H, D) & \longrightarrow\left(\operatorname{Hom}_{\mathcal{C}_{q}}, \mu_{q}^{1}\right), \\
h & \longmapsto(h, 0,-E h), \quad \text { for } \quad h \in B \widehat{\otimes} H .
\end{aligned}
$$

We are now ready to check that both $\varphi$ and $\psi$ are chain maps. Indeed, we have

$$
\varphi\left(\mu_{q}^{1}\left(h, \mu_{q}^{1}\left(r^{\prime}\right), r\right)\right)=\varphi\left(D h, \mu_{q}^{1}(E h)+\mu_{q}^{1}(r), F h\right)=D h=D \varphi\left(h, \mu_{q}^{1}\left(r^{\prime}\right), r\right)
$$

and

$$
\psi(D h)=(D h, 0,-E D h)=(D h, 0, F h)=\mu_{q}^{1}(h, 0,-E h)=\mu_{q}^{1}(\psi(h))
$$

In the above calculations, we have written elements of the direct sum $B \widehat{\otimes} H \oplus \mu_{q}^{1}(B \widehat{\otimes} R) \oplus B \widehat{\otimes} R$ as tuples instead of sums.

The next step is to calculate $\varphi \psi$ and $\psi \varphi$ and verify that they descend to the identity on cohomology. For $\varphi \psi$, this is trivial since $\varphi \psi=\mathrm{id}$. For the other composition $\psi \varphi$, we pick an element $x \in \operatorname{Ker}\left(\mu_{q}^{1}\right)$
and check whether $\psi(\varphi(x))-x$ lies in the image of $\mu_{q}^{1}$. First of all, write $x=\left(h, \mu_{q}^{1}\left(r^{\prime}\right), r\right)$ and note that $x \in \operatorname{Ker}\left(\mu_{q}^{1}\right)$ implies $D h=0, r+E h=0$ and $F h=0$. We obtain

$$
\psi(\varphi(x))-x=\psi\left(h, \mu_{q}^{1}\left(r^{\prime}\right), r\right)-\left(h, \mu_{q}^{1}\left(r^{\prime}\right), r\right)=(h, 0,-E h)-\left(h, \mu_{q}^{1}\left(r^{\prime}\right), r\right)=\left(0, \mu_{q}^{1}\left(r^{\prime}\right), 0\right)
$$

The expression on the right is simply the image $\mu_{q}^{1}\left(0,0, r^{\prime}\right)$. We conclude that $\varphi$ and $\psi$ are quasi-inverse to each other. This establishes the desired quasi-isomorphism 8.8). In case $D=0$ and $H(X, Y)$ is finitedimensional, the cohomology is simply $B \otimes H(X, Y)$ which is projective. This finishes the proof.

With the help of Lemma 8.39, we can also reformulate the condition $D=0$ to a more intuitive statement. Let us distinguish between true and actual cohomology. By true cohomology of $\mathcal{C}_{q}$, we mean the flat tensor products $B \widehat{\otimes} \mathrm{H}_{\operatorname{Hom}_{\mathcal{C}}}(X, Y)$. It only depends on the non-deformed category itself. By actual cohomology, we mean the directly observed cohomology $\mathrm{H}\left(\operatorname{Hom}_{\mathcal{C}_{q}}, \mu_{\mathcal{C}_{q}}^{1}\right)$. The two cohomologies typically differ:

$$
\begin{gathered}
\mathbb{C} \xrightarrow{0} \mathbb{C} \\
\text { original differential } \\
\text { cohomology: } \mathbb{C}[0] \oplus \mathbb{C}[1]
\end{gathered}
$$

$$
\mathbb{C} \llbracket q \rrbracket \xrightarrow{q} \mathbb{C} \llbracket q \rrbracket
$$

$\cdots \sim \sim$
deformed differential
actual cohomology:
$0[0] \oplus \frac{\mathbb{C}\lfloor q \mathbb{1}}{(q)}[1]$ true cohomology: $\quad \mathbb{C} \llbracket q \rrbracket[0] \oplus \mathbb{C} \llbracket q \rrbracket[1]$

Lemma 8.39 quantifies the difference between true and actual cohomology. True and actual cohomology of $\mathcal{C}_{q}$ are equal if $D=0$ and fail to be canonically equal if $D \neq 0$. Among curvature-free deformations, we can summarize our observations without any claim to rigor very roughly as follows:

$$
\begin{aligned}
\text { Classical Kadeishvili applies to } \mathcal{C}_{q} & \Longleftrightarrow \operatorname{HHom}_{\mathcal{C}_{q}}(X, Y) \text { projective } B \text {-modules } \\
\Longleftrightarrow \quad \text { true cohomology }=\text { actual cohomology } & \Longleftrightarrow \quad D=0 \quad \Longleftrightarrow \quad \mu_{\mathrm{H} \mathcal{C}_{q}}^{1}=0
\end{aligned}
$$

## 9 Uncurving of strings and bands

In this section, we show how to remove curvature of band objects in derived deformed gentle algebras. First, we set up general theory for removing curvature of objects in deformd $A_{\infty}$-categories. Then we recall how objects of $\mathrm{Tw} \operatorname{Gtl} \mathcal{A}$ can be classified geometrically as strings and bands, due to 35 . All objects of $\mathrm{Tw} \operatorname{Gtl} \mathcal{A}$ can naturally be interpreted as objects in $\mathrm{Tw}_{\mathrm{Gtl}}^{q} \boldsymbol{\mathcal { A }}$. However, they do not satisfy the Maurer-Cartan equation of $\mathrm{Tw} \mathrm{Gtl}_{q} \mathcal{A}$ itself and become curved objects. In the present section, we introduce a method to reduce curvature of these curved objects, which we call the "complementary angle trick".

$$
\underset{(X, \delta) \in \operatorname{Tw~Gtl} \mathcal{A}}{\text { Band object }} \xrightarrow{\text { upon deformation }} \underset{\longrightarrow}{\text { Curved object }}(X, \delta) \in{\mathrm{Tw} \mathrm{Gtl}_{q} \mathcal{A}}^{l} \xrightarrow{\text { uncurving }} \quad \begin{gathered}
\text { Uncurved object } \\
\left(X, \delta_{q}\right) \in \mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} \mathcal{A}
\end{gathered}
$$

We show that the complementary angle trick succeeds in removing curvature from band objects which satisfy a technical condition. Subsequently, we show how to drop this condition. While not an ideal term, we will constantly refer to removing curvature as "uncurving". The starting point is a full arc system with [NMDC]. We find that uncurvability differs between band objects and string objects and depends also on the topology of the objects. Our findings can be summarized as follows:

- Uncurvability of band objects: Our main criterion states that a band object can be uncurved in case the underlying curve, regarded as a curve in $S$, is not contractible and does not include a teardrop.
- Uncurvability of string objects: The general rule is that string objects cannot be uncurved. There are exceptions, for example a string where both ends touch each other may be uncurvable if the deformation parameter lies in $\mathfrak{m}^{2}$.


### 9.1 The theory of uncurving

In this section, we recollect uncurving theory for $A_{\infty}$-deformations. The starting point is an arbitrary deformed $A_{\infty}$-category with curvature. We are interested in the question which objects become curvaturefree once we apply a gauge functor to the category. In the present section, we explain this question and define terminology. We prove elementary properties. Uncurving has been studied in the literature, for instance under the name of the "curvature problem" in 50.

It is our aim to explain in how far a gauge equivalence can change the curvature of a deformation. As a starting point, let $\mathcal{C}$ be an $A_{\infty}$ category. Let $\mathcal{C}_{q}$ and $\mathcal{C}_{q}^{\prime}$ be two deformations of $\mathcal{C}$ connected by a gauge equivalence $F: \mathcal{C}_{q} \rightarrow \mathcal{C}_{q}^{\prime}$. The curvature of $\mathcal{C}_{q}$ and $\mathcal{C}_{q}^{\prime}$ are related by the zeroth (curved) $A_{\infty}$-functor relation:

$$
F^{1}\left(\mu_{\mathcal{C}_{q}}^{0}\right)=\mu_{\mathcal{C}_{q}^{\prime}}^{0}+\mu^{1}\left(F^{0}\right)+\mu^{2}\left(F^{0}, F^{0}\right)+\ldots
$$

The map $F^{1}$ has leading term the identity and is then a linear isomorphism by Lemma 5.13. This means that the curvature $\mu_{\mathcal{C}_{q}}^{0}$ depends only on $F^{0}$ and $F^{1}$, and not on the higher components $F^{\geq 2}$. If we approximate $F^{1}$ as the identity, we conclude that uncurving essentially depends only on the choice of $F^{0}$. This gives rise to the following definition:

Definition 9.1. Let $\mathcal{C}$ be an $A_{\infty}$ category and $\mathcal{C}_{q}$ a deformation. Let $X \in \mathcal{C}$. Then $X$ is uncurvable if there exists an $S \in \mathfrak{m} \operatorname{End}^{1}(X)$ such that

$$
\mu_{X}^{0}+\mu_{\mathcal{C}_{q}}^{1}(S)+\mu_{\mathcal{C}_{q}}^{2}(S, S)+\ldots=0
$$

We will now prove several basic properties regarding uncurvable objects.
Lemma 9.2. Let $F: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ be a functor of deformed $A_{\infty}$-categories. If $X \in \mathcal{C}_{q}$ is uncurvable, then so is $F(X)$.

Proof. Let $S \in \mathfrak{m} \operatorname{End}^{1}(X)$ be the uncurving morphism, that is

$$
\mu_{X}^{0}+\mu^{1}(S)+\mu^{2}(S, S)+\ldots=0
$$

Now set

$$
T:=F_{X}^{0}+F^{1}(S)+F^{2}(S, S)+\ldots \in \mathfrak{m} \operatorname{End}^{1}(F(X))
$$

We claim that $T$ is an uncurving morphism for $F(X)$. Indeed,

$$
\begin{aligned}
\mu_{F(X)}^{0}+\mu^{1}(T)+\mu^{2}(T, T)+\ldots & =\left(\mu_{F(X)}^{0}+\mu^{1}\left(F_{X}^{0}\right)+\mu^{2}\left(F_{X}^{0}, F_{X}^{0}\right)+\ldots\right) \\
& +\left(\mu^{1}\left(F^{1}(S)\right)+\mu^{2}\left(F^{1}(S), F_{X}^{0}\right)+\mu^{2}\left(F_{X}^{0}, F^{1}(S)\right)+\ldots\right) \\
& +\left(\mu^{1}\left(F^{2}(S, S)\right)+\mu^{2}\left(F^{1}(S), F^{1}(S)\right)+\mu^{2}\left(F^{2}(S, S), F_{X}^{0}\right)+\ldots\right)+\ldots
\end{aligned}
$$

We apply the curved $A_{\infty}$ rule to these terms and continue

$$
\begin{aligned}
= & F^{1}\left(\mu_{X}^{0}+\mu^{1}(S)+\mu^{2}(S, S)+\ldots\right)+F^{2}\left(\mu_{X}^{0}+\mu^{1}(S)+\mu^{2}(S, S)+\ldots, S\right) \\
& +F^{2}\left(S, \mu_{X}^{0}+\mu^{1}(S)+\mu^{2}(S, S)+\ldots\right)+\ldots \\
= & F^{1}(0)+F^{2}(0, S)+F^{2}(S, 0)+F^{3}(0, S, S)+\ldots=0
\end{aligned}
$$

This shows that $T$ is an uncurving morphism for $F(X)$.
Lemma 9.3. Let $\mathcal{C}$ be an $A_{\infty}$ category and $\mathcal{C}_{q}$ a deformation. Assume $\mathcal{C}$ is minimal, $X \cong Y$ in $\mathcal{C}$ and $X$ is uncurvable. Then so is $Y$.

Proof. Regard the embedding

$$
\{Y\}_{q} \rightarrow\{X, Y\}_{q}
$$

where both sides are defined as full subcategories of $\mathcal{C}_{q}$. The embedding is an equivalence, since it reduces to the inclusion $\{Y\} \rightarrow\{X, Y\}$ which is essentially surjective in cohomology. By Lemma 5.57 there is a quasi-equivalence in opposite direction

$$
\{X, Y\}_{q} \rightarrow\{Y\}_{q}
$$

In particular, it maps $X$ to $Y$. Since $X$ is uncurvable, an application of Lemma 9.2 shows that $Y$ is uncurvable.

The next lemma concerns uncurving of objects in minimal models. The notation $\mathrm{H} \mathcal{C}_{q}$ denotes any minimal model of $\mathrm{H} \mathcal{C}_{q}$ according to Definition 8.11. A minimal model comes with a choice of quasiisomorphism $F_{q}: \mathcal{C}_{q} \rightarrow \mathrm{H} \mathcal{C}_{q}$. Correspondingly, the objects of $\mathcal{C}_{q}$ and $\mathrm{H} \mathcal{C}_{q}$ are identified via $F_{q}$.
Lemma 9.4. Let $\mathcal{C}$ be an $A_{\infty}$ category and $\mathcal{C}_{q}$ a deformation. Then an object $X$ is uncurvable in $\mathcal{C}_{q}$ if and only if it is uncurvable in $\mathrm{H} \mathcal{C}_{q}$.

Proof. Let $F_{q}: \mathcal{C}_{q} \rightarrow \mathrm{H} \mathcal{C}_{q}$ be a quasi-isomorphism and let $X \in \mathcal{C}_{q}$. We need to show that $X$ is uncurvable if and only if $F_{q}(X)$ is uncurvable. By Lemma $9.2 F_{q}(X)$ is clearly uncurvable if $X$ is uncurvable. For the other direction, regard the restriction $\left.F_{q}\right|_{\{X\}}:\{X\}_{q} \rightarrow\left\{F_{q}(X)\right\}_{q}$. It is a quasi-isomorphism and by Lemma 5.57 there exists a quasi-isomorphism $\left\{F_{q}(X)\right\}_{q} \rightarrow\{X\}_{q}$ in opposite direction. Now if $F_{q}(X)$ is uncurvable, then by Lemma 9.2 also $X$ is uncurvable. This finishes the proof.

Corollary 9.5. Let $\mathcal{C}$ be an $A_{\infty}$ category and $\mathcal{C}_{q}$ a deformation. If $X$ and $Y$ are quasi-isomorphic in $\mathcal{C}$, then $X$ is uncurvable if and only if $Y$ is uncurvable.

Proof. Pick a minimal model $\mathrm{H}_{q}$. The objects $X$ and $Y$ are isomorphic in HC . Combining Lemma 9.3 and 9.4 we conclude

$$
X \in \mathcal{C}_{q} \text { uncurvable } \Leftrightarrow X \in \mathrm{H} \mathcal{C}_{q} \text { uncurvable } \Leftrightarrow Y \in \mathrm{H} \mathcal{C}_{q} \text { uncurvable } \Leftrightarrow Y \in \mathcal{C}_{q} \text { uncurvable. }
$$

This chain of equivalences proves the claim.
Corollary 9.5 might be slightly surprising. It is entirely irrelevant for uncurvability how $X$ and $Y$ get deformed themselves, the only relevant measure is whether they are quasi-equivalent in $\mathcal{C}$. This supports Definition 5.16 where we defined two objects $X, Y \in \mathcal{C}_{q}$ to be quasi-isomorphic already if they are quasi-isomorphic in $\mathcal{C}$.

The next basic property deals with twisted complexes. It is a useful preparation for section 9.4 Recall that cones in $A_{\infty}$-categories are merely specific twisted complexes, more precisely the cone over $f \in \operatorname{Hom}^{0}(X, Y)$ is defined as

$$
\operatorname{cone}(f)=\left(Y[1] \oplus X,\left(\begin{array}{ll}
0 & f \\
0 & 0
\end{array}\right)\right)
$$

Corollary 9.6. Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation. If $X, Y \in \mathcal{C}_{q}$ are uncurvable, then so is every cone $(f)$ for $f \in \operatorname{Hom}_{\mathcal{C}}^{0}(X, Y)$. More generally, let $X_{1}, \ldots, X_{k}$ be uncurvable. Then any twisted complex $\left(X_{1}\left[s_{1}\right] \oplus \ldots \oplus X_{k}\left[s_{k}\right], \delta\right) \in \operatorname{Tw} \mathcal{C}_{q}$ is uncurvable.
Proof. Although a direct proof should be possible, combining the uncurving elements $S$ for each of the objects involved with the $\delta$ matrix, we give an abstract proof using Lemma 9.2. The idea is to uncurve the objects involved first, and then form twisted complex anew so that it automatically becomes an object without curvature.

Regard the full subcategory $\mathcal{D}:=\left\{X_{1}, \ldots, X_{k}\right\} \subseteq \mathcal{C}$, together with its deformation $\mathcal{D}_{q}$. All objects in $\mathcal{D}_{q}$ are uncurvable, which provides an isomorphism

$$
F: \mathcal{D}_{q} \xrightarrow{\sim} \mathcal{E}_{q}
$$

to a curvature-free deformation $\mathcal{E}_{q}$ of $\mathcal{D}$. This induces an isomorphism

$$
\tilde{F}: \quad \operatorname{Tw} \mathcal{D}_{q} \xrightarrow{\sim} \operatorname{Tw} \mathcal{E}_{q} .
$$

This isomorphism maps the twisted complex $C:=\left(\bigoplus X_{i}\left[s_{i}\right], \delta\right)$ to some object $\tilde{F}(C)$ without curvature. The inverse functor $\tilde{F}^{-1}$ maps $\tilde{F}(C)$ back to $C$. According to Lemma 9.2 the object $C$ is then uncurvable in $\operatorname{Tw} \mathcal{D}_{q}$ and therefore also in $\operatorname{Tw} \mathcal{C}_{q}$.

### 9.2 Strings and bands

In this section, we recall the classification of objects in $\mathrm{HTw} \mathrm{Gtl} \mathcal{A}$. This classification is due to Haiden, Katzarkov and Kontsevich and categorizes the objects into two classes, the so-called string and band objects. Roughly speaking, a string object is a non-closed curve running between two punctures of $\mathcal{A}$ and a band object is a closed curve that avoids the punctures of $\mathcal{A}$. In the present section, we recall the precise classification and how to realize string and band objects explicitly as twisted complexes.

Originally, the gentle algebra $\operatorname{Gtl} \mathcal{A}$ was introduced in 18 to provide a combinatorial description of the wrapped Fukaya category of $S \backslash M$. It was shown in that paper's appendix that $\mathrm{Gtl} \mathcal{A}$ indeed embeds


Figure 9.1: A string object and a band object on the 4-punctured sphere


Figure 9.2: Smooth versus discrete
into $\mathrm{wFuk}(S \backslash M)$. The question arose whether this embedding is essentially surjective, upon passing to the derived category $\operatorname{HTw} \operatorname{Gtl} \mathcal{A}$. If one puts suitable restrictions on the geometry of the objects allowed in $\mathrm{wFuk}(S \backslash M)$ and works with a $\mathbb{Z}$-grading, the embedding is indeed essentially surjective. The fact that an arc system suffices to generate the wrapped Fukaya category has apparently been folklore for longer, and was affirmed by Haiden, Katzarkov and Kontsevich 35 .

Until now, we have defined $\operatorname{Gtl} \mathcal{A}$ as a $\mathbb{Z} / 2 \mathbb{Z}$-graded $A_{\infty}$-category. In order to state and discuss the classification of objects, we need to recall $\mathbb{Z}$-gradings $\mathrm{Gtl}^{\mathbb{Z}} \mathcal{A}$ on $\mathrm{Gtl} \mathcal{A}$. The procedure to upgrade $\mathrm{Gtl} \mathcal{A}$ to a $\mathbb{Z}$-graded $A_{\infty}$-category $\mathrm{Gtl}^{\mathbb{Z}} \mathcal{A}$ is as follows: Choose a line field on the surface $S$, with singularities allowed at the punctures. Choose a grading of the $\operatorname{arcs} a \in \mathcal{A}$ relative to the line field. Define the degree of an angle $\alpha: a \rightarrow b$ as the rounding off of the amount it turns relative to the grading of $a$ and $b$.

A precise description of this procedure, including definitions of the notions of line field, arc grading, degree of an angle can be found in 35 . With this in mind, we are ready to recall the classification of objects.

Theorem 9.7 ( 35$)$. Let $\mathcal{A}$ be a full arc system with [NMD]. Up to isomorphism, the objects of $\mathrm{HTw} \mathrm{Gtl}{ }^{\mathbb{Z}} \mathcal{A}$ can be classified as direct sums of the following:

- String objects: graded curves in $S$, starting and ending at two punctures, but otherwise avoiding punctures and not bounding a teardrop in $S \backslash M$. The curve is to be considered up to homotopy, keeping endpoints fixed.
- Band objects: closed graded curves in $S$ with indecomposable local system, avoiding punctures and not bounding a teardrop in $S \backslash M$. The curve is to be considered up to homotopy and the local system up to isomorphism.

An example string and band object are depicted in Figure 9.1 Actual representatives of string and band objects as twisted complexes in $\mathrm{HTw} \mathrm{Gtl}^{\mathbb{Z}} \mathcal{A}$ can be provided by an explicit construction, which we now describe in some detail. The input datum of both constructions is a graded curve of one of the two types above.

A string object can be constructed by approximating the curve by a sequence of arcs $a_{1}, \ldots, a_{k}$, and adding up these arcs to form a twisted complex

$$
\left(a_{1}\left[s_{1}\right] \oplus \ldots \oplus a_{k}\left[s_{k}\right], \delta\right) \in \operatorname{Tw~}_{\operatorname{Gtl}}{ }^{\mathbb{Z}} \mathcal{A}
$$

The shifts $s_{i} \in \mathbb{Z}$ are defined such that the degree of $a_{i}\left[s_{i}\right]$ with respect to the line field equals the inherent grading of the curve. Let us explain how $\delta$ is found: At every endpoint between two consecutive arcs $a_{i}, a_{i+1}$ of the sequence, determine whether the curve runs to the left or to the right of the puncture. The curve either follows an angle $a_{i} \rightarrow a_{i+1}$ or an angle $a_{i+1} \rightarrow a_{i}$. Insert this angle into the $\delta$ matrix. Possibly, the order of the arcs as summands of the twisted complex needs to be reordered to make $\delta$ upper triangular. In summary, the $\delta$ matrix indicates how the arcs $a_{1}, \ldots, a_{k}$ are stitched together. The procedure is depicted in Figure 9.2

Remark 9.8. Let us record the following properties: The twisted differential $\delta$ automatically becomes homogeneous of degree 1 due to the chosen shifts $s_{i}$. Identity angles $\alpha_{i}$ are not allowed, and can in fact be avoided when choosing the approximation by arcs. Two consecutive angles $\alpha_{i}, \alpha_{i+1}$ are never composable because they turn around the opposite ends of their common arc. One might wonder about composability in case this arc is a loop. Indeed the two consecutive angles may be composable in $\mathrm{Gtl} \mathcal{A}$, but are not composable in the order enforced by the ordering on the angles in the twisted complex.

Band objects can be constructed in a fashion similar to strings. In contrast to strings, their ends are however also stitched together by an angle, and the local system is manifested in the $\delta$ matrix. Let us explain these steps in more detail.

- Insert not only the angles between two consecutive arcs $a_{i}$ and $a_{i+1}$, but also the angle between the last arc $a_{k}$ and the first arc $a_{1}$ into the $\delta$ matrix. Reorder the arcs to make $\delta$ upper triangular. In contrast to the case of strings, such a reordering need not exist. This happens if all arcs are connected entirely cyclically. Abort in this case, and approximate the curve by a different sequence of arcs. It is shown below that this is possible, without inserting identities into $\delta$.
- The shifts are chosen such that $\delta$ has degree 1 . In contrast to strings which have an entry less in their $\delta$ matrix, it requires a check that this can be done in a consist manner. After walking around the curve one full cycle, do we end up with the same degree shift as we started with? The answer is yes, and the reason is that the curve was required to be graded with respect to the line field.
- If the local system is of dimension $d>1$, duplicate all arcs in the twisted complex so that each arc appears $d$ times. Also duplicate the angles so each angle appears $d$ times, running between the $i$-th copy of some arc and the $i$-th copy of the next (without running from one copy to the other).
- If the local system is non-trivial, represent it as a matrix $M=\left(m_{p q}\right) \in \mathbb{C}^{d \times d}$ and insert it into the $\delta$ matrix as follows: Choose two consecutive arcs $a_{i}, a_{i+1}$ in the representation of the curve by arcs. Then change the $\delta$ entry running from the $q$-th copy of $a_{i}$ to the $p$-th copy of $a_{i+1}$ to $m_{p q}$. For instance in case $d=2$, the part of $\delta$ matrix between $a_{i}$ and $a_{i+1}$ shall look like

$$
\left(\begin{array}{c|cc} 
& m_{11} \alpha_{i} & m_{12} \alpha_{i} \\
& m_{21} \alpha_{i} & m_{22} \alpha_{i} \\
\hline & &
\end{array}\right) .
$$

This overwrites the default identity matrix at those entries written to $\delta$ in step (2). In case the angle between $a_{i}$ an $a_{i+1}$ runs from $a_{i+1}$ to $a_{i}$ instead, use the inverse of $M$ instead of $M$ itself. It does not matter which pair of consecutive arcs is chosen. In fact, $M$ could be arbitrarily factorized into matrices, one for each pair of consecutive arcs, and the values could be written to $\delta$ per pair. It does not matter which factorization we choose: The isomorphism class of the resulting object in $\mathrm{HTw} \mathrm{Gtl}{ }^{\mathbb{Z}} \mathcal{A}$ only depends on the product of the factors.

Lemma 9.9. Let $\mathcal{A}$ be a full arc system with [NMD]. Then every string and band has an approximation by arcs where no angle is an identity and all arcs can be ordered such that $\delta$ is upper triangular.

Proof. First, choose some arbitrary approximation of the curve such that no connecting angle is the identity. This is always possible. The rest of the proof consists of tweaking this approximation such that $\delta$ becomes upper triangular.

For strings, there is always an ordering of the arcs in which $\delta$ is upper triangular, and we are done. For bands however, such an ordering need not exist. That is, the arcs might be connected cyclically. The remaining task in this proof is to break the cyclicity in the band case by tweaking the arc collection.

We may assume that one of the angles $\alpha_{i}$ consists of at least three indecomposable components. Otherwise, choose a different approximation by arcs where one angle winds a little more around some puncture.

Regard such an angle $\alpha_{i}$ that consists of at least three indecomposable components, and split it into a product $\alpha_{i}=\alpha_{i}^{3} \alpha_{i}^{2} \alpha_{i}^{1}$ of three non-empty angles such that $\alpha_{i}^{2}$ is indecomposable. In particular, $\alpha_{i}^{2}$ is an interior angle of some polygon. We now modify the angle sequence $\alpha_{1}, \ldots, \alpha_{k}$ by flipping $\alpha_{i}$ over to the opposite side of this polygon, see Figure 9.3 This tweak yields a non-cyclic approximation where all angles are still non-empty.

Let us discuss this classification in the context of the grading question. What are the objects up to isomorphism of $\mathrm{H} \operatorname{Tw} \operatorname{Gtl} \mathcal{A}$ ? Here $\operatorname{Gtl} \mathcal{A}$ and its twisted completion are taken as usual with $\mathbb{Z} / 2 \mathbb{Z}$-grading. The answer is, there are both more and less objects. For example, twisted complexes differing only by


Figure 9.3: Removing cyclicity

$\mathbb{Z}$-graded $\mathrm{Gtl}^{\mathbb{Z}} \mathcal{A}$ $\mathbb{Z}$-graded $\mathrm{HTw} \mathrm{Gtl}^{\mathbb{Z}} \mathcal{A}$

$\downarrow \quad$ Strings and bands can also be constructed $\mathbb{Z} / 2 \mathbb{Z}$-graded

## $\mathbb{Z} / 2 \mathbb{Z}$-graded Gtl $\mathcal{A}$ <br> $\imath$

$\mathbb{Z} / 2 \mathbb{Z}$-graded H Tw Gtl $\mathcal{A}$
classification unknown

Figure 9.4: Overview on objects of $\mathbb{Z}$-vs. $\mathbb{Z} / 2 \mathbb{Z}$-graded gentle algebras
even shifts are unequal in the $\mathbb{Z}$-grading, but are identified in the $\mathbb{Z} / 2 \mathbb{Z}$-grading. On the other hand, some twisted complexes that can be made with respect to the $\mathbb{Z} / 2 \mathbb{Z}$-grading cannot be constructed in the $\mathbb{Z}$-grading. One might say, the $\mathbb{Z} / 2 \mathbb{Z}$-graded $\mathrm{HTw} \mathrm{Gtl} \mathcal{A}$ category contains all objects of $\mathrm{HTw} \mathrm{Gtl}^{\mathbb{Z}} \mathcal{A}$ for every possible $\mathbb{Z}$-grading, plus additional objects that cannot be obtained from a $\mathbb{Z}$-graded version, modulo identifying objects differing by shifts. This is depicted in the overview Figure 9.4 We are however not aware of a concise classification of the objects of $\mathrm{H} \operatorname{Tw} \mathrm{Gtl} \mathcal{A}$.

A broad class of objects in $\mathrm{HTw} \operatorname{Gtl} \mathcal{A}$ can however be constructed by forming $\mathbb{Z} / 2 \mathbb{Z}$-graded strings and bands, corresponding to curves on $S$ without grading requirements. Let us define what we mean by this:

- $\mathbb{Z} / 2 \mathbb{Z}$-graded string objects: Stitch arcs together as in the $\mathbb{Z}$-graded case, and shift arcs with $s_{i} \in \mathbb{Z} / 2 \mathbb{Z}$ such that the degree of $\delta$ is odd.
- $\mathbb{Z} / 2 \mathbb{Z}$-graded band objects: Stitch arcs together as in the $\mathbb{Z}$-graded case, shift arcs with $s_{i} \in \mathbb{Z} / 2 \mathbb{Z}$ such that the degree of $\delta$ is odd. The shifts are consistent: After cycling around the curve once, we end up with the same shift because the curve is orientable.


### 9.3 Complementary angle trick

In this section, we introduce a method to uncurve objects of $\operatorname{Tw~} \mathrm{Gtl}_{q} \mathcal{A}$. The starting point is the classification of objects in $\operatorname{Tw} \operatorname{Gtl} \mathcal{A}$ recalled in section 9.2 The idea to uncurve these objects is to infinitesimally deform their $\delta$-matrix by inserting infinitesimal multiples of the complements of the angles already present in the $\delta$-matrix. We therefore call this method the "complementary angle trick". In the present section, we show that this trick successfully uncurves $\mathbb{Z} / 2 \mathbb{Z}$-graded band objects under certain conditions.

We will start the setup in a slightly more general approach: We take $\mathcal{A}$ to denote a full arc system with [NMDC] and we take the category $\mathrm{Gtl}_{r} \mathcal{A}$ to be the associated deformed gentle algebra constructed in Paper I This deformed gentle algebra depends on a parameter

$$
r=r_{0} 1+\sum_{\substack{q \in M \\ n \geq 1}} r_{q, n} \ell_{q}^{n} \in \mathfrak{m} Z(\operatorname{Gtl} \mathcal{A})
$$

Here $B$ is a chosen deformation base with maximal ideal $\mathfrak{m}$. We say that $r$ is without 1-component if $r_{0}=0$. For simplicity, we write $\mu_{q}$ for the product of $\operatorname{Gtl}_{r} \mathcal{A}$.
Example 9.10. The category $\mathrm{Gtl}_{r} \mathcal{A}$ may simply be the standard deformation $\mathrm{Gtl}_{r} \mathcal{A}=\mathrm{Gtl}_{q} \mathcal{A}$ over $B=\mathbb{C} \llbracket M \rrbracket$, detailed in section 6.6 It is determined by the specific parameter

$$
r=\sum_{q \in M} q \ell_{q} \in(M) Z(\operatorname{Gtl} \mathcal{A})
$$

This parameter is without 1-component.

Recall the notion of uncurvability: Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathcal{C}_{q}$ a deformation of $\mathcal{C}$. Then an object $X \in \mathcal{C}_{q}$ is uncurvable if there is an odd $S \in \mathfrak{m} \operatorname{End}_{\mathcal{C}}(X)$ such that $\mu_{q, X}^{0}+\mu_{q}^{1}(S)+\mu_{q}^{2}(S, S)+\ldots=0$. For twisted complexes $X=\left(\sum X_{i}\left[s_{i}\right], \delta\right)$, this means to find an infinitesimal deformation $\delta_{q}$ of $\delta$ such that the curvature of the twisted complex becomes zero:

$$
\mu_{q, X}^{0}+\mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{1}\left(\delta_{q}\right)+\mu_{\operatorname{Add~Gt1}_{r} \mathcal{A}}^{2}\left(\delta_{q}, \delta_{q}\right)+\ldots=0
$$

The infinitesimal part of $\delta_{q}$ is allowed to lie anywhere in the matrix, not restricted to the upper-triangular part.

Let us now describe our "complementary angle trick". Regard a band object $X=\left(a_{1}\left[s_{1}\right] \oplus \ldots, \delta\right)$. To simplify the discussion, we assume its local system is one-dimensional with transition value simply equal to 1 . In the twisted complex $X$, every arc then only appears once (apart from arcs appearing multiple times in the approximation), and every angle $\alpha_{i}$ appearing in $\delta$ appears with a coefficient of +1 .

The curvature $\mu_{q, X}^{0}$ of $X$ consists by definition of the sum of the curvatures $\mu_{a_{i}}^{0}$ of the constituent arcs of $X$. Since we are regarding a standard deformation $\operatorname{Gtl}_{r} \mathcal{A}$ of $\operatorname{Gtl} \mathcal{A}$, we know this curvature explicitly: Every arc carries an infinitesimal amount of turns around both of its endpoints as curvature. Since $X$ is a band object, both endpoints of every arc $a_{i}$ are connected to the predecessor or successor arc $a_{i-1}$ and $a_{i}$ by angles $\alpha_{i}$.

The trick to uncurving is to add the complements $\alpha_{i}^{\prime}$ of these angles to the $\delta$ matrix, depicted in Figure 9.5 c. Generically denote by $\ell$ a full turn around any puncture. We denote by $r \ell^{-1} \alpha_{i}^{\prime}$ the element of $B \otimes \operatorname{Gtl} \mathcal{A}$ obtained from $r$ by extracting the part that winds around the same puncture as $\alpha_{i}^{\prime}$, shortening all angles by one full turn, and multiplying by $\alpha_{i}^{\prime}$. Naturally, the element $r \ell^{-1} \alpha_{i}^{\prime}$ can be interpreted as

Definition 9.11. Let $\mathcal{A}$ be a full arc system with [NMDC], $B$ a deformation base and $r \in \mathfrak{m} Z(\mathrm{Gtl} \mathcal{A})$ a parameter without 1-component. Regard a band object $X=\left(a_{1}\left[s_{1}\right] \oplus \ldots, \delta\right)$ with trivial 1-dimensional local system. Assume its $\delta$-angles $\alpha_{1}, \ldots, \alpha_{k}$ are all shorter than a full turn and not identities. Then the complementary angle trick associates to $X$ the twisted complex $\left(a_{1}\left[s_{1}\right] \oplus \ldots, \delta_{q}\right) \in \operatorname{Tw}^{\prime} \operatorname{Gtl}_{q} \mathcal{A}$ with $\delta_{q}$ given by

$$
\delta_{q}:=\delta+\delta^{\prime}=\sum_{i=1}^{k} \alpha_{i}+r \ell^{-1} \alpha_{i}^{\prime}
$$

Example 9.12. Regard the deformation $\operatorname{Gtl}_{r} \mathcal{A}$ with parameter $r=q \ell_{p}$ over $B=\mathbb{C} \llbracket q \rrbracket$, with $\ell_{p}$ denoting the central element consisting of single turns around the puncture $p$. Then the deformation entry added to the $\delta$ matrix is simply $r \ell^{-1} \alpha_{i}^{\prime}=q \alpha_{i}^{\prime}$ for every $\alpha_{i}$ which winds around $p$.
Remark 9.13. The deformation of $\delta$ to $\delta_{q}$ happens precisely on the opposite side of the diagonal of the matrix: If $\alpha_{i}$ is in the $\delta$ matrix as angle from $a_{i}$ to $a_{i+1}$, then the angle $r \ell^{-1} \alpha_{i}^{\prime}$ is inserted within the $\delta$ matrix as morphism from $a_{i+1}$ to $\alpha_{i}$. If $\alpha_{i}$ was an angle from $a_{i+1}$ to $a_{i}$, then $r \ell^{-1} \alpha_{i}^{\prime}$ is inserted as morphism from $a_{i}$ to $a_{i+1}$. This way we obtain a matrix $\delta_{q}$ with an infinitesimal lower-triangular part.
Remark 9.14. The complementary angle trick also works when the local system is higher-dimensional and non-trivial. Recall that the $\delta$ matrix encodes the transition matrix $M$ of a higher-dimensional local system by carrying its entries $m_{p q}$ in front of the $\alpha_{i}$ morphism from the $q$-th copy of $a_{i}$ to the $p$-th copy of $a_{i+1}$. In case the angle runs in the opposite direction, the $\delta$ matrix encodes $M$ by carrying the entries $m^{p q}:=\left(M^{-1}\right)_{p q}$ of the inverse matrix.

In order to uncurve this band object, we include the inverse matrix $M^{-1}$ in the uncurving deformation of $\delta$, or $M$ in case the angle runs in opposite direction. For instance if $d=2$ and $\alpha_{i}$ runs from $a_{i}$ to $a_{i+1}$, the part of the $\delta_{q}$ matrix between $a_{i}$ and $a_{i+1}$ shall read

$$
\text { part of } \delta_{q}=\left(\begin{array}{cc|cc} 
& & m_{11} \alpha_{i} & m_{12} \alpha_{i} \\
m_{21} \alpha_{i} & m_{22} \alpha_{i} \\
\hline m^{11} r \ell^{-1} \alpha_{i}^{\prime} & m^{12} r \ell^{-1} \alpha_{i}^{\prime} & & \\
m^{21} r \ell^{-1} \alpha_{i}^{\prime} & m^{22} r \ell^{-1} \alpha_{i}^{\prime} & &
\end{array}\right)
$$

We are now ready to check that the complementary angle trick succeeds in uncurving band objects. The trick however comes with strict conditions. We use the following terminology:
Definition 9.15. An indexed arc on $X$ is one of the $\operatorname{arcs} a_{i}$ of $X$, remembering the index $i$. An segment of indexed arcs on $X$ is a sequence of consecutive arcs $a_{i}, a_{i+1}, \ldots, a_{i+j}$, remembering the indices. An indexed segment is contractible in $S$ if it returns to the same puncture as it started from and the loop defined this way is contractible in the closed surface $S$.

(a) Teardrop

(b) No teardrop

(c) Complementary angle trick

(d) Contractible segment

Figure 9.5: Illustration of the complementary angle tick and its technicalities

Remark 9.16. A band object which bounds a teardrop in $S$ is depicted in Figure 9.5a Meanwhile, the shape depicted in Figure 9.5b does not constitute a teardrop. A contractible segment of indexed arcs is depicted in Figure 9.5d The existence of a contractible segment of arcs on $X$ does not imply that the underlying curve of $X$ has a teardrop.

As we shall see in Lemma 9.17 the complementary angle trick succeeds in uncurving $X \in \operatorname{Tw~} \mathrm{Gtl}_{r} \mathcal{A}$ when we assume the following three conditions on $X$ :

- The underlying curve of $X$, regarded as a curve in the closed surface $S$, is not contractible and does not bound a teardrop.
- All angles $\alpha_{i}$ in the $\delta$ matrix are non-identities and strictly smaller than a full turn.
- No segment of indexed arcs of $X$ is contractible.

In section B, we explain how to abandon the condition that segments of indexed arcs are not contractible. Without the condition, one has to add further angles to $\delta_{q}$ for every location where $X$ comes close to itself, other than only the complementary angles $\alpha_{i}^{\prime}$. It is interesting to note that geometrically these additional angles can be interpreted as "complementary to segments of indexed arcs" of $X$.

Lemma 9.17. Let $\mathcal{A}$ be a full arc system with [NMDC]. Regard a standard deformation $\mathrm{Gtl}_{r} \mathcal{A}$ of $\mathrm{Gtl} \mathcal{A}$ by some $r \in \mathfrak{m} Z(\operatorname{Gtl} \mathcal{A})$ without 1-component. Let $X \in \operatorname{Tw} \operatorname{Gtl} \mathcal{A}$ be a $\mathbb{Z} / 2 \mathbb{Z}$-graded band object whose underlying curve in $S$ is not contractible and does not bound a teardrop. Assume that all angles in $X$ are non-identities and shorter than full turns. Then $X \in \operatorname{Tw} \operatorname{Gtl}_{r} \mathcal{A}$ is uncurvable.

Proof. Without loss of generality, we assume that $X$ has one-dimensional local system and all transition values in $\delta$ are 1. The case with contractible segments is dealt with in section B We shall therefore assume that $X$ has no contractible segments of indexed arcs.

Now let us prove that the complementary angle trick successfully uncurves $X$. This entails checking that the curvature of $\left(\oplus a_{i}\left[s_{i}\right], \delta_{q}\right) \in \mathrm{Tw}^{\prime} \mathrm{Gtl}_{r} \mathcal{A}$ vanishes. Explicitly, the curvature is

$$
\sum_{k \geq 0} \mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{k}\left(\delta_{q}, \ldots, \delta_{k}\right)
$$

The summand at $k=0$ is the curvature $\mu_{X, q}^{0}$, explicitly the sum of the curvatures of the individual arcs. Note that $\mu_{q}^{1}$ vanishes due to [NMDC]. In the first step of the proof, we show that $\mu_{X, q}^{0}$ precisely cancels $\mu_{\operatorname{AddGtl}_{r} \mathcal{A}}^{2}\left(\delta_{q}, \delta_{q}\right)$. In the second step of the proof, we show that all the higher terms $\mu_{q}^{\geq 3}\left(\delta_{q}, \ldots\right)$ vanish.

Let us analyze $\mu_{q}^{2}\left(\delta_{q}, \delta_{q}\right)$. Since $\delta_{q}=\delta+\delta^{\prime}$ is the sum of the original $\delta$ and the modification $\delta^{\prime}$ due to the complementary angle trick, we need to check the original part $\mu_{q}^{2}(\delta, \delta)$ and the new components $\mu_{q}^{2}\left(\delta, \delta^{\prime}\right), \mu_{q}\left(\delta^{\prime}, \delta\right)$ and $\mu_{q}^{2}\left(\delta^{\prime}, \delta^{\prime}\right)$. Recall also that the product $\mu_{q}^{2}$ is not deformed: It is merely the $B$-linear extension of the original $\mu^{2}$ by assumption of [NMDC]. As observed in Remark 9.8, we have $\mu_{q}^{2}(\delta, \delta)=0$.


Meanwhile, we have

$$
\mu_{\mathrm{Add} \mathrm{Gtl}_{r} \mathcal{A}}^{2}\left(\delta, \delta^{\prime}\right)=\sum_{i=1}^{k}-\alpha_{i} r \ell^{-1} \alpha_{i}^{\prime}=-r \in \mathfrak{m} Z(\operatorname{Gtl} \mathcal{A})
$$

where $r$ on the right-hand side is interpreted as linear combination of powers of full turns starting at those arc ends of the arc approximation where the angles $\alpha_{i}$ enter. The minus sign comes from the sign


Figure 9.6: The horizontal band in the 1-punctured torus. This band only has a suitable twisted complex representation when the 4 -gon is divided into triangles.
convention for $\operatorname{Add}_{\operatorname{Gtl}}^{r} \mathcal{A}$. Similarly,

$$
\mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{2}\left(\delta, \delta^{\prime}\right)=\sum_{i=1}^{k}-r \ell^{-1} \alpha_{i}^{\prime} \alpha_{i}=-r \in \mathfrak{m} Z(\operatorname{Gtl} \mathcal{A})
$$

where $r$ on the right-hand side is interpreted as linear combination of powers of full turns starting at those arc ends of the arc approximation where the angles $\alpha_{i}$ leave.

We have used that all angles in the twisted differential of $X$ are non-identities and shorter than full turns: Identities would give extra terms in the products $\mu^{2}\left(\delta, \delta^{\prime}\right)$ and $\mu^{2}\left(\delta^{\prime}, \delta\right)$. Moreover, taking complementary angles is only possible if all angles in $X$ are at most full turns. A precise full turn would in turn give an identity in $\delta^{\prime}$, hence an undesired contribution to e.g. $\mu^{2}\left(\delta^{\prime}, \delta\right)$. In short, we assumed just the right condition so that nothing but the right terms appears in $\mu^{2}\left(\delta_{q}, \delta_{q}\right)$. We conclude

$$
\mu_{X, q}^{0}+\mu_{\mathrm{Add} \mathrm{Gtl}_{r} \mathcal{A}}^{2}\left(\delta_{q}, \delta_{q}\right)=0
$$

For the second step of the proof, we show that $\mu_{q}^{k \geq 3}\left(\delta_{q}, \ldots, \delta_{q}\right)=0$. Assume $D$ is an orbigon contributing to this product. Then the boundary of $D$ is a contractible indexed segment of $X$, in contradiction to the assumption that there are no contractible indexed segments. This shows that $\mu_{q}^{k \geq 3}\left(\delta_{q}, \ldots, \delta_{q}\right)=0$. Finally, we conclude that the curvature of the deformed twisted complex $X_{q}=\left(\oplus a_{i}\left[s_{i}\right], \delta_{q}\right)$ vanishes. This finishes the proof.

### 9.4 The uncurvable objects

In this section, we show that most band objects in $\operatorname{Tw~} \operatorname{Gtl}_{q} \mathcal{A}$ are uncurvable. The starting point is the classification of band objects recalled in section 9.2 and the complementary angle trick defined in section 9.3 . The goal is to show that a band object is uncurvable if its underlying curve in $S$ is topologically nontrivial and does not bound a teardrop. We have already shown in Lemma 9.17 that the complementary angle trick succeeds in uncurving these objects under the technical condition that all angles in the $\delta$-matrix of $X$ are non-identities and shorter than full turns. In the present section, we show how to abandon this technical condition.

Our starting point is again a full arc system $\mathcal{A}$ with $[\mathrm{NMDC}]$ and a deformation $\mathrm{Gtl}_{r} \mathcal{A}$ with $r \in$ $\mathfrak{m} Z(\operatorname{Gtl} \mathcal{A})$ a deformation parameter without 1-component. It is our wish to apply the complementary angle trick to every band object whose underlying curve in $S$ is topologically nontrivial and does not bound a teardrop. Combining Corollary 9.5 and Lemma 9.17 we would be done if every such band object has a twisted complex representation where all angles are non-identities and shorter than full turns. This is however not the case:

Remark 9.18. There are arc systems in which some bands fail to have representatives which satisfy the requirements of Lemma 9.17 For example, regard the 1-punctured torus with two arcs depicted in Figure 9.6 Its horizontal, or vertical, band cannot be represented as a twisted complex with all angles non-identities and shorter than a full turn. As soon as we divide the 4 -gon into two triangles, the band suddenly has a desired representation.

There are however arc systems which guarantee the existence of representatives suitable for Lemma 9.17 for every band object whose underlying curve in $S$ is topologically nontrivial and does not bound a teardrop. Following Remark 9.18, the idea is to simply require that the arc system contains only of triangles:

Lemma 9.19. Let $\mathcal{A}$ be a full arc system with [NMDC] and assume all polygons in $\mathcal{A}$ are triangles. Let $X \in \mathrm{HTw} \operatorname{Gtl} \mathcal{A}$ be a string or band object whose underlying curve in $S$ is nontrivial and does not bound a teardrop. Then $X$ has a twisted complex representation in which all angles $\alpha_{i}$ are non-identities and shorter than a full turn.


Figure 9.7: Removing cyclicity in the proof of Lemma 9.19

Proof. We proceed as in the proof of Lemma 9.9, and go a little further. Let us repeat the steps: Choose an initial approximation of $X$ by arcs $a_{i}$ such all angles $\alpha_{i}$ are non-identities and strictly shorter than a full turn. The rest of the proof is concerned with tweaking the approximation so as to make $\delta$ upper triangular. In case of a string object, we are done. Let us inspect the given arc collection $\alpha_{1}, \ldots, \alpha_{k}$ with its connecting angles $\alpha_{1}, \ldots, \alpha_{k}$. By assumption, all angles $\alpha_{i}$ go from arc $a_{i}$ to $a_{i+1}$, or all the other way around. Let us assume the former is the case: that $\alpha_{i}$ runs from $a_{i}$ to $a_{i+1}$. Moreover, any pair of consecutive $\operatorname{arcs} \alpha_{i}, \alpha_{i+1}$ runs at opposite ends of the $\operatorname{arc} a_{i+1}$, and all are non-identities.

Our strategy to break cyclicity is to pull some consecutive arcs with their angles following the interior of a polygon to the opposite side of the polygon. Let us make this concrete and distinguish the following hierarchy of cases: (a) there is an angle $\alpha_{i}$ with at least three indecomposable components, (b) $k=1$, (c) there are two consecutive decomposable angles, (d) $k=2$, (e) one angle $\alpha_{i}$ is indecomposable. By this hierarchy of cases, we mean that case (b) shall include that (a) does not hold; (c) shall include that (a) and (b) do not hold, etc. Samples for all cases are depicted in Figure 9.7

Regard case (a). Then we can flip a part of $\alpha_{i}$ to the other side of a triangle.
Regard case (b). If $\alpha_{1}$ has just one indecomposable component, we have an immediate contradiction. If $\alpha_{1}$ has two indecomposable components, then the arc in the middle of $\alpha_{1}$ appears twice in the triangle, with equal orientation. This is also a contradiction, since the triangle is then not embedded anymore: In Figure 9.7, the two dots inside the triangle would need to be equal, rendering the triangle non-embedded.

Regard case (c). Regard two consecutive decomposable angles $\alpha_{i}$ and $\alpha_{i+1}$. Then the last indecomposable part of $\alpha_{i}$ and the first indecomposable part of $\alpha_{i+1}$ are interior angles of a triangle. We can now flip these parts of the angles to the opposite side of the triangle. As depicted in the figure, this suffices to break cyclicity.

Regard case (d). It is impossible that both $\alpha_{1}$ and $\alpha_{2}$ are indecomposable. Indeed, this would mean that the curve partially winds around the interior of a triangle. Since we are excluding case (c), we can assume that $\alpha_{1}$ is indecomposable and $\alpha_{2}$ consists of two indecomposable parts. Then $\alpha_{2}$ crosses both the head and the tail side of some arc, which is impossible.

Regard case (e). Let $\alpha_{i}$ be the indecomposable angle and regard $\alpha_{i-1}$ as well as $\alpha_{i+1}$. Since the curve does not bound a teardrop, the angle $\alpha_{i-1}$ enters the polygon from outside and $\alpha_{i+1}$ leaves the triangle. In other words, both are longer than the preceding and succeeding interior angles of the triangle. The tweak we apply to the angle sequence is to cut away the parts of $\alpha_{i-1}$ and $\alpha_{i+1}$ lying inside the triangle and deleting $\alpha_{i}$. A shorter sequence remains to be dealt with.

In every step, the angles that are already present become only shorter. Moreover, all angles that are inserted new in a step are interior angles of a polygon by choice. Since all polygons in $\mathcal{A}$ are triangles and all $\operatorname{arcs}$ in $\mathcal{A}$ are non-contractible in $S$, all interior angles are shorter than a full turn. In total, we end up with an approximation where all angles are non-identities and shorter than full turns, as well as $\delta$ being upper triangular.

Theorem 9.20. Let $\mathcal{A}$ be a full arc system with [NMDC]. Let $r \in \mathfrak{m} Z(\mathrm{Gtl} \mathcal{A})$ be a deformation parameter without 1-component. Then all $\mathbb{Z} / 2 \mathbb{Z}$-graded band objects whose underlying curves in $S$ are topologically nontrivial and do not bound a teardrop are uncurvable.

Proof. The proof consists of two steps. The first is to observe that we have already proven the case when all polygons in $\mathcal{A}$ are triangles. The second is to extend to the general case.

For the first step, let us assume that all polygons in $\mathcal{A}$ are triangles. Then by Lemma 9.19, the band object $X$ has a twisted complex representation where all angles are non-identities and shorter than full turns. By Lemma 9.17, the complementary angle trick now successfully removes the curvature of $X$.

For the second step, cut all polygons that are not triangles yet into pieces by adding arcs. Let us denote the resulting marked surface with arc system by $\mathcal{A}^{\prime}$. We have an embedding $\mathrm{Gtl} \mathcal{A} \subseteq \operatorname{Gtl} \mathcal{A}^{\prime}$, and correspondingly

$$
i: \mathrm{Tw} \operatorname{Gtl} \mathcal{A} \subseteq \mathrm{Tw} \operatorname{Gtl} \mathcal{A}^{\prime}
$$

It is well-known that this map is actually a quasi-equivalence, see 18 . The reason is that all additional $\operatorname{arcs}$ of $\mathcal{A}^{\prime}$ can be built up to quasi-isomorphism as twisted complexes of $\operatorname{arcs}$ in $\mathcal{A}$. Correspondingly, we also have a quasi-isomorphism

$$
\pi: \operatorname{Tw} \operatorname{Gtl} \mathcal{A}^{\prime} \rightarrow \operatorname{Tw} \operatorname{Gtl} \mathcal{A}
$$

that sends an object $i(X)$ to some object $\pi(i(X))$ quasi-isomorphic to $X$. Now choose $X$ to be a band object as in the hypothesis, i.e. topologically nontrivial in $S$ and not bounding a teardrop. Then $i(X) \in$ $\operatorname{Tw} \operatorname{Gtl} \mathcal{A}^{\prime}$ is uncurvable by the first step of the proof. According to Lemma 9.2, also $\pi(i(X))$ is uncurvable. Since $X$ itself is quasi-isomorphic to $\pi(i(X))$, it is uncurvable as well by Corollary 9.5 This finishes the proof.

Remark 9.21. Let us get back to the 1-punctured torus with two arcs of Remark 9.18. We have seen that the horizontal band has no twisted complex representation that is suitable for the uncurving trick. According to Theorem 9.20 it can be uncurved nevertheless and by Corollary 9.5 this must in fact be possible for every chosen twisted complex representation.

Denote the angles in the 1-punctured torus by $\alpha, \beta, \gamma, \delta$ as in Figure 9.6 Pick the twisted complex representation

$$
X=\left(b \oplus a \oplus b[1],\left(\begin{array}{ccc}
0 & \gamma & \text { id } \\
0 & 0 & \delta \\
0 & 0 & 0
\end{array}\right)\right)
$$

By experimenting, we have found the uncurved twisted complex

$$
X_{q}=\left(b \oplus a \oplus b[1],\left(\begin{array}{ccc}
0 & \gamma & \mathrm{id} \\
-q \beta \alpha \delta & q \beta \alpha & \delta \\
-q \alpha \delta \gamma \beta & q \gamma \beta \alpha & 0
\end{array}\right)\right)
$$

In other words, apart from the expected complementary angles $q \beta \alpha \delta, q \gamma \beta \alpha$ and $q \alpha \delta \gamma \beta$, we also have to insert $q \beta \alpha$. This difficulty is the reason we restricted the complementary angle trick to the case where all polygons are triangles.

Remark 9.22. There are bands objects which are uncurvable but do not fall under the requirements of Theorem 9.20 Deriving finer criteria is however increasingly difficult. For instance, uncurvability of band objects representing curves with a teardrop depends on whether the deformation parameter includes $\ell^{s}$ for low $s$.

For string objects, the situation is complicated as well. Most string objects cannot be uncurved, but there are exceptions. The underlying curve of a string object can typically be interpreted as an arc candidate for some full arc system if it has no self-intersections. In this case, it is not uncurvable for general deformation parameter $r$. A string object whose underlying curve is a loop may however be uncurvable if $r \in \mathfrak{m}^{2} Z(\operatorname{Gtl} \mathcal{A})$.

## 10 The category of zigzag paths

In this section, we define the category $\mathbb{L}$ and construct for it an explicit homological splitting. This category $\mathbb{L}$ is a new, discrete analog of the smooth zigzag category studied by Cho, Hong and Lau 26 . We follow their idea of including all zigzag curves with chosen spin structure into a category, except that we realize the curves as twisted complexes over the gentle algebra instead of objects in the Fukaya category.

After defining this category $\mathbb{L}$, the second step in this section is to analyze the morphisms between the objects. We introduce terminology to handle locations where two zigzag paths come close to each other: situations of type A, B, C and D. We show that every morphism between two zigzag paths can be written as a linear combination of angles, and that each angle comes from a unique $\mathrm{A}, \mathrm{B}, \mathrm{C}$ or D type situation.

Finally, we use this classification of morphisms to provide an explicit homological splitting for the category $\mathbb{L}$. This entails identifying a cohomology space $H$ for every hom space, and finding complementary


Figure 10.1: Illustration of zigzag paths and small angles
spaces $R$ for every pair of zigzag paths. The splitting of individual morphisms into $H, I$ and $R$ parts is collected in Table 10.5 which will be referred to throughout the paper.

### 10.1 Category of zigzag paths

In this section, we define the category $\mathbb{L}$ of zigzag paths. The idea is to turn zigzag paths in $Q$ into twisted complexes in $\mathrm{Tw} \operatorname{Gtl} Q$. The construction of the twisted complexes requires the additional input data of a spin structure for every zigzag path. We fix terminology and notation for these spin structures. The result of the construction is the subcategory $\mathbb{L} \subseteq \operatorname{Tw} \mathrm{Gtl} Q$.

Zigzag paths in $Q \sim$ Category of zigzag paths $\mathbb{L} \subseteq \operatorname{Tw} \operatorname{Gtl} Q$
Generally, arcs as objects in $\mathrm{Gtl} Q$ can be stitched together to form twisted complexes which model curves in the Fukaya category. This is a well-known method, explicit in 16, Section 9.2] and implicit in 35. We have detailed it in section 9.2 and shall now apply it specifically to zigzag paths. Recall that a zigzag path has necessarily even length, because it alternates between turning left and right.

Given a zigzag path $L$, we sometimes need to refer to specific angles surrounding $L$. We set up this terminology as follows: Assume the consecutive arcs of the zigzag path are $a_{1}, a_{2}, \ldots$. Let us assume $Q$ is geometrically consistent, so that every puncture in $Q$ has valency at least four. Thus, the mere data of the arc sequence $a_{1}, a_{2}, \ldots$ already determines for every $i \in \mathbb{N}$ whether $L$ turns left or right at (the head or tail of) $a_{i}$. For those $i \in \mathbb{N}$ where $L$ turns left at $a_{i}$, denote by $\alpha_{i}$ the angle that winds around the common puncture $h\left(a_{i}\right)=t\left(a_{i+1}\right)$, is shorter than a full turn, starts at the head of $a_{i}$ and ends at the tail of $a_{i+1}$. For those $i \in \mathbb{N}$ where $L$ turns right at $a_{i}$, denote by $\alpha_{i}$ the angle that winds around the common puncture $h\left(a_{i}\right)=t\left(a_{i+1}\right)$, is shorter than a full turn, starts at the tail of $a_{i+1}$ and ending at the head of $a_{i}$. The notation is depicted in Figure 10.1a. We also call $\alpha_{i}$ the small angle in $L$ between $a_{i}$ and $a_{i+1}$.
Remark 10.1. Every (indexed) arc $a_{i}$ either has precisely two small angles leaving it and no small angle ending at it, or two small angles ending at it and no small angles leaving it. In fact, the arcs of those types alternate along $L$. The reader can easily convince himself of this fact by regarding Figure 10.1a.

Turning a zigzag path into a twisted complex requires the datum of a spin structure. For our purposes, this simply entails choosing a sign $(-1)^{\# \alpha_{i}}$ for every small angle $\alpha_{i}$ in the zigzag path. Writing the sign additively, we fix the terminology as follows:

Definition 10.2. A spin structure on a zigzag path $L$ is a choice of signs $\# \alpha_{i} \in \mathbb{Z} / 2 \mathbb{Z}$ for each of its small angles $\alpha_{i}$.

The notation $\# \alpha_{i}$ makes sense: If the small angle between $a_{i}$ and $a_{i+1}$ is equal to the small angle between $a_{j}$ and $a_{j+1}$, then $i$ and $j$ differ precisely by a period of $L$. In short, every small angle of $L$ appears only once as small angle of $L$ and therefore the notation $\# \alpha_{i}$ makes sense for the scope of a single zigzag path.

Giving a spin structure simultaneously for all zigzag paths of $Q$ is equivalent to giving a sign $\# \alpha \in$ $\mathbb{Z} / 2 \mathbb{Z}$ for all indecomposable angles $\alpha$ in $Q$. Indeed, if two zigzag paths $L_{1}$ and $L_{2}$ share a small angle, then $L_{1}$ and $L_{2}$ are actually equal. This fact is depicted in Figure 10.1b. In that figure, the dashed arcs indicate that the drawn angle shall be an interior angle of a polygon. In summary, giving a collection of spin structures for all zigzag paths is equivalent to choosing a sign $\# \alpha \in \mathbb{Z} / 2 \mathbb{Z}$ for every indecomposable angle $\alpha$ in $Q$.
Definition 10.3. Let $Q$ be a geometrically consistent dimer and $L$ a zigzag path with spin structure. Write the arcs of $L$ as $a_{1}, \ldots, a_{2 k}$, chosen such that $L$ turns left from $a_{1}$ to $a_{2}$. Let $\alpha_{1}, \alpha_{2}, \ldots$ be the small angles in $L$ between $a_{1}$ and $a_{2}$, etc. Then the twisted complex associated with $L$ is given by

$$
L=\left(a_{1} \oplus a_{3} \oplus \ldots \oplus a_{k} \oplus a_{2} \oplus \ldots \oplus a_{2 k}, \delta\right)
$$

with twisting differential

$$
\delta=\left[\right]
$$

In Definition 10.3, we have introduced abuse of notation twice: Using the letter " $L$ " both for a zigzag path and its twisted complex, and simply calling both uses a "zigzag path". The intention behind this abuse is to switch seamlessly between between both uses. A typical sentence in this paper will be: "Regard the endomorphisms of some zigzag path $L$." In that sentence, it is clear that $L$ shall be a zigzag path and we regard the endomorphisms of its associated twisted complex.

The twisted complex defined in Definition 10.3 is indeed a well-defined object of Tw Gtl $Q$, i.e. $\delta$ satisfies the Maurer-Cartan equation:

Lemma 10.4. Let $L$ be a zigzag path with spin structure. Then its twisting differential $\delta$ satisfies the Maurer-Cartan equation, so that $L$ indeed lies in $\mathrm{Tw} \operatorname{Gtl} Q$.

Proof. The Maurer-Cartan equation reads

$$
\mu_{\mathrm{Add} \mathrm{Gtl} Q}^{1}(\delta)+\mu_{\operatorname{Add~Gtl} Q}^{2}(\delta, \delta)+\mu_{\operatorname{Add~Gtl} Q}^{3}(\delta, \delta, \delta)+\ldots=0
$$

Proving the Maurer-Cartan equation therefore boils down to showing that for any sequence of compatible angles $\alpha_{1}, \ldots, \alpha_{k}$ appearing in the $\delta$ matrix of $L$ we have

$$
\mu_{\mathrm{Gt1} Q}^{k}\left(\alpha_{k}, \ldots, \alpha_{1}\right)=0
$$

Let us check all such terms. Since the differential $\mu_{\mathrm{Gtl} Q}^{1}$ vanishes, we can assume $k \geq 2$. Assume $a_{1}$ starts at arc $a_{1}$ and ends at $a_{2}$. In order to have any nonzero contribution $\mu_{\mathrm{Gt1} Q}^{k}\left(\ldots, \alpha_{1}\right)$, there would need to be a small angle starting at $a_{2}$. According to Remark 10.1. every arc however admits either only incoming or only outgoing small angles. Since $\alpha_{1}$ is already an incoming angle for $a_{2}$, we conclude that $a_{2}$ has no outgoing angles. Therefore the product $\mu_{\mathrm{Gtl} Q}^{k}\left(\ldots, \alpha_{1}\right)$ vanishes. The reader can convince themself of this visually by drawing a zigzag path together with its small angles and trying to draw an immersed disk bounded solely by small angles. We conclude that $\delta$ satisfies the Maurer-Cartan equation.

Definition 10.5. Choose a spin structure for all zigzag paths $L_{1}, \ldots, L_{N}$ in $Q$. Then the category of zigzag paths $\mathbb{L} \subseteq \operatorname{Tw} \operatorname{Gtl} Q$ is the (full) subcategory $\mathbb{L}=\left\{L_{1}, \ldots, L_{N}\right\}$ of $\mathrm{Tw} \mathrm{Gtl} Q$ consisting of all zigzag paths with their single chosen spin structure.

Every zigzag path only appears once in $\mathbb{L}$. We do not allow the same zigzag path multiple times in $\mathbb{L}$ with different spin structures, since this is not the goal of this paper and it would make calculations more complicated. Since $\mathbb{L}$ depends on $Q$ and the choice of spin structures, denoting this category by the letter $\mathbb{L}$ denotes a slight abuse of notation.

### 10.2 ABCD situations

In this section, we provide a basis for the hom spaces between zigzag paths. We depart from two zigzag paths $L_{1}$ and $L_{2}$ and analyze their hom space in the category $\mathbb{L}$. The first step is to introduce a notion of elementary morphisms from $L_{1}$ to $L_{2}$. This way, every morphism from $L_{1}$ to $L_{2}$ can be written as a linear combination of elementary morphisms. We then classify elementary morphisms according to the geometry of $L_{1}$ and $L_{2}$ in the surroundings of the morphisms. This gives rise to a classification into four types A, B, C, D. Elementary morphisms associated with these four types provide a basis for $\operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)$. Ultimately, the A, B, C, D types will accompany us during our entire journey to the minimal model.

Regard two zigzag paths $L_{1}$ and $L_{2}$. It is our aim to provide a basis for $\operatorname{Hom}\left(L_{1}, L_{2}\right)$. Recall from section 6.9 that an indexed arc of a zigzag path consists of an arc lying on the zigzag path, together with the datum of whether the zigzag path turns left or right at the head (equivalently tail) of the arc.


Figure 10.2: All elementary morphisms $\varepsilon: L_{1} \rightarrow L_{2}$ are contained in one of these situations.

Recall from Definition 10.3 that every zigzag path comes with an associated twisted complex over Gtl $Q$. Therefore any morphism $\varepsilon \in \operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)$ can be uniquely written as a linear combination of angles from indexed arcs of $L_{1}$ to indexed arcs of $L_{2}$. In other words, every angle from an indexed arc of $L_{1}$ to an indexed arc of $L_{2}$ gives rise to a morphism $\varepsilon \in \operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)$, and all morphisms can be obtained as sums of such "elementary morphisms". Let us make this precise:

Definition 10.6. An elementary morphism $\varepsilon: L_{1} \rightarrow L_{2}$ is an angle from an indexed arc of $L_{1}$ to an indexed arc in $L_{2}$, interpreted as morphism between twisted complexes.

We now move on to defining A, B, C and D situations. As a preparation, regard two consecutive arcs $a, b$ in an elementary polygon of $Q$. This pair defines a zigzag path $L$, namely the one starting with $a$ and turning then to $b$. This way, $a$ and $b$ become indexed arcs of $L$. Indeed, the zigzag path never traverses $a$ followed by $b$ again, except after cycling once fully through $L$.

Definition 10.7. Let $Q$ be a geometrically consistent dimer and $L_{1}$ and $L_{2}$ be zigzag paths.
An A situation from $L_{1}$ to $L_{2}$ consists of two consecutive indexed $\operatorname{arcs}$ of $L_{1}$ and two consecutive indexed arcs of $L_{2}$ such that both midpoints (head of the first arc, equally tail of the second arc) are equal and all four arcs provide distinct incidences at this point. As in Figure 10.2a the two arcs of $L_{1}$ are cyclically denoted 1,2 . The two arcs of $L_{2}$ are cyclically denoted 3,4 . The angles involved are denoted $\alpha, \beta, \gamma$ and $\beta^{\prime}$. An elementary angle of this $\mathbf{A}$ situation is a product of such angles running from $1 / 2$ to $3 / 4$. Concretely, these are the angles in the A section of Table 10.5.

A B (resp. C) situation from $L_{1}$ to $L_{2}$ consists of an indexed arc 2 of $L_{1}$ and an indexed arc 5 of $L_{2}$ such that $2=5$ as arcs in $Q, L_{1}$ turns left (resp. right) at the head and tail of 2 , and $L_{2}$ turns right (resp. left) at the head and tail of 2. As in Figure 10.2b (resp. 10.2c), the neighboring indexed arcs are denoted $1,3,4,6$. An elementary angle of this B (resp. C) situation is a composition of angles in the figure, including $\mathrm{id}_{2 \rightarrow 5}$, that runs from $1 / 2 / 3$ to $4 / 5 / 6$. Concretely, these are the angles in the B (resp. C) section of Table 10.5

If $L_{1}=L_{2}$, then a $\mathbf{D}$ situation from $L_{1}=L_{2}$ to itself consists of a single indexed arc of $L_{1}=L_{2}$. If $L_{1}=L_{2}$ turns right at the head of the arc, this arc is denoted 1 and the next indexed arc is denoted 2 . If $L_{1}=L_{2}$ turns left, the arc is denoted 2 and the next indexed arc is denoted 1 . The angles are named as in Figure 10.2d An elementary angle of this $\mathbf{D}$ situation is a composition of angles in the figure that runs from the first arc to itself or to the second, from the second to the first, or from the second to itself by at least one full turn.

An elementary morphism may be annotated with its type of situation to enhance clarity: $\beta$ (A), $\alpha_{3}$ (B), id (C), id (D), etc.

Remark 10.8. Any arcs involved in a situation are allowed to be equal. The distinction and ordering of arcs only concerns the local behavior of the arc ends at the common punctures.

We show that the $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D situations classify elementary morphisms uniquely. One may think that, for example, an elementary morphism $\alpha_{3} \alpha_{4}$ of a B situation equals an elementary morphism $\beta^{\prime} \alpha_{2}$ of a C situation, or even $\alpha_{2} \alpha_{1}$ of the same B situation. We now explain that this is not the case.

Proposition 10.9. Let $Q$ be a geometrically consistent dimer and let $L_{1}, L_{2}$ be zigzag paths in $Q$. Then any elementary morphism $\varepsilon: L_{1} \rightarrow L_{2}$ is an elementary morphism of precisely one $\mathrm{A}, \mathrm{B}, \mathrm{C}$ or D situation. Moreover, it is assigned only once as an elementary morphism of that situation.


Figure 10.3: Examining case-by-case which situation $\varepsilon$ belongs to. Here $\varepsilon: L_{1} \rightarrow L_{2}$ is an elementary morphism strictly shorter than a full turn. Each of the cases corresponds to the option whether $L_{1}$ and $L_{2}$ turns left or right at $\varepsilon$ and how many indecomposable angles $\varepsilon$ consists of. In the figures, the latter number is indicated below the target arc. The number above the target arc indicates the number of indecomposable angles in the complement of $\varepsilon$. Due to zigzag consistency, the sum of the two numbers is at least 4.

Proof. We show that any elementary morphism $\varepsilon: L_{1} \rightarrow L_{2}$ appears in an $\mathrm{A}, \mathrm{B}, \mathrm{C}$ or D situation. In case $\varepsilon$ is strictly shorter than a full turn, a case-by-case study is performed in Figure 10.3. Now if $\varepsilon$ is a full turn or longer, write $\varepsilon=\varepsilon^{\prime} \ell^{n}$, where $\ell^{n}$ denotes a number of full turns. Then $\varepsilon^{\prime}$ itself is shorter than a full turn and hence is an elementary morphism of an $\mathrm{A}, \mathrm{B}, \mathrm{C}$ or D situation. Then by definition also $\varepsilon$ is an elementary morphism of the same situation. Note this also applies in case $\varepsilon^{\prime}$ is id (B), id (C) or id (D), since the turns $\left(\alpha_{1} \beta^{\prime} \alpha_{2}\right)^{i}(\mathrm{~B}),\left(\alpha_{4} \beta \alpha_{3}\right)^{i}(\mathrm{~B}),\left(\alpha_{1} \beta^{\prime} \alpha_{2}\right)^{i}(\mathrm{C}),\left(\alpha_{4} \beta \alpha_{3}\right)^{i}(\mathrm{C}), \alpha \alpha^{\prime}(\mathrm{D})$ and $\alpha^{\prime} \alpha$ (D) are also elementary morphisms of $\mathrm{B}, \mathrm{C}$ and D situations.

For uniqueness, realize that for any elementary morphism, as distinguished in Figure 10.3 we can read off the entire situation and which of its elementary morphisms it concerns by inspecting which arc ends coincide, where the zigzag path turns at $\varepsilon$ and in which directions the arrows point.

Classifying elementary morphisms into A, B, C, D situations is extremely handy. During this paper, we often need to indicate generic morphisms of these four types. For instance, we may say that a certain morphism is an " $\alpha_{3}$ morphism". By this we mean that it is an $\alpha_{3}$ morphism of one certain B situation.

### 10.3 Homological splitting

In this section, we introduce a homological splitting for the category $\mathbb{L}$ of zigzag paths. The starting point is the category $\mathbb{L}$ together with the description of its hom spaces according to section 10.2 The first step in this section is to fix once and for all the requirements on $Q$ and additional data we assume for the rest of the paper. The second step to define the splitting $H \oplus I \oplus R$ by giving an explicit basis in terms of the $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ situations in $Q$. The idea is to reflect the geometry of the associated zigzag curves as far as possible. It is worth memorizing some basis elements, for instance $\beta$ (A) morphisms always belong to $R$.

Let us fix once and for all the requirements on $Q$ and the description of additional data we require. Apart from assuming that $Q$ is a fixed geometrically consistent dimer with spin structures chosen for every of its zigzag paths, we also require the data of what we call idenity and co-identity locations. The idea is that the choice of homological splitting in this paper is not entirely canonical, but depends on these two kinds of choices. The datum of an identity location on a zigzag path $L$ entails the choice of an indexed arc $a_{0}$ on $L$. The midpoint of $a_{0}$ is to be thought of as location of the identity intersection point between the associated zigzag curve $\tilde{L}$ and its Hamiltonian deformation. The datum of a co-identity location on a zigzag path $L$ entails the choice of a small angle $\alpha_{0}$ on $L$. The midpoint of this angle is to be thought of as the location of the co-identity intersection point between the associated zigzag curve $\tilde{L}$
and its Hamiltonian deformation. For the visual meaning of identity and co-identity locations, we refer to Figure 7.6 and 13.3 . With these use cases in mind, we can state our convention as follows:
Convention 10.10. $Q$ is a geometrically consistent dimer equipped with a choice of spin structure, identity location $a_{0}$ and co-identity location $\alpha_{0}$ on every zigzag path. The category $\mathbb{L}$ contains every zigzag path once, with the chosen spin structure. The co-identity $\alpha_{0}$ shall be chosen to lie in a counterclockwise polygon.

We assume Convention 10.10 throughout section 11,12 and 13 without further notice. Only in the statement of the main theorems will we mention again that the convention is assumed. In section C the convention is assumed as well, while section D specifically deals with the sphere case where we abandon the convention. The requirement that $\alpha_{0}$ shall lie in a counterclockwise polygon is required to make certain calculations work. For more details, we refer to the discussion in section F. 3

We are now ready to define the homological splitting. For $H$, the idea is to reflect the intersection geometry of the associated zigzag curves as far as possible. The basis elements for $H$ defined in this section will be used throughout the paper and we refer to them as cohomology basis elements. For $R$, we have to make slightly arbitrary choices of basis elements.
Definition 10.11. Let $L_{1}$ and $L_{2}$ be two zigzag paths. Denote by $H \subseteq \operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)$ the subspace spanned by the cohomology basis morphisms

- for every B situation, $(-1)^{\# \alpha_{3}+1} \alpha_{3}+(-1)^{\# \alpha_{4}} \alpha_{4}$,
- for every C situation, $\mathrm{id}_{2 \rightarrow 5}$,
- if $L_{1}=L_{2}$, then $(-1)^{\# \alpha_{0}+1} \alpha_{0}$ and $\sum_{a} \mathrm{id}_{a}$.

Denote by $R \subseteq \operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)$ the subspace spanned by the following elements, collected from all situations from $L_{1}$ to $L_{2}$ :

- $\gamma \beta\left(\alpha \beta^{\prime} \gamma \beta\right)^{i}(\mathrm{~A}), \beta\left(\alpha \beta^{\prime} \gamma \beta\right)^{i}(\mathrm{~A})$,
- $\left(\alpha_{4} \beta \alpha_{3}\right)^{i}, i \geq 1$ (B), $\left(\alpha_{1} \beta^{\prime} \alpha_{2}\right)^{i}, i \geq 1$ (B), $\operatorname{id}_{2 \rightarrow 5}$ (B), $\alpha_{3}\left(\alpha_{4} \beta \alpha_{3}\right)^{i}(\mathrm{~B}), \alpha_{1}\left(\beta^{\prime} \alpha_{2} \alpha_{1}\right)^{i}$ (B),
- $\beta^{\prime}\left(\alpha_{2} \alpha_{1} \beta^{\prime}\right)^{i}(\mathrm{C}), \beta\left(\alpha_{3} \alpha_{4} \beta\right)^{i}(\mathrm{C}), \alpha_{1} \beta^{\prime}\left(\alpha_{2} \alpha_{1} \beta^{\prime}\right)^{i}(\mathrm{C}), \beta \alpha_{3}\left(\alpha_{4} \beta \alpha_{3}\right)^{i}(\mathrm{C})$,
- $\operatorname{id}_{a}, a \neq a_{0}(\mathrm{D}), \alpha^{\prime}\left(\alpha \alpha^{\prime}\right)^{i}(\mathrm{D}),\left(\alpha^{\prime} \alpha\right)^{i}, i \geq 1$ (D).

Denote by $I$ the image of the twisted differential $\mu_{\mathbb{L}}^{1}: \operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right) \rightarrow \operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)$. Then $R, I$ and $H$ form the (standard) splitting of $\operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)$.

The cohomology basis elements are to be interpreted as follows: The odd element $(-1)^{\# \alpha_{3}+1} \alpha_{3}+$ $(-1)^{\# \alpha_{4}} \alpha_{4}$ corresponds to the odd intersection between $\tilde{L}_{1}$ and $\tilde{L}_{2}$ at the midpoint of the arc $2=5$. In contrast, the even element $\operatorname{id}_{2 \rightarrow 5}$ corresponds to the even intersection between $\tilde{L}_{1}$ and $\tilde{L}_{2}$ at the midpoint of the arc $2=5$. The odd element $(-1)^{\# \alpha_{0}+1} \alpha_{0}$ corresponds to the co-identity element of the zigzag curve $\tilde{L}_{1}$. The even element $\sum_{a} \operatorname{id}_{a}$ corresponds to the identity element of the zigzag curve $\tilde{L}_{1}$. Two of the geometric interpretations are depicted in Figure 10.4

The splitting is not that hard to find and has at least been anticipated in 16, Section 9.2]. For more information we refer to the discussion in section F. 1 Our next step is to show that the standard splitting indeed provides a homological splitting of $\mathbb{L}$.

Lemma 10.12. Let $L_{1}$ and $L_{2}$ be zigzag paths. Then $\operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)=H+I+R$.
Proof. Any morphism $L_{1} \rightarrow L_{2}$ is a sum of elementary morphisms. By Proposition 10.9 any elementary morphisms belongs to an $\mathrm{A}, \mathrm{B}, \mathrm{C}$ or D situation. Given this case distinction, Table 10.5 shows that any such morphism can be written as a sum of elements in $R, I$ and $H$.

It is worth commenting on the fact that the equations in Table 10.5 actually hold true. This is due to diligent evaluation of the twisted differential

$$
\mu_{\mathbb{L}}^{1}(\varepsilon)=\mu^{1}(\varepsilon)+\mu^{2}(\delta, \varepsilon)+\mu^{2}(\varepsilon, \delta)+\ldots
$$

Let us examine the possible $\mu^{\geq 3}$ terms appear. For any elementary morphism $\varepsilon: L_{1} \rightarrow L_{2}$, at most one $\delta$ can be inserted upfront and at the back. Hence only $\mu^{3}$ disks can appear. Inspecting the direction of $\delta$ morphisms, we conclude that the only possible disks appear in the case of $\mu_{\mathbb{L}}^{1}(\beta)$ of situation A and $\mu_{\mathbb{L}}^{1}(\beta)$ and $\mu_{\mathbb{L}}^{1}\left(\beta^{\prime}\right)$ of situation C .

| Situation A | $1 \rightarrow 3$ | $\beta \alpha\left(\beta^{\prime} \gamma \beta \alpha\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha+1} \beta\left(\alpha \beta^{\prime} \gamma \beta\right)^{i}\right)+(-1)^{\# \alpha+\# \gamma+\\|\beta\\|} \gamma \beta\left(\alpha \beta^{\prime} \gamma \beta\right)^{i}$ without triangle degeneration |
| :---: | :---: | :---: |
|  | $1 \rightarrow 3$ | $\beta \alpha=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha+1} \beta\right)+(-1)^{\# \alpha+\# \gamma+\\|\beta\\|} \gamma \beta+(-1)^{\# \alpha+\# \gamma_{1}+\# \gamma_{2}} \operatorname{id}_{L_{1} \rightarrow L_{2}}$ in case of triangle degeneration |
|  | $1 \rightarrow 4$ | $\gamma \beta \alpha\left(\beta^{\prime} \gamma \beta \alpha\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha+1} \gamma \beta\left(\alpha \beta^{\prime} \gamma \beta\right)^{i}\right)$ |
|  | $2 \rightarrow 3$ | $\beta\left(\alpha \beta^{\prime} \gamma \beta\right)^{i} \in R$ |
|  | $2 \rightarrow 4$ | $\gamma \beta\left(\alpha \beta^{\prime} \gamma \beta\right)^{i} \in R$ |
| Situation B | $1 \rightarrow 5$ | $\alpha_{1}\left(\beta^{\prime} \alpha_{2} \alpha_{1}\right)^{i} \in R$ |
|  | $1 \rightarrow 6$ | $\alpha_{2} \alpha_{1}\left(\beta^{\prime} \alpha_{2} \alpha_{1}\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha_{2}+1} \alpha_{1}\left(\beta^{\prime} \alpha_{2} \alpha_{1}\right)^{i}\right)$ |
|  | $2 \rightarrow 4$ | $\alpha_{3}\left(\alpha_{4} \beta \alpha_{3}\right)^{i} \in R$ |
|  | $2 \rightarrow 5$ | $\left(\alpha_{1} \beta^{\prime} \alpha_{2}\right)^{i} \in R, i \geq 1$ |
|  | $2 \rightarrow 5$ | $\left(\alpha_{4} \beta \alpha_{3}\right)^{i} \in R, i \geq 1$ |
|  | $2 \rightarrow 5$ | $\mathrm{id}_{2 \rightarrow 5} \in R$ |
|  | $2 \rightarrow 6$ | $\alpha_{2}\left(\alpha_{1} \beta^{\prime} \alpha_{2}\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha_{2}}\left(\alpha_{1} \beta^{\prime} \alpha_{2}\right)^{i}\right)+(-1)^{\# \alpha_{1}+\# \alpha_{2}} \alpha_{1}\left(\beta^{\prime} \alpha_{2} \alpha_{1}\right)^{i}, i \geq 1$ |
|  | $2 \rightarrow 6$ | $\begin{aligned} & \alpha_{2} \underset{(-1)^{\# \alpha_{1}+\# \alpha_{2}} \alpha_{1}}{=}(-1)^{\# \alpha_{2}}\left((-1)^{\# \alpha_{3}+1} \alpha_{3}+(-1)^{\# \alpha_{4}} \alpha_{4}\right)+\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha_{2}} \mathrm{id}_{2 \rightarrow 5}\right)+ \\ & + \end{aligned}$ |
|  | $3 \rightarrow 4$ | $\alpha_{3} \alpha_{4}\left(\beta \alpha_{3} \alpha_{4}\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha_{4}+1} \alpha_{3}\left(\alpha_{4} \beta \alpha_{3}\right)^{i}\right)$ |
|  | $3 \rightarrow 5$ | $\alpha_{4}\left(\beta \alpha_{3} \alpha_{4}\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha_{4}+1}\left(\alpha_{4} \beta \alpha_{3}\right)^{i}\right)-\alpha_{3}\left(\alpha_{4} \beta \alpha_{3}\right)^{i}, i \geq 1$, |
|  | $3 \rightarrow 5$ | $\alpha_{4}=(-1)^{\# \alpha_{4}}\left((-1)^{\# \alpha_{3}+1} \alpha_{3}+(-1)^{\# \alpha_{4}} \alpha_{4}\right)+(-1)^{\# \alpha_{3}+\# \alpha_{4}} \alpha_{3}$ |
| Situation C | $1 \rightarrow 5$ | $\alpha_{1} \beta^{\prime}\left(\alpha_{2} \alpha_{1} \beta^{\prime}\right)^{i} \in R$ |
|  | $1 \rightarrow 6$ | $\beta^{\prime}\left(\alpha_{2} \alpha_{1} \beta^{\prime}\right)^{i} \in R$ |
|  | $2 \rightarrow 4$ | $\beta \alpha_{3}\left(\alpha_{4} \beta \alpha_{3}\right)^{i} \in R$ |
|  | $2 \rightarrow 5$ | $\left(\alpha_{1} \beta^{\prime} \alpha_{2}\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha_{2}+1} \alpha_{1} \beta^{\prime}\left(\alpha_{2} \alpha_{1} \beta^{\prime}\right)^{i-1}\right), i \geq 1$ |
|  | $2 \rightarrow 5$ | $\left(\alpha_{4} \beta \alpha_{3}\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha_{4}+1} \beta \alpha_{3}\left(\alpha_{4} \beta \alpha_{3}\right)^{i-1}\right), i \geq 1$ |
|  | $2 \rightarrow 5$ | $\mathrm{id}_{2 \rightarrow 5} \in H$, |
|  | $2 \rightarrow 6$ | $\beta^{\prime} \alpha_{2}\left(\alpha_{1} \beta^{\prime} \alpha_{2}\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha_{2}+1} \beta^{\prime}\left(\alpha_{2} \alpha_{1} \beta^{\prime}\right)^{i}\right)+(-1)^{\# \alpha_{1}+\# \alpha_{2}} \alpha_{1} \beta^{\prime}\left(\alpha_{2} \alpha_{1} \beta^{\prime}\right)^{i}$ |
|  | $3 \rightarrow 4$ | $\beta\left(\alpha_{3} \alpha_{4} \beta\right)^{i} \in R$ |
|  | $3 \rightarrow 5$ | $\alpha_{4} \beta\left(\alpha_{3} \alpha_{4} \beta\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha_{4}} \beta\left(\alpha_{3} \alpha_{4} \beta\right)^{i}\right)+(-1)^{\# \alpha_{3}+\alpha_{4}} \beta \alpha_{3}\left(\alpha_{4} \beta \alpha_{3}\right)^{i}$ |
| Situation D | $1 \rightarrow 1$ | $\mathrm{id}_{1 \rightarrow 1} \in R$ if $1 \neq a_{0}$ |
|  | $1 \rightarrow 1$ | $\mathrm{id}_{1 \rightarrow 1}=\sum \mathrm{id}_{a}-\sum_{a \neq a_{0}} \mathrm{id}_{a}$ if $1=a_{0}$ |
|  | $1 \rightarrow 1$ | $\left(\alpha^{\prime} \alpha\right)^{i} \in R, i \geq 1$ |
|  | $1 \rightarrow 2$ | $\alpha=\alpha_{0} \in H$ if $\alpha=\alpha_{0}$ |
|  | $1 \rightarrow 2$ | $\alpha=(-1)^{\# \alpha+\# \alpha_{0}+1} \alpha_{0}+\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha+1}\left(\operatorname{id}_{a_{1}}+\ldots+\operatorname{id}_{a_{k}}\right)\right)$ if $\alpha \neq \alpha_{0}$ in case 1 with $k$ odd |
|  | $1 \rightarrow 2$ | $\alpha=(-1)^{\# \alpha+\# \alpha_{0}} \alpha_{0}+\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha+1}\left(\operatorname{id}_{a_{1}}+\ldots+\operatorname{id}_{a_{k}}\right)\right)$ if $\alpha \neq \alpha_{0}$ in case 1 with $k$ even |
|  | $1 \rightarrow 2$ | $\alpha=(-1)^{\# \alpha+\# \alpha_{0}+1} \alpha_{0}+\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha}\left(\operatorname{id}_{a_{1}}+\ldots+\operatorname{id}_{a_{k}}\right)\right)$ if $\alpha \neq \alpha_{0}$ in case 2 with $k$ odd |
|  | $1 \rightarrow 2$ | $\alpha=(-1)^{\# \alpha+\# \alpha_{0}} \alpha_{0}+\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha+1}\left(\operatorname{id}_{a_{1}}+\ldots+\operatorname{id}_{a_{k}}\right)\right)$ if $\alpha \neq \alpha_{0}$ in case 2 with $k$ odd |
|  | $1 \rightarrow 2$ | $\alpha\left(\alpha^{\prime} \alpha\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha}\left(\alpha^{\prime} \alpha\right)^{i}\right), i \geq 1$ |
|  | $2 \rightarrow 1$ | $\alpha^{\prime}\left(\alpha \alpha^{\prime}\right)^{i} \in R$ |
|  | $2 \rightarrow 2$ | $\mathrm{id}_{2 \rightarrow 2} \in R$ if $2 \neq a_{0}$ |
|  | $2 \rightarrow 2$ | $\mathrm{id}_{2 \rightarrow 2}=\sum \mathrm{id}_{a}-\sum_{a \neq a_{0}} \mathrm{id}_{a}$ if $2=a_{0}$ |
|  | $2 \rightarrow 2$ | $\left(\alpha \alpha^{\prime}\right)^{i}=\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha+1} \alpha^{\prime}\left(\alpha \alpha^{\prime}\right)^{i-1}\right)-\left(\alpha^{\prime} \alpha\right)^{i}, i \geq 1$ |

Table 10.5: Verification of the homological splitting


Figure 10.4: The cohomology elements $(-1)^{\# \alpha_{3}} \alpha_{3}+(-1)^{\# \alpha_{4}+1} \alpha_{4}$ (B) and id (C) are the odd resp. even generators of Floer cohomology.

First, let us examine the case of $\mu_{\mathbb{L}}^{1}(\beta)$ in situation A. Denote the corresponding $\delta$-angles by $\gamma_{1}$ and $\gamma_{2}$, and the next arcs of $L_{1}$ and $L_{2}$ by 5 and 6 . Then traversing $6, \gamma_{1}, 2, \beta, 3, \gamma_{2}, 5$ must bound a disk and hence 5 and 6 are equal arcs and the disk is a simple polygon. Then indeed $\mu_{\mathbb{L}}^{1}(\beta)$ includes an id ${ }_{5 \rightarrow 6}$ morphism. See Figure 10.6

Next, we rule out the possibility that $\mu_{\mathbb{L}}^{1}(\beta)$ or $\mu_{\mathbb{L}}^{1}\left(\beta^{\prime}\right)$ of situation C has a disk contribution. Carry out the same analysis as in situation A and find that $\beta$ or $\beta^{\prime}$ is an elementary polygon angle, respectively. Together with the fact that the neighboring angles $\alpha_{3}$ and $\alpha_{4}$ are also elementary polygon angles comprising a full turn around puncture $t(2)$, this contradicts the fact that $Q$ is a dimer. We conclude that the differential $\mu_{\mathbb{L}}^{1}(\beta)$ in situation A remains as the only one that may include non-obvious terms.

Let us explain the meaning of case 1 and 2 in Table 10.5. This case distinction appears when we try to write an $\alpha$ angle with $\alpha \neq \alpha_{0}$ in terms of $H, I$ and $R$. Obviously, precisely one of the following two is the case:

1. The segment of $L_{1}$ starting with the target 2 of $\alpha$ and continuing in the direction of $\alpha$ first hits the source or target of $\alpha_{0}$ before hitting $a_{0}$.
2. The segment of $L_{1}$ starting with the source 1 of $\alpha$ and continuing in the opposite direction of $\alpha$ first hits the source or target of $\alpha_{0}$ before hitting $a_{0}$.

The two cases are depicted in Figure 10.7
In case 1 , put $a_{1}=2$, the indexed target of $\alpha$. Denote by $a_{1}, \ldots, a_{k}$ the segment of $L$ until $a_{k}$ is either the indexed source or target of $\alpha_{0}$, whichever comes first. In case 2 , put $a_{1}=1$, the indexed source of $\alpha$. Denote by $a_{1}, \ldots, a_{k}$ the segment of $L$ until $a_{k}$ is either the indexed source or target of $\alpha_{0}$, whichever comes first.

In both cases, let $\alpha_{i}$ be the angle from $a_{i-1}$ to $a_{i}$ if $i$ is odd, or from $a_{i}$ to $a_{i-1}$ if $i$ is even. In particular, put $\alpha_{1}=\alpha$ and $\alpha_{k+1}=\alpha_{0}$.

In case 1, we have

$$
\begin{aligned}
& \mu_{\mathbb{L}}^{1}\left(\mathrm{id}_{a_{i}}\right)=(-1)^{\# \alpha_{i}+1} \alpha_{i}+(-1)^{\# \alpha_{i+1}+1} \alpha_{i+1}, \text { if } i \text { odd, } \\
& \mu_{\mathbb{L}}^{1}\left(\operatorname{id}_{a_{i}}\right)=(-1)^{\# \alpha_{i}} \alpha_{i}+(-1)^{\# \alpha_{i+1}} \alpha_{i+1}, \quad \text { if } i \text { even }
\end{aligned}
$$

Adding these up, we obtain

$$
\begin{aligned}
& \mu_{\mathbb{L}}^{1}\left(\operatorname{id}_{a_{1}}+\ldots+\operatorname{id}_{a_{k}}\right)=(-1)^{\# \alpha+1} \alpha+(-1)^{\# \alpha_{0}+1} \alpha_{0}, \text { if } k \text { odd, } \\
& \mu_{\mathbb{L}}^{1}\left(\operatorname{id}_{a_{1}}+\ldots+\operatorname{id}_{a_{k}}\right)=(-1)^{\# \alpha+1} \alpha+(-1)^{\# \alpha_{0}} \alpha_{0}, \quad \text { if } k \text { even. }
\end{aligned}
$$

In case 2 , we similarly get

$$
\begin{aligned}
& \mu_{\mathbb{L}}^{1}\left(\mathrm{id}_{a_{1}}+\ldots+\operatorname{id}_{a_{k}}\right)=(-1)^{\# \alpha} \alpha+(-1)^{\# \alpha_{0}} \alpha_{0}, \quad \text { if } k \text { odd }, \\
& \mu_{\mathbb{L}}^{1}\left(\operatorname{id}_{a_{1}}+\ldots+\operatorname{id}_{a_{k}}\right)=(-1)^{\# \alpha} \alpha+(-1)^{\# \alpha_{0}+1} \alpha_{0}, \text { if } k \text { even. }
\end{aligned}
$$

This precisely verifies the equations in Table 10.5 concerning $\alpha$ (D).


Figure 10.6: Triangle degeneration in situation A


Figure 10.7: Position of an angle $\alpha$ between $\alpha_{0}$ and $a_{0}$.

Proposition 10.13. Let $L_{1}$ and $L_{2}$ be zigzag paths. Then the standard splitting $H, I, R$ defines a homological splitting of $\operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)$.
Proof. We need to check that $I$ is the image of $\mu_{\mathbb{L}}^{1}$, that $H+I$ is the kernel of $\mu_{\mathbb{L}}^{1}$ and that $\operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)=$ $H \oplus I \oplus R$. First, note that $I$ is the image of $\mu_{\mathbb{L}}^{1}$ by definition. Next let us check that $\mu_{\mathbb{L}}^{1}(H)=0$. In our situational formalism, this is a simple calculation:

$$
\begin{aligned}
\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha_{3}+1} \alpha_{3}+(-1)^{\# \alpha_{4}} \alpha_{4}\right) & =\mu^{2}\left((-1)^{\# \alpha_{3}+1} \alpha_{3},(-1)^{\# \alpha_{4}} \alpha_{4}\right)+\mu^{2}\left((-1)^{\# \alpha_{3}} \alpha_{3},(-1)^{\# \alpha_{4}} \alpha_{4}\right)+\mu^{\geq 3}(\ldots) \\
& =(-1)^{\# \alpha_{3}+\# \alpha_{4}} \alpha_{3} \alpha_{4}+(-1)^{\# \alpha_{3}+\# \alpha_{4}+1} \alpha_{3} \alpha_{4}=0, \\
\mu_{\mathbb{L}}^{1}(\operatorname{id}(\mathrm{C})) & =\mu^{\geq 3}(\ldots, \operatorname{id}, \ldots)=0, \\
\mu_{\mathbb{L}}^{1}\left(\sum \operatorname{id}_{a}\right) & =\mu^{2}\left(\sum \operatorname{id}_{a}, \delta\right)+\mu^{2}\left(\delta, \sum \operatorname{id}_{a}\right)+\mu^{\geq 3}(\ldots, \operatorname{id}, \ldots)=-\delta+\delta=0, \\
\mu_{\mathbb{L}}^{1}\left((-1)^{\# \alpha_{0}+1} \alpha_{0}\right) & =(-1)^{\# \alpha_{0}+1} \mu^{\geq 3}\left(\ldots, \alpha_{0}, \ldots\right)=0 .
\end{aligned}
$$

Indeed, both id (C) and $\alpha_{0}(\mathrm{D})$ have no $\delta$ morphisms that can be multiplied upfront or at the back, and correspondingly also produce no $\mu^{\geq 3}$ disks. Since $I \subseteq \operatorname{Ker}\left(\mu_{\mathbb{L}}^{1}\right)$, we conclude $H+I \subseteq \operatorname{Ker}\left(\mu_{\mathbb{L}}^{1}\right)$.

Next, recall from Lemma 10.12 that we have $\operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)=H+I+R$. Let us verify by dimension counting that this sum is direct. For $N \in \mathbb{N}$, denote by $E_{N} \subseteq E:=\operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)$ the subspace spanned by elementary angles of at most $N$ full turns. This is a finite dimensional space.

Table 10.5 implies that any element of $E_{N}$ can be written as a sum of elements of $H \cap E_{N}, I \cap E_{N}$ and $R \cap E_{N}$ if $N \geq 1$. The sum of these three spaces is direct, since the sum of their dimensions is less than or equal to the dimension of $E_{N}$ :

$$
\begin{aligned}
& \operatorname{dim}\left(H \cap E_{N}\right)=\# B+\# C\left[+2 \text { if } L_{1}=L_{2}\right] \\
&+ \\
& \operatorname{dim}\left(I \cap E_{N}\right) \leq 2 N \# A+(4 N-1) \# B+4 N \# C\left[+2 N \# D-1 \text { if } L_{1}=L_{2}\right] \\
& \quad+ \\
& \operatorname{dim}\left(R \cap E_{N}\right)=2 N \# A+(4 N+1) \# B+4 N \# C\left[+(2 N+1) \# D-1 \text { if } L_{1}=L_{2}\right] \\
& \quad \mid \wedge \\
& \operatorname{dim} E_{N}=4 N \# A+(8 N+1) \# B+(8 N+1) \# C\left[+(4 N+1) \# D \text { if } L_{1}=L_{2}\right] .
\end{aligned}
$$

Here $\# A, \# B, \# C$ and $\# D$ denote the number of situations of type $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D that appear from $L_{1}$ to $L_{2}$.

The filtration of $E$ by $E_{N}$ is exhaustive. If the zero element of $E$ is a nontrivial sum of elements of $H, I$ and $R$, then by picking the maximum number of turns $N$ involved, we obtain a contradiction to $E_{N}$ being the direct sum of $H \cap E_{N}, I \cap E_{N}$ and $R \cap E_{N}$. We conclude that $E=H \oplus I \oplus R$.

Finally, let us argue that $\operatorname{Ker}\left(\mu_{\mathbb{L}}^{1}\right)=H \oplus I$. Indeed, the equivalent statement that $\mu_{\mathbb{L}}^{1}: R \rightarrow I$ is injective can be checked by an elementary situational calculation. Note that if $\varepsilon$ is from a certain situation, then $\mu_{\mathbb{L}}^{1}(\varepsilon)$ is from the same situation, apart from triangle degenerations and situation D morphisms.

Alternatively, note the cohomology of the complex $\operatorname{Hom}_{\mathbb{L}}\left(L_{1}, L_{2}\right)$ is precisely Floer cohomology: The category $\mathrm{HTw} \operatorname{Gtl} Q$ is nothing else than the wrapped Fukaya category of $Q$. Now Floer cohomology has a basis element for every intersection of $L_{1}$ and $L_{2}$, plus an identity and a co-identity if $L_{1}=L_{2}$. We conclude

$$
\operatorname{dim}\left(\operatorname{Ker}\left(\mu_{\mathbb{L}}^{1}\right) / I\right)=\# B+\# C\left[+2 \text { if } L_{1}=L_{2}\right]=\operatorname{dim} H
$$

This implies $\operatorname{Ker}\left(\mu_{\mathbb{L}}^{1}\right)=H \oplus I$ and finishes the proof.

## 11 The deformed category of zigzag paths

In this section, we define and study the deformed category $\mathbb{L}_{q}$ of zigzag paths. This category is the deformed version of $\mathbb{L}$. Its deformation comes from the deformation $\mathrm{Gtl}_{q} Q$ of the gentle algebra. Already in the definition of $\mathbb{L}_{q}$, we apply the complementary angle trick from section 9.4 in order to remove the curvature. We then analyze the differential $\mu_{\mathbb{L}_{q}}^{1}$, in fact investigate how $\mu_{\mathbb{L}_{q}}^{1}$ interacts with the homological splitting of $\mathbb{L}$ from section 10.3 We show that $\mathbb{L}_{q}$ falls under the regime of the simplified deformed Kadeishvili construction from section 8.8 We provide explicit formulas for the deformed codifferential $h_{q}$ and deformed projection $\pi_{q}$ in terms of "tails" of the morphisms involved.


These "tails" arise from applying the general Kadeishvili construction to the specific case of $\mathbb{L}_{q}$. According to the general deformed Kadeishvili theorem, we need to find for every cohomology basis element $h \in H$ a certain deformed version $\varphi^{-1}(h)$ such that $\varphi^{-1}(h)$ and $h$ only differ by $R$ terms. In case of $\mathbb{L}_{q}$, we can explicitly describe $\varphi^{-1}(h)$ for each of the cohomology basis elements in $H$. The explicit description forces us to define and make us of what we call the tails of morphisms. Tails work as follows: Let $\varepsilon: L_{1} \rightarrow L_{2}$ be an elementary morphism. Look at all locations where $L_{1}$ and $L_{2}$ come close to each other and bound a discrete immersed disk together with $\varepsilon$. It turns out that these locations of closeness have a hierarchical structure, which we organize in a tree. This tree is the tail of $\varepsilon$ and carries the data of many secondary $\mathrm{A}, \mathrm{B}, \mathrm{C}$ or D situations which lie far away from $\varepsilon$. We use the angles contained in this tail to construct explicitly the deformed cohomology basis elements $\varphi^{-1}$. In other words, tails are the right tool to convert the rather algebraic requirement of the deformed Kadeishvili theorem into a geometric interpretation in the specific case of $\mathbb{L}_{q}$.

Remark 11.1. From this section onwards, we typically write $\mu$ or $\mu_{q}$ for the product of $\operatorname{Add}^{\operatorname{Gtl}}{ }_{q} Q$ :

$$
\mu_{q}:=\mu:=\mu_{\mathrm{Add} \mathrm{Gtl}_{q} Q}
$$

The reason for this notation is that we frequently expand products of the twisted completion $\mathrm{Tw}_{\mathrm{Gtl}}^{q} \boldsymbol{Q}$ in terms of the products of the additive completion. This shorthand is meant to facilitate this expansion and mixes well with writing $\mu$ or $\mu_{q}$ for the product of $\operatorname{Gtl}_{q} Q$. The product of $\mathrm{Gtl} Q$ is irrelevant and is never meant. We keep the shorthand $\mu_{q}$ until section 13

### 11.1 Deformed zigzag paths

In this section, we define the category $\mathbb{L}_{q}$ of deformed zigzag paths. The starting point is the non-deformed category $\mathbb{L}$ of zigzag paths. Taking the same twisted complexes gives a subcategory of $\mathrm{Tw} \mathrm{Gtl}_{q} Q$. The aim of the entire paper is to compute a minimal model for this subcategory. According to the deformed Kadeishvili theorem, the first step is to gauge away as much curvature as possible. The aim of the present section is to conduct this uncurving procedure, and to define $\mathbb{L}_{q}$ to be the resulting uncurved category. The essential tool for uncurving is the complementary angle trick of section 9.4 .



Figure 11.1: Fictitious discrete immersed disks bounded by segments of zigzag paths


Recall that the zigzag paths are objects in $\operatorname{Tw} \operatorname{Gtl} Q$. They can also be interpreted as objects in $\mathrm{Tw} \mathrm{Gtl}_{q} Q$ by definition of the deformed twisted completion. As an object of $\mathrm{Tw} \mathrm{Gtl}_{q} Q$, every zigzag path has curvature. Our approach is to uncurve every zigzag path by means of the complementary angle trick of section 9.4 For every zigzag path, the trick gives rise to a twisted complex with also infinitesimal below-diagonal entries. By definition, this is an element of the category $\mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} Q$, see section 5.5 We shall call this object a deformed zigzag path because its $\delta$-matrix has been deformed. The precise definition for deformed zigzag paths reads as follows:

Definition 11.2. Let $L$ be a zigzag path of $Q$, with twisted complex

$$
L=\left(a_{1} \oplus a_{3} \oplus \ldots \oplus a_{k} \oplus a_{2} \oplus \ldots \oplus a_{2 k}, \delta\right)
$$

as in Definition 10.3 Then the corresponding deformed zigzag path is the following object of $\mathrm{Tw}^{\prime} \mathrm{Gtl}_{q}$, also denoted $L$ :

$$
\begin{aligned}
L & =\left(a_{1} \oplus a_{3} \oplus \ldots \oplus a_{k} \oplus a_{2} \oplus \ldots \oplus a_{2 k}, \delta\right), \\
\delta & =\left[\begin{array}{ccccc|c} 
\\
\hline(-1)^{\# \alpha_{1}} q_{1} \alpha_{1}^{\prime} & (-1)^{\# \alpha_{2}} q_{2} \alpha_{2}^{\prime} & 0 & \ldots & 0 & \text { ditto } \\
0 & (-1)^{\# \alpha_{3}} q_{3} \alpha_{3}^{\prime} & (-1)^{\# \alpha_{4}} q_{4} \alpha_{4}^{\prime} & \ldots & 0 & \\
\ldots & \ldots & \ldots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \ldots & (-1)^{\# \alpha_{2 k-2} q_{2 k-2} \alpha_{2 k-2}^{\prime}} & \\
(-1)^{\# \alpha_{2 k}} q_{2 k} \alpha_{2 k}^{\prime} & 0 & 0 & \ldots & (-1)^{\# \alpha_{2 k-1} q_{2 k-1} \alpha_{2 k-1}^{\prime}} &
\end{array}\right] .
\end{aligned}
$$

Here "ditto" denotes the same matrix entries as in Definition 10.3 The letter $q_{i}$ denotes the puncture around which $\alpha_{i}$ winds.

The deformed zigzag paths are objects of $\mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} Q$. The entries of their $\delta$-matrix are angles $\alpha_{i}$ and $\alpha_{i}^{\prime}$, which we call the inner respectively outer $\delta$-angles of $L$. The main interest of the present paper is in the subcategory of all deformed zigzag paths, each with their associated choice of spin structure. We give the category consisting of these objects a name:

Definition 11.3. The category of deformed zigzag paths is the full subcategory $\mathbb{L}_{q} \subseteq \operatorname{Tw}^{\prime} \mathrm{Gtl}_{q} Q$ consisting of the deformed zigzag paths.

The category $\mathbb{L}_{q}$ is a deformation of $\mathbb{L}$, as we have explained in Lemma 5.38 It is completely acceptable that we have inserted below-diagonal entries into the twisted complex. In what follows, we explain why $\mathbb{L}_{q}$ is indeed curvature-free. It is in fact a consequence of the more general property of the complementary angle trick Lemma 9.17 but we provide here a direct proof as well. The direct proof builds on the fact that segments of zigzag paths in geometrically consistent dimers cannot bound discrete immersed disks. In Figure 11.1 we have depicted a fictitious discrete immersed disk bounded by a zigzag path and we shall prove that this situation is indeed impossible:

Lemma 11.4. Let $Q$ be a zigzag consistent dimer and $L$ a zigzag path. Then a segment of $L$ cannot bound a discrete immersed disk.

Proof. Assume towards contradiction that $D: P_{k} \rightarrow|Q|$ is a discrete immersed disk bounded by a segment of $L$. Then at least one of the consecutive interior angles of the discrete immersed disk must be an outer $\delta$-angle $\alpha^{\prime}$ and hence consist of at least two indecomposable angles. The rest of the argument is a standard contradiction from geometric consistency theory: Construct a zigzag path $L^{\prime}$ at $\alpha^{\prime}$ that points inside the discrete immersed disk. Follow $L^{\prime}$ until it leaves the discrete immersed disk. To be precise, "leaving" refers not to the leaving an area in $|Q|$, but to touching the boundary $\partial P_{k}$ in the domain of the underlying polygon immersion $D: P_{k} \rightarrow|Q|$. Either way, the final arc $a$ before definitely leaving the discrete immersed disk lies on the boundary of the disk due to the zigzag nature of $L$. The $L$ and $L^{\prime}$ segments from $\alpha^{\prime}$ until $a$ are homotopic, since both lie in the discrete immersed disk. We obtain a contradiction with geometric consistency. This shows that $L$ cannot bound a discrete immersed disk.

Lemma 11.5. Every deformed zigzag path $L$ is curvature-free.
Proof. The curvature of $L$ as object in $\operatorname{Tw~}_{\operatorname{Gtl}_{q} Q}$ is

$$
\mu_{\operatorname{Add~}_{\operatorname{Gtl}_{q} Q, L}^{0}}^{0}+\mu_{\operatorname{Add}_{\operatorname{Gtl}_{q} Q}^{1}}^{1}(\delta)+\mu_{\operatorname{Add~Gtl}_{q} Q}^{2}(\delta, \delta)+\ldots
$$

It is our task to prove that this curvature vanishes. A first observation is that the two terms $\mu_{\operatorname{Add~Gtl}_{q} Q, L}^{0}$ and $\mu_{\operatorname{Add~Gtl}_{q} Q}^{2}(\delta, \delta)$ precisely cancel each other and we have $\mu_{\operatorname{Add~Gtl}_{q} Q}^{1}=0$. By Lemma 11.4 a segment of $L$ cannot bound a discrete immersed disk and we conclude that $\mu_{\text {Add Gtl }_{q}}^{\geq 3}(\delta, \ldots)=0$. Adding up all 4 terms, we see that the curvature of the deformed zigzag path vanishes. This finishes the proof.

### 11.2 EFGH disks

In this section, we develop elementary understanding of the differential $\mu_{\mathbb{L}_{q}}^{1}$. The starting point is the description of the category $\mathbb{L}_{q}$ by explicit twisted complexes. The differential $\mu_{\mathbb{L}_{q}}^{1}$ does not vanish, but counts those discrete immersed disks where apart from one single angle all interior angles stem from the $\delta$-matrix of the twisted complex. In this section, we classify these disks into four types which we call E , $\mathrm{F}, \mathrm{G}$ and H disks. The goal is to be able to say:

$$
\text { Terms occurring in } \mu_{\mathbb{L}_{q}}^{1}(\varepsilon) \quad \longleftrightarrow \quad \mathrm{E}, \mathrm{~F}, \mathrm{G}, \mathrm{H} \text { disks of } \varepsilon
$$

We start with a basic analysis of $\mu_{\mathbb{L}_{q}}^{1}$. Recall we write $\mu_{q}$ for $\mu_{\operatorname{AddGtl}_{q} Q}$. Let $\varepsilon: L_{1} \rightarrow L_{2}$ be an elementary morphism. Then we have

$$
\mu_{\mathbb{L}_{q}}^{1}(\varepsilon)=\sum_{k, l \geq 0} \mu_{q}(\underbrace{\delta, \ldots, \delta}_{k}, \varepsilon, \underbrace{\delta, \ldots, \delta}_{l}) .
$$

Here each $\delta$ insertion stands for the $\delta$-matrix of $L_{1}$ or $L_{2}$, depending on whether $\delta$ stands right or left of $\varepsilon$. The individual summands $\mu_{q}^{1}(\delta, \ldots, \varepsilon, \ldots, \delta)$ can again be expanded by writing the $\delta$-matrix as the sum of its entries, the $\delta$-angles. The elegant way to capture all the terms arising this way is as follows: We say a disk made of $\mu_{q}(\delta, \ldots, \varepsilon, \ldots, \delta)$ is a final-out, first-out or all-in disk with angle sequence consisting of $\varepsilon$, preceded and succeeded by an arbitrary number of $\delta$-angles. This way, we have enumerated all terms contributing to $\mu_{\mathbb{L}_{q}}^{1}(\varepsilon)$.

In Definition 11.6 we categorize the disks made of $\mu_{q}(\delta, \ldots, \varepsilon, \ldots, \delta)$ into four types. As a starting point, every disk made of $\mu_{q}(\delta, \ldots, \varepsilon, \ldots, \delta)$ is by definition a discrete immersed disk, together with possibly an outside morphism $\beta$ or $\gamma$ in the terminology of Definition 6.50. In the categorization and its proof, we make heavy use of the slots, concluding puncture and concluding arc terminology introduced in section 6.9. For instance, we may say that $L_{1}$ "turns right towards the concluding puncture of the disk". As an example, in Figure 11.2a the zigzag path $L_{1}$ turns right towards the concluding puncture and $L_{2}$ turns left towards the concluding puncture. The categorization is obtained by a case distinction based on the behavior of $L_{1}$ and $L_{2}$ towards the concluding puncture.

Definition 11.6. Let $\varepsilon: L_{1} \rightarrow L_{2}$ be an elementary morphism. Then a disk that can be made of $\mu_{q}(\delta, \ldots, \varepsilon, \ldots, \delta)$ is of

- type $\mathbf{E}$ if it is some-out, $L_{1}$ turns right towards the concluding puncture, $L_{2}$ turn left towards the concluding puncture, and there are at least 3 slots outside the disk,
- type $\mathbf{F}$ if it is some-out, there are at least two slots both inside and outside the disk, and $L_{1}$ turns left and $L_{2}$ right towards the concluding puncture, or the other way around,


Figure 11.2: Illustration of type E, F, G, H disks. The type F disk is depicted in case $L_{1}$ turns outside and $L_{2}$ turns inside the discrete immersed disk at the concluding puncture. For the type G disk, the naming of $\alpha_{1}$ and $\alpha_{2}$ is in case the arc outside the disk points upwards (type G1). For the type H disk, the naming of $\alpha_{1}$ and $\alpha_{3}$ are in case the concluding arc points to the right.

- type $\mathbf{G}$ if it is some-out, $L_{1}$ turns right towards the concluding puncture and $L_{2}$ turns left towards to concluding puncture and there are 2 slots outside the disk,
- type $\mathbf{H}$ if it is all-in; or if it is first-out, $L_{2}$ turns right towards the concluding puncture and there is only 1 slot inside the disk; or if it is final-out, $L_{1}$ turns left towards the concluding puncture and there is only 1 slot inside the disk.
A disk of type G is of type G1 if the first, shared, arc of $L_{1}$ and $L_{2}$ at the concluding puncture, outside the disk, is oriented towards the concluding puncture, and of type G2 if the arc is oriented away from the puncture.

The terminology is depicted in Figure 11.2 We now show that the types E, F, G, H indeed provide an exhaustive classification of disks that can be made of $\mu_{q}(\delta, \ldots, \varepsilon, \ldots, \delta)$. During the proof, we will frequently use slots terminology from section 6.9. We also show that some of the disk types come in pairs or triples. For instance, type E disks come in pairs. By this, we mean that every type E disk comes naturally with a distinct partner also of type $E$.
Lemma 11.7. Let $\varepsilon: L_{1} \rightarrow L_{2}$ be an elementary morphism. Then type E, F, G, H provide an exhaustive classification of disks that can be made from $\mu_{q}(\delta, \ldots, \varepsilon, \ldots, \delta)$. Type E disks come naturally in pairs, type F disks come alone, type G disks come in pairs and type H disks come in triples.

Proof. We first prove that every disk that can be made of $\mu_{q}(\delta, \ldots, \varepsilon, \ldots, \delta)$ falls under one of the four types and then comment on the pairs and triples.

Let $D$ be a disk that can be made of $\mu_{q}(\delta, \ldots, \varepsilon, \ldots, \delta)$. Then $D$ is either first-out, final-out or all-in. We shall in all three cases that $D$ falls under our classification. The simplest way to understand the proof is by trying to recognize the properties we derive about $D$ in Figure 11.2

Assume $D$ is first-out. Then the first morphism must be a $\delta$ insertion and not $\varepsilon$, otherwise $L_{2}$ would bound a discrete immersed disk, in conflict with geometric consistency. The $\delta$-insertion necessarily concerns an outer $\delta$-angle, as opposed to an inner $\delta$-angle, and we conclude that $L_{1}$ turns right at the concluding puncture. There are at least two slots outside of $D$ at the concluding puncture. If $L_{2}$ turns left at the concluding puncture, then $D$ is of type E or G , depending on the number of slots outside $D$. If $L_{2}$ turns right at the concluding puncture, then $D$ is of type F or H , depending on the number of slots inside $D$.

A similar classification holds if $D$ is final-out. Finally assume $D$ is all-in. Then the concluding arc belongs to both $L_{1}$ and $L_{2}$. Including the concluding arc, the $L_{1}$ and $L_{2}$ segments bounding the disk are at least 2 arcs long, since otherwise the $L_{2}$ or $L_{1}$ segment would bound a discrete immersed disk in conflict with geometric consistency. Let us analyze how $L_{1}$ and $L_{2}$ continue beyond the concluding arc of $D$, away from their segments that bound $D$. Imagine that $L_{1}$ turns left at the head (and tail) of the concluding arc. Then $L_{1}$ enters the interior of $D$, in conflict with geometric consistency, or lands on the arc of $L_{2}$ before the concluding arc, which would render $L_{1}=L_{2}$ and mean that $L_{1}$ bounds a discrete immersed disk. We conclude that $L_{1}$ turns right and $L_{2}$ turns left at the concluding arc. In particular, there are at least two slots on the outside of the disk. This constitutes a type H disk. We have finished the first part of the proof.


Figure 11.3: If $L_{1}$ and $L_{2}$ intersect above or at $\varepsilon$, then $\varepsilon$ has only type E disks.

For the second part of the proof, let us comment on the pairs and triples. We shall here restrict to the case of type E disks, since the other cases are similar. To show that type E disks come in pairs, the idea is to simply match two type E disks with each other by swapping the $\delta$ insertions: Regard a first-out type E disk. Then its first angle is a $\delta$ insertion. Remove this $\delta$ insertion and instead append a $\delta$ insertion as final angle. The result is a final-out disk, the desired partner disk. The partner disk is also of type E. The first angle of the first-out disk and the final angle of its final-out partner disk are depicted in Figure 11.2a as well. Both disks have the same underlying discrete immersed disk, up to cyclically permuting the inputs and the output by one. The other types F, G, H are similar, and the relevant first/final angles are depicted in Figure 11.2 as well. This finishes the proof.

### 11.3 Deformed differential

In this section, we investigate the precise shape of the differential $\mu_{\mathbb{L}_{q}}^{1}$. The starting point is the description of possible output of $\mu_{\mathbb{L}_{q}}^{1}$ in terms of E, F, G, H disks, according to section 11.2 For the purposes of the deformed Kadeishvili theorem, this description would not be sufficient. We therefore trace the shape of $\mu_{\mathbb{L}_{q}}^{1}$ even further. As announced, this leads to the data structure which we call the tail of an elementary morphism.

Let $\varepsilon: L_{1} \rightarrow L_{2}$ be an elementary morphism. We have seen that the disks that can be made of $\mu_{q}(\delta, \ldots, \varepsilon, \ldots, \delta)$ are of type $\mathrm{E}, \mathrm{F}, \mathrm{G}, \mathrm{H}$ and come in groups, which we call the disk shapes of $\varepsilon$. In order to describe $\mu_{\mathbb{L}_{q}}^{1}$, we need to capture the situation near the concluding puncture or arc of the disk shapes. For instance for a type E disk shape $D$, we shall assign to $D$ the A situation at the concluding puncture of $D$. An A situation is already given by specifying the its angles $\alpha, \beta, \gamma$ and $\beta^{\prime}$ as in Figure 10.2. In generality, we fix the following notation:

Definition 11.8. Let $\varepsilon: L_{1} \rightarrow L_{2}$ be an elementary morphism.

- For each type E disk shape $D$ of $\varepsilon$, denote by $\left(\alpha^{D}, \beta^{D}, \gamma^{D}, \beta^{\prime D}\right)$ the A situation at the concluding puncture. Let $s^{D} \in \mathbb{Z}$ be the sum of the total \# signs of all $\delta$ insertions along the disk, including both $\# \alpha^{D}$ and $\# \gamma^{D}$.
- For each type F disk shape $D$ of $\varepsilon$, denote by $\left(\alpha^{D}, \beta^{D}, \gamma^{D}, \beta^{\prime D}\right)$ the A situation at the concluding puncture. Let $s^{D} \in \mathbb{Z}$ be the sum of the total \# signs of all $\delta$ insertions along the disk, including $\# \alpha_{D}$ if $D$ is first-out and including $\# \gamma_{D}$ if $D$ is final-out.
- For each type G disk shape $D$ of $\varepsilon$, denote by $\left(\alpha_{1}^{D}, \alpha_{2}^{D}, \alpha_{3}^{D}, \alpha_{4}^{D}, \beta^{D}, \beta^{D}\right)$ the B situation at the concluding puncture. Let $s^{D} \in \mathbb{Z}$ be the sum of the total \# signs of all $\delta$ insertions along the disk, including both $\# \alpha_{1}$ and $\# \alpha_{2}$ or $\# \alpha_{3}$ and $\# \alpha_{4}$, depending on the orientation of the arc $2^{D}=5^{D}$.
- For each type H disk shape $D$ of $\varepsilon$, denote by $\left(\alpha_{1}^{D}, \alpha_{2}^{D}, \alpha_{3}^{D}, \alpha_{4}^{D}, \beta^{D}, \beta^{D}\right)$ the C situation at the concluding puncture. Let $s^{D} \in \mathbb{Z}$ be the sum of the total $\#$ signs of all $\delta$ insertions along the disk, including $\# \alpha_{1}$ and $\# \alpha_{3}$ or $\# \alpha_{2}$ and $\# \alpha_{4}$, depending on the orientation of the concluding arc.
In all cases, the element $q^{D} \in \mathbb{C} \llbracket Q_{0} \rrbracket$ is the product of all punctures covered by the discrete immersed disk, including those punctures on the boundary whose $\delta$ insertion is an outer $\delta$ angle, and including the concluding puncture or both endpoints of the concluding arc.

In Definition 11.9, we define tails of elementary morphisms. The motivation is as follows: To apply the deformed Kadeishvili theorem to $\mathbb{L}_{q}$, we need to provide the deformed counterparts of the cohomology basis elements from section 10.3 For instance, for a given situation B cohomology basis element $h=$ $(-1)^{\# \alpha_{3}+1} \alpha_{3}+(-1)^{\# \alpha_{4}} \alpha_{4} \in H$ we need to find a deformed counterpart $\varphi^{-1}(h)$ such that $\varphi^{-1}(h)$ and $h$ only differ by infinitesimal $R$ terms and $\mu_{\mathbb{L}_{q}}^{1}\left(\varphi^{-1}(h)\right)=0$. The first step is to note that $\mu_{\mathbb{L}_{q}}^{1}(h)$ is described, among others, by the type E disks which can be made of its two components $\alpha_{3}$ and $\alpha_{4}$. Every


Figure 11.4: Illustration of tails
type E disk shape $D$ of $\alpha_{3}$ gives a contribution to $\mu_{\mathbb{L}_{q}}^{1}\left(\alpha_{3}\right)$ of

$$
\begin{equation*}
(-1)^{s^{D}+\# \gamma^{D}+\left|\gamma^{D} \beta^{D}\right|} q^{D} \gamma^{D} \beta^{D}+(-1)^{s^{D}+\# \alpha^{D}} q^{D} \beta^{D} \alpha^{D} . \tag{11.1}
\end{equation*}
$$

Similarly, every type E disk of $\alpha_{4}$ gives a contribution to $\mu_{\mathbb{L}_{q}}^{1}\left(\alpha_{4}\right)$. Counting these contributions together, we see that $\mu_{\mathbb{L}_{q}}^{1}(h)$ already contains two terms for every type E disk shape of $\alpha_{3}$ plus two terms for every type E disk shape of $\alpha_{4}$. We see that $\mu_{\mathbb{L}_{q}}^{1}(h)$ is far from zero. The deformed counterpart $\varphi^{-1}(h)$ is given by adding $R$ terms to $h$ such that $\mu_{\mathbb{L}_{q}}^{1}$ eventually becomes zero. We observe that adding a multiple of $\beta^{D}$ for every disk shape $D$ of $\alpha_{3}$ or $\alpha_{4}$ does the trick in that it kills the two terms in 11.1. However, $\mu_{\mathbb{L}_{q}}^{1}\left(\beta^{D}\right)$ does not equate only to the two terms in 11.1, but also to terms coming from the $\mathrm{E}, \mathrm{F}, \mathrm{G}, \mathrm{H}$ disks which can be made of $\beta^{D}$ itself. In turn, we have to kill these terms by adding yet more $R$ terms, and every time we add $R$ terms we obtain new $R$ terms which we kill again. This gives rise to a recursive terms killing process which we can fortunately organize in a hierarchical structure, the tail of $\alpha_{3}$ and $\alpha_{4}$. The precise definition for general elementary morphisms reads as follows:

Definition 11.9. Let $\varepsilon: L_{1} \rightarrow L_{2}$ be an elementary morphism. Then its tail is the tree $T$ defined as follows. Insert $\varepsilon$ as root. For each disk shape $D$ of $\varepsilon$, attach $D$ as a child, annotated additionally with the type of $D$. Continue inductively: For each leaf $D \in T$ of type E , attach all disk shapes of $\beta^{D}$ as children, annotated with their types.

Let $D \in T$ be a node of type $E$. Denote by $D_{0}=\varepsilon, \ldots, D_{n}=D$ be the sequence of nodes from the root till $D$. Set

$$
S^{D}=\sum_{i=1}^{n} s^{D_{i}}, \quad Q^{D}=\prod_{i=1}^{n} q^{D_{i}}
$$

The morphism $\varepsilon$ is E-preserving if its tail $T$ only consists of type E disks, apart from $\varepsilon$ itself.
The schematic of tails is depicted in Figure 11.4a Roughly speaking, a tail collects sequences of type E disks where every $\beta^{D}$ morphism serves as $\varepsilon$ for the next item in the sequence. The tail also collects type F, G or H disks but does not trace them any further. A sample elementary morphism together with its tail is depicted in Figure 11.4. In drawing the elementary morphism and its disks, we have neglected the zigzag nature of the zigzag paths. In drawing the tail, we have only depicted the tree structure and the type indication on all nodes and ignored the discrete immersed disk and situation data.

The typical tail is best imagined as a linear chain of type E disks with possibly an $\mathrm{F}, \mathrm{G}$ or H disk at the end. Theoretically, nonlinear tails exist, but they require an angle sequence which bounds more than a single discrete immersed disks. Such angle sequences exist, but are very large, see also Remark 6.32.

Depending on the further knowledge of an elementary morphism $\varepsilon$, we can say a more about the structure of its tail. In fact, every morphism from a B or C situation is E-preserving by virtue of geometric consistency. The following lemma makes this precise. Its premise is depicted in Figure 11.3

Lemma 11.10. Let $\varepsilon: L_{1} \rightarrow L_{2}$ be an elementary morphism. Suppose that above $\varepsilon$ the zigzag paths $L_{1}$ and $L_{2}$ intersect and their segments from $\varepsilon$ until the intersection are homotopic. Or suppose that at $\varepsilon$, the zigzag path $L_{1}$ turns to the the target arc of $\varepsilon$ or $L_{2}$ turns to the source arc of $\varepsilon$. Then $\varepsilon$ is E-preserving. In particular, this applies if $\varepsilon$ is a morphism from a B or C situation.

Proof. The first observation is that $\varepsilon$ cannot have disk shapes of type G or H , since these would create a digon with the intersection above $\varepsilon$. We argue that $\varepsilon$ can also not have type F disk shapes. Indeed, the ray of $L_{1}$ or $L_{2}$ that turns into the interior of a discrete immersed disk would leave the disk at some point, creating a contractible self-intersection of $L_{1}$ or $L_{2}$ or a digon with the intersection above $\varepsilon$. We conclude that $\varepsilon$ has only disks of type E . The same argument can now be applied inductively to all children of $\varepsilon$. Ultimately, the entire tail of $\varepsilon$ consists only of type $E$ disk shapes and we conclude $\varepsilon$ is E-preserving.

Elementary morphisms of a B or C situation automatically satisfy the premises of the lemma, simply because the involved zigzag paths intersect at the arc which we called $2=5$ in Figure 10.2. This finishes the proof.

As we will see, tails are indeed the right tool to describe the deformed counterparts of cohomology basis elements. In Lemma 11.11. we prepare for this by explicitly decomposing $\mu_{\mathbb{L}_{q}}^{1}$ with respect to the decomposition

$$
\operatorname{Hom}_{\mathbb{L}_{q}}\left(L_{1}, L_{2}\right)=\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} H\right) \oplus \mu_{\mathbb{L}_{q}}^{1}\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R\right) \oplus\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R\right)
$$

Here $H$ and $R$ refer to the standard splitting for $\mathbb{L}$ defined in section 10.3 .
Lemma 11.11. Let $\varepsilon: L_{1} \rightarrow L_{2}$ be an elementary morphism. Then modulo $R$ we have

$$
\begin{align*}
\mu_{\mathbb{L}_{q}}^{1}(\varepsilon) & =\mu^{2}(\varepsilon, \delta)+\mu^{2}(\delta, \varepsilon) \\
& +\mu_{\mathbb{L}_{q}}^{1}\left(\sum_{\substack{D \in T \\
\text { of type } \\
D \neq \varepsilon}}(-1)^{S^{D}+1} Q^{D} \beta^{D}+\sum_{\substack{D \in T \\
\text { of type G1 }}}(-1)^{S^{D}+1} Q^{D} \mathrm{id}_{2^{D} \rightarrow 5^{D}}\right) \\
& +\sum_{\substack{D \in T}}(-1)^{S^{D}+1} Q^{D}\left((-1)^{\# \alpha_{3}^{D}+1} \alpha_{3}^{D}+(-1)^{\# \alpha_{4}^{D}} \alpha_{4}^{D}\right)  \tag{11.2}\\
& +\sum_{\substack{D \in T \\
\text { of type } 10}}(-1)^{S^{D}} Q^{D}\left((-1)^{\# \alpha_{3}^{D}+1} \alpha_{3}^{D}+(-1)^{\# \alpha_{4}^{D}} \alpha_{4}^{D}\right) \\
& +\sum_{\substack{D \in T \\
\text { of type } \mathrm{H}}}(-1)^{S^{D}} Q^{D} \mathrm{id}_{2^{D} \rightarrow 5^{D}} .
\end{align*}
$$

Proof. Let us evaluate the right-hand side. We have

$$
\begin{aligned}
& \mu_{\mathbb{L}_{q}}^{1}\left(\sum_{\substack{D \in T \\
\text { of type G1 }}}(-1)^{S^{D}+1} Q^{D} \operatorname{id}_{2^{D} \rightarrow 5^{D}}\right) \\
& \quad=\sum_{\substack{D \in T \\
\text { of type G1 }}}(-1)^{S^{D}+1} Q^{D}\left((-1)^{\# \alpha_{1}^{D}+1} \alpha_{1}^{D}+(-1)^{\# \alpha_{2}^{D}} \alpha_{2}^{D}+(-1)^{\# \alpha_{3}^{D}} \alpha_{3}^{D}+(-1)^{\# \alpha_{4}^{D}+1} \alpha_{4}^{D}\right)
\end{aligned}
$$

Further for $D \in T \backslash\{\varepsilon\}$ of type E , we have modulo $R$ that

$$
\begin{aligned}
\mu_{\mathbb{L}_{q}}^{1}\left((-1)^{S^{D}+1} Q^{D} \beta^{D}\right) & =(-1)^{S^{D}+\# \gamma^{D}+\left|\beta^{D}\right|+1} Q^{D} \gamma^{D} \beta^{D}+(-1)^{S^{D}+\# \alpha^{D}} Q^{D} \beta^{D} \alpha^{D} \\
& +\sum_{\substack{E \in C_{T}(D) \\
\text { of type E }}}(-1)^{S^{D}+1} Q^{D}\left((-1)^{s^{E}+\# \gamma^{E}+\left|\beta^{E}\right|+1} q^{E} \gamma^{E} \beta^{E}+(-1)^{s^{E}+\# \alpha^{E}} q^{E} \beta^{E} \alpha^{E}\right) \\
& +\sum_{\substack{E \in C_{T}(D) \\
\text { of type G1 }}}(-1)^{S^{D}+1} Q^{D}\left((-1)^{s^{E}+\# \alpha_{2}^{E}+1} q^{E} \alpha_{2}^{E}+(-1)^{s^{E}+\# \alpha_{1}^{E}} q^{E} \alpha_{1}^{E}\right) \\
& +\sum_{\substack{E \in C_{T}(D) \\
\text { of type G2 }}}(-1)^{S^{D}+1} Q^{D}\left((-1)^{s^{E}+\# \alpha_{3}^{E}+1} q^{E} \alpha_{3}^{E}+(-1)^{s^{E}+\# \alpha_{4}^{E}} q^{E} \alpha_{4}^{E}\right) \\
& +\sum_{\substack{E \in C_{T}(D) \\
\text { of type H }}}(-1)^{S^{D}+1} Q^{D}(-1)^{s^{E}} q^{E} \operatorname{id}_{2^{E} \rightarrow 5^{E}} .
\end{aligned}
$$

Here, we have stripped off type F disks and the two first- and final-out type H disks. Both yield multiples of $\beta(\mathrm{A}), \beta(\mathrm{C})$ and $\beta^{\prime}(\mathrm{C})$, which lie in $R$. Let us now add up the right-hand side of 11.2 . This becomes a telescopic sum: The E, G2 and H terms cancel pairwise and the G2 terms cancel in triples. Only terms coming directly from the root remain. Modulo $R$, the right-hand side of 11.2 now reads

$$
\begin{aligned}
& \mu_{q}^{2}(\varepsilon, \delta)+\mu_{q}^{2}(\delta, \varepsilon) \\
& +\sum_{\substack{D \in C_{T}(\varepsilon) \\
\text { of type E }}}(-1)^{S^{D}+\# \gamma^{D}+\left|\beta^{D}\right|+1} Q^{D} \gamma^{D} \beta^{D}+(-1)^{S^{D}+\# \alpha^{D}} Q^{D} \beta^{D} \alpha^{D} \\
& +\sum_{\substack{D \in C_{T}(\varepsilon) \\
\text { of type G1 }}}(-1)^{S^{D}+\# \alpha_{2}^{D}+1} Q^{D} \alpha_{2}^{D}+(-1)^{S^{D}+\# \alpha_{1}^{D}} Q^{D} \alpha_{1}^{D} \\
& +\sum_{\substack{D \in C_{T}(\varepsilon) \\
\text { of type G2 }}}(-1)^{S^{D}+\# \alpha_{3}^{D}+1} Q^{D} \alpha_{3}^{D}+(-1)^{S^{D}+\# \alpha_{4}^{D}} Q^{D} \alpha_{4}^{D} \\
& +\sum_{\substack{D \in C_{T}(\varepsilon) \\
\text { of type H }}}(-1)^{S^{D}} Q^{D} \operatorname{id}_{2^{D} \rightarrow 5^{D}} .
\end{aligned}
$$

Modulo $R$, this is precisely $\mu_{\mathbb{L}_{q}}^{1}(\varepsilon)$. Indeed, the terms missing for $\mu_{\mathbb{L}_{q}}^{1}(\varepsilon)$ are type F disks and the two first- and final-out type H disks, which again lie in $R$. This finishes the proof.

### 11.4 Deformed cohomology basis elements

In this section, we compute the deformed cohomology basis elements of $\mathbb{L}_{q}$. The starting point is the homological splitting $H \oplus I \oplus R$ for $\mathbb{L}$ fromsection 10.3 . This splitting itself is not a homological splitting for the deformed category $\mathbb{L}_{q}$. Rather, we show that $\mathbb{L}_{q}$ together with $H \oplus I \oplus R$ falls under the " $D=0$ " case of our deformed Kadeishvili theorem studied in section 8.8. Accordingly, the category $\mathbb{L}_{q}$ comes with an associated homological splitting, including a list of deformed counterparts $\varphi^{-1}(h)$ of the cohomology basis elements $h \in H$. In the present section, we compute all these deformed cohomology basis elements.

Proposition 11.12. For the category $\mathbb{L}_{q} \subseteq \mathrm{Tw}_{\mathrm{wtl}}^{q}$ we have

$$
\mu_{\mathbb{L}_{q}}^{1}(H) \subseteq \mu_{\mathbb{L}_{q}}^{1}\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R\right)
$$

Hence the deformed Kadeishvili construction of section 8.8 applies to $\mathbb{L}_{q}$. It produces a deformed homological splitting

$$
\mathbb{L}_{q}=H_{q} \oplus \mu_{\mathbb{L}_{q}}^{1}\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R\right) \oplus\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R\right)
$$

and a minimal model $\mathrm{H} \mathbb{L}_{q}$. For each cohomology basis element $h \in H$, we obtain a deformed counterpart $\varphi^{-1}(h)$, explicitly given as follows:

- The deformed counterpart of $(-1)^{\# \alpha_{3}+1} \alpha_{3}+(-1)^{\# \alpha_{4}} \alpha_{4}(\mathrm{~B})$ is

$$
\begin{align*}
& (-1)^{\# \alpha_{3}+1} \alpha_{3}+\sum_{D \in T\left(\alpha_{3}\right) \backslash\left\{\alpha_{3}\right\}}(-1)^{\# \alpha_{3}+S^{D}+1} Q^{D} \beta^{D} \\
& +(-1)^{\# \alpha_{4}} \alpha_{4}+\sum_{D \in T\left(\alpha_{4}\right) \backslash\left\{\alpha_{4}\right\}}(-1)^{\# \alpha_{4}+S^{D}} Q^{D} \beta^{D} . \tag{11.3}
\end{align*}
$$

- The deformed counterpart of $\mathrm{id}_{2 \rightarrow 5}(\mathrm{C})$ is

$$
\begin{align*}
\mathrm{id}_{2 \rightarrow 5} & +(-1)^{\# \alpha_{1}+\# \alpha_{2}} q_{1}\left(\beta^{\prime}+\sum_{D \in T\left(\beta^{\prime}\right) \backslash\left\{\beta^{\prime}\right\}}(-1)^{S^{D}} Q^{D} \beta^{D}\right)  \tag{11.4}\\
& +(-1)^{\# \alpha_{3}+\# \alpha_{4}} q_{2}\left(\beta+\sum_{D \in T(\beta) \backslash\{\beta\}}(-1)^{S^{D}} Q^{D} \beta^{D}\right) .
\end{align*}
$$

- The deformed counterpart of $\sum_{a} \mathrm{id}_{a}(\mathrm{D})$ is

$$
\sum \mathrm{id}_{a}
$$

- The deformed counterpart of $(-1)^{\# \alpha_{0}+1} \alpha_{0}(\mathrm{D})$ is

$$
(-1)^{\# \alpha_{0}+1} \alpha_{0}+(-1)^{\# \alpha_{0}} q \alpha_{0}^{\prime}
$$

In the case of $\mathrm{id}_{2 \rightarrow 5}$, the punctures $q_{1}, q_{2} \in Q_{0}$ are the head and tail of arc 2 . In the case of $(-1)^{\# \alpha_{0}+1} \alpha_{0}$, the puncture $q$ is the one around which $\alpha_{0}$ turns. The codifferential is denoted $h_{q}$ and the projection onto $H_{q}$ is denoted $\pi_{q}$.

Proof. All the $\beta(\mathrm{A})$ and $\alpha_{0}^{\prime}(\mathrm{D})$ morphisms added in the claimed deformed cohomology basis elements lie in $\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R$. In order to show $\mu_{\mathbb{L}_{q}}^{1}(H) \subseteq \mu_{\mathbb{L}_{q}}^{1}\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R\right)$, it therefore suffices to show that $\mu_{\mathbb{L}_{q}}^{1}$ vanishes on all the four types of claimed deformed cohomology basis elements. We check this for all four types individually.

For the situation B type the vanishing amounts to applying Lemma 11.11 to $\alpha_{3}$ and $\alpha_{4}$, and adding the results. Since $\alpha_{3}$ and $\alpha_{4}$ are E-preserving, the complicated sums over type G and H disks vanish.

For the situation C type, note that $\beta(\mathrm{C})$ and $\beta^{\prime}(\mathrm{C})$ are E-preserving. Therefore two applications of Lemma 11.11 and a direct computation give the following results, whose sum renders the desired differential indeed zero:

$$
\begin{aligned}
\mu_{\mathbb{L}_{q}}^{1}\left(\beta^{\prime}+\sum_{D \in T\left(\beta^{\prime}\right) \backslash\left\{\beta^{\prime}\right\}}(-1)^{S^{D}} Q^{D} \beta^{D}\right)= & (-1)^{\# \alpha_{1}} \alpha_{1} \beta^{\prime}+(-1)^{\# \alpha_{2}+1} \beta^{\prime} \alpha_{2} \\
\mu_{\mathbb{L}_{q}}^{1}\left(\beta+\sum_{D \in T(\beta) \backslash\{\beta\}}(-1)^{S^{D}} Q^{D} \beta^{D}\right)= & (-1)^{\# \alpha_{4}} \alpha_{4} \beta+(-1)^{\# \alpha_{3}+1} \beta \alpha_{3} \\
\mu_{\mathbb{L}_{q}}^{1}\left(\mathrm{id}_{2 \rightarrow 5}\right)= & (-1)^{\# \alpha_{2}+1} q_{1} \alpha_{1} \beta^{\prime}+(-1)^{\# \alpha_{1}} q_{1} \beta^{\prime} \alpha_{2} \\
& +(-1)^{\# \alpha_{3}+1} q_{2} \alpha_{4} \beta+(-1)^{\# \alpha_{4}} q_{2} \beta \alpha_{3}
\end{aligned}
$$

For the situation D identity $\sum_{a} \mathrm{id}_{a}(\mathrm{D})$, note that no disk sequences can be made with an identity. The ordinary product of $\mathrm{id}_{a}$ with a neighboring $\alpha_{i}^{\prime}$ each appears twice in $\mu_{\mathbb{L}_{q}}^{1}$ and they cancel each other:

$$
\mu_{\mathbb{L}_{q}}^{1}\left(\sum \mathrm{id}_{a}\right)=0,
$$

For the situation D co-identity $(-1)^{\# \alpha_{0}+1} \alpha_{0}(\mathrm{D})$, note that no disks can be made of $\mu_{q}\left(\delta, \ldots, \alpha_{0}^{\prime}, \ldots, \delta\right)$ due to consistency. We get

$$
\mu_{\mathbb{L}_{q}}^{1}\left((-1)^{\# \alpha_{0}+1} \alpha_{0}+(-1)^{\# \alpha_{0}} q \alpha_{0}^{\prime}\right)=-\left(-q \alpha_{0} \alpha_{0}^{\prime}-q \alpha_{0}^{\prime} \alpha_{0}+q \alpha_{0} \alpha_{0}^{\prime}+q \alpha_{0}^{\prime} \alpha_{0}\right)=0
$$

This finishes the proof.

### 11.5 Deformed codifferential and projection

In this section, we compute part of the deformed codifferential and deformed projection for $\mathbb{L}_{q}$. The starting point is the homological splitting $H \oplus I \oplus R$ for $\mathbb{L}$. In section 11.4 we have verified that $\mathbb{L}_{q}$ satisfies the requirements of the deformed Kadeishvili construction of section 8.8 so that we obtain a deformed homological splitting $H_{q} \oplus \mu_{\mathbb{L}_{q}}^{1}\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R\right) \oplus\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R\right)$. According to Definition 8.17 there is an associated deformed codifferential $h_{q}$ given by projecting morphisms onto $\mu_{\mathbb{L}_{q}}^{1}\left(\mathbb{C} \llbracket Q_{0} \rrbracket \otimes R\right)$ and finding the $R$ preimage element under $\mu_{\mathbb{L}_{q}}^{1}$. In the present section, we examine this procedure for the morphisms $\beta \alpha$ and $\beta$ of A situations and indicate how one proceeds for other types of morphisms.

Let us recall how the (non-deformed) codifferential for $\mathbb{L}$ works. Regard an A situation, given by angles $\left(\alpha, \beta, \gamma, \beta^{\prime}\right)$. Then $\beta$ lies in $R$, while $\beta \alpha$ lies in $I+R$. According to Table 10.5 we have $h(\beta \alpha)=$ $(-1)^{\# \alpha+1} \beta$. For the deformed codifferential, we however have to add terms. It is namely not true that $\mu_{\mathbb{L}_{q}}^{1}\left((-1)^{\# \alpha+1} \beta\right)=\beta \alpha+(-1)^{\# \alpha+\# \gamma+\|\beta\|+1} \gamma \beta$. Rather, the differential $\mu_{\mathbb{L}_{q}}^{1}(\beta)$ is computed by the formula (11.2). The formula implies we have to subtract terms from the expression $(-1)^{\# \alpha+1} \beta$ in order to make its $\mu_{\mathbb{L}_{q}}^{1}$ image up to $R$-terms equal to $\beta \alpha$. In the following proposition, we compute these terms.

Proposition 11.13. Let $\left(\beta, \alpha, \gamma, \beta^{\prime}\right)$ denote an A situation $L_{1} \rightarrow L_{2}$. Denote by $T(\beta)$ the tail of $\beta$. Then we have

$$
\begin{aligned}
h_{q}(\beta \alpha)= & (-1)^{\# \alpha+1} \beta+\sum_{\substack{D \in T(\beta) \backslash\{\beta\} \\
\text { of type E }}}(-1)^{S^{D}+\# \alpha+1} Q^{D} \beta^{D}+\sum_{\substack{D \in T(\beta) \\
\text { of type G1 }}}(-1)^{S^{D}+\# \alpha+1} Q^{D} i d_{2^{D} \rightarrow 5^{D}}, \\
\varphi \pi_{q}(\beta \alpha)= & \sum_{\substack{D \in T(\beta) \\
\text { of type G1 }}}(-1)^{S^{D}+\# \alpha+1} Q^{D}\left((-1)^{\# \alpha_{3}^{D}+1} \alpha_{3}^{D}+(-1)^{\left.\# \alpha_{4}^{D} \alpha_{4}^{D}\right)}\right. \\
& +\sum_{\substack{D \in T(\beta) \\
\text { of type G2 }}}(-1)^{S^{D}+\# \alpha} Q^{D}\left((-1)^{\# \alpha_{3}^{D}+1} \alpha_{3}^{D}+(-1)^{\# \alpha_{4}^{D}} \alpha_{4}^{D}\right) \\
& +\sum_{\substack{D \in T(\beta)) \\
\text { of type H }}}(-1)^{S^{D}+\# \alpha} Q^{D} \operatorname{id}_{2^{D} \rightarrow 5^{D}} .
\end{aligned}
$$

Proof. Apply $h_{q}$ and $\varphi \pi_{q}$ on both sides of Lemma 11.11 with $\varepsilon=\beta$.
Remark 11.14. The morphisms of type $\beta \alpha(\mathrm{A})$ are the most important morphisms to which we would like to apply the deformed codifferential $h_{q}$. There are only very few cases where we need to apply $h_{q}$ to other morphisms. In fact, $h_{q}$ vanishes by definition on all deformed cohomology basis elements and all elements of $R$. The only interesting elementary morphisms which do not lie in $H_{q} \oplus\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R\right)$ are $\alpha_{4} \beta$ (C), $\beta^{\prime} \alpha_{2}(\mathrm{C})$ and $\alpha_{3} \alpha_{4}(\mathrm{~B})$. For these three types of morphisms, one obtains formulas for their $h_{q}$ and $\pi_{q}$ values by applying $\sqrt{11.2}$ to $\varepsilon=\beta(\mathrm{C})$ or $\varepsilon=\beta^{\prime}(\mathrm{C})$ or $\varepsilon=\alpha_{3}(\mathrm{~B})$. The results are formulas very similar to Proposition 11.13 In contrast to $\beta$ (A), these three morphisms have the benefit of being E-preserving. Therefore all complicated G and H terms on the right-hand side of 11.2 do not even appear and only $\mu^{2}(\delta, \varepsilon)+\mu^{2}(\varepsilon, \delta)$ and the sum over type E nodes remain.

## 12 Result components of Kadeishvili trees

In this section, we develop a first glance at the minimal model $H \mathbb{L}_{q}$. The starting point is the knowledge of $\mathbb{L}_{q}$ established in section 11 and the deformed Kadeishvili construction established insection 8. According to the deformed Kadeishvili construction, the deformed $A_{\infty}$-structure on the minimal model $\mathrm{H} \mathbb{L}_{q}$ is determined by Kadeishvili trees. The outstanding task is therefore to enumerate and analyze all results from all possible Kadeishvili trees.

In section 12.1 we explain which products are to be computed and set up notation. In section 12.2 we list possible types of morphisms resulting from Kadeishvili trees. In section 12.3 we introduce a notion of "result components" which allows us to systematically track terms arising from Kadeishvili trees. In section 12.4 we conclude the section with a semi-explicit, inductive characterization of result components:

## Individual output terms

of $\mu_{\mathrm{H} \mathbb{L}_{q}}\left(h_{k}, \ldots, h_{1}\right)$


## Result components

classified by Table 12.5

### 12.1 Kadeishvili trees

This section explains how our minimal model theorem applies to $\mathbb{L}_{q}$ specifically. We explain which trees need to be investigated, and which not. We also set up specific terminology. Note that we keep writing $\mu:=\mu_{q}:=\mu_{\text {Add }^{\prime} \operatorname{Gtl}_{q} Q}$, see Remark 11.1

The deformed Kadeishvili construction instructs us to start with the hom spaces. If $L_{1}$ and $L_{2}$ are zigzag paths, then $\operatorname{Hom}_{H \mathbb{L}_{q}}\left(L_{1}, L_{2}\right)=\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} H$, where $H$ denotes the cohomology in the standard splitting of $\mathrm{Tw} \mathrm{Gtl} Q$. The higher products on $\mathrm{H}_{q}$ are obtained as outputs of Kadeishvili trees, with $\varphi^{-1}: \mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} H \rightarrow H_{q}$ applied at all leaves, $h_{q} \mu_{\mathbb{L}_{q}}$ applied at all non-leaf nodes, and $\varphi \pi_{q} \mu_{\mathbb{I}_{q}}$ applied at the root.

We do not need to calculate all trees. In fact, we observe directly from the Kadeishvili construction that $\mu_{\mathrm{H} \mathbb{L}_{q}}$ is strictly unital, with the same unit morphisms as $\mathbb{L}_{q}$. More precisely, we already know that for every zigzag path $L$ and compatible morphism $h$ we have

$$
\begin{array}{r}
\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}=0, \\
\mu_{\mathbb{H}_{\mathbb{L}_{q}}}^{\geq 3}\left(\ldots, \mathrm{id}_{L}, \ldots\right)=0,
\end{array}
$$

$$
\begin{aligned}
& \mu_{\mathrm{H} \mathbb{L}_{q}}^{2}\left(h, \operatorname{id}_{L}\right)=h \\
& \mu_{\mathrm{H} \mathbb{L}_{q}}^{2}\left(\operatorname{id}_{L}, h\right)=(-1)^{|h|} h .
\end{aligned}
$$

It therefore suffices to regard Kadeishvili trees whose inputs are all non-identity cohomology basis elements. At this point, it is clever to have terminology available to study not only entire Kadeishvili trees, but also their subtrees. We fix the following terminology:

Definition 12.1. An h-tree is an ordered tree $T$ where each non-leaf node has at least two children, together with non-identity deformed basis cohomology elements $h_{1}, \ldots, h_{N}$ on the leaves from right to left, with $h_{i}: L_{i} \rightarrow L_{i+1}$.

A $\pi$-tree is an ordered tree $T$ with at least three nodes where each non-leaf node has at least two children, together with non-identity deformed basis cohomology elements $h_{1}, \ldots, h_{N}$ on the leaves from right to left, with $h_{i}: L_{i} \rightarrow L_{i+1}$.

Both h-trees and $\pi$-trees can be evaluated. They have results or outputs. When evaluating an h-tree, we put $h_{q} \mu$ on every non-leaf node. When evaluating a $\pi$-tree, we put $h_{q} \mu$ on every non-leaf non-root node, and $\varphi \pi_{q} \mu$ on the root.

Remark 12.2. Often we will make statements about "products". The datum of a "product" shall then typically include all of its inputs. For example "a product $\mu^{\geq 3}(\ldots)$ " refers to a choice of arity $k \geq 3$, a collection of compatible morphisms $a_{1}, \ldots, a_{k}$ and the result of the product itself.
Remark 12.3. Since we abbreviate $\mu=\mu_{q}=\mu_{\operatorname{AddGtl}_{q} Q}$, a product $\mu^{2}(a, b)$ stands simply for the product of angles and does not include discrete immersed disk terms like $\mu^{3}(\delta, a, b)$ stemming from twisted completion. Similarly, a product $\mu^{\geq 3}$ always stands for a single discrete immersed disk, and following the rule explained in Remark 12.2 it includes the datum of input morphisms some of which may be $\delta$-morphisms. Note that $\delta$-morphisms are always spelt out as $\alpha(\mathrm{D})$ or $\alpha^{\prime}(\mathrm{D})$.

Remark 12.4. We occasionally group $\beta(\mathrm{C})$ and $\beta^{\prime}(\mathrm{C})$ as $\beta / \beta^{\prime}(\mathrm{C})$ due to their similar nature. We ignore signs and deformation parameters $q \in \mathbb{C} \llbracket Q_{0} \rrbracket$ in this section. For example, we may say that a product like $\mu^{\geq 3}\left(\alpha, \alpha^{\prime}, \alpha_{3}, \ldots\right)$ is id ( D ), meaning that it is equal to some arc identity, possibly multiplied by a sign and deformation parameters. We abbreviate a situation B cohomology basis element $(-1)^{\# \alpha_{3}+1} \alpha_{3}+$ $(-1)^{\# \alpha_{4}} \alpha_{4}$ simply as $\alpha_{3}+\alpha_{4}$.

### 12.2 Possible tree output

In this section, we analyze the possible types of output of Kadeishvili trees. The starting point is the observation that the result of a Kadeishvili tree is a linear combination of elementary morphism, but not every elementary morphism can actually appear. In the present section, we compose a tight list of possible elementary morphisms that can result from Kadeishvili trees.

By nature, a $\pi$-tree can only have a linear combination of cohomology basis elements as output. Similarly, an h-tree can only have a linear combination of $R$ basis morphims as output. Every node of an h- or $\pi$-tree carries an evaluation result itself. To further narrow down on the possible output of the tree, we have to investigate what happens at every node in the tree. For instance, we claim that angle length cannot grow to infinity as we go from leaves to root. In fact, we claim there is a list $S$ of morphisms which is stable under evaluations, in the sense that any $h_{q} \mu$ applied to a sequence of morphisms from $S$ yields a morphism from $S$ again. The explicit list reads as follows:

$$
S:=\beta(\mathrm{A}), \operatorname{id}(\mathrm{B}), \alpha_{3}(\mathrm{~B}), \alpha_{4}(\mathrm{~B}), \mathrm{id}(\mathrm{C}), \beta(\mathrm{C}), \beta^{\prime}(\mathrm{C}), \mathrm{id}(\mathrm{D}), \alpha_{0}(\mathrm{D}) \text { and } \alpha_{0}^{\prime}(\mathrm{D})
$$

We claim that this list $S$ is preserved under evaluations $h_{q} \mu$. Before we prove this, let us prepare reasoning. For all three cases of $h_{q} \mu^{2}$, first-out $h_{q} \mu^{\geq 3}$ and final-out $h_{q} \mu^{\geq 3}$ evaluation, we have set up product schemes which indicate the type of output from in principle any kind of evaluations with arbitrary inputs from $S$.

These product schemes are found in Table 12.1 Figure 12.2 and 12.3 They are generally structured by the three keys $\mu, h_{q}$ and $\varphi \pi_{q}$. The schemes should universally be read as follows: A product of morphisms of given types may yield only the types of morphisms indicated in the $\mu$ row. Application of the codifferential yields the morphism indicated in the $h_{q}$ row. Of course, vanishing products are also possible. For later use, the possible results of $\varphi \pi_{q} \mu$ have been collected in the $\varphi \pi_{q}$ row. With these product schemes in mind, we are ready to prove that the list $S$ is preserved:

Lemma 12.5. Let $T$ be an h-tree. Then its output contains only $\beta$ (A), id (B), $\alpha_{3}(\mathrm{~B}), \alpha_{4}(\mathrm{~B})$, id (C), $\beta(\mathrm{C}), \beta^{\prime}(\mathrm{C}), \mathrm{id}(\mathrm{D}), \alpha_{0}(\mathrm{D})$ and $\alpha_{0}^{\prime}(\mathrm{D})$ terms.

| $m_{2} \backslash m_{1}$ |  | $\beta(\mathrm{A})$ | $\alpha_{4}(\mathrm{~B})$ | $\alpha_{3}(\mathrm{~B})$ | $\beta / \beta^{\prime}(\mathrm{C})$ | $\alpha_{0}(\mathrm{D})$ | $\alpha_{0}^{\prime}(\mathrm{D})$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\beta$ | $\mu^{2}=$ | imp | imp | $\beta(\mathrm{A}) / \beta^{\prime}(\mathrm{C})$ | imp | $\beta \alpha(\mathrm{A})$ | imp |
|  | $h_{q}=$ | 0 | 0 | 0 | 0 | $\beta(\mathrm{A})+\mathrm{E} / \mathrm{G}$ | 0 |
|  | $\varphi \pi_{q}=$ | 0 | 0 | 0 | 0 | G/H | 0 |
| $\alpha_{4}$ | $\mu^{2}=$ | $\beta^{\prime}(\mathrm{C}) / \beta(\mathrm{A})$ | imp | imp | $\alpha^{\prime}(\mathrm{D})$ | imp | imp |
|  | $h_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
|  | $\varphi \pi_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\alpha_{3}$ | $\mu^{2}=$ | imp | imp | imp | imp | imp | imp |
|  | $h_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
|  | $\varphi \pi_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\beta / \beta^{\prime}$ | $\mu^{2}=$ | imp | imp | $\alpha^{\prime}$ | imp | $\beta \alpha_{3} / \mathrm{imp}$ | imp |
|  | $h_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
|  | $\varphi \pi_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\alpha_{0}$ | $\mu^{2}=$ | $\gamma \beta$ | $\alpha_{3} \alpha_{4}$ | imp | $\operatorname{imp} / \alpha_{1} \beta^{\prime}$ | imp | $\alpha_{0} \alpha_{0}^{\prime}$ |
|  | $h_{q}=$ | 0 | $\alpha_{3}+\mathrm{E}$ | 0 | 0 | 0 | $\alpha_{0}^{\prime}$ |
|  | $\varphi \pi_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\alpha_{0}^{\prime}$ | $\mu^{2}=$ | imp | imp | $\alpha_{4} \beta \alpha_{3}$ | imp | $\alpha_{0}^{\prime} \alpha_{0}$ | imp |
|  | $h_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
|  | $\varphi \pi_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
| id(B) |  | imp | imp | $\beta(\mathrm{A})$ | imp | $\alpha_{1}$ | imp |
|  | $h_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
|  | $\varphi \pi_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
| id(C) | $\mu^{2}=$ | $\gamma \beta / \alpha_{2} / \alpha_{3}$ | $\alpha$ (D) | imp | $\gamma \beta$ | imp | $\alpha_{4} \beta$ |
|  | $h_{q}=$ | id(B) | $\mathrm{id}(\mathrm{D})$ | 0 | 0 | 0 | $\beta(\mathrm{C})+\mathrm{E}$ |
|  | $\varphi \pi_{q}=$ | $0 / \alpha_{3}+\alpha_{4}$ | $\alpha_{0}$ | 0 | 0 | 0 | 0 |
| id(D) | $\mu^{2}=$ | $\beta(\mathrm{A})$ | $\alpha_{4}(\mathrm{~B})$ | $\alpha_{3}(\mathrm{~B})$ | $\beta / \beta^{\prime}(\mathrm{C})$ | $\alpha_{0}(\mathrm{D})$ | $\alpha_{0}^{\prime}(\mathrm{D})$ |
|  | $h_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
|  | $\varphi \pi_{q}=$ | 0 | $\alpha_{3}+\alpha_{4}$ | 0 | 0 | $\alpha_{0}(\mathrm{D})$ | 0 |

Table 12.1: Multiplication scheme

Proof. Let $S$ be the set of all elementary morphisms of these types. We prove the claim by induction on the tree size. For an h-tree with just one node, leaf and root at the same time, there is nothing to show, since the result of this tree is the deformed basis cohomology element itself which only contains terms from $S$.

Now for any arbitrarily large tree, output components are of the form

$$
\begin{equation*}
h_{q}\left(\mu^{\geq 3}\left(m_{k}, \ldots, m_{1}\right)\right) \text { and } h_{q}\left(\mu^{2}\left(m_{2}, m_{1}\right)\right), \tag{12.1}
\end{equation*}
$$

where by induction hypothesis each $m_{i}$ is from $S$, or may in case of $\mu^{\geq 3}$ also be a $\delta$ insertion. We will now check all possible terms that can occur in 12.1.

In the case of a discrete immersed disk $\mu^{\geq 3}\left(m_{k}, \ldots, m_{1}\right)$, assume the disk is all-in. Then the $\mu^{\geq 3}$ result is an identity from situation $\mathrm{B}, \mathrm{C}$ or D . Its image under $h_{q}$ vanishes in all three cases.

Assume now the disk is first-out. Then the first morphism is by assumption one in $S$. In particular, it is strictly smaller than one full turn. Since the disk is first-out, the first morphism $m_{1}$ consists of at least two indecomposable angles. We can now compute the $\mu^{\geq 3}$ result on a case-by-case basis, distinguishing after the type of $m_{1}$. The results are shown in Figure 12.2. The figures omit the case of $\alpha_{0}^{\prime}$ which is similar to that of an outer $\delta$ insertion. Hatching indicates the interior of the disk. Similarly, the results for final-out disks are shown in Figure 12.3

We conclude that for any discrete immersed disk, the output $h_{q}\left(\mu^{\geq 3}\left(m_{k}, \ldots, m_{1}\right)\right)$ consists only of terms lying in $S$. Next, we check the terms occurring in a simple composition $\mu^{2}\left(m_{2}, m_{1}\right)$ ). Table 12.1 contains the results of such multiplications, "imp" denoting an impossible combination, hence vanishing product. The table also lists their images under $h_{q}$, abbreviating tail terms $\beta(\mathrm{A})$ as +E and tail terms id (B) as +G . We conclude that as long as factors lie in $S$, their image under $h_{q} \mu^{2}$ also has components only in $S$.

In the product tables, the expression $+G / H$ appears under the $\varphi \pi_{q}$ key. We have used this abbreviation

| $m_{2} \backslash m_{1}$ |  | id (B) | id (C) | id (D) |
| :---: | :---: | :---: | :---: | :---: |
| $\beta$ (A) | $\mu^{2}=$ | imp | $\beta \alpha / \alpha_{1} / \alpha_{4}$ | $\beta$ (A) |
|  | $h_{q}=$ | 0 | $\beta(\mathrm{A})+\mathrm{E} / \mathrm{G}$ | 0 |
|  | $\varphi \pi_{q}=$ | 0 | $\mathrm{G} / \mathrm{H} / \alpha_{3}+\alpha_{4}$ | 0 |
| $\alpha_{4}$ | $\mu^{2}=$ | $\beta$ (A) | imp | $\alpha_{4}$ |
|  | $h_{q}=$ | 0 | 0 | 0 |
|  | $\varphi \pi_{q}=$ | 0 | 0 | $\alpha_{3}+\alpha_{4}$ |
| $\alpha_{3}$ | $\mu^{2}=$ | imp | $\alpha$ (D) | $\alpha_{3}$ |
|  | $h_{q}=$ | 0 | id (D) | 0 |
|  | $\varphi \pi_{q}=$ | 0 | $\alpha_{0}$ | 0 |
| $\beta / \beta^{\prime}(\mathrm{C})$ | $\mu^{2}=$ | imp | $\beta \alpha$ | $\beta / \beta^{\prime}(\mathrm{C})$ |
|  | $h_{q}=$ | 0 | $\beta(\mathrm{A})+\mathrm{E} / \mathrm{G}$ | 0 |
|  | $\varphi \pi_{q}=$ | 0 | $\mathrm{G} / \mathrm{H}$ | 0 |
| $\alpha_{0}$ | $\mu^{2}=$ | $\alpha_{3}$ | imp | $\alpha_{0}$ |
|  | $h_{q}=$ | 0 | 0 | 0 |
|  | $\varphi \pi_{q}=$ | 0 | 0 | $\alpha_{0}$ |
| $\alpha_{0}^{\prime}$ | $\mu^{2}=$ | imp | $\beta^{\prime} \alpha_{2}$ | $\alpha_{0}^{\prime}$ |
|  | $h_{q}=$ | 0 | $\beta^{\prime}(\mathrm{C})+\mathrm{E}$ | 0 |
|  | $\varphi \pi_{q}=$ | 0 | 0 | 0 |
| id(B) | $\mu^{2}=$ | imp | id (D) | id (B) |
|  | $h_{q}=$ | 0 | 0 | 0 |
|  | $\varphi \pi_{q}=$ | 0 | 1 if $a=a_{0}$ | 0 |
| id(C) | $\mu^{2}=$ | id (D) | imp | id (C) |
|  | $h_{q}=$ | 0 | 0 | 0 |
|  | $\varphi \pi_{q}=$ | 1 if $a=a_{0}$ | 0 | id (C) |
| id(D) | $\mu^{2}=$ | id (B) | id (C) | id (D) |
|  | $h_{q}=$ | 0 | 0 | 0 |
|  | $\varphi \pi_{q}=$ | 0 | id (C) | 1 if $a=a_{0}$ |

Table 12.1: Multiplication scheme (continued)
$+\mathrm{G} / \mathrm{H}$ to denote the terms of $\pi_{q}(\beta \alpha)$ according to Proposition 11.13. In other words, $+\mathrm{G} / \mathrm{H}$ simply denotes tail terms of the form $\alpha_{3}+\alpha_{4}(\mathrm{~B})$ and id (C).

### 12.3 Result components

In this section, we introduce our notion of "result components". The reason for this notion is that any evaluation $h_{q} \mu$ in a Kadeishvili tree may in principle yield a large number of terms. Any single of these terms may yield multiple terms again upon the next evaluation in the tree. The notion of result components serves to get grip on these terms. After the definition, we provide some terminology and a few examples.

Before we state the precise definition, let us illustrate the idea: Regard an evaluation task like computing $(3 x+5 y)(2 x+3 y)$ or even $(3 x+5 y)(2 x+3 y)(x-y)$. These evaluations can be represented by the trees


The result expression $6 x^{2}+19 x y+15 y^{2}$ is concise, but does not include information on how it was derived from the individual factors. The idea behind result components is to retain this information instead. For example, the left tree should have four result components:

(a) $\delta$ insertion
$\mu=\gamma \beta$
$h_{q}=0$
$\varphi \pi_{q}=0$

(e) $\beta$ or $\beta^{\prime}$ (C)
$\mu=\gamma \beta$
$h_{q}=0$
$\varphi \pi_{q}=0$


(b) $\delta$ insertion
$\mu=\alpha_{2}$ or $\alpha_{3}$ (B)
$h_{q}=$ id (B) or 0
$\varphi \pi_{q}=\alpha_{3}+\alpha_{4}$

(f) $\beta$ or $\beta^{\prime}(\mathrm{C})$
$\mu=\alpha_{2}$ or $\alpha_{3}$
$h_{q}=\mathrm{id}$ (B) or 0
$\varphi \pi_{q}=\alpha_{3}+\alpha_{4}$


$$
\begin{array}{ll}
(\mathrm{j}) \beta(\mathrm{A}) & (\mathrm{k}) \beta(\mathrm{A}) \\
\mu=\alpha_{2} \text { or } \alpha_{3} & \mu=\beta(\mathrm{A}) \\
h_{q}=\mathrm{id}(\mathrm{~B}) \text { or } 0 & h_{q}=0 \\
\varphi \pi_{q}=\alpha_{3}+\alpha_{4} & \varphi \pi_{q}=0
\end{array}
$$


(c) $\delta$ insertion
$\mu=\beta$ (A)
$h_{q}=0$
$\varphi \pi_{q}=0$

(g) $\beta$ or $\beta^{\prime}(\mathrm{C})$
$\mu=\beta$ (A)
$h_{q}=0$
$\varphi \pi_{q}=0$


(d) $\delta$ insertion
$\mu=\beta$ or $\beta^{\prime}(\mathrm{C})$
$h_{q}=0$
$\varphi \pi_{q}=0$

(h) $\beta$ or $\beta^{\prime}$ (C)
$\mu=\beta$ (A)
$h_{q}=0$
$\varphi \pi_{q}=0$


Figure 12.2: Possible result components of first-out disks in Kadeishvili trees
$6 x^{2}$ derived from multiplying $3 x$ and $2 x$, $10 x y$ derived from multiplying $5 y$ and $2 x$,
$9 x y$ derived from multiplying $3 x$ and $3 y$,
$15 y^{2}$ derived from multiplying $5 y$ and $3 y$.

In other words, the left tree has four distinct result components, even though the result can be abbreviated to only three terms. The tree on the right has just four result terms, while there are eight distinct result components. For example, one of these eight result components consists of the choice of $3 x$ on the leftmost leaf, $2 x$ on the middle leaf and $-y$ on the rightmost leaf.

Let us prepare for result components of Kadeishvili trees: In contrast to the simple multiplication trees above, the leaves of a Kadeishvili tree are labeled by deformed cohomology basis elements. These elements consist of a finite or infinite amount of additive components. For example, let $\alpha_{3}+\alpha_{4}$ denote a certain cohomology basis element and assume both $\alpha_{3}$ and $\alpha_{4}$ have a tail each consisting of one type E disk with $\beta$ morphisms denoted $\beta_{1}$ and $\beta_{2}$ respectively. Then the deformed cohomology basis element reads $\alpha_{3}+\alpha_{4}+\beta_{1}+\beta_{2}$ and is defined to have four distinct additive components, even though technically it may be possible that $\beta_{1}=\beta_{2}$.

Similarly, node evaluations in Kadeisvhili trees may yield a large amount of additive components. For example, an evaluation $h_{q}(\beta \alpha)$ may yield an expression like $\beta+\beta_{1}+\beta_{2}+\mathrm{id}(\mathrm{B})$ according to Proposition 11.13 This evaluation is defined to have four distinct additive components. In other words, an additive component always refers to one of the main terms or a choice of one of the tail terms. We are finally ready to define result components of Kadeishvili trees:

Definition 12.6. The restriction of an h-tree or $\pi$-tree $\left(T, h_{1}, \ldots, h_{N}\right)$ at a non-root node $P \in T$ is the h-tree from $P$ up to all leaves, together with the corresponding subset of $\left(h_{1}, \ldots, h_{N}\right)$. A result component of an h-tree or $\pi$-tree is defined inductively as follows:

- A result component of an h-tree with only one node consists of an additive component appearing in the corresponding $h_{1}$.

(a) $\delta$ insertion
$\mu=\beta \alpha$
$h_{q}=\beta(\mathrm{A})+\mathrm{E}$ tail $\varphi \pi_{q}=\mathrm{G} / \mathrm{H}$

(i) $\beta$ (A)
$\mu=\beta \alpha$
$h_{q}=\beta(\mathrm{A})+\mathrm{E}$ tail $\varphi \pi_{q}=\mathrm{G} / \mathrm{H}$

(b) $\delta$ insertion
$\mu=\alpha_{1}$ or $\alpha_{4}$
$h_{q}=0$
$\varphi \pi_{q}=0$ or $\alpha_{3}+\alpha_{4}$

(f) $\beta / \beta^{\prime}(\mathrm{C})$
$\mu=\alpha_{1}$ or $\alpha_{4}$
$h_{q}=0$
$\varphi \pi_{q}=0$ or $\alpha_{3}+\alpha_{4}$

(j) $\beta$ (A)
$\mu=\alpha_{1}$ or $\alpha_{4}$
$h_{q}=0$
$\varphi \pi_{q}=0$ or $\alpha_{3}+\alpha_{4}$

(c) $\delta$ insertion
$\mu=\beta$ ( A )
$h_{q}=0$
$\varphi \pi_{q}=0$

(g) $\beta / \beta^{\prime}(\mathrm{C})$
$\mu=\beta$ (A)
$h_{q}=0$
$\varphi \pi_{q}=0$

$(\mathrm{k}) \beta(\mathrm{A})$
$\mu=\beta(\mathrm{A})$
$h_{q}=0$
$\varphi \pi_{q}=0$

(d) $\delta$ insertion
$\mu=\beta / \beta^{\prime}$ (C)
$h_{q}=0$
$\varphi \pi_{q}=0$

(h) $\beta / \beta^{\prime}$ (C) $\mu=\beta$ (A)
$h_{q}=0$
$\varphi \pi_{q}=0$


Figure 12.3: Possible result components of final-out disks in Kadeishvili trees

- A result component of an h-tree with at least three nodes consists of result components $r_{1}, \ldots, r_{k}$ of the restrictions at the ordered children of the root, together with choices $n_{0}, \ldots, n_{k} \geq 0$ of $\delta$ insertions, and an additive component appearing in

$$
h_{q} \mu_{q}(\underbrace{\delta, \ldots, \delta}_{n_{k}}, r_{k}, \ldots, r_{1}, \underbrace{\delta, \ldots, \delta}_{n_{0}})
$$

- A result component of a $\pi$-tree consists of result components $r_{1}, \ldots, r_{k}$ of the restrictions at the ordered children of the root, together with choices $n_{0}, \ldots, n_{k} \geq 0$ of $\delta$ insertions, and an additive component appearing in

$$
\varphi \pi_{q} \mu_{q}(\underbrace{\delta, \ldots, \delta}_{n_{k}}, r_{k}, \ldots, r_{1}, \underbrace{\delta, \ldots, \delta}_{n_{0}}) .
$$

- For $\pi$-trees, the result components $(-1)^{\# \alpha_{3}} \alpha_{3}$ and $(-1)^{\# \alpha_{4}+1} \alpha_{4}$ shall be grouped together as one result component. Also, the result components id (D) shall be grouped together as one result component.
- Additive components arising from different tail nodes in the evaluation of $h_{q}$ or $\varphi \pi_{q}$ shall be kept distinct as result components.
A direct morphism is a result component of a one-node h-tree. A result component derives from the result components $r_{1}, \ldots, r_{k}$, and from all the result components they derive from themselves. A direct morphism derives from nothing. A tail result component is one that comes from a tail additive component of the final $h_{q}$ or $\varphi \pi_{q}$ evaluation. Tail additive components of direct morphisms are also counted as tail result components. Any other result component is a main result component. The class of all result components of $\pi$-trees is denoted Result ${ }_{\pi}$.



Figure 12.4: A $\pi$-tree with a concrete result component

(a) Tree with $\alpha_{3}$ as result component

(c) Trees with $\beta$ (C) as result component.

(b) Trees with $\alpha_{0}^{\prime}$ as result component

$$
\geq 0
$$


(d) Trees with $\beta^{\prime}(\mathrm{C})$ as result component

Figure 12.6: Classification of $\alpha_{3}, \alpha_{0}^{\prime}, \beta(\mathrm{C})$ and $\beta^{\prime}(\mathrm{C})$ result components

Example 12.7. Regard an evaluation $h_{q}(\beta \alpha)$. Its result consists of a tower of $\beta$ (A) morphisms. They need not be distinct as morphisms, but shall be treated as distinct result components. The $\beta$ (A) with the lowest $q$ power is the main result component, all others are tail result components.

Example 12.8. A sample $\pi$-tree with a concrete result component is depicted in Figure 12.4 . The inputs of this $\pi$-tree are four $\beta_{3}+\beta_{4}$ morphisms, corresponding to the four intersection points between the zigzag curves. One of the zigzag paths is denoted $L$. The first input morphism departs from $L$ and the fourth ends on $L$. The angles depicted are the main $\alpha_{4}$ components of the first, third and fourth input, as well as the first tail component of the second input. The double stroke on the rightmost arrow indicated the separation between the first and the fourth morphisms of the sequence. In case the rightmost arc is the identity location of the first/final zigzag path, then the output of the $\pi$-tree is the identity. Otherwise the output vanishes. This identity result component is derived from the main components of the first, the first tail component of the second, and the main components of the third and fourth input morphisms, with $n_{1}=3$ many $\delta$ 's after the first morphism and $n_{2}=2$ many $\delta$ 's after the second morphism. This example illustrates a nontrivial result component of a $\pi$-tree and shows how tail components lead to results. In contrast, no single result component of this $\pi$-tree derives from the main components of all four input morphisms.

### 12.4 Classification of result components

In this section, we provide a semi-explicit, inductive characterization of result components of Kadeishvili trees. To understand what this means, recall from section 12.2 that only certain types of morphisms can appear as result components. For each of these types, we will describe all possible in which it is derived from simpler result components. This description is recursive, and has to remain so until we match result components with pieces of smooth immersed disks later on.

For example, regard a result component $\alpha_{3}$ of an h-tree. We are interested in how this $\alpha_{3}$ can possibly have been derived. A glance at the multiplication and disk tables 12.1 and $12.2,12.3$ reveals that it must be a product $h_{q} \mu^{2}\left(\alpha_{0}, \alpha_{4}\right)$. In turn, we are interested in how $\alpha_{0}$ and $\alpha_{4}$ could have been derived. Another glance at the multiplication and disk tables reveals that both are necessarily direct. We conclude that any result component $\alpha_{3}$ of an h-tree is necessarily the result component of the tree $h_{q} \mu^{2}\left(\alpha_{0}, \alpha_{3}+\alpha_{4}\right)$ with leaves $\alpha_{0}$ and $\alpha_{3}+\alpha_{4}$.


Figure 12.7: Trees with $\beta(\mathrm{A})$ as result component. In the first and second tree, the framed part is essential and further id (C) and $\alpha_{0}$ inputs are optional. In the third tree, at least one id (C) or $\alpha_{0}$ is required and further ones are optional. The $\beta(\mathrm{A})$ on the left is supposed to be a direct, $h_{q} \mu^{\geq 3}$ or tail $h_{q} \mu^{2}$ result component.


Figure 12.8: Trees with $\alpha_{3}+\alpha_{4}$ as main result component of $\pi_{q} \mu^{2}$

$$
\begin{array}{cccc}
\mathrm{id}(\mathrm{C}) \beta(\mathrm{A}) & \beta(\mathrm{A}) \mathrm{id}(\mathrm{C}) & \beta / \beta^{\prime} \mathrm{id}(\mathrm{C}) & \beta(\mathrm{A}) \quad \alpha_{0} \\
h_{q} \mu^{2}=\mathrm{id}(\mathrm{~B})(\text { main }) & h_{q} \mu^{2}=\mathrm{id}^{\prime}(\mathrm{B})(\mathrm{tail}) & h_{q} \mu^{2}=\mathrm{id}^{\prime}(\mathrm{B})(\text { tail }) & h_{q} \mu^{2}=\mathrm{id}^{\prime}(\mathrm{B})(\text { tail })
\end{array}
$$

Figure 12.9: Trees with id (B) as result component. No subdisk is assigned, but the trees are used for trees of id (D).

(a) This tree makes use of an id (B) result component. In the subdisks depicted here, the id (B) component comes from a first-out disk or $h_{q} \mu^{2}(\mathrm{id}(\mathrm{C}), \beta(\mathrm{A}))$ or $h_{q} \mu^{2}(\beta(\mathrm{~A}), \mathrm{id}(\mathrm{C}))$. Other options are impossible.

(b) The same tree as in Figure 12.10a but with id (C) at the end. The subdisk is depicted in case id (B) comes from a first-out disk or a product $h_{q} \mu^{2}(\beta(\mathrm{~A})$, id (C)).

(c) An all-in disk produces an id (D) morphism if its concluding arc is $a_{0}$. The subdisk is obtained by tying all handles together along the disk and inserting a short segment with an output mark between beginning and end. The two pictures show how we insert this segment, depending on whether the zigzag path turns left or right at the concluding arc. For orientation, the first and final interior angles of the disk are drawn. Note that due to Remark 13.4 their handles approach the concluding arc indeed in the way drawn, and the subdisk becomes smooth.

Figure 12.10: Trees with id (D) as result component

(a) Two trees with very similar subdisks: the first tree has the upper output mark, the second tree has the lower output mark. The subdisk itself is a strip between a zigzag curve and its Hamiltonian deformation. The position of its left and right boundary depend on the id (D) component chosen as result component in $\mu^{2}\left(\alpha_{3}\right.$, id (C)). A long version, where the left and right boundary are maximally distant from each other, is depicted in the upper image, and a short version in the lower.

(b) The first tree has the upper output mark, the second tree has the lower output mark. A long and a short version are depicted.

| $\alpha_{4} \quad \alpha_{3} \quad$ id (C) | $\alpha_{4} \quad$ id (C) $\quad \alpha_{4}$ | $\alpha_{3} \quad$ id (C) id (C) | id (C) $\alpha_{4}$ | id (C) |
| :---: | :---: | :---: | :---: | :---: |
| id (D) | >id $\left(\begin{array}{l}(\mathrm{D})\end{array}\right.$ | id (D) / | $\text { id }(\mathrm{D})$ | $/$ |
| $\varphi \pi_{q} \mu^{2}=\alpha_{3}+\alpha_{4}$ | $\varphi \pi_{q} \mu^{2}=\alpha_{3}+\alpha_{4}$ | $\varphi \pi_{q} \mu^{2}=\mathrm{id}(\mathrm{C})$ | $\varphi \pi_{q} \mu^{2}$ | id (C) |

Figure 12.11: A group of 8 trees with $\alpha_{3}+\alpha_{4}$ and id (C) as result components that produce degenerate subdisks. The subdisks are depicted for the first four trees. Subdisks of the other four trees are similar.

| Type | Cue | Possible ways of derivation |
| :---: | :---: | :---: |
| $\alpha_{0}$ | h | direct |
| id (C) | h | direct |
| $\alpha_{4}$ | h | direct |
| $\alpha_{3}$ | h | direct or Figure 12.6a |
| $\alpha_{0}^{\prime}$ | h | direct or Figure 12.6b |
| $\beta / \beta^{\prime}(\mathrm{C})$ | h | direct or Figure 12.6 or 12.6 d |
| $\beta$ (A) | direct | tail of some $\alpha_{3}, \alpha_{4}, \beta(\mathrm{C})$ or $\beta^{\prime}(\mathrm{C})$ |
| $\beta$ (A) | main $h_{q} \mu^{\geq 3}$ | final-out disk, with final morphism an outer $\delta$ insertion, $\beta(\mathrm{A}), \beta / \beta^{\prime}(\mathrm{C})$ or $\alpha_{0}^{\prime}$ |
| $\beta$ (A) | tail $h_{q} \mu$ | comes with corresponding main result component $\beta$ (A), $\beta / \beta^{\prime}(\mathrm{C})$ or $\alpha_{3}$, example see Figure 13.6 b |
| $\beta$ (A) | main $h_{q} \mu^{2}$ | Figure 12.7 |
| id (D) | h | Figure 13.7a or 13.7b |
| id (B) | h | Figure 12.2 or 12.9 |
| $\alpha_{3}+\alpha_{4}$ | main $\varphi \pi_{q} \mu^{\geq 3}$ | Figure 12.2 or 12.3 |
| $\alpha_{3}+\alpha_{4}$ | main $\varphi \pi_{q} \mu^{2}$ | Figure 12.8 or 12.11 |
| $\alpha_{3}+\alpha_{4}$ | tail $\varphi \pi_{q} \mu$ | tail of a certain $\varphi \pi_{q}(\beta \alpha)$, with $\beta \alpha$ itself being a $\mu^{2}$ product or a disk of Figure 12.3 |
| id (C) | main $\varphi \pi_{q} \mu^{2}$ | Figure 12.11 |
| id (C) | main $\varphi \pi_{q} \mu^{\geq 3}$ | all-in disk of type H , whose inner morphisms may be $\delta$ insertions, $\beta$ (A), $\alpha_{3}(\mathrm{~B}), \alpha_{4}(\mathrm{~B}), \beta / \beta^{\prime}(\mathrm{C}), \alpha_{0}(\mathrm{D})$, $\alpha_{0}^{\prime}(\mathrm{D})$, example see Figure 13.6e |
| id (C) | tail $\varphi \pi_{q} \mu$ | tail of a certain $\varphi \pi_{q}(\beta \alpha)$ evaluation |
| id (D) | $\pi$ | Figure 12.10 |
| $\alpha_{0}$ | $\pi$ | Figure 12.12 |

Table 12.5: Classification of result components

We have conducted this investigation for all types of morphisms, resulting in the classification of Table 12.5 Let us explain here how to read this table: The first and second column specify a type of result component. More precisely, the first column fixes the type of morphism. The second column sets further conditions on the type of result component. For example, the second column may indicate that only result components of h-trees shall be considered, or only tail result components of $\varphi \pi_{q} \mu^{\geq 3}$. The third column then provides a list of ways in which a result component of the specified type can be derived.

For example, we have seen before that an $\alpha_{3}$ result component of an h-tree is necessarily direct or derived from the tree in Figure 12.6a This is reflected in the fourth row of the classification table. As another example, we read off from the table that an id (C) main result component of $\varphi \pi_{q} \mu^{2}$ is necessarily a result component of one of the four id (C) trees in Figure 12.11.

As a final example, our classification of $\beta$ (A) tail result components of h-trees is relatively implicit: These result components come from an evaluation $h_{q}(\beta \alpha), h_{q}\left(\alpha_{4} \beta\right), h_{q}\left(\beta^{\prime} \alpha_{2}\right)$ or $h_{q}\left(\alpha_{3} \alpha_{4}\right)$. These evaluations produce a $\beta(\mathrm{A}), \beta(\mathrm{C}), \beta^{\prime}(\mathrm{C})$ or $\alpha_{3}(\mathrm{~B})$ main result component. The tail $\beta$ (A) result component then sits at the tail of these morphisms. In other words, whenever we encounter a tail $\beta$ (A) result component of an h-tree, we will make reference to its associated main result components for further inspection.

Remark 12.9. Let us comment on two specific cases of Table 12.5 Both id (C) and $\alpha_{3}+\alpha_{4}$ tail result component of $\varphi \pi_{q} \mu$ necessarily come from a certain $\varphi \pi_{q}(\beta \alpha)$ evaluation. The result component $\beta \alpha$ itself produces also main and tail result components from $h_{q}(\beta \alpha)$. We will use this observation later as a tool to abbreviate the construction of subdisks.

Remark 12.10. The figures referenced in Table 12.5 show more than only trees: They depict trees and subdisks side-by-side. At present, the subdisks may simply be ignored and only trees count. We have

(a) The inputs of these two trees consists $\alpha_{3}$ (B) and id (C) lying on the same arc, and the co-identity. The evaluation of $h_{q}$ in the tree gives a sum of identities id (D) ranging over all arcs lying between the arc and the co-identity. The subdisk is depicted for the second tree. It is a wedge lying between the zigzag curve and its Hamiltonian deformation. Since $\alpha_{0}$ is the first input, the wedge lies on the side of $\alpha_{0}$ where $\alpha_{0}$ points to. In the case of the figure it lies to the right of $\alpha_{0}$.

(b) The subdisk of the first tree is depicted. Since $\alpha_{0}$ is the last input, the wedge lies on the opposite side of where $\alpha_{0}$ points to. In the case of the figure it lies to the left of $\alpha_{0}$.

$$
\begin{aligned}
\text { id (C) } \begin{array}{ll}
\alpha_{4} & \alpha_{3} \\
\backslash & \text { id (C) } \\
\vdots \\
\varphi \pi_{q} \mu^{2}=\alpha_{0} & \varphi \pi_{q} \mu^{2}=\alpha_{0}
\end{array}
\end{aligned}
$$


(c) These two trees have only two inputs and yield the co-identity directly. The subdisk is depicted for the second tree and features a sample case where the two inputs and the output lie maximally far apart, namely at the identity and at the co-identity.

(d) This tree has the special feature that all its angles neighbor one single arc. The subdisk is tiny and concentrated around the midpoint of this arc.

Figure 12.12: Trees with $\alpha_{0}$ as result component
chosen this way of presentation to facilitate retrospection during the reading of section 13
Lemma 12.11. The result components classification of Table 12.5 is complete: The named types of result components can only be derived in the given way.

Proof. The checks are detailed in section C. 1

## 13 From trees to disks

In this section, we show how to transform a result component of a $\pi$-tree into a kind of smooth immersed disk. Simply speaking, we draw all intersection points and connect them in a way dictated by the result component. The result is a matching between result components and certain types of smooth immersed disks. It leads to our main theorem which is a precise characterization of the minimal model $\mathrm{H} \mathbb{L}_{q}$ in terms of smooth immersed disks.

Result components with inputs $h_{1}, \ldots, h_{N}$
$\stackrel{\text { Subdisk mapping D }}{\longleftrightarrow}$

CR, ID, DS, DW disks<br>with inputs $h_{1}, \ldots, h_{N}$

Our bijection between result components of $\pi$-trees and smooth immersed disks is denoted D. The domain of the mapping D is the set Result $\mathrm{t}_{\pi}$ of result components of $\pi$-trees. The precise terminology is that D sends a result component $r \in$ Result $_{\pi}$ to its associated "subdisk" $\mathrm{D}(r)$. We define the mapping D inductively over tree size, by also defining subdisks for result components of h-trees: If a result component $r$ derives from certain result components $r_{1}, \ldots, r_{N}$ closer to the leaves, then the subdisk of $r$ is defined by gluing the subdisks of $r_{1}, \ldots, r_{N}$.

In section 13.1 we introduce a protocol which lays down how subdisks may be glued together along handles. This way, subdisks of $\pi$-trees are closed disks, while subdisks of h-trees rather look like half a disk. In section 13.2 we introduce a precise container class Disk ${ }_{\text {SL }}$ meant to capture the shape of subdisks of $\pi$-trees. In section 13.3, we show explicitly how to draw the subdisk associated with a result component. In section 13.4 we define the subdisk mapping D : Result $\rightarrow$ Disk $_{\text {SL }}$ and classify its image. The class of disks reached by D decomposes into four visually distinguished types: CR, ID, DS and DW. In section 13.5 , we finish our computational journey and state our precise description of the minimal model $H \mathbb{L}_{q}$. In section 13.6, we state our main theorem. The proofs of intermediate classification results and sign computations have been placed in section C We continue using the shorthand $\mu:=\mu_{q}:=\mu_{\text {Add Gtl }_{q} Q}$, see Remark 11.1

### 13.1 The subdisk protocol

In this section, we introduce our protocol for subdisk handles. The purpose of this protocol is to give an accurate description of the handles with which we will glue subdisks together. Recall from section 7.5 that every zigzag path $L$ comes with an associated zigzag curve $\tilde{L}$. The subdisk of a result component of an h-tree should consist of a sequence of intersection points and segments of the zigzag curves involved, filled with half a disk. While the intersection points and zigzag segments can be located anywhere on the dimer, both endpoints of the sequence should be located near the value of the result component itself. For a given type of result component, we wish that the endpoints follow a predictable pattern to facilitate gluing of subdisks. The protocol presented here is meant to define this local pattern, although we will give no precise definition what kind of object a subdisk of an h-tree is from a global view.

Remark 13.1. The subdisk protocol enjoys the following characteristics:

- The protocol applies to every type of morphism that can appear as result component of h-trees, namely $\alpha_{3}, \alpha_{4}, \beta / \beta^{\prime}(\mathrm{C}), \alpha_{0}, \alpha_{0}^{\prime}$ and $\beta(\mathrm{A})$.
- For each morphism $\varepsilon: L_{1} \rightarrow L_{2}$ of these types, the protocol defines a germ (small interval) of the zigzag curves $\tilde{L}_{1}$ and $\tilde{L}_{2}$.
- Every germ comes with a handle. A handle is an indication which endpoint is its gluable outside, and an indication which surface side is regarded as disk inside and which as disk outside.

With these characteristics in mind, the protocol is defined in Figure 13.1. The germ intervals are drawn thickly, the gluable endpoints are drawn by dots and the disk inside is drawn hatched. Only $\beta$ (A) comes in two variants: a short and a long version. We use the short version for tail components of $h_{q} \mu^{2}$ and all components of $h_{q} \mu^{\geq 3}$, and the long version for direct morphisms and inputs of $h_{q} \mu^{\geq 3}$. We will explain the reason of this distinction in Remark 13.2

Remark 13.2. The distinction between short and long version of $\beta$ ( A ) protocol is due to a general phenomenon of subdisks. Namely, we will glue subdisks together by prolonging and subsequently connecting their handles. Sometimes, morphisms lie so close to each other that their handles connect without need for prolongation. In fact, if we drew every $\beta(\mathrm{A})$ result component as the long version, it would strictly speaking not be possible to draw the right subdisks in some cases.

The best example is Figure 12.7 The co-identities can be drawn one after another next to the connector of the $\beta$ (A) input. With a long version, we would have to shorten the connector before drawing the co-identities. In convening a short and long version, we prioritized the convention that handles can always be prolonged and never shortened.

Remark 13.3. The most easily imaginable result components come from discrete immersed disks $h_{q} \mu^{\geq 3}$. In these disks, some angles may be result components, while some are $\delta$-morphisms. By the subdisk protocol, the result components have handles assigned. These handles do not immediately connect to each other. Instead, the result components lie apart by as many arcs as the number $n_{i}$ of $\delta$-morphisms between them. To facilitate smooth connections, we need to connect the handles by means of the angle cutting procedure laid out in Definition 7.29 .

We are now ready to use the protocol for the first time:
Lemma 13.4. Let $r_{1}, \ldots, r_{N}$ be a sequence of result components $r_{i}: L_{i} \rightarrow L_{i+1}$. Assume the values of these result components are the consecutive angles of a discrete immersed disk when complemented with $\delta$-morphisms. Then their subdisk handles and the cuttings of the $\delta$-angles connect smoothly. Here, all short $\beta$ (A) handles shall be extended to long ones first.


Figure 13.1: Subdisk protocol

Proof. The sequence of angles of the discrete immersed disk is a mix of result components and $\delta$ morphisms. To check that everything connects smoothly, it suffices to check two neighbors at a time. These may be either two $\delta$-morphisms, two result components, or one $\delta$-morphism to the left or right of a result component. The first case of two $\delta$-morphisms is trivial.

The second case of two result components is checked in Figure 13.2 In this figure, all possible pairs of consecutive disk angles are checked for smoothness.

The third case of one result component and one $\delta$-morphism is an automatic feature of the subdisk protocol. The example case of $\alpha_{3}$ is depicted by dotted lines in Figure 13.1

### 13.2 Shapeless disks

In this section, we introduce shapeless disks as a container type for subdisks of $\pi$-trees. Recall that we intend to define a mapping between result components of $\pi$-trees and certain types of smooth immersed disks. In the present section, we define a suitable codomain for this map. Our solution is a broad container format, which we call shapeless disks. A shapeless disk consists of intersection points of zigzag curves with curve segments in between, filled by a disk immersion up to reparametrization. The specialty of shapeless disks is that intersection points may occur multiple times, with zero distance between each other. The set Disk ${ }_{\text {SL }}$ of all shapeless disks will serve as codomain of the map $\mathrm{D}:$ Result $_{\pi} \rightarrow$ Disk $_{\text {SL }}$.
Remark 13.5. In the definition of SL disks, it is essential that all cohomology basis elements are understood as intersection points between the associated zigzag curves. This correspondence is defined in section 7.5. In particular, an identity $\mathrm{id}_{L}$ is viewed as the even intersection point of $\tilde{L}$ and its Hamiltonian deformation $\tilde{L}^{\prime}$, located at the midpoint of the identity location arc $a_{0}$ of $L$. The co-identity $\alpha_{0}$ is viewed as the odd intersection point between $\tilde{L}$ and $\tilde{L}^{\prime}$, located at the midpoint of the chosen co-identity angle $\alpha_{0}$. The Hamiltonian deformation $\tilde{L}^{\prime}$ goes right of $\tilde{L}$ at $\alpha_{0}$ and left of $\tilde{L}$ at $a_{0}$, see Figure 13.3

We are now ready to define shapeless disks.
Definition 13.6. Let $N \geq 0$ and let $L_{1}, \ldots, L_{N+1}$ be a sequence of zigzag paths. Let $h_{i}: L_{i} \rightarrow L_{i+1}$ be cohomology basis elements. An SL disk (shapeless disk) consists of

- an output cohomology basis element $t: L_{N+1} \rightarrow L_{1}$,
- a possibly empty $\tilde{L}_{1}$ segment from $t$ to $h_{1}$,
- for every $i=1, \ldots, N$ a possibly empty $\tilde{L}_{i+1}$ segment from $h_{i}$ to $h_{i+1}$,
- a possibly empty $\tilde{L}_{N+1}$ segment from $h_{N}$ to $t$,
- an oriented polygon immersion $D: P_{N+1} \rightarrow|Q|$ up to reparametrization,
such that $D$ has convex corners and traces the segments of $\tilde{L}_{1}, \ldots, \tilde{L}_{N+1}$ one after another. More precisely, $D$ shall map the $i$-th corner to $h_{i}$, the $N+1$-th corner to $t$, the edge between $i$-th and $i+1$-th corner to the $\tilde{L}_{i+1}$ segment and the edge between $N+1$-th and 1 st corner to the $\tilde{L}_{1}$ segment, lying on the right side of this chain of segments. The mapping $D$ need not be an immersion on the boundary. The disk may have infinitesimally small area. The class of SL disks is denoted Disk ${ }_{\text {SL }}$.
Remark 13.7. The notion of SL disks is depicted in Figure 13.4. We may refer to an empty segment also as a segment of infinitesimally small length and say that the two endpoints of the segment are infinitesimally close.


Figure 13.2: Two subdisks along an immersed disk are connected by tying their handles together as shown. Instead of only handles, we have drawn for $\alpha_{3}, \alpha_{4}, \beta / \beta^{\prime}, \alpha_{0}$ and $\alpha_{0}^{\prime}$ their entire subdisks as if they were direct morphisms, for sake of legibility. In the first two pictures, hatching indicates the disk interior. For all other pictures, there should be no ambiguity about inside and outside.


Figure 13.3: A co-identity of $L$ in a subdisk is drawn as a switch from $L$ to its Hamiltonian deformation.


Figure 13.4: These figures illustrate SL disks with a given number of $N=0,1,6$ inputs. In the SL disk with six inputs, the two zigzag paths $L_{1}$ and $L_{2}$ are supposed to be equal and the input $h_{1}$ is supposed to be the co-identity of $L_{1}$. The three inputs $h_{2}, h_{3}, h_{4}$ lie infinitesimally close to each other. The way we have portrayed them is meant to imply $L_{2}=L_{4}$ and $L_{3}=L_{5}$. The morphisms $h_{2}, h_{3}, h_{4}$ change back and forth from $L_{2}$ to $L_{5}$. Allowing this distinctive behavior is the reason for our definition of SL disks.


Figure 13.5: Subdisks of direct morphisms

Remark 13.8. The definition of an SL disk entails the option of infinitesimally small area and empty zigzag curve segments. It is impossible to draw these accurately, so we have opted to visually inflate every infinitesimally small area and empty segments and draw them as substantial area and short but visible segments in all drawings. While the definition of SL disks does technically not involve any Hamiltonian deformations, we always draw co-identity and identity as switches from $\tilde{L}$ to Hamiltonian deformation, see Figure 13.3. We draw stacked co-identities as repeated switches from $\tilde{L}$ to $\tilde{L}^{\prime}$ to $\tilde{L}^{\prime \prime}$ etc. with infinitesimally small distance in between, in line with the Fukaya-theoretic viewpoint.

Remark 13.9. An SL disk is in principle allowed to have as few as zero or one inputs. An SL disk without inputs is a monogon, an SL disk with a single input is a digon. Under the present assumption that $Q$ is geometrically consistent, an SL disk automatically has a minimum of two inputs. There are a few exceptions: The monogon with infinitesimally small area, located at an arbitrary intersection point, constitutes an SL disk without inputs. The digon bounded by two infinitesimally small segments of two intersecting zigzag curves, located at a single intersection, constitutes an SL disk with a single input. The digon bounded by a zigzag path and itself with input the identity and output the co-identity constitutes an SL disk with a single input. For geometrically consistent $Q$, all SL disks with less than two inputs have infinitesimal area. They are an artifact of the definition and will not be used.

### 13.3 Constructing subdisks

In this section, we define subdisks for most Kadeishvili trees. As announced, the procedure is an inductive drawing construction, taking into account the way a given result component was derived. The reader has encountered many subdisk drawings already, spread out over figures from section 12. Here we will explain these drawings and add more.

Definition 13.10. Regard a $\beta(\mathrm{A}), \alpha_{3}(\mathrm{~B}), \beta / \beta^{\prime}(\mathrm{C})$ or $\alpha_{0}^{\prime}(\mathrm{D})$ result component of an h-tree or an $\alpha_{3}+\alpha_{4}(\mathrm{~B})$, id (C), $\sum \mathrm{id}_{a}(\mathrm{D})$ or $\alpha_{0}(\mathrm{D})$ result component of a $\pi$-tree. Then its subdisk is defined inductively by the catalog presented in the rest of this section.


Figure 13.6: Further examples of how to tie subdisks

Any $\alpha_{3}, \alpha_{4}, \beta / \beta^{\prime}(\mathbf{C}), \alpha_{0}, \alpha_{0}^{\prime}$ result component. Depending on whether direct or not, their subdisks are given in Figure 13.5, 12.6a 12.6 c 12.6 d and 12.6 b

Direct $\beta$ (A). Note it is necessarily part of a tail of some morphism $\varepsilon$, which is either $\alpha_{3}, \alpha_{4}$ or $\beta / \beta^{\prime}$ (C). The subdisk of $\beta(\mathrm{A})$ is now obtained by taking the subdisk of $\varepsilon$ and connecting it all the way around the tail disks by cutting the $\delta$ angles, continuing up until the given $\beta$ (A) component. Finish with the short subdisk version of $\beta$ (A).

A $\beta$ (A) main result component of $h_{q} \mu^{2}$. Its subdisk is shown in Figure 12.7.

A $\beta$ (A) main result component of $h_{q} \mu^{\geq 3}$. The given discrete immersed disk is necessarily finalout, with final morphism an outer $\delta$ insertion, $\beta(\mathrm{A})$ or $\beta / \beta^{\prime}(\mathrm{C})$. The result components that may be used in this higher product are $\delta$ insertions, $\beta(\mathrm{A}), \beta / \beta^{\prime}(\mathrm{C}), \alpha_{3}(\mathrm{~B}), \alpha_{4}(\mathrm{~B}), \alpha_{0}(\mathrm{D}), \alpha_{0}^{\prime}(\mathrm{D})$. All of them have subdisk handles assigned. Close all $\beta$ (A) handles. Connect the handles of all morphisms around the discrete immersed disk in clockwise order, following the $\delta$ insertions. Note that this produces a smooth curve according to Lemma 13.4 . Finish with the short version of $\beta$ (A). An example is shown in Figure 13.6a.

A $\beta$ (A) tail result component of $h_{q} \mu^{2}$ or $h_{q} \mu^{\geq 3}$. The corresponding main result component is a $\beta(\mathrm{A}), \beta / \beta^{\prime}(\mathrm{C})$ or $\alpha_{3}$. Now the subdisk of the $\beta(\mathrm{A})$ tail result component is obtained by taking the subdisk of the main result component, closing it if it is a $\beta(\mathrm{A})$, and connecting it all the way around the tail disk by cutting the $\delta$ angles, continuing up until the given $\beta$ (A) tail result component. Finish with the short subdisk version. An example is shown in Figure 13.6b

An $\alpha_{3}+\alpha_{4}$ main result component of $\varphi \pi_{q} \mu^{\geq 3}$. The disk is then one of Figure 12.2 or 12.3 In all cases, connect the handles all around the disk as in the $\beta$ (A) case. If the disk is first-out, cut the $\delta$ angle at the beginning of the disk. If the disk is final-out, cut the $\delta$ angle at the end of the disk. Finally, close the disk with an output mark. An example is shown in Figure 13.6c

$$
\begin{array}{cc}
\mathrm{id}(\mathrm{C}) & \alpha_{4} \\
\vdots & \vdots \\
h_{q} \mu^{2}= & \operatorname{id}(\mathrm{D})
\end{array}
$$

(a) We do not assign a subdisk to this tree.

$$
\begin{gathered}
\alpha_{3} \quad \text { id }(\mathrm{C}) \\
h_{q} \mu^{2}=\text { id (D) }
\end{gathered}
$$

(b) We do not assign a subdisk to this tree.

$$
\begin{gathered}
\backslash_{i d}^{\cdots} / \backslash_{1}^{\cdots} / \\
\varphi \pi_{q} \mu^{2}(\mathrm{D})
\end{gathered}
$$

(c) Close inspection shows this tree evaluates to zero.

Figure 13.7: Miscellaneous trees.


Figure 13.8: In Figure 12.10a, the id (B) component can impossibly come from $\mu^{2}\left(\beta / \beta^{\prime}(\mathrm{C})\right.$, id (C)) or $\mu^{2}\left(\beta(\mathrm{~A}), \alpha_{0}\right)$, because the arrow directions along the disk mismatch resp. because the arrow direction of $\alpha_{0}$ contradicts Convention 10.10. The four resulting trees have no id (D) result components. Of the four trees, the two with id (C) right after the output mark are depicted here.

An $\alpha_{3}+\alpha_{4}$ main result component of $\varphi \pi_{q} \mu^{2}$. The entire tree is then one of those in Figure 12.8 where the subdisks are also depicted.

An $\alpha_{3}+\alpha_{4}$ tail result component of $\varphi \pi_{q} \mu^{\geq 3}$ or $\varphi \pi_{q} \mu^{2}$. It comes from a type G disk in a certain $\varphi \pi_{q}(\beta \alpha)$ evaluation of a product $\mu^{2}$ or one of the disks $\mu^{\geq 3}$ of Figure 12.3 Note that this very same $\beta$ (A) appears as main result component of the $h_{q}(\beta \alpha)$ evaluation and we have already assigned a subdisk with short $\beta$ (A) version to it. Now obtain the subdisk of $\alpha_{3}+\alpha_{4}$ from the subdisk of $\beta$ (A) by closing the subdisk and connecting it all the way up around the disk by cutting the $\delta$ angles, and finally finishing with an output mark at the $2 / 5$ arc of the G situation. An example for $\varphi \pi_{q} \mu^{\geq 3}$ is shown in Figure 13.6d

An id (C) main result component of $\varphi \pi_{q} \mu^{2}$. Its subdisk is depicted in Figure 12.11
An id (C) main result component of $\varphi \pi_{q} \mu^{\geq 3}$. The disk is then all-in and of type H. Its inner morphisms may be $\delta$ insertions, $\beta(\mathrm{A}), \alpha_{3}(\mathrm{~B}), \alpha_{4}(\mathrm{~B}), \beta / \beta^{\prime}(\mathrm{C}), \alpha_{0}(\mathrm{D}), \alpha_{0}^{\prime}(\mathrm{D})$. Connect them all and finish with an output mark on the concluding $2 / 5 \mathrm{arc}$ of the disk. An example is shown in Figure 13.6e

An id (C) tail result component of $\varphi \pi_{q} \mu^{2}$ or $\varphi \pi_{q} \mu^{\geq 3}$. It comes from a type H disk in a certain $\varphi \pi_{q}(\beta \alpha)$ evaluation. Note that this $\beta(\mathrm{A})$ already appears as a main result component and has a subdisk assigned. Now obtain the subdisk of id (C) from closing the subdisk of $\beta$ (A) and connecting it all the way up until the $2 / 5$ concluding arc of the type H disk. Finish with an output mark.

An id (D) result component. Its subdisk is depicted in Figure 12.10
An $\alpha_{0}$ result component of $\varphi \pi_{q} \mu^{2}$ or $\varphi \pi_{q} \mu^{\geq 3}$. Its subdisk is depicted in Figure 12.12.
Remark 13.11. We have associated subdisks to all result components of all $\pi$-trees. Because they are difficult to draw consistently, we do not assign subdisks to id (B) result components and id (D) result components of h-trees.

The reader who has read the catalog of subdisk definitions may feel unsure what these subdisks actually are. To ease his pain, we remind him that subdisks are specific collections of data defined in section 13.1 and section 13.2 Let us explain and record that the subdisks defined in the catalog actually satisfy these conditions:

Lemma 13.12. Subdisks are well-defined. Subdisks of h-trees respect the subdisk protocol. Subdisks of $\pi$-trees are SL disks, providing a map D: Result ${ }_{\pi} \rightarrow$ Disk $_{\text {SL }}$.

Proof. This is easy and follows from induction on tree size. We shall not check all cases, but explain the line of argument. The base case of induction are the subdisks of the direct morphisms, which are depicted in Figure 13.5 and indeed respect the subdisk protocol.

As induction hypothesis, assume the subdisk of any result component of an h-tree with less than $N$ inputs already respects the subdisk protocol. Regard a result component $r$ of an h- or $\pi$-tree with $N$ inputs. Assume $r$ is derived from result components $r_{1}, \ldots, r_{k}$. Each of these result components $r_{i}$ has less than $N$ inputs and hence their subdisks respects the subdisk protocol.

According to the catalog, the subdisk of $r$ is constructed by gluing or extending the subdisks of $r_{1}, \ldots, r_{k}$. At these points, the catalog typically invokes Lemma 13.4 . This invokation is indeed possible since $r_{1}, \ldots, r_{k}$ all respect the subdisk protocol. The final step of the catalog entry is to finish the drawing somewhere near $r$ itself. This step is indicated in individual pictures, from which it is evident that the finish respects the subdisk protocol respectively is an SL disk. This completes the induction.

### 13.4 The four types of disks

In this section, we exhibit the image of $\mathrm{D}:$ Result $_{\pi} \rightarrow$ Disk $_{\text {SL }}$. More precisely, we group result components of $\pi$-trees into four different types, according to the shape of their subdisk. These four types of result components go by the name CR, ID, DS and DW result components. We will also define four types of shapeless disks, which are meant to coincide with the image of these types of result components under D:

| Geometry | Result component | Shapeless disk |
| :---: | :---: | :---: |
| Degenerate strip | DS result component | DS disk |
| Degenerate wedge | DW result component | DW disk |
| Identity degenerate | ID result component | ID disk |
| Co-identity rule | CR result component | CR disk |

Recall from section 13.3 that every result component of a $\pi$-tree comes with a subdisk assigned. All of these subdisks are SL disks, but some are more special than others. For example, the subdisks depicted in Figure 12.11 are all degenerate: There are two zigzag curve segments with infinitesimally small length. In contrast, all segments in the subdisk in Figure 13.6e are nonempty. We exploit these differences in subdisks to define four classes of result components:

Definition 13.13. A result component $r \in \operatorname{Result}_{\pi}$ is a

- DS result component if it is the result component of one of the 8 trees of Figure 12.11.
- DW result component if it is the result component of one of the 7 trees of Figure 12.12
- ID result component if it is a result component of one of the trees in Figure 12.10a or 12.10b or a result component of Figure 12.10c where the first angle of the discrete immersed disk is an $\alpha_{3}$ or the final angle of the disk is an $\alpha_{4}$.
- CR result component otherwise.

The classes of DS, DW, ID and CR result components are denoted Result ${ }_{\mathrm{DS}}$, Result ${ }_{\mathrm{DW}}$, Result ${ }_{\mathrm{ID}}$, Result $_{\mathrm{CR}} \subseteq$ Result $_{\pi}$ respectively.

Remark 13.14. We have chosen the acronyms to reflect the amount of degeneracy allowed in the subdisks: Subdisks of DS result components are "degenerate strips". Subdisks of DW result components are "degenerate wedges". Subdisks of ID result components are "identity degenerate", having an identity output and one of the inputs lying infinitesimally close to it. CR are mostly regular and satisfy the "co-identity rule".

The remainder of this section is devoted to defining the notions of CR, ID, DS and DW disks. These four classes are subsets of DisksL and meant to be explicitly constructible: For every imaginable SL disk, the reader should be able to determine whether it concerns a CR, ID, DS or DW disk or non of those. Ultimately, we will prove that these very explicit classes of disks are precisely the images of Result $\mathrm{CR}_{\mathrm{CR}}$, Result $_{\text {ID }}$, Result ${ }_{\text {DS }}$ and Result ${ }_{\text {DW }}$ under D.

Definition 13.15. A CR disk is an SL disk all of whose segments are nonempty, with the exception that multiple stacked co-identity inputs connected by infinitesimally short $\tilde{L}_{i}$ segments are allowed, as long as their zigzag curve is oriented clockwise with the disk. The class of CR disks is denoted Disk ${ }_{\mathrm{CR}}$.


Figure 13.9: This picture depicts the schematic of CR disks. The specific CR disk depicted here has twelve inputs, of which four are of type B or C and eight are co-identities. The eight co-identities come in two stacks, each consisting of four co-identities lying infinitesimally close to each other.

(a) Clockwise B input

(b) Counterclockwise B input

(c) C input

Figure 13.10: This picture depicts the schematic of ID disks, categorized according to whether the degenerate input is of type $B$ or $C$. Each of the specific ID disks depicted here has nine inputs, of which five are of type B or C and four consist of a stack of co-identities. The degenerate input is the one at the top corner. For the case of degenerate B input, we have depicted both the clockwise and the counterclockwise case. For the case of degenerate C input, we have depicted only the case where the degenerate input precedes the output mark.

Remark 13.16. The behavior of CR disks is depicted in Figure 13.9 We remark that in any CR disk, whenever a co-identity appears in the angle cut just before or after an intersection of type B, it appears only once due to arrow directions.

Definition 13.17. An ID disk is an SL disk satisfying the following conditions:

- The output is the identity of a zigzag path,
- Precisely one input, the degenerate input, is infinitesimally close to the output,
- The degenerate input is of type B or C,
- The disk becomes CR upon excision of the output and substitution of the output mark by the degenerate input,
- In case of a degenerate B input, it precedes respectively succeeds the output mark if $L_{1}$ is oriented clockwise respectively counterclockwise with the disk,
- In case of a degenerate C input, the source zigzag path of the degenerate input is counterclockwise and the target zigzag path is clockwise.
The class of ID disks is denoted Disk ${ }_{\text {ID }}$.
Remark 13.18. The two conditions of Definition 13.17 specific to the $B$ and $C$ case can be formulated in more relaxed terms. In case of a degenerate B input, the three zigzag paths given by the source and target of the degenerate input and the output all have the same orientation. We can make the precedence of degenerate input and output therefore dependent on any of the three, instead of $L_{1}$. In case of a degenerate C input, the requirement regarding orientations equivalently requires that the source and target zigzag path of the degenerate input are always oriented "towards" the disk, instead of "away from" the disk. This is visually depicted in Figure 13.10.

Definition 13.19. A DS disk is an immersed strip fitting into one of the two digons bounded by a zigzag curve $\tilde{L}$ and its Hamiltonian deformation $\tilde{L}^{\prime}$. More precisely, the strip is a 4 -gon bounded by $\tilde{L}$,
$\tilde{L}^{\prime}$ and two (indexed) arcs $a$ and $b$ lying on $L$. The arc $b$ is the one lying closer to the co-identity. Two inputs lie on the midpoint of the arc $a$ and one input on the midpoint of the arc $b$. The output mark lies on the midpoint of the arc $b$. There are corner cases in which additional conditions apply:

- If $a=a_{0}$, then $L$ is oriented away from the co-identity.
- If $a=b$, then either (a) $L$ is oriented away from the co-identity and turns left at $a=b$, and the input at the $b$ side is odd/final, or (b) $L$ is oriented towards the co-identity and turns right at $a=b$, or (c) $L$ is oriented towards the co-identity and turns left at $a=b$, and the $b$ side input is odd/final.
- If $a=b=a_{0}$, then both conditions must be met: $L$ is oriented away from the co-identity, turns left at $a=b$ and the input at the $b$ side is odd/final.
The class of DS disks is denoted Disk ${ }_{\text {DS }}$.
The behavior of DS disks is best observed in Figure 12.11. In the definition of DS disks, we have used terminology that $\tilde{L}$ may be oriented towards or away from the co-identity. Indeed, a strip lies in the digon between identity and co-identity. This brings a distinction whether $\tilde{L}$ is oriented towards the co-identity and away from the identity, or away from the co-identity and towards the identity. Of course, the whole definition with its corner cases is designed to capture precisely the result components of Figure 12.11.

Definition 13.20. A DW disk is one of the following:

- A 3-gon sitting between a zigzag curve $\tilde{L}$ and its Hamiltonian deformation, bounded on one side by an arc $a$ of $L$ and on the other side by the co-identity. The output mark is placed at the co-identity. It is allowed that $a=a_{0}$ if $L$ is oriented away from the co-identity.
- A 4-gon, obtained from the first option by inserting an additional co-identity input infinitesimally preceding the output. The condition is that $L$ is oriented away from the co-identity and that $a \neq t\left(\alpha_{0}\right)$.
- A 4-gon, obtained from the first option by inserting an additional co-identity input infinitesimally succeeding the output. The condition is that $L$ is oriented towards the co-identity.
The set of DW disks is denoted by Disk ${ }_{D W}$.
The behavior of DW disks is best observed in Figure 12.12 The definition distinguishes three types of DW disks. To be more precise with the conditions, observe that a DW disk the second type is allowed to have $a=a_{0}$ while a DW disk of the third type is required to have $a \neq a_{0}$. In the second type, the $\operatorname{arc} a$ is supposed to be not the tail arc $t\left(\alpha_{0}\right)$ of the co-identity angle $\alpha_{0}$. In the third type, the assertion $a \neq t\left(\alpha_{0}\right)$ holds automatically, since the co-identity angle $\alpha_{0}$ is located in a counterclockwise polygon of $Q$ and $L$ is supposed to be oriented towards the co-identity. All DW disks have infinitesimal area, but precisely two nonempty zigzag segments. In fact, the distance between the midpoint of the arc $a$ and the midpoint of the co-identity angle $\alpha_{0}$ is at least half an angle in size.

We have constructed the definitions of CR, ID, DS and DW disks such that the subdisk of a CR result component is a CR disk, and so on:

Lemma 13.21. The subdisk of a CR, ID, DS or DW result component is a CR, ID, DS or DW disk, respectively.

Proof. The inspection is performed in section C.2

In fact, we will prove and discuss later that our definition of CR, ID, DS and DW disks is also sharp: Every CR, ID, DS and DW disk is actually reached as a subdisk of some result component.

Remark 13.22. It is very pleasant that most subdisks are rather regular in the sense that their zigzag curve segments are non-empty. The only irregularities are found in stacked co-identities of CR and ID disks, the degenerate output of ID disks, and the two irregular types of DS and DW disks. Viewed geometrically, this is not really a surprise: In Figure 13.11 we argue that smooth immersed disks with non-transversal intersections lying infinitesimally close to each other are very thin. The conclusion is that within the Fukaya category, one expects only very few irregular disks between zigzag curves. The DS and DW disks provide the exact representation-theoretic witness of this phenomenon.


Figure 13.11: This picture explains that one expects only very few non-transversal disks among zigzag curves in the Fukaya category. We depict a smooth immersed disk bounded by zigzag curves and assume it has two transversal intersections $h_{1}, h_{2}$ on the boundary which lie infinitesimally close to each other. Due to the zigzag nature and transversality, $h_{1}$ and $h_{2}$ must be the even and odd intersection points located at the midpoint of one single arc of $Q$. This means that the source zigzag path of $h_{1}$ is the target zigzag path of $h_{2}$ and the entire disk is then very thin.

### 13.5 The minimal model

In this section, we tie together the computation of $\mathrm{H} \mathbb{L}_{q}$ : Its $A_{\infty}$-structure is defined in terms of $\pi$-trees. All the result components of a $\pi$-tree can again be matched with CR, ID, DS and DW disks:

$$
\begin{array}{cc}
\mathrm{H}_{q} \\
\text { Minimal model }
\end{array} \stackrel{\text { Kadeishvili }}{\longleftrightarrow} \quad \begin{gathered}
\text { Result }_{\pi} \\
\text { Result components }
\end{gathered} \longleftrightarrow \begin{gathered}
\mathrm{D}
\end{gathered} \begin{gathered}
\text { CR, ID, DS, DW } \\
\text { Immersed disks }
\end{gathered}
$$

This correspondence allows us to express the minimal model $H \mathbb{L}_{q}$ in terms of disks. The present section is meant to spell out the details and provide intuition.

Our first step is to get more grip on the subdisk mapping $D:$ Result $_{\pi} \rightarrow$ Disk $_{\text {SL }}$. We have already seen in section 13.4 that D sends CR result components to CR disks, and so on. In the following lemma, we affirm that all CR, ID, DS and DW disks are actually reached by D.

Lemma 13.23. The classes of CR, ID, DS and DW result components are disjoint, as are the classes of CR, ID, DS and DW disks. The subdisk mapping D bijectively sends each of the four result component classes to its disk counterpart:

$$
\begin{array}{ccccccl}
\text { Result }_{\pi}=\text { Result }_{\mathrm{CR}} & \dot{U} & \text { Result }_{\mathrm{ID}} & \dot{U} & \text { Result }_{\mathrm{DS}} & \dot{U} \text { Result }_{\mathrm{DW}} \\
& \mathrm{D} \downarrow 2 & \mathrm{D} \downarrow 2 & \mathrm{D} \downarrow 2 & \mathrm{D} \downarrow 2 & & \\
& \text { Disk }_{\mathrm{CR}} & \dot{U} & \text { Disk }_{\mathrm{ID}} & \dot{U} & \text { Disk }_{\mathrm{DS}} & \dot{\cup} \\
& \text { Disk }_{\mathrm{DW}} & \subseteq & \text { Disk }_{\mathrm{SL}}
\end{array}
$$

Proof. The inclusions are checked and an explicit inverse map is constructed in section C
At this point, we can already describe the minimal model $\mathrm{H} \mathbb{L}_{q}$ in a rigged way by means of disks: Let $r$ be a result component of a $\pi$-tree. Then its subdisk $\mathrm{D}(r)$ comes with a designated output mark, in particular we can read off its output morphism $\mathrm{t}(\mathrm{D}(r))$. In fact, the value of $r$ is equal to $\mathrm{t}(\mathrm{D}(r))$, at least when sign and $q$-parameters are stripped off. For example, a subdisk of an $\alpha_{3}+\alpha_{4}$ result component has output mark at this very same $\alpha_{3}+\alpha_{4}$ morphism, by construction. Even though we currently have to recover signs and $q$-parameters from result component instead of disk, this enables us to largely describe the product in terms of disks:

$$
\begin{align*}
& \mu^{N}\left(h_{N}, \ldots, h_{1}\right)=\sum_{\substack{r \in \operatorname{Result}_{\pi}, \ldots, h_{N} \\
r \text { has inputs } h_{1}, \ldots, h_{N}}} r \\
& =\sum_{\substack{D \in \text { Disk }_{C R} \text { U.Disk } \\
D \text { has in } \text { indisuts }^{h_{1}}, \ldots, h_{N}}} \mathrm{D}^{-1}(D) \tag{13.1}
\end{align*}
$$

Here $\mathrm{t}(D)$ denotes the output morphism of the disk $D$, and $\operatorname{sgn}(r)$ and qparam $(r)$ temporarily denote the sign and $q$-parameter of a result component. As announced, the sign $(-1)^{\operatorname{sgn}\left(\mathrm{D}^{-1}(D)\right)}$ and $q$-parameter qparam $\left(\mathrm{D}^{-1}(D)\right)$ in this formula are only recovered from the result component $\mathrm{D}^{-1}(D)$ instead of $D$ itself.

Our next step is to write signs and $q$-parameters in terms of $D$ instead of recovering them from $\mathrm{D}^{-1}(D)$. In fact, Lemma 13.25 will show that the sign is precisely the Abouzaid sign of $D$, which the reader may recall from the context of Fukaya categories in section 7.1. Moreover, the $q$-parameter is precisely the product of all punctures covered by $D$, counted with multiplicities. Before we make these statements, let us fix the Abouzaid sign terminology in our present context:

Definition 13.24. Let $D$ be an SL disk. Then its Abouzaid sign $\operatorname{Abou}(D) \in \mathbb{Z} / 2 \mathbb{Z}$ is the sum of all \# signs around $D$, plus the number of odd inputs $h_{i}: L_{i} \rightarrow L_{i+1}$ where $L_{i+1}$ is oriented counterclockwise with $D$, plus one if its output $t: L_{1} \rightarrow L_{N+1}$ is odd and $L_{N+1}$ is oriented counterclockwise. The $\boldsymbol{q}$ parameter $\operatorname{Punc}(D) \in \mathbb{C} \llbracket Q_{0} \rrbracket$ of $D$ is defined as the product of all punctures covered by $D$, counted with multiplicities.

Lemma 13.25. Let $r \in$ Result $_{\pi}$ be the result component of a $\pi$-tree. Then its sign is equal to the Abouzaid sign of its subdisk $\mathrm{D}(r)$ and its $q$-parameter $\in \mathbb{C} \llbracket Q_{0} \rrbracket$ is equal to the product of all punctures covered by $\mathrm{D}(r)$, counted with multiplicities:

## In terms of $\mathbf{r}$

$$
\begin{array}{ccc}
\operatorname{Sign} \operatorname{sgn}(r) \in \mathbb{Z} / 2 \mathbb{Z} & = & \operatorname{Sign} \operatorname{Abou}(D) \in \mathbb{Z} / 2 \mathbb{Z} \\
q \text {-parameter } \operatorname{qparam}(r) \in \mathbb{C} \llbracket Q_{0} \rrbracket & = & q \text {-parameter } \operatorname{Punc}(D) \in \mathbb{C} \llbracket Q_{0} \rrbracket
\end{array}
$$

Proof. Both checks can be performed in an inductive fashion. The signs checks are detailed in section C.8. The checks for $q$-parameters are easier and left to the reader.

With the help of this lemma, we are ready to translate the rigged formula 13.1 into the soothing description of the minimal model purely in terms of disks. For simplicity we denote the identity element of a zigzag path $L$ by $\mathrm{id}_{L}=\sum_{a} \mathrm{id}_{a}$.
Theorem 13.26. Let $Q$ be a geometrically consistent dimer. Regard the category $\mathbb{L}_{q} \subseteq \mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} Q$ of deformed zigzag paths according to Convention 10.10. Then the $A_{\infty}$-structure of the minimal model $\mathrm{H} \mathbb{L}_{q}$ is described as follows:

- The curvature and differential vanish:

$$
\mu_{\mathrm{H} \mathbb{L}_{q}}^{0}=\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}=0
$$

- The minimal model is unital: For every cohomology basis element $h: L_{1} \rightarrow L_{2}$ we have

$$
\begin{array}{r}
\mu_{\mathrm{H} \mathbb{L}_{q}}^{\geq 3}\left(\ldots, \operatorname{id}_{L_{1}}, \ldots\right)=0 \\
\mu_{\mathrm{H} \mathbb{L}_{q}}^{2}\left(h, \operatorname{id}_{L_{1}}\right)=(-1)^{|h|} \mu_{\mathrm{H} \mathbb{L}_{q}}^{2}\left(\operatorname{id}_{L_{2}}, h\right)=h .
\end{array}
$$

- The products are given by CR, ID, DS and DW disks: Let $N \geq 2$ and let $h_{1}, \ldots, h_{N}$ be a sequence of non-identity cohomology basis morphisms with $h_{i}: L_{i} \rightarrow L_{i+1}$. Then their product is given by

$$
\mu^{N}\left(h_{N}, \ldots, h_{1}\right)=\sum_{\substack{D \in \text { Disk }_{C R} \text { U.DiskID UUDisk } \\ D \text { has inputs } h_{1}, \ldots, h_{N}}}(-1)^{\text {Abou }(D)} \operatorname{Punc}(D) \mathrm{t}(D)
$$

Proof. This is a summary of our journey. As we have observed earlier, the minimal model $H \mathbb{L}_{q}$ has vanishing differential and curvature. It is also unital with the same identities as $\mathbb{L}$. When $h_{1}, \ldots, h_{N}$ are cohomology basis elements, the rigged formula (13.1) gives

In the second row, we have inserted Lemma 13.25. This finishes the proof.

### 13.6 Main result

In this section, we present our main result. It ties together the "discrete relative Fukaya category" $\mathrm{H} \mathbb{L}_{q}$ and the "smooth relative Fukaya category" relFuk $Q$ :

Discrete relative
$\left(\mathrm{H} \mathbb{L}_{q}\right)_{\mathrm{tr}}$

Smooth relative
relFuk $\left.^{\text {pre }} Q\right|_{\text {Ob } \mathbb{L}}$

The starting point on the discrete side is the explicit description of the minimal model $\mathrm{H}_{\mathbb{L}_{q}}$ due to Theorem 13.26 The starting point on the smooth side is the explicit description of the subcategory relFuk $\left.{ }^{\text {pre }} Q\right|_{\text {ObL }}$ from Corollary 7.39 The main result entails a strict isomorphism between the transversal part $\left(\mathrm{H}_{q}\right)_{\mathrm{tr}}$ on one side and relFuk $\left.{ }^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}$ on the other side. In what follows, we recall a few specific properties of relFuk $\left.{ }^{\text {pre }}\right|_{\mathrm{Ob} \mathbb{L}}$ and a few similarities with $\mathrm{H} \mathbb{L}_{q}$.

Remark 13.27. In section 7 we have elaborated on the construction of Fukaya categories. More specifically, we have defined the categories $\operatorname{Fuk}^{\text {pre }} Q, \operatorname{Fuk} Q, \operatorname{relFuk}^{\text {pre }} Q, \operatorname{relFuk} Q$ and their subcategories given by zigzag curves. Most importanty, recall from Definition 7.23 that relFuk ${ }^{\text {pre }} Q$ denotes the relative Fukaya pre-category of $Q$. In section 7.5 , we have provided an extensive elaboration on how zigzag paths can be interpreted as objects in relFuk ${ }^{\text {pre }} Q$. In particular, a zigzag path $L \in \mathrm{H} \mathbb{L}_{q}$ corresponds to a zigzag curve $\tilde{L} \in$ relFuk $^{\text {pre }} Q$. Recall from Definition 7.31 that relFuk $\left.{ }^{\text {pre }} Q\right|_{\text {ObL }}$ is the $A_{\infty}$-pre-category defined as the subcategory of relFuk ${ }^{\text {pre }} Q$ given by zigzag curves $\tilde{L}$, together with the spin structure dictated by $L \in \mathbb{L}$. As we have seen in Lemma 7.36 , a sequence of zigzag curves $\left(\tilde{L}_{1}, \ldots, \tilde{L}_{N+1}\right)$ is transversal in relFuk $\left.{ }^{\text {pre }} Q\right|_{\text {ob } \mathbb{L}}$ if and only if the zigzag paths $L_{i}$ are pairwise distinct. We have described the subcategory relFuk $\left.{ }^{\text {pre }} Q\right|_{\text {ObL }}$ more explicitly in Corollary 7.39 .
Remark 13.28. In Lemma 7.34, we have identified basis elements for the hom spaces $\operatorname{Hom}$ Fuk $Q\left(\tilde{L}_{1}, \tilde{L}_{2}\right)$ with intersection points of $L_{1}$ and $\tilde{L}_{2}$. In case $\tilde{L}_{1}=\tilde{L}_{2}$, the intersection points only refer to the transversal self-intersections, plus the identity and co-identity self-intersections. In section 10.3, we have seen that basis elements for $\operatorname{Hom}_{\mathbb{H}}\left(L_{1}, L_{2}\right)$ are identified with intersection points between $L_{1}$ and $\tilde{L}_{2}$ as well. In case $L_{1}=L_{2}$, the intersection points only refer to the transversal self-intersections, plus identity and co-identity self-intersections:

| Category of zigzag paths | Geometry | Fukaya category |
| :---: | :---: | :---: |
| Zigzag path | Zigzag curve | Zigzag curve |
| $L$ | $\tilde{L}$ | $\tilde{L}$ |
| Cohomology basis element | Intersection point | Basis element |
| $h: L_{1} \rightarrow L_{2}$ | $p \in \tilde{L}_{1} \cap \tilde{L}_{2}$ | $p: \tilde{L}_{1} \rightarrow \tilde{L}_{2}$ |

Remark 13.29. As laid out in Lemma 7.35 the relative Fukaya category relFuk $Q$ is a deformation of Fuk $Q$. As such, its hom spaces are the $B$-enlargement of the hom spaces of Fuk $Q$, see also Lemma 7.35

$$
\operatorname{Hom}_{\text {relFuk } Q}\left(\tilde{L}_{1}, \tilde{L}_{2}\right)=B \widehat{\otimes} \operatorname{Hom}_{\text {Fuk } Q}\left(\tilde{L}_{1}, \tilde{L}_{2}\right)
$$

Similarly, $\mathrm{H} \mathbb{L}_{q}$ is a deformation of $\mathrm{H} \mathbb{L}$ by construction. Its hom spaces are the $B$-enlargement of the hom spaces of $H \mathbb{L}$ :

$$
\operatorname{Hom}_{\mathrm{H} \mathbb{L}_{q}}\left(L_{1}, L_{2}\right)=B \widehat{\otimes} \operatorname{Hom}_{\mathrm{H} \mathbb{L}}\left(L_{1}, L_{2}\right) .
$$

The identification of the basis elements of $\operatorname{Hom}_{\text {Fuk } Q}\left(\tilde{L}_{1}, \tilde{L}_{2}\right)$ and $\operatorname{Hom}_{H \mathbb{L}_{q}}\left(L_{1}, L_{2}\right)$ provides an explicit $B$-linear identification of the hom spaces $\operatorname{Hom}_{\operatorname{relFuk} Q}\left(\tilde{L}_{1}, \tilde{L}_{2}\right)$ and $\operatorname{Hom}_{H \mathbb{L}_{q}}\left(L_{1}, L_{2}\right)$.

In Lemma 13.30, we examine CR, DS, ID and DS disks in the case that the sequence of input zigzag paths is transversal. The notable outcome is that only CR disks remain, which can in turn be interpreted directly as smooth immersed disks. This establishes the desired link between the minimal model $\mathrm{H} \mathbb{L}_{q}$ and relFuk $\left.{ }^{\text {pre }} Q\right|_{\text {ObL }}$ which we will expand in Theorem 13.31 .
Lemma 13.30. Let $L_{1}, \ldots, L_{N+1}$ be a sequence of zigzag paths in $Q$ such that $\tilde{L}_{1}, \ldots, \tilde{L}_{N+1}$ is a transversal sequence. Let $h_{i}: L_{i} \rightarrow L_{i \pm 1}$ for $1 \leq i \leq N$ and $h: L_{1} \rightarrow L_{N+1}$ be cohomology basis elements in $\mathrm{H} \mathbb{L}$. Denote by $p_{i}: \tilde{L}_{i} \rightarrow \widetilde{L}_{i+1}$ and $\bar{p}: \overline{\tilde{L}}_{1} \rightarrow \tilde{L}_{N+1}$ the corresponding basis elements in Fuk $\left.Q\right|_{\mathrm{Ob} \mathbb{L}}$. Then:

1. There are no ID, DS and DW disks with inputs $h_{1}, \ldots, h_{N}$.
2. There is a bijection

$$
\Phi:\left\{\begin{array}{c}
\text { CR disks } \\
\text { with inputs } h_{1}, \ldots, h_{N} \\
\text { and output } h
\end{array}\right\} \longrightarrow \sim\left\{\begin{array}{c}
\text { Smooth immersed disks } \\
\text { with inputs } p_{1}, \ldots, p_{N} \\
\text { and output } p
\end{array}\right\}
$$

3. The Abouzaid signs agree: $\operatorname{Abou}(D)=\operatorname{Abou}(\Phi(D))$.
4. The $q$-paramaters agree: $\operatorname{Punc}(D)=\operatorname{Punc}(\Phi(D))$.

Proof. We explain the four statements one after another. For the first statement, let $D$ be an ID, DS or DW disk with inputs $h_{1}, \ldots, h_{N}$. Then necessarily at least two of the zigzag curves $L_{1}, \ldots, L_{N+1}$ are equal. Therefore $\left(\tilde{L}_{1}, \ldots, \tilde{L}_{N+1}\right)$ is not a transversal sequence, in contradiction with the assumption. This shows that there are no ID, DS or DW disks with input $h_{1}, \ldots, h_{N}$. In other words, there can only be CR disks with inputs $h_{1}, \ldots, h_{N}$ among the four types of disks.

For the second statement, pick a CR disk with inputs $h_{1}, \ldots, h_{N}$ and output $h$. Since all zigzag paths $L_{1}, \ldots, L_{N+1}$ are pairwise distinct, the sequence $h_{1}, \ldots, h_{N}$ does not contain any co-identities. Therefore all zigzag curve segments involved in the CR disk $D$ are non-empty. This way $D$ immediately constitutes a smooth immersed disk in the sense of Definition 7.21 We denote this smooth immersed disk by $\Phi(D)$. The smooth immersed disk $\Phi(D)$ has inputs $p_{1}, \ldots, p_{N}$ and output $p$, precisely as desired. This sets up the desired mapping $\Phi$. The map $\Phi$ is clearly injective, since a CR disk contains as much information about the polygon immersion $D: P_{N+1} \rightarrow|Q|$ as does a smooth immersed disk. For instance, the two notions of CR disks and smooth immersed disks both identify immersions related by reparametrization. The map $\Phi$ is also surjective, since a smooth immersed disk with inputs $p_{1}, \ldots, p_{N}$ and output $p$ can immediately be interpreted as a CR disk. This shows that $\Phi$ is a bijection.

For the third statement, let $D$ be a CR disk with inputs $h_{1}, \ldots, h_{N}$ and output $h$. According to Definition 13.24 the Abouzaid sign $\operatorname{Abou}(D) \in \mathbb{Z} / 2 \mathbb{Z}$ is the sum of all \# signs on the boundary of $D$, plus the number of odd inputs $h_{i}$ where $L_{i+1}$ is oriented counterclockwise with $D$, plus one if the output $h: L_{1} \rightarrow L_{N+1}$ is odd and $L_{N+1}$ is oriented counterclockwise. This is exactly the same as the definition of the Abouzaid sign of $\Phi(D)$, see Definition 7.22 and 7.12. This shows $\operatorname{Abou}(D)=\operatorname{Abou}(\Phi(D))$.

For the fourth statement, let $D$ be a CR disk with inputs $h_{1}, \ldots, h_{N}$ and output $h$. According to Definition 13.24, the $q$-parameter $\operatorname{Punc}(D) \in \mathbb{C} \llbracket Q_{0} \rrbracket$ is the product of all punctures covered by $D$, counting punctures multiple times if they are covered multiple times. This is exactly the same as the definition of the $q$-parameter of $\Phi(D)$, see Definition 7.22 This finishes the proof.

Our main theorem shows that the transversal part of $\mathrm{H} \mathbb{L}_{q}$ agrees with the subcategory relFuk $\left.{ }^{\text {pre }} Q\right|_{\text {ObL }}$ of the relative Fukaya pre-category. For sake of logical independence, we repeat the setup here: The starting point is a geometrically consistent dimer $Q$. We assume Convention 10.10 We denote by $\mathbb{L}_{q} \subseteq \mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} Q$ the category of deformed zigzag paths according to Definition 11.3 We denote by $\mathrm{H} \mathbb{L}_{q}$ the minimal model of $\mathbb{L}_{q}$, described explicitly in Theorem 13.26 We denote by relFuk $\left.{ }^{\text {pre }} Q\right|_{\text {Ob } \mathbb{L}}$ the subcategory of the relative Fukaya pre-category of $Q$, described explicitly in Corollary 7.39 We denote by $\left(\mathrm{H}_{L_{q}}\right)_{\mathrm{tr}}$ the transversal part of $\mathrm{H}_{q}$ with respect to relFuk $\left.{ }^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}$, as defined in Definition 7.24. The notion of strict isomorphism of deformed $A_{\infty}$-pre-categories is provided in Definition 7.25 Under this terminology, we state our main theorem as follows:

Theorem 13.31. Let $Q$ be a geometrically consistent dimer and assume Convention 10.10 Then there is a strict isomorphism of deformed $A_{\infty}$-pre-categories

$$
F_{q}:\left.\left(\mathrm{HI}_{q}\right)_{\mathrm{tr}} \xrightarrow{\sim} \operatorname{relFuk}^{\mathrm{pre}} Q\right|_{\mathrm{Ob} \mathbb{L}} .
$$

The functor $F_{q}$ sends a zigzag path $L \in \mathrm{H} \mathbb{L}_{q}$ to the associated zigzag curve $\tilde{L}$ and a cohomology basis element $h: L_{1} \rightarrow L_{2}$ to the associated intersection point $p: \tilde{L}_{1} \rightarrow \tilde{L}_{2}$.
Proof. This follows directly from Theorem 13.26 but we state the details. The starting point is the description of the $A_{\infty}$-deformation $\mathrm{H}_{L_{q}}$ of HL from Theorem 13.26 and the description of the $A_{\infty^{-}}$ pre-category deformation relFuk $\left.{ }^{\text {pre }} Q\right|_{\mathrm{ObL}}$ of Fuk $^{\text {pre }} Q \mathrm{Ob} \mathbb{L}^{\text {from Corollary } 7.39}$ We have detailed the definition of the transversal part $\left(\mathrm{H}_{q}\right)_{\text {tr }}$ with respect to Fuk $\left.{ }^{\text {pre }} Q\right|_{\mathrm{ObL}}$ in Definition 7.24 .

To construct the functor $F_{q}$ according to Definition 7.25 we have to execute four steps: (1) to set up a bijection between the objects of $\mathrm{H} \mathbb{L}$ and $\left.\mathrm{Fuk}^{p r e} Q\right|_{\mathrm{Ob} \mathbb{L}}$, (2) to show that the transversal sequences of $(\mathrm{H} \mathbb{L})_{\text {tr }}$ and $\left.\mathrm{Fuk}^{\text {pre }} Q\right|_{\text {ob } \mathbb{L}}$ agree under the bijection on objects, (3) to set up a $\mathbb{C} \llbracket Q_{0} \rrbracket$-linear degree 0 isomorphism between the hom spaces of $\left(\mathrm{H}_{q}\right)_{\operatorname{tr}}$ and $\left.\operatorname{relFuk}^{\mathrm{pre}} Q\right|_{\mathrm{Ob} \mathbb{L}}$, (4) to show that the higher products of $\left(\mathrm{H} \mathbb{L}_{q}\right)_{\mathrm{tr}}$ and relFuk $\left.{ }^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}$ agree under the identification of hom spaces.

For step (1), the bijection between objects of $\underset{\tilde{L}}{\mathbb{L}}$ and $\left.\mathrm{Fuk}^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}$ consists simply of mapping a zigzag path $L \in \mathrm{H} \mathbb{L}$ to its associated zigzag curve $\left.\tilde{L} \in \operatorname{Fuk}^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}$ :

$$
\begin{array}{rl}
F_{q}: \mathrm{Ob}(\mathrm{HL}) & \longrightarrow \\
L & \mathrm{Ob}\left(\left.\mathrm{Fuk}^{\text {pre }} Q\right|_{\mathrm{ObL}}\right) \\
& \longmapsto \tilde{L}
\end{array}
$$

For step (2), we have to explain that the transversal sequences of $(H \mathbb{L})_{\text {tr }}$ are precisely the transversal sequences of $\left.\mathrm{Fuk}^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}$ under the identification of $L$ with $\tilde{L}$. In fact, this is immediate from the definition of $\left(H \mathbb{L}_{q}\right)_{\mathrm{tr}}$ as transversal part of $\mathrm{H} \mathbb{L}_{q}$ with respect to relFuk $\left.{ }^{\text {pre }} Q\right|_{\mathrm{Ob} \mathbb{L}}$ under the identification
of $L$ with $\tilde{L}$. Explicitly, a sequence $\left(L_{1}, \ldots, L_{N}\right)$ in $(\mathrm{H} \mathbb{L})_{\mathrm{tr}}$ is by definition transversal if and only if $\left(\tilde{L}_{1}, \ldots, \tilde{L}_{N}\right)$ is transversal.

For step (3), we have to set up a $\mathbb{C} \llbracket Q_{0} \rrbracket$-linear identification between the hom spaces of $\left(\mathrm{H} \mathbb{L}_{q}\right)_{\text {tr }}$ and relFuk $\left.{ }^{\text {pre }} Q\right|_{\mathrm{ObL}}$. In order to define this identification, let $\left(L_{1}, L_{2}\right) \in(\mathrm{H} \mathbb{L})_{\mathrm{tr}}^{2}$. Then we have $L_{1} \neq L_{2}$. We now set up the identification by sending a cohomology basis element $h: L_{1} \rightarrow L_{2}$ to its associated intersection point $p: \tilde{L}_{1} \rightarrow \tilde{L}_{2}$, as described in Remark 13.29

$$
\begin{aligned}
F_{q}^{1}: \operatorname{Hom}_{\left(\mathrm{H}_{q}\right)_{\mathrm{tr}}}\left(L_{1}, L_{2}\right) & \sim \operatorname{Hom}_{\mathrm{relFuk}^{\text {pre }}} \text { |ob⿺} \\
h & \left(\tilde{L}_{1}, \tilde{L}_{2}\right) \\
& \longmapsto p .
\end{aligned}
$$

For step (4), we have to show that $F_{q}$ preserves the products of $\left(\mathrm{H}_{q}\right)_{\mathrm{tr}}$ and relFuk $\left.{ }^{\text {pre }} Q\right|_{\text {Ob } \mathbb{L}}$. Pick $N \geq 1$ and let $\left(L_{1}, \ldots, L_{N+1}\right) \in(H \mathbb{L})_{\mathrm{tr}}^{N+1}$. Pick basis elements $h_{i} \in \operatorname{Hom}_{\mathrm{H} \mathbb{L}}\left(L_{i}, L_{i+1}\right)$ and let $p_{i} \in$ $\operatorname{Hom}_{\text {Fuk }}{ }^{\text {pre }} Q$ Obı $\left(\tilde{L}_{i}, \tilde{L}_{i+1}\right)$ be the associated intersection points. We get

$$
\begin{aligned}
& F_{q}^{1}\left(\mu_{\left(\mathrm{H} \mathbb{L}_{q}\right)_{\mathrm{tr}}}\left(h_{N}, \ldots, h_{1}\right)\right)=F_{q}^{1}\left(\sum_{\substack{D \in \text { DiskcrádiskiDUUDiskDSUUDiskDw } \\
D \text { has inputs } h_{1}, \ldots, h_{N}}}(-1)^{\operatorname{Abou}(D)} \operatorname{Punc}(D) \mathrm{t}(D)\right) \\
& =\sum_{\substack{D \in \text { Disk }_{C R} \\
D \text { has inputs } h_{1}, \ldots, h_{N}}}(-1)^{\operatorname{Abou}(D)} \operatorname{Punc}(D) F_{q}^{1}(\mathrm{t}(D)) \\
& =\sum_{\substack{h: L_{1} \rightarrow L_{N+1}}} \sum_{\begin{array}{c}
D \in \text { Disk }_{C R} \\
D \text { has input } h_{1}, \ldots, h_{N} \\
\text { and output } h
\end{array}}(-1)^{\operatorname{Abou}(D)} \operatorname{Punc}(D) F_{q}^{1}(h) \\
& =\sum_{p \in \tilde{L}_{1} \cap \tilde{L}_{N+1}} \sum_{D \in M_{q}\left(p_{1}, \ldots, p_{N}, p\right)}(-1)^{\operatorname{Abou}\left(\Phi^{-1}(D)\right)} \operatorname{Punc}\left(\Phi^{-1}(D)\right) p \\
& =\sum_{p \in \tilde{L}_{1} \cap \tilde{L}_{N+1}} \sum_{D \in M_{q}\left(p_{1}, \ldots, p_{N}, p\right)}(-1)^{\operatorname{Abou}(D)} \operatorname{Punc}(D) p \\
& =\left.\mu_{\text {relFuk }}{ }^{\text {pre }} Q\right|_{\text {ObL }}\left(p_{N}, \ldots, p_{1}\right) \text {. }
\end{aligned}
$$

In the first row, we have inserted the description of $\mu_{\mathrm{H} \mathbb{L}_{q}}$ from Theorem 13.26 In the second row, we have pulled $F_{q}^{1}$ into the sum and used that there are no ID, DS and DW disks with inputs $h_{1}, \ldots, h_{N}$ according to Lemma 13.30 In the third row, we have turned the sum into a double sum ranging over the possible output basis elements $h$. The notation $h: L_{1} \rightarrow L_{N+1}$ used is a slight abuse: The sum is intendend to run over the basis elements $h \in \operatorname{Hom}_{\mathrm{HL}}\left(L_{1}, L_{N+1}\right)$. In the fourth row, we have re-enumerated the summands as smooth immersed disks instead of CR disks. This enumeration uses the bijection $\Phi$ set up in Lemma 13.30 In the fifth row, we have used that $\Phi^{-1}(D)$ and $D$ have the same Abouzaid sign and $q$-parameter, according to Lemma 13.30 In the sixth row, we have inserted the definition of the products of relFuk $\left.{ }^{\text {pre }} Q\right|_{\text {Ob L }}$, according to Corollary 7.39 .

This proves step (4) and finishes the construction of the strict isomorphism $F_{q}$ between $\left(\mathrm{H} \mathbb{L}_{q}\right)_{\mathrm{tr}}$ and relFuk $\left.{ }^{\text {pre }} Q\right|_{\text {ObL }}$. By construction, $F_{q}$ sends $L$ to $\tilde{L}$ and a cohomology basis element $h: L_{1} \rightarrow L_{2}$ to the associated intersection point $p: \tilde{L}_{1} \rightarrow \tilde{L}_{2}$. This finishes the proof.

Remark 13.32. The main result shows that $\mathrm{HTw}_{\mathrm{Gtl}}^{q}$ $Q$ is a candidate for a relative wrapped Fukaya category. Also, the subcategory of $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ given by $\mathbb{Z} / 2 \mathbb{Z}$-graded band objects is a candidate model for relFuk $Q$ in the sense of Definition 7.26 .

It seems likely that $\mathrm{H} \mathbb{L}_{q}$ is (gauge equivalent to) the subcategory of zigzag paths in (any model for) relFuk $Q$. Our main result is however no guarantee for this, since taking subcategories and lifting precategories to categories need not commute: Every subcategory of a lift is a lift of the subcategory, but not the other way around. For further discussion we refer to section F.2.2

## A Examples

We provide here an example of a CR and an ID disk together with their matching result components. The aim is to demonstrate in practice how one finds the preimage of a given CR or ID disk under the subdisk mapping. The examples illustrate the strong geometric aspect of the subdisk construction and its inverse construction. On the other hand, the examples demonstrate the sheer amount of case distinctions and precision work required for reconstructing the result component from a given CR or ID disk.


Figure A.1: The immersed disk: a hexagon


Figure A.2: The Kadeishvili tree, with result components depicted at each node

## A. 1 ID disk

We present here an example pair of an ID disk and its matching result component. We depart from the view of the ID disk and construct by inspection the corresponding Kadeishvili tree and result component. The ID disk we present is very small. The reader can use this example to get a feeling how the presence of the degenerate input and the small size of the disk are translated into the Kadeishvili tree.

The example ID disk is presented in Figure A.1 It is situated at a puncture with six incident arcs and six incident polygons. Every pair of two neighboring incident arcs makes for a zigzag path, and the smoothed zigzag curves $\tilde{L}_{1}, \ldots, \tilde{L}_{6}$ have six intersections around the puncture. These intersections alone bound a hexagon.

Recall that every zigzag curve is supposed to have locations assigned of identity and co-identity morphisms. In our example, the identity morphism of the sixth curve is supposed to lie on the curve's second arc when reading clockwise around the puncture. The disk we present makes use of this feature in order to be a disk of ID type with six inputs and an identity output.

Let us explain precisely the data of this disk: The disk's inputs are the six intersection points $h_{1}, \ldots, h_{6}$ of $\tilde{L}_{1}, \ldots, \tilde{L}_{6}$ around the puncture, and the output is the identity of $\tilde{L}_{6}$. The identity output lies infinitesimally close to the first input. More precisely, the first input is the degenerate input of this ID disk and succeeds the output mark. The arcs are oriented so that $\tilde{L}_{1}$ is oriented clockwise and $\tilde{L}_{6}$ is counterclockwise with the disk. This turns the disk into an ID disk according to definition.

To interpret the meaning of this disk, we have to understand its features: First, the disk is very small. The difficulty for us lies instead in the fact that the disk has no direct discrete analog: There is no space between the arcs to form a higher product of $\mathrm{Gtl}_{q} Q$. The disk is however an ID disk and as such has a nonzero value in our explicit model $H \mathbb{L}_{q}$. Second, the disk is unpredictable in the relative Fukaya category. Indeed, the sequence $\tilde{L}_{1}, \ldots, \tilde{L}_{6}, \tilde{L}_{1}$ is not transversal. The disk even has an input lying infinitesimally close to the output. All this means the higher product in the relative Fukaya is unpredictable. The smallness and unpredictability features are nicely illustrated by the fact that the simplest possible Kadeishvili tree $\varphi \pi_{q} \mu\left(h_{6}, \ldots, h_{1}\right)$ simply gives a zero result.

There exists is a single Kadeishvili $\pi$-tree that yields a nonzero output for the input sequence $h_{1}, \ldots, h_{6}$. In fact, there is only a single result component, depicted in Figure A.2 We explain here the precise data of this result component and why it is the only result component.

Step 1: Combining $h_{5}$ and $h_{4}$ : The deformed cohomology basis element $h_{5}$ is the sum of an arc identity and a $\beta^{\prime}(\mathrm{C})$ morphism. The deformed cohomology basis element $h_{4}$ is the sum of an arc identity and a $\beta$ (C) morphism. Combining both into the product $\mu_{q}^{2}\left(h_{5}, h_{4}\right)$ gives the sum of two terms, the first of which comes from the arc identity of $h_{5}$ and the $\beta(\mathrm{C})$ morphism of $h_{4}$, the second comes from the $\beta^{\prime}(\mathrm{C})$ morphism of $h_{5}$ and the arc identity of $h_{4}$. The product morphisms are both situation A morphisms, the first is of type $\gamma \beta$ (A) and the second of type $\beta \alpha$ (A). Applying the codifferential $h_{q}$ to the first gives zero, since $\gamma \beta$ (A) lies in the $R$-part of $\operatorname{Hom}_{\mathbb{L}_{q}}\left(L_{3}, L_{5}\right)$. Applying of the codifferential $h_{q}$ to the second product gives a certain angle $\beta_{45}$, depicted explicitly in the figure. In summary, only the second term survives and is used for the result component.
Step 2: Combining with $h_{3}$ : The deformed cohomology basis element $h_{3}$ consists of an arc identity and a $\beta^{\prime}(\mathrm{C})$ morphism. Combining with $\beta_{45}$ gives the product $\mu_{q}^{2}\left(\beta_{45}, \mathrm{id}(\mathrm{C})\right)$ which is a situation A morphism of type $\beta \alpha(\mathrm{A})$. Applying the codifferential $h_{q}$ gives a certain angle $\beta_{345} \in \operatorname{Hom}_{\mathbb{L}_{q}}\left(L_{2}, L_{5}\right)$, depicted explicitly in the figure.
Step 3: Combining with $h_{2}$ : Analogous to the previous step, gives angle $\beta_{2345} \in \operatorname{Hom}_{\mathbb{L}_{q}}\left(L_{1}, L_{5}\right)$.
Step 4: Combining with $h_{6}$ : The deformed cohomology basis element $h_{6}$ consists of an arc identity and a $\beta(\mathrm{C})$ morphism. Combining with $\beta_{2345}$ gives the product $\mu_{q}^{2}\left(\mathrm{id}(\mathrm{C}), \beta_{2345}\right)$ which is a situation B morphism of type $\alpha_{2}(\mathrm{~B})$. Application of the codifferential $h_{q}$ gives the identity angle id (B) $\in$ $\operatorname{Hom}_{\mathbb{L}_{q}}\left(L_{1}, L_{6}\right)$.
Step 5: Combining with $h_{1}$ : The deformed cohomology basis element $h_{1}$ consists of an arc identity id $(\mathrm{C})$ and a $\beta^{\prime}(\mathrm{C})$ morphism. Combining with id $(\mathrm{B})$ gives the product $\mu_{q}^{2}(\mathrm{id}(\mathrm{B}), \mathrm{id}(\mathrm{C}))$ which is the arc identity on the second arc of $L_{6}$ read in clockwise direction. By assumption, this second arc is the identity location of $L_{6}$. Application of the projection $\pi_{q}$ gives the identity morphism $\operatorname{id}_{L_{6}} \in \operatorname{Hom}_{\mathbb{L}_{q}}\left(L_{6}, L_{6}\right)$. This is the final result component of the Kadeishvili $\pi$-tree.
$q$-parameters: In the past five steps, we have ignored signs and $q$-parameters in the result component. The $q$-parameter in fact consists of the single puncture located at the center of the hexagon. This parameter enters the result component in Step 1 by means of the $\beta^{\prime}(\mathrm{C})$ morphism of $h_{5}$.

The result component described above and depicted in Figure A. 2 is the only result component of the sequence $h_{1}, \ldots, h_{6}$, at least as far as displayed in the figure. For the specific $\pi$-tree, the single result component is the only result component. Indeed we have exhausted in every step all possible products $\mu_{q}^{2}$ or $\mu_{q}^{\geq 3}$ and all terms in their codifferential, apart from possible tail terms which lie far away and are not visible in the figure. Other $\pi$-trees with the same input sequence $h_{1}, \ldots, h_{6}$ do not yield any result components. Indeed, one might for instance try to combine $\beta_{2345}$ with $h_{1}$ before combining with $h_{6}$. However, the product $\mu_{q}^{2}\left(\beta_{2345}, \mathrm{id}(\mathrm{C})\right)$ gives a situation B morphism $\alpha_{1}(\mathrm{~B})$ whose codifferential vanishes. This explains how the result component depicted in Figure A. 2 is the only result component and illustrates the delicate nature of matching disks with result components.

## A. 2 CR disk

We present here an example of a CR disk together with its Kadeishvili tree and result component.
The CR disk we present is depicted in Figure A.3. Its data has the following properties: There are eleven input morphisms. The first is an odd morphism (type B), the second and fifth are even (type C), the third and fourth are stacked co-identities, the sixth and seventh are odd (type B), the eighth is a co-identity, the ninth is even (type C), the tenth and eleventh are stacked co-identities. The output is an odd morphism (type B). In total, the boundary of the CR disk includes seven (in principle) distinct zigzag curves.


Figure A.3: A large CR disk with 11 inputs


Figure A.4: Narrow tree of the CR disk

The CR disk consists visually of three connected arms. It has a total of five indecomposable narrow locations A, B, C, D, E. The narrow locations A, B lie on the first arm, the narrow locations C, D lie on the second arm, the narrow location E lies on the third arm. The narrow tree is depicted in Figure A.4. In the terminology of the narrow tree, the morphism $h_{1}$ lies below A and B. The morphisms $h_{2}, h_{3}, h_{4}$, $h_{5}$ lie below C and D . The morphism $h_{6}, h_{7}, h_{8}$ lie below E . The morphisms $h_{9}, h_{10}, h_{11}$ lie left-within E . The narrow locations C and E are (direct) siblings in the narrow tree, indicated in the figure by a dotted line.

The figure does not depict the entire dimer. In fact, the zigzag segments and narrow locations have been depicted accurately, but the drawing misses some arcs. For instance, regard the puncture $q$ which is the connecting puncture of the narrow locations C and E . Simply put, $q$ is the puncture just to the right of $h_{9}, h_{10}$. The figure suggests there are seven incident polygons, preventing $Q$ from being a dimer. This explains that the visible arcs are not sufficient for the CR disk to exist and further arcs need to be added in the interior of the CR disk.

According to Lemma 13.23 proved in section C.6 the CR disk is the subdisk of a single CR result component. The construction of this result component is depicted in Figure A.5 till A.9. We explain the construction as follows:


Figure A.5: Combining 5, 4, 3, 2

Step 1: Combining $h_{5}, h_{4}, h_{3}, h_{2}$ : This step is depicted in Figure A. 5 The morphisms $h_{4}$ and $h_{3}$ are identical, namely a co-identity. Both consist of an $\alpha_{0}$ and an $\alpha_{0}^{\prime}$ term. Take the product $\mu_{q}^{2}\left(\alpha_{0}, \alpha_{0}^{\prime}\right)$. Applying the codifferential $h_{q}$ gives $\alpha_{0}^{\prime}$. The morphism $h_{5}$ consists of an id (C) and a $\beta$ $(\mathrm{C})$ term. Take the product $\mu_{q}^{2}\left(\mathrm{id}(\mathrm{C}), \alpha_{0}^{\prime}\right)$. Applying the codifferential $h_{q}$ gives a $\beta(\mathrm{C})$ morphism. The morphism $h_{2}$ consists of an id (C) and a $\beta(\mathrm{C})$ term. Take the product $\mu_{q}^{2}(\beta(\mathrm{C})$, id (C)). This gives a result of type $\beta \alpha(\mathrm{A})$. Applying the codifferential $h_{q}$ gives a $\beta$ (A) main result component, depicted without label in the figure. The codifferential also yields a $\beta$ (A) tail result component, labeled as such in the figure. The codifferential yields one further tail result component visible in the figure, but we do not use it.
Step 2: Combining $h_{8}$ and $h_{7}$ : This step is depicted in Figure A.6. The morphism $h_{8}$ is a co-identity and comes with terms $\alpha_{0}$ and $\alpha_{0}^{\prime}$. The morphism $h_{7}$ is an type B morphism and comes with terms $\alpha_{3}$ and $\alpha_{4}$. Take the product $\mu_{q}^{2}\left(\alpha_{0}, \alpha_{4}\right)$. Applying the codifferential $h_{q}$ gives an $\alpha_{3}(\mathrm{~B})$ main result component.

Step 3: Combining $h_{11}, h_{10}, h_{9}$ : This step is depicted in Figure A. 7 The morphisms $h_{11}$ and $h_{10}$ are


Figure A.6: Combining 8, 7


Figure A.7: Combining 11, 10, 9


Figure A.8: Combining $11+10+9,8+7,6$ and $5+4+3+2$
identical co-identities. Both consist of terms $\alpha_{0}$ and $\alpha_{0}^{\prime}$. Take the product $\mu_{q}^{2}\left(\alpha_{0}, \alpha_{0}^{\prime}\right)$. Applying the codifferential $h_{q}$ gives an $\alpha_{0}^{\prime}$ term. The morphism $h_{9}$ is a type C morphism and among others consists of an id (C) term. Take the product $\mu_{q}^{2}\left(\alpha_{0}^{\prime}\right.$, id (C)). Applying the codifferential $h_{q}$ gives a $\beta(\mathrm{C})$ result component.
Step 4: Assembly, part I: This step is depicted in Figure A.8 In this step, we combine the second and third arm of the CR disk. On morphism level, we combine the result components obtained in step 1,2 and 3 . We start as follows: Pick the result component $\beta(\mathrm{C})$ from step 3 and the result component $\alpha_{3}$ (B) from step 2. Take the product $\mu_{q}^{8}\left(\beta(\mathrm{C}), \delta, \delta, \alpha_{3}, \alpha_{4}, \delta, \delta, \delta\right)$. Here the letter $\delta$ denotes inner and outer $\delta$ insertions. The result is a situation A morphism $\beta \alpha$ (A). Applying the codifferential gives a term $\beta$ (A). Now pick the $\beta$ (A) result component stemming from step 1 and take the product $\mu_{q}^{8}(\beta(\mathrm{~A}), \delta, \delta, \delta, \beta(\mathrm{A}), \delta, \delta, \delta)$ of both. The result is a situation $\mathrm{A} \beta \alpha(\mathrm{A})$ morphism. Applying the codifferential $h_{q}$ gives a $\beta$ (A) result component.
Step 5: Assembly, part II: This step is depicted in Figure A.9 In this step, we combine the second and third with the first arm of the CR disk. On morphism level, we combine the morphism $h_{1}$ with the result component of $h_{2}, \ldots, h_{11}$ obtained in step 4. The morphism $h_{1}$ is a type B morphism and consists of $\alpha_{3}$ and $\alpha_{4}$ terms, but also includes two $\beta$ (A) tail result components. Of these two tail result components, take the higher-order one and combine with the other result component $\beta$ (A) from step 4 to obtain the product

$$
\mu_{q}^{16}(\delta, \delta, \delta, \delta, \beta(\mathrm{~A}), \delta, \delta, \delta, \delta, \delta, \beta(\mathrm{A}), \delta, \delta, \delta, \delta, \delta)
$$

The discrete immersed disk which computes this product is final-out. Its result includes an $\alpha_{3}$ term. Applying the projection $\pi_{q}$ gives the final $\alpha_{3}+\alpha_{4}$ result component of the tree.


Figure A.9: Combining $(11+10+9)+(8+7)+6+(5+4+3+2)$ and 1

## B Uncurving of band objects

In this section, we prove Lemma 9.17, which concerns uncurvability of band objects. It is our task to explicitly construct a deformed twisted complex such that a given band object becomes curvature-free. More precisely, the starting point consists of:

- a punctured surface $(S, M)$,
- a full arc system $\mathcal{A}$ with [NMDC],
- a standard deformation $\operatorname{Gtl}_{r} \mathcal{A}$ where $r \in \mathfrak{m} Z(\operatorname{Gtl} \mathcal{A})$ is without 1-component,
- a band object organized in a twisted complex $X=\left(\oplus a_{i}\left[s_{i}\right], \delta\right)$,
such that the underlying curve of $X$ in the closed surface $S$ is not contractible and does not bound a teardrop, and every connecting angle $\alpha_{i}$ in $\delta$ is longer than an identity and shorter than a full turn. It is our aim to construct a deformation $\delta_{q} \in \operatorname{Hom}_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{1}(X, X)$ such that $X_{q}=\left(\oplus a_{i}\left[s_{i}\right], \delta_{q}\right)$ has vanishing curvature.

The present section is logically independent of the computation of the minimal model $\mathrm{H}_{q}$. However, it builds directly on its methods in three ways: First, we introduce a notion of situations to characterize types of angles between arcs of $X$, similar to the notion of situations for morphisms between zigzag paths. Second, we build a protocol which describes what kind of angles we may encounter while gathering the complementary angles, similar to the E, F, G, H disks or the subdisk protocol for zigzag paths. Third,


Figure B.1: An example which requires four additional complementary angles
we build up the list of complementary angles in a recursive way, a bit similar to the way we work with tails of morphims or subdisks for zigzag paths.

The idea to construct $\delta_{q}$ is to add not only complementary angles for all connecting angles $\alpha_{i}$ of $X$, but also complementary angles at locations where the strands of $X$ come close to each other. An example for $X$ is depicted in Figure B. 1 Apart from the complementary angles for the connecting angles $\alpha_{i}$, this example requires four additional angles to be added to $\delta_{q}$. In terms of the situational formalism of section B. 1 the four angles are of type id ( $\mathrm{A}^{\prime}$ ), $\beta\left(\mathrm{A}^{\prime}\right), \beta(\mathrm{A})$ and $\beta(A)$.

In section B.1. we inspect the possible configurations of arcs and angles around $X$ and introduce notions of situations. In section B.2 we define a temporary type of possible angles we may add in order to produce $\delta_{q}$, and investigate their possible products. In section B.3 we construct $\delta_{q}$ in a recursive way and show that the curvature of $X_{q}=\left(\oplus a_{i}\left[s_{i}\right], \delta_{q}\right)$ vanishes.

## B. 1 Situations

In this section, we examine the possible configurations of arcs and angles along $X$. We capture these types configurations in the notion of A, A', A", ID and D situations. The main difference with the notion of situations for zigzag paths is that two different indexed arcs which are equal as arcs of $\mathcal{A}$ need not determine an intersection of the underlying curve. For this reason, we obtain a slightly larger amount of different situations.

Definition B.1. Regard the band object $X=\left(\oplus a_{i}\left[s_{i}\right], \delta\right)$. An indexed $\operatorname{arc}$ on $X$ is a choice of arc $a_{i}$, remembering the index $i$. If $a_{i}$ is an indexed arc of $X$, then the strand of $X$ at $a_{i}$ refers to the portion of $X$ given by the neighboring indexed arcs $\ldots, a_{i-1}, a_{i}, a_{i+1}, \ldots$. An elementary morphism $\varepsilon: a_{i} \rightarrow a_{j}$ on $X$ is a single angle between two indexed arcs $a_{i}, a_{j}$ on $X$, interpreted as $\varepsilon \in \operatorname{Hom}_{\text {Add Gtl } \mathcal{A}}(X, X)$. The source strand or target strand of an elementary morphism $\varepsilon: a_{i} \rightarrow a_{j}$ is the strand of $X$ at $a_{i}$ or $a_{j}$, respectively.

Two angles in $\mathcal{A}$ which wind around a common puncture have an overlap if they contain a shared indecomposable angle. Otherwise, the two angles are disjoint. For every indexed arc $a_{i}$, the object $X$ contains a distinction whether $X$ turns left or right towards a given endpoint of $a_{i}$.

Recall that an arc incidence at a puncture is slightly different from an arc incident at a puncture: An arc incidence includes the datum whether it concerns the head or tail of the arc. A loop incident at a puncture gives rise to two arc incidences. We generically denote a full turn by $\ell$. We may call the $\alpha_{i}$ angles of $X$ also the turning angles.

Definition B.2. We define the following types of situations on $X$ :

- A type A situation consists of a puncture together with incidences of two indexed arc $a_{i}$ and $a_{j}$ such that (a) $X$ turns left at $a_{i}$ towards the puncture, and (b) $X$ turns right at $a_{j}$ towards the puncture, and (c) the turning angles do not overlap. The associated angles of the situation are denoted $\alpha, \beta, \gamma$ and $\beta^{\prime}$, as in Figure B.2b
- A type A' situation consists of a puncture $q$ together with incidences of two distinct indexed $\operatorname{arcs} a_{i}$ and $a_{j}$ which are equal as arc of $\mathcal{A}$, such that the strand of $a_{i}$ turns right at $a_{i}$ towards $q$ and the strand of $a_{j}$ turns left towards $q$. The associated angles are denoted $\alpha, \beta, \gamma$ and id, as in Figure B.2c
- A type A" situation consists of a puncture $q$ together with incidences of two distinct indexed $\operatorname{arcs} a_{i}$ and $a_{j}$ which are equal as arcs of $\mathcal{A}$, such that the strand of $a_{i}$ turns left at $a_{i}$ towards $q$


Figure B.2: Illustration of A, A', A", ID and D situations
and the strand of $a_{j}$ turns right towards $q$. The associated angles are denoted $\alpha$, id, $\gamma, \beta^{\prime}$, as in Figure B.2d

- A type ID situation consists of a puncture $q$ together with incidences of two distinct indexed arcs $a_{i}$ and $a_{j}$ which are equal as arcs of $\mathcal{A}$, such that (a) the strand of $a_{i}$ turns left at $a_{i}$ towards $q$, and (b) the strand of $a_{j}$ turns right towards $q$, and (c) the turning angles together form a full turn. The associated angles are denoted $\alpha$, id and $\gamma$, as in Figure B.2e
- A type D situation consists of a pair of consecutive arcs of $X$. It gives rise to $\alpha$ and $\alpha^{\prime}$ angles: A type $\alpha$ angle of $X$ is one of the $\alpha_{i}$ angles of $X$. A type $\alpha^{\prime}$ angle is one of the $\alpha_{i}^{\prime}$ angles of $X$, see Figure B.2f

Remark B.3. When an angle comes from a situation of a certain type, we typically indicate the type in brackets for clarity. For instance, a certain angle may qualify as a " $\beta$ (A) angle". When working with situations, we may from time to time indicate the situations by its associated angles, for instance referring to a "type A situation $\left(\alpha_{R}, \beta_{R}, \gamma_{r}, \beta_{R}^{\prime}\right)$ ".

The difference with zigzag paths is that the arcs of $X$ are not oriented in the same direction, and the $\alpha(\mathrm{D})$ angles are not necessarily indecomposable. Another important difference is that a zigzag segment is never contractible, while a segment of $X$ may be contractible. In our uncurving construction, we form $\delta_{q}$ by inserting angles whenever a contractible segment lies "above" the angle. In order to make this precise, we use the following terminology:

Definition B.4. For an A/A'/A"/ID situation, above refers to tracing $X$ in the opposite direction of $\alpha$ and in the direction of $\gamma$.

The direction "above" in the situation figures Figure B.2b till B.2e is the natural upwards direction on paper.

## B. 2 Uncurving protocol

In this section, we examine possible products which can be made from angles of $\mathrm{A}, \mathrm{A}^{\prime}, \mathrm{A}$ ", ID and D situations. The core tool is a notion of angles with balloons. The datum of balloons makes it possible to safely examine possible products $\mu^{2}$ and $\mu_{\bar{r}}^{\geq 3}$ between angles. All of the angles we later insert into $\delta_{q}$ are of this type, but come with additional data which is irrelevant and not accessible at the current stage of the construction. This way, angles with balloons serve as a "protocol" which greatly facilitates the construction of $\delta_{q}$.

The aim of the examination is to draw maximally strong conclusions on the configurations of arcs and angles from the fact that the underlying curve of $X$ is not contractible and does not bound a teardrop.

(a) $\beta(\mathrm{A})$ with balloon

(b) $\beta\left(\mathrm{A}^{\prime}\right)$ with balloon

(c) id (A") with balloon

Figure B.3: Illustration of angles with balloons


Figure B.4: Illustration of orbigons made of angles with balloons

The notion of angles with balloons is the cheapest way to incorporate this property of $X$ into individual angles. Angles with balloons are elementary angles whose $X$ strand lying above the angle is contractible:
Definition B.5. An angle with balloon on $X$ is a $\beta \ell^{m}$ (A/A') or id $\ell^{m}$ (ID/A") angle whose $X$ segment above $\beta$ or id is contractible. We also count $\alpha(\mathrm{D})$ and $\alpha^{\prime} \ell^{m}(\mathrm{D})$ as angles with balloons.

Angles with balloons are depicted in Figure B.3 The balloons facilitate a lot of tricks and desirable properties. With the above/below terminology from Definition B.4 we can say that the balloon always lies above the angle. Moreover, an id angle with balloon directly determines the turning directions of its source and target strands:
Lemma B.6. Let id (ID/A") be an identity angle with balloon. Then above, its source strand turns left and its target strand turns right. Similarly below, its source strand turns right and its target strand turns left.

Proof. In both ID and A" situations, the turning on the above side is predetermined. For the below side, note that turning in the opposite direction would immediately constitute a teardrop in the underlying curve of $X$, contradicting the assumption that there are no teardrops. This finishes the proof.

We now examine products and higher products of angles with balloons. When $h_{1}, \ldots, h_{k}$ are elementary angles, then an (additive) contribution to a product $\mu_{r}^{k \geq 2}\left(h_{k}, \ldots, h_{1}\right)$ is simply the product itself in case $k=2$, or an orbigon contributing to the product in case $k \geq 3$. As a first step, we can show that when a higher product of angles with balloons are taken, no contribution to the higher product is all-in:

Lemma B.7. Let $h_{1}, \ldots, h_{k}$ be a sequence of $k \geq 3$ angles with balloons. Then any contribution to $\mu_{r}^{k}\left(h_{k}, \ldots, h_{1}\right)$ is first-out or final-out. At the concluding puncture, the first and final strands of $X$ both turn outside the disk and their turning angles are disjoint.

Proof. Regard a given contribution, denoted $D$. It is our task to show that $D$ is not all-in. We have depicted a fictitious all-in contribution in Figure B.4a. With this figure, the reason that $D$ cannot be all-in is immediate: All interior angles of the orbigon are angles with balloons. Correspondingly, the $k$-many $X$ segments above the interior angles are all contractible. Since the orbigon itself is contractible, we conclude that the curve underlying $X$ is contractible, in contradiction with our assumption.

We conclude that every contribution is first-out or final-out. It is our task to show that the first and final strands of the orbigon both outside the disk at the concluding puncture, instead of inside. This is



(a) Type $\mu(\beta(\mathrm{A}), \ldots)$


(c) Type $\mu(\ldots, \beta(\mathrm{A}))$

or

(e) Type $\mu\left(\alpha^{\prime}, \ldots\right)$

(g) Type $\mu^{2}(\beta(\mathrm{~A})$, id)

(i) Type $\mu^{2}(\mathrm{id}, \beta(\mathrm{A}))$

or
(b) Type $\mu\left(\beta\left(\mathrm{A}^{\prime}\right), \ldots\right)$


or
(d) Type $\mu\left(\ldots, \beta\left(\mathrm{A}^{\prime}\right)\right)$

(f) Type $\mu\left(\ldots, \alpha^{\prime}\right)$

(h) Type $\mu^{2}\left(\beta\left(\mathrm{~A}^{\prime}\right)\right.$, id $)$


(j) Type $\mu^{2}\left(\mathrm{id}, \beta\left(\mathrm{A}^{\prime}\right)\right)$

Figure B.5: Possible configurations of products
a standard argument concerning contractibility. We have depicted the desired situation in Figure B.4b, Towards a contradiction, assume that one of the strands turns inside the disk. View the orbigon as a bracketed discrete immersed disk. Denote the polygon immersion of the discrete disk by $D: P_{k} \rightarrow S$. Then at the concluding puncture, the strand of $X$ turns inside the interior of $P_{k}$. Tracing the strand further, at some point it necessarily intersects itself or leaves $P_{k}$. This constitutes a teardrop for the underlying curve of $X$, in contradiction with the assumption that $X$ has no teardrop. This shows that both strands turns outwards.

Finally, let us explain why the two turning angles outside the disk are disjoint. Indeed, if the two turning angles outside the disk are not disjoint, the orbigon with its interior angles constitutes a teardrop for the underlying curve of $X$, in contradiction with the assumption. This finishes the proof.

According to Lemma B.7, the situation around the concluding puncture of an orbigon gives rise to an A/A" situation, which we call the concluding situation of the contribution. Thanks to this characterization, we can inspect the contributions to $\mu_{\bar{r}}^{>2}\left(h_{k}, \ldots, h_{1}\right)$ in more detail, depending on the type of the angles $h_{1}, \ldots, h_{k}$. We have expressed this inspection in Lemma B. 8 There, we have detailed only the statement of the first item, because the others statements are analogous and can be interpreted from the figures.

Lemma B.8. Let $h_{1}, \ldots, h_{k}$ be a sequence of angles with balloons. Let $D$ be a contribution to the product $\mu_{r}^{\geq 2}\left(h_{k}, \ldots, h_{1}\right)$. Then $D$ can impossibly concern a product of the type $\mu^{2}\left(\beta \ell^{m}, \beta \ell^{n}\right), \mu^{2}\left(\beta \ell^{m}, \alpha^{\prime} \ell^{n}\right)$, $\mu^{2}\left(\alpha^{\prime} \ell^{m}, \beta \ell^{n}\right)$ or $\mu^{2}\left(\mathrm{id} \ell^{m}, \mathrm{id} \ell^{n}\right)$. Instead, it falls under one of the following cases, and we make the following refining statements:

- If $D$ concerns a product $\mu_{\vec{r}}^{\geq 3}\left(\beta \ell^{m}(\mathrm{~A}), \ldots\right)$ with $\beta \ell^{m}$ final-out: Let $\left(\alpha_{1}, \beta_{1}, \gamma_{1}\right)$ be the angles associated with the situation of $\beta$. The concluding situation $\left(\alpha_{2}, \beta_{2}, \gamma_{2}\right)$ of $D$ is necessarily of type A. The angles $\beta_{1} \alpha_{1}$ and $\alpha_{2}$ are disjoint. Depending on the size of the angle between $\beta_{1}$ and $\alpha_{2}$, we obtain a well-defined A situation $\left(\alpha_{R}, \beta_{R}, \gamma_{R}\right)$ or an A" situation $\left(\alpha_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$ Figure B.5a.
- If $D$ concerns a product $\mu_{\bar{r}}^{\geq^{3}}\left(\beta \ell^{m}\left(\mathrm{~A}^{\prime}\right), \ldots\right)$ with $\beta \ell^{m}$ final-out: The concluding situation of $D$ is of type A and we obtain a well-defined A situation $\left(\alpha_{R}, \beta_{R}, \gamma_{R}\right)$ or an A" situation $\left(\alpha_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$ Figure B.5b).
- If $D$ concerns a product $\mu_{\bar{r}}^{\geq 3}\left(\ldots, \beta \ell^{m}(\mathrm{~A})\right)$ with $\beta \ell^{m}$ first-out: The concluding situation of $D$ is of type A and we obtain a well-defined A situation $\left(\alpha_{R}, \beta_{R}, \gamma_{R}\right)$ or A" situation $\left(\alpha_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$ Figure B.5c.
- If $D$ concerns a product $\mu_{\vec{r}}{ }^{3}\left(\ldots, \beta \ell^{m}\left(\mathrm{~A}^{\prime}\right)\right)$ with $\beta \ell^{m}$ first-out: The concluding situation of $D$ is of type A and we obtain a well-defined A situation $\left(\alpha_{R}, \beta_{R}, \gamma_{R}\right)$ or A" situation $\left(\alpha_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$ Figure B.5d.
- If $D$ concerns a product $\mu_{\bar{r}}^{2}\left(\alpha^{\prime} \ell^{m}, \ldots\right)$ with $\alpha^{\prime} \ell^{m}(\mathrm{D})$ final-out: We obtain a well-defined A or A" situation Figure B.5e.
- If $D$ concerns a product $\mu_{r}^{\geq 3}\left(\ldots, \alpha^{\prime} \ell^{m}\right)$ with $\alpha^{\prime} \ell^{m}(\mathrm{D})$ final-out: We obtain a well-defined A or A" situation Figure B.5f).
- If $D$ concerns a product $\mu^{2}\left(\operatorname{id} \ell^{m}(\mathrm{ID} / \mathrm{A} "), \beta \ell^{n}(\mathrm{~A})\right)$ : We obtain a well-defined A situation or A" situation Figure B.5g.
- If $D$ concerns a product $\mu^{2}\left(\mathrm{id} \ell^{m}(\mathrm{ID} / \mathrm{A} "), \beta \ell^{n}\left(\mathrm{~A}^{\prime}\right)\right)$ : We obtain a well-defined A situation or A" situation Figure B.5h.
- If $D$ concerns a product $\mu^{2}\left(\beta \ell^{m}(\mathrm{~A})\right.$, id $\left.\ell^{n}(\mathrm{ID} / \mathrm{A} ")\right)$ : We obtain a well-defined A situation or A" situation Figure B.5i).
- If $D$ concerns a product $\mu^{2}\left(\beta \ell^{m}\left(\mathrm{~A}^{\prime}\right), \mathrm{id} \ell^{n}(\mathrm{ID} / \mathrm{A} ")\right)$ : We obtain a well-defined A situation or A" situation Figure B.5j).
- $D$ may also concern a product of one of the types

$$
\begin{aligned}
& \mu^{2}\left(\alpha, \alpha^{\prime} \ell^{m}\right), \quad \mu^{2}\left(\alpha^{\prime} \ell^{m}, \alpha\right), \quad \mu^{2}\left(\alpha, \operatorname{id} \ell^{m}(\operatorname{ID} / \mathrm{A} ")\right), \quad \mu^{2}\left(\mathrm{id} \ell^{m}(\mathrm{ID} / \mathrm{A} "), \alpha\right), \\
& \mu^{2}\left(\alpha^{\prime} \ell^{m}, \mathrm{id} \ell^{n}(\mathrm{ID} / \mathrm{A} ")\right), \quad \mu^{2}\left(\mathrm{id} \ell^{m}\left(\mathrm{ID} / \mathrm{A}^{\prime}\right), \alpha^{\prime} \ell^{n}\right), \quad \mu^{2}\left(\beta \ell^{m}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right), \alpha\right) \quad \text { or } \quad \mu^{2}\left(\alpha, \beta \ell^{m}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right)\right) .
\end{aligned}
$$

In all cases, the $\beta$ angle of the resulting A or $\mathrm{A} "$ situation comes again with a balloon.
Proof. Our first task is to show that there are no products of the type $\mu^{2}\left(\beta \ell^{m}, \beta \ell^{n}\right)$. But this is obvious from the definition and exactly the same as in the case of zigzag paths: The two $\beta$ angles would need to wind around the same puncture $q$. In order to have the product $\mu^{2}\left(\beta \ell^{m}, \beta \ell^{n}\right), X$ necessarily turns right at $q$ when viewed from $\beta \ell^{m}$, but needs to turn left when viewed from $\beta \ell^{n}$. This shows there is no product $\mu^{2}\left(\beta \ell^{m}, \beta \ell^{n}\right)$. Similarly, there are no products $\mu^{2}\left(\alpha^{\prime} \ell^{m}, \beta \ell^{n}\right)$ and $\mu^{2}\left(\beta \ell^{m}, \alpha^{\prime} \ell^{n}\right)$. It is also easy to see that there is no product of type $\mu^{2}\left(\mathrm{id} \ell^{m}, \mathrm{id} \ell^{n}\right)$. Indeed, from Lemma B. 6 we obtain two contradicting statements regarding the turning of the target strand of the first, equivalently source strand of the second identity.

Let us now filter out a few possible contributions to higher products. Thanks to Lemma B.7 we already know that there are no all-in contributions. Furthermore, the two strands both turn outside the disk, so there are no options to form a $\mu_{\vec{r}}^{\geq 3}(\alpha, \ldots)$ or $\mu_{r}^{\geq 3}(\ldots, \alpha)$ product.

We now dedicate ourselves to working through the list of viable products and verifying their properties. We highly recommend taking the figures as a visual aid for the arguments.

We start by regarding a result $D$ of a product $\mu_{r}^{\geq 3}\left(\beta \ell^{m}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right), \ldots\right)$ with $\beta \ell^{m}$ final-out. The strand of the first arc of the orbigon turns right at the concluding puncture. Its turning angle is clearly at most as long as the remaining part of $\beta$, because otherwise the orbigon $D$ and the balloon of $\beta$ would constitute a teardrop. We obtain a well-defined child: If the turning angle is strictly shorter, we obtain an A situation. If the turning angle is equally long, we obtain instead an $\mathrm{A} "$ situation. In both cases, the resulting $\beta_{R}$ or $\mathrm{id}_{R}$ comes with a balloon, obtained as the joining of the balloon of $\beta$ and the orbigon $D$. The case of $\mu_{\bar{r}}^{\geq 3}(\ldots, \beta)$ with $\beta$ first-out is very similar to the case of $\mu_{\bar{r}}^{\geq 3}(\beta, \ldots)$.

Regard now a result $D$ of a product $\mu_{r}^{\geq 3}\left(\alpha^{\prime} \ell^{m}, \ldots\right)$ with $\alpha^{\prime} \ell^{m}$ final-out or $\mu_{r}^{\geq 3}\left(\ldots, \alpha^{\prime} \ell^{m}\right)$ with $\alpha^{\prime} \ell^{m}$ first-out. The either case, we obtain an A or A" situation at the concluding puncture by Lemma B. 7 In case of final-out $\alpha^{\prime} \ell^{m}$, the result $D$ is of the form $\beta_{R} \alpha_{R} \ell^{m}(\mathrm{~A})$ or $\alpha_{R} \ell^{m}(\mathrm{~A} ")$. In case of first-out $\alpha^{\prime} \ell^{m}$, the result $D$ is of the form $\gamma_{R} \beta_{R} \ell^{m}(\mathrm{~A})$ or $\gamma_{R} \ell^{m}(\mathrm{~A} ")$. In either case, the $\beta_{R}$ or id ${ }_{R}$ angle of the resulting situation comes with a balloon.

Now regard a product $\mu^{2}\left(\beta \ell^{m}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right), \mathrm{id} \ell^{n}\right)$. By definition, the identity comes with a balloon. The $\beta$ angle does not wind around the above end of id, but around the below end of id, because the target strand of id turns right at the above puncture, rendering a composition with $\beta$ impossible. Next, we note that the target strand of the identity turns right at the below side of the source strand and the turning angle is at most $\beta$, for otherwise the combination of the balloons of the identity and $\beta$ would constitute a teardrop. As a result, space remains for defining a type A situation ( $\alpha_{R}, \beta_{R}, \gamma_{R}$ ) or type A" situation $\left(\alpha_{R}, \mathrm{id}_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$. Its $\beta_{R}$ or $\mathrm{id}_{R}$ angle comes with a balloon again, namely the combination of the balloons of $\beta$ and id. The case of $\mu^{2}\left(\mathrm{id} \ell^{m}, \beta \ell^{n}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right)\right)$ is similar.

For the products $\mu^{2}\left(\alpha, \alpha^{\prime} \ell^{m}\right)$ and $\mu^{2}\left(\alpha^{\prime} \ell^{m}, \alpha\right)$, we are not supposed to define anything. Note that it is clear that for this product to exist, $\alpha$ and $\alpha^{\prime}$ need to be precisely complementary angles. For a product $\mu^{2}\left(\beta \ell^{m}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right), \alpha\right)$ or $\mu^{2}\left(\alpha, \beta \ell^{m}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right)\right)$, it is interesting to note that $\alpha$ must be the $\alpha$ or $\gamma$ angle from the same situation as $\beta$. This finishes the proof.

## B. 3 Flowers

In this section, we construct the deformed twisted differential $\delta_{q}$ which uncurves $X$. We start by introducing flowers, a combinatorical gadget that recursively keeps track of all contractible segments of $X$. Such a flower includes by construction all the terms we need to insert into $\delta_{q}$. More precisely, we define $\delta_{q}$ as the sum over the values of all flowers of $X$ and show that $X_{q}=\left(\oplus a_{i}\left[s_{i}\right], \delta_{q}\right)$ is indeed curvature-free.

$$
\text { Flowers of } X \quad \longrightarrow \quad \text { Terms for } \delta_{q}
$$

The core idea of our construction is best explained as follows. We start with $\delta_{q}$ containing only the angles $\alpha_{i}$ and their complements $r \ell^{-1} \alpha_{i}^{\prime}$. This already makes $\mu_{X}^{0}$ and $\mu^{2}\left(\delta_{q}, \delta_{q}\right)$ cancel, however we get a potentially unlimited amount of orbigon contributions from $\mu_{r}^{k \geq 3}\left(\delta_{q}, \ldots, \delta_{q}\right)$. For each of these terms, we need to insert an additional term into $\delta_{q}$ in order to make it cancel out. The new terms inserted into $\delta_{q}$ however can give rise to further disturbing terms in $\mu_{r}^{k \geq 3}\left(\delta_{q}, \ldots, \delta_{q}\right)$ and we need to iteratively repeat this process.

In every step of the process, we should remember the entire history of how a given term was formed, much like the notion of tails or result components for zigzag paths. The tool of flowers which we define here systematically keeps track of the appearing terms. Since a term typically appears recursively for every orbigon that can be formed from already existing terms, the orbigons get stitched together much like a flower. All the terms we insert into $\delta_{q}$ come from orbigons, in particular we can ensure inductively that they come naturally with balloons. This way Lemma B. 8 applies and facilitates the friction-free definition of flowers.

Not only orbigons need to be taken into account. It is possible that at the concluding puncture of an orbigon, the two strands which were separated by the orbigon now come together and keep traveling in parallel for a while, as in Figure B.6a On such occasions, we have to insert a whole sequence of identities into $\delta_{q}$. After a while, the two strands may separate again and wildly continue forming orbigons.

The rule of thumb could be memorized as follows:

- id (A") needs to be inserted when the strands come together.
- id (ID) needs to be inserted when the strands keep running together.
- $\beta$ ( $\left.\mathrm{A}^{\prime}\right)$ needs to be inserted when the strands separate.
- $\beta$ (A) needs to be inserted when the strands come together and immediately separate again.

The construction of flowers is so technical because of the large amount of complexity observed while constructing $\delta_{q}$. Explicitly, it concerns the following complications: First, segments of $X$ can bound multiple orbigons. This makes that there is no linear way of enumerating the terms we need to add to $\delta_{q}$. Instead, the terms will "cross-pollinate" each other. Second, for every orbigon that can be made of terms already present in $\delta_{q}$, we typically add a new $\beta$ (A) angle to $\delta_{q}$. This $\beta$ angle already creates two new products $\mu^{2}(\beta, \alpha)$ and $\mu^{2}(\gamma, \beta)$. Cancelling them entails working with four terms in total. Third, the identities need separate creation and cancellation procedures.

(a) An ID stem flower

(b) An orbigon flower

(c) A compound flower

Figure B.6: Illustration of flowers

Remark B.9. We write

$$
r=\sum_{\substack{m \geq 1 \\ q \in M}} r_{q, m} \ell_{q}^{m} \in \mathfrak{m} Z(\operatorname{Gtl} \mathcal{A}) .
$$

Recall that an orbigon $D$ comes with a deformation parameter $r_{D}$. The element $r_{D}$ lies in the deformation base $B$ and is defined as the product of all $r_{q, m}$ ranging over all orbifold points of $D$.

Definition B.10. A flower comes with the datum of an concluding situation, which is an A, A', A" or ID situation. A flower always comes with the datum of a value, which is a $B$-multiple of id $\ell^{m}$ (A"), id $\ell^{m}$ (ID), $\beta \ell^{m}(\mathrm{~A})$ or $\beta \ell^{m}$ ( $\mathrm{A}^{\prime}$ ) for some $m \geq 0$, depending on the type of the concluding situation.

We distinguish five types of flowers, namely $\alpha$ flowers, $\alpha^{\prime}$ flowers, orbigon flowers, ID stem flowers and compound flowers. The orbigon flowers remember which flowers their orbigon is made of and the compound flowers each remember which flowers they form a compound of. The complete recursive definition of flowers is given in the catalog below. Wherever a flower $F$ is operated on in a formula, its value is meant.
$\alpha$ flowers: Any $\alpha$ (D) angle of $X$ determines a flower $F$.

- The concluding situation of $F$ is the type D situation determined by $\alpha$.
- The value of $F$ is $\alpha$.
$\alpha^{\prime}$ flowers: Any complementary angle $\alpha^{\prime} \ell^{m-1}(\mathrm{D})$ with $m \geq 1$ determines a flower $F$.
- The concluding situation of $F$ is the type D situation determined by $\alpha^{\prime}$.
- The value of $F$ is $r_{q, m} \alpha^{\prime} \ell^{m-1}$.

Orbigon flowers: Let $F_{1}, \ldots, F_{k}$ be a sequence of flowers, together with the datum of a type A situation $\left(\alpha_{R}, \beta_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$ or a type A" situation $\left(\alpha_{R}, \mathrm{id}_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$, two integers $m \geq 0$ and $n \geq 1$ and an orbigon $D$ whose interior angles are the values of $F_{1}, \ldots, F_{k}$ together with $\beta_{R}^{\prime} \ell^{m}$. Then this data defines a new flower $F$.

- The concluding situation of $F$ is the type A situation $\left(\alpha_{R}, \beta_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$ or the A" situation $\left(\alpha_{R}, \mathrm{id}_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$.
- The value of $F$ is a multiple of $\beta_{R} \ell^{n}(\mathrm{~A})$ or $\operatorname{id}_{R} \ell^{n}(\mathrm{~A} ")$, depending on whether the concluding puncture of $F$ is of type A or A". Denote by $q$ the puncture in the middle of the concluding situation and by $r_{D} \in B$ the deformation parameter of the orbigon $D$. Let $\left|F_{i}\right|_{B} \in B$ denote the coefficient of the value of $F_{i}$, stripping away the information which angle it concerns.
In case the given situation is of type A, the precise value of $F$ is defined as

$$
(-1)^{\left\|\beta_{R} \alpha_{R} \beta_{R}^{\prime}\right\| \sum_{i=1}^{k}\left\|F_{i}\right\|+\sum_{i<j}\left\|F_{i}\right\|\left\|F_{j}\right\|+\left\|\beta_{R}\right\|\|\alpha\|+\|\alpha\|} r_{D} r_{m+n+1, q}\left|F_{k}\right|_{B} \ldots\left|F_{1}\right|_{B} \beta_{R} \ell^{n} .
$$

In case the given situation is instead of type A ", the precise value of $F$ is defined as

$$
(-1)^{\left\|\alpha_{R} \beta_{R}^{\prime}\right\| \sum_{i=1}^{k}\left\|F_{i}\right\|+\sum_{i<j}\left\|F_{i}\right\|\left\|F_{j}\right\|} r_{D} r_{q, m+n+1}\left|F_{k}\right|_{B} \ldots\left|F_{1}\right|_{B} \operatorname{id}_{R} \ell^{n} .
$$

ID stem flowers: Let $F_{1}$ be a flower whose concluding situation is an ID situation ( $\mathrm{id}_{R}, \alpha_{R}, \gamma_{R}$ ) or an A" situation $\left(\alpha_{R}, \mathrm{id}_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$. Depending on whether the source and target strands of $\mathrm{id}_{R}$ separate below the situation, we obtain a new A' situation $\left(\alpha_{S}, \beta_{S}, \gamma_{S}\right)$ or ID situation $\left(\alpha_{S}\right.$, id $\left._{S}, \gamma_{S}\right)$. Let $F_{2}$ be an $\alpha^{\prime}$ flower whose value is the complement of $\gamma_{S}$. This defines a new flower $F$.

- The concluding situation of $F$ is the A' situation $\left(\alpha_{S}, \beta_{S}, \gamma_{S}\right)$ or the ID situation $\left(\alpha_{S}, \operatorname{id}_{S}, \gamma_{S}\right)$.
- The value of $F$ is a multiple of $\beta \ell^{m}\left(\mathrm{~A}^{\prime}\right)$ or $\mathrm{id}_{S} \ell^{m}$ (ID), depending on whether the concluding situation is of type $A^{\prime}$ or ID. In case the situation is of A' type, the precise value of $F$ is

$$
(-1)^{\left\|F_{2}\right\|\left\|F_{1}\right\|+\left|F_{1}\right|++\left\|\beta_{S}\right\|\left\|\alpha_{S}\right\|+\left|\alpha_{S}\right|} F_{2} F_{1} \alpha_{S}^{-1}
$$

In case the situation is of ID type, the precise value of $F$ is

$$
(-1)^{\left\|F_{2}\right\|\left\|F_{1}\right\|+\left|F_{1}\right|} F_{1} F_{2} \alpha_{S}^{-1} .
$$

Compound flowers: Let $F_{1}, \ldots, F_{k}$ be a sequence of $k \geq 2$ flowers of ID stem or orbigon type with concluding situations $\left(\alpha_{i}, \beta_{i}, \gamma_{i}\right)\left(\mathrm{A} / \mathrm{A}^{\prime}\right)$, such that (a) $\alpha_{i}$ agrees with $\gamma_{i+1}$ and (b) the angle $\beta_{R}^{\prime}$ given by the union of the $\beta_{i}^{\prime}$ ( or $\mathrm{id}_{i}$ ) angles, including all $\alpha_{i}$ and $\gamma_{i}$, is at most a full turn. Then this data defines a new compound flower $F$.

- The concluding situation of $F$ is the type A or A " situation $\left(\alpha_{R}, \beta_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$ or $\left(\alpha_{R}, \mathrm{id}_{R}, \gamma_{R}, \beta_{R}^{\prime}\right)$ which contains the $\beta_{R}^{\prime}$ just constructed.
- The value of $F$ is a multiple of $\beta_{R} \ell^{n}(\mathrm{~A})$ or $\operatorname{id}_{R} \ell^{n}(\mathrm{~A} ")$, depending on whether the concluding situation is of type A or A". To determine the precise coefficient, we shall use a trick by applying recursion. Regard the flowers $F_{1}, \ldots, F_{k-1}$. In case $k=2$, this is the single ID stem or orbigon flower $F_{1,1}:=F_{1}$. In case $k \geq 3$, this sequence determines a (smaller) compound flower $F_{1, k-1}$. In either case, we can assume that we already know the value of $F_{1, k-1}$.
We distinguish two similar cases: Assume $F_{k}$ itself is an orbigon flower whose interior angles come from the flowers $G_{1}, \ldots, G_{l}$. Let $m \geq 0$ be such that $\beta_{R}^{\prime} \ell^{m}$ together with $G_{1}, \ldots, G_{l}$ are the interior angles of the orbigon $F_{k}$. Then we define the value of $F$ as

$$
(-1)^{\left\|F_{1, k-1}\right\| \sum_{j=1}^{l}\left\|G_{j}\right\|+\left\|\beta_{R}\right\|\left\|\alpha_{R}\right\|+\left|\alpha_{R}\right|} F_{1, k-1}\left(\beta_{R}^{\prime} \ell^{m}\right)^{-1}
$$

Assume otherwise that $F_{k}$ itself is an ID stem flower. Then we define the value of $F$ as

$$
(-1)^{\left\|F_{1, k-1}\right\|+\left\|\beta_{R}\right\|\| \| \alpha_{R} \|+\left|\alpha_{R}\right|} F_{1, k-1} \alpha_{R}^{-1} .
$$

With this sophisticated construction of flowers, we can define $\delta_{q}$ simply as the sum over the values of all flowers of $X$ :

Definition B.11. We put

$$
\delta_{q}=\sum_{F \text { flower of } X} F \in \quad \operatorname{Hom}_{\text {Add }^{4} \operatorname{Gtl}_{r} \mathcal{A}}^{1}(X, X)
$$

Lemma B.12. The element $\delta_{q}$ is well-defined and its leading term is $\delta$.
Proof. Let us explain why $\delta_{q}$ is well-defined. The first observation is that there are only finitely many orbigons for a given sequence of interior angles. In conclusion, for any $C>0$ there is only a finite number of flowers $F$ for which it has taken at most $C$ recursive steps to define $F$. The second observation is that the value of any flower lies in a certain power of the maximal ideal $\mathfrak{m} \subseteq B$. Every time a new flower is formed, its $\mathfrak{m}$-adic exponent increases. Together, both observations show that the series which defines $\delta_{q}$ converges in the $\mathfrak{m}$-adic topology.

It is very easy to see that the leading term of $\delta_{q}$ is $\delta$ : The only flowers whose value does not lie in a power of the maximal ideal are the $\alpha$ flowers. Since $\delta$ is just the sum of all $\alpha$ flowers, this finishes the proof.

We aim at showing that $\sum_{k \geq 0} \mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{k}\left(\delta_{q}, \ldots, \delta_{q}\right)=0$. In order to flexibly cancel terms in this sum, we introduce an obvious notion of result components:

Definition B.13. A result component of $\sum_{k \geq 0} \mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{k}\left(\delta_{q}, \ldots, \delta_{q}\right)$ consists of a sequence of $k \geq 0$ flowers $F_{1}, \ldots F_{k}$ together with an additive component of $\mu_{\text {Add Gtl }_{r} \mathcal{A}}^{k}\left(F_{k}, \ldots, F_{1}\right)$. More precisely,


Figure B.7: Compound cancellation

- for $k=0$ this entails the curvature $\mu_{X}^{0}$,
- for $k=2$ this entails a nonvanishing product $\mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{k}\left(F_{k}, \ldots, F_{1}\right)$,
- for $k \geq 3$ this entails a choice of orbigon contributing to $\mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{k}\left(F_{k}, \ldots, F_{1}\right)$.

Proposition B.14. We have $\sum_{k \geq 0} \mu_{\mathrm{Add} \mathrm{Gtl}_{r} \mathcal{A}}^{k}\left(\delta_{q}, \ldots, \delta_{q}\right)=0$. Therefore $X_{q}=\left(\oplus a_{i}\left[s_{i}\right], \delta_{q}\right)$ is curvaturefree.

Proof. We shall provide a list of cancellations and then check that every result component is contained in this list. We essentially distinguish two types of cancellations, namely the simple and compound cancellations. A sample compound cancellation is depicted in Figure B.7 The precise list of cancellations reads as follows:

- The curvature $\mu_{X}^{0}$ and the $\mu^{2}\left(\alpha, \alpha^{\prime}\right)$ and $\mu^{2}\left(\alpha^{\prime}, \alpha\right)$ result components.
- simple cancellation: Let $F_{1}, \ldots, F_{k}$ be a sequence of flowers. Furthermore, let ( $\alpha_{R}, \beta_{R}, \gamma_{R}, \beta_{R}^{\prime}$ ) be an A situation or $\left(\alpha_{R}, \beta_{R}^{\prime}, \gamma_{R}\right)$ be an A" situation. Then for every orbigon $D$ with interior angles $F_{1}, \ldots, F_{k}, \beta_{R}^{\prime} \ell^{m}$, we have the cancellation

$$
\mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{k}\left(F_{k}, \ldots, F_{1}, \alpha_{R}^{\prime} \ell^{m+n}\right)+\mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{k}\left(\gamma_{R}^{\prime} \ell^{m+n}, F_{k}, \ldots, F_{1}\right)=0
$$

By abuse of notation, we have written $\mu(\ldots)$ where we actually refer to the contribution of the very specific orbigon $D$. We have also denoted by $\alpha_{R}^{\prime}(\mathrm{D})$ and $\gamma_{R}^{\prime}(\mathrm{D})$ the complementary angles of $\alpha_{R}$ and $\gamma_{R}$, in the sense that $\alpha_{R}^{\prime} \alpha_{R}=\ell=\gamma_{R}^{\prime} \gamma_{R}$.

- compound cancellation: Let $F$ be a compound flower consisting of flowers $F_{1}, \ldots, F_{k}$.

Denote by $G_{1}, \ldots, G_{l}$ the flowers that the flower $F_{k}$ is derived from, and by $H_{1}, \ldots, H_{n}$ the flowers that $F_{1}$ is derived from. Denote the concluding situation of $F_{i}$ by $\left(\alpha_{i}, \beta_{i}, \gamma_{i}, \beta_{i}^{\prime}\right)$ or $\left(\alpha_{i}, \beta_{i}, \gamma_{i}, \mathrm{id}_{i}\right)$, depending on whether it concerns an A situation or an A' situation.
Regard the compound flower $F_{1, k-1}$ consisting of $F_{1}, \ldots, F_{k-1}$ and the compound flower $F_{2, k}$ consisting of $F_{2}, \ldots, F_{k}$. In case $k=2$, the flowers $F_{1,1}=F_{1}$ and $F_{2,2}=F_{2}$ are simply ID stem or orbigon flowers, instead of compound flowers. Either way, we have the cancellations

$$
\begin{align*}
\mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{\geq 3}\left(F_{1, k-1}, G_{l}, \ldots, G_{1}\right)+\mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{2}\left(F, \alpha_{k}\right) & =0  \tag{B.1}\\
\mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{\geq 3}\left(H_{n}, \ldots, H_{1}, F_{2, k}\right)+\mu_{\operatorname{Add~Gtl}_{r} \mathcal{A}}^{2}\left(\gamma_{1}, F\right) & =0
\end{align*}
$$

By abuse of notation, $\left.\mu\left(F_{1, k-1}\right), \ldots\right)$ actually refers to the specific orbigon given by $F_{k}$. Similarly, $\mu\left(\ldots, F_{2, k}\right)$ refers to the specific orbigon given by $F_{1}$.
In B.1), we have silently assumed that $F_{1}$ and $F_{k}$ are orbigon flowers. In case $F_{1}$ is instead an ID stem flower, the term $\mu_{\mathrm{Add} \mathrm{Gtl}_{r} \mathcal{A}}^{\geq 3}\left(\ldots, F_{2, k}\right)$ should read $\mu^{2}\left(\mathrm{id}_{1}, F_{2, k}\right)$ instead. Similarly, in case $F_{k}$ is an ID stem flower, $\mu_{\mathrm{Add} \mathrm{Gtl}_{r} \mathcal{A}}^{\geq 3}\left(F_{1, k-1}, \ldots\right)$ should read $\mu^{2}\left(F_{1, k-1}, \mathrm{id}_{k}\right)$ instead.
Finally, let us explain why all possible result components of $\sum_{k \geq 0} \mu_{r}^{k}\left(\delta_{q}, \ldots, \delta_{q}\right)$ are captured in the above cancellation list. Indeed, all results of flowers naturally come with balloons. Therefore Lemma B. 8 applies and any result component falls under one of the following cases:

- Regard a contribution $D$ to $\mu^{k \geq 3}\left(\beta \ell^{m}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right), \ldots\right)$ with $\beta \ell^{m}$ final-out or $\mu^{k \geq 3}\left(\ldots, \beta \ell^{m}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right)\right)$ with $\beta \ell^{m}$ first-out. By construction, $\beta$ comes from the concluding situation of a flower. Whether it concerns an orbigon, ID stem or compound flower, the two terms fall under the compound cancellation.
- Regard a contribution $D$ to $\mu^{k \geq 3}\left(\alpha^{\prime} \ell^{m}(\mathrm{D}), \ldots\right)$ with $\alpha^{\prime} \ell^{m}$ final-out or $\mu^{k \geq 3}\left(\ldots, \alpha^{\prime} \ell^{m}(\mathrm{D})\right)$ with $\alpha^{\prime} \ell^{m}$ first-out. Then all the other interior angles of the orbigon are also the values of flowers and $D$ falls directly under the simple cancellation.
- A contribution $\mu^{2}\left(\beta \ell^{m}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right)\right.$, $\left.\mathrm{id} \ell^{n}\right)$ or $\mu^{2}\left(\mathrm{id} \ell^{m}, \beta \ell^{n}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right)\right)$ falls under the compound cancellation.
- A contribution $\mu^{2}\left(\alpha^{\prime} \ell^{m}(\mathrm{D}), \mathrm{id} \ell^{n}\right)$ or $\mu^{2}\left(\mathrm{id} \ell^{m}, \alpha^{\prime} \ell^{n}(\mathrm{D})\right)$ falls under the compound cancellation.
- Regard a contribution $\mu^{2}\left(\alpha(\mathrm{D}), \beta \ell^{m}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right)\right)$ or $\mu^{2}\left(\beta \ell^{m}\left(\mathrm{~A} / \mathrm{A}^{\prime}\right), \alpha(\mathrm{D})\right)$. Let $F$ be the flower that $\beta$ comes from. Depending on whether $F$ is an orbigon or ID stem flower or a compound flower, the term falls under the simple or compound cancellation.

This shows that all terms have been canceled. It is a basic inspection that all terms have been canceled only once. This shows that $\sum_{k \geq 0} \mu_{\mathrm{Add} \mathrm{Gtl}_{r} \mathcal{A}}^{k}\left(\delta_{q}, \ldots, \delta_{q}\right)=0$ and finishes the proof.

## C Classification of result components

We collect here a few deferred proofs: In section C.1 we prove Lemma 12.11 which classifies result components. In section C.2 we prove Lemma 13.21 which concerns subdisks of CR, ID, DS and DW result components. We spend section C.3 till C.7 with a proof of Lemma 13.23 which classifies the image of D. In section C.8, we prove Lemma 13.25 concerning the signs of result components.

## C. 1 Shape of result components

We prove here Lemma 12.11 which claims that Table 12.5 is an exhausting classification of result components. Recall the situation: We are given a result component of an h- or $\pi$-tree and are supposed to analyze how it is derived. It is not necessary to find the entire tree it is derived from, but only so far that we recognize it fits the scheme of Table 12.5 . We will now go through Table 12.5 case-by-case:
$\alpha_{0}$, id (C), $\alpha_{4}$ from h-trees: Note that $\alpha_{0}$ and id (C) belong to $H$ and hence only appear as direct morphism or result component of a $\pi$-tree. Also, $\alpha_{4}$ does not appear in the disk and multiplication tables as result component of an h-tree and therefore any $\alpha_{4}$ is either direct or a result component of a $\pi$-tree in combination with $\alpha_{3}$.
$\alpha_{3}$ from h-trees: The angle $\alpha_{3}$ does not appear in the disk table as result component of a disk $h_{q} \mu^{\geq 3}$. Whenever it appears as the result component of a product $h_{q} \mu^{2}$, it must be as $h_{q} \mu^{2}\left(\alpha_{0}, \alpha_{4}\right)$ and the involved $\alpha_{0}$ and $\alpha_{4}$ are direct.
$\alpha_{0}^{\prime}$ from h-trees: The angle $\alpha_{0}^{\prime}$ only appears in the disk and multiplication table as $h_{q} \mu^{2}\left(\alpha_{0}, \alpha_{0}^{\prime}\right)$. The $\alpha_{0}$ involved is necessarily direct, and inductively we conclude that the $\alpha_{0}^{\prime}$ is the result component of one of the trees of Figure 12.6b
$\beta(\mathrm{C})$ from h-trees: The angle $\beta(\mathrm{C})$ is necessarily direct or the result component of $h_{q} \mu^{2}\left(\mathrm{id}(\mathrm{C}), \alpha_{0}^{\prime}\right)$. The id (C) is necessarily direct and we already know $\alpha_{0}^{\prime}$ is the result component of one of the trees in Figure 12.6b
$\beta^{\prime}(\mathrm{C})$ from h-trees: Same as $\beta(\mathrm{C})$.
$\beta$ (A) from h-trees: It appears as tail component in the deformed cohomology basis elements by Proposition 11.12 It appears as tail component of $\alpha_{3}$ and $\alpha_{4}$ in the deformed version of $(-1)^{\# \alpha_{3}} \alpha_{3}+$ $(-1)^{\# \alpha_{4}+1} \alpha_{4}$, and as tail component of $\beta(\mathrm{C})$ and $\beta^{\prime}(\mathrm{C})$ in the deformed version of id (C).
$\beta$ (A) main result component of $h_{q} \mu^{\geq 3}$ : Its classification follows directly from the disk tables 12.2 and 12.3 Note that an all-in disk does not produce a $\beta$ (A) result component either.
$\beta$ (A) main result component of $h_{q} \mu^{2}$ : According to Table Table 12.1 this product is either of the type $h_{q} \mu^{2}\left(\beta(\mathrm{~A}), \alpha_{0}\right), h_{q} \mu^{2}\left(\beta(\mathrm{~A})\right.$, id (C)) or $h_{q} \mu^{2}\left(\beta / \beta^{\prime}(\mathrm{C})\right.$, id (C)). In the first two cases, we inductively trace the $\beta(\mathrm{A})$ involved. In the third case, note that we already know the entire tree of $\beta / \beta^{\prime}$ (C). Ultimately, we end up either with $\beta(\mathrm{C})$ or $\beta^{\prime}(\mathrm{C})$ or a $\beta(\mathrm{A})$ that is a direct, $h_{q} \mu^{\geq 3}$ or tail $h_{q} \mu^{2}$ result component, plus multiple compositions with $\alpha_{0}$ or id (C) on the right. All three kinds of trees are depicted in Figure 12.7
$\beta$ (A) tail result component: If it is the tail result component of an $h_{q} \mu^{2}$, then it is necessarily one of the tail result components marked by +E in Table 12.1 All options come with a corresponding main result component $\beta(\mathrm{A}), \beta / \beta^{\prime}(\mathrm{C})$ or $\alpha_{3}$. If it is the tail result component of an $h_{q} \mu^{\geq 3}$, then it is the result component of one of the final-out disks in Figure 12.3 and comes with a corresponding $\beta$ (A).
id (D) result component of an h-tree: According to the disk and multiplication tables, this concerns a product $h_{q} \mu^{2}\left(\mathrm{id}(\mathrm{C}), \alpha_{4}\right)$ or $h_{q} \mu^{2}\left(\alpha_{3}, \mathrm{id}(\mathrm{C})\right)$. The $\alpha_{4}$ and id (C) involved are necessarily direct. Furthermore, the $\alpha_{3}$ is direct or the result component of $h_{q} \mu^{2}\left(\alpha_{0}, \alpha_{4}\right)$. In the latter case, we conclude that $\mu^{2}\left(\alpha_{3}, \mathrm{id}(\mathrm{C})\right)$ equals the co-identity $\alpha_{0}$ itself, and therefore $h_{q} \mu^{2}\left(\alpha_{3}, \mathrm{id}(\mathrm{C})\right)$ vanishes. The id (D) is therefore necessarily a result component of $h_{q} \mu^{2}\left(\mathrm{id}(\mathrm{C}), \alpha_{4}\right)$ or $h_{q} \mu^{2}\left(\alpha_{3}\right.$, id (C)) with all involved $\alpha_{3}, \alpha_{4}$ or id (C) being direct.
id (B) result component of an h-tree: According to the disk and multiplication tables, this is a first-out disk or one of the trees in Figure 12.9.
$\alpha_{3}+\alpha_{4}$ main result components of $\varphi \pi_{q} \mu^{\geq 3}$ : Obvious.
$\alpha_{3}+\alpha_{4}$ main result component of $\varphi \pi_{q} \mu^{2}$ : It necessarily concerns a result component of a product of the type $\varphi \pi_{q} \mu^{2}(\mathrm{id}(\mathrm{C}), \beta(\mathrm{A})), \varphi \pi_{q} \mu^{2}(\beta(\mathrm{~A}), \mathrm{id}(\mathrm{C})), \varphi \pi_{q} \mu^{2}\left(\mathrm{id}(\mathrm{D}), \alpha_{4}\right)$ or $\varphi \pi_{q} \mu^{2}\left(\alpha_{4}, \mathrm{id}(\mathrm{D})\right)$. The first and second case are depicted in Figure 12.8. The third and fourth case are depicted in Figure 12.11
$\alpha_{3}+\alpha_{4}$ tail result component of a $\pi$-tree: It comes either from the G components in Figure 12.3 or from the G components in Table 12.1. In all cases, this concerns a tail component of a certain $\varphi \pi_{q}(\beta \alpha)$. Then the corresponding $h_{q}(\beta \alpha)$ indeed has a corresponding $\beta$ (A) main result component.
id (C) main result component of $\varphi \pi_{q} \mu^{2}$ : It comes from a product $\mu^{2}(\mathrm{id}(\mathrm{C}), \mathrm{id}(\mathrm{D}))$ or $\mu^{2}(\mathrm{id}(\mathrm{D}), \mathrm{id}(\mathrm{C}))$. Since this id (D) is necessarily the result component of the tree in Figure 13.7a or 13.7b we obtain the four id (C) trees in Figure 12.11
id (C) main result component of $\varphi \pi_{q} \mu^{\geq 3}$ : It comes from an all-in disk where the first zigzag path turns right at the concluding arc and the final zigzag path turns left at the concluding arc. An example is depicted in Figure 13.6e
id (C) tail result component of a $\pi$-tree: It comes either from the H components in Figure 12.3 or from the H components in Table 12.1. In all cases, this concerns a tail component of a certain $\varphi \pi_{q}(\beta \alpha)$. Then the corresponding $h_{q}(\beta \alpha)$ indeed has a corresponding $\beta$ (A) main result component.
id (D) result component of a $\pi$-tree: It is either (a) the $\varphi \pi_{q} \mu^{\geq 3}$ of an all-in disk with equal first and final zigzag path, or (b) the result component of $\varphi \pi_{q} \mu^{2}(\mathrm{id}(\mathrm{C}), \mathrm{id}(\mathrm{B}))$ or (c) $\varphi \pi_{q} \mu^{2}(\mathrm{id}(\mathrm{B}), \mathrm{id}(\mathrm{C}))$ or (d) $\varphi \pi_{q} \mu^{2}\left(\mathrm{id}(\mathrm{D})\right.$, id (D)). Option (d) is impossible, since nonvanishing of $\varphi \pi_{q} \mu^{2}(\mathrm{id}(\mathrm{D}), \mathrm{id}(\mathrm{D}))$ implies that both identities involved are the identity id $a_{a_{0}}$ at the zigzag path's identity location, which is in contradiction to the fact that this involved id (D) lies in $R$. Options (a), (b) and (c) are possible and depicted in Figure 12.10 Note the id (B) component involved in (b) and (c) can impossibly come from $\mu^{2}\left(\beta / \beta^{\prime}(\mathrm{C})\right.$, id (C)) or $\mu^{2}\left(\beta(\mathrm{~A}), \alpha_{0}\right)$, because the arrow directions along the disk mismatch resp. because the arrow direction of $\alpha_{0}$ contradicts Convention 10.10. This is reviewed in Figure 13.8
$\alpha_{0}$ result component of a $\pi$-tree: A glance at the multiplication and disk tables reveals that it comes from a product $\pi_{q} \mu^{2}\left(\mathrm{id}(\mathrm{C}), \alpha_{4}\right)$ or $\pi_{q} \mu^{2}\left(\alpha_{3}, \mathrm{id}(\mathrm{C})\right)$ or $\pi_{q} \mu^{2}\left(\alpha_{0}, \mathrm{id}(\mathrm{D})\right)$ or $\pi_{q} \mu^{2}\left(\mathrm{id}(\mathrm{D}), \alpha_{0}\right)$. In the first case, both inputs are necessarily direct. In the second case, id (C) is definitely direct. Meanwhile, $\alpha_{3}$ may be direct or come from $h_{q} \mu^{2}\left(\alpha_{0}, \alpha_{4}\right)$ with both $\alpha_{0}$ and $\alpha_{4}$ direct. In the third and fourth case, $\alpha_{0}$ is direct. Meanwhile, id (D) may come from $h_{q} \mu^{2}\left(\alpha_{3}, \operatorname{id}(\mathrm{C})\right)$ or $h_{q} \mu^{2}\left(\operatorname{id}(\mathrm{C}), \alpha_{4}\right)$ with both $\alpha_{3} / \alpha_{4}$ and id (C) direct. This results in 7 options in total, depicted in Figure 12.12

We have checked all cases of Table 12.5. This finishes the proof of Lemma 12.11.

## C. 2 The shape of subdisks

In this section, we prove Lemma 13.21 Subdisks of CR, ID, DS and DW result components are CR, ID, DS and DW disks, respectively.

Lemma C.1. Subdisks of a CR result components are CR disks: $D\left(\right.$ Result $\left._{\mathrm{CR}}\right) \subseteq$ Disk $_{\mathrm{CR}}$.
Proof. Let $r$ be a CR result component. We show that $\mathrm{D}(r)$ is a CR disk. First, the corners of $\mathrm{D}(r)$ are convex by construction. Second, since the subdisk is obtained from gluing smaller subdisks, this inductively provides that $\mathrm{D}(r)$ is indeed the boundary of some immersed disk. Third, stacked co-identities lie infinitesimally close to each other as in Figure 12.7. but all other input morphisms and the output morphism lie apart.

Regard a stack of co-identities on a zigzag path $L$ used in $r$. We show that $L$ is oriented clockwise with $\mathrm{D}(r)$. Recall from Convention 10.10 that co-identities lie in angles with puncture to the right of the zigzag curve in its natural orientation. The claim now follows from inspection of Figure $12.6 \mathrm{c}, 12.6 \mathrm{~d}$,
12.6 b 12.7 In all cases, our convention implies that the zigzag curve is oriented clockwise with the subdisk.

Lemma C.2. Subdisks of ID result components are ID disks: $\mathrm{D}\left(\right.$ Result $\left._{\text {ID }}\right) \subseteq$ Disk $_{\text {ID }}$.
Proof. First, a subdisk of Figure 12.10 a or 12.10 b has degenerate C input and first and final zigzag paths are oriented towards the interior of the disk. This makes an ID disk.

Second, a subdisk of Figure 12.10 c has degenerate B input. Let us inspect the situation at the concluding arc of the all-in disk involved. Assume $\alpha_{3}$ is the first angle of the disk. Then the degenerate input directly succeeds the output mark and no further input follows at infinitesimally small distance. The final angle of the disk can impossibly be $\alpha_{3}$ or $\alpha_{4}$ due to orientation of the concluding arc, therefore no input precedes the output at infinitesimally small distance. We see that the first, equivalently final zigzag path $L_{1}=L_{N+1}$ is oriented counterclockwise with the subdisk. We conclude that the subdisk is an ID disk. A similar conclusion holds in case $\alpha_{4}$ is the final angle.

Before we tackle DS and DW result components, let us recall the nasty results $h_{q}(\alpha(\mathrm{D}))$. Denote by $S_{\alpha}$ the sequence of arcs running from the source or target of $\alpha$ to the source or target of $\alpha_{0}$, whichever are closer, without hitting $a_{0}$. The codifferential $h_{q}(\alpha)$ is then equal to the signed sum of these arc identities:

$$
h_{q}(\alpha)=\sum_{c \in S_{\alpha}} \operatorname{id}_{c}
$$

We are now ready to deal with DS result components.
Lemma C.3. Subdisks of DS result components are DS disks: $\mathrm{D}\left(\right.$ Result $\left._{\mathrm{DS}}\right) \subseteq$ Disk $_{\mathrm{DS}}$.
Proof. Let $T$ be one of the 8 trees in Figure 12.11 and $r$ a result component. We need to check that its subdisk falls under the condition of DS disks. Indeed, it is bounded by two arcs $a$ and $b$ and lies between a zigzag curve $\tilde{L}$ and its Hamiltonian deformation $\tilde{L}^{\prime}$. We only need to check two borderline conditions: The first condition is that $a \neq a_{0}$ if $\tilde{L}$ is oriented towards the co-identity. The second condition is that the strip has positive width if $\tilde{L}$ is oriented away from the co-identity.

Assume $r$ is a result component with $a=a_{0}$. We will show by inspection that $S_{\alpha}$ runs away from $a_{0}$ in oppposite direction of the orientation of $\tilde{L}$. For this, consider the two cases that the inner product is $\mu^{2}\left(\alpha_{3}, \operatorname{id}(\mathrm{C})\right)$ or $\mu^{2}\left(\operatorname{id}(\mathrm{C}), \alpha_{4}\right)$ and distinguish further regarding arrow directions. The following graphics depicts all four cases, with the horizontal zigzag path being $L$ :


In all four cases, we conclude that the arc sequence $S_{\alpha}$ runs away from $a_{0}$ against the orientation of $\tilde{L}$. This means that $\tilde{L}$ is oriented from $b$ to $a$, in other words: away from the co-identity. This proves the first condition. The second condition is checked similarly.

Lemma C.4. Subdisks of DW result components are DW disks: $\mathrm{D}\left(\right.$ Result $\left._{\mathrm{DW}}\right) \subseteq$ Disk $_{\mathrm{DW}}$.
Proof. This is similar to Lemma C. 3 Let us elaborate nevertheless: By definition, DW disks are a collection of three similar types of disks with $\alpha_{0}$ output. By definition, DW result components are the $\alpha_{0}$ result components of the 7 trees in Figure 12.12. It is our task to check for every of these 7 trees that their subdisks fall under one of the three types of DW disks.

Of the 7 trees, the two trees without $\alpha_{0}$ input fall under the triangle DW disk type. The two trees with an $\alpha_{0}$ input at the beginning fall under the 4 -gon DW disk type with $\alpha_{0}$ succeeding the output mark. The two trees with an $\alpha_{0}$ at the end, as well as the tree with the infinitesimally small subdisk, fall under the 4 -gon DW type with $\alpha_{0}$ preceding the output mark. The additional conditions are checked in the same way as for DS result components. This case distinction finishes the proof.


Figure C.1: Data structures for section C.3 till C. 7

## C. 3 Narrow locations

In this section, we begin proving Lemma 13.23 which states that all CR, ID, DS and DW disks lie in the image of $D$. Let us recall the situation: In section C. 2 we have already shown that D maps only to CR, ID, DS and DW disks. Starting in the present section and ending insection C.7. we show that every CR, $\mathrm{ID}, \mathrm{DS}$ and DW disk is actually reached by D.

Our strategy is to construct an explicit inverse map: In the present section, we analyze the shape of a given CR disk $D \in$ Disk $_{\text {CR }}$. In section C.4 we build a candidate tree $\mathrm{T}(D)$. In section C.5 we build a result component $\mathrm{R}(D)$ of $\mathrm{T}(D)$. In section C.6. we verify that its subdisk $\mathrm{D}(\mathrm{R}(D))$ is equal to $D$ again. We will finish our line of proof in section C.7 by applying similar arguments to ID disks and checking the cases of DS and DW disks combinatorially. The essential data structures for the course of these sections are collected in Figure C. 1 .

Here is the observation that drives our strategy: Imagine $r$ is a result component of a $\pi$-tree with subdisk $D=\mathrm{D}(r)$. The subdisk catalog orders us to draw the subdisk stroke around all inputs of the $\mu^{\geq 3}$ disk and end up near its first/final arc, on both sides of the stroke. The subdisk becomes narrow there! The reader finds examples in Figure 13.6.

Conversely, assume a CR disk $D$ is given without further knowledge. In order to guess a tree $T$, we simply need to record all the narrow locations of $D$ We are ready for a precise definition of narrow locations. The reader may already have a glance at Figure C.2 where all upcoming notions are depicted.

Definition C.5. Let $D$ be a CR disk. Index the angles that the boundary of $D$ cuts, in clockwise order. A narrow location of $D$ consists of two indices $m>l$ on the boundary such that:

- Both $m$ and $l$ lie in angles whose centers (which are punctures) lie on the inside of the disk.
- Let $p_{m}$ be the path connecting $m$ to its puncture and $p_{l}$ the path connecting $l$ to its puncture. Both lift to paths $\tilde{p}_{m}$ and $\tilde{p}_{l}$ in the unit disk model. Then require that $\tilde{p}_{m}$ and $\tilde{p}_{l}$ actually meet.
In particular both punctures are equal. This is the connecting puncture. The union of $\tilde{p}_{m}$ and $\tilde{p}_{l}$ is the connector of $(l, m)$. Identify two narrow locations if they only differ by rotation around the respective punctures (which are co-identities or situation C morphisms around these punctures). A narrow location $(l, m)$ is trivial if $l$ and $m$ only differ by rotation around their connecting puncture. A narrow location $(l, m)$ is indecomposable if it is nontrivial and there does not exist an index $l<n<m$ such that both $(l, n)$ and $(n, m)$ are nontrivial narrow locations. Two narrow locations $(l, m)$ and $\left(l^{\prime}, m^{\prime}\right)$ are disjoint if $l^{\prime} \geq m$ or $l \geq m^{\prime}$ (up to rotation around the connecting punctures). A decomposition of a narrow location $(l, m)$ into indecomposables consists of indecomposable disjoint narrow locations whose union is $(l, m)$.

Lemma C.6. Let $D$ be a CR disk. Then any two indecomposable narrow locations $(l, m)$ and $\left(l^{\prime}, m^{\prime}\right)$ are either nested or disjoint. Any narrow location decomposes uniquely into indecomposables.

Proof. To prove the first claim, assume $(l, m)$ and $\left(l^{\prime}, m^{\prime}\right)$ are neither nested nor disjoint. Without loss of generality, we have $l<l^{\prime}<m<m^{\prime}$. Looking at the unit disk model, the connector of $\left(l^{\prime}, m^{\prime}\right)$ then has to pass through the lift of the connecting puncture of $(l, m)$. In particular the lifts of the connecting


Figure C.2: Illustrations of narrow locations and their terminology
punctures of $(l, m)$ and $\left(l^{\prime}, m^{\prime}\right)$ are actually equal. Moreover, since the paths $\tilde{p}_{l}, \tilde{p}_{l^{\prime}}, \tilde{p}_{m}, \tilde{p}_{m^{\prime}}$ now all meet, we have that all of $\left(l, l^{\prime}\right),\left(l^{\prime}, m\right),\left(m, m^{\prime}\right)$ are actually narrow locations, contradicting indecomposability.

For the second claim, existence of a decomposition is clear (one keeps decomposing until the components are indecomposable). Uniqueness follows from the first claim.

## C. 4 Narrow trees

In this section, we introduce narrow trees. The idea is to capture all narrow locations of a given CR disk in a structured way. Since we have already seen that narrow locations are nested or disjoint, the most natural structure to capture them is a tree.

Definition C.7. Let $D$ be a CR disk with boundary of length $|D|$. Then its narrow tree is the ordered decorated tree defined as follows:

- The nodes are all indecomposable narrow locations.
- The nodes are connected according to inclusion.
- The nodes are ordered horizontally from high $(l, m)$ to low $(l, m)$.
- The nodes are decorated with their narrow location $(l, m)$.
- Except in the case where $(1,|D|)$ is a narrow location, insert a root standing for the artificial narrow location $(1,|D|)$. The root's children are the maximal indecomposable narrow locations.

An schematic example of a disk and its narrow tree is shown in Figure C.2i. In this example, the root is named R. Note that the decorations $(l, m)$ have been ignored in the narrow tree. In fact, it is not even possible to given concrete numbers for the decorations $(l, m) \in \mathbb{N} \times \mathbb{N}$, since we have only drawn the disk schematically.

Definition C.8. Let $D$ be a CR disk and $(l, m)$ a narrow location (other than the artificial narrow location). Note that $(l, m)$ contains multiple identified pairs $\left(l^{\prime}, m^{\prime}\right)$. The connector of minimal $l^{\prime}$ and maximal $m^{\prime}$ is the upper connector and the connector of maximal $l^{\prime}$ and minimal $m^{\prime}$ the lower connector.

Let above $(l, m)$ refer to the portion of $D$ minus the disk bounded by the upper connector and the corresponding $D$ boundary segment, within $(l, m)$ refer to the portion of $D$ between upper and lower connector, and below $(l, m)$ refer to the portion of $D$ bounded by the lower connector and the corresponding $D$ boundary segment. The segment of the boundary of $D$ within $(l, m)$ splits into two components, which are the left-within and right-within the narrow location. The upper boundary of $(l, m)$ consists of the two situation A arcs just above $(l, m)$ and the lower boundary of $(l, m)$ consists of the two situation A arcs just below $(l, m)$.

Now let $(l, m)$ be an indecomposable narrow location of an immersed disk $D$. The stray morphisms are the input morphisms that are below $(l, m)$, but above all of its children. A (direct) left (resp. right) sibling consists of a sibling $\left(l^{\prime}, m^{\prime}\right)$ of $(l, m)$ in the narrow tree such that $l^{\prime}=m$ up to rotation around the connecting puncture (resp. $m^{\prime}=l$ ). A (direct) sibling is a direct left or right sibling.

A stack in $(l, m)$ is one of the following: (a) a combination of a stray morphism $\alpha_{3}+\alpha_{4}$ directly followed by an $\alpha_{0}$ within the next angle being cut, or (b) the combination of multiple $\alpha_{0}$ and id (C) differing only by rotation around a puncture, or (c) a child together with all siblings and their within morphisms, or (d) any stray morphism that is not part of one of these combinations. The narrow location is 2-rich if it has at least two stacks. It is 1-rich if it has precisely one stack, and if this stack contains a child then it has a morphism within or a sibling. The narrow location is $\mathbf{0}$-rich if it has precisely one stack, and this stack is a child without morphisms within and without direct sibling.

Remark C.9. Due to zigzag consistency, any narrow location has at least one child or at least one stray morphism. This means that an indecomposable narrow location $(l, m)$ is either 2 -rich, 1 -rich or 0 -rich.

## C. 5 Subresults

In this section, we build a result component from any given CR disk. More precisely, we associate to every CR disk $D$ a $\pi$-tree $\mathrm{T}(D)$ with a result component $\mathrm{R}(D)$, in the hope that $\mathrm{D}(\mathrm{R}(D))$ equals $D$ again. We will call $\mathrm{T}(D)$ the subtree and $\mathrm{R}(D)$ the subresult associated with $D$.

The strategy of constructing $\mathrm{T}(D)$ is to take the narrow tree as a starting point and keep inserting $h_{q} \mu^{2}$ nodes to bind together morphisms that lie directly next to each other. We also have to put special attention to the relation of narrow locations: Siblings have to inserted in a specific order, sometimes irregularly.

Let us start from regarding a narrow location $(l, m)$. Whenever $(l, m)$ has a left direct sibling or a morphism left-within, we can easily make a final-out $\mu^{\geq 3}$ disk of $(l, m)$, whose final morphism stems from the direct sibling and the further morphisms left-within $(l, m)$. Indeed, $(l, m)$ has at least one child or a stray morphism, therefore this really yields a $\mu_{\mathrm{Tw} \mathrm{Gtl}_{q} Q}^{\geq 2}$.

However in case $(l, m)$ has no left direct sibling and no morphism left-within, we have to distinguish whether $(l, m)$ is 2 -rich, 1 -rich or 0 -rich. In the 2 -rich case, we can proceed with an ordinary $\mu^{\geq 3}$ and obtain a main result component. In the 1-rich case, we can proceed with $\mu^{2}$ and obtain a first-order tail result component. In the 0 -rich case, we interpret $(l, m)$ as a tail node of the first 1 -rich or 2 -rich narrow location we arrive at when tracing the tree from $(l, m)$ towards the leaves.

This is our basic recipe of turning narrow locations into trees. Let us record a lemma affirming that this construction works.

Lemma C.10. Let $D$ be a CR disk and let $(l, m)$ be either:

- an indecomposable narrow location. Then let $E$ be the sequence of zigzag segments below ( $l, m$ ) and above all children, together with the lower boundary of $(l, m)$ and the upper boundary of all children.
- the root $(1,|D|)$ of the narrow tree of $D$, and assume it is of type id (C). Then let $E$ denote the sequence of zigzag segments starting at the output mark, staying above the children of $(1,|D|)$, and ending at the output mark, including the $2 / 5$ arc at the output mark.
- the root $(1,|D|)$ of the narrow tree of $D$, and assume it is of type $\alpha_{3}+\alpha_{4}$. Then let $E$ denote the sequence of zigzag segments starting at the output mark, staying above the children of $(1,|D|)$, and ending at the output mark, excluding the $2 / 5$ arc at the output mark.
Then $E$ bounds a discrete immersed disk. Upon reversing Figure 13.5 and the stack figures thereafter, the sequence of stacks injects into the sequence of interior boundary angles of $E$. The complement of the image consists of $\delta$ insertions and the $\beta$ (A) morphisms at the children.

In exception to this assignment, a stack directly after or before a child (or the output mark in case 1) shall map to the entire corresponding $\beta$ (A) morphism in $F$, not only the $\beta$ (A) morphism surrounding the left or right part. This means that if there is both a stack directly after and a stack directly before the output mark (or the output mark in case 1), they map in particular to the same $\beta$ ( A ), in exception to injectivity.

Proof. Note that all $D$ boundary segments involved can be split into small pieces cutting through indecomposable angles. We shall now construct the discrete immersed disk $F$ as follows. At each of these angles whose puncture lies outside $D$, flow $E$ outwards to the puncture.

At each of the angles whose puncture lies inside $D$, flow $E$ inwards to the puncture. At all angles whose puncture lies outside $D$, this procedure enlarges the disk and in particular keeps it immersed. If $F$ becomes non-immersed as discrete disk, then this is due to two angles whose punctures lie inside $D$ and meet. In other words, loss of immersedness constitutes a narrow location $\left(l^{\prime}, m^{\prime}\right)$ of $D$.

Let us check the possible locations of $\left(l^{\prime}, m^{\prime}\right)$. Note that $\left(l^{\prime}, m^{\prime}\right)$ cannot equal $(1,|D|)$. Indeed, in case $(1,|D|)$ is a narrow location at all, it is nontrivial by assumption, its lower boundary consists of merely two arcs and it does not pose an obstruction to immersedness at all. Similarly, deduce that $\left(l^{\prime}, m^{\prime}\right)$ is not a trivial narrow location.

Since $F$ cuts all children away and children are indecomposable, any child is either contained in $\left(l^{\prime}, m^{\prime}\right)$ or disjoint. Splitting $\left(l^{\prime}, m^{\prime}\right)$ into indecomposable narrow locations then necessarily yields a chain of direct siblings, which are children of $(l, m)$. By construction of $F$, such a chain of direct siblings children does not constitute an obstruction to immersedness at all.

The second part of the statement consists of generically checking whether any two consecutive stacks occurring along the boundary of $D$ may fall into the same interior boundary angle. For all subdisks in Figure 13.2 this is definitely not the case. We shall therefore check that Figure 13.2 actually displays all possible consecutive input stacks. We will illustrate this in case of an $\alpha_{3}+\alpha_{4}$ input, whose $2 / 5$ arc is oriented counterclockwise with the disk, and an arbitrary successor and predecessor.

Since $D$ has convex corners, it cuts the two angles $\alpha_{2}$ and $\alpha_{3}$ before respectively after $\alpha_{3}+\alpha_{4}$. By arrow directions, possible $\alpha_{0}$ inputs may occur both on $\alpha_{2}$ and $\alpha_{3}$. Since we assumed the co-identity rule, there is at most one $\alpha_{0}$ on both angles. By construction, we assign to the combination of $\alpha_{3}$ and possibly one $\alpha_{0}$ the corresponding $\alpha_{3}$ interior boundary morphism. Regardless of its precise nature, the predecessor stack will definitely not map to $\alpha_{3}$ at the same time.

It remains to check the the successor stack. If the successor lies directly at the target of $\alpha_{3}$, then due to arrow direction that morphism produces the next $\alpha_{3}$ type interior boundary angle indeed lying one arc apart. If the successor lies farther apart, it does not neighbor with $\alpha_{3}$ anymore and will not yield this as interior boundary angle, which is also in line with injectivity. The three combinations that do occur are those in Figures $13.2113 .2 \mathrm{~m}, 13.2 \mathrm{o}$

We are now ready for the complete definition of subresults.
Definition C.11. Let $D$ be a CR disk. Then its subresult is the result component $\mathrm{R}(D) \in \operatorname{Result}_{\pi}$ on the subtree $\mathrm{T}(D)$ constructed below.

Basic structure of $\mathrm{T}(D)$ and $\mathrm{R}(D)$ :
Inductively for every node in the narrow tree of $D$, we construct a corresponding h-tree (and finally $\pi$-tree) and explain how it shall be inserted into $\mathrm{T}(D)$. We refer to any h-tree being constructed during this process as a subtree. We select inductively for every node of $\mathrm{T}(D)$ a single result component. This gives a final result component $\mathrm{R}(D)$ of $\mathrm{T}(D)$. The construction of $\mathrm{T}(D)$ is intended to inverse the process of taking the subdisk of result components. One can keep an eye on this during the construction of $\mathrm{T}(D)$.

Non-root 0 -rich nodes $(l, m)$ without left direct sibling and without morphisms left-within:
Interpret the narrow location $(l, m)$ as tail node. Trace the tree down to the first narrow location that is 1-rich or 2-rich. Then this narrow location produces a $\beta$ (A) tail result component. Use this as result component of the subtree. An example is depicted in Figure C.3a
Non-root 1-rich nodes $(l, m)$ without left direct sibling and without morphisms left-within:
Now $(l, m)$ has precisely one stack, and it is either a stack of $\alpha_{4}$ and $\alpha_{0}$, or of multiple $\alpha_{0}$ and id (C), or a combination of at least two direct siblings or within morphisms, or just a single stray morphism. In all cases, binding the stack together yields the desired first-order $\beta$ (A) first-order tail result component. Note that in the case of a single stray morphism, the $\beta$ (A) is actually an additive component of the deformed cohomology input element, and constitutes a leaf node of $\mathrm{T}(D)$. An example is depicted in Figure C.3b
Non-root nodes $(l, m), 2$-rich or with a left direct sibling or a morphism left-within:
Bind all stacks of $(l, m)$ together. They will serve as inputs for a final-out disk $h_{q} \mu^{\geq 3}$. Now let us prepare the final-out morphism. In case $(l, m)$ has a left direct sibling, the final-out morphism is the $\beta(\mathrm{A})$ output of the left direct sibling. If there are morphisms left-within but no left direct sibling, bind them together in a $\beta(\mathrm{A})$ or $\beta / \beta^{\prime}(\mathrm{C})$ tree and use this as final-out morphism. If there are no left direct siblings and no left-within morphisms, then use a simple $\delta$ insertion as as final-out morphism. Note that we have ensured that in terms of $\mathrm{Tw} \mathrm{Gtl} l_{q}$, this node is really $\mu_{\mathrm{Tw}_{\mathrm{w}}}^{\geq 2} \mathrm{Gt}_{q}$ and not $\mu_{\mathrm{Tw} \mathrm{Gt1}}^{q}$. After making the $h_{q} \mu^{\geq 3}$ disk, compose it afterwards with any id (C) or $\alpha_{0}$ morphisms that may lie right-within $(l, m)$. An example is depicted in Figure C.3c

The $\alpha_{3}+\alpha_{4}$ case. If the output is of type $\alpha_{3}+\alpha_{4}$, then $(1,|D|)$ itself is a narrow location. We will treat the case where $(1,|D|)$ is trivial first and the non-trivial cases afterwards.
Output of type $\alpha_{3}+\alpha_{4}$ with $(1,|D|)$ being a trivial narrow location:
If $(1,|D|)$ is a trivial narrow location, then $D$ revolves around a single puncture, with only id (C) and $\alpha_{0}$ inputs. The corresponding $\pi$-tree is found by composing from left to right.
Output of type $\alpha_{3}+\alpha_{4}$ with $2 / 5$ arc pointing away from $D$, with $(1,|D|)$ having a right-within morphism:
Due to arrow direction, there does not lie an $\alpha_{0}$ input in the very first angle cut by $D$, after the output mark. This means the very first input of $D$ is an id (C) just one cut angle after the output mark. Build the subtree of $(1,|D|)$ as if it were a non-root node, only putting finally $\varphi \pi_{q} \mu^{2}$ instead of $h_{q} \mu^{2}$. The final $\mu^{2}$ is a product of $\beta(\mathrm{A})$ and id (C). Since the $2 / 5$ arc points away from $D$, this $\mu^{2}$ has a $\alpha_{4}$ component and its $\varphi \pi_{q} \mu^{2}$ has the desired $\alpha_{3}+\alpha_{4}$ main result component. An example is depicted in Figure C.3d
Output of type $\alpha_{3}+\alpha_{4}$ with $2 / 5$ arc pointing away from $D$, with $(1,|D|)$ without right-within morphisms, but decomposable or with a left-within morphism:

Build the subtree of $(1,|D|)$ as if it were a non-root node, only putting finally $\varphi \pi_{q} \mu^{\geq 3}$ instead of $h_{q} \mu^{\geq 3}$. Let us check that this indeed yields a $\alpha_{3}+\alpha_{4}$ result component. Indeed, the final evaluation $\mu^{\geq 3}$ yields $\alpha_{4}$ and $\varphi \pi_{q} \mu^{\geq 3}$ has the desired $\alpha_{3}+\alpha_{4}$ main result component. An example is depicted in Figure C.3e.
Output of type $\alpha_{3}+\alpha_{4}$ with $2 / 5$ arc pointing away from $D$, with $(1,|D|)$ without morphisms within, indecomposable and 0 -rich:

Interpret the narrow location $(l, m)$ as tail node, do not insert a node into $T$, and continue with the child.
Output of type $\alpha_{3}+\alpha_{4}$ with $2 / 5$ arc pointing away from $D$, with $(1,|D|)$ without morphisms within, indecomposable and 1-rich:

Then take the subtree of $(1,|D|)$ as if it were not the root node, but put $\varphi \pi_{q} \mu^{2}$ instead of $h_{q} \mu^{2}$ and $\varphi \pi_{q} \mu^{\geq 3}$ instead of $h_{q} \mu^{\geq 3}$ as root. As established in the multiplication and disk tables, this indeed yields an $\alpha_{3}+\alpha_{4}$ tail result component.
Output of type $\alpha_{3}+\alpha_{4}$ with $2 / 5$ arc pointing towards $D$, with $(1,|D|)$ having a morphism left-within:
Note that in the final angle being cut before the output mark, there can not lie an $\alpha_{0}$ input be-
cause of the arc direction. Build the subtree of $(1,|D|)$ now as if $(1,|D|)$ were not the root node, excluding the final id (C) for the moment. This subtree has a $\beta(\mathrm{A})$ result component that is in fact an indecomposable angle. Finally, compose the remaining id (C) with this $\beta$ (A) and obtain $\alpha_{3}+\alpha_{4}$ as main result component.
Output of type $\alpha_{3}+\alpha_{4}$ with $2 / 5$ arc pointing towards $D$, with $(1,|D|)$ without morphism left-within, but decomposable or with morphism right-within:

Decompose $(1,|D|)$ into indecomposables $C_{1}, \ldots, C_{k}$ (note $k \geq 1$ ). First combine $C_{2}, \ldots, C_{k}$ from left to right, together with their morphisms within and the morphisms right-within $C_{1}$. By assumption there are direct siblings to be combined or there is at least one morphism right-within $C_{k}$. This yields a $\beta(\mathrm{A})$ or $\beta / \beta^{\prime}(\mathrm{C})$ result component. Now use this as first-out morphism to bind a $\varphi \pi_{q} \mu^{\geq 3}$ disk of $C_{1}$. This disk yields a result $\mu^{\geq 3}=\alpha_{2}$ and hence a main result component $\varphi \pi_{q} \mu^{\geq 3}=\alpha_{3}+\alpha_{4}$. An example is depicted in Figure C.3f
Output of type $\alpha_{3}+\alpha_{4}$ with $2 / 5$ arc pointing towards $D$, with $(1,|D|)$ indecomposable without morphisms within and 0 -rich:

Then $\alpha_{3}+\alpha_{4}$ actually appears as tail result component of a $\varphi \pi_{q} \mu^{2}$ or $\varphi \pi_{q} \mu^{\geq 3}$ further downwards. Output of type $\alpha_{3}+\alpha_{4}$ with $2 / 5$ arc pointing towards $D$, with $(1,|D|)$ indecomposable without morphisms within and 1-rich:

By zigzag consistency, the stack of $(1,|D|)$ cannot consist of a single morphism and also not of the combination of $\alpha_{4}$ and $\alpha_{0}$. Therefore binding this stack together yields a final $\mu^{2}$ component equal to $\beta \alpha$ (A), $\beta^{\prime} \alpha_{2}$ or $\beta \alpha_{3}$. Now its $\varphi \pi_{q} \mu^{2}$ has the desired $\alpha_{3}+\alpha_{4}$ first-order tail result component.
Output of type $\alpha_{3}+\alpha_{4}$ with $2 / 5$ arc pointing towards $D$, with ( $1,|D|$ ) indecomposable without morphisms within and 2-rich:

Then make a $\varphi \pi_{q} \mu^{\geq 3}$ first-out disk of $(1,|D|)$ with a $\delta$-insertion as first morphism. Note that due


The id (C) case. If the output is of type id (C), then $(1,|D|)$ is not a narrow location itself. Building the disk is a little easier.

Output of type id (C), with $(1,|D|) 0$-rich:
Then trace the tree from $(1,|D|)$ towards the leaves. Pick the subtree of the first narrow location that is 1 -rich or 2 -rich. Note it has a $\beta$ (A) main result component. Changing its root from $h_{q} \mu^{\geq 3}$ to $\varphi \pi_{q} \mu^{\geq 3}$ or from $h_{q} \mu^{2}$ to $\varphi \pi_{q} \mu^{2}$ then yields the desired id (C) tail result component.
Output of type id (C), with $(1,|D|)$ 1-rich:
In case of a single child with morphism within, it has a subtree with $h_{q} \mu^{2}$ main result component $\beta$ (A) associated. The corresponding $\varphi \pi_{q} \mu^{2}$ version has the desired id (C) first-order tail result component. In case of a stack, this stack cannot consist of a single stray morphism, since this would contradict zigzag consistency. Instead, it must be a stack of $\alpha_{4}$ and $\alpha_{0}$ or multiple $\alpha_{0}$ and id (C). Such a stack comes with a subtree $h_{q} \mu^{2}$ and main result component $\beta$ (A) or $\beta / \beta^{\prime}$ (C). Upon replacing $h_{q} \mu^{2}$ by $\varphi \pi_{q} \mu^{2}$, we obtain the desired id (C) main result component.
Output of type id (C), with (1, $|D|)$ 2-rich:
We now have at least one of the following: two children, one child and one stray morphism, or no child and two stray morphisms. Pick the subtrees of all children, binding direct siblings and morphisms within together as usual. Bind all stacks of stray morphisms together. Finally, tie everything into a $\mu^{\geq 3}$ disk.

## The id (D) case.

Output of type id (D), with $(1,|D|) 0$-rich:
Trace the tree downwards to the leaves. Pick the subtree of the first narrow location that is 1-rich or 2 -rich. Note it has a $\beta$ (A) main result component. Changing its root from $h_{q} \mu^{\geq 3}$ to $\varphi \pi_{q} \mu^{\geq 3}$ or from $h_{q} \mu^{2}$ to $\varphi \pi_{q} \mu^{2}$ then yields the desired id (D) tail result component.

Output of type id (D), with $(1,|D|)$ 1-rich:
Then $(1,|D|)$ contains a single stack, and this stack is not a single child without morphisms within. This means the stack produces a $\beta(\mathrm{A})$ or $\beta / \beta^{\prime}(\mathrm{C})$ main result component and the corresponding $\varphi \pi_{q} \mu^{2}$ or $\varphi \pi_{q} \mu^{\geq 3}$ version includes the desired id (D) first-order tail result component.
Output of type id (D), with $(1,|D|)$ 2-rich:
Bind together all stacks in $(1,|D|)$ and finally take $\varphi \pi_{q} \mu^{\geq 3}$. This is possible due to Lemma C. 10

## C. 6 Verifying the inverse

In this section, we verify that D maps CR result components bijectively to CR disks. Believe it or not, we have defined $\mathrm{R}(D)$ in such a way that its subdisk is $D$ again. This already shows that D reaches all CR disks. Proving injectivity of $D$ on Result ${ }_{C R}$ is harder and requires further constructions.
Lemma C.12. Assigning subresults provides a map $R:$ Disk $_{C R} \rightarrow$ Result $_{\pi}$. We have $\mathrm{D} \circ \mathrm{R}=\left.\mathrm{Id}\right|_{\text {Disk }_{C R}}$.
Proof. This follows inductively from the construction of $\mathrm{R}(D)$.
The above lemma already shows that D reaches all CR disks. Prove injectivity of D on Result ${ }_{\mathrm{CR}}$ is harder. It requires us to show how to reconstruct the basic structure of $r$ from $\mathrm{D}(r)$. By basic structure, we mean a very specific notion: the evaluation tree.

Definition C.13. Let $r$ be a result component of a Kadeishvili $\pi$-tree ( $T, h_{1}, \ldots, h_{N}$ ). Then its evaluation tree is the decorated ordered tree defined as follows:

- There is a node for every tail result component of type $h_{q} \mu^{2}$ or $\varphi \pi_{q} \mu^{2}$ and every result component of type $h_{q} \mu^{\geq 3}$ or $\varphi \pi_{q} \mu^{\geq 3}$ used in $r$.
- For every node, insert as many subsequent nodes above as the used tail part is long.
- Connect the nodes according to the tree structure of $T$.
- Order the nodes horizontally according to their horizontal appearance in $T$.
- Regarding decoration of a node $X$, note that $X$ determines a result component of a subtree on its own and comes with a $\beta(\mathrm{A})$ morphism. This determines a narrow location of $D(r)$ and is the decoration of $X$.
Lemma C.14. Let $r$ be a result component of a $\pi$-tree. Then the narrow tree of $\mathrm{D}(r)$ is equal to the evaluation tree of $r$.

Proof. All nodes in the evaluation tree of $r$ stand for taking immersed disks and yielding $\beta$ (A) morphisms and hence determine narrow locations. The nodes of the evaluation tree are also connected according to inclusion. Now let us show by induction on the height that this inclusion of the evaluation tree in the narrow tree is actually surjective. Let $N$ be a node in the narrow tree all of whose children appear in the evaluation tree. We will show that then also $N$ appears in the evaluation tree. In fact, $r$ needs to bind all children $C_{1}, \ldots, C_{k}$ together in some immersed disk. This determines a node $M$ in the evaluation tree, and also a narrow location $M$. But now the children are nested in both $M$ and $N$, which means one of $M$ and $N$ is included in the other. Now if $M$ is strictly included in $N$, then all children $C_{1}, \ldots, C_{k}$ are not direct children of $M$, in contradiction to our assumption. If $N$ is strictly included in $M$, then $M$ can impossibly be an immersed disk since a version shorter on both sides already bounds a disk. Therefore $M=N$ as narrow locations and since $M$ appears in the evaluation tree, we have that $N$ appears in the evaluation tree, which was to be shown. Finally given the equal structure of the trees, their decorations are also equal.

Equipped with this characterization, we are ready to prove D injective on Result ${ }_{\mathrm{CR}}$.
Lemma C.15. The map D : Result ${ }_{\mathrm{CR}} \rightarrow$ Disk $_{\mathrm{CR}}$ is injective, and hence $\mathrm{D}:$ Result ${ }_{\mathrm{CR}} \xrightarrow{\sim}$ Disk $_{\mathrm{CR}}$.
Proof. Let $r_{1}, r_{2} \in$ Result $_{\mathrm{CR}}$ denote two $\pi$-trees together with result components whose subdisks $\mathrm{D}\left(r_{1}\right)$ and $\mathrm{D}\left(r_{2}\right)$ are equal. By Lemma C.14 the evaluation trees of $r_{1}$ and $r_{2}$ are then equal. This means that these trees differ only by the order in which stacks are bound and the order in which the nodes of the evaluation trees are linked together and bound together with morphisms within. It is readily checked using the disk and multiplication schemes of Table 12.1, 12.2 and 12.3 that there is only a unique way to bind these combinations.

(a) A non-root node of the narrow tree. This example has no siblings, no morphisms left-within, and is 0 -rich. Whatever is below the dots, $\beta$ becomes a tail result component.

(b) A non-root node of the narrow tree. This example has no siblings, no morphisms left-within, and is 1 -rich. $\beta$ becomes a tail result component of the stack of the 3 indicated inputs.

(c) A non-root node of the narrow tree. This example has siblings, and those are bound left to right.

(d) An $\alpha_{3}+\alpha_{4}$ output with $2 / 5$ arc pointing away from $D$. Siblings are bound left-to-right. In this example, sibling 1 has no morphisms left-within. Its subtree is a $\mu^{\geq 3}$ disk or a tail result component.

(e) An $\alpha_{3}+\alpha_{4}$ output with $2 / 5$ arc pointing away from $D$. Siblings are bound left-to-right. In this example, sibling 1 is bound as a final-out disk with the stack formed by its three morphisms left-within.

(f) An $\alpha_{3}+\alpha_{4}$ outputs with $2 / 5$ arc pointing towards $D$. Siblings 2 and higher are bound first. The result is then used as first morphism for a first-out disk of sibling 1 . In this example, sibling 2 is assumed to be 2 -rich and has no morphisms left-within, therefore a $\delta$ insertion is used.

Figure C.3: Examples of subtree construction.

For example, a stack of $\alpha_{4}$ and $\alpha_{0}$ can only be bound in one way, the choice of result component is clear from the next higher node in the evaluation tree. A stack of $\alpha_{0}$ and id (C) morphisms can only be bound in one way. A stack of direct siblings and $\alpha_{0}$ and id (C) morphisms can also be bound only in one way. Essentially, all steps in the construction of $\mathrm{T}(D)$ are the unique way to obtain a result component. In other words, there is only one way to compose a result component whose evaluation tree equals the narrow tree of $\mathrm{D}\left(r_{1}\right)=\mathrm{D}\left(r_{2}\right)$. We conclude that $r_{1}=r_{2}$.

## C. 7 The case of ID, DS and DW disks

In this section, we verify that D sends ID, DS and DW result components bijectively to ID, DS and DW disks. For ID disks, we sketch inverse constructions similar to the CR case. For DS and DW disks, the statement reduces to combinatorics.

Let us first dedicate ourselves to ID result components.
Lemma C.16. The map D : Result ${ }_{\text {ID }} \rightarrow$ Disk $_{\text {SL }}$ is injective and its image is precisely Disk ${ }_{\text {ID }}$.
Proof. Injectivity is similar to the case of CR result components. We now show that all ID disks are reached by D. Let $D$ be an ID disk. We provide a preimage through explicit construction. In all cases it can be checked that its subdisk is $D$ again. We shall distinguish the "regular" inputs from the degenerate ones.

First, assume the degenerate input is of C type. Then the procedure is similar to the CR case. We construct a result component of Figure 12.10a or 12.10b Let us apply the formalism of result components to $D$. Regard the narrow location $(1,|D|)$ itself.

If $(1,|D|)$ is trivial, then evaluate from left to right all identities except the degenerate input and the final regular one. Then compose with the final regular id (C) and note that this precisely produces an id (B) result component of Figure 12.10 a or 12.10 b . Note that $D$ may have stacked $\alpha_{0}$ inputs directly after the output mark. They are welcome in our construction: They are composed one after another with the ultimate $\beta(\mathrm{A})$.

If $(1,|D|)$ is nontrivial, decompose it into narrow locations $C_{1}, \ldots, C_{k}$. As in the CR case, generally from left to right. In particular, if there is no regular identity left-within $C_{1}$, then evaluate $C_{2}, \ldots, C_{k}$ first and use their result for a first-out disk of $C_{1}$. If there is however a regular identity left-within $C_{1}$, then evaluate $C_{1}, \ldots, C_{k}$ entirely from left to right. In both cases this yields $h_{q}\left(\alpha_{2}\right)=\mathrm{id}(\mathrm{B})$. Finally compose with the degenerate id (C) and obtain the desired id (C) result component. Again, note that stacked $\alpha_{0}$ inputs directly after the output mark are welcome, and that the orientation of the $2 / 5$ arc at the degenerate id $(\mathrm{C})$ is relevant.

Second, assume the degenerate input is of B type. We construct an all-in disk as in Figure 12.10c Denote by $L$ the source and target zigzag path of $D$. Denote by $a_{0}$ its identity arc. The degenerate B input just before the output mark dictates that $L$ turns right at $a_{0}$.

If $L$ is oriented clockwise with $D$, then the B input preceding the output identity gives rise to an $\alpha_{4}$. As in Lemma C.10 the stacks of $D$ inject into the boundary angles of an all-in disk. The tree corresponding to $D$ is obtained by binding all stacks into trees, and then evaluating their all-in disk with $\mu^{\geq 3}=\operatorname{id}(\mathrm{D})$ and hence $\varphi \pi_{q} \mu^{\geq 3}=\mathrm{id}(\mathrm{D})$.

If $L$ is oriented counterclockwise with $D$, then the B input succeeding the output identity gives rise to an $\alpha_{3}$. Again, an all-in disk yields the desired id (D) result component.

Next are DS result components. Since both DS result components and DS disks are defined by combinatorics, this reduces to simple checks.
Proposition C.17. The map D : Result ${ }_{\mathrm{DS}} \rightarrow$ Disk $_{\mathrm{SL}}$ is injective and its image is precisely Disk ${ }_{\mathrm{DS}}$.
Proof. Let a DS disk of $L$ and $(a, b)$ be given. We recapitulate how to reconstruct its result component.
Depending on whether $L$ turns right or left at $a$, let the first factor of the inner multiplication be the corresponding id (C) or $\alpha_{3}+\alpha_{4}$. Correspondingly the second factor will be $\alpha_{3} / \alpha_{4}$ or id (C). This indeed produces an id (B) result component, since $b$ lies in the arc sequence $S_{\alpha}$ starting at $a$. Similarly depending on whether $L$ turns right or left at $b$, let the other factor of the outer multiplication be the corresponding id (C) or $\alpha_{3} / \alpha_{4}$.

Now if the two inputs at $a$ come first in $D$, insert the inner multiplication on the right side of the tree. If $a$ comes last, insert the inner multiplication on the left side of the tree.

Finally, note that $a$ may equal $a_{0}$. The $\alpha(\mathrm{D})$ angle involved lies on the opposite side of where $L$ goes. Therefore if $S$ lies in the same direction as $L$, then the angle lies on the opposite side and its $h_{q}$ involves the identity on $i$, hence $a=b$ really yields a result component. If $S$ lies in the opposite direction of where $L$ goes, then the angle lies on the side of $S$ and the possible set of $b$ indices does not include $a$.

Lemma C.18. The map D : Result ${ }_{D W} \rightarrow$ Disk $_{S L}$ is injective and its image is precisely Disk ${ }_{D W}$.
Proof. This is similar to the DS case. The reader may find the Lemma C. 4 and its proof helpful.

## C. 8 Signs and $q$-parameters

In this section, we prove Lemma 13.25 which claims that the sign of a result component is precisely the Abouzaid sign of its subdisk. Our strategy is to start from direct inputs and work our way up. In particular, we first compute signs of result components of h-trees. This approach requires that we define Abouzaid signs also for subdisks of h-trees. Note we treat signs additively everywhere.

Definition C.19. Let $D$ be a subdisk of a result component of an h-tree or $\pi$-tree. Then its Abouzaid $\operatorname{sign} \operatorname{Abou}(D) \in \mathbb{Z} / 2 \mathbb{Z}$ is the sum of all \# signs around $D$, plus the number of odd inputs $h_{i}: L_{i} \rightarrow L_{i+1}$ where $L_{i+1}$ is oriented counterclockwise with $D$, plus one if it concerns a $\pi$-tree and its output $t: L_{1} \rightarrow$ $L_{N+1}$ is odd and $L_{N+1}$ is oriented counterclockwise. In case $r$ is a $\beta$ (A) result component, the long version of the subdisk shall be taken.

Recall that a result component of an h - or $\pi$-tree is not only a morphism $r \in \operatorname{Hom}_{H \mathbb{L}_{q}}\left(L_{1}, L_{N+1}\right)$, but also remembers how it was derived from the tree. In particular, the value of any result component of an h-tree or $\pi$-tree does not carry any scalars, except signs. It is of the form

$$
\begin{equation*}
\pm Q \varepsilon \text { resp. } \pm Q t \tag{C.1}
\end{equation*}
$$

where $\pm$ is a sign, $Q=q_{1} \ldots q_{k} \in \mathbb{C} \llbracket Q_{0} \rrbracket$ is a pure product of punctures, $\varepsilon: L_{1} \rightarrow L_{N+1}$ is an elementary morphism resp. $t$ is a cohomology basis element of $\mathbb{L}$. Note C.1 means we measure the sign relative to the natural signs of the cohomology basis elements.

Let us now show that any result component comes precisely with the Abouzaid sign. For a result component $r$ of a $\pi$-tree or h-tree, we denote by $S(r)$ the Abouzaid sign of the subdisk of $r$. We proceed by induction.

## Signs of direct inputs

Proposition C.20. Let $r$ be a direct result component of an h-tree or $\pi$-tree that has a subdisk associated. Then the sign of $r$ as in C.1 equals the Abouzaid sign of the subdisk of $r$.

Proof. The sign is indeed correct for direct inputs. A co-identity $\alpha_{0}(\mathrm{D})$ comes sign $\# \alpha_{0}+1$. Its subdisk is by definition on the $\alpha_{0}$ side (instead of the $\alpha_{0}^{\prime}$ side) and by Convention 10.10 the zigzag paths runs counterclockwise, which makes the Abouzaid sign of the co-identity also equal to $\# \alpha_{0}+1$. A direct $\alpha_{3}$ input comes with sign $\# \alpha_{3}+1$. Its subdisk consists of an input on the $2 / 5$ arc and cutting the $\alpha_{3}$ angle, which means the Abouzaid sign is also $\# \alpha_{3}+1$. Similarly, a direct $\alpha_{4}$ comes with sign $\# \alpha_{4}$, a direct $\beta$ (C) comes with sign $\# \alpha_{3}+\# \alpha_{4}+S^{D}$, a direct $\beta^{\prime}(\mathrm{C})$ with $\operatorname{sign} \# \alpha_{1}+\# \alpha_{2}+S^{D}$, a direct $\alpha_{0}^{\prime}$ with sign $\# \alpha_{0}$. Finally, a direct $\beta(\mathrm{A})$ as tail component of an $\alpha_{3}+\alpha_{4}$ input comes with sign $\# \alpha_{3}+1+S^{D}$ if it is from the $\alpha_{3}$ part of (11.3) (and hence $L_{i+1}$ is counterclockwise) and with sign $\# \alpha_{4}+S^{D}$ if it is from the $\alpha_{4}$ part of 11.3 (and hence $L_{i+1}$ is clockwise). A direct $\beta$ (A) as tail component of an id (C) input comes with $\operatorname{sign} \# \alpha_{1}+\# \alpha_{2}+S^{D}$ if it is from the $\beta^{\prime}$ part of 11.4 and with sign $\# \alpha_{3}+\# \alpha_{2}+S^{D}$ if it is from the $\beta$ part of 11.4 . All these signs are equal to the Abouzaid signs.

## Signs of h-trees

Next, let us check the signs for result components of h-trees.
Lemma C.21. Let $r$ be a result component of an h-tree or $\pi$-tree that has a subdisk associated. Then the sign of $r$ as in C.1 equals the Abouzaid sign of the subdisk of $r$.
Proof. The signs are collected in Table C.4. This table informs about the applicable $h_{q}$ rule, the sum of the input signs, the sum of signs due to $\mu$ applications, the sum of signs due to $h_{q}$ applications and the total Kadeishvili sign. Note that by induction assumption, the inputs already have the correct Abouzaid sign.

In those rows of Table C. 4 that concern a disk result component, the $m_{1}, \ldots, m_{k}$ refer to the inner angles of the disk. That is, the disk is supposed to be $\mu^{k \geq 3}\left(m_{k}, \ldots, m_{1}\right)$. Some $m_{i}$ may be $\delta$ insertions. Their $S\left(m_{i}\right)$ shall stand for zero. For $\beta(\mathrm{A})$ result components, $S^{D}$ shall stand for zero if it concerns a main result component, while equal to the sum of the hash signs around the tail if it concerns a tail result component. For the $\alpha_{0}^{\prime}, \beta / \beta^{\prime}(\mathrm{C})$ result components, $N_{\alpha_{0}}$ denotes the number of $\alpha_{0}$ inputs in the result

| Component | $\alpha_{0}^{\prime}$ of Figure 12.6b | $\beta(\mathrm{C})$ of Figure 12.6c | $\beta^{\prime}(\mathrm{C})$ of Figure 12.6d |
| :---: | :---: | :---: | :---: |
| $h_{q}$ rule | $h_{q}\left(\alpha \alpha^{\prime}\right)$ | $h_{q}\left(\alpha \alpha^{\prime}\right), h_{q}\left(\alpha_{4} \beta\right)$ | $h_{q}\left(\alpha \alpha^{\prime}\right), h_{q}\left(\beta^{\prime} \alpha_{2}\right)$ |
| i-sign | $N_{\alpha_{0}}\left(\# \alpha_{0}+1\right)-1$ | $N_{\alpha_{0}}\left(\# \alpha_{0}+1\right)-1$ | $N_{\alpha_{0}}\left(\# \alpha_{0}+1\right)-1$ |
| $\mu$-sign | $N_{\alpha_{0}}-1$ | $N_{\alpha_{0}}$ | $N_{\alpha_{0}}-1$ |
| $h_{q}$-sign | $\left(N_{\alpha_{0}}-1\right)\left(\# \alpha_{0}+1\right)$ | $\left(N_{\alpha_{0}}-1\right)\left(\# \alpha_{0}+1\right)+\# \alpha_{4}$ | $\left(N_{\alpha_{0}}-1\right)\left(\# \alpha_{0}+1\right)+\# \alpha_{2}+1$ |
| K-sign | $N_{\alpha_{0}}-1$ | $N_{\alpha_{0}}$ | $N_{\alpha_{0}}$ |
| Component | $\beta(\mathrm{A})$ of $h_{q} \mu^{2}\left(\tilde{\beta}(\mathrm{~A}), \alpha_{0}\right)$ | $\beta(\mathrm{A})$ of $h_{q} \mu^{2}\left(\alpha_{0}, \alpha_{4}\right)$ | $\beta(\mathrm{A})$ of $h_{q} \mu^{2}\left(\mathrm{id}(\mathrm{C}), \alpha_{0}^{\prime}\right)$ |
| $h_{q}$ rule | $h_{q}(\beta \alpha)$ | $h_{q}\left(\alpha_{3} \alpha_{4}\right)$ | $h_{q}\left(\alpha_{4} \beta\right)$ |
| i-sign | $S(\tilde{\beta})+\# \alpha_{0}+1$ | $\# \alpha_{0}+1+\# \alpha_{4}$ | $S\left(\alpha_{0}^{\prime}\right)$ |
| $\mu$-sign | 1 | 1 | 1 |
| $h_{q}$-sign | $S^{D}+\# \alpha_{0}+1$ | $S^{D}+\# \alpha_{4}+1$ | $S^{D}+\# \alpha_{4}$ |
| K-sign | +1 | +1 | +1 |
| Component | $\beta(\mathrm{A})$ of $h_{q} \mu^{2}(\tilde{\beta}(\mathrm{~A}), \mathrm{id}(\mathrm{C}))$ | $\beta(\mathrm{A})$ of $h_{q} \mu^{2}\left(\beta / \beta^{\prime}(\mathrm{C}), \mathrm{id}(\mathrm{C})\right)$ | $\beta(\mathrm{A})$ of $h_{q} \mu^{2}\left(\alpha_{0}^{\prime}, \operatorname{id}(\mathrm{C})\right)$ |
| $h_{q}$ rule | $h_{q}(\beta \alpha)$ | $h_{q}(\beta \alpha)$ | $h_{q}(\beta \alpha)$ |
| i-sign | $S(\tilde{\beta})$ | $S\left(\beta / \beta^{\prime}\right)$ | $S\left(\alpha_{0}^{\prime}\right)$ |
| $\mu$-sign | 0 | 0 | 0 |
| $h_{q}$-sign | $S^{D}+\# \alpha+1$ | $S^{D}+\# \alpha+1$ | $S^{D}+\# \alpha+1$ |
| K-sign | +1 | +1 | +1 |
| Component | $\beta$ (A) of final-out $h_{q} \mu^{\geq 3}$ | $\alpha_{3}(\mathrm{~B})$ of Figure 12.6a |  |
| $h_{q}$ rule | $h_{q}(\beta \alpha)$ | $h_{q}\left(\alpha_{3} \alpha_{4}\right)$ |  |
| i-sign | $\sum S\left(m_{i}\right)$ | $\# \alpha_{0}+1+\# \alpha_{4}$ |  |
| $\mu$-sign | 0 | 1 |  |
| $h_{q}$-sign | $S^{D}+\# \alpha+1$ | $\# \alpha_{4}+1+S^{D}$ |  |
| K-sign | +1 | +1 |  |

Table C.4: Signs of most result components of h-trees
component (including the $\alpha_{0}^{\prime}$ used). A single time the notation $\tilde{\beta}$ was used to distinguish two different $\beta$ (A) result components.

Let us examine the example of an $\beta$ (A) result component of $h_{q} \mu^{2}\left(\alpha_{0}, \alpha_{4}\right)$ in detail. Both inputs $\alpha_{0}$ and $\alpha_{4}$ are necessarily direct. The morphism $\alpha_{0}$ comes with sign $\# \alpha_{0}+1$ and $\alpha_{4}$ comes with sign $\# \alpha_{4}$. This gives a total input sign of $\# \alpha_{0}+1+\# \alpha_{4}$. We have $\mu^{2}\left(\alpha_{0}, \alpha_{4}\right)=-\alpha_{3} \alpha_{4}$, which gives a sign of 1 due to application of $\mu$. According to Lemma 11.11, we have

$$
h_{q}\left(\alpha_{3} \alpha_{4}\right)=(-1)^{\# \alpha_{4}+1}\left(\alpha_{3}+\sum_{D \in T\left(\alpha_{3}\right) \backslash \varepsilon}(-1)^{S^{D}} Q^{D} \beta^{D}\right) .
$$

This gives a sign of $\# \alpha_{4}+1+S^{D}$ due to application of $h_{q}$. A Kadeishvili sign of 1 is added. The total sign the $\beta(\mathrm{A})$ tail result component is now

$$
\# \alpha_{0}+1+\# \alpha_{4}+1+\# \alpha_{4}+1+S^{D}+1 \equiv \# \alpha_{0}+S^{D}
$$

Let us compare with the Abouzaid sign $S(\beta)$. This sign consists of all \# signs around the tail, including $\# \alpha_{0}$, plus two because both odd inputs $\alpha_{0}$ and $\alpha_{3}+\alpha_{4}$ are counterclockwise. This is precisely the same.

Note Table C. 4 does not treat explicitly the $\beta(\mathrm{A})$ result component of Figure 12.7. This result component is however a combination of $\beta / \beta^{\prime}(\mathrm{C})$ or $\beta(\mathrm{A})$ and $\alpha_{0}$ and id (C) compositions on the right, which are already present in Table C. 4 .

| Component | $\alpha_{3}+\alpha_{4}$ of final-out | $\alpha_{3}+\alpha_{4}$ of first-out $\mu^{k \geq 3}\left(m_{k}, \ldots, m_{1}\right)$ |
| :--- | :--- | :--- |
| $\mu^{\geq 2}$ | $\alpha_{4}$ | $\alpha_{2}$ |
| i-sign | $\sum S\left(m_{i}\right)$ | $\sum S\left(m_{i}\right)$ |
| $\mu$-sign | 0 | 1 |
| $\varphi \pi_{q}$-sign | $\# \alpha_{4}+1$ | $\# \alpha_{2}+1$ |
|  |  |  |
| Component | $\alpha_{3}+\alpha_{4}$ of $\mu^{2}(\beta(\mathrm{~A})$, id $(\mathrm{C}))$ | $\alpha_{3}+\alpha_{4}$ of $\mu^{2}(\mathrm{id}(\mathrm{C}), \beta(\mathrm{A}))$ |
| $\mu^{\geq 2}$ | $\alpha_{4}$ | $\alpha_{2}$ |
| i-sign | $S(\beta)$ | $S(\beta)$ |
| $\mu$-sign | 0 | 1 |
| $\varphi \pi_{q}$-sign | $\# \alpha_{4}+1$ | $\# \alpha_{2}+1$ |
|  |  |  |
| Component | id $(\mathrm{C})$ of all-in |  |
| $\mu^{\geq 2}$ | id $(\mathrm{C})$ |  |
| i-sign | $\sum S\left(m_{i}\right)$ |  |
| $\mu$-sign | 0 |  |
| $\varphi \pi_{q}$-sign | 0 |  |

Table C.5: Signs of some $\pi$-trees

## Signs of $\pi$-trees

Lemma C.22. Let $r$ be a direct result component of a $\pi$-tree that has a subdisk associated. Then the sign of $r$ as in C.1 equals the Abouzaid sign of the subdisk of $r$.

Proof. The most signs of $\pi$-trees are checked in Table C.5. Let us treat the others manually.
Let us check the case where $\alpha_{3}+\alpha_{4}$ is the $G$ tail result component of some $\varphi \pi_{q}(\beta \alpha)$. We could theoretically check this by going through all the cases. It is easier to rely on what we already have. Namely $-h_{q}(\beta \alpha)$ has an associated main result component $\beta$, which comes out of $h_{q} \mu^{\geq 2}$ with the correct $\operatorname{sign} S(\beta)$. This means the $\mu^{\geq 2}$ must have had sign $S(\beta)+\# \alpha+1+1$ in front of $\beta \alpha$. Then its $\alpha_{3}+\alpha_{4}$ G tail result component comes with an additional sign of $S^{D}+\# \alpha+1$ in case of G1 and $S^{D}+\# \alpha$ in case of G2. In total, we get a sign of $S(\beta)+S^{D}+1$ in case of a G1 tail result component and $S(\beta)+S^{D}$ in case of a G2 result component, precisely the Abouzaid sign.

Let us check the case where id (C) is the H tail result component of some $\varphi \pi_{q}(\beta \alpha)$. Then $-h_{q}(\beta \alpha)$ has an associated result component $\beta$, which comes out of $h_{q} \mu^{\geq 2}$ with the correct sign $S(\beta)$. This means the $\mu^{\geq 2}$ must have had sign $S(\beta)+\# \alpha+1+1$ in front of $\beta \alpha$. Then its id (C) H tail result component comes with an additional sign of $S^{D}+\# \alpha$. In total, we get a sign of $S(\beta)+S^{D}$, precisely the Abouzaid sign.

Checks for the id (D) result components of Figure 12.10 are contained in Table C. 6 For example, regard the case the id (B) comes from a first-out disk $\mu^{k \geq 3}\left(m_{k}, \ldots, m_{1}\right)$. According to Figure 12.2 the outside part of $m_{1}$ is $\alpha_{2}$ and hence odd. We get $\mu^{k \geq 3}\left(m_{k}, \ldots, m_{1}\right)=-\alpha_{2}$ and evaluation of $h_{q}\left(\alpha_{2}\right)$ gives another sign of $(-1)^{\# \alpha_{2}}$. Together with the Kadeishvili sign we obtain that id (D) has sign $S\left(m_{k}\right)+\ldots+$ $S\left(m_{1}\right)+\# \alpha_{2}$ as result component, precisely the Abouzaid sign.

Finally, let us check the $\alpha_{3}+\alpha_{4}$ and id (C) result components of the 8 trees of Figure 12.11. Recall such a result component a degenerate strip on a zigzag path $L$ as subdisk.

Let us investigate the inner product first. In case of $\mu^{2}\left(\alpha_{3}, \mathrm{id}(\mathrm{C})\right)$ resp. $\mu^{2}\left(\mathrm{id}(\mathrm{C}), \alpha_{4}\right)$, the inner product has sign $\# \alpha_{3}+1$ resp. $\# \alpha_{4}+1$. Regard the infinitesimally short stem of the strip between the two factors of the inner product. If the angle $\alpha_{3}$ resp. $\alpha_{4}$ as morphism $L \rightarrow L$ falls under case 1 of Figure 10.7, then $h_{q}$ adds a sign of $\# \alpha_{3}+1$ resp. $\# \alpha_{4}+1$. Together with the Kadeishvili sign, the inner product has a total sign of 1 . Indeed, the stem is counterclockwise in this case. If $\alpha_{3}$ resp. $\alpha_{4}$ falls under case 2 , then $h_{q}$ adds a sign of $\# \alpha_{3}$ resp. $\# \alpha_{4}$. Together with the Kadeishvili sign, the inner product has a total sign of 0 . Indeed, the stem is clockwise in this case.

Let us now investigate the outer product. Regard the infinitesimally short stem at the output mark. When id (C) or $\alpha_{3}+\alpha_{4}$ comes at the end of the strip, $\varphi \pi_{q} \mu$ adds no sign (in case of $\alpha_{3}+\alpha_{4}$, the

| Component | all-in disk $\mu^{k \geq 3}\left(m_{k}, \ldots, m_{1}\right)$ | $\varphi \pi_{q} \mu^{2}\left(h_{q} \mu^{2}(\mathrm{id}(\mathrm{C}), \beta(\mathrm{A})), \mathrm{id}(\mathrm{C})\right)$ |
| :--- | :--- | :--- |
| $h_{q}$ | - | $h_{q}\left(\alpha_{2}\right)$ |
| i-sign | $\sum S\left(m_{i}\right)$ | $S(\beta)$ |
| $\mu / h_{q} / \varphi \pi_{q}$-sign | 0 | $1+\# \alpha_{2}$ |
| K-sign | 0 | 1 |
|  |  |  |
| Component | $\varphi \pi_{q} \mu^{2}\left(h_{q} \mu^{2}(\beta(\mathrm{~A}), \mathrm{id}(\mathrm{C})), \mathrm{id}(\mathrm{C})\right)$ | $\varphi \pi_{q} \mu^{2}\left(\mathrm{id}(\mathrm{C}), h_{q} \mu^{2}(\mathrm{id}(\mathrm{C}), \beta(\mathrm{A}))\right)$ |
| $h_{q}$ | $h_{q}(\beta \alpha)$ | $h_{q}\left(\alpha_{2}\right)$ |
| i-sign | $S(\beta)$ | $S(\beta)$ |
| $\mu / h_{q} / \varphi \pi_{q}$-sign | $S^{D}+\# \alpha+1$ | $1+\# \alpha_{2}$ |
| K-sign | 1 | 1 |
|  |  |  |
| Component | $\varphi \pi_{q} \mu^{2}\left(\mathrm{id}(\mathrm{C}), h_{q} \mu^{2}(\beta(\mathrm{~A}), \mathrm{id}(\mathrm{C}))\right)$ | $\varphi \pi_{q}\left(\mathrm{id}(\mathrm{C}), h_{q} \mu^{k \geq 3}\left(m_{k}, \ldots, m_{1}\right)\right)$ |
| $h_{q}$ | $h_{q}(\beta \alpha)$ | $h_{q}\left(\alpha_{2}\right)$ |
| i-sign | $S(\beta)$ | $\sum S\left(m_{i}\right)$ |
| $\mu / h_{q} / \varphi \pi_{q}$-sign | $S^{D}+\# \alpha+1$ | $1+\# \alpha_{2}$ |
| K -sign | 1 | 1 |
| Component | $\varphi \pi_{q}\left(h_{q} \mu^{k \geq 3}\left(m_{k}, \ldots, m_{1}\right), \mathrm{id}(\mathrm{C})\right)$ |  |
| $h_{q}$ | $h_{q}\left(\alpha_{2}\right)$ |  |
| i-sign | $\sum S\left(m_{i}\right)$ |  |
| $\mu / h_{q} / \varphi \pi_{q}$-sign | $1+\# \alpha_{2}$ | 1 |
| K -sign | 1 |  |

Table C.6: Signs of id (D) result components of Figure 12.10
intrinsic sign $\# \alpha_{4}$ stays correctly until the end). In case $\alpha_{3}+\alpha_{4}$ comes at the end of the strip, this indeed constitutes an odd intersection and an odd output and both add the same sign since both refer to the orientation of the stem. In case id (C) comes at the end of the strip, this indeed produces an even intersection and an even output mark.

When id (C) comes at the beginning of the strip, $\varphi \pi_{q} \mu$ gets no sign. Indeed, the output is then also even. When $\alpha_{3}+\alpha_{4}$ comes at the beginning of the strip, then $\varphi \pi_{q} \mu$ gets a single extra sign. The output is then also $\alpha_{3}+\alpha_{4}$ and their claimed signs refer to the orientation of $L$ and of its Hamiltonian deformation $L^{\prime}$. Since both point in the same direction, but lie on opposite sides of the strip, they contribute to the Abouzaid sign with 1.

## D The case of punctured spheres

In this section, we redo our entire minimal model computation in the case of specific punctured spheres. In particular, our treatise includes the 3 -punctured sphere, also known as pair of pants. The simplest yet instructive example of mirror symmetry for punctured surfaces, it would be a shame not to know the minimal model of its deformed zigzag category. However, no dimer on a sphere is consistent and Theorem 13.26 therefore fails to apply. This is the reason we redo the entire calculation in the case of specific sphere dimers with $M \geq 3$ punctures.

In section D.1, we give an overview of what goes wrong for non-consistent dimers. In section D. 2 , we focus on specific sphere dimers $Q_{M}$ with an odd number of punctures $M \geq 3$. We choose a specific type of spin structure, tailored to the use in mirror symmetry. In section D.3, we choose a homological splitting. In section D.4 we describe the deformed zigzag category $\mathbb{L}_{q}$. In section D.5 we compute the deformed decomposition of $\mathbb{L}_{q}$. In section D.6 we introduce the suitable notion of result components. In section D.7. we assemble the minimal model $\mathrm{H} \mathbb{L}_{q}$. In section D.8, we comment on the case of $Q_{M}$ for even $M$.

As we state in Proposition D.18 and D.22, the minimal model H $\mathbb{L}_{q}$ can be described explicitly by means of CR, ID, DS and DW disks. This description also accurately captures the curvature und residual differential on $H \mathbb{L}_{q}$. Explicitly, for odd $M$ and specific choice of spin structure the category $\mathrm{H} \mathbb{L}_{q}$ is curvature-free and has a residual differential. For even $M$ the category $H \mathbb{L}_{q}$ has curvature and residual differential.

## D. 1 Absence of consistency

In this section, we list consequences of the lack of geometric consistency. Within section 10 until 13 we have used geometric consistency heavily. Is geometric consistency actually a necessity? To find an answer, we collect the most important statements which we proved using consistency. For every statement, we explain how it depends on consistency and whether it can be partially recovered when dropping the consistency assumption.
Zigzag segments do not bound disks. We proved this statement directly using geometric consistency. Upon dropping geometric consistency, zigzag segments can easily bound disks.
Deformed zigzag paths are uncurvable. We proved this by explicit uncurving, which succeeds because zigzag segments do not bound disks. When zigzag segments bound disks, two issues can occur: If the zigzag segment ends in an A situation, we need to adapt the uncurving procedure, but it provides no hindrance to uncurving. If the zigzag curve actually bounds a teardrop, then the zigzag path can inherently not be uncurved.
A, B, C, D situations exhaust all angles. This statement mainly uses that every puncture has at least 4 arc incidences. In fact, a non-consistent dimer may have punctures with only 2 arc incidences. This gives rise to new types similar to B and C situations, with the difference that the head or tail of the shared arc $2=5$ may have 2 arc incidences. More practically, this would push $\beta(\mathbf{C})$ or $\beta^{\prime}$ (C) to be empty angles.
$\mathbb{L}_{q}$ satisfies $\mu_{\mathbb{L}_{q}}^{1}(H) \subseteq \mu_{\mathbb{L}_{q}}^{1}\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R\right)$. If $Q$ is so nonconsistent that $\mathbb{L}_{q}$ has inherent curvature, then this statement is already not applicable any more. Namely, $\mu_{\mathbb{L}_{\sigma}}^{1}$ does not square to zero anymore and we cannot invoke the simplified Kadeishvili theorem of section 8.8. If $Q$ is only so nonconsistent that $\mathbb{L}_{q}$ can be uncurved (albeit by an adapted procedure), then the inclusion typically does not hold. Nevertheless, the deformed Kadeishvili theorem applies and yields a minimal model with residue differential. It is clear that the description of $H_{q}$ will be very complicated.
$\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$ disks as classification of tail terms. The shape of the terms in $\mu_{q}^{\geq 3}(\delta, \ldots, \varepsilon, \ldots, \delta)$ is analyzed by zooming in at the concluding puncture. In distinguishing E, F, G, H disks, we have used that the concluding puncture has at least 4 arc incidences. When dropping consistency, the resulting terms need not be of $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ type, but also of the variants explained above.

Situation B/C cohomology basis elements have only type $\mathbf{E}$ tail. This is proved directly using geometric consistency: An intersection between $L_{1}$ and $L_{2}$ renders it impossible to find type F , $\mathrm{G}, \mathrm{H}$ disks when tracing $L_{1}$ and $L_{2}$ away from the intersection. Upon dropping consistency, B/C cohomology basis elements acquire tail also from F, G, H disks. This raises additional complexity: The G and H disks will contribute result components, forcing us into capturing them.
Every narrow locations has at least one below morphism. A narrow location without below morphism constitutes a zigzag segment bounding a disk. Upon dropping consistency, imagine a segment of a zigzag path $L$ that bounds a disk. According to our explanations above, the $\delta$-matrix should already be adapted to facilitate uncurving. In fact, the new $\delta$-matrix of $L$ will have a situation A morphism for every disk bounded by one of its segments and more situation A morphisms inserted on its tail. These situation A morphisms compensate for the lack of below morphisms. This modification allows us to construct result components for a given CR/ID disk $D$ even if a narrow location $(l, m)$ of $D$ has no below morphism.
The calculation for the sphere dimers $Q_{M}$ basically proceeds as in the geometrically consistent case. Based on the above list of issues, we can however point out a few differences: In the consistent case, $\mathbb{L}_{q}$ is always uncurvable. In the sphere case, for odd $M$ and specific choice of spin structure it is uncurvable, for even $M$ it is not uncurvable. In the consistent case, minimal models constructed by our deformed Kadeishvili construction are based on a deformed cohomology space $H_{q}$ satisfying $\mu_{\mathbb{L}_{q}}^{1}\left(H_{q}\right)=0$. In the sphere case, we only achieve $\mu_{\mathbb{L}_{q}}^{1}\left(H_{q}\right) \subseteq H_{q}$. In the consistent case, the deformed counterpart $\varphi^{-1}(h)$ of $h=\mathrm{id}(\mathrm{C})$ morphisms includes tails of $\beta / \beta^{\prime}(\mathrm{C})$. In the sphere case, the deformed counterpart $\varphi^{-1}(h)$ of $h=\mathrm{id}(\mathrm{C})$ morphisms includes no tails, but a single nearby id (B) morphism.


Figure D.1: The sphere dimer and its zigzag curves

## D. 2 The sphere and its zigzag category

In this section, we define specific sphere dimers and define their category of zigzag paths. The dimers we pick are those also used as A-side for commutative mirror symmetry in 3 . The dual dimers of these spheres are consistent and therefore suited for noncommutative mirror symmetry of 18 .

The dimer we regard is the sphere dimer $Q_{M}$ for $M \geq 3$ depicted in Figure D.1a. This dimer has $M$ punctures and $M$ arcs. It has two polygons, namely the clockwise front side and the counterclockwise rear side. We shall briefly discuss the differences between the cases of odd and even $M$, and then focus on the odd case. In order to apply our findings to deformed mirror symmetry later on, we shall define one specific spin structure.

The zigzag curves of this sphere dimer $Q_{M}$ are described as follows: In case $M$ is odd, there is precisely one zigzag curve. It cycles around the arcs once, and then cycles around the arcs again with opposite $\delta$ angles. In case $M$ is even, there are precisely two zigzag curves, each of them cycling around the arcs once. The smooth zigzag curves in both cases are depicted in Figure D.1c. In the picture, the arc system has been pulled to the front side of the sphere so that the zigzag curves become clearly visible.

Let us now focus on the case of odd $M \geq 3$ and fix spin structure as follows.
Convention D.1. The letter $Q=Q_{M}$ stands for the sphere dimer with $M \geq 3$ odd. The spin structure of the zigzag path is chosen by assigning $\# \alpha=1$ to an odd number of interior angles on the rear side of $Q_{M}$, and $\# \alpha=0$ to all other angles. The co-identity locations $\alpha_{0}$ are supposed to lie on the rear side and the identity locations $a_{0}$ are arbitrary indexed arcs.

In Definition D.2, we define the category of zigzag paths $\mathbb{L} \subseteq \operatorname{Tw~Gtl} Q_{M}$ as in the case of geometrically consistent dimers. In the case $Q=Q_{M}$, the only object in the category is the single zigzag path $L$.

Definition D.2. The category of zigzag paths $\mathbb{L} \subseteq \operatorname{Tw} \operatorname{Gtl} Q_{M}$ is the full subcategory consisting of the single zigzag path.

We intend to write down the explicit twisted complex for $L$. The main issue consists of numbering all indexed arcs of $L$ and the angles between them. In fact, a zigzag path consists of indexed arcs, as opposed to purely arcs of $Q_{M}$. We shall therefore denote the arcs in sequence by $a_{1}, \ldots, a_{2 M}$, with the convention that $h\left(a_{1}\right)=q_{M}$ and $t\left(a_{1}\right)=q_{1}$ and $L$ turns right at the head of $a_{1}$.

The indexed small angles of $L$ shall be denoted by $\alpha_{1}, \ldots, \alpha_{2 M}$, such that $\alpha_{1}: a_{1} \rightarrow a_{2}$ and $\alpha_{2}: a_{3} \rightarrow$ $a_{2}$ and so on. In other words, $\alpha_{i}$ runs at the head of $a_{i}$ if $i$ odd and at the tail of $a_{i}$ if $i$ is even. In other words, we have $\alpha_{2 i}: a_{2 i+1} \rightarrow a_{2 i}$ and $\alpha_{2 i+1}: \alpha_{2 i+1} \rightarrow \alpha_{2 i+2}$. These angles are depicted in Figure D.2a

We count all indices modulo $2 M$. In contrast, an index shift of $M$ typically turns a situation from left to right and from right to left. For example, we have $\alpha_{i+M} \neq \alpha_{i}$. Compare Figure D.2a and D.2b We also have $a_{i}=a_{i+M}$ as arcs of $Q_{M}$, but not as indexed arcs of $L$.

For $i=1, \ldots, 2 M$, we denote the complementary angle to $\alpha_{i}$ by $\alpha_{i}^{\prime}$. For instance if $i$ is odd, then $\alpha_{i}$ runs from $a_{i+1}$ to $a_{i}$. We denote the arc identity of $a_{i}$ by $\mathrm{id}_{i}$, and the arc identity $a_{i} \rightarrow a_{i+M}$ by id ${ }_{i}^{\prime}$. We set $\beta_{i}:=\operatorname{id}_{j}^{\prime} \alpha_{i}$ and $\beta_{i}^{\prime}:=\operatorname{id}_{j}^{\prime} \alpha_{i}^{\prime}$, where $j$ is chosen so that the composition makes sense. We are now ready to define the deformed category of zigzag paths:

Generically denoting a full turn by $\ell$, these angles together with their multiples with $\ell$ powers form a basis of the hom space $\operatorname{Hom}_{\mathbb{L}}(L, L)$. Examples of these angles are depicted in Figure D.2c


Figure D.2: Numbering of punctures, arcs and angles

## D. 3 Homological splitting

In this section, we provide a homological splitting for $\mathbb{L}$ in case of odd $M \geq 3$. We first explain the analogy of all basis morphisms with the consistent case. Then, we write down an explicit choice of cohomology basis elements and an explicit choice of $R$. We explain why it constitutes a homological splitting.

We have seen that $\operatorname{End}_{\mathbb{L}}(L, L)$ has a basis given by basis morphisms of the kind $\mathrm{id}_{i}, \operatorname{id}_{i}^{\prime}, \alpha_{i}, \alpha_{i}^{\prime}, \beta_{i}$ and $\beta_{i}^{\prime}$. Let us compare with the consistent case. The angle $\mathrm{id}_{i}$ is an arc identity. In terms of $\mathrm{A}, \mathrm{B}, \mathrm{C}$, D situations, we denoted it as id (D). The angle $\mathrm{id}_{i}^{\prime}$ is comparable to an id (C) morphism for odd $i$, and comparable to an id (B) morphism for even $i$. The angle $\alpha_{i}$ is simply $\alpha(\mathrm{D})$, and similarly $\alpha_{i}^{\prime}$ is $\alpha^{\prime}(\mathrm{D})$. There are ambiguities of interpreting $\beta_{i}$ and $\beta_{i}^{\prime}$ in terms of $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ situations. A possible choice is matching $\beta_{i}$ with $\alpha_{4}(\mathrm{~B})$ for odd $i$ and with $\alpha_{1}(\mathrm{~B})$ for even $i$, and matching $\beta_{i}^{\prime}$ with $\alpha_{3}$ (B) for odd $i$ and with $\alpha_{2}$ (B) for even $i$. In short, the two differences are that we have explicit indices $i$ instead of using $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ situations as enumeration tools and that we have less morphisms overall since ever puncture has only 2 arc incidences.

We now define a candidate splitting $H \oplus I \oplus R$, modeled after the consistent case:

Definition D.3. Denote by $H \subseteq \operatorname{Hom}_{\mathbb{L}}(L, L)$ the space spanned by the cohomology basis elements

- $i d_{i}^{\prime}$ for $i$ odd
- $(-1)^{\#(i+M)+1} \beta_{i}^{\prime}+(-1)^{\# i} \beta_{i}$ where $i$ odd,
- $\sum_{i=1}^{2 M} \mathrm{id}_{a_{i}}$,
- $(-1)^{\# \alpha_{0}+1} \alpha_{0}$.

Choose the space $R \subseteq \operatorname{Hom}_{\mathbb{L}}(L, L)$ to be spanned by $\beta_{i}$ for even $i, \beta_{i}^{\prime}$ for odd $i,\left(\alpha_{i}^{\prime} \alpha_{i}\right)^{k+1}$ for all $i$ and $k \geq 0, \beta_{i} \ell^{k+1}$ for even $i$ and $k \geq 0, \beta_{i}^{\prime} \ell^{k+1}$ for odd $i$ and $k \geq 0, \operatorname{id}_{i}$ for $i \neq i_{0}, \operatorname{id}_{i}^{\prime}$ for even $i, \beta_{i} \alpha_{i}^{\prime} \ell^{k}$ for odd $i$ and $k \geq 0, \alpha_{i}^{\prime} \ell^{k}$ for all $i$ and $k \geq 0, \beta_{i} \alpha_{i}^{\prime} \ell^{k}$ for even $i$ and $k \geq 0$. The spaces $H$ and $R$, together with $I:=\operatorname{Im}\left(\mu_{\mathbb{L}}^{1}\right)$, constitute the standard splitting for $\mathbb{L}$.

In Table D. 3 we have checked that every morphism in $\operatorname{End}_{\mathbb{L}}(L, L)$ can be written in terms of $H, I$ and $R$. The table also serves as a convenient reference for the definition of $H$ and $R$. In analogy to the consistent case, the sum $H+I+R$ is in fact direct. Let us record this as follows:

Lemma D.4. The spaces $H, I=\operatorname{Im}\left(\mu_{\mathbb{L}}^{1}\right)$ and $R$ provide a homological splitting for $\mathbb{L}$.

## D. 4 Deformed category of zigzag paths

In this section, we define the category $\mathbb{L}_{q}$ of deformed zigzag paths. As in the consistent case, its definition is based on the complementary angle trick. In contrast to the consistent case, we must expect that uncurving fails. Due to our specific spin structure, curvature cancels nevertheless.

Definition D.5. Regard a sphere dimer $Q=Q_{M}$ with odd $M \geq 3$. Let \# denote a choice of spin structure as in Definition D.2. Then the deformed category of zigzag paths is category $\mathbb{L}_{q} \subseteq$

$$
\begin{aligned}
\mathrm{id}_{i} & =\operatorname{id}_{i} \in R, \text { if } i \neq i_{0}, \\
\mathrm{id}_{i} & =\sum_{j} \mathrm{id}_{j}-\sum_{j \neq i} \mathrm{id}_{j}, \text { if } i=i_{0}, \\
\mathrm{id}_{i}^{\prime} & =\operatorname{id}_{i}^{\prime} \in R, \text { even } i, \\
\mathrm{id}_{i}^{\prime} & =\operatorname{id}_{i}^{\prime} \in H, \text { odd } i, \\
\alpha_{i} & =\alpha_{i} \in H, \text { if } \alpha_{i}=\alpha_{0}, \\
\alpha_{i} & =\mu^{1}\left( \pm \mathrm{id}_{a_{j}} \pm \ldots \pm \mathrm{id}_{a_{i}}\right) \pm \alpha_{0}, \\
\alpha_{i}^{\prime} & =\alpha_{i}^{\prime} \in R, \\
\beta_{i} & =\beta_{i} \in R, \text { even } i, \\
\beta_{i} & =(-1)^{\# i}\left((-1)^{\#(i+M)+1} \beta_{i}^{\prime}+(-1)^{\# i} \beta_{i}\right)+(-1)^{\# i+\#(i+M)} \beta_{i}^{\prime} \text { odd } i, \\
\beta_{i}^{\prime} & \left.=(-1)^{\#(i+M)} \mu^{1} \text { idid }_{i}^{\prime}\right)+(-1)^{\#(i+M)}\left[(-1)^{\#(i+M-1)+1} \beta_{i-1}^{\prime}+(-1)^{\#(i-1)} \beta_{i-1}\right]+ \\
& (-1)^{\# i+\# \#(i+M)} \beta_{i}, \text { even } i, \\
\beta_{i}^{\prime} & =\beta_{i}^{\prime} \in R, \text { odd } i, \\
\alpha_{i} \ell^{k} & =(-1)^{\# i} \mu^{1}\left(\left(\alpha_{i}^{\prime} \alpha_{i}\right)^{k}\right), \\
\alpha_{i}^{\prime} \ell^{k} & =\alpha_{i}^{\prime} \ell^{k} \in R, \\
\beta_{i} \ell^{k} & =\beta_{i} \ell^{k} \in R, \text { even } i, \\
\beta_{i} \ell^{k} & =(-1)^{\# i+1} \mu^{1}\left(\beta_{i} \alpha_{i}^{\prime} \ell^{k-1}\right)+(-1)^{\# i+\#(i+M)} \beta_{i}^{\prime} \ell^{k}, \text { odd } i, \\
\beta_{i}^{\prime} \ell^{k} & =\beta_{i}^{\prime} \ell^{k} \in R, \text { odd } i, \\
\beta_{i}^{\prime} \ell^{k} & =(-1)^{\#(i+M)} \mu^{1}\left(\beta_{i} \alpha_{i}^{\prime} \ell^{k-1}\right)+(-1)^{\#(i+M)+\# i} \beta_{i} \ell^{k}, \text { even } i, \\
\left(\alpha_{i}^{\prime} \alpha_{i}\right)^{k} & =\left(\alpha_{i}^{\prime} \alpha_{i}\right)^{k} \in R, \\
\left(\alpha_{i} \alpha_{i}^{\prime}\right)^{k} & =(-1)^{\# i+1} \mu^{1}\left(\alpha_{i}^{\prime} \ell^{k-1}\right)+(-1)\left(\alpha_{i}^{\prime} \alpha_{i}\right)^{k}, \\
\beta_{i} \alpha_{i}^{\prime} \ell^{k} & =\beta_{i} \alpha_{i}^{\prime} \ell^{k} \in R, \\
\beta_{i}^{\prime} i_{i} \ell^{k} & =(-1)^{\# i+1} \mu^{1}\left(\beta_{i}^{\prime} \ell^{k}\right), \text { odd } i, \\
\beta_{i}^{\prime} \alpha_{i} \ell^{k} & =(-1)^{\#(i+M)+1} \mu^{1}\left(\beta_{i} \ell^{k}\right), \text { even } i .
\end{aligned}
$$

Table D.3: Decomposing arbitrary morphisms into $H, I$ and $R$
$\mathrm{Tw} \mathrm{Gtl}_{q} Q_{M}$ consisting of the single deformed zigzag path

$$
\begin{aligned}
L & =\left(a_{1} \oplus \ldots \oplus a_{2 M}, \delta\right), \\
\delta & =\left[\begin{array}{ccccc}
0 & (-1)^{\# 1} q_{1} \alpha_{1}^{\prime} & 0 & \cdots & 0 \\
(-1)^{\# 1} \alpha_{1} & 0 & (-1)^{\# 2} \alpha_{2} & \cdots & 0 \\
0 & (-1)^{\# 2} q_{2} \alpha_{2}^{\prime} & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & (-1)^{\#(2 M)} \alpha_{2 M} \\
0 & 0 & 0 & (-1)^{\#(2 M)} q_{2 M} \alpha_{2 M}^{\prime} & 0
\end{array}\right] .
\end{aligned}
$$

Let us introduce the following shorthand notation for $k \in \mathbb{Z}$ :

$$
\begin{align*}
\# \# k & :=\# k+\ldots+\#(k+M-1), \\
Q_{i} & :=q_{i} q_{i+2} \ldots q_{i+M-3}, \\
Q^{\text {odd }}(k) & :=\prod_{\substack{s=k \\
s \text { odd } \\
k+M-1}} q_{s},  \tag{D.1}\\
Q^{k+M-1} q_{s} . & :=\prod_{\substack{s=k \\
s \text { even }}} q_{s} .
\end{align*}
$$

We now come to our first meaningful calculation in the sphere case: the curvature of $\mathbb{L}_{q}$. We expect curvature in principle, since $L$ is contractible when regarded in the closed surface $|Q|$. With our specific spin structure, the curvature contributions from front and rear side however cancel each other.

Lemma D.6. The curvature of $L$ vanishes, we have $\mu_{\mathbb{L}_{q}}^{0}=0$.
Proof. We have to evaluate $\mu_{\operatorname{Add~}_{\operatorname{Gtl}_{q} Q}}^{0}+\mu_{\operatorname{Add~}_{\operatorname{Gtl}}^{q} Q}^{2}(\delta, \delta)+\mu_{\mathrm{Add} \mathrm{Gtl}_{q} Q}^{\geq 3}(\delta, \ldots, \delta)$. As in the geometrically consistent case, the first two terms cancel each other. In contrast, the term $\mu^{M}(\delta, \ldots, \delta)$ yields two individual contribution for each index $i=1, \ldots, 2 M$, one from the front and one from the rear side:

$$
\mu^{M}(\delta, \ldots, \delta)=\sum_{i=1}^{2 M}(-1)^{\# \# i} Q_{i}^{\text {even }} \mathrm{id}_{i}^{\prime}+\sum_{i=1}^{2 M}(-1)^{\# \#(i-M)} Q_{i-M}^{\text {odd }} \mathrm{id}_{i}^{\prime}=0
$$

We have used that $Q_{i}^{\text {even }}=Q_{i-M}^{\text {odd }}$ and $\# \# i+\# \#(i-M) \equiv 1 \in \mathbb{Z} / 2 \mathbb{Z}$ by assumption on the spin structure.

## D. 5 Deformed decomposition

In this section, we provide the deformed decomposition for $\mathbb{L}_{q}$. More precisely, the category $\mathbb{L}_{q}$ is curvature-free but does not satisfy $\mu_{q}^{1}(B \widehat{\otimes} H) \subseteq \mu_{q}^{1}(B \widehat{\otimes} R)$. The deformed Kadeishvili theorem nevertheless applies and defines a deformed decomposition $H_{q} \oplus \mu_{q}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R)$ with $\mu_{q}^{1}\left(H_{q}\right) \subseteq H_{q}$. In this section, we compute $H_{q}$ explicitly, together with a few values of the deformed codifferential.

As a preparation, we perform here a few calculations of $\mu_{\mathbb{L}_{q}}^{1}$ :

$$
\begin{aligned}
\mu_{\mathbb{L}_{q}}^{1}\left(\mathrm{id}_{i}^{\prime}\right) & =\mu^{2}\left(\mathrm{id}_{i}^{\prime},(-1)^{\# \alpha_{i-1}} \alpha_{i-1}+(-1)^{\# \alpha_{i}} \alpha_{i}\right)+\mu^{2}\left((-1)^{\# \alpha_{i+M}} \alpha_{i+M}+(-1)^{\# \alpha_{i+M-1}} \alpha_{i+M-1}, \mathrm{id}_{i}^{\prime}\right) \\
& =(-1)^{\#(i-1)+1} \beta_{i-1}+(-1)^{\# i+1} \beta_{i}+(-1)^{\#(i+M)} \beta_{i}^{\prime}+(-1)^{\#(i+M-1)} \beta_{i-1}^{\prime}
\end{aligned}
$$

for even $i$,

$$
\begin{aligned}
\mu_{\mathbb{L}_{q}}^{1}\left(\mathrm{id}_{i}^{\prime}\right)= & \mu^{2}\left(\mathrm{id}_{i}^{\prime},(-1)^{\# i} q_{i} \alpha_{i}^{\prime}+(-1)^{\#(i-1)} q_{i-1} \alpha_{i-1}^{\prime}\right) \\
& \quad+\mu^{2}\left((-1)^{\#(i+M)} q_{i+M} \alpha_{i+M}^{\prime}+(-1)^{\#(i+M-1)} q_{i+M-1} \alpha_{i+M-1}^{\prime}, \mathrm{id}_{i}^{\prime}\right) \\
= & (-1)^{\# i+1} q_{i} \beta_{i}^{\prime}+(-1)^{\#(i-1)+1} q_{i-1} \beta_{i-1}^{\prime}+(-1)^{\#(i+M)} q_{i} \beta_{i}+(-1)^{\#(i+M-1)} q_{i-1} \beta_{i-1}
\end{aligned}
$$

for odd $i$,

$$
\begin{aligned}
\mu_{\mathbb{L}_{q}}^{1}\left(\alpha_{i} \ell^{k}\right) & =\mu^{2}\left(\alpha_{i} \ell^{k},(-1)^{\# i} q_{i} \alpha_{i}^{\prime}\right)+\mu^{2}\left((-1)^{\# i} q_{i} \alpha_{i}^{\prime}, \alpha_{i} \ell^{k}\right)+\mu^{M}\left(\delta, \ldots, \alpha_{i} \ell^{k}, \ldots, \delta\right) \\
& =(-1)^{\# i+1} q_{i}\left[\left(\alpha_{i} \alpha_{i}^{\prime}\right)^{k+1}+\left(\alpha_{i}^{\prime} \alpha_{i}\right)^{k+1}\right]+\mu^{M}\left(\delta, \ldots, \alpha_{i} \ell^{k}, \ldots, \delta\right), \\
\mu_{\mathbb{L}_{q}}^{1}\left(\alpha_{i}^{\prime} \ell^{k}\right) & =\mu^{2}\left(\alpha_{i}^{\prime} \ell^{k},(-1)^{\# i} \alpha_{i}\right)+\mu^{2}\left((-1)^{\# i} \alpha_{i}, \alpha_{i}^{\prime} \ell^{k}\right)+\mu^{M}\left(\delta, \ldots, \alpha_{i}^{\prime} \ell^{k}, \ldots, \delta\right) \\
& =(-1)^{\# i+1}\left[\left(\alpha_{i} \alpha_{i}^{\prime}\right)^{k+1}+\left(\alpha_{i}^{\prime} \alpha_{i}\right)^{k+1}\right]+\mu^{M}\left(\delta, \ldots, \alpha_{i}^{\prime} \ell^{k}, \ldots, \delta\right), \\
\mu_{\mathbb{L}_{q}}^{1}\left(\beta_{i} \ell^{k}\right) & =\mu^{2}\left(\beta_{i} \ell^{k},(-1)^{\# i} q_{i} \alpha_{i}^{\prime}\right)+\mu^{2}\left((-1)^{\#(i+M)} \alpha_{i+M}, \beta_{i} \ell^{k}\right)+\mu^{M}\left(\delta, \ldots, \beta_{i} \ell^{k}, \ldots, \delta\right) \\
& =(-1)^{\# i+1} q_{i} \beta_{i} \alpha_{\alpha}^{\prime} \ell^{k}+(-1)^{\#(i+M)+1} \beta_{i}^{\prime} \alpha_{i} \ell^{k}+\mu^{M}\left(\delta, \ldots, \beta_{i} \ell^{k}, \ldots, \delta\right), \\
\mu_{\mathbb{L}_{q}}^{1}\left(\beta_{i}^{\prime} \ell^{k}\right) & =\mu^{2}\left(\beta_{i}^{\prime} \ell^{k},(-1)^{\# i} \alpha_{i}\right)+\mu^{2}\left((-1)^{\#(i+M)} q_{i} \alpha_{i+M}^{\prime}, \beta_{i}^{\prime} \ell^{k}\right)+\mu^{M}\left(\delta, \ldots, \beta_{i}^{\prime} \ell^{k}, \ldots, \delta\right) \\
& =(-1)^{\# i+1} \beta_{i}^{\prime} \alpha_{i} \ell^{k}+(-1)^{\#(i+M)+1} q_{i} \beta_{i} \alpha_{i}^{\prime} \ell^{k}+\mu^{M}\left(\delta, \ldots, \beta_{i}^{\prime} \ell^{k}, \ldots, \delta\right) .
\end{aligned}
$$

Remark D.7. The shorthand notation (D.1) has the property that

$$
Q_{k}^{\text {even }}=q_{k} Q_{k+1}^{\text {even }} \text { for even } k, \quad Q_{k}^{\text {odd }}=q_{k} Q_{k+1}^{\text {odd }} \text { for odd } k .
$$

For even $k$ we have

$$
Q_{k-M+1}^{\text {odd }}=Q_{k+2}^{\text {even }}=q_{k+2} \ldots q_{k+M-2}, \quad Q_{k}^{\text {even }}=q_{k} \ldots q_{k+M-1}=q_{k} \cdot\left(q_{k+2} \ldots q_{k+M-2}\right)=q_{k} \cdot Q_{k-M+1}^{\text {odd }} .
$$

We have used that $\# \#(k+1)=\# \# k+\#(k+M)-\# k$.
In order to apply the deformed Kadeishvili theorem, we need to compute the space $H_{q}$ according to Lemma 8.16 In other words, we shall calculate the deformed counterparts $h-E h$ of the cohomology basis elements $h \in H$. By abuse of notation, let us write $\# \alpha_{0}$ for the $\#$ sign associated with $\alpha_{0}$ and $q_{\alpha_{0}}$ for the puncture around which $\alpha_{0}$ winds.

Lemma D.8. The deformed cohomology basis elements are given by

$$
(-1)^{\#(i+M)+1} \beta_{i}^{\prime}+(-1)^{\# i} \beta_{i} \text { for odd } i
$$

and

$$
\operatorname{id}_{i}^{\prime}+(-1)^{\#(i-1)+\#(i+M-1)} q_{i-1} \operatorname{id}_{i-1}^{\prime} \text { for odd } i
$$

and

$$
\sum_{i=1}^{2 M} \mathrm{id}_{i} \quad \text { and } \quad(-1)^{\# \alpha_{0}+1} \alpha_{0}+(-1)^{\# \alpha_{0}} q_{\alpha_{0}} \alpha_{0}^{\prime}
$$

The differentials $\mu_{\mathbb{L}_{q}}^{1}$ for these morphisms are given by

$$
\mu_{\mathbb{L}_{q}}^{1}\left((-1)^{\#(i+M)+1} \beta_{i}^{\prime}+(-1)^{\# i} \beta_{i}\right)=(-1)^{\# \#(i+1)+1} Q_{i+2} \operatorname{id}_{L} \in H_{q}
$$

and

$$
\begin{aligned}
& \mu_{\mathbb{L}_{q}}^{1}\left(\mathrm{id}_{i}^{\prime}+(-1)^{\#(i-1)+\#(i+M-1)} q_{i-1} \operatorname{id}_{i-1}^{\prime}\right)=(-1)^{\# i+\#(i+M)} q_{i}\left((-1)^{\#(i+M)+1} \beta_{i}^{\prime}+(-1)^{\# i} \beta_{i}\right) \\
& +(-1)^{\#(i-1)+\#(i+M-1)+1} q_{i-1}\left((-1)^{\#(i-2+M)+1} \beta_{i-2}^{\prime}+(-1)^{\#(i-2)} \beta_{i-2}\right) \in H_{q}
\end{aligned}
$$

and

$$
\mu_{\mathbb{L}_{q}}^{1}\left(\sum_{i=1}^{2 M} \mathrm{id}_{i}\right)=0
$$

and

$$
\begin{aligned}
& \mu_{\mathbb{L}_{q}}^{1}\left((-1)^{\# \alpha_{0}+1} \alpha_{0}+(-1)^{\# \alpha_{0}} q_{\alpha_{0}} \alpha_{0}^{\prime}\right)= \\
& \quad \sum_{\substack{j=0 \\
j \text { even }}}^{M-1}(-1)^{\# \#(i+j-M+1)+1} Q_{i+j-M+1}^{\text {odd }}\left[\operatorname{id}_{i+j+1}^{\prime}+(-1)^{\#(i+j)+\#(i+j-M)} q_{i+j-M} \operatorname{id}_{i+j}^{\prime}\right] \\
& \quad+\sum_{\substack{j=2 \\
j \text { even }}}^{M-1}(-1)^{\# \#(i-j+1)} Q_{i-j+1}^{\text {even }}\left[\mathrm{id}_{i-j+1}^{\prime}+(-1)^{\#(i-j+M)+\#(i-j)} q_{i-j} \operatorname{id}_{i-j}^{\prime}\right] \in H_{q} .
\end{aligned}
$$

Proof. We need to check two things: First, all added infinitesimal terms lie in $\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} R$. Second, the map $\mu_{\mathbb{L}_{q}}^{1}$ sends all the deformed basis elements to $H_{q}$.

The first step is an easy observation: Indeed id ${ }_{i-1}^{\prime}$ for odd $i$ and $\alpha_{0}^{\prime}$ lie in $R$. For the second part, we need to evaluate $\mu_{q}^{1}$ on the deformed cohomology basis elements and check that the result belongs to $H_{q}$. We execute all calculations in order:

First, we regard the morphism $(-1)^{\#(i+M)+1} \beta_{i}^{\prime}+(-1)^{\# i} \beta_{i}$ for odd $i$. We compute

$$
\begin{aligned}
& \mu_{\mathbb{L}_{q}}^{1}\left((-1)^{\#(i+M)+1} \beta_{i}^{\prime}+(-1)^{\# i} \beta_{i}\right)=(-1)^{\#(i+M)+\# i} \beta_{i}^{\prime} \alpha_{i}+q_{i} \beta_{i} \alpha_{i}^{\prime}-q_{i} \beta_{i} \alpha_{i}^{\prime}+(-1)^{\# i+\#(i+M)+1} \beta_{i}^{\prime} \alpha_{i} \\
& \quad+\sum_{j=0}^{M-1} \mu^{M}\left((-1)^{\#(i+j+1)}\left[q_{i+j+1}\right] \alpha_{i+j+1}\left[{ }^{\prime}\right], \ldots,(-1)^{\#(i+M-1)} q_{i+M-1} \alpha_{i+M-1},\right. \\
& \left.\quad(-1)^{\#(i+M)+1} \beta_{i}^{\prime},(-1)^{\#(i+1)} \alpha_{i+1},(-1)^{\#(i+2)} q_{i+2} \alpha_{i+2}^{\prime}, \ldots,(-1)^{\#(i+j)}\left[q_{i+j}\right] \alpha_{i+j}\left[^{\prime}\right]\right) \\
& \quad+\sum_{j=0}^{M-1} \mu^{M}\left((-1)^{\#(i+2 M-1-j)}\left[q_{i+2 M-1-j}\right] \alpha_{i+2 M-1-j} l^{\prime}\right], \ldots,(-1)^{\#(i+M+1)} \alpha_{i+M+1}, \\
& \left.\left.\quad(-1)^{\# i} \beta_{i},(-1)^{\#(i-1)} \alpha_{i-1}^{\prime}, \ldots,(-1)^{\#(i-j)}\left[q_{i-j}\right] \alpha_{i-j} l^{\prime}\right]\right) \\
& =\sum_{j=0}^{M-1}(-1)^{\# \#(i+1)+1} Q_{i+2} \operatorname{id}_{i+j+1}+\sum_{j=0}^{M-1}(-1)^{\# \#(i+M+1)} Q_{i+2} \mathrm{id}_{i-j} \\
& =(-1)^{\# \#(i+1)+1} Q_{i+2} \mathrm{id}_{L} \in H_{q} .
\end{aligned}
$$

We have used that $\# \#(k+M)+\# \#(k)$ is the total number of $\#$ signs in the dimer, which is odd by assumption.

Second, we regard the morphism $\operatorname{id}_{i}^{\prime}+(-1)^{\#(i-1)+\#(i+M-1)} q_{i-1} \mathrm{id}_{i-1}^{\prime}$ for odd $i$. We compute

$$
\begin{aligned}
\mu_{\mathbb{L}_{q}}^{1} & \left(\mathrm{id}_{i}^{\prime}+(-1)^{\#(i-1)+\#(i+M-1)} q_{i-1} \mathrm{id}_{i-1}^{\prime}\right) \\
= & (-1)^{\# i+1} q_{i} \beta_{i}^{\prime}+(-1)^{\#(i-1)+1} q_{i-1} \beta_{i-1}^{\prime} \\
& +(-1)^{\#(i+M)} q_{i} \beta_{i}+(-1)^{\#(i+M-1)} q_{i-1} \beta_{i-1} \\
& +(-1)^{\#(i-2)+\#(i-1)+\#(i+M-1)+1} q_{i-1} \beta_{i-2}+(-1)^{\#(i+M-1)+1} q_{i-1} \beta_{i-1} \\
& +(-1)^{\#(i-1)} q_{i-1} \beta_{i-1}^{\prime}+(-1)^{\#(i+M-2)+\#(i-1)+\#(i+M-1)} q_{i-1} \beta_{i-2}^{\prime} \\
= & (-1)^{\# i+1} q_{i} \beta_{i}^{\prime}+(-1)^{\#(i+M)} q_{i} \beta_{i} \\
& +(-1)^{\#(i-2)+\#(i-1)+\#(i+M-1)+1} q_{i-1} \beta_{i-2}+(-1)^{\#(i+M-2)+\#(i-1)+\#(i+M-1)} q_{i-1} \beta_{i-2}^{\prime} \\
= & (-1)^{\# i+\#(i+M)} q_{i}\left((-1)^{\#(i+M)+1} \beta_{i}^{\prime}+(-1)^{\# i} \beta_{i}\right) \\
& +(-1)^{\#(i-1)+\#(i+M-1)+1} q_{i-1}\left((-1)^{\#(i-2+M)+1} \beta_{i-2}^{\prime}+(-1)^{\#(i-2)} \beta_{i-2}\right) \in H_{q} .
\end{aligned}
$$

Third, we regard the identity $\operatorname{id}_{L}=\sum_{i} \operatorname{id}_{i}$ and compute

$$
\mu_{q}^{1}\left(\mathrm{id}_{L}\right)=\mu^{2}\left(\operatorname{id}_{L}, \delta\right)+\mu^{2}\left(\delta, \mathrm{id}_{L}\right)=0
$$

Fourth, we deal with the co-identity $(-1)^{\# \alpha_{0}+1} \alpha_{0}+(-1)^{\# \alpha_{0}} q_{\alpha_{0}} \alpha_{0}^{\prime}$. Let $i$ be the index where $\alpha_{0}$ is located, such that $\alpha_{i}=\alpha_{0}$. Note that $i$ is even, since $\alpha_{0}$ is supposed to lie on the counterclockwise side of $Q_{M}$. In evaluating $\mu^{1}$ on the deformed co-identity, there appear two types of terms: four $\mu^{2}$ terms and many $\mu^{M}$ terms. The $\mu^{2}$ terms cancel each other as in the case for consistent dimers:

$$
\begin{aligned}
& \mu^{2}\left((-1)^{\# \alpha_{0}+1} \alpha_{0},(-1)^{\# \alpha_{0}} q_{\alpha_{0}} \alpha_{0}^{\prime}\right)+\mu^{2}\left((-1)^{\# \alpha_{0}} q_{\alpha_{0}} \alpha_{0}^{\prime},(-1)^{\# \alpha_{0}} \alpha_{0}\right) \\
& \quad+\mu^{2}\left((-1)^{\# \alpha_{0}} q_{\alpha_{0}} \alpha_{0}^{\prime},(-1)^{\# \alpha_{0}+1} \alpha_{0}\right)+\mu^{2}\left((-1)^{\# \alpha_{0}} \alpha_{0},(-1)^{\# \alpha_{0}} q_{\alpha_{0}} \alpha_{0}^{\prime}\right) \\
& =q_{\alpha_{0}} \alpha_{0} \alpha_{0}^{\prime}-q_{\alpha_{0}} \alpha_{0}^{\prime} \alpha_{0}+q_{\alpha_{0}} \alpha_{0}^{\prime} \alpha_{0}-q_{\alpha_{0}} \alpha_{0} \alpha_{0}^{\prime}=0 .
\end{aligned}
$$

We are now ready to calculate the $\mu^{M}$ terms:

$$
\begin{aligned}
& \sum_{j=0}^{M-1} \mu^{M}\left((-1)^{\#(i+j-M+1)}\left[q_{i+j-M+1}\right] \alpha_{i+j-M+1}^{\left[{ }^{\prime}\right]}, \ldots,(-1)^{\#(i-1)} q_{i-1} \alpha_{i-1}^{\prime},\right. \\
& \left.(-1)^{\# \alpha_{0}+1} \alpha_{0},(-1)^{\#(i+1)} q_{i+1} \alpha_{i+1}^{\prime}, \ldots,(-1)^{\#(i+j)}\left[q_{i+j}\right] \alpha_{i+j}^{\left[{ }^{\prime}\right]}\right) \\
& +\sum_{j=0}^{M-1} \mu^{M}\left((-1)^{\#(i-j+M-1)}\left[q_{i-j+M-1}\right] \alpha_{i-j+M-1}^{\left[{ }^{\prime}\right]}, \ldots,(-1)^{\#(i+1)} \alpha_{i+1},\right. \\
& \left.(-1)^{\# \alpha_{0}} q_{\alpha_{0}} \alpha_{0}^{\prime},(-1)^{\#(i-1)} \alpha_{i-1}, \ldots,(-1)^{\#(i-j)}\left[q_{i-j}\right] \alpha_{i-j}^{\left[{ }^{\prime}\right]}\right) \\
& =\sum_{j=0}^{M-1}(-1)^{\# \#(i+j-M+1)+1} Q_{i+j-M+1}^{\text {odd }} \operatorname{id}_{i+j+1}^{\prime}+\sum_{j=0}^{M-1}(-1)^{\# \#(i-j)} Q_{i-j}^{\text {even }} \operatorname{id}_{i-j}^{\prime} \\
& =(-1)^{\# \#(i-M+1)+1} Q_{i-M+1}^{\text {odd }} \operatorname{id}_{i+1}^{\prime}+(-1)^{\# \#(i)} Q_{i}^{\text {even }} \mathrm{id}_{i}^{\prime} \\
& +\sum_{\substack{j=2 \\
j \text { even }}}^{M-1}(-1)^{\# \#(i+j-M+1)+1} Q_{i+j-M+1}^{\text {odd }} \operatorname{id}_{i+j+1}^{\prime}+\sum_{\substack{j=2 \\
j \text { even }}}^{M-1}(-1)^{\# \#(i+j-M)+1} Q_{i+j-M}^{\text {odd }} \mathrm{id}_{i+j}^{\prime} \\
& +\sum_{\substack{j=2 \\
j \text { even }}}^{M-1}(-1)^{\# \#(i-j)} Q_{i-j}^{\text {even }} \operatorname{id}_{i-j}^{\prime}+\sum_{\substack{j=2 \\
j \text { even }}}^{M-1}(-1)^{\# \#(i-j+1)} Q_{i-j+1}^{\text {even }} \operatorname{id}_{i-j+1}^{\prime} \\
& =\sum_{\substack{j=0 \\
j \text { even }}}^{M-1}(-1)^{\# \#(i+j-M+1)+1} Q_{i+j-M+1}^{\mathrm{odd}}\left[\mathrm{id}_{i+j+1}^{\prime}+(-1)^{\#(i+j)+\#(i+j-M)} q_{i+j-M} \mathrm{id}_{i+j}^{\prime}\right] \\
& +\sum_{\substack{j=2 \\
j \text { even }}}^{M-1}(-1)^{\# \#(i-j+1)} Q_{i-j+1}^{\text {even }}\left[\operatorname{id}_{i-j+1}^{\prime}+(-1)^{\#(i-j+M)+\#(i-j)} q_{i-j} \operatorname{id}_{i-j}^{\prime}\right] \in H_{q} .
\end{aligned}
$$

We have used Remark D. 7 and $\# \#(i+1) \equiv \# \#(i-M+1)+1 \in \mathbb{Z} / 2 \mathbb{Z}$. These calculations show that the claimed elements are indeed the deformed cohomology basis elements, which finishes the proof.

The deformed Kadeishvili theorem lays out the following procedure: We have already uncurved the category $\mathbb{L}_{q}$ successfully. We have the full space $H_{q}$ in our hands. Next we have to compute the deformed codifferential $h_{q}: \operatorname{Hom}_{\mathbb{L}_{q}}(L, L) \rightarrow B \widehat{\otimes} R$. After that, we will be able to evaluate Kadeishvili trees and derive the minimal model.

It is not necessary to compute the entire codifferential $h_{q}$. Instead, the most important cohomology basis morphisms are of the form $\beta_{i}+\beta_{i}^{\prime}$ and $\alpha_{0}+q \alpha_{0}^{\prime}$ and $\mathrm{id}_{i}^{\prime}+q \mathrm{id}_{i-1}^{\prime}$. Any product $\mu_{q}^{\geq 3}$ of these can only produce an identity. Any product $\mu_{q}^{2}$ of these can only produce $\beta_{i} \alpha_{i}^{\prime}$ or $\beta_{i}^{\prime} \alpha_{i}$ or $\alpha_{0} \alpha_{0}^{\prime}$ or $\alpha_{0}^{\prime} \alpha_{0}$ or $\beta_{i}$ or $\beta_{i}^{\prime}$ or $\alpha_{i}$ or $\alpha_{i}^{\prime}$. It suffices to calculate the codifferential of these morphisms.

Let us analyze all the easy cases before we calculate the harder ones: For those morphisms lying in $R$, the codifferential immediately vanishes. Moreover, for odd $i$ the element $\beta_{i}+\beta_{i}^{\prime}$ lies in $H_{q}$ and $\beta_{i}^{\prime}$ lies in $R$, thus $h_{q}\left(\beta_{i}\right)=0$ for odd $i$. The nontrivial cases are as follows:
Lemma D.9. We have the following values of the codifferential:

$$
\begin{aligned}
h_{q}\left(\alpha_{i}\right) & = \pm \operatorname{id}_{a_{j}}+\ldots \pm \operatorname{id}_{a_{i}} \text { for } \alpha_{i} \neq \alpha_{0}, \\
h_{q}\left(\beta_{i}^{\prime}\right) & =(-1)^{\#(i+M)} \operatorname{id}_{i}^{\prime} \text { for even } i, \\
h_{q}\left(\beta_{i}^{\prime} \alpha_{i}\right) & =(-1)^{\# i+1} \beta_{i}^{\prime} \text { for odd } i, \\
h_{q}\left(\beta_{i}^{\prime} \alpha_{i}\right) & =(-1)^{\#(i+M)+1} \beta_{i} \text { for even } i, \\
h_{q}\left(\alpha_{0} \alpha_{0}^{\prime}\right) & =(-1)^{\# i+1} \alpha_{0}^{\prime} .
\end{aligned}
$$

Proof. The first two cases are simple: The value of $\mu_{q}^{1}$ on identities equals the value of $\mu^{1}$ and therefore decomposition of $\alpha_{i}$ and $\beta_{i}^{\prime}$ from Table D. 3 remains valid. We remark that for $\alpha_{i}$ the right-hand side needs to be written as $\alpha_{0}+q \alpha_{0}^{\prime}-q \alpha_{0}^{\prime}$, but since $\alpha_{0}^{\prime} \in R$ this is no issue.

The third case consists of checking $\beta_{i}^{\prime} \alpha_{i}$ for odd $i$ :

$$
\mu_{q}^{1}\left(\beta_{i}^{\prime}\right)=(-1)^{\# i+1} \beta_{i}^{\prime} \alpha_{i}+(-1)^{\#(i+M)+1} q_{i} \beta_{i} \alpha_{i}^{\prime} \ell^{k}+\mu^{M}(\ldots)
$$

The term $\beta_{i} \alpha_{i}^{\prime}$ lies in $R$. The terms resulting from $\mu^{M}(\ldots)$ are all of the form $\mathrm{id}_{i}$. The $h_{q}$ of such terms necessarily vanishes, and we deduce the above codifferential equation.

The fourth case of $\beta_{i}^{\prime} \alpha_{i}$ for even $i$ is similar. Finally, we check the fifth case of $\alpha_{0} \alpha_{0}^{\prime}$ :

$$
\mu_{q}^{1}\left(\alpha_{0}^{\prime}\right)=(-1)^{\# i+1}\left[\left(\alpha_{0} \alpha_{0}^{\prime}\right)+\left(\alpha_{0}^{\prime} \alpha_{0}\right)\right]+\mu^{M}(\ldots)
$$

The terms resulting from $\mu^{M}(\ldots)$ are all of the form $\operatorname{id}^{\prime}$. These either lie directly in $R$ or they lie in $H$ when combined with additional $\mathrm{id}_{i-1}^{\prime} \in R$.

In Lemma D.9. we have saved ourselves from computing the describing the correct signs of $h_{q}\left(\alpha_{i}\right)$. In fact, the signs are analogous to those presented in section 10.3

## D. 6 Result components

In this section, we analyze result components of $\mathrm{H} \mathbb{L}_{q}$ and match them with CR, ID, DS and DW disks. The starting point is the category $\mathbb{L}_{q}$. In section D. 5 we have already computed the deformed cohomology basis elements and the deformed codifferential. Here, we regard Kadeishvili trees, analyze the shape of their outputs and introduce a suitable notion of result components. We introduce a suitable notion of CR, ID, DS and DW disks and match all result components with smooth disks of these four types.

As in the classical case, we start by computing a multiplication table for important endomorphisms of $L \in \mathbb{L}_{q}$. The multiplication table is found in Table D.4
Remark D.10. Most values in Table D. 4 are checked easily using Table D. 3 and more specifically Lemma D.9. They can be grouped essentially in three types: those multiplications which always yield a particular value (with respect to $\mu^{2}, h_{q} \mu^{2}$ and $\pi_{q} \mu^{2}$ ), those which vanish if $i$ is even or odd and yield a nonzero value if $i$ is odd respectively even, and those which involve $\alpha_{0}$ where only close inspection proves them to vanish. The products $\mu^{2}\left(\alpha_{0}, \mathrm{id}_{i}^{\prime}\right)$ and $\mu^{2}\left(\mathrm{id}_{i}^{\prime}, \alpha_{0}\right)$ notably fall in the latter category. Let us digest this in case of $\mu^{2}\left(\alpha_{0}, \mathrm{id}_{i}^{\prime}\right)$ : From the fact that $\alpha_{0}$ lies in the counterclockwise polygon, we deduce that the source arc of $\alpha_{0}$ is odd and therefore $i$ is even. The result $\mu^{2}\left(\alpha_{0}, \operatorname{id}_{i}^{\prime}\right)$ then equals $\beta_{i-1}^{\prime}$. Now $i-1$ is odd, the element $\beta_{i-1}^{\prime}$ lies in $R$ and we conclude $h_{q}\left(\beta_{i-1}^{\prime}\right)=\pi_{q}\left(\beta_{i-1}^{\prime}\right)=0$. This explains the entry of $m_{2}=\alpha_{0}$ and $m_{1}=\mathrm{id}_{i}^{\prime}$ in Table D.4.

| $m_{2}$ | $\backslash m_{1}$ | $\mathrm{id}_{i}$ | $\mathrm{id}^{\prime}{ }^{\prime}$ | $\beta_{i}$ | $\beta_{i}^{\prime}$ | $\alpha_{0}$ | $\alpha_{i}^{\prime}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{id}_{i}$ | $\mu^{2}=$ | $\mathrm{id}_{i}$ | $\mathrm{id}^{\prime}{ }_{i}$ | $\beta_{i}$ | $\beta_{i}^{\prime}$ | $\alpha_{0}$ | $\alpha_{i}^{\prime}$ |
|  | $i=$ | any | odd | odd | even | any | any |
|  | $h_{q}=$ | 0 | 0 | 0 | $\mathrm{id}_{i}^{\prime}$ | 0 | 0 |
|  | $\pi_{q}=$ | $\left(\mathrm{id}_{i}\right)$ | $\mathrm{id}^{\prime}$ | $\beta_{i}+\beta_{i}^{\prime}$ | $\beta_{i-1}+\beta_{i-1}^{\prime}$ | $\alpha_{0}$ | 0 |
| $\mathrm{id}_{i}^{\prime}$ | $\mu^{2}=$ | $\mathrm{id}_{i}^{\prime}$ | $\mathrm{id}_{i}$ | $\alpha_{i}$ | $\alpha_{i}^{\prime}$ | $\beta_{i}$ | $\beta_{i}^{\prime}$ |
|  | $i=$ | odd | any | any | any | even | even |
|  | $h_{q}=$ | 0 | 0 | $\mathrm{id}_{i}$ | 0 | 0 | $\mathrm{id}^{\prime}$ |
|  | $\pi_{q}=$ | $\mathrm{id}_{i}^{\prime}$ | $\left(\mathrm{id}_{i}\right)$ | $\alpha_{0}$ | 0 | 0 | $\beta_{i-1}+\beta_{i-1}^{\prime}$ |
| $\beta_{i}$ | $\mu^{2}=$ | $\beta_{i}$ | $\alpha_{i}^{\prime}$ | $\alpha_{i}^{\prime} \alpha_{i}$ | imp | imp | $\beta_{i} \alpha_{i}^{\prime}$ |
|  | $i=$ | odd | any | any | any | any | any |
|  | $h_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
|  | $\pi_{q}=$ | $\beta_{i}+\beta_{i}^{\prime}$ | 0 | 0 | 0 | 0 | 0 |
| $\beta_{i}^{\prime}$ | $\mu^{2}=$ | $\beta_{i}^{\prime}$ | $\alpha_{i}$ | imp | $\alpha_{i} \alpha_{i}^{\prime}$ | $\beta_{i}^{\prime} \alpha_{0}$ | imp |
|  | $i=$ | even | even | any | any | even | any |
|  | $h_{q}=$ | $\mathrm{id}_{i}^{\prime}$ | $\mathrm{id}_{i}$ | 0 | $\alpha_{i}^{\prime}$ | $\beta_{i}$ | 0 |
|  | $\pi_{q}=$ | $\beta_{i-1}+\beta_{i-1}^{\prime}$ | $\alpha_{0}$ | 0 | 0 | 0 | 0 |
| $\alpha_{0}$ | $\mu^{2}=$ | $\alpha_{0}$ | $\beta_{i}^{\prime}$ | $\beta_{i}^{\prime} \alpha_{i}$ | imp | imp | $\alpha_{i} \alpha_{i}^{\prime}$ |
|  | $i=$ | any | even | odd | any | any | even |
|  | $h_{q}=$ | 0 | 0 | $\beta_{i}^{\prime}$ | 0 | 0 | $\alpha_{i}^{\prime}$ |
|  | $\pi_{q}=$ | $\alpha_{0}$ | 0 | 0 | 0 | 0 | 0 |
| $\alpha_{i}^{\prime}$ | $\mu^{2}=$ | $\alpha_{i}^{\prime}$ | $\beta_{i}$ | imp | $\beta_{i} \alpha_{i}^{\prime}$ | $\alpha_{i}^{\prime} \alpha_{i}$ | imp |
|  | $i=$ | any | odd | any | any | any | any |
|  | $h_{q}=$ | 0 | 0 | 0 | 0 | 0 | 0 |
|  | $\pi_{q}=$ | 0 | $\beta_{i}+\beta_{i}^{\prime}$ | 0 | 0 | 0 | 0 |

Table D.4: Multiplication table. Whenever the parity of $i$ is specified, this refers to the parity of the index of the $\mu^{2}$ result, instead of the indices of the inputs or $h_{q}$ and $\pi_{q}$ values.

Remark D.11. As in the case of consistent dimers, the multiplication table merely indicates possible products, as opposed to products that actually exist. For example, the three indices " $i$ " in a product rule like $\beta_{i}=\mu^{2}\left(\operatorname{id}_{i}, \beta_{i}\right)$ are not meant to denote the same index, but rather indicate the type of morphism: The first is an indexed $\beta$ morphism, the second an indexed arc identity and the third again an indexed $\beta$ morphism. The table merely implies that any actually existing product is of the form $\beta_{i}=\mu^{2}\left(\mathrm{id}_{j}, \beta_{k}\right)$ for some combination of indices $(i, j, k)$ allowed. Of course, we can check which combinations actually yield nonvanishing results: Those are precisely $\beta_{i}=\mu^{2}\left(\mathrm{id}_{i+M+1}, \beta_{i}\right)$ for odd $i$ and $\beta_{i}=\mu^{2}\left(\operatorname{id}_{i+M}, \beta_{i}\right)$ for even $i$. We will refer to precise combinations of indices $(i, j, k)$ that yield nonvanishing results as precise shape of the product. We may also refer to precise shapes when referring to $h_{q}$ or $\pi_{q}$ evaluations like $\beta_{i}+\beta_{i}^{\prime}=\pi_{q} \mu^{2}\left(\mathrm{id}_{i}, \beta_{i}\right)$. In any case, the precise shape is understood to link all indices involved.

Definition D.12. Kadeishvili $h$-trees, $\pi$-trees and their result components are defined as in the consistent case. In particular, a tree is supposed to have at least two leaves. The grouping rule for result components specifically reads as follows: The $\pi$-tree result components $(-1)^{\#(i+M)+1} \beta_{i}^{\prime}$ and $(-1)^{\# i} \beta_{i}$ shall be grouped together as one result component. Also, the result components $\mathrm{id}_{i}$ shall be grouped together as one result component.

We now analyze which result components are possible. As in the consistent case, we can assume that the inputs of a Kadeishvili tree do not include the identity element $\mathrm{id}_{L}=\sum_{i=1}^{2 M} \mathrm{id}_{i}$. We analyze all possible result components of $h$-trees first, before proceeding to $\pi$-trees. Their inputs may be deformed basis elements of type $\alpha_{0}+q \alpha_{0}^{\prime}, \beta_{i}+\beta_{i}^{\prime}$ and $\mathrm{id}_{i}^{\prime}$.

A Kadeishvili $h$-tree is decorated by $h_{q} \mu_{\mathbb{L}_{q}}$ on all its non-leaf nodes. A Kadeishvili $\pi$-tree is decorated by $h_{q} \mu_{\mathbb{L}_{q}}$ on all its non-leaf non-root nodes, and $\pi_{q} \mu_{\mathbb{L}_{q}}$ on the root. Our notation $\mu_{q}^{2}$ or $\mu_{q}^{\geq 3}$ refers to the products of $\operatorname{Add~}_{\mathrm{Gtl}_{q} Q}$.

As a first clue towards our analysis, we claim that $\mu_{\bar{q}}^{\geq 3}$ can only be applied at the root. Indeed a disk


Figure D.6: The $\beta_{i}^{\prime}$ trees


Figure D.7: The $\alpha_{0}^{\prime}$ trees
$\mu_{\bar{q}}^{\geq 3}$ can only yield $\mathrm{id}_{i}$ and $\mathrm{id}_{i}^{\prime}$. In both cases, their $h_{q}$-values vanish. Their $\pi_{q}$-values are given by

$$
\pi_{q}\left(\mathrm{id}_{i}\right)=\left\{\begin{array}{ll}
\mathrm{id}_{L} & \text { if } i=i_{0} \\
0 & \text { else }
\end{array} \quad \text { and } \quad \pi_{q}\left(\mathrm{id}_{i}^{\prime}\right)= \begin{cases}\operatorname{id}_{i}^{\prime}+q_{i-1} \mathrm{id}_{i-1}^{\prime} & \text { if } i \text { odd } \\
0 & \text { else }\end{cases}\right.
$$

This shows that $\mu_{\bar{q}}^{\geq 3}$ can only be applied at the root. The decoration at all other nodes necessarily concerns a $\mu_{q}^{2}$.
Lemma D.13. Any result component $\alpha_{0}$ or $\beta_{i}$ of an $h$-tree is direct. Any result components $\beta_{i}^{\prime}, \alpha_{i}^{\prime}$, $\mathrm{id}_{i}^{\prime}$, $\mathrm{id}_{i}$ of $h$-trees are derived from one of the trees in Figure D.6, D.7, D.9, D. 10

Proof. We start with explaining the first statement, and then delve into the second one. Our first observation is that $\alpha_{0}$ does not appear in the multiplication table D. 4 at all and therefore any result component $\alpha_{0}$ is necessarily direct. Regard now a result component $\beta_{i}$ and assume it is non-direct. According to the multiplication table, it must be derived from a product of the form $\mu^{2}\left(\beta_{i}^{\prime}, \alpha_{0}\right)$. Let us investigate the precise shape of this product: As $\alpha_{0}$ is located on the rear side, the index $i$ of the morphism $\beta_{i}^{\prime}$ is necessarily even. Therefore $\beta_{i}^{\prime}$ cannot be direct, while a glance at the multiplication table simultaneously reveals that $\beta_{i}^{\prime}$ with even $i$ cannot be produced as a non-direct result component either. We conclude that no single result component can be derived from a product $\mu^{2}\left(\beta_{i}^{\prime}, \alpha_{0}\right)$. Finally, this means that any result component $\beta_{i}$ is direct. This proves the first desired statement.

Regard now a result component $\beta_{i}^{\prime}$ and assume it is non-direct. We have already seen that $i$ is necessarily odd and $\beta_{i}^{\prime}$ is derived from a product of the form $\mu^{2}\left(\alpha_{0}, \beta_{i}\right)$. The precise shape of this product is $\mu^{2}\left(\alpha_{0}, \beta_{i}\right)$ with $i$ being equal to the index of the co-identity angle $\alpha_{0}$ incremented by $M$. Finally, we also realize that both $\alpha_{0}$ and $\beta_{i}$ are direct. The tree is depicted in Figure D. 6 .

Regard a result component $\alpha_{i}^{\prime}$ and assume it is non-direct. According to the multiplication table, it is derived from $\mu^{2}\left(\beta_{i}^{\prime}, \beta_{i}^{\prime}\right)$ or $\mu^{2}\left(\alpha_{0}, \alpha_{i}^{\prime}\right)$. Let us examine both cases separately. In the first case, the precise shape of the product is $\mu^{2}\left(\beta_{i+M}^{\prime}, \beta_{i}^{\prime}\right)$. In particular either $i$ or $i+M$ is even, while there are in fact no result component $\beta_{i}^{\prime}$ with even $i$. The first case is therefore impossible. In the second case, the precise shape is $\mu^{2}\left(\alpha_{0}, \alpha_{0}^{\prime}\right)$. We recall that $\alpha_{0}$ is necessarily direct, and $\alpha_{0}^{\prime}$ may either be direct or be derived from a product $\mu^{2}\left(\alpha_{0}, \alpha_{0}^{\prime}\right)$ again. This gives a recursion on how $\alpha_{0}^{\prime}$ is derived. Solving this recursion gives the tree in Figure D. 7.

Regard a result component $\mathrm{id}_{i}^{\prime}$ and assume it is non-direct. According to the multiplication table, it is derived from $\mu^{2}\left(\mathrm{id}_{i}, \beta_{i}^{\prime}\right)$ or $\mu^{2}\left(\beta_{i}^{\prime}, \mathrm{id}_{i}\right)$ or $\mu^{2}\left(\mathrm{id}_{i}^{\prime}, \alpha_{i}^{\prime}\right)$. Let us examine all three cases. In the first and second case, $\beta_{i}^{\prime}$ needs even index in order to have nonvanishing $h_{q}$. However, we have already seen that result components $\beta_{i}^{\prime}$ all have odd index. This means there is no result component derived from the first or second case. In the third case, the precise shape of the product is $\mathrm{id}_{i}^{\prime}=h_{q} \mu^{2}\left(\mathrm{id}_{i+1}^{\prime}, \alpha_{i}^{\prime}\right)$ and $i$ is even. We conclude that $\mathrm{id}_{i+1}^{\prime}$ is direct. Even better, the result component $\alpha_{i}^{\prime}$ is necessarily $\alpha_{0}^{\prime}$ and is derived from one of the trees in Figure D.7. This gives rise to the tree in Figure D.9.

Regard a result component $\mathrm{id}_{i}$. It is necessarily non-direct, since we excluded the identity cohomology elements from the tree inputs. According to the multiplication table, it is derived from $\mu^{2}\left(\mathrm{id}_{i}^{\prime}, \beta_{i}\right)$ or $\mu^{2}\left(\beta_{i}^{\prime}, \mathrm{id}_{i}^{\prime}\right)$. Let us explore both cases. In the first case, the precise shape is $\operatorname{id}_{j}=h_{q} \mu^{2}\left(\mathrm{id}_{i+M+1}^{\prime}, \beta_{i}\right)$ and $i$ is odd. Note that $j$ is free, and in fact $h_{q} \mu^{2}\left(\mathrm{id}_{i+M+1}^{\prime}, \beta_{i}\right)$ produces many arc identities at once. Finally, due to parity both $\mathrm{id}_{i+M+1}^{\prime}$ and $\beta_{i}$ are necessarily direct. This yields one tree. In the second case, the precise shape is $\alpha_{i}=\mu^{2}\left(\beta_{i+M}^{\prime}, \mathrm{id}_{i+1}^{\prime}\right)$ with even $i$. We already know that $\mathrm{id}_{i+1}^{\prime}$ is necessarily direct. In contrast, $\beta_{i+M}^{\prime}$ may be either direct or derived from $\mu^{2}\left(\alpha_{0}, \beta_{i+M}\right)$. The latter case however entails that $\alpha_{i}=\alpha_{0}$, hence $\mu^{2}\left(\beta_{i+M}^{\prime}, \mathrm{id}_{i+1}^{\prime}\right)=\alpha_{0}$ and $h_{q} \mu^{2}\left(\beta_{i+M}^{\prime}, \operatorname{id}_{i+1}^{\prime}\right)=0$. We conclude that $\beta_{i+M}^{\prime}$ is necessarily direct instead. This yields one single tree. In total, both trees producing $\mathrm{id}_{i}$ result components are depicted in Figure D. 10

We are now ready to approach result components of $\pi$-trees.


Figure D.9: The id ${ }_{i}^{\prime}$ trees


Figure D.10: The $\mathrm{id}_{i}$ trees

Lemma D.14. All $\pi_{q} \mu^{2}$ result components $\mathrm{id}_{i}^{\prime}, \beta_{i}^{\prime}+\beta_{i}, \mathrm{id}_{i}, \alpha_{0}$ are derived from one of the trees in Figure D.11, D.12, D.13, D.14.
Proof. The idea is to systematically read off from the multiplication table D.4 all possible ways these result components may be be derived from result components of $h$-trees. All result components of $h$ trees falls under the regime of Lemma D.13 allowing us to make statements on how they are derived themselves. In each case, we acquire full knowledge of the entire $\pi$-tree.

Regard an $\mathrm{id}_{i}^{\prime}$ result component of a $\pi$-tree. According to the multiplication table D .4 it is necessarily derived from $\mu^{2}\left(\mathrm{id}_{i}, \mathrm{id}_{i}^{\prime}\right)$ or $\mu^{2}\left(\mathrm{id}_{i}^{\prime}, \mathrm{id}_{i}\right)$. In the first case, the precise shape is $\mathrm{id}_{i}^{\prime}=\pi_{q} \mu^{2}\left(\mathrm{id}_{i+M}, \mathrm{id}_{i}^{\prime}\right)$ with $i$ odd. We realize that $\mathrm{id}_{i}^{\prime}$ is necessarily direct, while $\mathrm{id}_{i+M}$ may come from two possible trees. In the second case, the precise shape is $\operatorname{id}_{i}^{\prime}=\pi_{q} \mu^{2}\left(\mathrm{id}_{i}^{\prime}, \mathrm{id}_{i}\right)$ with $i$ odd. We realize that $\mathrm{id}_{i}^{\prime}$ is direct, while $\mathrm{id}_{i}$ may again come from two possible trees. In total, the four possible trees are depicted in Figure D.11.

Regard an $\beta_{i}^{\prime}+\beta_{i}$ result component. According to the multiplication table, it is derived from $\mu^{2}\left(\mathrm{id}_{i}, \beta_{i}\right)$ or $\mu^{2}\left(\mathrm{id}_{i}, \beta_{i}^{\prime}\right)$ or $\mu^{2}\left(\mathrm{id}_{i}^{\prime}, \alpha_{0}^{\prime}\right)$ or $\mu^{2}\left(\beta_{i}, \mathrm{id}_{i}\right)$ or $\mu^{2}\left(\beta_{i}^{\prime}, \mathrm{id}_{i}\right)$ or $\mu^{2}\left(\alpha_{0}^{\prime}, \mathrm{id}_{i}^{\prime}\right)$. Let us investigate all these six cases. In the first case, the precise shape is $\beta_{i}^{\prime}+\beta_{i}=\pi_{q} \mu^{2}\left(\mathrm{id}_{i+M+1}, \beta_{i}\right)$ with $i$ odd. We realize that $\beta_{i}$ is direct and $\mathrm{id}_{i+M+1}$ may come from two possible trees. In the second case, the precise shape is $\beta_{i-1}^{\prime}+\beta_{i-1}=\pi_{q} \mu^{2}\left(\mathrm{id}_{i+M}, \beta_{i}^{\prime}\right)$ with $i$ even. Since no result component $\beta_{i}^{\prime}$ with even $i$ exists, this case is impossible. In the third case, the precise shape is $\beta_{i-1}^{\prime}+\beta_{i-1}=\pi_{q} \mu^{2}\left(\mathrm{id}_{i+1}^{\prime}, \alpha_{0}^{\prime}\right)$ with $i$ even. We realize that $\mathrm{id}_{i+1}^{\prime}$ is direct and $\alpha_{0}^{\prime}$ comes from the known standard type of tree. In the fourth case, the precise shape is $\beta_{i}^{\prime}+\beta_{i}=\pi_{q} \mu^{2}\left(\beta_{i}, \mathrm{id}_{i}\right)$ with $i$ odd. We realize that $\beta_{i}$ is direct and $\mathrm{id}_{i}$ may come from two possible trees. In the fifth case, the precise shape is $\beta_{i-1}^{\prime}+\beta_{i-1}=\pi_{q} \mu^{2}\left(\beta_{i}^{\prime}, \mathrm{id}_{i}\right)$ with $i$ even. Since no result component $\beta_{i}^{\prime}$ with odd $i$ exists, this case is impossible. In the sixth case, the precise shape is $\beta_{i+M}^{\prime}+\beta_{i+M}=\pi_{q} \mu^{2}\left(\alpha_{0}^{\prime}, \mathrm{id}_{i+M}^{\prime}\right)$ with $i=h\left(\alpha_{0}\right)$ even. We realize that $\mathrm{id}_{i+M}^{\prime}$ is direct and $\alpha_{0}^{\prime}$ comes from the known tree. In total, all six trees are depicted in Figure D. 12

Regard an $\mathrm{id}_{i}$ result component. According to the multiplication table, it is derived from $\mu^{2}\left(\mathrm{id}_{i}, \mathrm{id}_{i}\right)$ or $\mu^{2}\left(\mathrm{id}_{i}^{\prime}, \mathrm{id}_{i}^{\prime}\right)$. In the first case, the precise shape is $\mathrm{id}_{i_{0}}=\pi_{q} \mu^{2}\left(\mathrm{id}_{i_{0}}, \mathrm{id}_{i_{0}}\right)$. Since the arc identity $\mathrm{id}_{i_{0}}$ never appears as result component of an $h$-tree, this case is however vacuous. In the second case, the precise shape is $\mathrm{id}_{i_{0}}=\pi_{q} \mu^{2}\left(\mathrm{id}_{i_{0}+M}^{\prime}, \mathrm{id}_{i_{0}}^{\prime}\right)$. This tree is depicted in Figure D. 13 .

Regard an $\alpha_{0}$ result component. According to the multiplication table, it is derived from $\mu^{2}\left(\mathrm{id}_{i}, \alpha_{0}\right)$ or $\mu^{2}\left(\alpha_{0}, \mathrm{id}_{i}\right)$ or $\mu^{2}\left(\mathrm{id}_{i}^{\prime}, \beta_{i}\right)$ or $\mu^{2}\left(\beta_{i}^{\prime}, \mathrm{id}_{i}^{\prime}\right)$. Let us investigate all four cases. In the first case, the precise shape is $\alpha_{0}=\pi_{q} \mu^{2}\left(\mathrm{id}_{i}, \alpha_{0}\right)$ with $i=h\left(\alpha_{0}\right)$ even. We realize that $\alpha_{0}$ is direct, while $\mathrm{id}_{i}$ may come from two possible trees. In the second case, the precise shape is $\alpha_{0}=\pi_{q} \mu^{2}\left(\alpha_{0}, \mathrm{id}_{i+1}\right)$ with $i=h\left(\alpha_{0}\right)$ even. We realize that $\alpha_{0}$ is direct, while $\mathrm{id}_{i+1}$ may come from two possible trees. In the third case, the precise shape is $\alpha_{0}=\pi_{q} \mu^{2}\left(\mathrm{id}_{i+M}^{\prime}, \beta_{i}\right)$ with $i$ even or $\alpha_{0}=\pi_{q} \mu^{2}\left(\mathrm{id}_{i+M+1}^{\prime}, \beta_{i}\right)$ with $i$ odd. The former case is impossible, since there is no result component $\beta_{i}$ with $i$ even. In the latter case, both $\operatorname{id}_{i+M+1}^{\prime}$ and $\beta_{i}$ are direct. In the fourth case, the precise shape is $\alpha_{0}=\pi_{q} \mu^{2}\left(\beta_{i+M}^{\prime}, \mathrm{id}_{i+1}^{\prime}\right)$ with $i$ even. We realize that $\mathrm{id}_{i+1}^{\prime}$ is direct, while $\beta_{i+M}^{\prime}$ may be direct or derived from $\mu^{2}\left(\alpha_{0}, \beta_{i+M}\right)$. We recall that in the latter case it is necessary that $i=h\left(\alpha_{0}\right)$. The total collection of seven trees is depicted in Figure D. 14 .


$$
\pi_{q}=\mathrm{id}_{i}^{\prime}
$$

$i$ odd, $j$ even
Figure D.11: $\pi$-trees for $\mathrm{id}^{\prime}{ }_{i}$



$\pi_{q}=\beta_{i}^{\prime}+\beta_{i}$
$i$ odd, $j$ odd




$$
\begin{gathered}
\pi_{q}=\beta_{i-1}^{\prime}+\beta_{i-1} \\
i=h\left(\alpha_{0}\right) \text { even }
\end{gathered}
$$

$\geq 0$


Figure D.12: $\pi$-trees for $\beta_{i}^{\prime}+\beta_{i}$

$$
\begin{array}{r}
\mathrm{id}_{i_{0}+M}^{\prime} \operatorname{idd}_{i_{0}}^{\prime} \\
\pi_{q}=\operatorname{id}_{L}
\end{array}
$$

Figure D.13: $\pi$-trees for $\mathrm{id}_{L}=\sum \mathrm{id}_{i}$

We now introduce the suitable version of CR, ID, DS and DW disks.
Definition D.15. CR, ID, DS and DW disks are defined as in the case of geometrically consistent dimers. More specifically, the definitions read as follows: A CR disk is an SL disk where all inputs and the output lie apart, with the exception that stacks of $\alpha_{0}$ inputs are allowed if the SL disk lies on the front side. An ID disk is an SL disk where all inputs and the output lie apart, with the exception that stacks of $\alpha_{0}$ inputs are allowed if the disk lies on the front side, and a $\beta_{i}^{\prime}+\beta_{i}$ may infinitesimally precede respectively succeed the output mark if the disk lies on the front respectively rear side. A DS disk is one of the particular types of degenerate strips fitting between $L$ and its Hamiltonian deformation. A DW disk is one of the particular types of degenerate wedges fitting fitting between $L$ and its Hamiltonian deformation, with one corner being the co-identity of $L$. The collections of CR/ID/DS/DW disks are denoted Disk ${ }_{C R}$, Disk ${ }_{I D}$, Disk ${ }_{\text {DS }}$ and Disk ${ }_{\text {DW }}$ respectively.

Subdisks of result components of $\pi$-trees are defined in the same way as in the consistent case. A few peculiarities of the subdisk construction are depicted in Figure D.15 and Figure D. 16 As in the


Figure D.14: $\pi$-trees for $\alpha_{0}$
consistent case, associating subdisks provides a bijection between result components and CR, ID, DS and DW disks. Since a $\pi$-tree has at least two inputs by definition, the subdisk mapping only reaches CR, ID, DS and DW disks which have at least two inputs as well. We denote these classes of CR, ID, DS and DW disks by Disk $\underset{\mathrm{CR}}{\geq 2}$, Disk ${\underset{\mathrm{ID}}{2}}_{\geq 2}^{2}$, Disk $\geq_{\mathrm{DS}}^{22}$ and Disk $\geq_{\mathrm{DW}}^{\geq 2}$. We record the bijectivity statement as follows:

Lemma D.16. The subdisk mapping $D$ is a bijection

$$
\text { D }: \operatorname{Result}_{\pi} \xrightarrow{\sim} \operatorname{Disk}_{\mathrm{CR}}^{\geq 2} \dot{U} \text { Disk }_{\mathrm{ID}}^{\geq 2} \dot{U} \operatorname{Disk}_{\mathrm{DS}}^{\geq 2} \dot{U} \text { Disk }_{\mathrm{DW}}^{\geq 2} .
$$

Proof. Injectivity should be clear. Proving surjectivity entails recovering for every CR, ID or DS disk $D$ a result component $r$ whose drawing $\mathrm{D}(r)$ is $D$. We will not prove this in detail. In fact, the cases to be studied are merely a subset of the cases of the case of consistent dimers.

## D. 7 Minimal model

In this section we provide our minimal model for $\mathbb{L}_{q}$. The assembly works as follows: In section D.6. we have already enumerated all result components for the products $\mu_{\mathrm{H} \mathbb{L}_{q}}^{\geq 2}$ in terms of CR, ID, DS and DW disks. In section D.5, we have computed the differential $\mu_{\mathbb{L}_{q}}^{1}$ on the deformed cohomology basis elements. In the present section, we assemble the minimal model $H \mathbb{L}_{q}$. In particular, we show that not only the higher products $\mu_{\overline{\mathrm{H}} \mathbb{L}_{q}}^{\geq 2}$ are computed by CR, ID, DS and DW disks, but also the differential $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}$. We offer an explicit list of the CR, ID, DS and DW disks that contribute to the differential.

In Proposition D.18, we claim that the differential $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}$ is enumerated accurately by CR, ID, DS and DW disks. It makes sense to compile a list of these disks in advance. Recall that the an SL disk with a single input is a digon: a smooth immersed disks with two corners. In what follows, we try to spot and list all digons of which the input is a given morphism $h$. We can already ignore DS and DW disks, since they have at least two inputs. The following is our sphere digon list:

Digons for the odd morphism $h=\beta_{i}^{\prime}+\beta_{i}$ : There is precisely one single digon with input $h$. It is a CR or ID disk and its output is $\mathrm{id}_{L}$. This type of digon is depicted in Figure D.17a
In general, one spots this digon as follows: The intersection point $h$ cuts the zigzag curve $\tilde{L}$ into two segments. One segment departs towards the front side at $h$ and the other depars towards the rear side. The identity $\mathrm{id}_{L}$ lies by choice on one of these two segments. Whether it lies on the frontor rear-bound segment determines the location of the claimed digon. Specifically, the digon lies on the front side if departing at $h$ towards the front we hit the identity id ${ }_{L}$ before returning to $h$, and on the rear side if departing at $h$ towards the rear side we hit the identity $\operatorname{id}_{L}$ before returning to $h$.


Figure D.15: $\mathrm{id}_{L}$ disk result components and their subdisks. The first five disks lie on the rear side and the second five on the front side. The first and final arc is supposed to be the identity arc $i_{0}$ of $L$ and is highlighted by a crossing double line. This double line also indicates the separation between the first and the final angle of the disk.

Digons for the even morphism $h=\mathrm{id}_{i}^{\prime}$ : There are precisely two digons with input $\mathrm{id}_{i}^{\prime}$. Both are CR disks with output type $\beta_{i}^{\prime}+\beta_{i}$. They simply reach around the punctures neighboring the input $\mathrm{id}_{i}^{\prime}$. They are depicted in Figure D.17b
Digons for the co-identity $h=\alpha_{0}$ : We spot $M$ digons contributing to $\mu^{1}\left((-1)^{\# \alpha_{0}+1} \alpha_{0}\right)$, namely ( $M-$ 1) $/ 2$ on the front side and $(M+1) / 2$ on the rear side. These digons are all CR disks and have output of type $\mathrm{id}_{i}^{\prime}$. In case of $M=5$, these digons are all heart-shaped and depicted in Figure D.17c
In general, one spots these digons as follows: Of the $M$-many self-intersection point $p \in L \cap L$, fix an arbitrary one. We shall construct from this data one certain digon that has corners $h$ and $p$. For this, note that $p$ cuts the zigzag curve $\tilde{L}$ into two segments. Only one of these two segments contains the co-identity location $\alpha_{0}$. The digon associated with $p$ is then the digon bounded by this segment. In other words, if the segment containing $\alpha_{0}$ departs to the front side at $h$, then the digon lies on the front side. If the segment containing $\alpha_{0}$ departs to the rear side at $h$, then the digon lies
$\alpha_{0} / \alpha_{i}$

 rear $\uparrow$, front

Figure D.16: $\mathrm{id}_{i}^{\prime}$ disk result components of $\pi$-trees and how to draw their subdisks

(a) Digon between $\beta_{i}^{\prime}+\beta_{i}$ and $\operatorname{id}_{L}$

(b) Digons between $\mathrm{id}_{i}^{\prime}$ and $\beta_{i}^{\prime}+\beta_{i}$

(c) Digon between $\alpha_{0}$ and $\operatorname{id}_{i}^{\prime}$

Figure D.17: Illustration of digons in $Q_{5}$
on the rear side. This determines a digon contributing to $\mu^{1}(h)$ for every of the self-intersection points $p \in L \cap L$.

Before we reach the main theorem, we shall comment on the signs of result components. Recall that the Abouzaid sign of an SL disk is defined in Definition 13.24 and allows an arbitrary nonnegative number of inputs. The definition of the Abouzaid sign carries over without change to the case of $Q=Q_{M}$. In analogy to Lemma 13.25 the sign of a result component agrees with the Abouzaid sign of its subdisk:

Lemma D.17. Let $r$ be the result component of a $\pi$-tree. Then the sign of $r$, relative to the signs of the output value, equals the Abouzaid sign of its subdisk $\mathrm{D}(r)$. The $q$-parameter $\in \mathbb{C} \llbracket Q_{0} \rrbracket$ is equal to Punc $(D)$, the product of all punctures covered by $\mathrm{D}(r)$ counted with multiplicities.

In contrast to the consistent case, the category $\mathrm{H} \mathbb{L}_{q}$ has a residual differential. We can in fact describe the differential by means of CR and ID digons, the sign being equal to the Abouzaid sign. In contrast to the consistent case, the definition of the Abouzaid sign rule is here also used for digons. We are now ready to formulate our freshly built interpretation as a description of the minimal model.
Proposition D.18. Let $Q_{M}$ be the standard sphere dimer with an odd number $M \geq 3$ of punctures. Let $h_{1}, \ldots, h_{N}$ be a sequence of $N \geq 0$ non-identity basis morphisms with $h_{i}: L_{i} \rightarrow L_{i+1}$. Then their product is given by

$$
\mu_{\mathrm{H} \mathbb{L}_{q}}^{N}\left(h_{N}, \ldots, h_{1}\right)=\sum_{\substack{D \in \text { Disk }_{\text {CR }} \text { UDisk } \\ D \text { has inputs } h_{1}, \ldots, h_{N}}}(-1)^{\operatorname{Abou}(D)} \operatorname{Punc}(D) \mathrm{t}(D) .
$$

More explicitly,

- The curvature $\mu_{\mathrm{H} \mathbb{L}_{q}}^{0}$ vanishes.
- The differential $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}$ is given by the digons in the sphere digon list, with Abouzaid sign rule.
- We have $(-1)^{|h|} \mu_{\mathrm{H} \mathbb{L}_{q}}^{2}\left(\mathrm{id}_{L}, h\right)=\mu_{\mathbb{L}_{q}}^{2}\left(h, \mathrm{id}_{L}\right)=h$ and $\mu_{\mathrm{H} \mathbb{L}_{q}}^{3}\left(\ldots, \mathrm{id}_{L}, \ldots\right)=0$.
- Products $\mu_{\mathrm{H}}^{\geq 2} \mathbb{L}_{q}$ on all sequences of non-identity inputs are given by the CR, ID, DS and DW disks, with Abouzaid sign rule.

Proof. The minimal model $\mathrm{H}_{\mathbb{L}_{q}}$ is given by our deformed Kadeishvili theorem. To start with, recall that we have already computed the deformed cohomology basis elements in Lemma D. 8 The deformed Kadeishvili theorem then provides curvature, differential and products for $\mathrm{H} \mathbb{L}_{q}$. We shall put focus on checks for the differential, since the structure of the products is very similar to the structure observed in the geometrically consistent case.

We have noted in Lemma D.6 that the curvature $\mu_{\mathbb{L}_{q}}^{0}$ already vanishes. According to our Kadeishvili theorem the curvature $\mu_{\mathrm{H} \mathbb{L}_{q}}^{0}$ then vanishes as well. This already proves the first statement. We check the three remaining statements in order.

For the second statement, regard the differential $\mu_{\mathbb{L}_{q}}^{1}$. According to the deformed Kadeishvili theorem, the differential $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}$ is given by the composition of $\mu_{\mathbb{L}_{q}}^{1}$ and the projection to deformed cohomology $\pi_{H_{q}}$.

We have computed the differential $\mu_{\mathbb{L}_{q}}^{1}$ already in Lemma D.8 and observed that $\mu_{\mathbb{L}_{q}}^{1}\left(H_{q}\right) \subseteq H_{q}$. The projection is therefore without effect and we have $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}(h)=\mu_{\mathbb{L}_{q}}(h)$ for deformed cohomology basis morphisms $h$. For example, we have

$$
\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}\left((-1)^{\#(i+M)+1} \beta_{i}^{\prime}+(-1)^{\# i} \beta_{i}\right)=(-1)^{\# \#(i+1)+1} Q_{i+2} \operatorname{id}_{L}
$$

It remains to show that for every deformed cohomology basis element $h$, its differential $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}(h)$ is enumerated accurately by CR, ID, DS and DW disks. We have listed all CR, ID, DS and DW disks in the sphere digon list. For any deformed cohomology basis element $h$ of one of the three types $\beta_{i} / \beta_{i}^{\prime}, \mathrm{id}_{i}^{\prime}$ and $\alpha_{0}$, it remains to interpret the result terms $r$ of $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}(h)$ as enumeration over the digons presented in the sphere digons list. The crucial part is to prove the sign of every term $r$ equal to the Abouzaid sign $\operatorname{Abou}(D)$ of the corresponding digon $D$.

First, regard the odd morphism $h=\beta_{i}^{\prime}+\beta_{i}$, with odd $i$. The calculation of Lemma D. 8 gives one single output term, namely the identity $\mathrm{id}_{L}$. This is exactly the result enumerated by the single digon $D$ from the sphere digons list. It remains to compare the sign $(-1)^{\# \#(i+1)+1}$ with the Abouzaid sign of $D$. The sign of $D$ is computed as follows: In case the digon lies on the front side, the $\#$ signs to be summed up are $\# i, \ldots, \#(i-M+1)$. Their sum amounts to $\# \#(i-M+1)$, which has equal parity with $\# \#(i+1)+1$, since the total number of $\#$ signs in $Q_{M}$ is assumed to be odd. In case the digon lies on the rear side, the $\#$ signs to be summed up are $\#(i+M), \ldots, \#(i+1)$. Their sum amounts to $\# \#(i+1)$. An additional sign flip is due, since $h$ is odd and lies counterclockwise with respect to the rear side. Ultimately, both front and rear cases give the sign $\# \#(i+1)+1$. This sign agrees with our calculation of $\mu_{\mathbb{L}_{q}}^{1}(h)$ in Lemma D. 8 .

Second, regard the even morphism id ${ }_{i}^{\prime}$. There are two digons in our digon list contributing to the differential $\mu^{1}\left(\mathrm{id}_{i}^{\prime}\right)$, namely the two small digons reaching around the neighboring punctures. The Abouzaid sign rule predicts a sign of $(-1)^{\# i+\#(i+M)}$ for the upper puncture and $(-1)^{\#(i-1)+\#(i+M-1)}$ for the lower puncture, exactly as calculated in Lemma D. 8 .

Third, regard the co-identity $(-1)^{\# \alpha_{0}+1} \alpha_{0}$. There are $M$ digons in our digon list contributing to the differential $\mu^{1}\left((-1)^{\# \alpha_{0}+1} \alpha_{0}\right)$, namely $(M-1) / 2$ on the front side and $(M+1) / 2$ on the rear side. For the front side disk with output $\mathrm{id}_{i-j+1}^{\prime}$, the Abouzaid sign rule predicts a sign of $(-1)^{\# \#(i-j+1)}$. For the rear side disk with output $\mathrm{id}_{i+j+1}^{\prime}$, the Abouzaid sign rule predicts a sign of $(-1)^{\# \#(i+j-M+1)+1}$, the absolute sign flip $(-1)$ coming from the odd co-identity whose zigzag path runs counterclockwise with respect to the rear side.

We conclude that the differential $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}(h)$ is computed accurately by the digons from the sphere digons list for any of the three types of morphisms $h$. This finishes the checks for the second statement of the proposition.

The third statement of the proposition is trivial, following immediately from unitality of $\mathbb{L}_{q}$ and the choice that $\operatorname{id}_{L} \in H$. The fourth statement follows from Lemma D.16 and Lemma D.17, in a way entirely analogous to the geometrically consistent case. This finishes the proof.

## D. 8 The case of even $M$

In this section, we comment on the category of zigzag paths of $Q_{M}$ for even $M$. We define the category $\mathbb{L}$ of zigzag paths and explain how to obtain a homological splitting. We explain how to run the curvature optimization for the corresponding subcategory of $\mathrm{Tw}_{\mathrm{Gtl}}^{q}$ $Q$ and define the category of deformed zigzag paths $\mathbb{L}$. Finally, we provide a minimal model for $\mathrm{H}_{q}$.

The dimer $Q_{M}$ for even $M \geq 4$ has two zigzag paths, each consisting of $M$ arcs. There are $M$ intersections between the zigzag curves. The front side of the dimer is a clockwise polygon, the rear side is a counterclockwise polygon.

Convention D.19. The letter $Q=Q_{M}$ with $M \geq 4$ even denotes the standard sphere dimer with $M$ punctures. The spin structure is chosen by assigning an even number of \# signs on the rear side of $Q_{M}$ and none on the front side. The identity locations are arbitrary chosen, and the co-identity location is chosen to lie on the rear side of $Q_{M}$.

Definition D.20. The category $\mathbb{L} \subseteq \mathrm{Tw} \operatorname{Gtl} Q_{M}$ is the category consisting of the two zigzag paths in $Q_{M}$. The standard splitting $H \oplus I \oplus R$ for $\mathbb{L}$ is defined in the analogous way as for odd $M$. The category $\mathbb{L}_{q} \subseteq \mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} Q_{M}$ is the category consisting of the two zigzag paths with deformed twisted differential analogous to Definition D. 5 .

A priori, it is our task to compute a minimal model of the category of zigzag paths in $\mathrm{Tw} \mathrm{Gtl}_{q} Q$ consisting of the same twisted complexes as $\mathbb{L} \subseteq \operatorname{Tw} \operatorname{Gtl} Q$. As usual, we are allowed to apply gauge in order to optimize the curvature. In contrast to the case of geometrically consistent $Q$ or $Q_{M}$ for odd $M$, the category $\mathbb{L}_{q}$ is not curvature-free, but its curvature is optimal nevertheless:

Lemma D.21. The curvatures of both zigzag paths $L_{1}, L_{2} \in \mathbb{L}_{q}$ are multiple of their respective identities $\operatorname{id}_{L_{1}} \in H$ and $\operatorname{id}_{L_{2}} \in H$. In particular, $\mathbb{L}_{q}$ has optimal curvature.

The deformed decomposition $H_{q} \oplus \mu_{q}^{1}(B \widehat{\otimes} R) \oplus(B \widehat{\otimes} R)$ of $\mathbb{L}_{q}$ is similar to the case of odd $M$. The differential does not vanish and maps to $H_{q} \oplus(B \widehat{\otimes} R)$. According to the deformed Kadeishvili construction, we can compute $\mathrm{H} \mathbb{L}_{q}$ by setting

$$
\begin{aligned}
\mu_{\mathrm{H} \mathbb{L}_{q}}^{0} & =\mu_{\mathbb{L}_{q}}^{0}, \\
\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}(h) & =\pi_{H_{q}} \mu_{\mathbb{L}_{q}}^{1}(h), \\
\mu_{\mathrm{H} \mathbb{L}_{q}}^{N \geq 2}\left(h_{N}, \ldots, h_{1}\right) & =\sum_{T \in \mathcal{T}_{N}}(-1)^{N_{T}} \operatorname{Res}\left(T, h_{1}, \ldots, h_{N}\right) .
\end{aligned}
$$

The computation for $\mu_{\mathrm{H} \mathbb{L}_{q}}^{N \geq 2}$ is similar to the case of odd $M$. The computation for $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}$ is similar to the case of odd $M$ as well, with the difference that $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}$ does not cancel because of the different choices of \# signs. The computation for $\mu_{\mathrm{H} \mathbb{L}_{q}}^{0}$ is elementary. As in the case of odd $M$, it turns out that the entire $A_{\infty}$-structure of the minimal model can be described through CR, ID, DS and DW disks:

Proposition D.22. Let $Q_{M}$ be the standard sphere dimer with an even number $M \geq 4$ of punctures. Let $h_{1}, \ldots, h_{N}$ be a sequence of $N \geq 0$ non-identity basis morphisms with $h_{i}: L_{i} \rightarrow L_{i+1}$. Then their product is given by

$$
\mu_{\mathrm{H} \mathbb{L}_{q}}^{N}\left(h_{N}, \ldots, h_{1}\right)=\sum_{\substack{D \in \text { Disk }_{\mathrm{CR}} \dot{\operatorname{USisk}} \\ D \text { has inputs }_{\mathrm{ID}} \operatorname{hisk}_{1}, \ldots, h_{N}}}(-1)^{\operatorname{Abou}(D)} \operatorname{Punc}(D) \mathrm{t}(D)
$$

## E Calculating the mirror objects

The aim of this section is to perform further minimal model calculations which we need for the third paper in the series. In section E.1 we explain which products in the minimal model need to be computed and why. In section E.2, we describe the input data of the minimal model construction. In section E.3 we construct a homological splitting. Insection E.4 we compute the deformed decomposition. In section E.5. we introduce a suitable notion of result components and classify them into two types which we call MD and MT result components. Insection E.6, we show how to match MD/MT result components with disks of two types which we call MD/MT disks. In Proposition E. 19 we finally describe the desired products in terms of MD and MT disks.

## E. 1 Mirror symmetry for punctured surfaces

In this section, we explain the reason we need to perform further minimal model calculations. The starting point is a brief recapitulatation of mirror symmetry for punctured surfaces. We then explain the idea of the deformed Cho-Hong-Lau construction and describe which products we need to compute.

Mirror symmetry for punctured surfaces 18 entails a quasi-isomorphism

$$
F: \operatorname{Gtl} Q \rightarrow \operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)
$$

Here $Q$ is a dimer and $\check{Q}$ is its mirror dimer which is assumed to be zigzag consistent. The vertices of $\check{Q}$ are identified with the zigzag paths in $Q$. The algebra $\operatorname{Jac} \check{Q}$ is the so-called Jacobi algebra of the dimer and is explicitly defined as the quiver algebra $\mathbb{C} \check{Q}$ modulo relations. The element $\ell \in \operatorname{Jac} \check{Q}$ is a central element known as the potential. The category $\operatorname{MF}(A, \ell)$ denotes the dg category of so-called matrix factorizations of $(A, \ell)$. The category $\operatorname{mf}(\operatorname{Jac} \mathscr{Q}, \ell)$ denotes one a certain small subcategory of $\operatorname{MF}(\operatorname{Jac} \check{Q}, \ell)$, specific to mirror symmetry.

The deformed mirror symmetry which we prove in the third paper entails a quasi-isomorphism of deformed $A_{\infty}$-categories

$$
F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)
$$

The category $\operatorname{Gtl}_{q} Q$ on the left-hand side has an object for every $\operatorname{arc} a \in Q_{1}$. The category on the righthand side is what we will call a deformed category of matrix factorizations. It has one object for every arc $a \in Q_{1}$ as well. The functor $F_{q}$ matches the arc $a \in \operatorname{Gtl}_{q} Q$ with an deformed matrix factorization $F_{q}(a)$.

In the third paper, we compute the deformed algebra $\mathrm{Jac}_{q} \check{Q}$ and deformed potential $\ell_{q}$. The starting point is the category of deformed zigzag paths $\mathbb{L}_{q} \subseteq \mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} Q$. Thanks to the description of the minimal model $\mathrm{H} \mathbb{L}_{q}$ which we provided in the present paper in Section 13.5 , we express in the third paper the deformed algebra $\mathrm{Jac}_{q} \check{Q}$ and the deformed potential $\ell_{q}$ explicitly in terms of combinatorical data of $Q$. Viewed the other way around, the present paper is the technical cornerstone for the third paper.

In the third paper, we also compute the precise shape of the deformed matrix factorizations contained in $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$. According to the deformed Cho-Hong-Lau construction, the precise shape is given by certain products in $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ which not only involve zigzag paths, but also the arc objects. The description of these products is not included in section 13 Therefore, the we have devoted the present section E to supplementing these products.

The objects of $\operatorname{mf}\left(\mathrm{Jac}_{q}, \ell_{q}\right)$ are explicitly of the form

$$
\begin{aligned}
F_{q}(a) & =\left(\bigoplus_{L \in \mathbb{L}} \operatorname{Hom}_{\mathrm{HTw} \mathrm{GtI} Q}(L, a) \otimes\left(\operatorname{Jac}_{q} \check{Q}\right) v_{L}, \delta\right) \\
\delta(m) & =\sum_{N \geq 0} \mu_{\mathrm{HTw} \mathrm{Gtl}}^{q} \boldsymbol{Q}
\end{aligned}(m, \underbrace{b, \ldots, b}_{N}) .
$$

Here $a$ denotes any arc in $Q$, the letter $v_{L}$ denotes the vertex of $\mathbb{C} Q \check{Q}$ defined by the zigzag path $L$, and $m$ denotes an element of $\operatorname{Hom}_{\mathrm{HTw} \operatorname{Gt1} Q}(L, a)$. The element $b$ denotes essentially a formal sum over all type B cohomology basis elements between zigzag paths in $Q$. Geometrically, the element $b$ includes all odd transversal intersections between zigzag curves.

The essential calculation which we shall therefore perform in the present section E consists of determining the hom space $\operatorname{Hom}_{\mathrm{HTw} \operatorname{Gtl} Q}(L, a)$ and computing all possible kinds of products of the form $\mu_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}\left(m, h_{N}, \ldots, h_{1}\right)$. Here $h_{i}: L_{i} \rightarrow L_{i+1}$ are type B cohomology basis elements between zigzag paths and $m: L_{N+1} \rightarrow a$ is a cohomology basis element from $L_{N+1}$ to an arc $a \in Q_{1}$.

## E. 2 The desired products

In this section, we examine which minimal model $\mathrm{HTw}_{\mathrm{Gtl}}^{q}$ $Q$ we shall compute. In principle, we are free to choose any minimal model. When computing the mirror Jacobi algebra $\mathrm{Jac}_{q} Q$ and potential $\ell_{q}$, we have however already made a choice for minimal model of $\mathbb{L}_{q} \subseteq \operatorname{Tw~}_{\mathrm{wtl}}^{q}$ $Q$. The minimal model we compute here needs to be compatible with these earlier choices.

Convention E.1. $Q$ is a geometrically consistent dimer or a standard sphere dimer $Q=Q_{M}$ with $M \geq 3$. The dimer is equipped with choices of spin structure, identity location $a_{0}$ and co-identity location $\alpha_{0}$ for every zigzag path. In case $Q=Q_{M}$ with $M$ odd, the spin structure is chosen by assigning $\# \alpha=1$ to an odd number of interior angles $\alpha$ on the rear side and $\# \alpha=0$ to all other angles. In case $Q=Q_{M}$ with
$M$ even, the spin structure is chosen by assigning $\# \alpha=1$ to an even number of interior angles $\alpha$ on the rear side and $\# \alpha=0$ to all other angles. The co-identity $\alpha_{0}$ shall be chosen to lie in a counterclockwise polygon.
Definition E.2. We denote by $\mathbb{L} \subseteq \mathrm{Tw} \operatorname{Gtl} Q$ and $\mathbb{L}_{q} \subseteq \mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} Q$ the categories of zigzag paths and of deformed zigzag paths, defined as follows:

- If $Q$ is geometrically consistent, $\mathbb{L}$ is the category of zigzag paths as defined in section 10.1 and $\mathbb{L}_{q}$ is the category of deformed zigzag paths as defined in section 11.1
- If $Q=Q_{M}$ for odd $M \geq 3$, then $\mathbb{L}$ is the category of zigzag paths as defined in section D. 2 and $\mathbb{L}_{q}$ is the category of deformed zigzag paths as defined in section D. 4
- If $Q=Q_{M}$ for even $M \geq 3$, then $\mathbb{L}$ is the category of zigzag paths as defined in section D. 8 and $\mathbb{L}_{q}$ is the category of deformed zigzag paths as defined in section D. 8

To construct a deformed mirror functor $F_{q}: \operatorname{HTw}_{\operatorname{Gtl}}^{q}$ $Q \rightarrow \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$, we need a choice of minimal model of the entire category $\mathrm{Tw} \mathrm{Gtl}_{q} Q$. According to our deformed Kadeishvili theorem, we can obtain such a minimal model by optimizing curvature for all objects and performing a familiar Kadeishvili tree construction. The input data for this construction consists only of a homological splitting for every hom space in $\mathrm{Tw} \mathrm{Gtl} Q$. All other steps are automatic.

Remark E.3. Our deformed Kadeishvili construction has the property that the restriction of a minimal model $\mathrm{H} \mathcal{C}_{q}$ to a subcategory $\mathcal{D} \subseteq \mathcal{C}$ is the same as the minimal model $\mathrm{H} \mathcal{D}$, when the homological splitting chosen for $\mathcal{D}$ is the restriction of the homological splitting chosen for $\mathcal{C}$ :

$$
\left.\mathrm{HC}\right|_{\mathrm{Ob} \mathcal{D}}=\mathrm{H} \mathcal{D} .
$$

To construct the restriction of this functor to $\operatorname{Gtl}_{q} Q$, we however do not need to compute the entire minimal model $\mathrm{HTw} \operatorname{Gtl}_{q} Q$ explicitly. It suffices to know products of the kind $\mu\left(h_{N}, \ldots, h_{1}\right)$ and $\mu\left(m, h_{N}, \ldots, h_{1}\right)$, where $h_{1}, \ldots, h_{N}$ are morphisms between zigzag paths and $m$ is a morphism from a zigzag path to an arc. By Remark E.3 it suffices to compute a minimal model of the category $Q_{1} \cup \mathbb{L}_{q}$, which is defined as the subcategory of $\mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} Q$ consisting of arcs and deformed zigzag paths. We define this category precisely as follows:

Definition E.4. The subcategory of $\operatorname{Tw} \operatorname{Gtl} Q$ given by the union of $\operatorname{Gtl} Q$ and $\mathbb{L}$ is denoted

$$
Q_{1} \cup \mathbb{L} \subseteq \operatorname{Tw} \operatorname{Gtl} Q
$$

The subcategory of $\operatorname{Tw~} \operatorname{Gtl}_{q} Q$ given by the union of $\operatorname{Gtl}_{q} Q$ and $\mathbb{L}_{q}$ is denoted

$$
Q_{1} \cup \mathbb{L}_{q} \subseteq \operatorname{Tw~}_{\operatorname{Gtl}_{q}} Q
$$

Remark E.5. Applying the Kadeishvili construction to $Q_{1} \cup \mathbb{L}_{q}$ involves choosing a homological splitting for $Q_{1} \cup \mathbb{L}$. The deformed mirror symmetry construction in the third paper departs from a single minimal model model $\mathrm{HTw} \mathrm{Gtl}_{q} Q$. In consequence, it is not allowed to compute the products of the two kinds $\mu_{\mathrm{Tw} \mathrm{Gtl}_{q} Q}\left(h_{N}, \ldots, h_{1}\right)$ and $\mu_{\mathrm{Tw}_{\mathrm{Gtl}}^{q}} Q\left(m, h_{N}, \ldots, h_{1}\right)$ via different homological splittings of $\mathbb{L}$. Instead, the homological splitting for $Q_{1} \cup \mathbb{L}$ needs to extend the homological splitting already chosen for $\mathbb{L}$.

## E. 3 Homological splitting

In this section, we construct a homological splitting for the category $Q_{1} \cup \mathbb{L}$. The starting point is the definition of the category $Q_{1} \cup \mathbb{L}$ in section E.2. According to Remark E.5, we have to define the homlogical splitting for $Q_{1} \cup \mathbb{L}$ in the following way:

- For the hom spaces $\operatorname{Hom}_{\mathrm{Tw} \operatorname{Gtl} Q}\left(L_{1}, L_{2}\right)$ between two zigzag paths $L_{1}, L_{2}$, the homological splitting is the homological splitting already established for $\mathbb{L}$. In case $Q$ is geometrically consistent, this refers to the homological splitting of section 10.3. In case $Q=Q_{M}$ for odd $M \geq 3$, this refers to the homological splitting of section D. 3 In case $Q=Q_{M}$ for even $M \geq 4$, this refers to the analog of section D. 3 indicated in section D. 7
- For the hom spaces $\operatorname{Hom}_{\mathrm{Tw} \operatorname{Gtl} Q}(L, a)$ between a zigzag path $L$ and $\operatorname{an} \operatorname{arc} a \in Q_{1}$, we are free to choose a homological splitting.
- For the hom spaces $\operatorname{Hom}_{\mathrm{Tw} \operatorname{GtI} Q}(a, L)$ between an arc $a \in Q_{1}$ and a zigzag path $L$, we are free to choose a homological splitting. In practice, this choice is irrelevant for the calculation of the products $\mu_{\mathrm{Tw} \operatorname{GtI} Q}\left(m, h_{N}, \ldots, h_{1}\right)$, so we will merely assume any arbitrary splitting has been chosen.

- For the hom spaces $\operatorname{Hom}_{\operatorname{Tw} \operatorname{Gtl} Q}(a, b)$ between two arcs $a, b \in Q_{1}$, the homological splitting is predetermined as $H=\operatorname{Hom}_{\mathrm{Tw} \operatorname{Gt1} Q}(a, b)$ and $I=R=0$ by the fact that $\mu_{\mathrm{Gt1} Q}^{1}=0$.
According to this list, the only remaining task is to choose a homological splitting for $\operatorname{Hom}_{\mathrm{Tw} \operatorname{GtI} Q}(L, a)$ whenever $L$ is a zigzag path and $a$ an arc. We start by classifying morphisms $L \rightarrow a$ into three types of situations.

As in the case of $\mathbb{L}$, let an elementary morphism $\varepsilon: L \rightarrow a$ refer to a morphism between twisted complexes consisting of a single angle of $Q$. We shall associate with every elementary morphism a situation. As usual, the terminology is as follows: Every elementary morphism belongs to a unique situation, and every situation is of one given type. Running out of letters, we will denote the situation types by S1, S2, S3. Every situation is defined by a certain collection of arcs and angles. The other way around, every situation has a collection of elementary morphisms associated, constructed from its defining angles.

Definition E.6. An $\mathbf{S} 1$ situation consists of a zigzag path $L$ and an indexed arc $a$ of $L$ such that $L$ turns left at the head of $a$. The nearby angles of the situation are denoted $\alpha, \alpha^{\prime}, \beta, \beta^{\prime}$ as in Figure E.1a. The elementary morphisms belonging to an S 1 situation are the morphisms $\varepsilon: L \rightarrow a$ given by $\mathrm{id}_{L \rightarrow a}, \beta \ell^{k}$, $\beta \beta^{\prime} \ell^{k}, \alpha \ell^{k}, \alpha \alpha^{\prime} \ell^{k}$.

An S2 situation consists of a zigzag path $L$ and an indexed arc $a$ such that $L$ turns right at the head of $a$. The nearby angles of the situation are denoted $\alpha, \alpha^{\prime}, \beta, \beta^{\prime}$ as in Figure E.1b. The elementary morphisms belonging to an S 2 situation are the morphisms $\varepsilon: L \rightarrow a$ given by $\mathrm{id}_{L \rightarrow a}, \alpha^{\prime} \ell^{k}, \alpha^{\prime} \alpha \ell^{k}, \beta^{\prime} \ell^{k}$, $\beta^{\prime} \beta \ell^{k}$.

An S3 situation consists of two consecutive indexed arcs on a zigzag path $L$ and an arc $a$ such that $a$ is incident at the common puncture of the two arcs but not equal to either of them. The nearby angles of the situation are denoted $\alpha, \beta, \gamma$ as in Figure E.1c. The elementary morphisms belonging to an S3 situation are the morphisms $\varepsilon: L \rightarrow a$ given by $\beta \ell^{k}, \beta \alpha \ell^{k}$.

We have constructed these definitions so that the situations exhaust all elementary morphisms in $\operatorname{Hom}_{\mathrm{Tw} \operatorname{GtI} Q}(L, a)$. We record this as follows:

Lemma E.7. Let $L$ be a zigzag path and $a$ an arc. Then any elementary angle $\varepsilon: L \rightarrow a$ belongs to precisely one S1, S2 or S3 situation.

We are now ready to construct our homological splitting for $\operatorname{Hom}_{\operatorname{Tw} \operatorname{Gt1} Q}(L, a)$. This means to provide a choice of basis elements for $H$ and $R$.

Definition E.8. Let $L$ be a zigzag path and $a$ an arc. We let $R \subseteq \operatorname{Hom}_{\mathrm{Tw} \operatorname{Gtl} Q}(L, a)$ be the subspace spanned by:

- for every S1 situation the morphisms $\operatorname{id}_{L \rightarrow a}, \alpha \alpha^{\prime} \ell^{k}$ and $\beta \beta^{\prime} \ell^{k}$,
- for every S2 situation the morphisms $\alpha^{\prime} \ell^{k}$ and $\beta^{\prime} \ell^{k}$,
- for every S 3 situation the morphism $\beta \ell^{k}$.

The space $H \subseteq \operatorname{Hom}_{\mathrm{Tw} \operatorname{Gt1} Q}(L, a)$ is spanned by:

- for every S1 situation the morphism $(-1)^{\# \beta} \beta$,
- for every S2 situation the morphism $\mathrm{id}_{L \rightarrow a}$.

Setting $I=\mu_{\mathrm{Tw} \mathrm{Gtl} Q}^{1}(R)$, we refer to $H, I, R$ as the (standard) splitting for $\operatorname{Hom}_{\mathrm{Tw} \mathrm{Gtl} Q}(L, a)$.
It is an elementary check that the standard splitting is indeed a homological splitting.
Lemma E.9. Let $a$ be an arc and $L$ a zigzag path. Then the standard splitting indeed forms a homological splitting for $\operatorname{Hom}_{\mathrm{Tw} \mathrm{Gtl} Q}(L, a)$.

## E. 4 Deformed decomposition

In this section, we determine the relevant part of the deformed decomposition of $Q_{1} \cup \mathbb{L}_{q}$. The starting point is the category $Q_{1} \cup \mathbb{L}_{q}$ defined in section E.2 and the homological splitting for $Q_{1} \cup \mathbb{L}$ defined in section E. 3 In the present section, we show that $Q_{1} \cup \mathbb{L}_{q}$ has optimal curvature. We determine explicitly the deformed decomposition of the hom spaces $\operatorname{Hom}_{Q_{1} \cup \mathbb{L}_{q}}(L, a)$, where $L \in \mathbb{L}$ is a zigzag path and $a \in Q_{1}$ an arc.

Lemma E.10. The category $Q_{1} \cup \mathbb{L}_{q}$ has optimal curvature.
Proof. If $Q$ is geometrically consistent, then $\mathbb{L}_{q}$ is curvature-free by Lemma 11.5. If $Q=Q_{M}$ with $M$ odd, then $\mathbb{L}_{q}$ is curvature-free by Lemma D.6. If $Q=Q_{M}$ with $M$ even, then $\mathbb{L}_{q}$ has optimal curvature by Lemma D.21. The subcategory $Q_{1} \subseteq Q_{1} \cup \mathbb{L}_{q}$ has optimal curvature by nature, so we conclude that $Q_{1} \cup \mathbb{L}_{q}$ has optimal curvature.

Since $Q_{1} \cup \mathbb{L}_{q}$ already has optimal curvature, the products $\mu_{\mathrm{HTw}_{\mathrm{Gtl}}^{q}}\left(m, h_{N}, \ldots, h_{1}\right)$ can be obtained by computing the deformed decomposition of $\operatorname{Hom}_{Q_{1} \cup \mathbb{L}_{q}}(L, a)$ and evaluating Kadeishvili $\pi$-trees. As next step, we shall therefore focus on finding the deformed decomposition for $\operatorname{Hom}_{Q_{1} \cup \mathbb{L}_{q}}$. More precisely, we are interested in the deformed cohomology basis elements.

Finding the deformed decomposition entails finding for every cohomology basis element $h \in H$ a deformed cohomology basis element $\varphi^{-1}(h)=h+r$ such that $r \in B \widehat{\otimes} R$ and

$$
\mu_{\mathrm{Tw} \mathrm{Gtl}_{q}}^{1}(h+r) \in(B \widehat{\otimes} H) \oplus(B \widehat{\otimes} R)
$$

For the cohomology basis elements $h$ of the hom space $\operatorname{Hom}_{\mathrm{Tw} \mathrm{GtI} Q}(L, a)$, we can compute $\varphi^{-1}(h)$ explicitly:

Lemma E.11. Let $a \in Q_{1}$ be an arc and $L$ a zigzag path. Then the space $H$ satisfies

$$
\mu_{\mathrm{Tw} \mathrm{Gtl}_{q} Q}^{1}(H) \subseteq(B \widehat{\otimes} H) \oplus(B \widehat{\otimes} R)
$$

In particular, we have $H_{q}=B \widehat{\otimes} H$ and the map $\varphi: H_{q} \rightarrow B \widehat{\otimes} H$ is the identity.
Proof. Let us start by checking for every cohomology basis element $h$ that $\mu_{\mathrm{Tw} \mathrm{Gtl}_{q} Q}^{1}(h)$ lies in $(B \widehat{\otimes} H) \oplus$ $(B \widehat{\otimes} R)$. Let $L$ be a zigzag path and $a$ an arc. Denote by $q=h(a)$ the puncture at the head of $a$ and by $p=t(a)$ the puncture at the tail of $a$. Denote by $\delta \in \operatorname{Hom}_{\operatorname{Add}^{1} \operatorname{Gtt}_{q} Q}(L, L)$ the twisted differential of $L \in \mathbb{L}_{q}$.

Regard an S1 situation between $L$ and $a$. Denote by $\beta, \beta^{\prime}$ the angles associated with the S 1 situation. We want to compute the differential of the cohomology basis element $h=(-1)^{\# \beta} \beta$. We have

$$
\begin{aligned}
& \mu_{\mathrm{Tw} \mathrm{Gti}}^{q} Q \\
& 1 \\
&(\beta)=\sum_{k \geq 0} \mu_{\operatorname{Add~Gtl}_{q} Q}^{k+1}(\beta, \delta, \ldots, \delta) \\
&=p \beta \beta^{\prime} \beta \quad[ \pm \operatorname{id}(\mathrm{S} 2) \pm \mathrm{id}(\mathrm{~S} 1) \pm \beta(\mathrm{S} 3)] \\
& \in(B \widehat{\otimes} H) \oplus(B \widehat{\otimes} R)
\end{aligned}
$$

In the first row, we have simply spelt out the definition of $\mu_{\mathrm{Tw} \mathrm{Gtl}_{q} Q}^{1}$. In the second row, we have evaluated all products. The first term $p \beta \beta^{\prime} \beta$ arises from $k=1$. Further terms may arise from $k \geq 2$, depending on the situation. If $Q$ is geometrically consistent, then $\beta$ (S3) terms may appear, stemming from first-out discrete immersed disks. If $Q$ is not geometrically consistent, also id (S1) and id (S2) terms can appear. Either way, we see $\mu_{\mathrm{Tw} \mathrm{Gtl}_{q} Q}^{1}(\beta)$ lies in $(B \widehat{\otimes} H) \oplus(B \widehat{\otimes} R)$.

Regard an S2 situation between $L$ and $a$. Denote by $\alpha^{\prime}$ and $\beta^{\prime}$ the associated angles. Then

$$
\mu_{\mathrm{Tw} \mathrm{Gtl}}^{q} \text { } Q\left(\operatorname{id}_{L \rightarrow a}\right)=-q \alpha^{\prime}-p \beta^{\prime} \in B \widehat{\otimes} R
$$

This proves the claimed inclusion $\mu_{\mathrm{Tw} \mathrm{Gtl}_{q} Q}^{1}(H) \subseteq B \widehat{\otimes} R$. In the terminology of Lemma 8.15 this means $E=0$. According to Lemma 8.16 we conclude

$$
H_{q}=\{h-E h \mid h \in B \widehat{\otimes} H\}=B \widehat{\otimes} H
$$

According to Definition 8.17 the $\operatorname{map} \varphi: H_{q} \rightarrow B \widehat{\otimes} H$ is the identity. This finishes the proof.

The deformed decomposition for the hom space $\operatorname{Hom}_{Q_{1} \cup \mathbb{L}_{q}}\left(L_{1}, L_{2}\right)$ between two zigzag paths $L_{1}, L_{2}$ is simply the deformed decomposition described earlier. In case $Q$ is geometrically consistent, this deformed decomposition was computed in Proposition 11.12 In case $Q=Q_{M}$ with odd $M$, it was computed in Lemma D. 8 and in case $Q=Q_{M}$ with even $M$, it was indicated in section D. 8 .

The deformed decomposition for the hom space $\operatorname{Hom}_{Q_{1} \cup \mathbb{L}_{q}}(a, b)$ between two arcs $a, b \in Q_{1}$ is trivially

$$
\left(H_{q}, \mu_{Q_{1} \mathbb{L}_{q}}^{1}(B \widehat{\otimes} R), B \widehat{\otimes} R\right)=\left(B \widehat{\otimes} \operatorname{Hom}_{\mathrm{Gt1} Q}(a, b), 0,0\right)
$$

The deformed decomposition for the hom space $\operatorname{Hom}_{Q_{1} \cup \mathbb{L}_{q}}(a, L)$ between an arc $a \in Q_{1}$ and a zigzag path $L$ depends on the choice one makes for the homological splitting of $\operatorname{Hom}_{\mathrm{Tw}} \mathrm{GtI} Q(a, L)$, but is entirely irrelevant to the present computation.

## E. 5 Result components

In this section, we define and analyze result components for the products $\mu_{\mathrm{HTw} \mathrm{Gtl}}^{q}$ Q $\left(m, h_{N}, \ldots, h_{1}\right)$. The starting point is the description of the deformed decomposition fromsection E.4. In the present section, we introduce a notion of result components suitable for computing the products $\mu_{\mathrm{H} \mathrm{Tw} \mathrm{Gtl}_{q} Q}\left(m, h_{N}, \ldots, h_{1}\right)$. We show that all result components fall into two classes which we call MD and MT result components.

According to the deformed Kadeishvili theorem of section 8, the product of the morphisms $h_{1}, \ldots, h_{N}, m$ in the minimal model $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ is described in terms of Kadeishvili $\pi$-trees. Here the sequence $h_{1}, \ldots, h_{N}$ denotes type B cohomology basis elements $h_{i}: L_{i} \rightarrow L_{i+1}$ and $m$ denotes a cohomology basis element $m: L_{N+1} \rightarrow a$. It is our task to evaluate all Kadeishvili $\pi$-trees $T$ with inputs $h_{1}, \ldots, h_{N}, m$. For sake of convenience, we use the notation of section 10.2 and 10.3 to denote angles, as opposed to the notation from section D. 2 specific to $Q=Q_{M}$. For instance, we denote the type B cohomology basis elements by $\alpha_{3}+\alpha_{4}$. As usual, we start with a description of the possible terms that may possibly appear during evaluation of a Kadeishvili $\pi$-tree:

Lemma E.12. Let $\left(T, h_{1}, \ldots, h_{N}, m\right)$ be a Kadeishvili $\pi$-tree with a type B cohomology basis elements $h_{1}, \ldots, h_{N}$ with $h_{i}: L_{i} \rightarrow L_{i+1}$ and $m$ a cohomology basis element $m: L_{N+1} \rightarrow a$. Then:

- A proper subtree of $T$ whose input morphisms only cover the morphisms between zigzag paths may only have result component $\beta$ (A).
- A proper subtree of $T$ whose input morphisms also cover $m$ vanishes.

Any nonvanishing result component $r$ is derived either as a disk $\pi_{q} \mu^{\geq 3}(m, \ldots)$, or as a product $\pi_{q} \mu^{2}$ (id (S2), $\left.\alpha_{3} / \alpha_{4}(\mathrm{~B})\right)$ with direct inputs.

Proof. The statement on the subtrees that only involve morphisms between zigzag paths is familiar from the calculation of $\mathrm{H}_{q}$.

Regard now a product $\mu_{\mathrm{Tw} \mathrm{Gtl}_{q} Q}\left(m, m_{k}, \ldots, m_{1}\right)$ where all $m_{1}, \ldots, m_{k}$ are of type $\beta$ (A) or $\alpha_{3} / \alpha_{4}$ (B) and $m$ is $\beta$ (S1) or id (S2). We claim this product lies in $R$, apart from the case of $\mu_{\operatorname{Add~Gtl}_{q} Q}^{2}\left(\mathrm{id}(\mathrm{S} 2), \alpha_{4}\right)$ and all-in disks $\mu_{\operatorname{Add~Gtl}_{q} Q}^{\geq 3}\left(\beta(\mathrm{~S} 1), m_{k}, \ldots, m_{1}\right)$. In these two exceptional cases, the product lies in $H$.

The first part of checking this claim is to regard the case the product is a $\mu^{2}$. The product is then of the form $\mu^{2}(\beta(\mathrm{~S} 1), \beta(\mathrm{A}))$ or $\mu^{2}\left(\beta(\mathrm{~S} 1), \alpha_{3} / \alpha_{4}(\mathrm{~B})\right)$ or $\mu^{2}(\mathrm{id}(\mathrm{S} 2), \beta(\mathrm{A}))$ or $\mu^{2}\left(\mathrm{id}(\mathrm{S} 2), \alpha_{3} / \alpha_{4}(\mathrm{~B})\right)$. The first case yields $\beta(\mathrm{S} 3) \in R$, the second and third type of composition are impossible, the fourth case yields $\beta(\mathrm{S} 1) \in H$.

The second part of checking the claim is to regard the case of a disk $\mu^{\geq 3}$. If it concerns an all-in disk, then the result is an arc identity $\operatorname{id}_{L \rightarrow a} \in R$ or $\mathrm{id}_{L \rightarrow a} \in H$. A final-out disk is impossible, since $\beta$ (S1) is an indecomposable angle. If it concerns a first-out disk, then the first angle of the disk may be a $\delta$-morphism or $\beta$ (A). In both cases, the result is of the type $\mu^{\geq 3}=\beta(\mathrm{S} 3) \in R$.

Finally, we draw two conclusions: Any h-tree consuming $m$ has vanishing result. A given $\pi$-tree $T$ with nonvanishing result must therefore consume $m$ directly at the root. This finishes the proof.

In analogy with section 12 we can define result components also for Kadeishvili $\pi$-trees with inputs $h_{1}, \ldots, h_{N}, m$. Virtually the same definition can be applied.

Definition E.13. Let $\left(T, h_{1}, \ldots, h_{N}, m\right)$ be a Kadeishvili $\pi$-tree with a type B cohomology basis elements $h_{1}, \ldots, h_{N}$ with $h_{i}: L_{i} \rightarrow L_{i+1}$ and $m$ a cohomology basis element $m: L_{N+1} \rightarrow a$. Then a result component of $\left(T, h_{1}, \ldots, h_{N}, m\right)$ is defined in analogy with Definition 12.6 The set of result components of all $\pi$-trees, ranging over all choices of $h_{1}, \ldots, h_{N}$ and $m$ and $T$, is denoted Result ${ }_{\pi \mathrm{M}}$.

(a) MD result component

(b) MT result component

Figure E.2: How MD and MT result components are derived

By Lemma E.12, result components of $\mu\left(m, h_{N}, \ldots, h_{1}\right)$ can be split into two types: those which are derived from a disk and those which are derived from a product $\pi_{q} \mu^{2}$ with direct inputs. In analogy with section 13.4 we shall give these two types the names mirror disks and mirror triangles, respectively.

Definition E.14. A result component $r \in \operatorname{Result}_{\pi \mathrm{M}}$ is a

- MD result component if it is derived as $\pi_{q} \mu^{\geq 3}(m, \ldots)$.
- MT result component if it derived as $\pi_{q} \mu^{2}\left(\mathrm{id}(\mathrm{S} 2), \alpha_{3} / \alpha_{4}(\mathrm{~B})\right)$.

We denote the set of MD and MT result components by Result ${ }_{\text {MD }}$ and Result ${ }_{\mathrm{MD}}$, respectively.
The distinction between MD and MT result components is depicted in Figure E.2. Result components of these two types compute the products $\mu_{\mathrm{HTw}_{\mathrm{Ttl}}^{q}} Q\left(m, h_{N}, \ldots, h_{1}\right)$. The two names MD (mirror disk) and MT (mirror triangle) have been chosen in order to reflect their use in computing the deformed mirror in the third paper of the present series.

## E. 6 The higher products

In this section, we compute the desired products of the kind $\mu_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}\left(m, h_{1}, \ldots, h_{N}\right)$. The starting point for the computation is the classification of result components from section E. 5 In the present section, we introduce MD and MT disks with the aim of expressing the products as an enumeration of disks. We define a matching, the subdisk mapping, between MD/MT result components and MD/MT disks. In Proposition E.19, we collect the desired description of the products.

The first step of the present section is to define the notion of MD and MT disks, meant to capture MD and MT result components geometrically. The key observation is that

- The final input of an MD result component is an odd morphism of type $m=\beta(\mathrm{S} 1): L_{N+1} \rightarrow a$ and the output is the morphism id (S2) : $L_{1} \rightarrow a$.
- The final input of an MT result component is an even morphism of type $m=\mathrm{id}(\mathrm{S} 1): L_{N+1}: a$ and the output is the morphism $\beta(\mathrm{S} 1): L_{1} \rightarrow a$.
In either case, we see that the two zigzag curves $\tilde{L}_{1}, \tilde{L}_{N+1}$ and the arc $a$ have a triple intersection at the midpoint of the arc $a$. In order to capture MD/MT result components by means of disks, the disk therefore also need to have nontransversal input sequence. This means the correct definition of MD/MT disks cannot be inferred from parallels with the relative Fukaya pre-category relFuk ${ }^{\text {pre }} Q$. Instead, the parallel needs to be drawn with the relative Fukaya category relFuk $Q$. The products of this category can be determined on a best-effort basis by performing Hamiltonian deformations on the involved curves. In the present context of the products $\mu\left(m, h_{N}, \ldots, h_{1}\right)$, this means we have to choose a Hamiltonian deformation of some of the zigzag paths or arcs in order to guess the correct notion of MD/MT disks.

There is one particular Hamiltonian deformation of the arcs that gives the correct notion of MD/MT disks: Push every arc $a$ a little into the neighboring clockwise polygon, leaving the zigzag curves in place. This specific Hamiltonian deformation simultaneously resolves all triple intersections between zigzag curves in $Q$ and arcs. It predicts us to find disks of two types, depicted in Figure E. 3 . We will verify in Lemma E.17 that it is the correct Hamiltonian deformation to capture the products $\mu_{\mathrm{HTw} \mathrm{Gtl}}^{q}$ Q $\left(m, h_{N}, \ldots, h_{1}\right)$. If we had chosen a different homological splitting in Definition E. 8 we would have needed a different Hamiltonian deformation.

In Definition E. 15 we provide a rigorous definition of MD/MT disks. For a given arc $a$, we have denoted by $L$ and $L^{\prime}$ the two zigzag paths which depart from $a$. Geometrically speaking, $\tilde{L}$ and $\tilde{L}^{\prime}$ are the two zigzag curves which intersect at the midpoint of $a$. This is depicted in Lemma E.17.

Definition E.15. An MD disk (mirror disk) is a CR disk with $N \geq 0$ inputs $h_{1}, \ldots, h_{N}$ whose

(a) Disk

(b) Triangle

Figure E.3: The two types of disks we expect to contribute to $\mu\left(m, h_{N}, \ldots, h_{1}\right)$

- inputs $h_{1}, \ldots, h_{N}$ are all odd and do not contain co-identities,
- output is even and not an identity,
- zigzag segments all run clockwise,
which has undergone the following surgery: The output mark, located at a certain arc $a$, has been cut off. The odd morphism at $a$ is added as final input, and the even morphism at $a$ is indicated as new output.

An MT disk (mirror triangle) is a triangle sitting between the deformed arc $a$ and the intersection of the two zigzag curves $\tilde{L}$ and $\tilde{L}^{\prime}$ intersecting at $a$.

We denote by Disk ${ }_{\text {MD }}$ and Disk ${ }_{\text {MT }}$ the sets of MD and MT disks, respectively.
In the remainder of this section, we show that the product $\mu_{\mathrm{HTw}_{\mathrm{Gtl}}^{q}} Q\left(m, h_{N}, \ldots, h_{1}\right)$ is indeed given by counting MD and MT disks with inputs $h_{1}, \ldots, h_{N}, m$. The first step is to map a given result component to an MD or MT disk. The description of this mapping is eased by the classification of result components given in Definition E.14 According to this classification, result components can be categorized into what we have called MD and MT result components. An MD result component $r \in$ Result $_{\pi \mathrm{M}}$ is necessarily derived as $\pi_{q} \mu_{\bar{q}}^{\geq 3}\left(\beta(\mathrm{~S} 1), m_{l}, \ldots, m_{1}\right)$ where all $m_{1}, \ldots, m_{k}$ are of type $\alpha_{3} / \alpha_{4}$ (B) or $\beta$ (A). In particular, every $m_{i}$ is the result component of an h-tree with inputs being a subsequence of $h_{1}, \ldots, h_{N}$. By section 13.3 every $m_{i}$ comes with an associated subdisk. To define the subdisk of the MD result component $r$, we essentially glue together the subdisks of the $m_{i}$. The precise definition of subdisks for MD and MT result components reads as follows:
Definition E.16. Let $r$ be a result component of a $\pi$-tree $\left(T, h_{1}, \ldots, h_{N}, m\right)$. Then its subdisk $\mathrm{D}(r)$ is the drawing defined as follows:

- If $T$ is derived as $\pi_{q} \mu^{2}\left(\mathrm{id}(\mathrm{S} 2), \alpha_{3}+\alpha_{4}\right)$ with both inputs being direct, then its subdisk $\mathrm{D}(r)$ is the infinitesimal triangle sitting between the Hamiltonian deformed arc $a$ and the intersection of the two input zigzag curves.
- If $T$ is derived as $\pi_{q} \mu^{\geq 3}\left(\beta(\mathrm{~S} 1), m_{k}, \ldots, m_{1}\right)$, then its subdisk $\mathrm{D}(r)$ is given by connecting the subdisks of $m_{1}, \ldots, m_{k}$ and finishing with the input $\beta$ (S1). The output mark is at id (S2) and lies infinitesimally apart from the final input $\beta$ (S1).

In Lemma E.17. we claim that every MD or MT disk is the subdisk of a unique single result component, in analogy with Lemma 13.23 We have to restrict MD disks to those with at least two inputs because subdisks of result components always have at least two inputs.

Lemma E.17. The subdisk of an MD result component is an MD disk. The subdisk of an MT result component is an MT disk. Denoting by Disk ${ }_{\text {MD }}^{\geq 2}$ the set of MD disks with at least two inputs, the map D : Result $\pi_{\mathrm{M}} \rightarrow$ Disk $_{\mathrm{MD}}^{\geq 2} \dot{\cup}$ Disk $_{\mathrm{MT}}$ is a bijection.

Proof. We divide the proof into two parts: First we comment on MT disks, then we comment on MD disks.

For MT result components and MT disks, there is not much to say: An MT disk $D$ is a small triangle located on the counterclockwise side of an. It immediately gives rise to two morphisms $\alpha_{3}+\alpha_{4}$ and $m=\mathrm{id}$ (S2) which multiply to $\pi_{q} \mu^{2}=\beta(\mathrm{S} 1)$. This gives an MT result component whose subdisk is $D$ again. This shows that D matches MT result components bijectively with MT disks.

For MD disks, it is our task to recover a tree and result component for a given MD disk $D \in$ Disk $_{\text {MD }}^{\geq 2}$. After our long journey in section C, we content ourselves with merely a brief description: Find the indecomposable narrow locations of $D$ and place them in a tree, ordered by inclusion. The root of this tree is the artifical narrow location, which we denoted $(1,|D|)$ in section C. 4 . According to section C.5. the children $C_{1}, \ldots, C_{k}$ of the root all come with subtrees $T_{1}, \ldots, T_{k}$ and subresults $r_{1}, \ldots, r_{k}$. Each tree
$T_{i}$ is an h-tree which consumes part of the inputs $h_{1}, \ldots, h_{N}$, and $r_{i}$ is a result component of $T_{i}$. In fact, all $r_{i}$ are of $\beta$ (A) type.

With this in mind, we are ready to associate with $D$ a tree $T$ and a result component $r$ of $T$, such that $\mathrm{D}(r)=D$. To construct the tree $T$, put all trees $T_{1}, \ldots, T_{k}$ next to each other, insert a root $\pi_{q} \mu^{\geq 3}$ and connect all outputs of $T_{1}, \ldots, T_{k}$ together with all remaining $b$ inputs and the input $m$ with the root of $T$. This gives the desired tree $T$. The result component $r$ is the $\pi_{q} \mu^{\geq 3}$ result component of id (S2) type simply given by the data of result components $r_{1}, \ldots, r_{k}$ on each $T_{i}$, bound together with the all remaining morphisms and $m$ following the geometry of $D$. The result component $r$ defined this way satisfies $\mathrm{D}(r)=D$. This shows that D maps surjectively onto Disk $\mathrm{MD}^{\geq 2} \dot{\cup}$ Disk $_{\mathrm{MT}}$.

We proceed by checking signs.
Lemma E.18. Let $r \in$ Result $_{\pi \mathrm{M}}$ be a result component. Then the absolute sign of $r$ equals the Abouzaid sign $\operatorname{Abou}(\mathrm{D}(r))$.

Proof. First we check the case of MD result components and second the case of MT result components, both focusing on the geometrically consistent case. Third we comment on the case of the sphere dimers $Q_{M}$.

For the first part, regard an MD result component $r$. Let $T$ be the $\pi$-tree from which $r$ stems. Then $T$ has shape as depicted in Figure E.2a. Let $T_{1}, \ldots, T_{k}$ be the children of the root of $T$, not counting the direct input $m$. Then $r$ is derived as $\pi_{q} \mu^{\geq 3}\left(m, r_{k}, \ldots, r_{1}\right)$, with $r_{1}, \ldots, r_{k}$ being resukt components of $T_{1}, \ldots, T_{k}$. It is our task to evaluate the sign of $r$. Our procedure is analogous to section C. 8

Let us compute the sign of the result component $r$. Since it is derived as the product $\mu^{\geq 3}\left(m, r_{k}, \ldots, r_{1}\right)$ and the disk is all-in, its total sign is the sum of: the \# signs of the $\delta$ insertions, the signs of the result components $r_{1}, \ldots, r_{k}$ and the sign of $m=(-1)^{\# \beta} \beta$.

On the other hand, let us compute the Abouzaid sign of $\mathrm{D}(r)$. By Definition E.16 the disk $\mathrm{D}(r)$ is formed by tying together the subdisks of $r_{1}, \ldots, r_{k}$. Correspondingly, its Abouzaid sign is the sum of the signs coming from odd counterclockwise intersections within the subdisks of $r_{1}, \ldots, r_{k}$, plus \# signs of the zigzag curve segments lying between two neighboring subdisks. By Lemma C. 21 and Definition C.19, the sign coming from odd counterclockwise intersections in the subdisk of $r_{i}$ is equal to the absolute sign of $r_{i}$. In other words, the total Abouzaid sign of $\mathrm{D}(r)$ is the sum of the absolute signs of $r_{1}, \ldots, r_{k}$ and the \# signs. This \# sign already includes sign $\# \beta$ of final input angle $\beta$. Finally, we conclude that both signs are equal.

For the second part, regard an MT result component $r$. It is derived as $\pi_{q} \mu^{2}\left(\mathrm{id}(\mathrm{S} 2), \alpha_{3} / \alpha_{4}(\mathrm{~B})\right)$, where id (S2) comes from an S2 situation and $\alpha_{3} / \alpha_{4}$ comes from a B situation such that both angles are composable. Recall that $\alpha_{3} / \alpha_{4}$ is merely an abbreviation for the morphism $\varphi^{-1}\left((-1)^{\# \alpha_{3}+1} \alpha_{3}+\right.$ $\left.(-1)^{\# \alpha_{4}} \alpha_{4}\right)$. The relevant result of the product is $\mu^{2}\left(\mathrm{id}(\mathrm{S} 2),(-1)^{\# \alpha_{4}} \alpha_{4}\right)=(-1)^{\# \beta+1} \beta(\mathrm{~S} 1)$, noting that $\beta$ is the same angle in $Q$ as $\alpha_{4}$. Relative to the sign of the cohomology basis element $(-1)^{\# \beta}$, the MT result component $r$ has a total sign of -1 .

On the other hand, regard the subdisk $\mathrm{D}(r)$ associated with $r$. It is a small triangle with two inputs and one output. The first input $\alpha_{3} / \alpha_{4}$ is odd but its zigzag curves are oriented clockwise with the triangle, the second input is even, and the output is odd and its target zigzag curve is oriented counterclockwise with the triangle. The Abouzaid sign of $\mathrm{D}(r)$ is therefore -1 . We conclude that both signs agree.

For the third part, let us comment on the case of $Q_{M}$. The choice of signs in the cohomology basis elements, $(-1)^{\#(i+M)+1} \beta_{i}^{\prime}+(-1)^{\# i} \beta_{i}$ and $i d_{i}^{\prime}$ is analogous to the choice of sign in the geometrically consistent case. In particular, the sign computations for MD and MT result components carry over to the case of $Q_{M}$ without change.

In Proposition E.19 we describe the desired products $\mu_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}\left(m, h_{N}, \ldots, h_{1}\right)$. It turns out that for any value of $N \geq 0$, this product is determined by enumerating MD and MT disks with inputs $h_{1}, \ldots, h_{N}, m$. In case $N \geq 1$, this description follows from Lemma E. 17 and Lemma E. 18 In case $N=0$, the description follows from inspection of the differential $\mu_{\mathrm{Tw} \mathrm{Gti}_{q} Q}^{1}(m)$. The notation $Q_{i+2} \in$ $\mathbb{C} \llbracket q_{1}, \ldots, q_{M} \rrbracket$ and $\# \#(i-M+1)$ appearing in Equation E. 2 is taken over from Equation D.1. The index $i$ denotes the index such that $\alpha_{i}$ is the same angle in $Q=Q_{M}$ as $\beta$.
Proposition E.19. Let $Q$ be a geometrically consistent dimer or standard sphere dimer $Q=Q_{M}$ with $M \geq 3$, as in Convention E. 1 Let $h_{1}, \ldots, h_{N}$ be a sequence of $N \geq 0$ type B cohomology basis elements $h_{i}: L_{i} \rightarrow L_{i+1}$. Let $m=(-1)^{\# \beta} \beta: L_{N+1} \rightarrow a$ be another cohomology basis element. Then we have

$$
\begin{equation*}
\mu_{\mathrm{H} \mathrm{Tw} \operatorname{Gtl}_{q} Q}\left(m, h_{N}, \ldots, h_{1}\right)=\sum_{\substack{D \in \operatorname{Disk}_{\mathrm{MD}} \text { U.Disk }_{\mathrm{MT}} \\ \text { with inputs } h_{1}, \ldots, h_{N}, m}}(-1)^{\operatorname{Abou}(D)} \operatorname{Punc}(D) \mathrm{t}(D) \tag{E.1}
\end{equation*}
$$

The differential $\mu_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}^{1}(m)$ is accurately described by this equality. More explicitly, it is given as follows:

- If $Q$ is a geometrically consistent dimer, then $\mu_{\mathrm{H} \mathrm{Tw} \mathrm{Gtl}}^{q}$ Q ${ }^{1}(m)$ vanishes.
- If $Q=Q_{M}$ with $M \geq 3$ odd, then for $m=\operatorname{id}_{L \rightarrow a}$ we have $\mu_{\mathrm{HTw} \operatorname{Gtl}_{q} Q}^{1}(m)=0$. For $m=(-1)^{\# \beta} \beta$ (S1), let $i$ be such that $a_{i}=t(\beta)$. Then

$$
\begin{equation*}
\mu_{\mathrm{HTw} \mathrm{Gtl}}^{q} \text { Q }\left((-1)^{\# \beta} \beta\right)=(-1)^{\# \#(i-M+1)} Q_{i+2} \mathrm{id}_{L \rightarrow a} \tag{E.2}
\end{equation*}
$$

- If $Q=Q_{M}$ with $M \geq 4$ even, then $\mu_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}^{1}(m)$ vanishes.

Proof. The entire computation is completely analogous to section 13.5 In section 13 we have shown how to match result components for $\mu_{\mathrm{H} \mathbb{L}_{q}}$ with CR/ID/DS/DW disks. In the present Lemma E.17, we have shown how to match result components for $\mu\left(m, h_{N}, \ldots, h_{1}\right)$ with MT/MD disks in case $N \geq 1$. In Lemma E.18, we have checked that the sign of a result component agrees with the Abouzaid sign of its associated subdisk. We conclude that the claimed product description E.1 in case $N \geq 1$ follows as in section 13.5

It remains to comment on the case $N=0$. This entails determining the differential $\mu_{\mathrm{HTw} \mathrm{Ttl}_{q} Q}^{1}(m)$ explicitly, where $m$ is a cohomology basis element $m: L \rightarrow a$. We need to distinguish cases on whether $Q$ is geometrically consistent or $Q=Q_{M}$. If $Q$ is geometrically consistent, then $\mu_{\mathrm{Tw} \mathrm{Gtl}_{q} Q}^{1}(m)$ lies in $B \widehat{\otimes} R$, as we have seen in the proof of Lemma E.11 Therefore

If $Q=Q_{M}$ with $M \geq 3$ odd, then we can calculate the differential easily by looking at the id (S2) terms appearing in $\mu_{\mathrm{Tw} \mathrm{Gtl}}^{q}$ Q $(m)$. For $m=\operatorname{id}_{L \rightarrow a}$ there are no such terms, but for $m=(-1)^{\# \beta} \beta$ we find a single type id (S2) term, namely

$$
\begin{aligned}
& \mu_{\operatorname{Add~Gtl}_{q} Q}^{M}\left((-1)^{\# \beta} \beta,(-1)^{\#(i-1)} q_{i-1} \alpha_{i-1}^{\prime},(-1)^{\#(i-2)} \alpha_{i-2}, \ldots,(-1)^{\#(i-M+1)} \alpha_{i-M+1}\right) \\
&=(-1)^{\# \#(i-M+1)} Q_{i+2} \mathrm{id}_{L \rightarrow a}
\end{aligned}
$$

The differential $\mu_{\mathrm{H} \mathrm{Tw} \operatorname{Gtl}_{q} Q}^{1}(m)$ is given by projecting this term to $H_{q}$. Since the term already lies in $H_{q}$, we conclude the desired formula for $\mu_{\mathrm{H} \mathrm{Tw} \mathrm{Gtl}_{q} Q}^{1}(m)$. This finishes the case distinction for $Q$ and proves the explicit description of $\mu_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}^{1}(m)$ in all cases.

It remains to check E.1) in case $N=0$. This entails reinterpreting the explicit description of $\mu_{\mathrm{HTw} \mathrm{Gtl}}^{q}$ Q ${ }^{1}(m)$ in terms of MD disks. This is an easy exercise and we finish the proof here.

This finishes the computation of the desired products $\mu_{\mathrm{HTw} \mathrm{Gtl}}^{q} \boldsymbol{Q}\left(m, h_{N}, \ldots, h_{1}\right)$.

## F Discussion

In this section, we provide more explanation on the results of the present paper.

## F. 1 Relation to the literature

In this section, we list a selection of existing papers and for each of them explain how they relate to ours. To start with, we specify five cornerstones of the present paper. Every paper in our list will then be discussed in the context of one of these cornerstones.

The first topic of this paper is the category $\operatorname{Gtl} Q$ and its deformation $\mathrm{Gtl}_{q} Q$. Next, we form the category of zigzag paths $\mathbb{L}_{q} \subseteq \operatorname{Tw} \mathcal{C}_{q}$ and immediately gauge away the curvature from this category. Then we apply our deformed Kadeishvili theorem, which forces us into lengthy calculations, which end up in a beautiful geometric interpretation. This structure in mind, we arrive at the following list of cornerstones:
O1 Theory of $A_{\infty}$-deformations
O2 Deformations of $\operatorname{Gtl} Q$
O3 The category of zigzag paths
O4 Uncurving of deformations
O5 A deformed Kadeishvili theorem

## O6 Calculations with $\operatorname{Gtl} Q$ and $\operatorname{Fuk} Q$

In the remainder of this section, we present a selection of modern papers, each of which associated with one of these cornerstones. For each paper, we will explain:

- what the paper proves,
- how our paper builds on that paper,
- what aftermath our paper provides to that paper.

Two of these cornerstones are not represented in the selected papers. Indeed, cornerstone (O2) is provided by the first paper Paper I of this series. Meanwhile, cornerstone (O5) is not related with any specific literature: While Kadeishvili's theorem nowadays exists in lots of variants 60], its use in deformations is apparently new.

Those papers dealing with the A-side of mirror symmetry all depart from either the geometric model Fuk $Q$ or the discrete model $\operatorname{Gtl} Q$. Moreover, we can classify the papers according to whether they work with deformations or not. This gives the following diagram:

| starting point | non-deformed | deformed |
| :---: | :---: | :---: |
| geometric <br> discrete | 18, Appendix B] | $63,46,47$ |
| $18,16,35$, | this paper |  |

The present paper fills this square by departing from a deformation of the discrete model $\mathrm{Gtl} Q$.

## F.1.1 Keller

In 40, Keller proves that Morita equivalences of dg-categories induce $L_{\infty}$-quasi-isomorphisms of the Hochschild DGLA. Thereby, Keller's work falls under cornerstone (O1).

Keller regards the Hochschild DGLA not as an $L_{\infty}$-algebra, but as $B_{\infty}$-algebra which is slightly stronger. Let $A$ and $B$ be Morita equivalent dg algebras, with Morita equivalence provided by the $A-B$ bimodule $M$. Then Keller's core argument for invariance is as follows: Embed $A$ and $B$ into the triangular dg algebra

$$
D:=\left(\begin{array}{cc}
A & M \\
0 & B
\end{array}\right)
$$

Both embeddings $A, B \subseteq D$ turn out to be Morita equivalences. Keller exploits the natural restriction maps $\mathrm{HC}(D) \rightarrow \mathrm{HC}(A)$ and $\mathrm{HC}(D) \rightarrow \mathrm{HC}(B)$, which automatically respect the $B_{\infty}$-structure. Departing from the knowledge that Hochschild cohomology of dg algebras is invariant under Morita equivalences as graded vector space, Keller concludes that both $\mathrm{HC}(D) \rightarrow \mathrm{HC}(A)$ and $\mathrm{HC}(D) \rightarrow \mathrm{HC}(B)$ are $B_{\infty}$-quasiisomorphisms. Correspondingly, $\mathrm{HC}(A)$ and $\mathrm{HC}(B)$ are $B_{\infty}$-quasi-isomorphic.

Unfortunately, Keller's result is currently restricted to the case dg algebras. To the present paper, this means that we simply assume as axioms that the theory extends to all $A_{\infty}$-categories.

The heavy use of $A_{\infty}$-deformations in our paper shows how imperative it is to extend Keller's paper to the $A_{\infty}$-context. According to private communication with Keller, these results can likely be obtained by the same triangular construction together with appropriate $A_{\infty}$-bimodule theory.

## F.1.2 Barmeier-Wang

In 8, Barmeier and Wang prove the power of $L_{\infty^{-}}$-morphisms in deformation theory of ordinary algebras. They depart from a quiver algebra with relations $A=\mathbb{C} Q / I$, where the ideal $I$ is supposed to come from a reduction system. In order to classify all deformations of $A$, they replace the Hochschild DGLA $\mathrm{HC}(A)$ by a quasi-isomorphic $L_{\infty}$-algebra $L(A)$. Their line of thinking has contributed heavily to our cornerstone (O1).

Barmeier and Wang study a type of quiver algebras where the ideal $I$ comes from the act of substituting paths. Let us describe this in more detail: Let $Q$ be a quiver and $S$ a finite set of paths in $Q$. Let us temporarily call a linear combination of paths $x=\lambda_{1} x_{1}+\ldots+\lambda_{k} x_{k} \in \mathbb{C} Q$ reducible if one of the paths $x_{i}$ contains an element of $S$ as subpath. Now for every $s \in S$ let $f_{s} \in \mathbb{C} Q$ be some "substitution" for $s$. Assume that every $f_{s}$ is irreducible and every $s \in S$ is irreducible (except containing itself as subpath). In the words of 8 , the system $\left(S,\left\{f_{s}\right\}_{s \in S}\right)$ is then a reduction system. The reduction system is supposed to satisfy the so-called diamond condition. Put $I:=\left(s-f_{s}\right)_{s \in S}$ and define the associated quiver algebra as $A=\mathbb{C} Q / I$. In this algebra, it indeed holds that $s$ can be substituted by $f_{s}$ :

$$
a s b=a f_{s} b \in A
$$

To define the Hochschild DGLA, we typically use the bar resolution. Barmeier and Wang demonstrate that this resolution can be substituted by any other one, in particular the very simple bimodule resolution $P$. of Chouhy-Solotar, see [8, Section 4.2]. The cochain map between resolutions immediately gives a quasi-isomorphism of complexes

$$
\begin{equation*}
\operatorname{Hom}_{A \otimes A^{\text {opp }}}\left(\bigoplus_{i \in \mathbb{N}} A \otimes A^{\otimes_{i}} \otimes A, A\right) \xrightarrow{\sim} \operatorname{Hom}_{A \otimes A^{\text {opp }}}(P \bullet, A) . \tag{F.1}
\end{equation*}
$$

The left-hand side already being a DGLA, the $L_{\infty}$-structure transfer theorem induces an $L_{\infty}$-structure on the right-hand side such that F.1 becomes a quasi-isomorphism of $L_{\infty}$-algebras.

Barmeier and Wang then compute part of the $L_{\infty}$-structure on the right-hand side, just enough to classify all of its Maurer-Cartan elements. It turns out that those Maurer-Cartan elements can be exactly identified with deformations of the substitutions $f_{s}$. Since Maurer-Cartan elements are preserved under the $L_{\infty}$-quasi-isomorphism, this simultaneously classifies all Maurer-Cartan elements of the left-hand side $\mathrm{HC}(A)$ :

$$
\text { deformations of } A=\mathbb{C} Q /\left(s-f_{s}\right)_{s \in S} \quad \longleftrightarrow \quad \text { deformations } \tilde{f}_{s}=f_{s}+g_{s}
$$

Barmeier and Wang's realization that $L_{\infty}$-quasi-isomorphisms transport Maurer-Cartan elements has greatly helped us shape the curved $A_{\infty}$-deformation theory of section 5 There, we make constant use of the fact that quasi-equivalences of $A_{\infty}$-categories induces $L_{\infty}$-quasi-isomorphisms of their Hochschild DGLAs:

$$
F: \mathcal{C} \xrightarrow{\sim} \mathcal{D} \quad \rightsquigarrow \quad F_{*}: \mathrm{HC}(\mathcal{C}) \xrightarrow{\sim} \mathrm{HC}(\mathcal{D}) .
$$

Correspondingly, their sets of Maurer-Cartan elements over any deformation base $B$ match:

$$
\overline{\mathrm{MC}}(\mathrm{HC}(\mathcal{C}), B) \xrightarrow{\sim} \overline{\mathrm{MC}}(\mathrm{HC}(\mathcal{D}), B) .
$$

This is the main principle that lets us push deformations to and fro between different categories. For example, it lets us seamlessly reduce an $A_{\infty}$-category to a skeleton of non-isomorphic objects without changing its deformation theory. It enable us to prove that if an object $X \in \mathcal{C}_{q}$ is uncurvable, then all objects $Y \in \mathcal{C}_{q}$ quasi-isomorphic to $X$ are uncurvable as well.

As aftermath of our paper, we would like to point out that Barmeier and Wang's paper is restricted to quiver algebras concentrated in degree zero with ideal given by a reduction system. Gentle algebras $\operatorname{Gtl} Q$ already fall wide outside of their scope. The paper of Barmeier and Wang shows that the existence of a reduction system renders all deformations inherently straightforward. Interesting geometries will however appear as soon as we pass to algebras without reduction system or those not in degree zero. For example, the Jacobi algebra $\operatorname{Jac}(Q)$ of a dimer does not possess a reduction system. It is a Calabi-Yau-3 algebra if $Q$ is a consistent dimer. This brings $\operatorname{Jac}(Q)$ into the regime of Calabi-Yau deformation theory, on which we comment in the third part of this paper series.

## F.1.3 Lowen-van den Bergh

In 50, Lowen and Van den Bergh explain how to remove curvature from $A_{\infty}$-deformations of dg categories. This contributes to (O4).

Lowen and Van den Bergh depart from a dg algebra $A$ together with an infinitesimally curved $A_{\infty}$ deformation $A_{q}$ over $\mathbb{C} \llbracket q \rrbracket$. Lowen and Van den Bergh observe that a category $\operatorname{Tw}\left(A_{q}\right)$ of twisted complexes over $A_{q}$ can be formed even with infinitesimal entries below the diagonal, just as in our Remark 5.37 Interpret $A_{q}$ as an $A_{\infty}$-deformation with a single object. Then the core observation of Lowen and Van den Bergh is that the following twisted complex has vanishing curvature:

$$
X:=\left(A \oplus A[1],\left(\begin{array}{cc}
0 & \mu_{q}^{0} / q  \tag{F.2}\\
q \operatorname{id}_{A} & 0
\end{array}\right)\right) \in \operatorname{Tw}\left(A_{q}\right) .
$$

This means that $B_{q}:=\operatorname{End}_{\operatorname{Tw}\left(A_{q}\right)}(X, X)$ is a curvature-free deformed $A_{\infty}$-algebra. What is its special fiber $B$ ? The higher products $\mu^{\geq 3}$ on $B_{q}$ are given by embracing $\mu_{A_{q}}$ with the matrix entries $\mu_{q}^{0} / q$ and $q \operatorname{id}_{A}$ :

$$
\mu_{B_{q}}^{k \geq 3}\left(a_{k}, \ldots, a_{1}\right)=\sum \mu_{\text {Add } A_{q}}^{\geq 3}\left(\delta, \ldots, \delta, a_{k}, \ldots, \delta, \ldots, \delta, a_{1}, \delta, \ldots, \delta\right) .
$$

Restricting this sum to $q=0$ yields only higher products $\mu^{\geq 3}$ of $A$. Since $A$ is a dg algebra, we deduce $\mu_{B}^{k \geq 3}=0$ and therefore $B$ is a dg algebra as well. We conclude: $B_{q}$ is a curvature-free $A_{\infty}$-deformation $B_{q}$ of some algebra $B$.

Lowen and van den Bergh prove that $A$ and $B$ are in fact related by Morita equivalence. This costs substantial effort and uses the assumption that the curvature $\mu_{A_{q}}^{0}$ is nilpotent in the cohomology of $A$. The result is however that $A$ and $B$ are Morita equivalent, and moreover that $B_{q}$ is the deformation of $B$ corresponding to the deformation $A_{q}$ of $A$ along this Morita equivalence:

$$
\begin{array}{ccc}
\text { dg algebra A } & \rightsquigarrow & \text { dg algebra } B \\
\text { curved } A_{\infty} \text {-deformation } A_{q} & \rightsquigarrow & \text { uncurved } A_{\infty} \text {-deformation } B_{q}
\end{array}
$$

The work of Lowen and Van den Bergh helped us understand that curvature is essential in the notion of $A_{\infty}$-deformations, but not an invariant on its own. While Lowen and Van den Bergh exchange the dg algebra itself to remove curvature, our section 9 provides an example where a mere gauge transformation suffices to remove curvature.

In our recollection of $A_{\infty}$-deformation theory, Lowen and Van den Bergh have greatly helped us understand how deformations can be transferred from one category to another. We have built on their understanding that the transfer should happen by means of a $L_{\infty}$-quasi-equivalence, while the corresponding map on Maurer-Cartan elements is always secondary:


As aftermath of our paper, we conclude that there is theoretically no hindrance to forming a derived category of a curved $A_{\infty}$-deformation: By section 5.5 and section 8.3 , a derived category H Tw $\mathcal{C}_{q}$ exists even for infinitesimally curved deformations. The statement of Lowen and Van den Bergh that a curved $A_{\infty}$-deformation has no classical derived category remains true, but our paper contends that the study of deformations profits greatly from permitting also these "non-classical derived categories" H Tw $\mathcal{C}_{q}$.

Our method in section 9.4 seems to be both a variant and alternative to Lowen and Van den Berghs uncurving construction (F.2). It is a closely related variant in that our uncurving procedure factorizes the curvature of $(X, \delta)$ into components of $\delta$ and new infinitesimal entries. By comparison, Lowen and Van den Bergh simply factorize $\mu^{0}=\left(\mu^{0} / q\right)\left(q \mathrm{id}_{A}\right)$. Our procedure is also an alternative in that we uncurve the twisted complex itself, without passing to a different category $\mathcal{D}$. This way there is no doubt that we have only performed a gauge equivalence, and checks for Morita equivalence are not required. Our method relies a lot on the fact that the twisted differential $\delta$ is very rich, and it would be interesting to know which other twisted complex categories have such property.

## F.1.4 Bocklandt-Abouzaid

In [18, Bocklandt introduces the $A_{\infty}$-structure on $\mathrm{Gtl} Q$ and proposes it as discrete model for the wrapped Fukaya category wFuk $Q$. In that paper's appendix, Abouzaid computes the minimal model of an arc system as part of wFuk $Q$ and obtains indeed the gentle algebra. We conclude that 18 contributes to cornerstone (O6). The paper approaches the A-side via both the non-deformed discrete and non-deformed geometric side.

As first step of the paper, Bocklandt defines an $A_{\infty}$-structure on $\mathrm{Gtl} Q$. Abouzaid then shows that this $\operatorname{Gtl} Q$ is in fact a discrete model for wFuk $Q$. He regards the wrapped Fukaya category, as defined in 2. He discovers that one can pass with relative ease to the minimal model if one restricts to those string objects given by an arc system. More concretely, he shows that on $A_{\infty}$-level one can get rid of the so-called continuation map.

With this in mind, the work of Bocklandt-Abouzaid is a non-deformed prototype for our result: If the discrete $\operatorname{Gtl} Q$ provides a model for the geometric wFuk $Q$, then the deformation $\operatorname{Gtl}_{q} Q$ is necessarily a model for a certain deformation of wFuk $Q$ (see section F.2.2.

In contrast to Abouzaid's appendix, our calculations have to depart from the discrete side. Indeed, a deformed wrapped Fukaya category does not exist as of yet, so that we cannot work ourselves from geometric to discrete (see section F.2.3).

Our paper provides a new proof of Abouzaid's appendix in 18, at least on the subset of zigzag paths. Indeed, we show that $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ matches with the relative Fukaya category. Both categories are deformations over the same deformation base $\mathbb{C} \llbracket Q_{0} \rrbracket$, i.e. they have one deformation parameter per puncture in $Q$. As soon as we restrict both $\mathrm{HTw}_{\operatorname{Gtl}}^{q} Q$ and relFuk $Q$ to the special fiber $q=0$, we hold in our hands an explicit matching between $\mathrm{HTw} \operatorname{Gtl} Q$ and Fuk $Q$, at least on the category of zigzag paths. This recovers part of the result of Bocklandt and Abouzaid.

## F.1.5 The lectures of Bocklandt

A recent textbook 16 of Bocklandt explains gentle algebras in detail, shows how to stitch arcs together to form bands, and how to move towards the Fukaya category. Bocklandt's book contributes heavily to cornerstone (O6), and departs from the discrete perspective without deformation.

In its Section 9, Bocklandt recollects the definition of the gentle algebra Gtl $Q$. Next, he shows how to stitch arcs together along shared angles. This procedure results in twisted complexes in Tw Gtl $Q$.

For us, Bocklandt's explicit stitching procedure makes it entirely transparent how zigzag paths should be realized as twisted complexes. A zigzag path does not have a unique twisted complex representation, but there is a particularly simple one which makes direct use of the path's zigzag nature. This point of view is facilitated heavily by Bocklandt's section 9.2.

With this in mind, we can state that the twisted complex construction for gentle algebras is not the only one where the result can be identified geometrically. In fact, also twisted complexes of curves in the Fukaya category or wrapped Fukaya category can be identified as being quasi-isomorphic to curves that result from gluing together the arcs involved, see the book's Section 6.4.1.

Bocklandt's textbook contains several more hints relevant to the present paper, namely how to recognize similarity of $\mathrm{HTw} \operatorname{Gtl} Q$ with the Fukaya category: In its section 9.2, the hom spaces in the minimal model of $\mathrm{Tw} \mathrm{Gtl} Q$ are computed. Bocklandt delivers a basis of representatives of the cohomology $\operatorname{HHom}(X, Y)$, in case $X, Y$ are twisted complexes model transversal curves, and in case $X=Y$ as well. He combines these ingredients into a description of some higher products of $\mathrm{H} \mathrm{Tw} \mathrm{Gtl} Q$.

These calculations of Bocklandt provide a direct starting shot for the present paper: They tell us how to choose cohomology representatives for $\operatorname{Hom}(X, Y)$ and indicate how to obtain the higher products. There are also vague indications as to how to build a homological splitting.

Our paper essentially completes the calculations of Bocklandt: First complete the cohomology basis elements of Bocklandt to an entire homological splitting, at least in the case of zigzag paths. Second, we compute the entire $A_{\infty}$-structure on $\mathrm{H} \mathbb{L}_{q}$, including on non-transversal sequences, where Bocklandt's calculations are lacking. Third, we extend $\operatorname{Gtl} Q$ to the deformed case and show how to obtain the relative Fukaya category. Our deformed case demonstrates how also complicated Kadeishvili trees can contribute to the higher products, in contrast to Bocklandt's non-deformed case where only the simplest Kadeishvili trees yield nonzero results. This renders our paper a powerful extrapolation of Bocklandt's method.

## F.1.6 Seidel

In 63, Seidel introduces the notion of relative Fukaya categories. He departs from the exact Fukaya category and explains how to work relative to a divisor. He foresees the necessity to use curvature for those Lagrangians that have teardrops intersecting with the divisor, while according to him all other Lagrangians would be free of curvature. This way, 63 contributes to cornerstone (O4).

While Seidel does not provide anything explicit in case of punctured surfaces, his ideas carry over without difficulty: The divisor becomes a finite collection of points, which in our paper correspond to the punctures $M \subseteq S$. Each immersed disk should be weighted with the power of a deformation parameter whose exponent is the intersection number of the disk with the divisor $D$.

Seidel envisions those Lagrangians to be infinitesimally curved which have teardrops intersecting the divisor. This expectation has fueled our expectations towards uncurvability of objects in $\operatorname{Tw} \mathcal{C}_{q}$ : According to Seidel, we should expect that those band objects which are topologically nontrivial and do not bound a teardrop in $S$ are uncurvable, while those with teardrop in $S$ are inherently curved.

Seidel's definition provides a deformed Fukaya category of pre-category style: Its higher products are only defined on transversal sequences. At the time of Seidel's paper, it was not clear how to turn this definition into an actual category. This was accomplished in general only 20 years later by Sheridan and Perutz 59. Yet, their construction relies on the Hamiltonian deformation approach, which renders the $A_{\infty}$-structure on the non-transversal sequences very complicated.

The aftermath of our paper is a very down-to-earth description of the relative Fukaya category, at least on the subset of zigzag paths: We describe explicitly all the immersed disks one needs for its definition, also on all non-transversal sequences. A small caution: Technically, we cannot prove that our explicit category $\mathrm{H}_{q}$ is indeed (a subcategory of) the relative Fukaya category, but its higher products on the transversal sequences suggest so.

We confirm Seidel's expectations regarding curvature in the relative Fukaya category insection 9 . We also extend the width of Seidel's deformation in that we use one deformation parameter per puncture. It would be interesting to reintroduce Seidel's relative Fukaya category with more deformation parameters even in the higher dimensional case.


Figure F.1: Markings on a boundary component after 35

## F.1.7 Barmeier-Schroll-Wang

Intriguingly, Barmeier, Schroll and Wang are working on $A_{\infty}$-deformations on Fukaya categories as well, in parallel to the present paper. The subject of their work is known to the author, so we would like to point out a few relations. The work falls under cornerstone (O2).

In the article 8, Barmeier and Wang investigate deformations of quiver algebras with relations. The idea behind the new work of Barmeier-Schroll-Wang is to apply their methods to topological Fukaya categories as well.

To understand their line of thought, we should look into the work 35 of Haiden, Katzarkov and Kontsevich. They define topological Fukaya categories $\operatorname{Fuk}(S, M)$ also for marked surfaces $(S, M)$ beyond our notion of punctured surfaces. Indeed, 35 allows the surface to have a boundary instead of punctured, and the boundary is supposed to consist alternatingly of markings and "boundary arcs". In the simplest case without boundary arcs, their notion is equivalent to our punctured surfaces. In the case with at least one boundary arc, the topological Fukaya category $\operatorname{Fuk}(S, M)$ however allows for a very explicit model: a graded algebra without differential and higher products. This is the point where the deformation theory of 8 comes into play.

The work of Barmeier-Schroll-Wang yields results complementary to ours, namely deformations in the case every boundary component has at least one boundary arc. Since our case of Gtl $Q$ is an $A_{\infty^{-}}$ localization of the case with boundary arcs, it will be interesting to speculate about the relations between our work and Barmeir-Schroll-Wang's.

## F.1.8 Haiden-Katzarkov-Kontsevich

In 35, Haiden, Katzarkov and Kontsevich famously analyze stability conditions on partially wrapped Fukaya categories. Twisted complexes of the gentle algebra $\mathrm{Gtl} Q$ serve as model for their actual work. Their work contributes to cornerstone (O6), departing from the non-deformed discrete side.

As first step, Haiden, Katzarkov and Kontsevich introduce a notion of marked surfaces. In a marked surface, each boundary component is supposed to consist alternatingly of markings and "boundary arcs". Those marked surfaces where every $S^{1}$ boundary component is fully marked are precisely the punctured surfaces we use in the present paper.

Using arc systems, they define topological Fukaya categories $\operatorname{Tw} \mathcal{F}_{A}(S)$. If one restricts to the case of marked surfaces where each $S^{1}$ boundary component is fully marked, this is just $\mathrm{Tw} \mathrm{Gtl} Q$ in our terminology. This is depicted in Figure F. 1.

By an explicit analysis of all possible twisted complexes, they classify the objects of Tw Fuk $(S, M)$ up to quasi-isomorphism. This yields two different classes, the string and band objects. We recall this classification in section 9.2 This classification led Bocklandt 16 to write down the explicit correspondence between curves and twisted complexes.

The paper 35 then continues to classify stability conditions on a subcategory of the topological Fukaya category. The result is that these can be identified with singular flat structures on the marked surface with given poles or zeros.

In our paper, we depart from a special case of the topological Fukaya categories of 35. Indeed, a dimer model $Q$ is a specific type of marked surface. Its topological Fukaya category in the sense of 35 is simply $\mathrm{Tw} \operatorname{Gtl} Q$.

The paper 35 also helps us in section 10 to skip a few checks. Let us recapitulate the claims in that section: Given two zigzag paths $L_{1}, L_{2} \in \operatorname{Tw} \operatorname{Gtl} Q$, we would like to compute the definition of $\operatorname{HHom}_{\mathrm{Tw}} \mathrm{Gtl} Q\left(L_{1}, L_{2}\right)$. While this could be checked by hand, we propose to exploit Bocklandt's equivalence 18

$$
F: \mathrm{H} \mathrm{Tw} \mathrm{Gtl} Q \xrightarrow{\sim} \mathrm{wFuk} Q .
$$

The zigzag paths $L_{1}, L_{2}$ live on the left-hand side, and hence

We are left with computing the right-hand side. For this, we need to know which curves the objects $F\left(L_{1}\right)$ and $F\left(L_{2}\right)$ are. Here 35 comes into play and suggests that $F\left(L_{1}\right)$ and $F\left(L_{2}\right)$ are simply the


Figure F.2: The curves of Lekili and Polishchuk in case of $n=3$ punctures
smoothed-out versions of $L_{1}$ and $L_{2}$. With this assumption, the right-hand side of (F.3) becomes simply the number of intersections between $L_{1}$ and $L_{2}$, plus two in case $L_{1}=L_{2}$. This finishes the calculation of the hom space, but has cheated slightly in the identification of $F\left(L_{1}\right)$ and $F\left(L_{2}\right)$.

Far away on the horizon, Bocklandt has suggested a conjecture regarding stability conditions versus deformations. The idea is as follows: If we reinterpret the flat structures of 35 as deformations of the complex structure, they should constitute deformations of the derived category of coherent sheaves of the marked surface:

$$
\operatorname{StabFuk} Q \cong \operatorname{Def} \operatorname{Coh} Q
$$

Mirror symmetry of punctured surfaces ensures that under some conditions there is a dual dimer such that $\operatorname{Fuk} Q \cong \operatorname{Coh} \check{Q}$ and $\operatorname{Coh} Q \cong \operatorname{Fuk} \check{Q}$. Here Coh is abuse of notation and mean a noncommutative version of coherent sheaves, e.g. matrix factorizations. We then arrive at

$$
\begin{align*}
\operatorname{Stab} \operatorname{Coh} \check{Q} & \cong \operatorname{Def} \operatorname{Fuk} \check{Q},  \tag{F.4}\\
\operatorname{Def} \operatorname{Fuk} Q & \cong \operatorname{Def} \operatorname{Coh} \check{Q},  \tag{F.5}\\
\operatorname{Stab} \operatorname{Fuk} Q & \cong \operatorname{Stab} \operatorname{Coh} \check{Q} . \tag{F.6}
\end{align*}
$$

Simply speaking, the conjecture arising from 35 is that mirror symmetry swaps stability conditions and deformations. To prove this monster conjecture, we need a solid understanding of deformations of Fukaya categories. Our series of three papers will set up a deformed version of mirror symmetry, providing an explicit realization of the correspondence (F.5). The present paper provides the preliminary step of equating deformations of the discrete model with those of the geometric side.

## F.1.9 Lekili-Polishchuk

In 46, Lekili and Perutz find a commutative mirror for the relative Fukaya category of the 1-punctured torus, apparently the first use of a relative Fukaya category in mirror symmetry. In 47, Lekili and Polishchuk generalize this result to the case of the $n$-punctured torus. They depart from a finite collection of split-generators of the Fukaya category and compute part of their deformed products in the relative Fukaya category. This way, they contribute to cornerstone (O6), with a viewpoint from the deformed geometric side.

Let $\mathbb{T}_{1}$ denote the 1-punctured torus. Lekili and Perutz depart from an explicit definition of the relative Fukaya pre-category relFuk $^{\text {pre }} T_{1}$ : Working over the local ring $\mathbb{Z} \llbracket q \rrbracket$, every immersed disk is weighted by the number it covers the single puncture.

Lekili and Polishchuk regard the $n$-punctured torus $\mathbb{T}_{n}$. One might expect that they use an explicit model of the relative Fukaya category relFuk $\mathbb{T}_{n}$ and then prove it equivalent to their commutative mirror. Instead, they pick a set of $n+1$ curves $L_{0}, \ldots, L_{n}$ in $\mathbb{T}_{n}$ which split-generate the wrapped Fukaya category wFuk $\mathbb{T}_{n}$.

They do not attempt to compute the higher $A_{\infty}$-products on this set of generators entirely, but rather show that the $A_{\infty}$-structure must come from the perfect complexes of some complex curve $T_{n} 47$, Theorem 1.1.1]. The rest of their argument is devoted to guessing which curve $T_{n}$ is the right one.

This deduction up to isomorphism yields a functor $F:\left\{L_{0}, \ldots, L_{n}\right\} \rightarrow \operatorname{Perf}\left(T_{n}\right)$. To extend this functor to all of the relative Fukaya category, Lekili and Polishchuk view all objects of the Fukaya category as modules over these curves. More precisely, they regard a fully faithful Yoneda functor $\operatorname{wFuk}\left(T_{n}\right) \rightarrow \operatorname{Mod}\left(\left\{L_{0}, \ldots, L_{n}\right\}\right)$. The right-hand side again maps to $\operatorname{Perf}\left(T_{n}\right)$ by an extension of $F$ to modules.

In the present paper, we have a very similar desire: to equate the $A_{\infty}$-structure on $\mathcal{C}_{q}:=\mathrm{H} \mathbb{L}_{q}$ and the zigzag subcategory $\mathcal{D}_{q}$ of the relative Fukaya category. If we tried to follow Lekili and Polishchuk's approach, we would start from the observation that the non-deformed versions $\mathcal{C}$ and $\mathcal{D}$ are isomorphic


Figure F.3: Zigzag paths of the standard 4-punctured torus
by 18. We would then compute a few deformed higher products of $\mathcal{C}_{q}$ and compare those with $\mathcal{D}_{q}$, just enough to prove that $\mathcal{C}_{q} \cong \mathcal{D}_{q}$.

In the present paper, we do not follow the approach of Lekili and Polishchuk. In a sense, is a pity we were not able to guess the right structure like they did.

As aftermath of our paper, we recover the meaning of the curves $L_{0}, \ldots, L_{n}$ of Lekili and Polishchuk. Indeed, let $Q$ denote the standard $n$-punctured torus of Figure 6.5b. Then the zigzag paths of $Q$ are depicted in Figure F. 3 . There are precisely $n$ diagonal, $n$ vertical and 1 horizontal zigzag paths. Out of these, the vertical and horizontal are precisely the collection of Lekili and Polishchuk.

Our paper completes Lekili-Polishchuk's understanding of the deformed $A_{\infty}$-structure on $\left\{L_{0}, \ldots, L_{n}\right\}$. Indeed, we compute an entire minimal model category $\mathcal{C}_{q}=\mathrm{H} \mathbb{L}_{q}$, which has the same deformed $A_{\infty^{-}}$ structure on transversal sequences as the relative Fukaya category. It is technically not legitimate, but we could assume that $\mathcal{C}_{q}$ indeed is a model for the relative Fukaya category. This would mean that we have computed all missing $A_{\infty}$-structure that Lekili and Polishchuk were looking for.

## F. 2 Why should it work?

This paper shows that the relative Fukaya category can be obtained from a small, discrete model. But why should such a small model exist? The question is why one expects the candidate we give indeed to be equivalent to the relative Fukaya category. In this section, we explain how one is led to believe from an a priori perspective that it should work, and explain why $\operatorname{Gtl}_{q} Q$ is suited as a candidate.

## F.2.1 The model question

In order to prove results concerning an $A_{\infty}$-category $\mathcal{C}$, one tries to switch between different models of $\mathcal{C}$. This means, one is interested in $A_{\infty}$-categories $\mathcal{D}$ that are isomorphic, quasi-isomorphic, quasi-equivalent or derived equivalent to the $\mathcal{C}$. If such a category $\mathcal{D}$ satisfies certain geometric or algebraic properties or size constraints, it is called a model (of the given kind) for $\mathcal{C}$ :

$$
\underset{\text { original }}{\mathcal{C}} \cong \underset{\text { model }}{\mathcal{D}} \underset{\text { better behaved }}{ }
$$

A standard question in symplectic geometry is then: Can we find a small model $\mathcal{D}$ for the Fukaya category $\mathcal{C}=\operatorname{Fuk} Q$ ? Ideally, this category $\mathcal{D}$ would have very few objects, and still generate the whole Fukaya category. It does not work however, because cones over a small set of band objects do not yield all other bands. The question arises how to relax the task so that a small model can still be achieved. A very natural alternative is to require only that $\mathcal{C}$ is contained in the model $\mathcal{D}$. Actually, one would not require $\mathcal{C} \subseteq \mathcal{D}$, because $\mathcal{D}$ itself is supposed to have few objects, but one would aim at:

$$
\begin{array}{ccc} 
& \curvearrowleft \text { better behaved } \\
\mathcal{C} & \operatorname{HTw}_{\text {original }} & \text { model }
\end{array}
$$

Thanks to Bocklandt and Abouzaid 18, it is now known that the Fukaya category is indeed contained in the derived category $\mathrm{HTw} \operatorname{Gtl} Q$ of the gentle algebra $\operatorname{Gtl} Q$. There is a quasi-fully-faithful inclusion

$$
\text { Fuk } Q \subseteq \mathrm{H} \operatorname{Tw} \operatorname{Gtl} Q
$$

In fact, the category $\mathrm{HTw} \mathrm{Gtl} Q$ is not all too large: It is quasi-equivalent to the wrapped Fukaya category wFuk $Q$. In other words, Bocklandt and Abouzaid resolve the (relaxed) model question for Fuk $Q$ positively.

Let us now pose the same model question for deformed $A_{\infty}$-categories: Given a category $\mathcal{C}$ with a deformation $\mathcal{C}_{q}$, can we find a better behaved category $\mathcal{D}$ with a deformation $\mathcal{D}_{q}$ such that $\mathcal{C}_{q}$ and $\mathcal{D}_{q}$ are isomorphic, quasi-isomorphic, quasi-equivalent or derived equivalent?


Let us discuss what this means. In the above sketch, we have used $\cong$ to indicate one of the four notions of equivalence. In either case, an equivalence on the level of deformations necessarily requires an equivalence on the non-deformed level. Conversely, one can transport deformations along equivalence of (non-deformed) categories. Let us summarize as follows:

- Let $\mathcal{D}$ be a model for $\mathcal{C}$, and let $\mathcal{C}_{q}$ be a deformation of $\mathcal{C}$. Then there exists a model $\mathcal{D}_{q}$ for $\mathcal{C}_{q}$, obtained as a deformation of $\mathcal{D}$.
- If $\mathcal{C}$ has no good model (of a certain kind), then $\mathcal{C}_{q}$ does not have a good model either.

Relative Fukaya categories were introduced by Seidel as a deformation of ordinary Fukaya categories. We may now ask: Is it possible to provide a small model for the relative Fukaya category? Unfortunately, this is not possible either. A small model for $\operatorname{relFuk} Q$ would also include a small model for $\operatorname{Fuk} Q$ itself, which does not exist. The right approach becomes apparent by relaxing the task again. Let us first spell this out in general:

Let $\mathcal{C}$ be a category with a relaxed model $\mathcal{D}$, and let $\mathcal{C}_{q}$ be a deformation. Does a deformation $\mathcal{D}_{q}$ exist such that it is a relaxed model for $\mathcal{C}_{q}$ ? The answer is that this does not necessarily exist. The reason is that deformations cannot necessarily be lifted from $\mathcal{C}$ to $\mathcal{D}$. In fact, the restriction map $\mathrm{HC}(\mathcal{D}) \rightarrow \mathrm{HC}(\mathcal{C})$ induced by the inclusion $\mathcal{C} \rightarrow \mathrm{HTw} \mathcal{D}$ does absolutely not have to be a quasi-isomorphism of $L_{\infty}$-algebras. An easy example is the inclusion of quivers


The center of the quiver algebra $\mathbb{C} Q_{1}$ on the left is of course $\mathbb{C}[a]$, while the center of the algebra $\mathbb{C} Q_{2}$ on the right is just $\mathbb{C}^{2}$, spanned by the two idempotents. We deduce that the map $\mathrm{HH}^{0}\left(\mathbb{C} Q_{2}\right) \rightarrow \mathrm{HH}^{0}\left(\mathbb{C} Q_{1}\right)$ is not surjective. Here $\mathrm{HH}^{0}$ denotes classical Hochschild cohomology, which is the same as $\mathrm{HH}^{-1}$ in the $A_{\infty}$-grading. In short, restriction maps between Hochschild cohomologies are far from surjective due to "global" phenomena.

Let us tie this back to the question of finding a small model for the relative Fukaya category. We have already discussed that $\operatorname{Gtl} Q$ provides a small relaxed model for the Fukaya category. As we have just seen, this does however not imply the existence of a relaxed model for relFuk $Q$ in the form of a deformation of $\mathrm{Gtl} Q$. One starting point for understanding the present paper is therefore:

To find an $A_{\infty}$-category $\mathcal{D}$ together with a deformation $\mathcal{D}_{q}$ such that $\operatorname{relFuk} Q$ embeds quasi-fully-faithfully into $\mathrm{HTw} \mathcal{D}_{q}$.

A priori it is not clear that such a category $\mathcal{D}$ and deformation $\mathcal{D}_{q}$ should exist. The reason is that the ordinary and wrapped Fukaya categories are not equivalent and have different deformation theory. For the same reason, such a pair is not uniquely determined. There are however several ways of trying to find such a pair:

## A1 Guessing,

A2 Trying out the candidate deformation $\mathcal{D}_{q}:=\operatorname{Gtl}_{q} Q$ of $\mathcal{D}:=\operatorname{Gtl} Q$.
A3 Extending the relative Fukaya category to a deformation $\mathrm{wFuk}_{q}$ of the wrapped Fukaya category.
In this paper, we succeed in approach A2: We show that $\mathrm{Gtl}_{q} Q$ is a relaxed small model for relFuk $Q$, at least on the subcategory of zigzag paths. In section F.2.2 we explain why approach A2 is plausible and in section F.2.3 we explain why approach A3 is promising for mathematicians who can handle wrapped symplectic geometry.

There are three reasons why the author picked approach A2 instead of A3. First, we already have a concrete $\mathrm{Gtl}_{q} Q$ available from Paper I Second, approach A2 comes only with combinatorial calculations,
as opposed to deforming and working with the wrapped Fukaya category in approach A3. The expertise in symplectic geometry on the side of the author was simply not enough. Third, this paper was originally written not in order to find a small model, but to compute the subcategory of zigzag paths in $\mathrm{H} \mathrm{Tw} \mathrm{Gtl}_{q} Q$. The interpretation as a small model for relFuk $Q$ has come out as a useful byproduct.

## F.2.2 The candidate $\operatorname{Gtl}_{q} Q$

The goal of this section is to describe why our candidate $\operatorname{Gtl}_{q} Q$ is plausible as a (relaxed) model for $\operatorname{relFuk} Q$. The category $\operatorname{Gtl}_{q} Q$ itself is a seemingly arbitrary choice defined in Paper I, so that it is a priori not clear why it should be a model for $\operatorname{relFuk} Q$. There are however reasons why one should expect $\mathrm{Gtl}_{q} Q$ to be a model, even before performing any calculations. In this section, we explain those reasons.

Bocklandt-Abouzaid showed that the gentle algebra $\mathrm{Gtl} Q$ is equivalent to the wrapped Fukaya category. In the words of section F.2.1, this implies that Gtl $Q$ is a relaxed model for Fuk $Q$. As we have seen in section F.2.1 it is however far from clear that a relaxed model for a deformation can be obtained as a deformation of a relaxed model. In other words, if $\operatorname{Gtl} Q$ is a model for Fuk $Q$, why should the deformation $\operatorname{Gtl}_{q} Q$ be a model for relFuk $Q$ ?

There are three reasons why one might expect $\operatorname{Gtl}_{q} Q$ to be a relaxed model for relFuk $Q$ :

- The derived category $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ exists by construction, and it is a deformation of $\mathrm{HTw} \mathrm{Gtl} Q$. In particular, it is equivalent to a deformation of $\mathrm{wFuk} Q$ and has a restriction to Fuk $Q$. In other words, it contains some deformation of Fuk $Q$. One may now speculate which deformation of Fuk $Q$ it concerns.
- A glance at the deformed higher products shows that $\mathrm{Gtl}_{q} Q$ closely resembles relFuk $Q$ : Although the objects of both categories are completely disjoint, every disk containing one puncture gets multiplied by that puncture. Every disk containing two punctures gets multiplied by both, etc. One easily becomes suspicious that the deformation of $\operatorname{Fuk} Q$ contained in $\mathrm{HTw}_{\mathrm{Gtl}}^{q}$ $Q$ is actually relFuk $Q$.
- Reasoning with the beauty of mathematics, one should expect that relFuk $Q$ is such a reasonable deformation that is extends to wFuk $Q$. By the Bocklandt-Abouzaid equivalence, it then induces a deformation on $\mathrm{Gtl} Q$, and one may now guess which one this is: probably isomorphic to $\mathrm{Gtl}_{q} Q$.
Against the second reason, one might object that similarity of deformations is not the same as equality. It might be possible that the deformation of Fuk $Q$ contained in $\mathrm{HTw}_{\mathrm{Gtl}}^{q}$ $Q$ is slightly off, even though the products of relFuk $Q$ and $\operatorname{Gtl}_{q} Q$ look so similar. For example, $\operatorname{Gtl}_{q} Q$ intrinsically multiplies disks by $q$. The process of deriving $\operatorname{Gtl}_{q} Q$ may change this factor however to $q+q^{2}$ instead. This would imply that $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ does not have the same higher products as relFuk.

The fact that this $\operatorname{Gtl}_{q} Q$ actually is a relaxed model for $\operatorname{relFuk} Q$ and the higher products on $\mathrm{HTw}_{\mathrm{Ttl}}^{q}$ $Q$ are identical to those of $\operatorname{relFuk} Q$ is therefore out of pure luck. We will comment on this fact in section F.3.3.

## F.2.3 Alternative via the wrapped Fukaya category

In this section, we explain another approach to obtain a small (relaxed) model for relFuk $Q$. Namely, we comment on the idea to deform the wrapped Fukaya category, labeled A3 in section F.2.1 We will see why it is realistic, and what the difficulties are.

Let us recall approach A3 as follows: One tries to lift the deformation of Fuk $Q$ given by relFuk $Q$ to a deformation $\mathrm{wFuk}_{q} Q$ of $\mathrm{wFuk} Q$. If one succeeds at this approach, then one immediately has relFuk $Q$ as a subcategory of $\mathrm{wFuk}_{q} Q$. Pick a generating set $X \subseteq \mathrm{wFuk} Q$, typically a collection of arcs that split the surface. Denote by $X_{q} \subseteq \mathrm{wFuk}_{q} Q$ the restriction of the deformation wFuk $Q$ to the generating set $X$. Since $X$ is a generating set for wFuk $Q$, we have a quasi-equivalence

$$
\mathrm{HTw} X \rightarrow \operatorname{wFuk} Q
$$

induced from the inclusion $X \subseteq \mathrm{wFuk} Q$. The deformation $\mathrm{wFuk}_{q} Q$ is therefore already determined by the deformation $X_{q}$. In other words, we have a quasi-equivalence

$$
\mathrm{HTw} X_{q} \rightarrow \mathrm{wFuk}_{q} Q .
$$

Since the right-hand side $\mathrm{wFuk}_{q} Q$ contains the relative Fukaya category, we conclude that

$$
\begin{array}{lc}
\text { HrelFuk } Q \subseteq & \mathrm{HTw} X_{q} \\
\text { original } & \text { model }
\end{array}
$$

In other words, $X_{q}$ is a small model for relFuk. We conclude: A lift of the deformation relFuk $Q$ to wFuk $Q$ solves the (relaxed) model question for relFuk $Q$. Such a lift does not need to exist a priori and it is not unique.

Let us explain how one may obtain a candidate deformation $\mathrm{wFuk}_{q} Q$ such that relFuk $Q \subseteq \mathrm{wFuk}_{q} Q$. We can already guess several of its properties:

- On band objects, the higher products are just given by disks multiplied by $q$-parameters, as in relFuk $Q$.
- String objects need to have curvature. There is no technical necessity for this, but it is likely from the point of view that our combinatorial model $\operatorname{Gtl}_{q} Q$ also has curvature.
- The definition of higher products through Hamiltonian deformations needs to be completely revised to be compatible with the curvature. Due to the new higher products, there now exist infinitesimal results of disks even on teardrops. The obstruction theory in the definition of the Fukaya category needs to be completely revised.
This list already highlights some of the difficulties. The author has no clue how to properly define such an extension.

Let us assume for a moment that the paper's result extends beyond zigzag paths. From this a posteriori perspective we can deduce that a lift from relFuk $Q$ to a deformation $\mathrm{wFuk}_{q} Q$ exists: Regard the Bocklandt-Abouzaid quasi-equivalence

$$
\mathrm{HTw} \mathrm{Gtl} Q \xrightarrow{\sim} \mathrm{wFuk} Q .
$$

Then the deformation $\mathrm{Gtl}_{q} Q$ of $\mathrm{Gtl} Q$ induces a deformation $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ of $\mathrm{HTw} \mathrm{Gtl} Q$, and by transport through the quasi-equivalence also a deformation $\mathrm{wFuk}_{q} Q$. Since $\mathrm{HTw}_{\mathrm{Gtl}}^{q} \boldsymbol{Q}$ contains the relative Fukaya category, we deduce that the same holds for $\mathrm{wFuk}_{q} Q$ :

$$
\text { relFuk } \subseteq \mathrm{HTw}_{\mathrm{Gtl}}^{q} \text { } Q \xrightarrow{\sim} \mathrm{wFuk}_{q} Q .
$$

In other words, if one believes for a moment that the result of this paper extends to all band objects, then a lift from relFuk $Q$ to wFuk $Q$ necessarily exists. Approach A3 does therefore have a solution, although it is unclear how to construct it explicitly.

## F. 3 Why does it work?

This paper shows that the relative Fukaya category can be obtained from a small, discrete model. But why does the calculation work out? What are the ingredients that make it work? In contrast to the a priori discussion in section F.2, we explain in the present section why it works from an a posteriori perspective. In particular, we discuss the role of choices and luck.

Let us paraphrase the methods of this paper. The starting point is the deformed gentle algebra $\operatorname{Gtl}_{q} Q$. The task is to prove that its derived category $\mathrm{H} \mathrm{Tw}_{\mathrm{Gtl}}^{q}$ $Q$ contains the relative Fukaya category. To achieve this, we need to realize all Lagrangians in the Fukaya category as specific twisted complexes over $\mathrm{Gtl} Q$, and show that the subcategory of these twisted complexes equals the relative Fukaya category up to quasi-equivalence of deformations.

How would we achieve an equivalence between this subcategory of $\mathrm{H} \mathrm{Tw}_{\mathrm{w}} \mathrm{Gtl}_{q} Q$ and the relative Fukaya category? The relative Fukaya category relFuk $Q$ has mostly vanishing differential $\mu^{1}$, while the category $\mathrm{Tw}_{\mathrm{Gtl}_{q} Q}$ has large hom spaces and non-vanishing differential. They are clearly far away, but the category $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ already comes closer to the relative Fukaya category. In the present paper, we show how to actually match them. During the calculations, four facilitating factors have come into play:

- Zigzags: Instead of proving the whole relative Fukaya category to lie inside $\mathrm{H}_{\mathrm{Tw}} \mathrm{Gtl}_{q} Q$, we only prove this for the subcategory $\mathbb{L}_{q}$ of zigzag paths.
- Choices: We choose a "natural" homological splitting of $\mathbb{L}_{q}$.
- Luck: During the calculation of the minimal model structure of $\mathbb{L}_{q}$, our choice of homological splitting proves to be right one both for efficient calculation and to obtain exactly the relative Fukaya category.
- Fearless calculations: Performing the model computation for $\mathbb{L}_{q}$ emits enormous amounts of data and requires us to construct a tower of data structures as depicted in Figure 4.1 Binding the discrete data structures together to form smooth disk requires us to work through hundreds of calculations and special cases in order to bring order into the chaos. Practically, lots of trees need to be classified and large multiplication tables need to be filled. This paper performs the calculation until the bitter end.

We explain these four facilitating factors in more detail in section F.3.1, F.3.2 and F.3.3

## F.3.1 Restriction to zigzag paths

The result presented in this paper is restricted to zigzag paths. In this section we explain how this restriction eases the calculations and how the general case may be obtained later on.

Recall that zigzag paths are paths in a dimer that alternatingly turn left and right. When we say "zigzag path", we frequently refer to their realization as twisted complex in $\mathrm{Tw} \mathrm{Gtl} Q$ or as a band object in Fuk $Q$. Zigzag paths are a small class out of a large set of objects in both categories. Three factors distinguish zigzag paths from other band objects in $\operatorname{Fuk} Q$ :

- The higher structure on zigzag paths is necessary to compute a mirror for $\operatorname{Gtl}_{q} Q$, according to Cho-Hong-Lau.
- The arcs in the twisted complex representation of zigzag paths have only small angles between each other, i.e. no full turns or larger angles. This makes it easy to get grip on the disks between zigzag paths.
- If one assumes that $Q$ is geometrically consistent, a mild requirement, then all zigzag paths in $Q$ bound neither discrete nor smooth immersed disks. This is very useful.
It appears possible that the restriction to zigzag paths be overcome in the future, even without redoing the calculations. Let us sketch how this will work. The first step is to prove mirror symmetry for $\mathrm{Gtl}_{q} Q$, and the second step is to realize that the mirror depends only on the higher structure on zigzag paths.

Indeed, both relFuk and $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ produce mirror functors

$$
\text { Mod relFuk } \rightarrow \operatorname{mf}\left(A_{q}, \ell_{q}\right) \quad \text { and } \quad \mathrm{HTw}_{\operatorname{Gtl}} Q \rightarrow \operatorname{mf}\left(A_{q}, \ell_{q}\right)
$$

Both mirrors $\mathrm{mf}\left(A_{q}, \ell_{q}\right)$ are equal, since the Cho-Hong-Lau construction only depends on the structure on the zigzag paths. The module category Mod relFuk contains quasi-fully-faithfully some deformed copy $(\operatorname{Gtl} Q)_{q}^{\prime}$ of $\operatorname{Gtl}_{q} Q$ and so does $\mathrm{HTw}_{\mathrm{Gtl}_{q}} Q$ contain the deformation $\mathrm{Gtl}_{q} Q$. Both are mapped quasiequivalently to the mirror. It seems that we can deduce this way that $(\mathrm{Gtl} Q)_{q}^{\prime} \cong \mathrm{Gtl}_{q} Q$ as deformations of $\operatorname{Gtl} Q$. Together with relFuk $\subseteq \operatorname{HTw}(\operatorname{Gtl} Q)_{q}^{\prime}$, we should be able to deduce that relFuk is simply a subcategory of $\mathrm{HTw} \mathrm{Gtl}_{q} Q$. In other words, this should imply that $\mathrm{Gtl}_{q} Q$ is a small model for relFuk.

## F.3.2 Choice

This paper presents a minimal model for (part of) $\mathrm{HTw}_{\mathrm{Gtl}}^{q}$ $Q$. Such a minimal model is by no means unique. In this section, we explain why our specific choice of homological splitting works so well.

Let $\mathcal{C}$ be an $A_{\infty}$-category. Recall that by a minimal model for $\mathcal{C}$ one means any other $A_{\infty}$-category $\mathcal{D}$ such that $\mathcal{D}$ is minimal and $\mathcal{C}$ and $\mathcal{D}$ are quasi-isomorphic:

$$
\mu_{\mathcal{D}}^{1}=0 \text { and } \mathcal{C} \cong \mathcal{D} .
$$

Given a category $\mathcal{C}$, one may look for minimal models simply by guessing. Such a guess involves

- Possibly identifying the cohomology $\operatorname{Hom}(X, Y)$ for every $X, Y \in \mathcal{C}$ with some explicit graded vector space $\mathcal{D}(X, Y)$.
- Guessing an $A_{\infty}$-structure on these spaces $\mathcal{D}(X, Y)$, turning them into an $A_{\infty}$-category $\mathcal{D}$.
- Finding an $A_{\infty}$-quasi-isomorphism $\mathcal{C} \rightarrow \mathcal{D}$ or $\mathcal{D} \rightarrow \mathcal{C}$.

Guessing minimal models requires an enormous imagination.
There are also systemic ways of finding minimal models. In fact, the Kadeishvili theorem which we recall in section 8.2 grants the existence of minimal models and provides an explicit way to construct them. The formula for the minimal model depends on the choice of a so-called homological splitting $\operatorname{Hom}_{\mathcal{C}}=H \oplus I \oplus R$.

Assume we have chosen a homological splitting $\operatorname{Hom}_{\mathcal{C}}=H \oplus I \oplus R$. Then the map $\mu^{1}: R \rightarrow I$ is bijective. One then defines the so-called codifferential $h: I \rightarrow R$ as the inverse of $\mu^{1}: R \rightarrow I$.

The Kadeishvili construction then describes the HC as follows: The objects are the same as in $\mathcal{C}$. The hom spaces are the chosen cohomology representatives $H$. The differential is defined as $\mu_{\mathrm{H} \mathcal{C}}^{1}:=0$. The interesting part in the definition are the (higher) products. They are defined as sums over trees of the form


For two inputs, there is precisely 1 such tree. For three inputs, there are 3 such trees. For four inputs, there are 11 such trees. The result of each tree shall be multiplied by a sign. The sign is given by $(-1)^{N_{T}}$, where $N_{T}$ is the number of nodes in the tree, excluding the root. In other words, $s$ is the number of nodes in the tree labeled $h \mu$. For instance, the product $\mu^{2}(a, b)$ for $a, b \in H$ is simply given by

$$
\mu_{\mathrm{H} \mathcal{C}}^{2}(a, b)=\pi \mu^{2}(a, b)
$$

The higher product $\mu^{3}(a, b, c)$ for $a, b, c \in H$ is given by

$$
\mu_{\mathrm{H} \mathcal{C}}^{3}(a, b, c)=\pi \mu^{3}(a, b, c)-\pi \mu^{2}\left(h \mu^{2}(a, b), c\right)-\pi \mu^{2}\left(a, h \mu^{2}(b, c)\right)
$$

Observing these formulas, we conclude that the minimal model does depend on the choice of $H$ and $R$. One may also say: The minimal model depends on the choice of codifferential.

In this paper, we select one concrete choice of a homological splitting for the category $\mathbb{L}$ of zigzag paths in $\mathrm{Tw} \operatorname{Gtl} Q$. The choice looks arbitrary, but has some sophistication behind it. Let us explain the philosophy behind the cohomology representatives $H$ in our choice:

- We know how many representatives we have to choose: as many as $\operatorname{Hom}\left(L_{1}, L_{2}\right)$ has dimension.
- The dimensions of $\operatorname{HHom}\left(L_{1}, L_{2}\right)$ and the dimension of the hom space in the Fukaya category are equal (either by calculation or by using Bocklandt-Abouzaid). Hom spaces in the Fukaya category are spanned by intersection points, therefore we should try to find one representatives of $\operatorname{H} \operatorname{Hom}\left(L_{1}, L_{2}\right)$ for every intersection point.
- For each intersection point $p \in L_{1} \cap L_{2}$, choose the representative in $H$ such that we have the best chance of obtaining the Fukaya category as minimal model. For example, a disk existing in the Fukaya category should be realizable as a product $\mu_{\mathbb{L}}^{\geq 3}$ of the corresponding basis elements in $H$.
- The signs of the elements in $H$ should be chosen such that in the minimal model we obtain exactly the Abouzaid sign rule, without further sign conversion.

Regard an endomorphism space $\operatorname{End}(L, L)$ of a zigzag path $L \in \mathbb{L}$. Our choice for $H$ consists of two morphism of $\operatorname{End}(L, L)$ : the identity and a co-identity. While the identity element of $\operatorname{End}(L, L)$ naturally stems from the unitality of $\operatorname{Gtl} Q$, the choice of co-identity involves a choice. We namely define the co-identity to be any of the angles involved in the $\delta$-matrix of $L$. In other words, we choose the connecting angle between an arbitrary pair of consecutive arcs in $L$.

Why is this a sensible choice? One of the reasons to use the identity for $H$ is that it is very natural and it provides a strict unit in the minimal model $\mathrm{H} \mathbb{L}$. This strict unit is simultaneously necessary to exist if we want to make HL equal to the zigzag paths in the Fukaya category.

A reason why we choose the other basis element of $H$ to be a small angle between two consecutive $\operatorname{arcs}$ of $L$ is that this angle is easily seen to lie in the kernel of $\mu^{1}: \operatorname{End}(L, L) \rightarrow \operatorname{End}(L, L)$. Moreover, we want to obtain the Fukaya category as minimal model, which means that we have to reflect the arbitrary location of the co-identity morphism of Fukaya categories an closely as possible by means of the combinatorical datum of an angle.

## F.3.3 Luck

A decent amount of luck has been involved in the functioning of the present paper. In this section, we present five specific occasions where luck is decisive. The reader instead interested in a technical explanation why our choice of $\operatorname{Gtl}_{q} Q$ and the homological splitting are wise choices is referred to section F.2.2 and section F.3.2

Transparency of deformed cohomology basis After building the homological splitting in the nondeformed case, we prepare in section 11.3 the calculation of the deformed differential $\mu_{q}^{1}$ on $\mathbb{L}_{q}$. It turns out that the differential $\mu_{q}^{1}$ of any morphisms falls apart in contributions of certain types E, F, G and H .

Cohomology basis elements come from type B and C situations which restrict the tail to type E disks. The entire tail of a cohomology basis element then becomes relatively simple: It depends only on type E disks, and its tail terms are all of the form $\beta$ (A). The description of the deformed cohomology basis elements becomes not only explicit this way, but also very homogeneous.

Requirements for deformed Kadeishvili theorem We are lucky that the deformed Kadeishvili theorem can be established in the full generality. From a technical point of view, the Kadeishvili theorem is the only part of the paper that is not straightforward. It form a bottleneck for the minimal model computation and without its working we could not have pursued the calculation.

Transparency of the deformed codifferential Luck comes into play in our computation of the deformed codifferential $h_{q}$ in section 11.5 As always in this paper, this computation is rather an enumeration in terms of disks than a calculation with a concrete output. The deformed codifferential that illustrates the impact of luck best is $h_{q}(\beta \alpha)$, where the angles $\alpha, \beta$ are from an A situation. In this case, we have to find a sum of angles in $R$ whose differential totals to $\beta \alpha$ plus possibly terms of $R$. The first-order guess is $\beta$ itself, however $\mu_{q}^{1}(\beta)$ may also contain disk terms from $\mathrm{E}, \mathrm{F}, \mathrm{G}$ and H disks.

We are double lucky. First, the F disks only produce $\beta$ angles from A situations, the G2 disks only produce $\alpha_{3}$ and $\alpha_{4}$ angles, and the H disks only produce $\beta$ and $\beta^{\prime}$ angles from C situations. All of these angles lie in the kernel of $h_{q}$. In other words, those angles are in fact irrelevant in order to compute $h_{q}(\beta \alpha)$. We conclude that only the type E and type G 1 disks are relevant for computing $h_{q}(\beta \alpha)$, which greatly reduces complexity.

As for $\alpha_{4}$, it can be written as a signed sum $\alpha_{4}= \pm h \pm \alpha_{3}$ of the cohomology basis element $h=$ $(-1)^{\# \alpha_{3}+1} \alpha_{3}+(-1)^{\# \alpha_{4}} \alpha_{4}$ and the angle $\alpha_{3}$ lying in $R$. As a cohomology basis element, $h$ in turn can be written as a sum $h=h^{\prime}+r$ of a deformed cohomology basis element $h^{\prime}$ and an remainder $r \in R$. All of $h^{\prime}, r$ and $\alpha_{3}$ have vanishing codifferentials $h_{q}$, so that we conclude $h_{q}\left(\alpha_{4}\right)=0$.

Second, the G1 disks yields result of the form $\alpha_{1} \pm \alpha_{2}$, where $\alpha_{1}$ and $\alpha_{2}$ are from a B situation. The angle $\alpha_{1}$ again lies in $R$, while $\alpha_{2}$ equals $d\left(\mathrm{id}_{2 \rightarrow 5}\right)$ modulo kernel of $h_{q}$. Since $\mu_{q}^{1}\left(\mathrm{id}_{2 \rightarrow 5}\right)=d\left(\mathrm{id}_{2 \rightarrow 5}\right)$, we can simply add $\operatorname{id}_{2 \rightarrow 5} \in R$ to $\beta \alpha$ and $\mu_{q}^{1}\left(\beta \pm \mathrm{id}_{2 \rightarrow 5}\right)$ will eliminate the $\alpha_{2}$ term. Ultimately, every G1 disk only adds in a simple B situation identity into the $h_{q}(\beta \alpha)$. This is the reason we obtain the comparatively neat formula in Proposition 11.13.

The chaos and order of result components In section 12, we introduce the notion of result components. The subsequent classification of result components, its matching with immersed disks and the analysis of the immersed disks obtained this way is a roller coaster ride of case distinctions. Despite the intermediate chaos, the result collapses into a manageable description: four types of immersed disks (CR, ID, DS), following more or less the same rules. This collapse is a very fortunate turn.

Just the right products in $\mathrm{H}_{\mathbb{L}_{q}}$ Even with a slightly different homological splitting, we might already have obtained a minimal model $\mathrm{H} \mathbb{L}_{q}$ that looks entirely different from the relative Fukaya category. It would be hopeless to compare even a slightly different result to the relative Fukaya category. We are very fortunate that we obtain the higher products of the relative Fukaya category up to strict isomorphism.

## F. 4 Which calculations can be reused?

The heart of the present paper is a long and very specific calculation. In fact, the starting point consists of a very concrete deformation $\mathrm{Gtl}_{q} Q$ of the gentle algebra and the specific subcategory of $\mathrm{Tw} \mathrm{Gtl}_{q} Q$ given by the zigzag paths. This raises the question how the calculations and the result presented here can ever be used by other mathematicians for their own purposes.

In this section, we would like to answer this question. We explain how one can use the gentle algebra, the specific deformation $\mathrm{Gtl}_{q} Q$, the homological splitting and the notion of result components in a modular way as standard tools in computations.

We are convinced that while the precise calculations apply only to the specific situation of $\mathrm{Gtl}_{q} Q$, the versatility lies in the manner of performing the calculations and matching their result with the expected outcome. We contend that the mathematical value of the present paper mainly lies in making Kadeishvili trees computable.

## F.4.1 The gentle algebra

The use of the gentle algebra to perform calculations in mirror symmetry of punctured surfaces is not yet standard, as of writing. For example, in 3 the notion is still implicit. Some newer work 35 however uses the notion actively. In this section, we would like to highlight how easy the gentle algebra makes it to describe intersection theory.

Second, the twisted complexes of $\mathrm{Gtl} Q$ can be classified up to quasi-isomorphism. Recall from section 9.2 that the twisted complexes of $\mathrm{Gtl} Q$ can be classified as string and band objects. Formulated the other way around, every twisted complex of $\mathrm{Gtl} Q$ can be obtained up to quasi-isomorphism by stitching together arcs along angles. Regard two twisted complexes $X, Y \in \mathrm{Tw} \mathrm{Gtl} Q$ stitched together from arcs. Then the hom space $\operatorname{Hom}(X, Y)$ is spanned by all angles from arcs of $X$ to arcs of $Y$.

Given a whole sequence $X_{1}, \ldots, X_{k+1}$ of twisted complexes in Tw Gtl $Q$ and angles $\alpha_{i}: X_{i} \rightarrow X_{i+1}$, how to evaluate the higher product $\mu^{k}\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ ? By definition, this product is taken in the $A_{\infty^{-}}$ category Tw Gtl $Q$ and as such is made up of $\delta$-insertions. For each $X_{i}$, the possible $\delta$-insertions are insertions of arbitrary angles used to stitch together the arcs of $X_{i}$. In total, this higher product $\mu^{k}\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ gives a result if the angle sequence $\alpha_{1}, \ldots, \alpha_{k}$ can be filled up with $\delta$-insertions to form an immersed disk.

We see that even the twisted completion $\mathrm{Tw} \mathrm{Gtl} Q$ is an utterly geometric object and can be used for geometric proofs.

## F.4.2 The deformation $\operatorname{Gtl}_{q} Q$

In Paper I we introduced the deformed gentle algebra $\mathrm{Gtl}_{q} Q$. In fact, we provided even broader deformations and proved that they exhaust all deformations of $\mathrm{Gtl} Q$ up to gauge equivalence. In this section, we would like to explain what makes $\operatorname{Gtl}_{q} Q$ so versatile for studying deformations of Fukaya categories and mirror symmetry.

First, $\mathrm{Gtl} Q$ itself is a small category itself and such ideally suited for computations. Its deformation $\operatorname{Gtl}_{q} Q$ can be described fairly easily. Already the crude insight that $\mathrm{Gtl}_{q} Q$ is a deformation capturing behavior similar to the relative Fukaya category makes $\operatorname{Gtl}_{q} Q$ an interesting A-side of mirror symmetry. For comments on the use of $\mathrm{Gtl}_{q} Q$ as model for the relative Fukaya category, see section F.2.1.

As we recall in section 9.2 the twisted complexes of $\mathrm{Gtl} Q$ can be classified as string and band objects. As we show in section 9.4 most band objects can be uncurved. The uncurving procedure adds in infinitesimal connecting angles into the $\delta$-matrix. Let us explain the effect of this procedure. Regard a sequence of uncurved twisted complexes $X_{1}, \ldots, X_{k+1}$ and angles $\alpha_{i}: X_{i} \rightarrow X_{i+1}$. Then the higher product $\mu^{k}\left(\alpha_{k}, \ldots, \alpha_{1}\right)$, taken in $\operatorname{Tw~}_{\mathrm{Gtl}_{q}} Q$ now includes $\delta$-insertions of the additional infinitesimal angles in the $\delta$-matrices of the $X_{i}$. This makes that also immersed disks count that are bounded by whole segments of the curves $X_{i}$, instead of only a single arc as is the case without deformation. In particular, immersed disks between the curves $X_{i}$ that also cover an arbitrary number of punctures now contribute to the product.

In the present paper, we match $\mathrm{HTw}_{\mathrm{Gtl}}^{q}$ $Q$ with $\operatorname{relFuk} Q$. In other words, the deformation $\mathrm{Gtl}_{q} Q$ of $\mathrm{Gtl} Q$ induces a deformation $\mathrm{H}_{L_{q}}$ of $\mathrm{H} \mathbb{L}$ that looks like the relative Fukaya category. It is interesting to speculate what happens if we start with other deformations on the Gtl $Q$ side. More precisely, recall from Paper I that Gtl $Q$ also permits deformations where "orbifold disks" contribute to the higher products. Such a deformation of course also induces a deformation on $H \mathbb{L}$. Since $H \mathbb{L}$ is a full subcategory of the Fukaya category, this makes a plausible case for new deformations of the entire Fukaya category. For sure, such deformations of the Fukaya category have not been discovered yet. Future readers may therefore find joy in experimenting with other deformations of $\operatorname{Gtl} Q$ and for example obtain deformed "Fukaya categories with orbifold points".

## F.4.3 The homological splitting

Whenever one wants to compute a minimal model of an $A_{\infty}$-category explicitly, one needs a homological splitting $R \oplus I \oplus H$ of the $A_{\infty}$-category. A homological splitting is by no means unique, and different homological splittings result in different but quasi-equivalent minimal models. The present paper deploys a specific choice of homological splitting for the category $\mathbb{L} \subseteq \mathrm{Tw}_{\mathrm{w}} \mathrm{Gtl} Q$ of zigzag paths. In this section, we explain why this homological splitting should be established as the standard splitting for $\mathbb{L}$. We also comment on how to extend it to curves other than zigzag paths.

The homological splitting we choose in this paper is very well suited for the category $\mathbb{L}$. This splitting is chosen under the expectation that $H \mathbb{L}$ is a full subcategory of the Fukaya category. The reader finds the definition of the homological splitting in section 10.3 and comments on why this particular splitting is suited in section F.3.2 In fact, the homological splitting is both the right splitting to simplify the

(a) The simplest tree detects the disk.

(b) The simplest tree detects no disk.

Figure F.4: Detecting disks with Kadeishvili trees
calculations, and the right one to prove $H \mathbb{L}_{q}$ equal to the relative Fukaya category without further hassle with gauge equivalence. There is no doubt that the homological splitting is the best one for $\mathbb{L}$.

It is clear that minor modifications to the homological splitting are possible. Most obviously, in the splitting we present the author is free to choose where to put identity and co-identity morphisms of each zigzag path (the choices of $a_{0}$ and $\alpha_{0}$ ). A few actual changes are also possible: For instance, regard a transversal odd crossing between two zigzag paths. In the words of section 10.3 this corresponds to a B situation. Our choice of cohomology basis elements consists of the angle sum $(-1)^{\# \alpha_{3}+1} \alpha_{3}+(-1)^{\# \alpha_{4}} \alpha_{4}$. Choosing $(-1)^{\# \alpha_{1}+1} \alpha_{1}+(-1)^{\# \alpha_{2}} \alpha_{2}$ instead is however possible just as well.

While basis morphisms in the Fukaya category have a unique "location" in the surface, cohomology basis morphisms of $\mathrm{Tw} \operatorname{Gtl} Q$ can only imitate this behavior. Basis morphism in the Fukaya category lie on arcs of $Q$, while odd cohomology basis morphisms of $\mathrm{Tw} \mathrm{Gtl} Q$ can only lie around punctures of $Q$. The quality of this imitation determines whether the minimal model computation yields a result in the desired shape or not.

In our choice of homological splitting, we consistently choose $\alpha_{3}+\alpha_{4}$ for every single B situation. This has the advantage that many immersed disks with intersection points $h_{1}, \ldots, h_{N}$ between zigzag paths can be imitated by the simplest possible Kadeishvili tree $\pi \mu\left(\beta_{N}, \ldots, \beta_{1}\right)$, where $\beta_{1}, \ldots, \beta_{N}$ denote the corresponding B situation cohomology morphisms of type $\alpha_{3}+\alpha_{4}$. More specifically, the simplest Kadeishvili tree is capable of capturing immersed disks where two situation B crossings follow each other within one arc distance. An illustration is shown in Figure F.4a

If we were to choose $\alpha_{3}+\alpha_{4}$ for some B situations and $\alpha_{1}+\alpha_{2}$ for other B situations, the simplest Kadeishvili tree would not recognize disks where B situations follow each other rapidly. An example is shown in Figure F.4b That figure depicts three curves and a piece of an immersed disk between them. For the upper B situation the morphism $h_{1}=\alpha_{3}+\alpha_{4}$ was chosen as cohomology basis representative, while for the lower B situation the morphism $h_{2}=\alpha_{1}+\alpha_{2}$ was chosen. It is impossible to form a disk $\mu^{\geq 3}\left(\ldots, h_{2}, h_{1}, \ldots\right)$ in $\mathrm{Tw}_{\mathrm{Gtl}}^{q} \boldsymbol{Q}$. We conclude that a random choice of cohomology basis morphisms makes the minimal model calculation much less tractable.

Let us put the versatility of our homological splitting in the context of result components. Any choice of homological splitting provides an automatic notion of result components. To exploit result components for a minimal model calculation, one however needs to enumerate all possible result components by some target set, see section F.4.4. This enumeration by a target set is not automatic and depends on situational insight.

In our case of computing $H \mathbb{L}$, the notion of result components only needs to be tweaked minimally in order to map bijectively to the target set of immersed disks. Upon choice of a very different homological splitting for $\mathbb{L}$, a notion of result components is still automatic, but the collection of result components does not biject to immersed disks anymore. Instead, it will biject to a complicated set of disk-like objects that requires far more detailed analysis. In other words, our choice of homological splitting has the advantage that its result components have a very simple target set.

It seems possible to find a homological splitting also objects in $\operatorname{Tw} \operatorname{Gtl} Q$ which are not zigzag paths. The idea is still to sort elementary morphisms into different kinds of situations and to define the spaces $H$ and $R$ explicitly. The difficulty is however that general string and band objects have no limit with regards to the kind of angles they involve between two arcs. This means it is hard to find explicit cohomology representatives and to check that it concerns a homological splitting.

## F.4.4 The notion of result components

Result components are a technical tool serving as the main carrier of information in this paper, see section 12 The idea is easy: A term like $(3 x+5 y)(2 x+3 y)$ has the result components $6 x^{2}, 9 x y, 10 x y$, $15 y^{2}$. In other words, there are four distinct result components, even though the result can be abbreviated to only three terms. Result components provide maximum insight into part of the result instead of the whole, and how it is obtained instead of what is obtained. In this section, we argue that result components provide a means to analyze complicated Kadeishvili trees.

Regard a Kadeishvili tree $T$ with $N$ leaves and let $h_{1}, \ldots, h_{N}$ be inputs for the tree that lie in cohomology. Then for every node $N \in T$, there is attached a set of result components. The set of result components is determined from choice of result components of all children of $N$. In other words, result components are an inductive notion.

Let us paraphrase how we use result components in the present paper. We map a set of result components to "open" smooth immersed disks, which are called subdisks in section 13 This map is defined inductively: Given a result component at a node $N$, it is analyzed how the result component was obtained from result components of the node's children. By induction hypothesis, every of the node's children already has a subdisk assigned. The subdisk associated with the result component at $N$ is then obtained by gluing together the subdisks of the children in a way specific to the type of result component. This provides an inductively defined map from the set of result components of a Kadeishvili tree to the set of immersed disks.

For some categories, result components are better suited than for others. If the reader suspects that its minimal model has limited higher products, result components will not provide any use since most Kadeishvili trees result in zero anyway. If he however suspects that the minimal model calculation will result in a certain infinite "hierarchy" of higher products, then result components capture the higher products effectively.

For the reader who wishes to calculate the minimal model of some $A_{\infty}$-category via result components, we suggest the following roadmap:

1. Find a homological splitting of the category. Typically, cohomology representatives must be found at the beginning and the rest space $R$ can be accumulated on the go. The next step is to perform a few test calculations of products $\mu^{k}\left(h_{k}, \ldots, h_{1}\right)$, where $h_{i}$ are cohomology basis elements. The typical node in a Kadeishvili tree has output covering one or multiple basis elements of $R$. This is the time to start accumulating basis elements into $R$. The reader would then try to evaluate some products $\mu^{k}(\ldots)$ where the inputs are mixed from both cohomology and $R$. Which inputs from $R$ multiply to a nonzero product and how does the product depend on its inputs?
2. Construct a notion of result components. The exact way to do this depends on the situation. In the easiest case, a result component would simply be defined inductively as an output term of the evaluation of $h \mu$ at each node, or $\pi \mu$ at the root. For other calculations like ours, it makes sense to distinguish or identify some output terms of $h \mu$ or $\pi \mu$ at every node (for example $\alpha_{3}$ and the corresponding $\alpha_{4}$ output are always collected as a combined result component $\alpha_{3}+\alpha_{4}$ ).
3. Analyze how result components are derived. It is by no means necessary to classify all result components directly. Rather, it is important to classify result components into different types and understand which result components of which type can be derived from result components of which other types.
4. Determine a "target structure" or "target set". The idea is to match result components with instances of some kind of better understood structure. For example, we have identified immersed disks as the correct target structure for result components $\mathbb{L}_{q}$. Upon commencing this step, a vague idea of what the target structure or target set will be may help. In either case, the target structure becomes clearer as the application of result components proceeds.
5. Matching result components with target objects. This step is hardest. But when performed successfully, this step ensures that the correspondence between result components and target objects can be written down explicitly and in a recursive manner.
6. Perform an inverse construction. The idea is to classify which instances of the target structure have been obtained via the identification. By constructing an explicit inverse mapping, it becomes clear which target objects have been reached and which not.
The hard part always lies in identifying the correct target structure and the right identification of result components with target objects. Depending on what is expected from the particular minimal model, it
might be possible to interpret the structure of a given Kadeishvili tree in a geometric way, so as to guess what the correct target object is.

The field of homological algebra requires us to perform a lot of minimal model calculations. Many minimal model calculations can be simplified vastly by choosing a clever homological splitting. However, minimal models are often not computed in their entirety. An example is Bocklandt's partial computation of $\operatorname{Hmf}(\operatorname{Jac} Q, \ell)$ in 18 , which is nevertheless sufficient to prove mirror symmetry for punctured surfaces. We are convinced that result components can facilitate the execution of complete minimal model calculations wherever a geometric outcome is expected.

## G Notation

The following is a list of heavily used notation specific to this paper:

| Notation | Meaning | Reference |
| :---: | :---: | :---: |
| $\mathcal{C}$ | $A_{\infty}$-category | Definition 5.1 |
| $\mathcal{C}_{q}$ | $A_{\infty}$-deformation of $\mathcal{C}$ | Definition 5.14 |
| $\mu_{q}$ | $\mu_{\mathcal{C}_{q}}$, more specifically $\mu_{\mathrm{Gtl}_{q} Q}$ or $\mu_{\mathrm{Add} \mathrm{Gt1}_{q} Q}$ | Definition 5.14 |
| $B$ | deformation base | Definition 5.4 |
| $\mathfrak{m}$ | maximal ideal of $B$ | Definition 5.4 |
| $B \widehat{\otimes} V$ | completed tensor product with vector space $V$ | Definition 5.6 |
| $\mathfrak{m} V$ | shorthand for $\mathfrak{m} \widehat{\otimes} V \subseteq B \widehat{\otimes} V$ | Definition 5.6 |
| Tw $\mathcal{C}_{q}$ | twisted completion of $\mathcal{C}_{q}$ | Definition 5.31 |
| $\mathrm{Tw}^{\prime} \mathcal{C}_{q}$ | liberal twisted completion of $\mathcal{C}_{q}$ | Remark 5.37 |
| H $\mathcal{C}_{q}$ | minimal model of $\mathcal{C}_{q}$ | Corollary 8.14 |
| $\mathcal{T}$ | set of Kadeishvili tree shapes | Definition 8.6 |
| $N_{T}$ | number of internal nodes of a tree | Definition 8.6 |
| $\varphi$ | bijection $H_{q} \rightarrow B \widehat{\otimes} H$ | Definition 8.17 |
| $F: \mathcal{C} \rightarrow \mathcal{D}$ | $A_{\infty}$-functor | Definition 5.17 |
| $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ | functor of deformed $A_{\infty}$-categories | Definition 5.18 |
| $\mathrm{MC}(L, B)$ | Maurer-Cartan elements of DGLA/ $L_{\infty}$-algebra | Definition 5.40 |
| $\overline{\mathrm{MC}}(L, B)$ | $\mathrm{MC}(L, B)$ modulo gauge equivalence/homotopy | Definition 5.47 |
| $\mathrm{HC}(\mathcal{C})$ | Hochschild DGLA of $\mathcal{C}$ | Definition 5.41 |
| (S, M) | punctured surface | Definition 6.1 |
| $\mathcal{A}$ | arc system | Definition 6.3 |
| $a$ | $\operatorname{arc}$ in $\mathcal{A}$ | Definition 6.3 |
| $h(a), t(a)$ | puncture at head/tail of arc $a$ | Definition 6.3 |
| $\alpha$ | angle in $\mathcal{A}$ | Definition 6.13 |
| $h(\alpha), t(\alpha)$ | arc at head/tail of angle $\alpha$ | Definition 6.3 |
| $Q$ | dimer, typically geometrically consistent | Definition 6.11 |
| $Q_{M}$ | standard sphere dimer | section D. 2 |
| $\operatorname{id}_{a}$ | arc identity | section 6.4 |
| $L$ | zigzag path | Definition 6.38 |
| $\mathbb{L}$ | zigzag category | Definition 10.5 |
| $\mathbb{L}_{q}$ | deformed zigzag category | Definition 11.3 |
| $a_{0}$ | identity location on zigzag path | Convention 10.10 |
| $\alpha_{0}$ | co-identity location on zigzag path | Convention 10.10 |
| $P_{k}$ | standard $k$-gon | section 6.5 |
| $\varepsilon$ | elementary morphism $\varepsilon: L_{1} \rightarrow L_{2}$ | section 6.9 |


| Notation | Meaning | Reference |
| :---: | :---: | :---: |
| $T$ | Kadeishvili tree shape | section 8.2 |
| $T$ | tail of a morphism $\varepsilon: L_{1} \rightarrow L_{2}$ | Definition 11.9 |
| Result $_{\pi}$ | class of result components of $\pi$-trees | Definition 13.6 |
| Disk ${ }_{\text {SL }}$ | class of shapeless disks | Definition 13.6 |
| Result $_{\text {CR }}$ | class of CR result components | Definition 13.13 |
| Result ${ }_{\text {ID }}$ | class of ID result components | Definition 13.13 |
| Result ${ }_{\text {DS }}$ | class of DS result components | Definition 13.13 |
| Result ${ }_{\text {DW }}$ | class of DW result components | Definition 13.13 |
| Disk ${ }_{\text {CR }}$ | class of CR disks | Definition 13.15 |
| Disk $_{\text {ID }}$ | class of ID disks | Definition 13.17 |
| Disk ${ }_{\text {DS }}$ | class of DS disks | Definition 13.19 |
| Disk ${ }_{\text {DW }}$ | class of DW disks | Definition 13.20 |
| D | subdisk mapping D : Result ${ }_{\pi} \rightarrow$ Disk $_{\text {SL }}$ | Lemma 13.12 |
| $\mathrm{t}(\mathrm{D})$ | target/output morphism of disk $D$ | Definition 13.6 |
| $\operatorname{Abou}(D)$ | Abouzaid sign of disk $D$ | Definition 13.24 |
| $\operatorname{Punc}(D)$ | product of punctures covered by $D$ | Definition 13.24 |

## Paper III

## Deformed Mirror Symmetry for Punctured Surfaces

## 14 Introduction

Mirror symmetry is the quest for equivalences between Fukaya categories and categories of coherent sheaves. Noncommutative mirror symmetry is the quest for equivalences between Fukaya categories and categories of sheaves on noncommutative spaces. In the present paper, our starting point is noncommutative mirror symmetry for punctured surfaces:

## Gentle algebra <br> Gtl $Q$



## Matrix factorizations

 $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$In the present paper, we pick one specific deformation $\operatorname{Gtl}_{q} Q$ of $\operatorname{Gtl} Q$ and find the corresponding deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$. The result is a deformed mirror equivalence:

Deformed gentle algebra
$\operatorname{Gtl}_{q} Q$


Deformed matrix factorizations $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$

In what follows, we explain our quest from different perspectives. We explain the philosophy of the specific deformation $\mathrm{Gtl}_{q} Q$, comment on the source of mirror functors from the construction of Cho, Hong and Lau 26, and explain a bottleneck concerning the question whether the deformed Jacobi algebra $\mathrm{Jac}_{q} \check{Q}$ is indeed a deformation of $\mathrm{Jac} \check{Q}$.

Deformation theory In $A_{\infty^{-}}$-deformation theory one studies possible modifications of a given $A_{\infty^{-}}$ structure which keep the $A_{\infty}$-relations intact. One possible line of study consists of formal (infinitesimally) curved $A_{\infty}$-deformations. The base ring for such deformations is a local algebra $B$ with a few additional properties.

An interesting question arises when one is given two equivalent $A_{\infty}$-categories $\mathcal{C}, \mathcal{D}$ and tries to transfer a deformation from $\mathcal{C}$ to $\mathcal{D}$. As a starting point, one is given a quasi-equivalence $F: \mathcal{C} \rightarrow \mathcal{D}$ and a deformation $\mathcal{D}_{q}$. Transferring the deformation $\mathcal{C}_{q}$ via $F$ then entails finding a deformation $\mathcal{D}_{q}$ of $\mathcal{D}$ together with a deformation $F_{q}$ of $F$ such that $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ is a functor of deformed $A_{\infty}$-categories.

The difficulty in transferring $A_{\infty}$-deformations lies in the character of $A_{\infty}$-theory. Indeed, both the $A_{\infty}$-products of $\mathcal{C}$ and $\mathcal{D}$ and the functor $F$ have higher components, which make it impossible to quickly to write down the corresponding deformation $\mathcal{D}_{q}$.

Nevertheless, it is known that a transfer of deformations along quasi-equivalences always exists. The clue is to interpret $A_{\infty}$-deformations of $\mathcal{C}$ as Maurer-Cartan elements of the Hochschild DGLA HC $(\mathcal{C})$. The quasi-equivalence $F: \mathcal{C} \rightarrow \mathcal{D}$ then gives rise to a non-canonical $L_{\infty}$-morphism $F_{*}: \mathrm{HC}(\mathcal{C}) \rightarrow \mathrm{HC}(\mathcal{D})$. By applying $F_{*}$ to a given deformation $\mathcal{C}_{q}$, viewed as Maurer-Cartan element, one obtains the corresponding deformation $\mathcal{D}_{q}$. While this abstract interpretation does make $\mathcal{D}_{q}$ computable, it sets the stage for the systematic quest of deformed mirror symmetry.

Gentle algebras A popular way of modeling wrapped Fukaya categories of punctured surfaces deploys gentle algebras 18. In this framework, one starts from an oriented closed surface $S$ with a finite set of punctures $M \subseteq S$. One chooses a system $\mathcal{A}$ of arcs which connect the punctures and divide the surface into polygons. To an arc system, one can associate a so-called gentle algebra $\mathrm{Gtl} \mathcal{A}$, reminiscent of the classical associative gentle algebras of 5]. The gentle algebra $\mathrm{Gtl} \mathcal{A}$ is actually an $A_{\infty}$-category whose higher products detect the topology of the punctured surface $S \backslash M$. It was shown in 18 that it accurately models the wrapped Fukaya category of $(S, M)$.

Dimers are specific kinds of arc systems which suit the purposes of mirror symmetry. We shall consider the specific mirror symmetry of punctured surfaces, built in 18 . This statement of mirror symmetry entails a quasi-isomorphism $F: \operatorname{Gtl} Q \rightarrow \operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$. The dimer $\check{Q}$ is the so-called dual dimer of $Q$ and can be built from $Q$ in a combinatorical way. In contrast, the mirror functor $F$ itself is only given non-constructively and built in an inductive way by solving cocycle equations. Whenever we are given a deformation $\mathrm{Gtl}_{q} Q$, it would be very hard to explicitly find the corresponding deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$.

Deformed Fukaya categories Seidel 63 has introduced the idea of deforming Fukaya categories relative to a divisor. The idea is to introduce a formal parameter $q$ and weight every pseudoholomorphic disk by $q^{s}$ where $s$ counts the number of intersections of the disk with the divisor. In Paper I we have transported this concept to the world of gentle algebras. The result is a deformation $\mathrm{Gtl}_{q} Q$, in which every puncture comes with its own deformation parameter. Whenever a disk covers the punctures $q_{1}, \ldots, q_{k}$, the contribution of this disk is weighted by the product $q_{1} \ldots q_{k}$.

We raised the hope that our candidate deformation $\mathrm{Gtl}_{q} Q$ would be the correct way to implement Seidel's idea on the side of gentle algebras. In Paper II we examined this expectation and computed a part of the $A_{\infty}$-structure of the derived category $\mathrm{HTw}_{\mathrm{Gtl}}^{q}$ $Q$. Very specifically, it concerns the subcategory $\mathrm{H} \mathbb{L}_{q} \subseteq \mathrm{HTw}_{\mathrm{Gtl}}^{q} \boldsymbol{Q}$ given by the zigzag paths in $Q$. The result is an explicit description of the $A_{\infty^{-}}$ structure of $\mathrm{H}_{L_{q}}$ in terms of certain types of immersed disks. While it is hard to determine values for products of non-transversal sequences in the relative Fukaya category, our description of $\mathrm{H} \mathbb{L}_{q}$ determines their values very accurately. Although our calculation is limited to the zigzag paths, we consider Paper II a crude verification that $\mathrm{Gtl}_{q} Q$ is the correct transport of Seidel's vision to gentle algebras and can be considered a "relative wrapped Fukaya category".

Relative Fukaya categories have already served as A-side of mirror symmetry before. For instance, Lekili and Perutz 46 find a commutative mirror for the relative Fukaya category of the 1-punctured torus, apparently the first use of a relative Fukaya category in mirror symmetry. In 47, Lekili and Polishchuk generalize this result to the case of the $n$-punctured torus. They depart from a finite collection of splitgenerators of the Fukaya category and compute part of their deformed products in the relative Fukaya category. Their mirror is then obtained by guessing the correct deformation on the B-side. Complete knowledge of the products in the relative Fukaya category or even a relative wrapped Fukaya category are not required in their approach.

Mirror functors A rich source of mirror functors is the recent construction of Cho, Hong and Lau 26. Their construction associates to a given $A_{\infty}$-category $\mathcal{C}$ with a suitable subcategory $\mathbb{L} \subseteq \mathcal{C}$ a Landau-Ginzburg model ( $\operatorname{Jac} Q^{\mathbb{L}}, \ell$ ) together with an $A_{\infty}$-functor

$$
F: \mathcal{C} \rightarrow \operatorname{MF}\left(\operatorname{Jac} Q^{\mathbb{L}}, \ell\right)
$$

The Cho-Hong-Lau construction can be applied to the category $\mathcal{C}=\mathrm{HTw} \operatorname{Gtl} Q$ by choosing $\mathbb{L}$ to be the subcategory given by so-called zigzag paths. This application yields back the original mirror symmetry for punctured surfaces $\operatorname{Gtl} Q \cong \operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ from 18 .

In the present paper, it is our aim to produce a deformation of $\operatorname{mf}(\operatorname{Jac} Q, \ell)$ which corresponds to $\mathrm{Gtl}_{q} Q$. Thanks to the Cho-Hong-Lau construction, this task becomes straightforward: The first step is to deform the Cho-Hong-Lau construction. The result is a procedure which generates mirror functors of the kind $\mathcal{C}_{q} \rightarrow \operatorname{MF}\left(\operatorname{Jac}_{q} Q^{\mathbb{L}}, \ell_{q}\right)$. The second step is to apply this deformed construction to the case of $\mathcal{C}_{q}=\mathrm{HTw} \mathrm{Gtl}_{q} Q$. The result is a quasi-isomorphism of deformed $A_{\infty}$-categories

$$
F_{q}: \operatorname{Gtl}_{q} Q \xrightarrow{\sim} \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right) .
$$

In particular, the deformation $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ is the desired deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ corresponding to $\operatorname{Gtl}_{q} Q$. It is possible to describe the algebra $\mathrm{Jac}_{q} \check{Q}$, the potential $\ell_{q}$ and the mirror objects $F_{q}(a)$ explicitly. This requires heavy computations in the minimal model H Tw $\mathrm{Gtl}_{q} Q$ which we have performed in Paper II Thanks to these earlier computations, we offer in the present paper an explicit description of $\mathrm{Jac}_{q} \dot{Q}, \ell_{q}$ and $F_{q}(a)$ in terms of combinatorics in $Q$.

Flatness of superpotential deformations A bottleneck in this paper is the question whether $\mathrm{Jac}_{q} \check{Q}$ is a (flat) deformation of $\operatorname{Jac} \mathscr{Q}$ as an algebra. Indeed, our deformed Cho-Hong-Lau construction leads to a definition of $\mathrm{Jac}_{q} \check{Q}$ as a mere quotient of $\mathbb{C} \check{Q} \llbracket Q_{0} \rrbracket$ by certain deformed relations. A quotient by deformed relations however need not be an algebra deformation in general. The question is whether the specific case of $\mathrm{Jac}_{q} \check{Q}$ is a deformation of $\operatorname{Jac} \check{Q}$ nevertheless. To resolve this question, we prove a flatness result for superpotential deformations of CY3 algebras.

Our flatness result is a culmination of a long sequence of improvements in the literature. Our starting point is the work of Berger, Ginzburg and Taillefer 11,12 which concerns PBW deformations of CY3 algebras. Like all previous results, their work requires the superpotential $W$ to be homogeneous. We translate their work to the setting of formal deformations and show that the homogeneity condition is superfluous and can be replaced by a mild boundedness condition. We obtain a flatness result for formal deformations of CY3 algebras with nonhomogeneous superpotential. In particular, it follows from this result that $\mathrm{Jac}_{q} \check{Q}$ is a flat deformation of $\operatorname{Jac} \check{Q}$ for almost all dimers $\check{Q}$.

Ultimately, our flatness result renders the category $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ a deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ and $F_{q}$ an equivalence of deformations. This proves noncommutative mirror symmetry for punctured surfaces.

Assembling deformed mirror symmetry The present paper is the final one in a series of three. We explain here the purpose of this series, which results have been obtained in the first papers and how we build on them in the present paper.

Our original motivation was to transport Seidel's idea of relative Fukaya categories to the world of gentle algebras and to use it as A-side in a deformed mirror symmetry for punctured surfaces. We realized that an effective way of constructing the deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ was to follow the Cho-Hong-Lau construction, inserting the deformation $\operatorname{Gtl}_{q} Q$ as an input instead of $\mathrm{Gtl} Q$. This approach requires an explicit and lengthy minimal model calculation, which is the reason we distributed the material into a series of three papers.

The first paper in the series is Paper I and concerns the deformation theory of the gentle algebras $\mathrm{Gtl} \mathcal{A}$ under certain assumptions on the $\operatorname{arc} \operatorname{system} \mathcal{A}$. One of the main results is a complete classification of the deformations of $\operatorname{Gtl} \mathcal{A}$.

The second paper in the series is Paper II and conducts all the necessary computations for applying the Cho-Hong-Lau construction. It focuses on the case of the specific deformation $\operatorname{Gtl}_{q} Q$ and defines the category of deformed zigzag paths $\mathbb{L}_{q}$. By means of a deformed Kadeishvili construction, it builds a minimal model $\mathrm{H} \mathbb{L}_{q}$ for $\mathbb{L}_{q}$. The main result is an explicit description of the deformed $A_{\infty}$-structure of $\mathrm{H} \mathbb{L}_{q}$ in terms of four types of immersed disks. These four types are labeled CR, ID, DS and DW disks and they agree precisely with the immersed disks one expects form the relative Fukaya category.

In the present paper, we tie the previous calculations together. We start by deforming the Cho-HongLau construction in general. Then, we apply this deformed construction to the special case of $\mathrm{Gtl}_{q} Q$ and obtain a deformed Jacobi algebra $\mathrm{Jac}_{q} \check{Q}$ and a deformed central element $\ell_{q}$. Thanks to the second paper, we have an explicit description of the $A_{\infty}$-structure on $\mathrm{H} \mathbb{L}_{q}$, giving an explicit description of $\mathrm{Jac}_{q} \check{Q}$ and $\ell_{q}$. This description is theoretically given in terms of the CR, ID, DS and DW disks from Paper II but simplifies a bit in the present paper because mostly products of transversal sequences are regarded. Apart from proving that $\mathrm{Jac}_{q} \check{Q}$ is indeed a deformation of $\operatorname{Jac} \check{Q}$, simply plugging in the results of Paper II already finishes deformed mirror symmetry.

Structure of the paper In section 15 we review $A_{\infty}$-categories and their deformations. We also introduce notation and terminology for treating algebra deformations, including the $\mathfrak{m}$-adic topology and flatness conditions. In section 16, we present Koszul duality and the relationship between cyclic $A_{\infty}$-algebras and Calabi-Yau dg algebras. We show how to tweak Koszul duality in order to obtain $A_{\infty}$-functors similar to the Cho-Hong-Lau construction. In section 17, we review dimers, gentle algebras and mirror symmetry of punctured surfaces. In section 18 we review the definition of the category of deformed zigzag paths $\mathbb{L}_{q}$ and description of its minimal model $\mathrm{H} \mathbb{L}_{q}$ from Paper II In section 19, we investigate deformations of Jacobi algebras given by deformations of the superpotential. We also consider the specific case Jac $\check{Q}$ of Jacobi algebras of dimers. In section 20 we motivate and review the Cho-HongLau construction. We provide an explicit deformed construction and resolve a few technicalities. In section 21, we apply the deformed Cho-Hong-Lau construction to the specific case of $\mathrm{Gtl}_{q} Q$. We provide explicit descriptions of the deformed Jacobi algebra $\mathrm{Jac}_{q} \check{Q}$, the central element $\ell_{q}$ and the deformed matrix factorizations $F_{q}(a)$. In section H, we work out deformed mirror symmetry for the examples of the 3 -punctured sphere and a 4 -punctured torus.

## 15 Preliminaries on $A_{\infty}$-categories

In this section, we recollect background material on $A_{\infty}$-categories and fix notation. In section 15.1, we recall $A_{\infty}$-categories, their functors, twisted completion and minimal models. In section 15.2 , we recall completed tensor products, deformations of $A_{\infty}$-categories and their functors. We very briefly comment on the construction of twisted completion and minimal model for $A_{\infty}$-deformations from Paper II. In section 15.3 , we introduce specific terminology and properties for submodules of $B \widehat{\otimes} X$ as preparation for section 19 In section 15.4 , we recall $\mathfrak{m}$-adically free modules. In section 15.5, we examine variants of our flatness condition for ideals.

## 15.1 $A_{\infty}$-categories

In this section we recall $A_{\infty}$-categories, completed tensor products, $A_{\infty}$-deformations and functors between $A_{\infty}$-deformations. The material is standard and can for instance be found in 16 . Throughout we work over an algebraically closed field of characteristic zero and write $\mathbb{C}$.
Definition 15.1. A ( $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded, strictly unital) $A_{\infty}$-category $\mathcal{C}$ consists of a collection of objects together with $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded hom spaces $\operatorname{Hom}(X, Y)$, distinguished identity morphisms id $X_{X} \in$ $\operatorname{Hom}^{0}(X, X)$ for all $X \in \mathcal{C}$, together with multilinear higher products

$$
\mu^{k}: \operatorname{Hom}\left(X_{k}, X_{k+1}\right) \otimes \ldots \otimes \operatorname{Hom}\left(X_{1}, X_{2}\right) \rightarrow \operatorname{Hom}\left(X_{1}, X_{k+1}\right), \quad k \geq 1
$$

of degree $2-k$ such that the $A_{\infty}$-relations and strict unitality axioms hold: For every compatible morphisms $a_{1}, \ldots, a_{k}$ we have

$$
\begin{aligned}
& \sum_{0 \leq j<i \leq k}(-1)^{\left\|a_{n}\right\|+\ldots+\left\|a_{1}\right\|} \mu\left(a_{k}, \ldots, \mu\left(a_{i}, \ldots, a_{j+1}\right), a_{j}, \ldots, a_{1}\right)=0, \\
& \mu^{2}\left(a, \operatorname{id}_{X}\right)=a, \mu^{2}\left(\operatorname{id}_{Y}, a\right)=(-1)^{|a|} a, \mu^{\geq 3}\left(\ldots, \operatorname{id}_{X}, \ldots\right)=0 .
\end{aligned}
$$

Next we recall the additive completion $\operatorname{Add} \mathcal{C}$ of an $A_{\infty}$-category $\mathcal{C}$. This category consists of formal sums of shifted objects. The hom space between two objects consists of matrices of morphisms between the summands.
Definition 15.2. Let $\mathcal{C}$ be an $A_{\infty}$ category with product $\mu_{\mathcal{C}}$. The additive completion $\operatorname{Add} \mathcal{C}$ of $\mathcal{C}$ is the category of formal sums of shifted objects of $\mathcal{C}$ :

$$
A_{1}\left[k_{1}\right] \oplus \ldots \oplus A_{n}\left[k_{n}\right] .
$$

The hom space between two such objects $X=\bigoplus A_{i}\left[k_{i}\right]$ and $Y=\bigoplus B_{i}\left[m_{i}\right]$ is

$$
\operatorname{Hom}_{\text {Add } \mathcal{C}}(X, Y)=\bigoplus_{i, j} \operatorname{Hom}_{\mathcal{C}}\left(A_{i}, B_{j}\right)\left[m_{j}-k_{i}\right]
$$

Here [-] denotes the right-shift. The products on Add $\mathcal{C}$ are given by multilinear extensions of

$$
\mu_{\text {Add } \mathcal{C}}^{k}\left(a_{k}, \ldots, a_{1}\right)=(-1)^{\sum_{j<i}\left\|a_{i}\right\| l_{j}} \mu_{\mathcal{C}}^{k}\left(a_{k}, \ldots, a_{1}\right)
$$

Here each $a_{i}$ lies in some $\operatorname{Hom}\left(X_{i}\left[k_{i}\right], X_{i+1}\left[k_{i+1}\right]\right)$. The integer $l_{i}$ denotes the difference $k_{i+1}-k_{i}$ between the shifts and the degree $\left\|a_{i}\right\|$ is the degree of $a_{i}$ as element of $\operatorname{Hom}_{\mathcal{C}}\left(X_{i}, X_{i+1}\right)$.

Next we recall the twisted completion $\operatorname{Tw} \mathcal{C}$ of an $A_{\infty}$-category $\mathcal{C}$. The objects of this category are virtual chain complexes of objects of $\mathcal{C}$ :
Definition 15.3. A twisted complex in $\mathcal{C}$ is an object $X \in \operatorname{Add} \mathcal{C}$ together with a morphism $\delta \in$ $\operatorname{Hom}_{\operatorname{Add} \mathcal{C}}^{1}(X, X)$ of degree 1 such that $\delta$ is strictly upper triangular and satisfies the Maurer-Cartan equation:

$$
\operatorname{MC}(\delta):=\mu^{1}(\delta)+\mu^{2}(\delta, \delta)+\ldots=0
$$

We may refer to the morphism $\delta$ as the twisted differential. Note that the upper triangularity ensures that this sum is well-defined. The twisted completion of $\mathcal{C}$ is the $A_{\infty}$-category Tw $\mathcal{C}$ whose objects are twisted complexes. Its hom spaces are the same as for the additive completion:

$$
\operatorname{Hom}_{T w} \mathcal{C}(X, Y)=\operatorname{Hom}_{\text {Add } \mathcal{C}}(X, Y)
$$

The products on $\operatorname{Tw} \mathcal{C}$ of $\mathcal{C}$ are given by embracing with $\delta$ 's:

$$
\mu_{\mathrm{Tw} \mathcal{C}}^{k}\left(a_{k}, \ldots, a_{1}\right)=\sum_{n_{0}, \ldots, n_{k} \geq 0} \mu_{\mathrm{Add} \mathcal{C}}(\underbrace{\delta, \ldots, \delta}_{n_{k}}, a_{k}, \ldots, a_{1}, \underbrace{\delta, \ldots, \delta}_{n_{0}})
$$

A functor between two $A_{\infty}$-categories is a mapping which matches the products of the two categories:
Definition 15.4. Let $\mathcal{C}$ and $\mathcal{D}$ be $A_{\infty}$-categories. Then a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ of $A_{\infty}$-categories consists of a map $F: \operatorname{Ob}(\mathcal{C}) \rightarrow \mathrm{Ob}(\mathcal{D})$ together with for every $k \geq 1$ a degree $1-k$ multilinear map

$$
F^{k}: \operatorname{Hom}_{\mathcal{C}}\left(X_{k}, X_{k+1}\right) \otimes \ldots \otimes \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, X_{2}\right) \rightarrow \operatorname{Hom}_{\mathcal{C}}\left(F X_{1}, F X_{k+1}\right)
$$

such that the $A_{\infty}$-functor relations hold:

$$
\begin{aligned}
& \sum_{0 \leq j<i \leq k}(-1)^{\left\|a_{j}\right\|+\ldots+\left\|a_{1}\right\|} F\left(a_{k}, \ldots, a_{i+1}, \mu\left(a_{i}, \ldots, a_{j+1}\right), a_{j}, \ldots, a_{1}\right) \\
&=\sum_{\substack{l \geq 0 \\
1=j_{1}<\ldots<j_{l} \leq k}} \mu\left(F\left(a_{k}, \ldots, a_{j_{l}}\right), \ldots, F\left(\ldots, a_{j_{2}}\right), F\left(\ldots, a_{j_{1}}\right)\right) .
\end{aligned}
$$

The functor $F$ is an isomorphism if $F: \operatorname{Ob}(\mathcal{C}) \rightarrow \operatorname{Ob}(\mathcal{D})$ is a bijection and $F^{1}: \operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow$ $\operatorname{Hom}_{\mathcal{D}}(F X, F Y)$ is an isomorphism for all $X, Y \in \mathcal{C}$. The functor $F$ is a quasi-isomorphism if $F$ : $\mathrm{Ob}(\mathcal{C}) \rightarrow \mathrm{Ob}(\mathcal{D})$ is a bijection and $F^{1}: \operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow \operatorname{Hom}_{\mathcal{D}}(F X, F Y)$ is a quasi-isomorphism of complexes for every $X, Y \in \mathcal{C}$.
Definition 15.5. When $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{E}$ are $A_{\infty}$-functors, then their composition is given by $G F: \mathrm{Ob}(\mathcal{C}) \rightarrow \mathrm{Ob}(\mathcal{E})$ on objects and

$$
(G F)\left(a_{k}, \ldots, a_{1}\right)=\sum G\left(F\left(a_{k}, \ldots\right), \ldots, F\left(\ldots, a_{1}\right)\right)
$$

Let us recall minimal models and their notation as follows:
Definition 15.6. An $A_{\infty}$-category $\mathcal{C}$ is minimal if $\mu_{\mathcal{C}}^{1}=0$. A minimal model of $\mathcal{C}$ is any minimal $A_{\infty}$-category $\mathcal{D}$ together with a quasi-isomorphism $F: \mathcal{D} \rightarrow \mathcal{C}$. A minimal model of $\mathcal{C}$ is generically denoted HC .

By the famous Kadeishvili theorem, every $A_{\infty}$-category has a minimal model. In fact, a minimal model can be constructed semi-explicitly by sums over trees.

### 15.2 Deformations of $A_{\infty}$-categories

In this section, we recall deformations of $A_{\infty}$-categories. We follow Paper II where also more detail can be found. We start by recalling completed tensor products. Then we recall $A_{\infty}$-deformations and their functors. We comment very briefly on the construction of the twisted completion and minimal models for $A_{\infty}$-deformations from Paper II

We recall now completed tensor products $B \widehat{\otimes} X$ with $B$ a local ring and $X$ a vector space. The letter $B$ will always denote a local ring with extra properties. We have decided to give this a name:

Definition 15.7. A deformation base is a complete local Noetherian unital $\mathbb{C}$-algebra $B$ with residue field $B / \mathfrak{m}=\mathbb{C}$. The maximal ideal is always denoted $\mathfrak{m}$.
Remark 15.8. By the Cohen structure theorem, every deformation base is of the form $\mathbb{C} \llbracket x_{1}, \ldots, x_{n} \rrbracket / I$ with $I$ denoting some ideal.

If $X$ is a vector spaces, then $B \widehat{\otimes} X=\lim \left(B / \mathfrak{m}^{k} \otimes X\right)$ denotes the completed tensor product over $\mathbb{C}$. For simplicity, we write $\mathfrak{m}^{k} X$ to denote the infinitesimal part $\mathfrak{m}^{k} X=\mathfrak{m}^{k} \widehat{\otimes} X \subseteq B \widehat{\otimes} X$. Recall that $B \widehat{\otimes} X$ is a $B$-module and comes with the $\mathfrak{m}$-adic topology, which turns $B \widehat{\otimes} X$ into a sequential Hausdorff space. For convenience, we may from time to time use expressions like $x=\mathcal{O}\left(\mathfrak{m}^{k}\right)$ to indicate $x \in \mathfrak{m}^{k} X$.
Definition 15.9. A map $\varphi: B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$ is continuous if it is continuous with respect to the $\mathfrak{m}$-adic topologies. A map $\varphi:\left(B \widehat{\otimes} X_{k}\right) \otimes \ldots \otimes\left(B \widehat{\otimes} X_{1}\right) \rightarrow B \widehat{\otimes} Y$ is continuous if for every $1 \leq i \leq k$ and every sequence of elements $x_{1}, \ldots, \hat{x}_{i}, \ldots, x_{k}$ the map

$$
\mu\left(x_{k}, \ldots,-, \ldots, x_{1}\right): B \widehat{\otimes} X_{i} \rightarrow B \widehat{\otimes} Y
$$

is continuous.
Remark 15.10. Every element in $B \widehat{\otimes} X$ can be written as a series $\sum_{i=0}^{\infty} m_{i} x_{i}$. Here $m_{i}$ is a sequence of elements $m_{i} \in \mathfrak{m}^{\rightarrow \infty}$ and $x_{i}$ is a sequence of elements $x_{i} \in X$. We have used the notation $m_{i} \in \mathfrak{m}^{\rightarrow \infty}$ to indicate that $m_{i} \in \mathfrak{m}^{k_{i}}$ for some sequence $\left(k_{i}\right) \subseteq \mathbb{N}$ with $k_{i} \rightarrow \infty$.

Remark 15.11. Every $B$-linear map $B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$ is automatically continuous (see Paper II for the argument), so is every every $B$-multilinear map $\left(B \widehat{\otimes} X_{k}\right) \otimes \ldots \otimes\left(B \widehat{\otimes} X_{1}\right) \rightarrow B \widehat{\otimes} Y$. Linear maps $X \rightarrow B \widehat{\otimes} Y$ can be uniquely extended to $B$-linear maps $B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$ and multilinear maps $X_{k} \otimes \ldots \otimes X_{1} \rightarrow B \widehat{\otimes} Y$ can be uniquely extended to $B$-multilinear maps $\left(B \widehat{\otimes} X_{k}\right) \otimes \ldots \otimes\left(B \widehat{\otimes} X_{1}\right) \rightarrow B \widehat{\otimes} Y$ (see Paper II).

Remark 15.12. The leading term of a $B$-linear $\operatorname{map} \varphi: B \widehat{\otimes} X \rightarrow B \widehat{\otimes} Y$ is the map $\varphi_{0}: X \rightarrow Y$ given by the composition $\varphi_{0}=\left.\pi \varphi\right|_{X}$, where $\pi: B \widehat{\otimes} Y \rightarrow Y$ denotes the standard projection. If the leading term $\varphi_{0}$ is injective or surjective, then $\varphi$ is injective or surjective itself (see Paper II for the argument).

We recall now $A_{\infty}$-deformations. When $\mathcal{C}$ is an $A_{\infty}$-category, the idea is to model its $A_{\infty}$-deformations on the collection of enlarged hom spaces $\left\{B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}(X, Y)\right\}_{X, Y \in \mathcal{C}}$. Any $B$-multilinear product on these hom spaces is automatically continuous. Similarly, functors of $A_{\infty}$-deformations will be defined as maps between tensor products of the enlarged hom spaces and will be automatically continuous as well.
$A_{\infty}$-deformations of $\mathcal{C}$ will always be allowed to have infinitesimal curvature. The reason is that only this way we get a homologically sensible notion: Whenever $\mu_{q}$ is an (infinitesimally) curved deformation, then $\nu=\mu-\mu_{q}$ is a Maurer-Cartan element of the Hochschild DGLA $\operatorname{HC}(\mathcal{C})$. We comment on this in more detail in Paper II

Definition 15.13. Let $\mathcal{C}$ be an $A_{\infty}$ category with products $\mu$ and $B$ a deformation base. An $A_{\infty}-$ deformation of $\mathcal{C}_{q}$ of $\mathcal{C}$ consists of

- The same objects as $\mathcal{C}$,
- Hom spaces $\operatorname{Hom}_{\mathcal{C}_{q}}(X, Y)=B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}(X, Y)$ for $X, Y \in \mathcal{C}$,
- $B$-multilinear products of degree $2-k$

$$
\mu_{q}^{k}: \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{k}, X_{k+1}\right) \otimes \ldots \otimes \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{1}, X_{2}\right) \rightarrow \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{1}, X_{k+1}\right), k \geq 1
$$

- Curvature of degree 2 for every object $X \in \mathcal{C}$

$$
\mu_{q, X}^{0} \in \mathfrak{m} \operatorname{Hom}_{\mathcal{C}_{q}}^{2}(X, X)
$$

such that $\mu_{q}$ reduces to $\mu$ once the maximal ideal $\mathfrak{m}$ is divided out, and $\mu_{q}$ satisfies the curved $A_{\infty}\left(c A_{\infty}\right)$ relations

$$
\sum_{k \geq l \geq m \geq 0}(-1)^{\left\|a_{m}\right\|+\ldots+\left\|a_{1}\right\|} \mu_{q}\left(a_{k}, \ldots, \mu_{q}\left(a_{l}, \ldots\right), a_{m}, \ldots, a_{1}\right)=0
$$

The deformation is unital if the deformed higher products still satisfy the unitality axioms

$$
\mu_{q}^{2}\left(a, \operatorname{id}_{X}\right)=a, \mu_{q}^{2}\left(\operatorname{id}_{Y}, a\right)=(-1)^{|a|} a, \mu_{q}^{\geq 3}\left(\ldots, \operatorname{id}_{X}, \ldots\right)=0
$$

It sometimes comes handy to work with deformations that include more objects than $\mathcal{C}$ does. We fix terminology as follows:

Definition 15.14. Let $\mathcal{C}$ be an $A_{\infty}$-category. Let $O$ be an arbitrary set of objects and $F: O \rightarrow \mathrm{Ob}(\mathcal{C})$ a map. Then the object-cloned version $F^{*} \mathcal{C}$ of $\mathcal{C}$ is the $A_{\infty}$-category given by object set $O$, hom spaces

$$
\operatorname{Hom}_{F^{*} \mathcal{C}}(X, Y)=\operatorname{Hom}_{\mathcal{C}}(F(X), F(Y)), \quad X, Y \in O,
$$

and products simply given by the same composition as in $\mathcal{C}$.
An $A_{\infty}$-deformation of $\mathcal{C}$ always gives an induced deformation of $F^{*} \mathcal{C}$. This provides a map of MaurerCartan elements $\operatorname{MC}(\mathrm{HC}(\mathcal{C}), B) \rightarrow \mathrm{MC}\left(\mathrm{HC}\left(F^{*} \mathcal{C}\right), B\right)$. In case $F$ is surjective, the categories $\mathcal{C}$ and $F^{*} \mathcal{C}$ are equivalent and the map of Maurer-Cartan elements becomes a bijection after dividing out gauge equivalence. However, the map on raw Maurer-Cartan elements is not a bijection itself. After these comments, we are ready for the following terminology:

Definition 15.15. Let $\mathcal{C}$ be an $A_{\infty}$-category and $B$ a deformation base. Let $O$ be an arbitrary set and $F: O \rightarrow \mathrm{Ob} \mathcal{C}$ a map. An object-cloning deformation is a deformation $\mathcal{D}_{q}$ of $\mathcal{D}=F^{*} \mathcal{C}$. The object-cloning deformation is essentially surjective if $F: \operatorname{Ob} \mathcal{D} \rightarrow \mathrm{Ob} \mathcal{C}$ reaches all objects of $\mathcal{C}$ up to isomorphism.

We are now ready to explain the natural extension of $A_{\infty}$-functors to the deformed case.

Definition 15.16. Let $\mathcal{C}, \mathcal{D}$ be two $A_{\infty}$-categories and $\mathcal{C}_{q}, \mathcal{D}_{q}$ deformations. A functor of deformed $A_{\infty}$-categories consists of a map $F_{q}: \operatorname{Ob}(\mathcal{C}) \rightarrow \operatorname{Ob}(\mathcal{D})$ together with for every $k \geq 1$ a $B$-multilinear degree $1-k$ map

$$
F_{q}^{k}: \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{k}, X_{k+1}\right) \otimes \ldots \otimes \operatorname{Hom}_{\mathcal{C}_{q}}\left(X_{1}, X_{2}\right) \rightarrow \operatorname{Hom}_{\mathcal{D}_{q}}\left(F_{q} X_{1}, F_{q} X_{k+1}\right)
$$

and infinitesimal curvature $F_{q, X}^{0} \in \mathfrak{m} \operatorname{Hom}_{\mathcal{D}}^{1}\left(F_{q} X, F_{q} X\right)$ for every $X \in \mathcal{C}$, such that the curved $A_{\infty}$-functor relations hold:

$$
\begin{aligned}
\sum_{0 \leq j \leq i \leq k}(-1)^{\left\|a_{j}\right\|+\ldots+\left\|a_{1}\right\|} F_{q}\left(a_{k}, \ldots,\right. & \left.a_{i+1}, \mu_{q}\left(a_{i}, \ldots, a_{j+1}\right), a_{j}, \ldots, a_{1}\right) \\
& =\sum_{\substack{l \geq 0 \\
1=j_{1}<\ldots<j_{l} \leq k}} \mu_{q}\left(F_{q}\left(a_{k}, \ldots, a_{j_{l}}\right), \ldots, F_{q}\left(\ldots, a_{j_{2}}\right), F_{q}\left(\ldots, a_{j_{1}}\right)\right)
\end{aligned}
$$

If $\mathcal{C}_{q}$ and $\mathcal{D}_{q}$ are strictly unital, then we say $F_{q}$ is strictly unital if $F_{q}^{1}\left(\mathrm{id}_{X}\right)=\operatorname{id}_{F_{q} X}$ for every $X \in \mathcal{C}$ and $F_{q}^{\geq 2}\left(\ldots, \operatorname{id}_{X}, \ldots\right)=0$.
Remark 15.17. Note that the functor $F_{q}$ itself is allowed to have a curvature component. The first two curved $A_{\infty}$-functor relations read

$$
\begin{aligned}
F_{q}^{0}+F_{q}^{1}\left(\mu_{\mathcal{C}_{q}, X}^{0}\right)= & \mu_{\mathcal{D}_{q}}^{1}\left(F_{q, X}^{0}\right) \\
F_{q}^{1}\left(\mu_{\mathcal{C}_{q}}^{1}(a)\right)+(-1)^{\|a\|} F_{q}^{2}\left(\mu_{\mathcal{C}_{q}, Y}^{0}, a\right)+F_{q}^{2}\left(a, \mu_{\mathcal{C}_{q}, X}^{0}\right)= & \mu_{\mathcal{D}_{q}}^{1}\left(F_{q}^{1}(a)\right)+\mu_{\mathcal{D}_{q}}^{2}\left(F_{q, Y}^{0}, F_{q}^{1}(a)\right) \\
& +\mu_{\mathcal{D}_{q}}^{2}\left(F_{q}^{1}(a), F_{q, X}^{0}\right), \quad \forall a: X \rightarrow Y .
\end{aligned}
$$

If $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ is a functor of $A_{\infty}$-deformations, then its leading term $F: \mathcal{C} \rightarrow \mathcal{D}$ is automatically a functor of $A_{\infty}$-categories.

Twisted completions of $A_{\infty}$-deformations exist. When $\mathcal{C}_{q}$ is a deformation of $\mathcal{C}$, we can form a twisted completion $\mathrm{Tw} \mathcal{C}_{q}$, as we have elaborated in Paper II This category $\mathrm{Tw} \mathcal{C}_{q}$ is a deformation of $\mathrm{Tw} \mathcal{C}$. The objects of $\operatorname{Tw} \mathcal{C}$ are defined in terms of twisted differentials as well, but the twisted differentials do not satisfy the Maurer-Cartan equation with respect to the deformed product $\mu_{\mathcal{C}_{q}}$. Instead, the failure to satisfy the Maurer-Cartan equation is captured in the object's curvature. For more details we refer to Paper II

Minimal models of $A_{\infty}$-deformations exist. When $\mathcal{C}_{q}$ is a deformation of $\mathcal{C}$, we can form a minimal model $\mathrm{H} \mathcal{C}_{q}$. This category $\mathrm{H} \mathcal{C}_{q}$ is a deformation of HC . The differential and curvature of $\mathrm{H} \mathcal{C}_{q}$ need not vanish. Instead, $\mathrm{H} \mathcal{C}_{q}$ carries an infinitesimal residue differential and curvature. For more details we refer to Paper II

### 15.3 Submodules of completed tensor products

We introduce here the notions of pseudoclosed and quasi-flat submodules of $B \widehat{\otimes} X$ which we use in section 19 We also comment on intersections between submodules.

Let us start by introducing pseudoclosed submodules. The rationale is that not all $B$-submodules of $B \widehat{\otimes} X$ are created equal: Some are closed under taking power series with increasing powers of $\mathfrak{m}$, some are not. As preparation, we define the following notation:
Definition 15.18. Let $X$ be a vector space and $Y \subseteq B \widehat{\otimes} X$ a subspace. Then we put

$$
\begin{aligned}
B Y & :=\operatorname{Im}(B \widehat{\otimes} Y \rightarrow B \widehat{\otimes} X) \\
\mathfrak{m}^{k} Y & :=\operatorname{Im}\left(\mathfrak{m}^{k} \widehat{\otimes} Y \rightarrow B \widehat{\otimes} X\right)
\end{aligned}
$$

Here, the maps $B \widehat{\otimes} Y \rightarrow B \widehat{\otimes} X$ and $\mathfrak{m}^{k} \widehat{\otimes} Y \rightarrow B \widehat{\otimes} X$ denote the multiplication maps which send for instance $b \otimes y \mapsto b y$ and $m \otimes y \mapsto m y$.
Remark 15.19. Explicitly, the spaces $B Y$ and $\mathfrak{m}^{k} Y$ are given by elements of $B \widehat{\otimes} X$ that can be written respectively as

$$
\begin{aligned}
& x=\sum_{i=0}^{\infty} m_{i} y_{i} \in B \widehat{\otimes} X, \quad m_{i} \in \mathfrak{m}^{\rightarrow \infty}, y_{i} \in Y, \\
& x=\sum_{i=0}^{\infty} m_{i} y_{i} \in B \widehat{\otimes} X, \quad m_{i} \in \mathfrak{m}^{\geq k, \rightarrow \infty}, y_{i} \in Y .
\end{aligned}
$$

Here we use the notation $m_{i} \in \mathfrak{m}^{\rightarrow \infty}$ to indicate that there is a sequence $\left(k_{i}\right) \subseteq \mathbb{N}$ converging to $\infty$ such that $m_{i} \in \mathfrak{m}^{k_{i}}$. The notation $m_{i} \in \mathfrak{m}^{\geq k, \rightarrow \infty}$ indicates that we shall have $k_{i} \geq \bar{k}$.
Remark 15.20. We warn that the notation $B Y$ is suggestive but should not be misunderstood. The rationale behind the notation is that $B Y$ should contain any $B$-linear multiples of $Y$ and power series of such elements in growing $\mathfrak{m}$-order. We emphasize that $B Y$ is not the same as the linear span of products $b \cdot y$ for $b \in B$ and $y \in Y$. Similarly, $B Y$ is not the same as the $\mathfrak{m}$-adic closure of this span. The analogous warning holds for $\mathfrak{m}^{k} Y$. The intention of the notation is to provide foundations for section 19 .
Definition 15.21. An $B$-submodule $M \subseteq B \widehat{\otimes} X$ is pseudoclosed if $B M \subseteq M$.
Example 15.22. If $Y \subseteq X$ is a linear subspace, then $B \widehat{\otimes} Y \subseteq B \widehat{\otimes} X$ is pseudoclosed. In contrast, the $B$-submodule $B \otimes X \subseteq \bar{B} \widehat{\otimes} X$ is not pseudoclosed if $X$ is infinite-dimensional, since $B(B \otimes X)=B \widehat{\otimes} X$.

Denote by $B \cdot Y \subseteq B \widehat{\otimes} X$ the space (finitely) spanned by elements of the form by with $b \in B$ and $y \in Y$. Denote by $\mathfrak{m}^{k} \cdot Y$ the space (finitely) spanned by elements of the form $m y$ with $m \in \mathfrak{m}^{k}$ and $y \in Y$. In general, the spaces $B Y$ and $\mathfrak{m}^{k} Y$ are not the same as $B \cdot Y$ and $\mathfrak{m}^{k} \cdot Y$. Pseudoclosed submodules are an exception:
Lemma 15.23. Let $Y \subseteq B \widehat{\otimes} X$ be a pseudoclosed $B$-submodule. Then $B Y=B \cdot Y$ and $\mathfrak{m}^{k} Y=\mathfrak{m}^{k} \cdot Y$. Proof. The first statement is obvious since $B Y \subseteq Y \subseteq B \cdot Y$. For the second statement, the idea is to exploit the Cohen structure theorem. Write $B=\mathbb{C} \llbracket q_{1}, \ldots, q_{n} \rrbracket / I$, and regard the maximal ideal $\mathfrak{m}=\left(q_{1}, \ldots, q_{n}\right)$. With this in mind, we can write any element $x \in B Y$ as a series

$$
x=\sum_{i=0}^{\infty} m_{i} \tilde{m}_{i} y_{i}
$$

Here $m_{i}$ is a monomial of degree $k$ in the variables $q_{1}, \ldots, q_{n}$, the letter $\tilde{m}_{i}$ denotes a sequence $\tilde{m}_{i} \in \mathfrak{m} \rightarrow \infty$, and $y_{i} \in Y$. We conclude

$$
x=\sum_{\substack{\text { monomials } M \\ \text { of degree } k}} M \sum_{\substack{i \geq 0 \\ m_{i}=M}} \tilde{m}_{i} y_{i}
$$

The outer sum is finite. For every monomial $M$ of degree $k$, the inner sum is an element of $Y$ since $Y$ is pseudoclosed by assumption. In conclusion, every summand of the outer sum lies in $\mathfrak{m}^{k} \cdot Y$, and hence $x \in \mathfrak{m}^{k} \cdot Y$. We have shown that $\mathfrak{m}^{k} Y \subseteq \mathfrak{m}^{k} \cdot Y$. The inverse inclusion is obvious. We conclude $\mathfrak{m}^{k} Y=\mathfrak{m}^{k} \cdot Y$, finishing the proof.

Example 15.24. A simple application is the case of the pseudoclosed submodule $Y=B X=B \widehat{\otimes} X$. In this case, the lemma states $\mathfrak{m}^{k} X=\mathfrak{m}^{k}(B X)=\mathfrak{m}^{k} \cdot(B X)$ as subsets of $B \widehat{\otimes} X$.
Remark 15.25. We interpret Lemma 15.23 as follows: The space $\mathfrak{m}^{k} \cdot Y$ makes only reference to the $B$ module structure, while the space $\mathfrak{m}^{k} Y$ references a mixture of the $B$-module structure with the topology of the ambient space $B \widehat{\otimes} X$. For pseudoclosed modules, the topological part is already captured by the algebraic structure.

We introduce now a flatness condition for $B$-submodules of $B \widehat{\otimes} X$. This flatness condition is particularly relevant in section 19. To distinguish the notion from existing notions of flatness, we have chosen to name it quasi-flatness.
Definition 15.26. A $B$-submodule $M \subseteq B \widehat{\otimes} X$ is quasi-flat if $M \cap \mathfrak{m} X \subseteq \mathfrak{m} M$.
Remark 15.27. The inverse inclusion $\mathfrak{m} M \subseteq M \cap \mathfrak{m} X$ holds automatically if $M$ is pseudoclosed.
Lemma 15.28. Let $M \subseteq B \widehat{\otimes} X$ be a $B$-submodule. Assume $M$ is quasi-flat and pseudoclosed. Let $\varphi: \pi(M) \rightarrow M$ be a linear section of the projection map $\pi: M \rightarrow \pi(M)$. Then the $B$-linear extension $\varphi: B \widehat{\otimes} \pi(M) \rightarrow M$ is a $B$-linear isomorphism.
Proof. Injectivity follows from Remark 15.12, since the leading term is the identity. For surjectivity, let $x \in M$. We construct sequences $\left(x_{k}\right)$ and $\left(y_{k}\right)$ such that $x=x_{1}+\ldots+x_{N}+y_{N}$ for every $N \in \mathbb{N}$ and $x_{k} \in \varphi\left(\mathfrak{m}^{k} \pi(M)\right)$ and $y_{k} \in \mathfrak{m}^{k+1} \cap M$.

To start with, write $x=\varphi\left(x_{1}\right)+y_{1}$ for some $x_{1} \in \pi(M)$ and $y_{1} \in M \cap \mathfrak{m} X$. For induction hypothesis, assume the sequences are given until index $k$. Then note $y_{k} \in M \cap \mathfrak{m}^{k+1} X$. By quasi-flatness, we get $y_{k} \in \mathfrak{m}^{k+1} M$. We can then write $y_{k}=x_{k+1}+y_{k+1}$ with $x_{k+1} \in \varphi\left(\mathfrak{m}^{k} \pi(M)\right)$ and $y_{k+1} \in M \cap \mathfrak{m}^{k+1} X$. This finishes the inductive construction of the sequences.

Finally, we have $x=\sum_{k=1}^{\infty} x_{k} \in \varphi(B \widehat{\otimes} \pi(M))$. We have shown that $\varphi$ is surjective. This finishes the proof.

The following is a useful criterion to find quasi-flat modules:
Proposition 15.29. Let $B$ be a deformation base and $X$ a vector space. Let $M_{1}, \ldots, M_{k} \subseteq B \widehat{\otimes} X$ be $B$-submodules. Then the following are equivalent:

- We have $B \widehat{\otimes} X=M_{1} \oplus \ldots \oplus M_{k}$.
- The modules $M_{i}$ are all pseudoclosed and quasi-flat, and $X=\pi\left(M_{1}\right) \oplus \ldots \oplus \pi\left(M_{k}\right)$.

Here $\pi: B \widehat{\otimes} X \rightarrow X$ denotes the canonical projection map.
Proof. We first show that $B \widehat{\otimes} X=M_{1} \oplus \ldots \oplus M_{k}$ implies that all $M_{i}$ are quasi-flat and $X=\pi\left(M_{1}\right) \oplus$ $\ldots \oplus \pi\left(M_{k}\right)$. After that, we show the converse.

For the first part, assume $B \widehat{\otimes} X=M_{1} \oplus \ldots \oplus M_{k}$. Let $1 \leq i \leq n$. We first show that $M_{i}$ is closed. Indeed, regard the projection $p_{i}: B \widehat{\otimes} X \rightarrow B \widehat{\otimes} X$ to the component $M_{i}$. This map is clearly $B$-linear, hence id $-p_{i}$ is $B$-linear. By Remark 15.11 the map id $-p_{i}$ is continuous. The kernel of id $-p_{i}$ is $M_{i}$ and we conclude that $M_{i}$ is closed. In particular, $M_{i}$ is pseudoclosed.

Next, we show that every $M_{i}$ is quasi-flat. Pick any $x \in \mathfrak{m} X \cap M_{i}$. Since $x \in \mathfrak{m} X$ and $B \widehat{\otimes} X=M_{1}+$ $\ldots+M_{k}$, we can write $x=y_{1}+\ldots+y_{k}$ with $y_{j} \in \mathfrak{m} M_{j}$. By pseudoclosedness we have $y_{j} \in \mathfrak{m} M_{j} \subseteq M_{j}$. Since the sum $M_{1} \oplus \ldots \oplus M_{k}$ is direct and $x \in M_{i}$, we get $y_{j}=0$ for $j \neq i$. We conclude $x=y_{i} \in \mathfrak{m} M_{i}$. This proves every $M_{i}$ quasi-flat.

Let us show that $X=\pi\left(M_{1}\right)+\ldots+\pi\left(M_{k}\right)$. Pick any $x \in X$. Since $x$ then also lies in $B \widehat{\otimes} X$, write $x=$ $y_{1}+\ldots+y_{k}$ with $y_{i} \in M_{i}$. We conclude $x=\pi(x)=\pi\left(y_{1}\right)+\ldots+\pi\left(y_{k}\right)$, therefore $x \in \pi\left(M_{1}\right)+\ldots+\pi\left(M_{k}\right)$. Since $x$ was arbitrary, this shows $X=\pi\left(M_{1}\right)+\ldots+\pi\left(M_{k}\right)$.

Let us show that the sum $\pi\left(M_{1}\right)+\ldots+\pi\left(M_{k}\right)$ is direct. Assume by contradiction there is a sequence $y_{1}, \ldots, y_{k}$ with $y_{i} \in M_{i}$ and $\pi\left(y_{1}\right)+\ldots+\pi\left(y_{k}\right)=0$. Then $\pi\left(\sum y_{i}\right)=0$, in other words $\sum y_{i} \in \mathfrak{m} X$. By assumption we have $B \widehat{\otimes} X=M_{1}+\ldots+M_{k}$, in particuar $\mathfrak{m} X \subseteq \mathfrak{m}\left(M_{1}+\ldots+M_{k}\right)$. Therefore we can write $\sum y_{i}=\sum z_{i}$ with $z_{i} \in \mathfrak{m} M_{i}$. Since the sum $M_{1} \oplus \ldots \oplus M_{k}$ is direct, we have $y_{i}=z_{i}$ for all $i$. We conclude $y_{i}=z_{i} \in \mathfrak{m} X$ and therefore $\pi\left(y_{i}\right)=0$ for all $i$. This shows that the sum $\pi\left(M_{1}\right)+\ldots+\pi\left(M_{k}\right)$ is direct. The first implication is proven, finishing the first part of the proof.

For the second part of the proof, assume every $M_{i}$ is pseudoclosed and quasi-flat and $X=\pi\left(M_{1}\right) \oplus$ $\ldots \oplus \pi\left(M_{k}\right)$. Choose a linear section $\pi\left(M_{i}\right) \rightarrow M_{i}$ of the projection $\pi: M_{i} \rightarrow \pi\left(M_{i}\right)$. According to Lemma 15.28 the $B$-linear extension $\varphi_{i}: B \widehat{\otimes} \pi\left(M_{i}\right) \rightarrow M_{i}$ is an isomorphism. Add up all $\varphi_{i}$ to arrive at the map

$$
\begin{aligned}
\varphi: B \widehat{\otimes}\left(\pi\left(M_{1}\right) \oplus \ldots \oplus \pi\left(M_{k}\right)\right) & \rightarrow B \widehat{\otimes} X, \\
\left(x_{1}, \ldots, x_{k}\right) & \mapsto \varphi_{1}\left(x_{1}\right)+\ldots+\varphi_{k}\left(x_{k}\right)
\end{aligned}
$$

Note we view $\pi\left(M_{1}\right) \oplus \ldots \oplus \pi\left(M_{k}\right)$ simply as vector space decomposition of $X$. The map $\varphi$ is $B$-linear and continuous. Its leading term is by construction the identity on $X$. Therefore $\varphi$ is an isomorphism.

Using the auxiliary map $\varphi$, we get that $M_{1}+\ldots+M_{k}=B \widehat{\otimes} X$ : By definition of $\varphi$, its image is necessarily contained in $M_{1}+\ldots+M_{k}$ and we conclude that $M_{1}+\ldots+M_{k}=B \widehat{\otimes} X$.

Let us now show that the sum $M_{1}+\ldots+M_{k}$ is direct. Pick any sequence $x_{1}, \ldots, x_{k}$ such that $x_{i} \in M_{i}$ and $x_{1}+\ldots+x_{k}=0$. Since $\varphi_{i}$ surjects on $M_{i}$, write $x_{i}=\varphi_{i}\left(y_{i}\right)$ with $y_{i} \in B \widehat{\otimes} \pi\left(M_{i}\right)$. We get that $\varphi\left(y_{1}+\ldots+y_{k}\right)=0$. Since $\varphi$ is injective, we get $y_{1}+\ldots+y_{k}=0$, hence $y_{i}=0$ for all $i$. This shows $x_{i}=\varphi_{i}\left(y_{i}\right)=0$ and we conclude that the sum $M_{1} \oplus \ldots \oplus M_{k}$ is direct. This finishes the proof.

We finish this section by explaining a property regarding intersections of $B$-submodules. Whenever $X, Y \subseteq B \widehat{\otimes} A$ are two subspaces, we may ask: Does it hold that

$$
\mathfrak{m} X \cap \mathfrak{m} Y=\mathfrak{m}(X \cap Y) ?
$$

This inclusion does not hold in general, but we shall give here the best possible variant in case one of the spaces $X, Y$ is not deformed. Let us start with an example where the inclusion fails, as well as the example $B=\mathbb{C} \llbracket q \rrbracket$.
Example 15.30. Regard $B=\mathbb{C} \llbracket p, q \rrbracket$ and $A=\mathbb{C}[X]$. Let $X=\operatorname{span}(p x) \subseteq B \widehat{\otimes} A$ and $Y=\operatorname{span}(q x) \subseteq$ $B \widehat{\otimes} A$. Then $p q x$ lies in $\mathfrak{m} X \cap \mathfrak{m} Y$, but not in $\mathfrak{m}(X \cap Y)$.
Example 15.31. Regard $B=\mathbb{C} \llbracket q \rrbracket$ and arbitrary $A$. Let $X, Y \subseteq B \widehat{\otimes} A$ be pseudoclosed. We claim that $(q) X \cap(q) Y \subseteq(q)(X \cap Y)$. Indeed, pick $z \in(q) X \cap(q) Y$. By definition of $(q) X$, we can write

$$
z=\sum q^{\geq 1, \rightarrow \infty} x_{i}=q \sum q^{\rightarrow \infty} x_{i}, \quad z=\sum q^{\geq 1, \rightarrow \infty} y_{i}=q \sum q^{\rightarrow \infty} y_{i}
$$

Since $X$ and $Y$ are pseudoclosed, the two sums $\sum q^{\rightarrow \infty} x_{i}$ and $\sum q^{\rightarrow \infty} y_{i}$ lie in $X$ and $Y$, respectively. Write $z=q z^{\prime}$. Then both sums are equal to $z^{\prime}$, hence $z^{\prime}$ lies in the intersection of $X$ and $Y$. We conclude $z=q z^{\prime} \in q(X \cap Y)$. This shows $(q) X \cap(q) Y \subseteq(q)(X \cap Y)$.

Let us now make a more general statement. The idea is to keep one of the spaces $X, Y$ non-deformed. In other words, one of them is simply of the form $B \widehat{\otimes} V$ for some subspace $V \subseteq A$. Let us make this precise as follows:

Proposition 15.32. Let $X$ be a vector space and $B$ a deformation base. Let $V \subseteq X$ be a subspace and $M \subseteq B \widehat{\otimes} X$ a pseudoclosed $B$-submodule. Then

$$
\mathfrak{m}^{k} M \cap\left(\mathfrak{m}^{k} V+\mathfrak{m}^{k+1} X\right) \subseteq \mathfrak{m}^{k}(M \cap(V+\mathfrak{m} X))+\mathfrak{m}^{k+1} M
$$

Proof. In the first part of the proof, we illustrate the statement in case of $B=\mathbb{C} \llbracket q \rrbracket$. In the second part, we build a commutative diagram which allows us to deduce the shape of elements of $\mathfrak{m}^{k} M \cap \mathfrak{m}^{k} V$ without choice of basis for $B$. In the third part, we conclude the desired statement.

For the first step, let us illustrate the case of $B=\mathbb{C} \llbracket q \rrbracket$. Let $q^{k} x \in(q)^{k} M \cap\left((q)^{k} V+(q)^{k+1} X\right)$. In particular, we have $q^{k} x \in(q)^{k} M$, hence $x \in M$ since $M$ is pseudoclosed. We also have $q^{k} x \in$ $(q)^{k} V+(q)^{k+1} X$, hence $x \in \mathbb{C} \llbracket q \rrbracket V+(q) X$. Together this shows $q^{k} x \in q^{k}(M \cap(V+(q) X))$. We conclude that in case $B=\mathbb{C} \llbracket q \rrbracket$ the claimed statement holds.

For the second step, we build the following commutative diagram of linear maps:


Let us explain the maps. The horizontal map $\varphi$ performs a projection to $\mathfrak{m}^{k} / \mathfrak{m}^{k+1}$ on the first tensor factor and an inclusion of $M$ into $B \widehat{\otimes} X$ followed by projection to $X$ and projection to $X / V$ on the second tensor factor. The left vertical map $\pi \circ c$ consists of the inclusion $M \subseteq B \widehat{\otimes} X$ on the second tensor factor, followed by multiplication with the first tensor factor and projection to the quotient by $\mathfrak{m}^{k+1} X+\mathfrak{m}^{k} V$. The right vertical map $\psi$ is induced from the multiplication map.

We claim that the diagram is commutative and $\psi$ is an isomorphism. To see commutativity, pick an element $m \otimes x$ with $m \in \mathfrak{m}^{k}$ and $x \in M$. Under $\varphi$ it is sent to $[m] \otimes[x]$, which under $\psi$ is sent to [ $\left.m x\right]$. Under $\pi \circ c$, the element $m \otimes x$ is also sent to [ $m x$ ]. This demonstrates commutativity. To see that $\psi$ is an isomorphism, recall that $\mathfrak{m}^{k} X$ and $\mathfrak{m}^{k} \widehat{\otimes} X$ are isomorphic by means of the splitting map $\mathfrak{m}^{k} X \rightarrow \mathfrak{m}^{k} \widehat{\otimes} X$. This splitting map induces a map

$$
\frac{\mathfrak{m}^{k} X}{\mathfrak{m}^{k+1} X+\mathfrak{m}^{k} V} \rightarrow \frac{\mathfrak{m}^{k}}{\mathfrak{m}^{k+1}} \otimes \frac{X}{V}
$$

This map is an inverse of $\psi$. This shows that $\psi$ is an isomorphism.
For the third part of the proof, we conclude the desired inclusion. Let us start with the remark that the $\operatorname{kernel}$ of $\varphi$ is equal to

$$
\begin{equation*}
\operatorname{Ker}(\varphi)=\mathfrak{m}^{k+1} \widehat{\otimes} M+\mathfrak{m}^{k} \widehat{\otimes}(M \cap(V+\mathfrak{m} X)) \tag{15.1}
\end{equation*}
$$

Let now $\sum_{i=0}^{\infty} m_{i} x_{i} \in \mathfrak{m}^{k} M \cap\left(\mathfrak{m}^{k} V+\mathfrak{m}^{k+1} X\right)$ with $m_{i} \in \mathfrak{m} \geq k, \rightarrow \infty$ and $x_{i} \in M$. Then

$$
(\pi \circ c)\left(\sum m_{i} \otimes x_{i}\right)=\sum m_{i} \pi_{0}\left(x_{i}\right)=0 .
$$

Since $\pi \circ c=\psi \varphi$ and $\psi$ is injective, we get $\varphi\left(\sum m_{i} \otimes x_{i}\right)=0$. Therefore $\sum m_{i} \otimes x_{i}$ lies in the kernel of $\varphi$, which explicitly reads

$$
\sum m_{i} \otimes x_{i} \in \mathfrak{m}^{k+1} \widehat{\otimes} M+\mathfrak{m}^{k} \widehat{\otimes}(M \cap(V+\mathfrak{m} X))
$$

Contracting the tensors gives

$$
\sum m_{i} x_{i} \in \mathfrak{m}^{k+1} M+\mathfrak{m}^{k}(M \cap(V+\mathfrak{m} X)) .
$$

Since $\sum m_{i} x_{i}$ was arbitrarily chosen, this finishes the proof.

## 15.4 m-adically free modules

In this section, we recall the notion of $\mathfrak{m}$-adically free modules and their use in $A_{\infty}$-deformations. The reason is that in section 20 it comes very handy to use $A_{\infty}$-deformations modeled on $B$-modules which are only noncanonically isomorphic to $B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}(X, Y)$. In this section, we first recall the definition of $\mathfrak{m}$-adically free modules, then tie it back to quasi-flat and pseudoclosed modules. After that, we provide the definition of $A_{\infty}$-deformations that makes use of $\mathfrak{m}$-adically free modules.

We start by recalling the notion of $\mathfrak{m}$-adically free modules. We pick the following definition, as in 73:
Definition 15.33. A $B$-module $M$ is $\mathfrak{m}$-adically free if there is a vector space $X$ such that $M \cong B \widehat{\otimes} X$ as $B$-modules.

An abstract $B$-module $M$ enjoys an $\mathfrak{m}$-adic topology given by the neighborhood basis $x+\mathfrak{m}^{k} \cdot M$ for every $x \in M$ and $k \in \mathbb{N}$. We claim that when $M \cong B \widehat{\otimes} X$, then this topology is automatically compatible with the $\mathfrak{m}$-adic topology on $B \widehat{\otimes} X$ :

Lemma 15.34. Let $M$ be a $B$-module and $\varphi: M \rightarrow B \widehat{\otimes} X$ be a $B$-linear isomorphism. Then $\varphi$ is a homeomorphism.

Proof. By bijectivity and $B$-linearity of $\varphi$ we have $\varphi\left(\mathfrak{m}^{k} \cdot M\right)=\mathfrak{m}^{k} \cdot(B \widehat{\otimes} X)$. By Lemma 15.23 , we have $\mathfrak{m}^{k} \cdot(B \widehat{\otimes} X)=\mathfrak{m}^{k} X$. This shows that $\varphi$ is a homeomorphism.

Lemma 15.35. Let $M, N$ be $\mathfrak{m}$-adically free $B$-modules and $\varphi: M \rightarrow N$ a $B$-linear map. Then $\varphi$ is automatically continuous.
Proof. Since $\varphi$ is $B$-linear, we have $\varphi\left(\mathfrak{m}^{k} \cdot M\right) \subseteq \mathfrak{m}^{k} \cdot N$. This proves $\varphi$ continuous.
The ad-hoc quasi-flatness condition $M \cap \mathfrak{m} X \subseteq \mathfrak{m} M$ is related to $\mathfrak{m}$-adic freeness. While the former is a condition that makes explicit reference to the ambient space, the latter depends only on the abstract $B$-module structure. Both are not equivalent, but we provide here the closest tie we can get.

Proposition 15.36. Let $M \subseteq B \widehat{\otimes} X$ be a $B$-submodule. Then the following are equivalent:

- $M$ is quasi-flat and pseudoclosed.
- There is a $B$-linear isomorphism $B \widehat{\otimes} Y \rightarrow M$ with injective leading term $Y \rightarrow X$.

Proof. Assume $M$ is quasi-flat and pseudoclosed. Then according to lemma we get a $B$-linear isomorphism $B \widehat{\otimes} \pi(M) \xrightarrow{\sim} M$ with leading term the identity. This shows the claim, in particular $M$ is $\mathfrak{m}$-adically free.

Conversely, assume $M$ is $\mathfrak{m}$-adically free, presented by an injective leading term. Then we have an isomorphism $\varphi: B \widehat{\otimes} Y \rightarrow M \subseteq B \widehat{\otimes} X$. The map is automatically continuous. We show that $M$ is pseudoclosed: Let $\sum m_{i} x_{i}$ be a series with $m_{i} \in \mathfrak{m}^{\rightarrow \infty}$ and $x_{i} \in M$. Then write $x_{i}=\varphi\left(y_{i}\right)$. We get $\sum m_{i} y_{i} \in B \widehat{\otimes} Y$ and $\varphi\left(\sum m_{i} y_{i}\right)=\sum m_{i} \varphi\left(y_{i}\right)=\sum m_{i} x_{i}$. This shows $\sum m_{i} x_{i} \in \operatorname{Im}(\varphi)=M$. Hence $M$ is pseudoclosed.

To show that $M$ is quasi-flat, pick an element of $M \cap \mathfrak{m} X$, written in the form $\varphi(x) \in M \cap \mathfrak{m} X$ with $x \in B \widehat{\otimes} Y$. We claim $x \in \mathfrak{m} Y$. Write $x=y+z$ with $y \in Y$ and $z \in \mathfrak{m} Y$. Then $\varphi(x)=\varphi(y)+\varphi(z)$. Hence $\varphi(y) \in \mathfrak{m} X$. In particular $y$ vanishes under the leading term $\pi \varphi: Y \rightarrow X$. By assumption, the leading term is injective and we get $y=0$. This shows $\varphi(x)=\varphi(z) \in \mathfrak{m} M$. This shows $M \cap \mathfrak{m} X \subseteq \mathfrak{m} M$.
Remark 15.37. It is not true that $M \subseteq B \widehat{\otimes} X$ is quasi-flat and pseudoclosed if and only if it is $\mathfrak{m}$-adically free. For instance, let $X=\operatorname{span}\left(x_{1}, x_{2}\right)$ and $B=\mathbb{C} \llbracket q \rrbracket$. Regard the space $M=(q) x_{1}+\mathbb{C} \llbracket q \rrbracket x_{2}$. The module $M$ is $\mathfrak{m}$-adically free through the isomorphism $B \widehat{\otimes} X \rightarrow M$ given by $x_{1} \mapsto q x_{1}$ and $x_{2} \mapsto x_{2}$. However $M$ is not quasi-flat since $q x_{1} \in((q) X \cap M) \backslash(q) M$.

One can use $\mathfrak{m}$-adically free modules to model $A_{\infty}$-deformations. To be more precise, we have so far defined $A_{\infty}$-deformations as (infinitesimally curved) $A_{\infty}$-structures on the completed tensor product $B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}(X, Y)$. It is possible to also allow arbitrary $\mathfrak{m}$-adically free $B$-modules instead, under the condition that the quotient by $\mathfrak{m}$ is $\operatorname{Hom}_{\mathcal{C}}(X, Y)$. We greatly profit from this variant in section 20
Definition 15.38. Let $\mathcal{C}$ be a $\mathbb{Z}$-graded (or $\mathbb{Z} / 2 \mathbb{Z}$-graded) $A_{\infty}$-category. A loose $A_{\infty}$-deformation of $\mathcal{C}$ is a collection of $\mathfrak{m}$-adically free $\mathbb{Z}$-graded (or $\mathbb{Z} / 2 \mathbb{Z}$-graded) $B$-modules $\left\{\operatorname{Hom}_{\mathcal{C}_{q}}(X, Y)\right\}_{X, Y \in \mathcal{C}}$ together with $B$-multilinear maps $\mu_{q}^{k \geq 0}$ of degree $2-k$ satisfying the curved $A_{\infty}$-relations, together with linear isomorphisms $\psi_{X, Y}: \operatorname{Hom}_{\mathcal{C}_{q}}(X, Y) /\left(\mathfrak{m} \cdot \operatorname{Hom}_{\mathcal{C}_{q}}(X, Y)\right) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(X, Y)$ for every $X, Y$, such that $\mathcal{C}$ is obtained by dividing out $\mathfrak{m}$ and identification via $\left\{\psi_{X, Y}\right\}$.

Definition 15.39. Let $\mathcal{C}, \mathcal{D}$ be two $A_{\infty}$-categories and $\mathcal{C}_{q}, \mathcal{D}_{q}$ be loose $A_{\infty}$-deformations. A functor of loose $A_{\infty}$-deformations $F_{q}: \mathcal{C}_{q} \rightarrow \mathcal{D}_{q}$ consists of maps $F_{q}$ : $\mathrm{Ob} \mathcal{C} \rightarrow \mathrm{Ob} \mathcal{D}$ together with $B$-multilinear maps $F_{q}^{k \geq 0}$ such that $F_{q}$ satisfies the curved $A_{\infty}$-functor relations and $F_{q, X}^{0} \in \mathfrak{m} \cdot \operatorname{Hom}_{\mathcal{C}_{q}}\left(F_{q}(X), F_{q}(X)\right)$ for every $X \in \mathcal{C}$. The leading term of $F_{q}$ is the functor $F: \mathcal{C} \rightarrow \mathcal{D}$ obtained by dividing out $\mathfrak{m}$ and identification via $\left\{\psi_{X, Y}\right\}$. We may also say that $F_{q}$ is a deformation of $F$ in this case.
Remark 15.40. In contrast to $A_{\infty}$-deformations modeled on $B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}(X, Y)$, a loose $A_{\infty}$-deformation does not directly give a Maurer-Cartan element of the Hochschild DGLA HC(C). Instead, one first needs to make a choice of $B$-linear identification $\varphi_{X, Y}: \operatorname{Hom}_{\mathcal{C}_{q}}(X, Y) \cong B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}(X, Y)$ for every $X, Y \in \mathcal{C}$. Of course, the identification needs to be compatible with $\psi_{X, Y}$ in the sense that its leading term must be the identity when identifying $\operatorname{Hom}_{\mathcal{C}_{q}}(X, Y) /\left(\mathfrak{m} \cdot \operatorname{Hom}_{\mathcal{C}_{q}}(X, Y)\right)$ via $\psi_{X, Y}$. In yet other words, the following diagram needs to commute:


Once choices have been made, one obtains a Maurer-Cartan element $\mu_{q, \varphi} \in \operatorname{MC}(\operatorname{HC}(\mathcal{C}), B)$. Let us explain why two different choices $\varphi, \varphi^{\prime}$ yield gauge-equivalent Maurer-Cartan elements: Both $\varphi_{X, Y}$ and $\varphi_{X, Y}^{\prime}$ have leading term the identity when identified via $\psi_{X, Y}$, hence the composition

$$
\varphi_{X, Y}^{\prime} \circ \varphi_{X, Y}^{-1}: B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow B \widehat{\otimes} \operatorname{Hom}_{\mathcal{C}}(X, Y)
$$

has leading term the identity (without any identification). This shows that the two Maurer-Cartan elements $\mu_{q, \varphi}$ and $\mu_{q, \varphi^{\prime}}$ are related by the strict gauge functor $\varphi^{\prime} \circ \varphi^{-1}:\left(B \widehat{\otimes} \mathcal{C}, \mu_{q, \varphi}\right) \rightarrow\left(B \widehat{\otimes} \mathcal{C}, \mu_{q, \varphi^{\prime}}\right)$.

In analogy to Definition 15.15, we fix the following terminology:
Definition 15.41. Let $\mathcal{C}$ be an $A_{\infty}$-category and $B$ a deformation base. Let $O$ be an arbitrary set and $F: O \rightarrow \mathrm{Ob} \mathcal{C}$ a map. A loose object-cloning deformation is a loose deformation $\mathcal{D}_{q}$ of $\mathcal{D}=F^{*} \mathcal{C}$. The loose object-cloning deformation is essentially surjective if $F: \mathrm{Ob} \mathcal{D} \rightarrow \mathrm{Ob} \mathcal{C}$ reaches all objects of $\mathcal{C}$ up to isomorphism.

### 15.5 On the quasi-flatness condition

In this section, we present two alternative ways to formulate the quasi-flatness condition that we studied in section 15.3. This serves as a preparation for later use in section 19 The two alternatives for the quasi-flatness inclusion $M \cap \mathfrak{m} A \subseteq \mathfrak{m} M$ read as follows:
Definition 15.42. Let $M \subseteq B \widehat{\otimes} X$ be a $B$-submodule. Then $M$ satisfies the

- weak quasi-flatness inclusion if for every $k \geq 1$ we have $M \cap \mathfrak{m}^{k} X \subseteq \mathfrak{m}^{k} M+\mathfrak{m}^{k+1} X$.
- strong quasi-flatness inclusion if for every $k \geq 1$ we have $M \cap \mathfrak{m}^{k} X \subseteq \mathfrak{m}^{k} M$.

We claim these alternative inclusions are indeed equivalent to quasi-flatness and moreover that any quasi-flat $B$-submodule $M \subseteq B \widehat{\otimes} X$ is automatically a closed subspace of $B \widehat{\otimes} X$ :
Proposition 15.43. Let $X$ be a vector space and $B$ a deformation base. Let $M \subseteq B \widehat{\otimes} X$ be a $B$ submodule. If $M$ is pseudoclosed, then the following are equivalent:

- $M$ is quasi-flat.
- $M$ is quasi-flat and closed.
- $M$ satisfies the weak quasi-flatness inclusion.
- $M$ satisfies the strong quasi-flatness inclusion.

In the following lemmas, we provide a proof of Proposition 15.43
Remark 15.44. The proofs are easier to understand if one has the example of $B=\mathbb{C} \llbracket q \rrbracket$ in mind. This case has the practical property that any element $x \in \mathfrak{m} X$ can automatically be written as $q y$ for some $y \in X$. If for example it now becomes known that $x \in \mathfrak{m} M$, then it is immediate that $y \in M$. To see this, write $x \in(q) M$ as a power series in elements of $M$ and divide by $q$ :

$$
q y=x=\sum_{n=0}^{\infty} q^{n+1} x_{n}=q \cdot \sum_{n=0}^{\infty} q^{\rightarrow \infty} x_{n}, \quad \text { hence } \quad y=\sum_{n=0}^{\infty} q^{n} x_{n} \in M
$$

Lemma 15.45. Let $M \subseteq B \widehat{\otimes} X$ be a pseudoclosed $B$-submodule. If $M$ satisfies the weak quasi-flatness inclusion, then $M$ is quasi-flat.

Proof. It is our task to show $M \cap \mathfrak{m} X \subseteq \mathfrak{m} M$. Pick $x \in M \cap \mathfrak{m} X$. Iterating the weak quasi-flatness inclusion in combination with pseudoclosedness gives

$$
x \in M \cap \mathfrak{m} X \subseteq \mathfrak{m} M+M \cap \mathfrak{m}^{2} X \subseteq \mathfrak{m} M+\mathfrak{m}^{2} M+M \cap \mathfrak{m}^{3} X \subseteq \ldots
$$

More precisely, write $x=x_{1}+y_{1}$ with $x_{1} \in \mathfrak{m} M$ and $y_{1} \in M \cap \mathfrak{m}^{2} X$. Then write $y_{1}=x_{2}+y_{2}$ with $x_{2} \in \mathfrak{m}^{2} M$ and $y_{2} \in M \cap \mathfrak{m}^{3} X$. Continuing this way, we obtain sequences $\left(x_{k}\right)$ and $\left(y_{k}\right)$ with the property that

$$
x=\left(x_{1}+\ldots+x_{N}\right)+y_{N}, \quad x_{k} \in \mathfrak{m}^{k} M, y_{k} \in M \cap \mathfrak{m}^{k+1} X
$$

Letting $N \rightarrow \infty$ we get within $B \widehat{\otimes} X$ that

$$
x=\sum_{k=1}^{\infty} x_{k}
$$

In principle, the right-hand side converges within the completion of $M$. Since we assumed that $M$ is pseudoclosed, we can however do better: Every summand $x_{k}$ lies in $\mathfrak{m}^{k} M$ and summation starts at $k=1$. Therefore the infinite sum lies in $\mathfrak{m} M$. We conclude that $x \in \mathfrak{m} M$. This shows $M \cap \mathfrak{m} X \subseteq \mathfrak{m} M$.

Lemma 15.46. Let $M \subseteq B \widehat{\otimes} X$ be a quasi-flat $B$-submodule. Then $M$ satisfies the strong quasi-flatness inclusion.

Proof. The proof consists of two parts: We first prove the auxiliary inclusion $\mathfrak{m}^{k} M \cap \mathfrak{m}^{k+1} X \subseteq \mathfrak{m}^{k+1} M$ for every $k \geq 1$. Second, we derive the strong quasi-flatness inclusion by iterating the auxiliary inclusion.

First, let us prove the auxiliary inclusion. Denoting by $\pi: B \widehat{\otimes} X \rightarrow X$ the standard projection, regard the subspace $\pi(M) \subseteq X$. We can choose a $B$-linear continuous map $\varphi: B \widehat{\otimes} \pi(M) \rightarrow M$ with leading term the identity. Then any $x \in M$ can be written as $x=\varphi(y)+z$ with $z \in M \cap \mathfrak{m} \subseteq \mathfrak{m} M$.

Let $x \in \mathfrak{m}^{k} M \cap \mathfrak{m}^{k+1} X$. Then we can write $x=\varphi(y)+z$ with $y \in \mathfrak{m}^{k} \pi(M)$ and $z \in \mathfrak{m}^{k+1} M$. We get that $y-\varphi(y) \in \mathfrak{m}^{k+1} X$ and $\varphi(y)=x-z \in \mathfrak{m}^{k+1} X$. Summing up, we get $y \in \mathfrak{m}^{k+1} X \cap \mathfrak{m}^{k} \pi(M)$, hence $y \in \mathfrak{m}^{k+1} \pi(M)$. In consequence, we have $\varphi(y) \in \mathfrak{m}^{k+1} M$. Finally, we get $x=\varphi(y)+z \in \mathfrak{m}^{k+1} M$. This proves the auxiliary inclusion.

Finally, we combine quasi-flatness with iterated applications of the auxiliary inclusion:

$$
\begin{aligned}
M \cap \mathfrak{m}^{k} X & =(M \cap \mathfrak{m} X) \cap \mathfrak{m}^{k} X \\
& \subseteq\left(\mathfrak{m} M \cap \mathfrak{m}^{2} X\right) \cap \mathfrak{m}^{k} X \\
& \subseteq\left(\mathfrak{m}^{2} M \cap \mathfrak{m}^{3} X\right) \cap \mathfrak{m}^{k} X \\
& \subseteq \ldots \subseteq \mathfrak{m}^{k} M
\end{aligned}
$$

This finishes the proof.
Lemma 15.47. Let $M \subseteq B \widehat{\otimes} X$ be a pseudoclosed and quasi-flat $B$-submodule. Then $M$ is closed.
Proof. Let $\sum x_{n}$ be a series of elements $x_{n} \in M$ that converges in $B \widehat{\otimes} X$. Then $x_{n} \in \mathfrak{m} \rightarrow \infty X \cap M$. By the strong quasi-flatness inclusion, we get $x_{n} \in \mathfrak{m}^{\rightarrow \infty} M$. The limit of the series hence lies in $B M$. Since $M$ is pseudoclosed, the limit lies in $M$. We conclude that $M$ is closed.

The combination of the above lemmas proves Proposition 15.43 For $B$-submodules, pseudoclosed and quasi-flat implies closed, and closed implies pseudoclosed. However closed does not imply quasi-flat, for instance the submodule $M=\mathbb{C} \llbracket q \rrbracket x_{1}+(q) x_{2} \subseteq \mathbb{C} \llbracket q \rrbracket \widehat{\otimes} \operatorname{span}\left(x_{1}, x_{2}\right)$ is closed but not quasi-flat.

## 16 Preliminaries on Koszul duality

In the present section, we present Koszul duality as a preparation for section 20 Koszul duality is a phenomenon which provides a rich source of nontrivial $A_{\infty}$-functors by matching $A_{\infty}$-algebras and dg algebras. Classical Koszul duality involves only ordinary (associative, non-dg) algebras and makes besteffort statements on their homological properties. Modern Koszul duality concerns $A_{\infty}$-algebras and dg algebras and largely recovers classical Koszul duality from a more elegant description.
A-side B-side

| Augmented $A_{\infty}$-algebra | DG algebra |
| :---: | :---: |
| $A=\left(\mathbb{C}\right.$ id $\left.\oplus \mathbb{C} X_{1} \oplus \ldots \oplus \mathbb{C} X_{n}, \mu_{A}\right)$ | $A^{!}=\left(\mathbb{C}\left\langle\left\langle x_{1}^{\vee}, \ldots, x_{n}^{\vee}\right\rangle\right\rangle, \mu_{A}^{\vee}\right)$ |
| Cyclic $A_{\infty}$-structure | Calabi-Yau dg structure |
| Degree $n$ | Dimension $n$ |
| $A$-coderivation | Linear map |
| $m: T(\bar{A}[1]) \otimes M \rightarrow T(\bar{A}[1]) \otimes N$ | $m^{\vee}: M \rightarrow N \otimes A^{!}$ |
| $A$-module | Twisted complex |
| $\left(M, \mu_{M}\right)$ | $F(M)=\left(M \otimes A^{!}, \mu_{M, 0}^{\vee}\right)$ |

Technically, modern Koszul duality consists of dualizing the $A_{\infty}$-axioms on vector space level. The results can be described both abstractly in terms of dual vector spaces and the bar construction, as well as concretely by means of a choice of basis and a construction of dg structure on a power series algebra. In the present section, we follow the works of Ginzburg 34, Van den Bergh 70, 13, Lu-Wu-Palmieri-Zhang 51 and 14 .

In section 16.1 we recall $A_{\infty}$-modules and their categories. In section 16.2 we recall Koszul duality between augmented $A_{\infty}$-algebras and dg algebras and the Koszul duality functor. Insection 16.3, we recall several classical properties of Koszul duality and explain how they survive in modern Koszul duality. In section 16.4 , we recall Calabi-Yau dg algebras. In section 16.5, we focus on ordinary (non-dg) Calabi-Yau algebras. In section 16.6, we focus on the special case of ordinary Calabi-Yau algebras of dimension $n=3$. In section 16.7. we recall the notion of cyclic $A_{\infty}$-algebras and explain the correspondence between cyclic $A_{\infty}$-algebras and dg Calabi-Yau algebras via Koszul duality. In section 16.8 we tweak Koszul duality statements in order to motivate the Cho-Hong-Lau construction.

Remark 16.1. In this section, we partially follow the dg sign convention rather than the $A_{\infty}$-sign convention:

$$
\begin{aligned}
\mu^{1}\left(\mu^{2}(f, g)\right) & =\mu^{2}\left(\mu^{1}(f), g\right)+(-1)^{|f|} \mu^{2}\left(f, \mu^{1}(g)\right) \\
\mu^{2}\left(f, \mu^{2}(g, h)\right) & =\mu^{2}\left(\mu^{2}(f, g), h\right)
\end{aligned}
$$

The precise distribution is as follows: The $A_{\infty}$-algebra $A$ follows $A_{\infty}$-signs. The dg algebra $A^{!}$, the dg categories $\operatorname{Mod}_{\text {right }}^{\mathrm{fd}} A$ and $\mathrm{Tw} A^{!}$as well as the dg functor $F: \operatorname{Mod}_{\mathrm{right}}^{\mathrm{fd}} A \rightarrow \mathrm{Tw} A^{!}$follow dg signs. We amend all signs to the $A_{\infty}$-setting insection 20

### 16.1 Modules

In this section, we recall modules over $A_{\infty}$-algebras. We start by recalling the interpretation of $A_{\infty^{-}}$ structures as coderivations on the bar construction. Then we recall $A_{\infty^{-}}$-modules and categories of $A_{\infty^{-}}$ modules.

We start by recalling the tensor coalgebra construction. We denote by [1] the left-shift.
Definition 16.2. Let $A$ be an $A_{\infty}$-algebra. Regard the tensor algebra

$$
T(A[1])=\bigoplus_{n \in \mathbb{N}} A[1]^{\otimes n}
$$

The canonical coproduct $\Delta: T(A[1]) \rightarrow T(A[1]) \otimes T(A[1])$ is given by

$$
\begin{equation*}
\Delta\left(a_{k} \otimes \ldots \otimes a_{1}\right)=\sum_{0 \leq i \leq k}\left(a_{i} \otimes \ldots \otimes a_{1}\right) \otimes\left(a_{k} \otimes \ldots \otimes a_{i+1}\right) \tag{16.1}
\end{equation*}
$$

A coderivation $m: T(A[1]) \rightarrow T(A[1])$ is a linear map satisfying the co-Leibniz rule

$$
\Delta \circ m=(m \otimes \mathrm{id}+\mathrm{id} \otimes m) \circ \Delta
$$

Here the maps $m \otimes$ id and id $\otimes m$ bear the sign given by the Koszul sign rule:

$$
\begin{aligned}
(m \otimes \mathrm{id})\left(\left(a_{k} \otimes \ldots \otimes a_{1}\right) \otimes\left(b_{l} \otimes \ldots \otimes b_{1}\right)\right) & =m\left(a_{k} \otimes \ldots \otimes a_{1}\right) \otimes\left(b_{l} \otimes \ldots \otimes b_{1}\right) \\
(\mathrm{id} \otimes m)\left(\left(a_{k} \otimes \ldots \otimes a_{1}\right) \otimes\left(b_{l} \otimes \ldots \otimes b_{1}\right)\right) & =(-1)^{\left\|a_{1}\right\|+\ldots+\left\|a_{k}\right\|}\left(a_{k} \otimes \ldots \otimes a_{1}\right) \otimes m\left(b_{l} \otimes \ldots \otimes b_{1}\right) .
\end{aligned}
$$

Remark 16.3. The $A_{\infty}$-product $\mu$ on $A$ can be interpreted as a coderivation $\mu_{A}: T(A[1]) \rightarrow T(A[1])$ given by

$$
\mu_{A}\left(a_{k} \otimes \ldots \otimes a_{1}\right)=\sum_{0 \leq j<i \leq k}(-1)^{\left\|a_{1}\right\|+\ldots+\left\|a_{j}\right\|} a_{k} \otimes \ldots \otimes \mu\left(a_{i}, \ldots, a_{j+1}\right) \otimes \ldots \otimes a_{1}
$$

The $A_{\infty}$-relations for $\mu$ are equivalent to the condition $\mu_{A}^{2}=0$. One easily checks that $\mu_{A}$ is indeed a coderivation with respect to the coproduct $\Delta$. This check explains the awkward flip used in 16.1). If one uses "Polish notation" $\mu\left(a_{1}, \ldots, a_{k}\right)$ instead of $\mu\left(a_{k}, \ldots, a_{1}\right)$, one can avoid this flip (see 62, 26).

Definition 16.4. Let $A$ be an $A_{\infty}$-algebra. Then the bar construction B $A$ is the dg coalgebra structure on $T(A[1])$ given by the canonical coproduct $\Delta$ together with the coderivation $\mu_{A}$.

Modules over dg algebras comes with only an action map $A \otimes M \rightarrow A$ and a differential $M \rightarrow M$. When $A$ is an $A_{\infty}$-algebra, one allows the action maps to have higher components. For sake of section 16.2, we restrict here to defining right $A$-modules. Left $A$-modules are defined analogously.

Definition 16.5. Let $A$ be an $A_{\infty}$-algebra. Then a right $A$-module is a graded vector space $M$ together with a degree 1 map $\mu: M \otimes T(A[1]) \rightarrow M$ of satisfying the $A_{\infty}$-relations when combined with the product $\mu$ of $A$ in a suitable way:

$$
\begin{aligned}
\sum_{0 \leq i \leq k}(-1)^{\left\|a_{1}\right\|+\ldots+\left\|a_{i}\right\|} & \mu\left(\mu\left(m, a_{k}, \ldots, a_{i+1}\right), a_{i}, \ldots, a_{1}\right) \\
& +\sum_{0 \leq j<i \leq k}(-1)^{\left\|a_{1}\right\|+\ldots+\left\|a_{j}\right\|} \mu\left(m, a_{k}, \ldots, a_{i+1}, \mu\left(a_{i}, \ldots, a_{j+1}\right), a_{j}, \ldots, a_{1}\right)=0
\end{aligned}
$$

We shall only regard unital $A$-modules, in the sense that $\mu(m$, id $)=m$ and $\mu^{\geq 3}(m, \ldots$, id, $\ldots)=0$.
An $A$-module can be captured elegantly as a coderivation, comparable to the way that the $A_{\infty}$-product on $A$ can be captured via a coderivation:

Definition 16.6. Let $A$ be an $A_{\infty}$-algebra and $M$ a graded vector space. Then we regard the comodule $\operatorname{map} \Delta_{M}: M \otimes T(A[1]) \rightarrow T(A[1]) \otimes(M \otimes T(A[1])$ given by

$$
\Delta_{M}\left(m \otimes a_{k} \otimes \ldots \otimes a_{1}\right)=\sum_{0 \leq i \leq k}\left(a_{i} \otimes \ldots \otimes a_{1}\right) \otimes\left(m \otimes a_{k} \otimes \ldots \otimes a_{i+1}\right)
$$

A map $f: M \otimes T(A[1]) \rightarrow N \otimes T(A[1])$ is a coderivation if

$$
\Delta_{N} \circ f=(\mathrm{id} \otimes f) \circ \Delta_{M}
$$

In the context of coderivations, we denote by $\mu_{A}$ also the map $M \otimes T(A[1]) \rightarrow M \otimes T(A[1])$ given by

$$
\mu_{A}\left(a_{k} \otimes \ldots \otimes a_{1} \otimes m\right)=\sum_{0 \leq j<i \leq k}(-1)^{\left\|a_{1}\right\|+\ldots+\left\|a_{j}\right\|} m \otimes a_{k} \otimes \ldots \otimes \mu\left(a_{i} \otimes \ldots \otimes a_{j+1}\right) \otimes \ldots \otimes a_{1}
$$

Remark 16.7. Whenever $f: M \otimes T(A[1]) \rightarrow N \otimes T(A[1])$ is a coderivation, we can consider its projection to $N$ which we denote by $f_{0}: M \otimes T(A[1]) \rightarrow N$. Conversely, if $f_{0}: M \otimes T(A[1]) \rightarrow N$ is a graded linear map, we can turn it into a coderivation $f: M \otimes T(A[1]) \rightarrow N \otimes T(A[1])$. The precise correspondence between $f$ and $f_{0}$ reads

$$
f\left(m \otimes a_{k} \otimes \ldots \otimes a_{1}\right)=\sum_{0 \leq i \leq k}(-1)^{\left|f_{0}\right|\left(\left\|a_{1}\right\|+\ldots+\left\|a_{i}\right\|\right)} f_{0}\left(m \otimes a_{k} \otimes \ldots \otimes a_{i+1}\right) \otimes a_{i} \otimes \ldots \otimes a_{1}
$$

In terms of Definition 16.6, a right $A$-module is simply a coderivation $\mu_{M}: M \otimes T(A[1]) \rightarrow M \otimes T(A[1])$ of degree 1 such that $\left(\mu_{M}+\mu_{A}\right)^{2}=0$. The use of the letter $\mu_{A}$ is clearly an abuse of notation, but we expect no confusion to arise.

Capturing an $A$-module in terms of a coderivation $\mu_{M}$ makes it particularly straightforward to define a category of $A$-modules:

Definition 16.8. Let $A$ be an $A_{\infty}$-algebra. Then $\operatorname{Mod}_{\text {right }}^{\mathrm{fd}} A$ is the dg category of finite-dimensional right $A$-modules, with structure specified as follows:

- The hom space $\operatorname{Hom}_{\text {Mod }_{\text {right }}^{\mathrm{fd}} A}(M, N)$ is the space of coderivations $M \otimes T(A[1]) \rightarrow N \otimes T(A[1])$.
- The product $\mu_{\text {Mod right }_{\text {fd }}^{\mathrm{fd}} A}^{2}$ is ordinary composition.
- The differential measures failure to be a module morphism:

$$
\mu_{\mathrm{Mod}_{\mathrm{right}}^{\mathrm{fd}} A}^{1}(f)=\left(\mu_{A}+\mu_{N}\right) \circ f-(-1)^{|f|} f \circ\left(\mu_{A}+\mu_{M}\right) .
$$

Remark 16.9. It is readily checked by hand that $\mu_{\operatorname{Mod}_{\mathrm{right}}^{\mathrm{ff}} A}^{1}(f)$ is a coderivation if $f$ is a coderivation.
Bimodules are another important tool in homological algebra. They can be defined in a way analogous to left or right $A$-modules, see for instance 62, Section 2]. We come back to bimodules in the dg case in section 16.4.

### 16.2 Koszul duality

In this section, we recapitulate Koszul duality as a preparation to the Cho-Hong-Lau construction. Koszul duality is a construction connecting an $A_{\infty}$-algebra $A$ with a dg algebra $A^{!}$, its Koszul dual. Surprisingly, this construction also induces a correspondence between $A_{\infty}$-modules over $A$ and twisted complexes over $A^{!}$:

$$
F: \operatorname{Mod}_{\mathrm{right}}^{\mathrm{fd}} A \longrightarrow \mathrm{Tw} A^{!}
$$

Our aim is to recall as fast as possible that Koszul duality produces functors. For further details we refer to 51 and 16, Section 12.5].
Definition 16.10. An augmented $A_{\infty}$-algebra is an $A_{\infty}$-algebra $A$ with a decomposition $A=\bar{A} \oplus \mathbb{C}$ id such that $\mu\left(a_{k}, \ldots, a_{1}\right) \in \bar{A}$ whenever $a_{1}, \ldots, a_{k} \in \bar{A}$.

Remark 16.11. If $A$ is an augmented $A_{\infty}$-algebra, many constructions for $A$ can be carried out by working with the "augmented" tensor coalgebra $T(\bar{A}[1])$ instead of the full tensor coalgebra $T(A[1])$. For instance, to define an $A$-module it suffices to provide the map $T(\bar{A}[1]) \otimes M \rightarrow M$ instead of $T(A[1]) \otimes M \rightarrow$ $M$ since we only work with unital modules.

The canonical coproduct $\Delta: T(\bar{A}[1]) \rightarrow T(\bar{A}[1]) \otimes T(\bar{A}[1])$ and for a graded vector space $M$ the comodule map $\Delta_{M}: T(\bar{A}[1]) \otimes M \rightarrow T(\bar{A}[1]) \otimes T(\bar{A}[1]) \otimes M$ are defined as in the non-augmented case, this time restricting to $T(\bar{A}[1])$ instead of $T(A[1])$.

Remark 16.12. If $M$ is a graded vector space, we denote its graded dual vector space by $M^{\vee}$. The dual of $T(\bar{A}[1])$ is equal to

$$
T(\bar{A}[1])^{\vee}=\prod_{n=0}^{\infty}\left(\bar{A}[1]^{\vee}\right)^{\otimes n}
$$

Here we make the identification that reverses the order of tensor components:

$$
\begin{align*}
& V_{1}^{\vee} \otimes \ldots \otimes V_{k}^{\vee} \sim \\
& \varphi_{1} \otimes \ldots \otimes \varphi_{k} \longrightarrow\left(V_{k} \otimes \ldots \otimes V_{1}\right)^{\vee},  \tag{16.2}\\
&(-1)^{\sum_{1 \leq s<t \leq k}\left|\varphi_{s}\right|\left|\varphi_{t}\right|}\left[\left(x_{k} \otimes \ldots \otimes x_{1}\right) \mapsto \varphi_{1}\left(x_{1}\right) \ldots \varphi_{k}\left(x_{k}\right)\right] .
\end{align*}
$$

The dual of a map $m: T(\bar{A}[1]) \rightarrow T(\bar{A}[1])$ has the shape $m^{\vee}: T(\bar{A}[1])^{\vee} \rightarrow T(\bar{A}[1])^{\vee}$.
Definition 16.13. Let $A$ be a finite-dimensional augmented $A_{\infty}$-algebra. Then its Koszul dual $A^{!}$is the dg algebra given by

$$
A^{!}=T(\bar{A}[1])^{\vee}
$$

Upon the identification of $\widehat{16.2}$, the product on $A^{!}$is defined as the standard product on $\left.T \widehat{(\bar{A}[1]}{ }^{\vee}\right)$. The differential on $A^{!}$is given by

$$
d v=(-1)^{|v|+1} v \circ \mu_{A}, \quad \forall v \in T(\bar{A}[1])^{\vee}=\operatorname{Hom}_{\mathbb{C}}(T(\bar{A}[1]), \mathbb{C})
$$

Here $\mu_{A}: T(\bar{A}[1]) \rightarrow T(\bar{A}[1])$ denotes the product of $A$.
Example 16.14. Pick a basis $x_{1}, \ldots, x_{k}$ for $A$. Then we have the dual elements $x_{i}^{\vee}$ of degree $\left|x_{i}^{\vee}\right|=$ $1-\left|x_{i}\right|$. Multiplication within $A^{!}$can be performed simply as $x_{i}^{\vee} \cdot x_{j}^{\vee}=x_{i}^{\vee} x_{j}^{\vee}$. To interpret this element as element of $T(\bar{A}[1])^{\vee}$, we have to apply the sign flip from 16.2 :

$$
x_{i}^{\vee} x_{j}^{\vee}=(-1)^{\left|x_{i}^{\vee}\right|\left|x_{j}^{\vee}\right|}\left(x_{j} \otimes x_{i}\right)^{\vee}
$$

Here we have written $\left(x_{j} \otimes x_{i}\right)^{\vee}$ for the element of $A^{!}$which sends the element $x_{j} \otimes x_{i}$ to 1 and all other basis tensors to zero. Now write the product of $A$ as

$$
\mu^{l}\left(x_{i_{l}}, \ldots, x_{i_{1}}\right)=\sum_{1 \leq j \leq k} c_{i_{l}, \ldots, i_{1}}^{j} x_{j}
$$

Then

$$
\begin{aligned}
d_{A^{!}}\left(x_{j}^{\vee}\right) & =(-1)^{\left|x_{j}^{\vee}\right|+1} \sum_{\substack{l \geq 1 \\
1 \leq i_{l}, \ldots, i_{1} \leq k}} c_{i_{l}, \ldots, i_{1}}^{j}\left(x_{i_{l}} \otimes \ldots \otimes x_{i_{1}}\right)^{\vee} \\
& =(-1)^{\left|x_{j}^{\vee}\right|+1} \sum_{\substack{l \geq 1 \\
1 \leq i_{l}, \ldots, i_{1} \leq k}} c_{i_{l}, \ldots, i_{1}}^{j}(-1)^{\sum_{1 \leq s<t \leq l}\left|x_{i_{s}}^{\vee}\right|\left|x_{i_{t}}^{\vee}\right|} x_{i_{1}}^{\vee} \ldots x_{i_{l}}^{\vee} .
\end{aligned}
$$

Remark 16.15. The Koszul dual $A^{!}$is indeed a dg algebra. Abstractly speaking, the dual of the operator $\Delta$ is the ordinary product $\Delta^{\vee}: T(\bar{A}[1])^{\vee} \otimes T(\bar{A}[1])^{\vee} \rightarrow T(\bar{A}[1])^{\vee}$. Dualizing $\mu_{A}^{2}=0$ gives $\left(\mu_{A}^{\vee}\right)^{2}=0$ and dualizing the co-Leibniz rule for $\mu_{A}$ with respect to $\Delta$ gives the Leibniz rule for $\mu_{A}$ with respect to $\Delta^{\vee}$. The signs can be checked by hand.

Definition 16.16. Let $A$ be an augmented $A_{\infty}$-algebra. Let $M$ and $N$ be graded vector spaces and $f: M \otimes T(\bar{A}[1]) \rightarrow N$ a linear map. Then the Koszul transform of $f$ is the partial dual map $f^{\vee}: M \rightarrow N \otimes A^{!}$. It is a graded linear map with characterizing property

$$
\forall m \in M, a \in T(\bar{A}[1]): \quad\left\langle f^{\vee}(m), a\right\rangle=f(m \otimes a)
$$

Here $\langle-,-\rangle$ liberally denotes the standard pairing between $T(\bar{A}[1])^{\vee}$ and $T(\bar{A}[1])$, in this case as map

$$
\langle-,-\rangle:\left(N \otimes T(\bar{A}[1])^{\vee}\right) \otimes T(\bar{A}[1]) \rightarrow N .
$$

We copy Definition 15.2 and adapt it slightly to the dg case.
Definition 16.17. Let $D$ be a dg algebra. Then the category $\operatorname{Add} D$ is the category of formal shifted sums of copies of $D$ with hom spaces given as spaces of matrices with entries $D$. The differential $\mu_{\text {Add } D}^{1}$ and product $\mu_{\mathrm{Add} D}^{2}$ are the linear and bilinear extension of differential and product of $D$, with a sign change. On single matrix entries, the sign change is as follows:

$$
\begin{aligned}
\mu_{\text {Add } D}^{1}(a) & =(-1)^{k-l} d_{D} a, \quad \forall a: D[k] \rightarrow D[l], \\
\mu_{\operatorname{Add} D}^{2}(a, b) & =(-1)^{|a|_{D}(k-l)} a b, \quad \forall a: D[l] \rightarrow D[m], b: D[k] \rightarrow D[l] .
\end{aligned}
$$

Definition 16.18. Let $D$ be a dg algebra. Then a twisted complex over $D$ is an element $X \in \operatorname{Add} D$ together with an element $\delta \in \operatorname{End}_{\operatorname{Add} D}^{1}(X)$ such that $\delta$ is upper triangular and satisfies the Maurer-Cartan equation:

$$
\mu_{\mathrm{Add} D}^{1}(\delta)+\mu_{\mathrm{Add} D}^{2}(\delta, \delta)=0
$$

The differential and product on $\operatorname{Tw} D$ follow the sign rule

$$
\begin{aligned}
\mu_{\mathrm{Tw} D}^{1}(f) & =\mu_{\mathrm{Add} D}^{1}(f)+\mu_{\mathrm{Add} D}^{2}(\delta, f)+(-1)^{|f|+1} \mu_{\mathrm{Add} D}^{2}(f, \delta), \\
\mu_{\mathrm{Tw} D}^{2}(f, g) & =\mu_{\mathrm{Add} D}^{2}(f, g) .
\end{aligned}
$$

The product $\mu_{\text {Add } D}^{2}$ is comparable to a matrix product and similarly $\mu_{\text {Add } D}^{1}$ acts as entry-wise differential. As preparation for the Koszul duality functor, we prove a few properties regarding the Koszul transform. We show that the Koszul transform of $f \circ g$ agrees with the matrix product $\mu_{\text {Add } D}^{2}$ of $f^{\vee}$ and $g^{\vee}$ and that taking the Koszul transform of $f \circ \mu_{A}$ amounts to taking the entry-wise differential $\mu_{\text {Add } D}^{1}$ of $f^{\vee}$. The precise statement is as follows:

Lemma 16.19. Let $A$ be a finite-dimensional augmented $A_{\infty}$-algebra. Let $f: M \otimes T(\bar{A}[1]) \rightarrow N \otimes T(\bar{A}[1])$ and $g: L \otimes T(\bar{A}[1]) \rightarrow M \otimes T(\bar{A}[1])$ be coderivations. Then

$$
\begin{align*}
\mu_{\operatorname{Add} D}^{2}\left(f_{0}^{\vee}, g_{0}^{\vee}\right) & =(f \circ g)_{0}^{\vee}  \tag{16.3}\\
\mu_{\text {Add } D}^{1}\left(f_{0}^{\vee}\right) & =(-1)^{|f|+1}\left(f \circ \mu_{A}\right)_{0}^{\vee} \tag{16.4}
\end{align*}
$$

Proof. The simplest way to evaluate both statements is by choosing bases for $A, L, M$ and $N$. Let $X, Y \in T(\bar{A}[1])$ be basis elements homogeneous with respect to both tensor degree and $\|\cdot\|_{\bar{A}[1]}$. Let $\varepsilon_{L} \in L$ and $\varepsilon_{M} \in M$ and $\varepsilon_{N} \in N$ be further basis elements. Let $f$ and $g$ be given coderivations. They are determined solely by their first component $f_{0}$ and $g_{0}$. Since the statement is linear in $f$ and $g$, we may assume that $g_{0}\left(\varepsilon_{L} \otimes X\right)=\varepsilon_{M}$ and $f_{0}\left(\varepsilon_{M} \otimes Y\right)=\varepsilon_{N}$ and that $f_{0}$ and $g_{0}$ vanish on all other basis elements for $L \otimes T(\bar{A}[1])$ and $M \otimes T(\bar{A}[1])$. The Koszul transforms of $f, g$ read

$$
\begin{aligned}
f^{\vee}\left(\varepsilon_{M}\right) & =\varepsilon_{N} \otimes Y^{\vee}, \\
g^{\vee}\left(\varepsilon_{L}\right) & =\varepsilon_{M} \otimes X^{\vee} .
\end{aligned}
$$

Let us now show 16.3). We note that $(f \circ g)_{0}$ vanishes on all basis elements, except

$$
(f \circ g)_{0}\left(\varepsilon_{L} \otimes X \otimes Y\right)=(-1)^{|g|\|Y\|} f\left(g\left(\varepsilon_{L} \otimes X\right) \otimes Y\right)=(-1)^{|g|\left|Y^{\vee}\right|} \varepsilon_{N}
$$

Therefore $(f \circ g)^{\vee}$ and $f^{\vee} \cdot g^{\vee}$ vanish on all basis elements of $L$, except

$$
\begin{aligned}
(f \circ g)^{\vee}\left(\varepsilon_{L}\right) & =(-1)^{|g|\left|Y^{\vee}\right|} \varepsilon_{N} \otimes(X \otimes Y)^{\vee} \\
& =(-1)^{|g|\left|Y^{\vee}\right|+\left|X^{\vee}\right|\left|Y^{\vee}\right|} \varepsilon_{N} \otimes Y^{\vee} X^{\vee} \\
& =(-1)^{\left(\left|\varepsilon_{L}\right|-\left|\varepsilon_{M}\right|\right)\left|Y^{\vee}\right|} \varepsilon_{N} \otimes Y^{\vee} X^{\vee} \\
& =(-1)^{\left(\left|\varepsilon_{L}\right|-\left|\varepsilon_{M}\right|\right)\left|Y^{\vee}\right|}\left(f^{\vee} \cdot g^{\vee}\right)\left(\varepsilon_{L}\right) \\
& =\mu_{\text {Add } D}^{2}\left(f^{\vee}, g^{\vee}\right)\left(\varepsilon_{L}\right) .
\end{aligned}
$$

In the third row we have used that $|g|+\left|\varepsilon_{L}\right|=\left|X^{\vee}\right|+\left|X^{\vee}\right|$. This proves 16.3).
We now prove 16.4. It is our aim to compute the composition $\left(f \circ \mu_{A}\right)_{0}=f_{0} \circ \mu_{A}$. Since $f_{0}$ vanishes on all basis elements except $\varepsilon_{M} \otimes Y$ and 16.4 is linear in $\mu_{A}$ itself, we may simply assume that $\mu_{A}\left(\varepsilon_{M} \otimes Z\right)=\varepsilon_{M} \otimes Y$ for some basis element $Z \in T(\bar{A}[1])$ and $\mu_{A}$ vanishes on all other basis elements. Then $\left(f \circ \mu_{A}\right)_{0}$ vanishes on all basis elements of $L \otimes T(\bar{A}[1])$, except

$$
\left(f \circ \mu_{A}\right)_{0}\left(\varepsilon_{M} \otimes Z\right)=f_{0}\left(\varepsilon_{M} \otimes Y\right)=\varepsilon_{N}
$$

We see that $\left(f \circ \mu_{A}\right)_{0}{ }^{\vee}$ vanishes on all basis elements of $M$, except

$$
\begin{aligned}
\left(f \circ \mu_{A}\right)_{0}^{\vee}\left(\varepsilon_{M}\right) & =\varepsilon_{N} \otimes Z^{\vee} \\
& =(-1)^{\left|Y^{\vee}\right|+1} \varepsilon_{N} \otimes d Y^{\vee} \\
& =(-1)^{|f|+1} \mu_{\operatorname{Add} D}^{1}\left(f^{\vee}\right)\left(\varepsilon_{M}\right)
\end{aligned}
$$

In the last row, we have used that $\left|\varepsilon_{M}\right|+|f|=\left|\varepsilon_{N}\right|+\left|Y^{\vee}\right|$. This finishes the proof.
Remark 16.20. In Lemma 16.19, it is essential that $f$ and $g$ be coderivations. For instance, $\mu_{A}$ : $M \otimes T(\bar{A}[1]) \rightarrow M \otimes T(A[1])$ is not a coderivation on its own and in fact its zeroth component $\mu_{M, 0}$ vanishes, while a composition $f \circ \mu_{A}$ may again have nonvanishing zeroth component and therefore nonvanishing Koszul transform.

In Corollary 16.21, we recall the Koszul duality functor between modules over $A$ and twisted complexes over $A^{!}$. The idea is to apply "Koszul transform" the action map of every module. The construction is functorial, therefore gives rise to a dg functor.
Corollary 16.21. Let $A$ be a finite-dimensional augmented $A_{\infty}$-algebra. Then the following defines a dg functor:

$$
\begin{aligned}
F: \operatorname{Mod}_{\mathrm{right}}^{\mathrm{fd}} A & \longrightarrow \mathrm{Tw} A^{!}, \\
\left(M, \mu_{M}\right) & \longmapsto\left(M \otimes A^{!}, \mu_{M, 0}^{\vee}\right), \\
f & f_{0}^{\vee} .
\end{aligned}
$$

## We call $F$ the Koszul duality functor of $A$.

Proof. There are three items to check. First, we show that $\mu_{M, 0}^{\vee}$ satisfies the Maurer-Cartan equation. Then, we check that $F$ preserves differential and product. We start with the Maurer-Cartan equation:

$$
\begin{aligned}
\mu_{\text {Add } A^{!}}^{1}\left(\mu_{M, 0}^{\vee}\right)+\mu_{\text {Add } A^{!}}^{2}\left(\mu_{M, 0}^{\vee}, \mu_{M, 0}^{\vee}\right) & =\left(\mu_{M, 0} \circ \mu_{A}\right)^{\vee}+\left(\mu_{M, 0} \circ \mu_{M}\right)^{\vee} \\
& =\left(\mu_{M} \circ \mu_{A}+\mu_{M} \circ \mu_{M}+\mu_{A} \circ \mu_{M}\right)_{0} \vee=0
\end{aligned}
$$

In the second row, we have used that $\left(\mu_{A} \circ \mu_{M}\right)_{0}=0$. Next, for every coderivation $f: M \otimes T(\bar{A}[1]) \rightarrow$ $N \otimes T(\bar{A}[1])$ we have

$$
\begin{aligned}
F\left(\mu^{1}(f)\right) & =\left(\left(\mu_{N}+\mu_{A}\right) \circ f-(-1)^{|f|} f \circ\left(\mu_{M}+\mu_{A}\right)\right)_{0}^{\vee} \\
& =\left(\mu_{N} \circ f\right)_{0}{ }^{\vee}-(-1)^{|f|}\left(f \circ \mu_{M}\right)_{0}{ }^{\vee}-(-1)^{|f|}\left(f \circ \mu_{A}\right)_{0}{ }^{\vee} \\
& =\mu_{\mathrm{Add} A^{!}}^{2}\left(\mu_{N, 0}^{\vee}, f_{0}^{\vee}\right)-(-1)^{|f|} \mu_{\mathrm{Add} A^{!}}^{2}\left(f_{0}^{\vee}, \mu_{M, 0}^{\vee}\right)+\mu_{\mathrm{Add} A^{!}}^{1}\left(f_{0}^{\vee}\right) \\
& =\mu_{\mathrm{Tw} A^{!}}^{1}(F(f)) .
\end{aligned}
$$

If additionally $g: L \otimes T(\bar{A}[1]) \rightarrow M \otimes T(\bar{A}[1])$ is a coderivation, then

$$
F\left(\mu^{2}(f, g)\right)=F(f \circ g)=(f \circ g)_{0}^{\vee}=\mu_{\mathrm{Add} A^{!}}^{2}\left(f^{\vee}, g^{\vee}\right)=\mu_{\mathrm{Tw} A^{!}}^{2}(F(f), F(g))
$$

This shows that $F$ is a dg functor and finishes the proof.
Remark 16.22. Strictly speaking, the object $F(M)=\left(M \otimes A^{!}, \mu_{M, 0}^{\vee}\right)$ only becomes a twisted complex upon choice of a graded basis for $M$. Furthermore, $\mu_{M, 0}^{\vee}$ need not be an upper triangular matrix. However, if $\bar{A}$ is concentrated in positive degrees, then sorting the basis elements of $M$ in order of descending degree makes $\mu_{M, 0}^{\vee}$ upper triangular.

### 16.3 Classical Koszual duals

In this section, we comment on the relations of the modern with the classical Koszul dual construction. Classical Koszul duality is namely a phenomenon known for ordinary algebras and we recall here its typical properties: First, the double Koszul dual $\left(A^{!}\right)^{!}$is $A$ again. Second, the Koszul dual algebra is formal. Third, the Koszul algebra is the Ext algebra of its simple module $\mathbb{C}=A / \bar{A}$. In the present section, we recall these statements in the classical context and recall how they translate to modern Koszul duality. A valuable source is 51 .

Koszul duality has classically been a correspondence between Koszul algebras, a class of ordinary algebras with quadratic relations:


The relations on either side are the "orthogonal complement" of the relations on the other side along the pairing $(V \otimes V) \otimes\left(V^{\vee} \otimes V^{\vee}\right) \rightarrow \mathbb{C}$. In particular, classical Koszul duality is an involution from the beginning.

Koszul duality for $A_{\infty}$-algebras is not a one-way street either. If $A$ is an augmented finite-dimensional $A_{\infty}$-algebra, we regard the double dual $\left(A^{!}\right)^{!}$. It is possible that this dg algebra is quasi-isomorphic to $A$ itself. However, the double Koszul dual construction applies vector space duals twice so that finiteness conditions are required to match $A$ exactly with $\left(A^{!}\right)^{!}$.

A sufficient finiteness criterion can be formulated if one assumes that the $A_{\infty}$-algebra $A$ has an additional grading, also referred to as Adams grading. The direct sum decomposition $A=\mathbb{C}$ id $\oplus \bar{A}$ is supposed to be compatible and the products $\mu^{k}$ are assumed to be homogeneous with respect to the Adams grading. One then says that $A$ is Adams connected if the homogeneous part of $\bar{A}$ with respect to any Adams degree $j \in \mathbb{Z}$ is finite-dimensional and vanishes either for all $j \leq 0$ or all $j \geq 0$ 51. This connectedness assumption is a sufficient finiteness criterion and ensures that the double $\operatorname{Koszul}\left(A^{!}\right)^{!}$is quasi-isomorphic to $A$ again:
Theorem $16.23(51)$. Let $A$ be an augmented $A_{\infty}$-algebra. If $A$ is Adams connected, then $\left(A^{!}\right)^{!}$and $A$ are quasi-isomorphic as $A_{\infty}$-algebras.

The classical Koszul dual is automatically an ordinary graded algebra, without the need to pass to cohomology. From a modern perspective, Koszul duals are dg algebras and only passing to cohomology H $A^{!}$and forgetting the $A_{\infty}$-structure gives an ordinary graded algebra. Koszul algebras are certain types of algebras distinguished by the property that their Koszul dual $A^{!}$tends to be a formal dg algebra. This way, one can forget the higher structure on $\mathrm{H} A^{!}$and recover classical Koszul duality. For sake of completeness, we recall the definition here:

Definition $16.24(9)$. A graded associative algebra $A$ is Koszul if it is positively graded $A=\bigoplus_{i \geq 0} A^{i}$, we have $A^{0}=\mathbb{C}$ id and $\mathbb{C}=A / \bigoplus_{i>0} A^{i}$ as an $A$-module has a resolution of graded $A$-modules

$$
\ldots \rightarrow P^{2} \rightarrow P^{1} \rightarrow P^{0} \rightarrow A \rightarrow 0
$$

in which every $P^{i}$ is generated by its degree $i$ component: $P^{i}=A P_{i}^{i}$.
In 51 a precise criterion was given for formality:
Theorem 16.25 (51, Corollary 2.7]). Let $A$ be an $(a, b)$-generated Koszul algebra in the sense of 51, Definition 2.6]. Then we have an $A_{\infty}$-quasi-isomorphism $A^{!} \cong \mathrm{H}^{0} A^{!}$.
Remark 16.26. Classical Koszul duality is full of correspondences between types of algebras. Folklore statements include that Koszul algebras correspond to Koszul algebras, Gorenstein corresponds to Gorenstein, Artin-Schelter regular corresponds to Frobenius 51, Corollary D, E].
Remark 16.27. The classical analog of $\operatorname{Tw} \mathrm{H}^{0} A^{!}$is $\operatorname{Perf} \mathrm{H}^{0} A^{!}$. This is the reason why Koszul duality is classically formulated as triangulated equivalence between categories of the form $\mathrm{D} \operatorname{Mod} A$ and $\operatorname{Perf} \mathrm{H}^{0} A^{!}$, see for instance 51, Theorem B]. These classical Koszul duality functors are typically complicated for the reason that they do not take the possible higher structures on $\mathrm{H} A^{!}$into account. In the classical world, this is "solved" by restricting to Koszul algebras. Thanks to modern Koszul duality, it is possible to recover classical Koszul duality from the functor $\operatorname{Mod}^{\mathrm{fd}} A \rightarrow \mathrm{Tw} A^{!}$. The idea is to replace $A^{!}$by its zeroth cohomology, which is an ordinary algebra. Abstractly, we aim for a functor $\mathrm{Tw} A^{!} \rightarrow \mathrm{Tw} \mathrm{H}^{0} A^{!}$ in order to precompose it with the Koszul duality functor $\operatorname{Mod}^{\mathrm{fd}} A \rightarrow \operatorname{Tw} A^{!}$. We follow this route in section 16.8

Another classical statement of Koszul duality is that $A^{!}$can be interpreted as Ext algebra of the simple module $\mathbb{C}=A / \bar{A}$ of $A$. As we recall in Lemma 16.28 this is also the case if $A$ is an $A_{\infty}$-algebra.
 an augmented $A_{\infty}$-algebra. Then we denote by $\mathbb{C}$ the simple right $A$-module $\mathbb{C}=A / \bar{A}$. Its action map $\mathbb{C} \otimes T(\bar{A}[1]) \rightarrow \mathbb{C}$ is simply zero. In these terms, $A^{!}$is just the dg algebra $\operatorname{Hom}_{r M o d f d A}(\mathbb{C}, \mathbb{C})$ :

Lemma 16.28. Let $A$ be an augmented $A_{\infty}$-algebra. Then we have an isomorphism of dg algebras

$$
A^{!} \cong \operatorname{Hom}_{\operatorname{Mod}_{\mathrm{right}}^{\mathrm{fd}}} A(\mathbb{C}, \mathbb{C})
$$

Proof. This follows easily from Corollary 16.21 Indeed, the Koszul duality functor $F$ sends the simple module $\mathbb{C}$ to the twisted complex $\left(A^{!}, 0\right) \in \mathrm{Tw} A^{!}$and therefore establishes a map between the two dg endomorphism algebras. It is easy to see that $F$ is fully faithful. Since the endomorphism algebra of $\left(A^{!}, 0\right)$ is just $A^{!}$, we are done.

Remark 16.29. Similar to Lemma 16.28 let $\left(M, \mu_{M}\right)$ be a finite-dimensional right $A$-module. Then we have an isomorphism of right $A^{!}$-modules

$$
M \otimes A^{!} \cong \operatorname{Hom}_{\operatorname{Mod}_{\mathrm{right}}^{\mathrm{fd}}}(\mathbb{C}, M)
$$

The differential on $M \otimes A^{!}$is induced from induced from $\operatorname{Mod}_{\text {right }}^{\mathrm{fd}} A$ and the action of $A^{!}$is by (signed)
 on the right.

### 16.4 Calabi-Yau algebras

In this section, we recall Calabi-Yau algebras. This class of algebras is now widely recognized as a noncommutative analog of Calabi-Yau manifolds. The definition brings several technical difficulties and correspondingly a wide range of adaptations and variants have been introduced in the literature. In the present section, we follow the original definition of Ginzburg 34, in particular we focus on the dg case.

We start by recalling opposite and enveloping algebras.
Definition 16.30. Let $A$ be a dg algebra. Then the opposite algebra $A^{\mathrm{op}}$ of $A$ is obtained by setting $A^{\mathrm{op}}=A$ as vector space and defining

$$
d_{A^{\mathrm{op}}}(a)=-d_{A}(a), \quad a \cdot A^{\mathrm{op}} b=(-1)^{|a||b|} b a .
$$

The enveloping algebra $A^{\mathrm{e}}=A \otimes A^{\mathrm{op}}$ is the tensor product of $A$ and $A^{\mathrm{op}}$. It is an dg algebra itself with product $(a \otimes b)(c \otimes d)=(-1)^{|b||c|} a c \otimes b d$ and differential $d(a \otimes b)=d a \otimes b+(-1)^{|a|} a \otimes d b$.

Definition 16.31. Let $A$ be a dg algebra. Then an $A$-bimodule is the same as an $A^{e}$-module. The space $A$ becomes naturally a bimodule over $A$ by putting $(a \otimes b) x=(-1)^{|b||x|} a x b$ for $a \otimes b \in A^{\text {op }}$ and $x \in A$. The space $A^{\mathrm{e}}$ is also an $A$-bimodule, but in two different ways. The default action is the outer action given by

$$
(a \otimes b) \cdot(x \otimes y)=(-1)^{|b||x|} a x \otimes y b, \quad a \otimes b \in A^{\mathrm{e}}, x \otimes y \in A^{\mathrm{e}}
$$

The alternative is the inner action given by

$$
(a \otimes b) \cdot(x \otimes y)=(-1)^{|b||x|+|a||b|+|a||x|} x b \otimes a y, \quad a \otimes b \in A^{\mathrm{e}}, x \otimes y \in A^{\mathrm{e}}
$$

The dg category of dg modules over a dg algebra can be defined similar to the module category over an $A_{\infty}$-algebra. If $D$ is a dg algebra, we denote this dg category by $\operatorname{Mod} D$. It is not the same as the ordinary category with hom spaces the morphisms of dg modules, but rather the differential measures failure to be a dg morphism. To define $\operatorname{Mod} D$, one turns the dg algebra into an honest $A_{\infty}$ by means of the sign flip $\mu^{1}(a)=(-1)^{|a|}$ and $\mu^{2}(a, b)=(-1)^{|b|} a b$ and then forms $\operatorname{Mod} D$ according to the recipe presented in section 16.1 One may express the result more explicitly which we will not attempt here.

In particular, we can form the category $\operatorname{Mod} A^{e}$. When $M \in \operatorname{Mod}_{A^{e}}$, we can regard the hom space

$$
M^{\mathrm{D}}=\operatorname{Hom}_{\operatorname{Mod} A^{\mathrm{e}}}\left(M, A^{\mathrm{e}}\right)
$$

For us, it is most important that $M^{\mathrm{D}}$ is again a $\operatorname{dg} A^{\mathrm{e}}$-module, often called the dual bimodule of $M$. Its differential is the differential $\mu_{\mathrm{Mod} A^{\mathrm{e}}}^{1}$ and the $A^{\mathrm{e}}$-action on $M^{\mathrm{D}}$ is the inner action on the codomain $A^{\mathrm{e}}$. It is elementary to check that $M^{\mathrm{D}}$ is indeed a dg module. According to Kontsevich and Soibelman, the bimodule $A^{\mathrm{D}}$ is to be viewed as "inverse dualizing bibundle" $F \mapsto F \otimes K_{X}^{-1}[-\operatorname{dim} X]$ of the noncommutative manifold defined by $A$ 42, Definition 8.1.6].

When $M$ is a dg module, the $n$-th left shift $M[n]$ also becomes an object of Mod $A^{\mathrm{e}}$. Reflecting the definition of Calabi-Yau manifolds in the commutative world, the dg algebra $A$ is called Calabi-Yau if the dual bimodule $A^{\mathrm{D}}$ is quasi-isomorphic to a shift of $A$ :
Definition 16.32. Let $A$ be a dg algebra. Then $A$ is Calabi-Yau of dimension $n \geq 1(\mathrm{CY} n)$ if $A^{\mathrm{D}}[n]$ and $A$ are quasi-isomorphic in the category $\operatorname{Mod} A^{\mathrm{e}}$.
Remark 16.33. The original definition [34, Definition 3.2.3] requires that the quasi-isomorphism is a self-dual morphism. Van den Bergh has shown in 70, Proposition C.1] that this condition is typically automatic.

The definition of the hom space $\operatorname{Hom}_{A^{e}}\left(A, A^{e}\right)$ takes the degrees of the chosen resolution of $A$ and the degrees of $A^{\mathrm{e}}$ into account. In particular, requiring $\operatorname{Hom}_{A^{\mathrm{e}}}\left(A, A^{\mathrm{e}}\right)$ and $A[n]$ to be quasi-isomorphic cannot be easily translated into a property regarding resolutions on $A$. If $A$ is however an ordinary algebra (concentrated in degree zero), then the definition simplifies as follows:

Lemma $16.34(34,(3.2 .5)])$. Let $A$ be an associative algebra which has a finite projective $A$-bimodule resolution of finitely generated bimodules. Then $A$ is Calabi-Yau of dimension $n \geq 1$ if and only if

$$
\operatorname{HH}^{k}\left(A, A^{\mathrm{e}}\right) \cong \begin{cases}A & \text { if } k=n  \tag{16.5}\\ 0 & \text { else }\end{cases}
$$

Here $\mathrm{HH}^{k}\left(A, A^{\mathrm{e}}\right)$ is equipped again with the inner $A$-bimodule action and the isomorphism is meant as $A$-bimodules.

Proof. By definition $\operatorname{HH}^{k}\left(A, A^{\mathrm{e}}\right)$ is the cohomology of $\operatorname{Hom}_{A^{\mathrm{e}}}\left(A, A^{\mathrm{e}}\right)$, together with the additional $A$ bimodule action. We now prove both directions.

If $A$ is CYn then $\operatorname{Hom}_{A^{e}}\left(A, A^{\mathrm{e}}\right)$ and $A[n]$ are quasi-isomorphic as objects of $\operatorname{Mod} A^{\mathrm{e}}$. There exist closed morphisms $f$ and $g$ between them and a morphism $h$ of degree -1 such that $f g=\operatorname{id}+d(h)$. In particular $f$ and $g$ define quasi-inverse morphisms of complexes and we conclude the cohomology of $\operatorname{Hom}_{A^{\mathrm{e}}}\left(A, A^{\mathrm{e}}\right)$ is $A$, concentrated in degree $n$.

Conversely, assume 16.5 holds. The conclusion is a general statement regarding complexes. Let $P^{\bullet}$ be a complex with homology concentrated in a single degree, namely $\varphi: A \xrightarrow{\sim} \mathrm{H}^{k} P$. Then there is a quasi-isomorphism $\psi: A \rightarrow P^{\bullet}$ of complexes given by $a \mapsto \varphi(a) \in \mathrm{H}^{k} P \subseteq P^{k}$. This map is indeed a map of chain complexes since the differential on $A$ and on the image $\psi(\bar{A})$ vanishes. The map is a quasi-isomorphism since on cohomology this map is just $\varphi$. This finishes the proof.

Remark 16.35. In contrast to dg algebras, two $A_{\infty}$-algebras can not be tensored easily to form a tensor $A_{\infty}$-algebra. Instead, the construction is difficult and has attracted various literature 49, 55. This includes an advanced construction of the tensor product $A \otimes A^{\text {op }}$ as $A_{\infty}$-algebra.

This way, $A \otimes A^{\mathrm{op}}$ itself becomes an $A$-bimodule. This construction should not be confused with the naive bimodule action of $A$ mentioned in 56, Theorem 1.8]. In this naive action, $A$ acts on the two tensor components of the graded vector space $A \otimes A^{\circ \mathrm{p}}$ separately. In the present section, we avoid all difficulties by restricting to the case of $A$ being a dg algebra whenever we need the algebra $A^{\mathrm{e}}$ or $A \otimes A^{\mathrm{op}}$ as $A$-bimodule.

In the definition of a category of modules $\operatorname{Mod} A$, one is not limited to choosing the specific resolution $T(A[1]) \otimes M$ of $M$. Instead one may choose an arbitrary "projective replacement". In case $A$ is an $A_{\infty}$-algebra, this is a topic of research, but if $A$ is a dg algebra, then the correct notion of "projective replacement" is to be K-projective:

Definition 16.36 ([51, Section 4.2]). Let $A$ be a dg algebra. Then a dg module $P$ is K-projective if the $\operatorname{dg}$ hom space $\operatorname{Hom}_{\operatorname{Mod} A}(P, Q)$ is acyclic whenever $Q$ is an acyclic dg module. A K-projective replacement of $M \in \operatorname{Mod}^{\mathrm{fd}} A$ is a K-projective dg module $P$ together with a quasi-isomorphism $P \rightarrow M$.

Remark 16.37. Keller shows in 41, Section 3.2] that K-projective replacements can be found by resolving a module into a complex of so-called dg-projective modules. A module $P$ is called dg-projective if it is a cofibrant object with respect to the projective model structure on the category of dg categories. Explicitly, $P$ is dg-projective if for every surjective quasi-isomorphism $L \rightarrow M$, every morphism $P \rightarrow M$ factors through $L$.

Remark 16.38. The hom spaces in $\mathrm{Mod}^{\mathrm{fd}} A$ enjoy various names in the literature, thanks to the fact that their cohomology can also be built as derived hom functor. Generally, the cohomology hom spaces $\mathrm{H}^{\bullet} \operatorname{Hom}_{\operatorname{Mod}^{\mathrm{fd}} A}(M, N)$ may be denoted $\operatorname{RHom}_{A}^{\bullet}(M, N)$ or $\operatorname{Ext}_{A}^{\bullet}(M, N)$. When $M=N$, the dg hom space $\operatorname{Hom}_{\text {Mod }}{ }^{\text {fd }} A(M, M)$ is actually a dg algebra itself and its minimal model becomes an $A_{\infty}$-algebra. This $A_{\infty}$-algebra is commonly denoted $\operatorname{RHom}_{A}^{\bullet}(M, M)$ as well, despite the fact that classical derived hom functors do not retain homotopy information.

Remark 16.39. If $A$ is a dg algebra, then the Hochschild cohomology $\operatorname{HH}^{\bullet}(A, M)$ with coefficients in an $A$-bimodule $M$ is by definition equal to $\mathrm{H}^{\bullet} \operatorname{Hom}_{\operatorname{Mod} A^{\mathrm{e}}}(A, M)$.

### 16.5 Van den Bergh and Serre duality

In this section, we recall Van den Bergh and Serre duality for ordinary (non-dg) Calabi-Yau algebras. As it turns out, the Calabi-Yau property for ordinary algebras is closely related to having a self-dual bimodule resolution. This fact has been observed and exploited in 14 .

Definition $16.40(14)$. Let $A$ be an algebra and $0 \rightarrow P^{n} \rightarrow \ldots \rightarrow P^{0} \rightarrow A$ be a projective $A$-bimodule resolution of $A$ by finitely generated bimodules. Then $P^{\bullet}$ is a self-dual resolution if $\operatorname{Hom}_{A^{\mathrm{e}}}\left(P^{\bullet}, A^{\mathrm{e}}\right) \cong$ $P^{n-\bullet}$ as complexes of $A$-bimodules:


Lemma 16.41. If $A$ has a selfdual projective bimodule resolution of length $n \geq 1$ then $A$ is CYn.
Proof. Let $P^{\bullet}$ be a selfdual bimodule resolution. By definition, the Hochschild cohomology $\operatorname{HH}^{k}\left(A, A^{\mathrm{e}}\right)$ is the homology in degree $k$ of the complex $\operatorname{Hom}_{A^{\mathrm{e}}}\left(P^{\bullet}, A^{\mathrm{e}}\right)$. By selfduality, this homology is just the homology in degree $n-k$ of $P^{\bullet}$, which is $A$ if $k=n$ and zero otherwise. This shows that $A$ is CYn.

A bimodule resolution of $A$ gives rise to resolutions for all left and right $A$-modules. The idea is to tensor the resolution on the right or left side with $M$, respectively:

Lemma 16.42 (12, Lemma 2.4]). Let $A$ be an algebra and $P^{\bullet} \rightarrow A$ a projective bimodule resolution. Let $M$ be a left $A$-module. Then $P^{\bullet} \otimes_{A} M \rightarrow M$ is a resolution for $M$.

Van den Bergh duality is a Poincaré-style theorem for Hochschild homology and cohomology of CYn algebras. This duality was first observed in 13 and we recall it as follows:

Theorem 16.43 ( $\boxed{13})$. Let $A$ be a CYn algebra. If $A$ has a finite projective resolution of finitely generated bimodules, then

$$
\operatorname{HH}^{k}(A, M) \cong \operatorname{HH}_{n-k}(A, M)
$$

Proof. We recall here the proof in the easy case where $A$ has a selfdual resolution $P^{\bullet}$. We compute

$$
\begin{aligned}
\mathrm{HH}^{k}(A, M) & =\mathrm{H}^{k} \operatorname{Hom}_{A^{\mathrm{e}}}\left(P^{\bullet}, M\right) \\
& =\mathrm{H}^{k}\left(\operatorname{Hom}_{A^{\mathrm{e}}}\left(P^{\bullet}, A^{\mathrm{e}}\right) \otimes_{A^{\mathrm{e}}} M\right) \\
& =\mathrm{H}^{k}\left(P^{n-\bullet} \otimes_{A^{\mathrm{e}}} M\right)=\operatorname{HH}_{n-k}(A, M)
\end{aligned}
$$

As we recapitulate in Lemma 16.44 a CYn algebra has the property that the $n$-th shift is a Serre functor for its derived category. In [14, CYn algebras were precisely defined by this characteristic property. It is unclear to which extent the definitions are equivalent.

Lemma 16.44 ( $\sqrt[12]{ }$, Proposition 2.3]). Let $A$ be an algebra with a finite projective resolution of finitely generated bimodules. If $A$ is CYn, then for all finite-dimensional $A$-modules $M, N$ there are natural isomorphisms of graded vector spaces

$$
\operatorname{Ext}_{A}^{\bullet}(M, N)^{\vee} \cong \operatorname{Ext}_{A}^{n-\bullet}(N, M)
$$

Proof. We recall here the proof from 12 . The first step of the proof is to realize that thanks to Lemma 16.42. $\operatorname{Ext}_{A}^{k}(M, N)$ equals the Hochschild cohomology $\operatorname{HH}^{k}\left(A, \operatorname{Hom}_{\mathbb{C}}(M, N)\right)$ :

$$
\operatorname{Ext}_{A}^{k}(M, N)=\mathrm{H}^{k} \operatorname{Hom}_{A}\left(P^{\bullet} \otimes M, N\right)=\mathrm{H}^{k} \operatorname{Hom}_{A^{\mathrm{e}}}\left(P^{\bullet}, \operatorname{Hom}_{\mathbb{C}}(M, N)\right)=\operatorname{HH}^{k}\left(A, \operatorname{Hom}_{\mathbb{C}}(M, N)\right)
$$

Here $\operatorname{Hom}_{\mathbb{C}}(M, N)$ is the $A$-bimodule with the left factor acting on $N$ and the right factor acting on $M$ : $(a \varphi b)(m)=a \varphi(b m)$.

The second step is to show $\operatorname{HH}^{k}\left(A, M^{\vee}\right) \cong \operatorname{HH}_{k}(A, M)^{\vee}$ for finite-dimensional $A$-bimodules $M$. Here the vector space dual $M^{\vee}$ is also an $A$-bimodule by letting the left factor act from the right on $M$ and the right factor act from the left on $M$, explicitly $(a \varphi b)(m)=\varphi(b m a)$. We compute

$$
\operatorname{HH}^{k}\left(A, M^{\vee}\right)=\mathrm{H}^{k} \operatorname{Hom}_{A^{\mathrm{e}}}\left(P^{\bullet}, \operatorname{Hom}_{\mathbb{C}}(M, \mathbb{C})\right)=\operatorname{Hom}_{\mathbb{C}}\left(P^{\bullet} \otimes_{A^{\mathrm{e}}} M, \mathbb{C}\right)=\operatorname{HH}_{k}(A, M)^{\vee}
$$

Combining the first two steps with Van den Berg duality, we conclude

$$
\begin{aligned}
\operatorname{Ext}_{A}^{k}(M, N) & \cong \operatorname{HH}^{k}\left(A, \operatorname{Hom}_{\mathbb{C}}(M, N)\right) \\
& =\operatorname{HH}^{k}\left(A, \operatorname{Hom}_{\mathbb{C}}(N, M)^{\vee}\right) \\
& \cong \operatorname{HH}_{k}\left(A, \operatorname{Hom}_{\mathbb{C}}(N, M)\right)^{\vee} \\
& \cong \operatorname{HH}^{n-k}\left(A, \operatorname{Hom}_{\mathbb{C}}(N, M)\right)^{\vee} \\
& \cong \operatorname{Ext}_{A}^{n-k}(N, M)^{\vee}
\end{aligned}
$$

This finishes the proof.

Remark 16.45. The duality between the two Ext spaces can also be interpreted as a pairing

$$
\langle-,-\rangle: \operatorname{Ext}_{A}^{\bullet}(M, N) \times \operatorname{Ext}_{A}^{n-\bullet}(N, M) \rightarrow \mathbb{C} .
$$

This pairing can also be expressed by means of the traces $\operatorname{Tr}_{M}, \operatorname{Tr}_{N}$ on $\operatorname{Ext}_{A}^{\bullet}(M, M)$ and $\operatorname{Ext}_{A}^{\bullet}(N, N)$ as

$$
\langle x, y\rangle=\operatorname{Tr}_{N}(x \circ y)=(-1)^{|x||y|} \operatorname{Tr}_{M}(y \circ x), \quad x \in \operatorname{Ext}_{A}^{\bullet}(M, N), \quad y \in \operatorname{Ext}_{A}^{n-\bullet}(N, M)
$$

Here $\circ$ denotes the composition of extensions, equivalently the product of $\operatorname{Hod} A$. The trace $\operatorname{Tr}_{M}$ : $\operatorname{Ext}_{A}^{\bullet}(M, M) \rightarrow \mathbb{C}$ is induced from the trace on the complex $\operatorname{Hom}_{A}\left(P^{\bullet} \otimes M, P^{\bullet} \otimes M\right)$ which computes Ext ${ }^{\bullet}(M, M)$. The correct signs were computed in 14, Appendix].

### 16.6 Jacobi algebras

In this section, we recall characterizations of ordinary (non-dg) CY3 algebras. The core idea is that most CY3 algebras are the Jacobi algebra of a quiver with superpotential. Conversely, most quivers with superpotential gives rise to a CY3 algebra. In the present section, we recall the precise criteria from 14. In particular, we recall superpotentials and their associated Jacobi algebras as well as a candidate bimodule resolution for Jacobi algebras. This section serves as a direct preparation for section 19

We start by fixing terminology for cyclic elements of quiver algebras as follows:
Definition 16.46. Let $Q$ be a quiver. A path in $Q$ is a cycle if it starts and ends at the same vertex. If $p$ is a cycle in $Q$, we denote by $p_{\text {cyc }} \in \mathbb{C} Q$ the sum of its cyclic permutations. We extend this assignment linearly to $\mathbb{C} Q$ and denote it by $p \mapsto p_{\text {cyc }}$ as well. An element $W \in \mathbb{C} Q$ is cyclic if it lies in the image of this map. Explicitly, $W$ is cyclic if it is a linear combination of cycles whose coefficients are invariant under cyclic permutation:

$$
W=\sum_{\text {cycles } a_{k} \ldots a_{1}} \lambda_{a_{k} \ldots a_{1}} a_{k} \ldots a_{1}, \quad \text { with } \quad \forall i=1, \ldots, k: \quad \lambda_{a_{k} \ldots a_{1}}=\lambda_{a_{i-1} \ldots a_{i+1} a_{i}} .
$$

A superpotential on a quiver $Q$ is defined as a cyclic element $W \in \mathbb{C} Q$ which is a linear combination of paths of length at least two:
Definition 16.47. A superpotential is a cyclic element $W \in \mathbb{C} Q_{\geq 2}$. Its relations are the elements

$$
\partial_{a} W=\sum_{\substack{\text { paths } a_{k} \ldots a_{1} \\ \text { with } a_{k}=a}} \lambda_{a_{k} \ldots a_{1}} a_{k-1} \ldots a_{1}, \quad a \in Q_{1}
$$

Its Jacobi algebra is given by

$$
\operatorname{Jac}(Q, W)=\frac{\mathbb{C} Q}{\left(\partial_{a} W\right)}
$$

Here $\left(\partial_{a} W\right)$ denotes the two-sided ideal generated by the partial derivatives $\partial_{a} W$ for $a \in Q_{1}$.
Remark 16.48. A typical assumption in the literature is that the paths are of length at least three. In fact, length two term gives rise to a single arrow being contained in one relation $\partial_{a} W$. The effect is that this arrow is killed in the Jacobi algebra.

The original paper of Ginzburg 14 formulated the expectation that all CY3 algebras "appearing in nature" are Jacobi algebras of quivers with superpotential. This expectation was largely verified in 14, with the core result that a quiver algebra with graded relations which is CY3 is necessarily of the form $\operatorname{Jac}(Q, W)$ :

Theorem 16.49 ( 14 , Theorem 3.1]). Let $Q$ be a quiver and let $A=\mathbb{C} Q / I$ be the quotient by a finitely generated graded ideal $I \subseteq \mathbb{C} Q_{\geq 2}$. If $A$ is CY3, then there exists a superpotential $W$ such that $\mathbb{C} Q / I \cong \operatorname{Jac}(Q, W)$.

The idea of the proof is to explore the structure of $I$ in terms of resolutions for the simple modules of $A$. These first bits of these resolutions are standard and do not depend on the ideal $I$. In contrast, the last bits depend on $I$ but can be guessed by consideration on the dimension of the Ext spaces based on the assumption that $A$ is CY3.

Conversely, not every Jacobi algebra is CY3. There is however a precise criterion due to 14 as well. The criterion is formulated in terms of a "candidate" bimodule resolution for $A$. We start with the following notation:
Definition 16.50. Let $W \in \mathbb{C} Q_{\geq 2}$ be a superpotential. Then the $\mathbb{C} Q_{0}$-bimodule generated by $W$ in $\mathbb{C} Q$ is denoted

$$
W=\mathbb{C} Q_{0} W Q_{0}=\bigoplus_{v \in Q_{0}} \mathbb{C} v W v \subseteq \mathbb{C} Q
$$

The relations space is denoted

$$
R=\operatorname{span}\left\{\partial_{a} W \mid a \in Q_{1}\right\}
$$

To introduce the candidate bimodule resolution, let $Q$ be a quiver, $W \in \mathbb{C} Q_{\geq 3}$ a superpotential. Let us temporarily write $A=\operatorname{Jac}(Q, W)$ for the Jacobi algebra. The candidate bimodule resolution has the shape

$$
\begin{equation*}
0 \rightarrow A \underset{\mathbb{C} Q_{0}}{\otimes} W \underset{\mathbb{C} Q_{0}}{\otimes} A \xrightarrow{g_{1}} A \underset{\mathbb{C} Q_{0}}{\otimes} R \underset{\mathbb{C} Q_{0}}{\otimes} A \xrightarrow{g_{2}} A \underset{\mathbb{C} Q_{0}}{\otimes} \underset{\mathbb{C} Q_{1}}{\mathbb{C} Q_{0}} \otimes \underset{\rightarrow}{\otimes} A \underset{\mathbb{C} Q_{0}}{\otimes} A \rightarrow A \rightarrow 0 \tag{16.6}
\end{equation*}
$$

Remark 16.51. The maps in the sequence 19.1 are described as follows:

- For the map $g_{1}$, let $w=\sum_{i \in I} r_{i} a_{i}=\sum_{j \in J} b_{i} r_{j} \in W$, where all $a_{i}$ an $b_{i}$ are arrows and the $r_{i}, r_{j} \in R$ are compatible elements. The map $g_{1}$ sends the element $1 \otimes w \otimes 1$ to

$$
g_{1}(1 \otimes w \otimes 1)=\sum_{i \in I} 1 \otimes r_{i} \otimes a_{i}-\sum_{j \in J} b_{j} \otimes r_{j} \otimes 1 .
$$

- For the map $g_{2}$, let $r \in R$ and for $d \geq 0$ write $r=\sum_{i \in I_{d}} p_{i}^{(d)} a_{i}^{(d)} q_{i}^{(d)} \in R$, where $p_{i}^{(d)}$ are paths of length $d$ and $a_{i}^{(d)}$ are arrows. The map $g_{2}$ sends the element $1 \otimes r$ to

$$
g_{2}(1 \otimes r \otimes 1)=\sum_{d \geq 0} \sum_{i \in I_{d}} p_{i}^{(d)} \otimes a_{i}^{(d)} \otimes q_{i}^{(d)}
$$

- The map $g_{3}$ is given by $1 \otimes a \otimes 1 \mapsto a \otimes 1-1 \otimes a$.
- The fourth map is simply the contraction map.

The sequence 19.1 is clearly a chain complex. Whether or not it is exact is an indicator on whether $A$ is CY3 or not:

Theorem 16.52 ( 14 , Theorem 4.3]). Let $Q$ be a quiver and $W \in \mathbb{C} Q_{\geq 3}$ a superpotential. Then the algebra $\operatorname{Jac}(Q, W)$ is CY3 if and only if the sequence 16.6 is a bimodule resolution for $A$.

### 16.7 Cyclic $A_{\infty}$-algebras

In this section, we recall the correspondence of cyclic $A_{\infty}$-algebras and Calabi-Yau dg algebras via Koszul duality. In Lemma 16.44 and 16.28 we have already seen that $A^{!}$is the endomorphism algebra of the simple module $\mathbb{C}$ and that modules over a CY $n$ algebra enjoy Serre duality of degree $n$. This is a strong indication that $\mathrm{H} A^{!}$has a cyclic structure of degree $n$. As we recall in this section, this is the norm and cyclic $A_{\infty}$-algebras correspond to Calabi-Yau algebras under Koszul duality:

$$
\begin{array}{ccc}
A & \text { Koszul } & \begin{array}{l}
A^{!} \\
\text {Cyclic }
\end{array} \quad \begin{array}{c} 
\\
\text { Calabi-Yau }
\end{array} .
\end{array}
$$

In this section, we start by recalling the notion of cyclic $A_{\infty}$-algebras. Then we explain that their Koszul duals are so-called deformed dg-preprojective algebras, which are Calabi-Yau. This explains the correspondence of cyclic $A_{\infty}$-algebras and Calabi-Yau dg algebras via Koszul duality. Our primary reference is 70 . We start by recalling cyclic $A_{\infty}$-algebras, a generalizations of Frobenius algebras:
Definition 16.53. Let $A$ be an $A_{\infty}$-algebra. Then $A$ is cyclic of degree $n$ if it is equipped with a bilinear nondegenerate pairing $\langle-,-\rangle: A \times A \rightarrow \mathbb{C}$ of degree $-n$ such that $\langle x, y\rangle=(-1)^{|x| y \mid}\langle y, x\rangle$ and

$$
\left\langle\mu^{k}\left(a_{k+1}, \ldots, a_{2}\right), a_{1}\right\rangle=(-1)^{\left\|a_{k+1}\right\|\left(\left\|a_{k}\right\|+\ldots+\left\|a_{1}\right\|\right)}\left\langle\mu^{k}\left(a_{k}, \ldots, a_{1}\right), a_{k+1}\right\rangle
$$

Remark 16.54. The pairing is a noncommutative and categorical manifestation of a symplectic form. This is one of the reasons that cyclic $A_{\infty}$-algebras are dominant in the theory of Fukaya categories. In our context, the notion of cyclic Fukaya categories is reserved for the A-side of mirror symmetry, while Calabi-Yau algebras are reserved for the B-side of mirror symmetry.

When discussing Koszul duality for cyclic $A_{\infty}$-algebras, we frequently need to choose a basis compatible with the pairing $\langle-,-\rangle$. We recall this piece of linear algebra as follows:
Lemma 16.55. Let $V$ be a finite-dimensional graded vector space over $\mathbb{C}$ and $\langle-,-\rangle: V \times V \rightarrow \mathbb{C}$ a nondegenerate bilinear form of degree $-n$ with $\langle x, y\rangle=(-1)^{|x||y|}\langle y, x\rangle$. Then there is a graded basis for $V$ in which $\langle-,-\rangle$ takes the form

$$
\begin{align*}
\langle-,-\rangle & =\left(\right)  \tag{16.7}\\
\pm I_{k} & =\operatorname{diag}\left((-1)^{(n+1)\left|x_{1}\right|}, \ldots,(-1)^{(n+1)\left|x_{k}\right|}\right) .
\end{align*}
$$

Here $k=\operatorname{dim} V_{<n / 2}$ and the first $k$ basis elements $x_{1}, \ldots, x_{k}$ can be freely chosen. The matrix $I_{l}$ appears only if $n$ is even and $n / 2$ is even. The symplectic part consisting of $I_{m}$ and $-I_{m}$ only appears if $n$ is even and $n / 2$ is odd. In these cases we have $l=\operatorname{dim} V_{n / 2}$ and $m=\operatorname{dim} V_{n / 2} / 2$, respectively.
Proof. Let $x_{1}, \ldots, x_{k}$ be a given graded basis for $V_{<n / 2}$. The restricted pairing $\langle-,-\rangle: V_{<n / 2} \times V_{>n / 2} \rightarrow \mathbb{C}$ gives an isomorphism $\varphi: V_{>n / 2} \xrightarrow{\sim} V_{<n / 2}^{\vee}$ and the basis $x_{1}, \ldots, x_{k}$ gives rise to a dual basis $x_{1}^{\vee}, \ldots, x_{i}^{\vee} \in$ $V_{<n / 2}^{\vee}$. Pick $x_{j}^{*} \in V_{>n / 2}$ such that $\varphi\left(x_{j}^{*}\right)=x_{j}^{\vee}$. Then

$$
\left\langle x_{i}, x_{j}^{*}\right\rangle=\varphi\left(x_{j}^{*}\right)\left(x_{i}\right)=x_{j}^{\vee}\left(x_{i}\right)=\delta_{i j}, \quad\left\langle x_{j}^{*}, x_{i}\right\rangle=(-1)^{\left(n-\left|x_{j}\right|\right)\left|x_{i}\right|} \delta_{i j}=(-1)^{(n+1)\left|x_{i}\right|} \delta_{i j} .
$$

Picking $x_{1}, \ldots, x_{k}, x_{1}^{*}, \ldots, x_{k}^{*}$ as first part of the basis sets up the first diagonal block of the matrix. If $n$ is odd, then $V=V_{<n / 2} \oplus V_{>n / 2}$ and we are done. If $n$ is even and $n / 2$ is even, then $\langle-,-\rangle$ is a nondegenerate symmetric bilinear form on $V_{n / 2}$ and we can find a basis in which $\langle-,-\rangle: V_{n / 2} \times V_{n / 2} \rightarrow \mathbb{C}$ is the identity matrix. If $n$ is even and $n / 2$ is odd, then $\langle-,-\rangle: V_{n / 2} \times V_{n / 2} \rightarrow \mathbb{C}$ is a symplectic form on $V_{n / 2}$ and we can choose a symplectic basis. This finishes the proof.
Definition 16.56. Let $V$ be a finite-dimensional graded vector space over $\mathbb{C}$ and $\langle-,-\rangle: V \times V \rightarrow \mathbb{C}$ a nondegenerate bilinear form of degree $-n$. Choose a basis for $V$ in which $\langle-,-\rangle$ takes the shape 16.7 :

$$
\begin{array}{ll}
x_{1}, \ldots, x_{k}, x_{1}^{*}, \ldots, x_{k}^{*} & \text { if } n \text { odd, } \\
x_{1}, \ldots, x_{k}, x_{1}^{*}, \ldots, x_{k}^{*}, x_{k+1}, \ldots, x_{k+l} & \text { if } n \text { even, } n / 2 \text { even, } \\
x_{1}, \ldots, x_{k}, x_{1}^{*}, \ldots, x_{k}^{*}, x_{k+1}, \ldots, x_{k+m}, x_{k+1}^{*}, \ldots, x_{k+m}^{*} & \text { if } n \text { even, } n / 2 \text { odd. }
\end{array}
$$

Then we call the pair of sequences

$$
\begin{array}{lll}
x_{1}, \ldots, x_{k} & \text { and } x_{1}^{*}, \ldots, x_{k}^{*} & \text { if } n \text { odd, } \\
x_{1}, \ldots, x_{k}, x_{k+1}, \ldots, x_{k+l} & \text { and } x_{1}^{*}, \ldots, x_{k}^{*}, x_{k+1}^{*}:=x_{k+1}, \ldots, x_{k+l}^{*}:=x_{k+l} & \text { if } n \text { even, } n / 2 \text { even, } \\
x_{1}, \ldots, x_{k}, x_{k+1}, \ldots, x_{k+m} & \text { and } x_{1}^{*}, \ldots, x_{k+m}^{*} & \text { if } n \text { even, } n / 2 \text { odd }
\end{array}
$$

a basis with dual basis for $V$ via $\langle-,-\rangle$.
Remark 16.57. For any $i$ we have $\left|x_{i}^{*}\right|=n-\left|x_{i}\right|$, regardless of whether $n$ is odd or even.
Van den Bergh investigated the correspondence between cyclic $A_{\infty}$-algebras and Calabi-Yau dg algebras via Koszul duality in 70. The technical core result is a structure characterization of Calabi-Yau dg algebras. Roughly speaking, a dg algebra is CYn if it is weakly equivalent to a so-called deformed dg-preprojective algebra $\Pi(Q, n, W)$ for some quiver $Q$ and superpotential $W \in \mathbb{C} Q_{\geq 3}$. Here $W$ is required to satisfy $\{W, W\}=0$ with respect to the so-called necklace bracket $\{-,-\}$, on which we will not elaborate here. We recall the construction of the algebras $\Pi(Q, n, W)$ as follows:

Definition 16.58. Let $Q$ be a graded quiver, $n \geq 3$ an integer and $W \in \mathbb{C} Q_{\geq 3}$ a superpotential homogeneous of degree $-n+3$ with respect to the grading on $Q$. Assume that $W$ is "graded cyclic", instead of cyclic. Assume $\{W, W\}=0$ with respect to the necklace bracket. Let $\tilde{Q}$ be the double quiver, obtained from $Q$ by inserting for every arrow $a$ an arrow $a^{*}$ in opposite direction. The arrow $a^{*}$ is assigned degree $-n+2-|a|_{Q}$. In the special case where an arrow $a$ is a loop with odd degree $|a|_{Q}=(2-n) / 2$, no arrow $a^{*}$ is adjoined and one puts $a^{*}=a$. Let $\bar{Q}$ be the quiver obtained from $\tilde{Q}$ by additionally inserting on every vertex a loop $z_{i}$ of degree $1-n$. Then the deformed dg-preprojective algebra

$$
\Pi(Q, n, W)=\left(\widehat{\mathbb{C}} \tilde{\tilde{Q}}, d_{\Pi}\right)
$$

is the dg algebra modeled on $\widehat{\mathbb{C}} \widehat{\tilde{Q}}$ with the following differential:

$$
\begin{aligned}
d_{\Pi}(a) & =(-1)^{(|a|+1)\left|a^{*}\right|} \partial_{a^{*}} W, \\
d_{\Pi}\left(a^{*}\right) & =(-1)^{|a|+1} \partial_{a} W, \\
d_{\Pi}\left(z_{i}\right) & =\sum_{h(a)=i}\left[a, a^{*}\right] .
\end{aligned}
$$

We recall Van den Bergh's structure result as follows:
Theorem 16.59 (70, Theorem 10.2.2]). Let $Q$ be a graded quiver, $n \geq 3$ an integer and $W \in \mathbb{C} \tilde{Q}_{\geq 3}$ a superpotential with $\{W, W\}=0$. Assume $W$ is homogeneous of degree $-n+3$ with respect to the grading on $Q$ and all arrows in $Q$ lie in degree range $[-n+2,0]$. Then $\Pi(Q, n, W)$ is CY $n$.

Van den Bergh also shows the converse statement that every CYn dg algebra under suitable finiteness conditions is equivalent to some deformed dg-preprojective algebra $\Pi(Q, n, W)$. It is an easy consequence of Theorem 16.59 that the Koszul dual of a minimal cyclic $A_{\infty}$-category is CYn:
Corollary 16.60. Let $A$ be a finite-dimensional augmented minimal $A_{\infty}$-algebra concentrated in nonnegative degrees with $A^{0}=\mathbb{C}$ id. Assume $A$ is cyclic of degree $n$. Then $A^{!}$is a deformed dg-preprojective algebra and is therefore CYn.
Proof. The idea is to represent $A^{!}$as a deformed dg-preprojective algebra $\Pi(Q, n, W)$ by letting the arrows of $Q$ stand for basis elements of $A$ and letting the superpotential $W$ record the product $\mu$. We ignore signs.

The first step is to choose a basis for $A$ by means of Lemma 16.55 . Since $A^{0}=\mathbb{C}$ id, we can include the element id in the the basis and obtain a basis with dual basis of the following form:

$$
A=\mathbb{C} \operatorname{id} \oplus \operatorname{span}\left\{x_{1}, \ldots, x_{k}\right\} \oplus \operatorname{span}\left\{x_{1}^{*}, \ldots, x_{k}^{*}\right\} \oplus \mathbb{C i d}^{*}
$$

Of course, we have to make adaptations in case $n$ is even: There may be self-dual basis elements of degree $n / 2$. This is not a problem and is adequately reflected in the definition of $\Pi(Q, n, W)$ by the intricate extra condition on loops. We shall for simplicity proceed with the assumption that $n$ is odd.

We define the quiver $Q$ to be a single vertex with $k$ arrows, such that the double quiver $\bar{Q}$ and the extended quiver $\tilde{Q}$ satisfy

$$
\begin{aligned}
& \widehat{\mathbb{C} \tilde{Q}}=\mathbb{C}\left\langle\left\langle x_{1}^{\vee}, \ldots, x_{k}^{\vee},\left(x_{1}^{*}\right)^{\vee}, \ldots,\left(x_{k}^{*}\right)^{\vee}\right\rangle\right\rangle, \\
& \widehat{\mathbb{C} \bar{Q}}=\mathbb{C}\left\langle\left\langle x_{1}^{\vee}, \ldots, x_{k}^{\vee},\left(x_{1}^{*}\right)^{\vee}, \ldots,\left(x_{k}^{*}\right)^{\vee},\left(\mathrm{id}^{*}\right)^{\vee}\right\rangle\right\rangle .
\end{aligned}
$$

In these graded algebras, the variables $x_{i}^{\vee}$ and $\left(x_{i}^{*}\right)^{\vee}$ have degrees $1-\left|x_{i}\right|$ and $1-\left|x_{i}^{*}\right|$, respectively. The variable $\left(\mathrm{id}^{*}\right)^{\vee}$ has degree $1-n$.

We define the superpotential $W \in \widehat{\mathbb{C} \tilde{Q}}$ by Koszul transforming the $A_{\infty}$-structure on $\mu$ :

$$
W=\sum_{\substack{1 \leq i_{0}, i_{1}, \ldots, i_{l} \leq k \\ 2^{l+1} \text { star options }}} \underbrace{\left\langle\mu\left(x_{i_{l}}^{[*]}, \ldots, x_{i_{1}}^{[*]}\right), x_{i_{0}}^{[*]}\right\rangle}_{\in \mathbb{C}} \cdot \underbrace{\left(x_{i_{l}}^{[*]}\right)^{\vee} \ldots\left(x_{i_{1}}^{[*]}\right)^{\vee}\left(x_{i_{0}}^{[*]}\right)^{\vee}}_{\in \mathbb{C} \tilde{Q}}
$$

Here we sum over choices of indices $i_{0}, \ldots, i_{l}$ as well as choices of starred or non-starred basis elements and variables. Minimality of $A$ ensures that $W$ lies in $\mathbb{C} Q_{\geq 3}$ and cyclicity of $A$ ensures that $W$ is graded cyclic. The $A_{\infty}$-relations for $A$ ensure that $\{W, W\}=0$.

The quiver $Q$, the cyclicity degree $n$ and the superpotential $W$ now yield a deformed dg-projective algebra $\Pi(Q, n, W)$. Explicitly, its differential reads

$$
\begin{aligned}
d_{\Pi} x_{i}^{\vee} & =(-1)^{\left(\left|x_{i}^{\vee}\right|+1\right) \mid\left(x_{i}^{*}\right)^{\vee}} \partial_{\left(x_{i}^{*}\right)^{\vee}} W, \\
d_{\Pi}\left(x_{i}^{*}\right)^{\vee} & =(-1)^{\left|x_{i}^{\vee}\right|+1} \partial_{x_{i}^{\vee}} W, \\
d_{\Pi}\left(\mathrm{id}^{*}\right)^{\vee} & =\sum_{i=1}^{k} x_{i}^{\vee}\left(x_{i}^{*}\right)^{\vee}-\left(x_{i}^{*}\right)^{\vee} x_{i}^{\vee} .
\end{aligned}
$$

In the remainder of the proof, we check that $\Pi(Q, n, W)$ equals $A^{!}$. In fact, we have already named all the variables $2 k+1$ variables in $\widehat{\mathbb{C} Q}$ precisely the way in which variables for $A^{!}$are named.

To see that also the differentials $d_{\Pi}$ and $d_{A^{\prime}}$ agree, note that the derivative $\partial_{x_{i}} \vee W$ precisely records the $x_{i}^{*}$ coefficient of all possible products $\mu(\ldots)$ of which the input is not the identity id or the co-identity id*. In principle, the differential of $A^{!}$also records those products $\mu(\ldots)$ which contain a co-identity, however those products vanish:

$$
\left|\mu^{l}\left(x_{i_{l}}^{[*]}, \ldots, \mathrm{id}^{*}, \ldots, x_{i_{1}}^{[*]}\right)\right| \geq(2-l)+(l-1)+n=n+1
$$

Here we have used the assumption that $A$ is concentrated in non-negative degrees and the co-idenity is in degree $n$. We draw the conclusion that such products vanish because $A$ is limited to degrees at most $n$. Ultimately, $d\left(x_{i}^{*}\right)^{\vee}=\partial_{x_{i}^{\vee}}$ agrees with the differential of $\left(x_{i}^{*}\right)^{\vee}$ in $A^{!}$.

Also on the variable $\left(\mathrm{id}^{*}\right)^{\vee}$, the differentials of $\Pi(Q, n, W)$ and $A^{!}$agree. Indeed, the differential of $A^{!}$records appearances of id* as result of products $\mu(\ldots)$. Thanks to cyclicity, we have

$$
\begin{aligned}
\left\langle\mu^{l \geq 3}\left(a_{l}, \ldots, a_{1}\right), \mathrm{id}\right\rangle & =0 \\
\left\langle\mu^{2}(a, b), \mathrm{id}\right\rangle+\left\langle\mu^{2}(b, \mathrm{id}), a\right\rangle+\left\langle\mu^{2}(\mathrm{id}, a), b\right\rangle & =0 .
\end{aligned}
$$

When plugging in $a=x_{i}$ and $b=x_{j}^{*}$, we precisely obtain $d_{A^{\prime}}\left(\left(\mathrm{id}^{*}\right)^{\vee}\right)=d_{\Pi}\left(\left(\mathrm{id}^{*}\right)^{\vee}\right)$. Ultimately, we conclude that $A^{!}=\Pi(Q, n, W)$. Thanks to Theorem 16.59 the algebra $\Pi(Q, n, W)$ is CY $n$ and therefore $A^{!}$is CYn itself. This finishes the proof.

### 16.8 Cho-Hong-Lau roadmap

In this section, we motivate the Cho-Hong-Lau construction from the perspective of Koszul duality. The idea is to pass the dg algebra $A^{!}$to cohomology. Regarded as ordinary algebra, the cohomology $\mathrm{H}^{0} A^{!}$ need not be a Calabi-Yau algebra itself. Under grading assumptions on $A$, it is however a Jacobi algebra and therefore a candidate to be CY3. We explain how to drop the grading requirements so that one can start from $A$ being $\mathbb{Z} / 2 \mathbb{Z}$-graded. The Koszul duality functor only survives this drop of requirements when we replace the actual cohomology $\mathrm{H}^{0} A^{!}$by a surrogate. At the end of the section, we provide this tweaked Koszul duality functor which comes close to the Cho-Hong-Lau construction.

Our first starting point is an $A_{\infty}$-category $A$ which is cyclic of degree $n$, concentrated in non-negative degree and has degree zero part $A^{0}=\mathbb{C}$ id:

$$
A=\mathbb{C} \operatorname{id} \oplus A^{1} \oplus \ldots \oplus A^{n-1} \oplus \mathbb{C i d}^{*}
$$

Our interest lies in the Koszul duality functor $\operatorname{Mod}_{\text {right }}^{\mathrm{fd}} A \rightarrow \mathrm{Tw} A^{!}$which we recalled in section 16.2 The essential step leading to the Cho-Hong-Lau construction consists of replacing the dg algebra $A^{!}$by its zeroth cohomology $\mathrm{H}^{0} A^{!}$, which is an ordinary associative algebra. In this section, we provide a simple explanation why this works particularly well if $A$ is cyclic of degree 3 .

We start with the observation that passing to the minimal model is unproblematic if $A$ is concentrated in non-negative degree. We denote the minimal model of the dg algebra $A^{!}$by $\mathrm{H} A^{!}$. Its degree zero part is $\mathrm{H}^{0} A^{!}$.

Lemma 16.61. Let $A$ be a non-negatively graded augmented $A_{\infty}$-algebra such that $A^{0}=\mathbb{C}$ id. Then there is an $A_{\infty}$-morphism $A^{!} \rightarrow \mathrm{H}^{0} A^{!}$.

Proof. By definition of the minimal model $\mathrm{H} A^{!}$, there is an $A_{\infty^{-}}$-quasi-isomorphism $A^{!} \rightarrow \mathrm{H} A^{!}$. In the remainder of the proof, we show that the projection $\pi_{0}: \mathrm{H} A^{!} \rightarrow \mathrm{H}^{0} A^{!}$is an $A_{\infty}$-morphism. Regard homogeneous elements $a_{1}, \ldots, a_{k} \in \mathrm{H} A^{!}$. Writing out the definition of $\mu_{\mathrm{H}^{0} A^{!}}\left(\pi_{0} a_{k}, \ldots, \pi_{0} a_{1}\right)$, it is our task to show

$$
\pi_{0} \mu_{\mathrm{H} A^{!}}\left(a_{k}, \ldots, a_{1}\right)= \begin{cases}\mu_{\mathrm{H} A^{!}}\left(\pi_{0} a_{2}, \pi_{0} a_{1}\right) & \text { if } k=2  \tag{16.8}\\ 0 & \text { if } k \neq 2\end{cases}
$$

The basic observation is that $A^{!}$is concentrated in non-positive degrees because $\bar{A}$ is concentrated in positive degrees. The cohomology $\mathrm{H} A^{!}$is then also concentrated in non-positive degrees. To check 16.8, we distinguish four cases according to the value of $k$ and the degrees of the input elements $a_{1}, \ldots, a_{k}$.

If $k=1$, both sides vanish. If $k=2$ and $a_{1}, a_{2}$ are both of degree zero and 16.8 holds. If $k=2$ and one of $a_{1}, a_{2}$ is of negative degree, then $\mu_{\mathrm{H} A^{!}}\left(a_{2}, a_{1}\right)$ is of negative degree and both sides of 16.8 vanish. If $k \geq 3$, then $\mu_{\mathrm{H}_{A^{!}}}\left(a_{k}, \ldots, a_{1}\right)$ has degree at most $2-k \leq-1$ and therefore both sides vanish as well. This finishes the proof.

We borrow the following notation from Definition 19.32
Definition 16.62. Let $V$ be a vector space and $R \subseteq V$ be a subspace. Then $I(R)_{\widehat{T(V)}}$ is the ideal of $\widehat{T(V)}$ given by the image of the map $\widehat{T(V)} \widehat{\otimes} R \widehat{\otimes} \widehat{T(V)} \rightarrow \widehat{T(V)}$. Explicitly, $I(R)_{\widehat{T(V)}}$ consists of all elements that can be written as a series

$$
\sum_{i=0}^{\infty} p_{i} r_{i} q_{i}, \quad p_{i} \in V^{\otimes \rightarrow \infty}, \quad r_{i} \in R, \quad q_{i} \in V^{\otimes \rightarrow \infty}
$$

Lemma 16.63. Let $A$ be an augmented $A_{\infty}$-algebra concentrated in non-negative degrees with $A^{0}=\mathbb{C}$ id. Then we have an algebra identification

$$
\mathrm{H}^{0} A^{!}=\frac{T \widehat{\left(\left(A^{1}\right)^{\vee}\right)}}{I\left(\left\{d_{A^{\prime}} x \mid x \in\left(A^{2}\right)^{\vee}\right\}\right)_{T\left(\widehat{\left(A^{1}\right)^{\vee}}\right)}} .
$$

Proof. Since $\bar{A}$ is concentrated in positive degrees, its left-shift $\bar{A}[1]$ is concentrated in non-negative degrees. We can write

$$
A^{!}=\prod_{i=0}^{\infty}(\underbrace{\left(A^{1}\right)^{\vee}}_{\operatorname{deg} 0} \oplus \underbrace{\left(A^{2}\right)^{\vee}}_{\operatorname{deg}-1} \oplus \ldots)^{\otimes i}
$$

In the remainder of the proof, we determine the degree zero cocycles and degree zero coboundaries of $A^{!}$.
For the degree zero cocycles, we note that the degree zero part of $A^{!}$is $\left(A^{1}\right)^{\vee}$ and the degree one part of $A^{!}$is zero. Together, the degree zero cocyles of $A^{!}$are just $T\left(A^{1}\right)^{\vee}$.

For the degree zero coboundaries, we determine first the degree -1 part of $A^{!}$. In fact, the part of degree -1 consists of products of degree zero elements with precisely one degree -1 element, more precisely

$$
\prod_{i=0}^{\infty} \sum_{j=0}^{i}\left(A^{1}\right)^{\vee} \otimes j \otimes\left(A^{2}\right)^{\vee} \otimes\left(A^{1}\right)^{\vee} \otimes i-j
$$

Now the degree zero coboundaries of $A^{!}$are precisely the differentials of elements in this space. Since $d_{A^{!}} x=0$ for $x \in\left(A^{1}\right)^{\vee}$, we immediately see that for $x_{1}, \ldots, x_{i} \in\left(A^{1}\right)^{\vee}$ and $y \in\left(A^{2}\right)^{\vee}$ we have

$$
d_{A^{\prime}}\left(x_{i} \otimes \ldots \otimes x_{i-j+1} \otimes y \otimes x_{i-j} \otimes \ldots \otimes x_{1}\right)=x_{i} \otimes \ldots \otimes x_{i-j+1} \otimes d_{A^{\prime}} y \otimes x_{i-1} \otimes \ldots \otimes x_{1}
$$

We conclude that the degree zero coboundaries of $A^{!}$are precisely elements of $I\left(\left\{d_{A^{!} y} \mid y \in\left(A^{2}\right)^{\vee}\right\}\right)$. This finishes the proof.

We shall continue writing $I$ for the ideal defined in Lemma 16.63
Definition 16.64. Let $A$ be an augmented $\mathbb{Z}$-graded finite-dimensional $A_{\infty}$-algebra cyclic of degree $n \geq 3$. Denote by $A^{1}$ and $A^{2}$ the homogeneous subspaces of degree 1 and 2 , respectively. Then we denote by $I$ the ideal

$$
\left.I=I\left(\left\{d_{A^{!}} x \mid x \in\left(A^{2}\right)^{\vee}\right\}\right)_{T \widehat{\left(\left(A^{1}\right)^{\vee}\right.}} \subseteq T \widehat{\left(\left(A^{1}\right)^{\vee}\right)}\right)
$$

It is in fact possible to say more about the ideal in case $A$ is cyclic of degree 3 :
Lemma 16.65. Let $A$ be an augmented finite-dimensional $A_{\infty}$-algebra. Assume $A$ is cyclic of degree 3 and concentrated in non-negative degrees with $A^{0}=\mathbb{C}$ id:

$$
A=\mathbb{C} \operatorname{id} \oplus A^{1} \oplus A^{2} \oplus \mathbb{C i d}^{*}
$$

Then we have a Koszul duality functor $\operatorname{Mod}_{\text {right }}^{\mathrm{fd}} A \rightarrow \operatorname{TwH} H^{0} A^{!}$. Upon choosing a basis $x_{1}, \ldots, x_{k}$ for $A^{1}$, the algebra $\mathrm{H}^{0} A^{!}$can be written as a Jacobi-type algebra and is naturally a candidate to be CY3:

$$
\begin{equation*}
\mathrm{H}^{0} A^{!}=\frac{\mathbb{C}\left\langle\left\langle x_{1}^{\vee}, \ldots, x_{k}^{\vee}\right\rangle\right\rangle}{I\left(\partial_{x_{i}^{\vee}} W\right)_{\mathbb{C}}\left\langle\left\langle x_{1}^{\vee}, \ldots, x_{k}^{\vee}\right\rangle\right\rangle} \tag{16.9}
\end{equation*}
$$

Proof. We divide the proof into three steps. In the first step, we explain the Koszul duality functor. In the second step, we examine the algebra $\mathrm{H}^{0} A^{!}$. In the third step, we comment on CY3-ness.

For the first step, pick the Koszul duality functor $\operatorname{Mod}_{\text {right }}^{\mathrm{fd}} A \rightarrow \mathrm{Tw} A^{!}$from Corollary 16.21. Thanks to Lemma 16.61, we have an additional morphism of $A_{\infty}$-algebras $A^{!} \rightarrow \mathrm{H}^{0} A$, which induces an $A_{\infty^{-}}$ functor $\mathrm{Tw} A^{!} \rightarrow \mathrm{Tw} \mathrm{H}^{0} A^{!}$. Composing these two, we obtain the desired Koszul duality functor

$$
\operatorname{Mod}_{\mathrm{right}}^{\mathrm{fd}} A \rightarrow \operatorname{Tw} A^{!} \rightarrow \operatorname{Tw~H}^{0} A^{!}
$$

For the second step of the proof, the starting point is the description $\mathrm{H}^{0} A^{!}=T \widehat{\left(\left(A^{1}\right)^{\vee}\right)} / I$ from Lemma 16.63 It is our task to examine the ideal $I$. Choose a basis $x_{1}, \ldots, x_{n}$ for $A^{1}$ and denote by $x_{1}^{*}, \ldots, x_{n}^{*}$ the corresponding dual basis for $A^{2}$ via $\langle-,-\rangle$. We construct the superpotential $\left.W \in \widehat{T\left(A^{1}\right)^{\vee}}\right)$ as follows:

$$
W=\sum_{1 \leq i_{1}, \ldots, i_{k}, i_{0} \leq n}\left\langle\mu\left(x_{i_{k}}, \ldots, x_{i_{1}}\right), x_{i_{0}}\right\rangle x_{i_{0}}^{\vee} x_{i_{1}}^{\vee} \ldots x_{i_{k}}^{\vee}
$$

This specific pairing has the chance of not vanishing only because $n=3$. In fact, the degree of $\mu(\ldots)$ is 2 while the degree of $x_{i_{0}}$ is 1 . The superpotential $W$ is cyclic since $\mu$ is assumed to be cyclic. We now claim that $\partial_{x_{i_{0}}} W=d_{A^{!}}\left(x_{i_{0}}^{*}\right)^{\vee}$. Indeed,

$$
\partial_{x_{i_{0}}^{\vee}} W=\sum_{1 \leq i_{1}, \ldots, i_{k} \leq n}\left\langle\mu\left(x_{i_{k}}, \ldots, x_{i_{1}}\right), x_{i_{0}}\right\rangle x_{i_{1}}^{\vee} \ldots x_{i_{k}}^{\vee}=d_{A^{\prime}}\left(x_{i_{0}}^{*}\right)^{\vee} .
$$

This proves the desired description of $\mathrm{H}^{0} A^{!}$. For the third part of the proof, we comment on the nontechnical statement regarding the claimed CY3 candidate status of $\mathrm{H}^{0} A^{!}$. As we have shown in the second part of the proof, the algebra $\mathrm{H}^{0} A^{!}$is the quotient of a noncommutative power series ring by derivatives of a superpotential. This does technically not imply that $\mathrm{H}^{0} A^{!}$is CY3. However, it is known that if the number of variables and the degree of the superpotential are high enough, then the typical superpotential does turn the quotient into a CY3 algebra 14 . Corollary 4.4]. This finishes the proof.

Remark 16.66. It is essential that the cyclicity degree of $A$ is 3 : Assume $A$ is cyclic of degree $n$ instead. In order to represent $d x^{\vee}$ as derivative of a superpotential $W \in T \widehat{\left(\left(A^{1}\right)^{\vee}\right)}$, the only natural way is by $d x^{\vee}$ being either the derivative $\partial_{x^{\vee}} W$ or $\partial_{\left(x^{*}\right)} \vee W$. However, the variable $x^{\vee}$ has degree -1 and the variable $\left(x^{*}\right)^{\vee}$ has degree $1-(n-2)=3-n$. We conclude that only for $n=3$ any of the variables, namely $\left(x^{*}\right)^{\vee}$, has a chance of appearing in $W$. This shows that cyclicity in degree $n=3$ is favorable for passing $A^{!}$to cohomology.

The reduction of the Koszul dual to zeroth cohomology in Lemma 16.65 makes room for further weakening of the assumptions on the side of the $A_{\infty}$-algebra $A$. As we show in Lemma 16.68 we can drop the requirement that $A$ be $\mathbb{Z}$-graded and concentrated in non-negative degrees. The idea is to circumvent $A^{!}$and work directly with the quotient algebra displayed on the right-hand side of 16.9 . When we restrict $\mu_{A}$ or $\mu_{M}$ to $T\left(\bar{A}^{1}\right)$, we shall use the letter $m$ instead of $\mu$ :

Definition 16.67. Let $A=\mathbb{C}$ id $\oplus \bar{A}^{1} \oplus \bar{A}^{2} \oplus \mathbb{C i d}^{*}$ be an augmented $\mathbb{Z} / 2 \mathbb{Z}$-graded finite-dimensional $A_{\infty^{-}}$ algebra cyclic of odd degree. We denote the restriction to $T\left(A^{1}\right)$ of the map $\mu_{A}: M \otimes T(\bar{A}) \rightarrow M \otimes T(\bar{A})$ and its partial dual $\mu_{A}^{\vee}$ by

$$
\begin{aligned}
m_{A}: & M \otimes T\left(\bar{A}^{1}\right) & \rightarrow M \otimes T\left(\bar{A}^{1}\right), \\
m_{A, 0}^{\vee}: & M & \rightarrow M \otimes T\left(\bar{A}^{1}\right)^{\vee}
\end{aligned}
$$

Let $M \in \operatorname{Mod}_{\text {right }}^{\mathrm{fd}} A$ be a finite-dimensional right $A$-module with action map $\mu_{M}: M \otimes T(A[1]) \rightarrow$ $M \otimes T(A[1])$. Then we denote the restriction to $T\left(A^{1}\right)$ of $\mu_{M}$ and $\mu_{M}^{\vee}$ by

$$
\begin{aligned}
m_{M}: & M \otimes T\left(\bar{A}^{1}\right) & \rightarrow M \otimes T\left(\bar{A}^{1}\right), \\
m_{M, 0}^{\vee}: & M & \rightarrow M \otimes T\left(\bar{A}^{1}\right)^{\vee} .
\end{aligned}
$$

We now formulate the Koszul duality where $A$ is not required to be $\mathbb{Z}$-graded. From the standpoint of Koszul duality, this statement is the closest to the Cho-Hong-Lau construction that lies within the framework of finite-dimensional modules and twisted complexes:

Lemma 16.68. Let $A=\mathbb{C}$ id $\oplus \bar{A}^{1} \oplus \bar{A}^{2} \oplus \mathbb{C i d}^{*}$ be an augmented $\mathbb{Z} / 2 \mathbb{Z}$-graded finite-dimensional $A_{\infty^{-}}$ algebra cyclic of odd degree. Write $J=T \widehat{\left(\left(A^{1}\right)^{\vee}\right)} / I$. Then we have a Koszul duality functor

$$
\begin{aligned}
F: \operatorname{Mod}_{\text {right }}^{\mathrm{fd}} A & \longrightarrow \mathrm{Tw} J, \\
\left(M, \mu_{M}\right) & \longmapsto\left(M \otimes J, m_{M, 0}^{\vee}\right), \\
f & \longmapsto f^{\vee} .
\end{aligned}
$$

Proof. We only provide a glimpse of the proof here, since the more general version is treated insection 20 For instance, it is instructive to explain why $\left(M, m_{M, 0}^{\vee}\right)$ is a twisted complex over $J$.

In order to recognize $\left(M, m_{M, 0}^{\vee}\right)$ as twisted complex over $J$, we have to check its Maurer-Cartan identity. Since $M$ is a module over $A$, we have $\left(m_{M} \circ m_{A}\right)_{0}+\left(m_{M} \circ m_{M}\right)_{0}=0$. The image of $\left(m_{M} \circ m_{A}\right)_{0} \vee$ lies in $M \otimes I$. This means that modulo $M \otimes I$, we have

$$
\mu_{\text {Add } J}^{2}\left(m_{M, 0}^{\vee}, m_{M, 0}^{\vee}\right)=\left(m_{M} \circ m_{M}\right)_{0}^{\vee}=0
$$

This shows that $m_{M, 0}^{\vee}$ satisfies the Maurer-Cartan equation in Add $J$ and ( $M \otimes J, m_{M, 0}^{\vee}$ ) is indeed a twisted complex.

Remark 16.69. The functor $F$ in Lemma 16.68 can be made explicit upon choice of basis for $A$. Choose a basis of a consisting of elements $x_{i}$ in odd degree and the dual elements $x_{i}^{*}$ in even degree:

$$
A=\underbrace{\mathbb{C} i d}_{\text {even }} \oplus \underbrace{\operatorname{span}\left\{x_{1}, \ldots, x_{k}\right\}}_{\text {odd }} \oplus \underbrace{\operatorname{span}\left\{x_{1}^{*}, \ldots, x_{k}^{*}\right\}}_{\text {even }} \oplus \underbrace{\mathbb{C i d}^{*}}_{\text {odd }}
$$

The algebra $J$ takes on the form

$$
J=\frac{\mathbb{C}\left\langle\left\langle x_{1}^{\vee}, \ldots, x_{k}^{\vee}\right\rangle\right\rangle}{I\left(\partial_{x_{i}^{\vee}} W\right)_{\mathbb{C}}\left\langle\left\langle x_{1}^{\vee}, \ldots, x_{k}^{\vee}\right\rangle\right\rangle}
$$

Formulated in terms of the basis, the fact that $m_{M}^{\vee}$ squares to zero is based on the observation that

$$
\begin{aligned}
0 & =\sum_{\substack{l \geq 0 \\
1 \leq i_{1}, \ldots, i_{l} \leq k}}(\mu \cdot \mu)\left(m, x_{i_{l}}, \ldots, x_{i_{1}}\right) \\
& =\sum_{\substack{l \geq 0 \\
1 \leq i_{1}, \ldots, i_{l} \leq k \\
0 \leq s \leq r \leq l}} \mu\left(m, x_{i_{l}}, \ldots, \mu\left(x_{i_{r}}, \ldots, x_{i_{s+1}}\right), \ldots\right)+\sum_{\substack{l \geq 0 \\
1 \leq i_{1}, \ldots, i_{l} \leq k \\
0 \leq r \leq l}} \mu\left(\mu\left(m, x_{i_{l}}, \ldots, x_{i_{r+1}}\right), \ldots, x_{1}\right) .
\end{aligned}
$$

Note that there are no signs since all $x_{i}$ are odd. When dualizing the above equality, the first summand on the second row lands in the ideal $I\left(\partial_{x_{i}} W\right)_{\mathbb{C}\left\langle\left\langle x_{1}, \ldots, x_{k}\right\rangle\right\rangle}$ and vanishes in $J$. Meanwhile, the second summand on the second row dualizes to $\left(m_{M, 0}^{\vee}\right)^{2}$. We conclude that $\left(m_{M, 0}^{\vee}\right)^{2}=0$, in other words $\left(M \otimes J, m_{M}^{\vee}\right)$ is a twisted complex.

The construction of Lemma 16.68 leads directly to the Cho-Hong-Lau construction. The idea is to extend the construction of the Koszul duality functor $\operatorname{Mod}_{\text {right }}^{\mathrm{fd}} A \rightarrow \mathrm{Tw} J$ to the case of non-augmented algebras. On the side of the Koszul dual, the required adaption consists of passing from the twisted completion to matrix factorizations. In section 20, we recall this Cho-Hong-Lau construction and in particular amend all signs to the rules for $A_{\infty}$-categories.

## 17 Preliminaries on mirror symmetry

In this section, we recollect preliminaries on mirror symmetry of punctured surfaces from 18 .

### 17.1 Gentle algebras

In this section, we recall the $A_{\infty}$-gentle algebra from 18 and its deformed version from Paper I In particular, we explain the philosophy of $\operatorname{Gtl}_{q} Q$ as a discrete relative Fukaya category.

Definition 17.1. A punctured surface is a closed oriented surface $S$ with a finite set of punctures $M \subseteq S$. We assume that $|M| \geq 1$, or $|M| \geq 3$ if $S$ is a sphere.

The assumptions on $|M|$ are cosmetical and explained in Paper II
Definition 17.2. Let $(S, M)$ be a punctured surface. An arc in $S$ is a not necessarily closed curve $\gamma:[0,1] \rightarrow S$ running from one puncture to another. An arc system on a punctured surface is a finite collection of arcs such that the arcs meet only at the set $M$ of punctures. Intersections and selfintersections are not allowed. The arc system satisfies the no monogons or digons condition [NMD] if

- No arc is a contractible loop in $S \backslash M$.
- No pair of distinct arcs is homotopic in $S \backslash M$.

The arc system satisfies the no monogons or digons in the closed surface condition [NMDC] if

- No arc is a contractible loop in $S$.
- No pair of distinct arcs is homotopic in $S$.

An arc system is full if the arcs cut the surface into contractible pieces, which we call polygons.
Definition 17.3. A dimer $Q$ is a full arc system on a punctured surface such that every polygon is bounded by at least three arcs and the arcs along the boundary of a polygon are all oriented in the same direction. The letter $Q_{0}$ denotes the set of punctures of the dimer, the letter $Q_{1}$ denotes the set of arcs, and $|Q|$ denotes the closed surface.
Remark 17.4. All polygons in a dimer are either bounded entirely clockwise, or entirely anticlockwise. A polygon neighboring a clockwise polygon is anticlockwise, and the other way around. A dimer can be interpreted as a quiver embedded in a surface, therefore quiver terminology applies and we may for instance refer to paths of $Q$.

(a) Standard polygon $P_{5}$

$\mu_{q}^{0}=q \ell_{q}+p \ell_{p}$
(b) Assigning curvature to an arc

$\mu_{q}^{6}\left(\alpha_{6}, \ldots, \alpha_{1}\right)=q \operatorname{id}_{a}$
(c) Deformed product $\mu_{q}^{6}$

Figure 17.1: Illustration for $\operatorname{Gtl} Q$ and $\operatorname{Gtl}_{q} Q$

The gentle algebra $\mathrm{Gtl} Q$ is an $A_{\infty}$-category defined as follows: Its objects are the arcs $a \in Q_{1}$. A basis for the hom space $\operatorname{Hom}_{\mathrm{Gt1} Q}(a, b)$ is given by the set of all angles around punctures from $a$ to $b$. This includes empty angles, which are the identities on the arcs. The hom spaces of Gtl $Q$ are not finite-dimensional, in contrast to what is classically called a gentle algebra. The $\mathbb{Z} / 2 \mathbb{Z}$-grading on Gtl $Q$ is given by declaring all interior angles of polygons to have odd degree. The differential $\mu^{1}$ is plainly set to zero, and the product $\mu^{2}(\alpha, \beta)$ of two angles $\alpha, \beta$ is defined as the concatenation $\alpha \beta$ of $\alpha$ and $\beta$ if both angles wind around the same puncture and $\alpha$ starts where $\beta$ ends:

$$
\mu^{1}:=0, \quad \mu^{2}(\alpha, \beta):=(-1)^{|\beta|} \alpha \beta .
$$

The higher products $\mu^{\geq 3}$ on $\operatorname{Gtl} Q$ are defined in terms of what we will call discrete immersed disks. Roughly speaking, a discrete immersed disk may either be a polygon, or a sequence of polygons stitched together. We make this precise by regarding immersions of the standard polygon $P_{k}$, depicted in Figure 17.1a
Definition 17.5. Let $Q$ be dimer. A discrete immersed disk in $Q$ consists of an oriented immersion $D: P_{k} \rightarrow|Q|$ of a standard polygon $P_{k}$ into the surface, such that

- The edges of the polygon are mapped to a sequence of arcs.
- The immersion does not cover any punctures.

The immersion mapping itself is only taken up to reparametrization. The sequence of interior angles of $D$ is the sequence of angles in $Q$ given as images of the interior angles of $P_{k}$ under the map $D$. An angle sequence $\alpha_{1}, \ldots, \alpha_{k}$ is a disk sequence if it is the sequence of interior angles of some discrete immersed disk.

We can now describe the higher products $\mu^{\geq 3}$ on $\operatorname{Gtl} Q$ as follows:
Definition 17.6. Let $Q$ be a dimer. Then the gentle algebra $\mathrm{Gtl} Q$ of $Q$ is the $A_{\infty}$-category with objects the arcs $a \in Q_{1}$, hom spaces spanned by angles, and $A_{\infty}$-product $\mu$ defined by $\mu^{1}=0$ and $\mu^{2}(\alpha, \beta)=(-1)^{|\beta|} \alpha \beta$. To define $\mu^{k \geq 3}$, let $\alpha_{1}, \ldots, \alpha_{k}$ be any disk sequence, let $\beta$ be an angle composable with $\alpha_{1}$, i.e. $\beta \alpha_{1} \neq 0$, and let $\gamma$ be an angle post-composable with $\alpha_{k}$, i.e. $\alpha_{k} \gamma \neq 0$. Then

$$
\mu^{k}\left(\beta \alpha_{k}, \ldots, \alpha_{1}\right):=\beta, \quad \mu^{k}\left(\alpha_{k}, \ldots, \alpha_{1} \gamma\right):=(-1)^{|\gamma|} \gamma
$$

The higher products vanish on all angle sequences other than these.
In Paper I we defined a deformation $\mathrm{Gtl}_{q} Q$ of $\mathrm{Gtl} Q$, hoping it to provide a discrete relative Fukaya category. In Paper II] we confirmed this hope. The starting point for the definition of $\mathrm{Gtl}_{q} Q$ is a dimer which satisfies the [NMDC] condition. The category $\mathrm{Gtl}_{q} Q$ is a curved $A_{\infty}$-deformation of Gtl $Q$ over the deformation base $B=\mathbb{C} \llbracket Q_{0} \rrbracket$, which has one deformation parameter per puncture. We denote the maximal ideal by $\mathfrak{m}=\left(Q_{0}\right) \subseteq \mathbb{C} \llbracket Q_{0} \rrbracket$.

The curvature $\mu_{q}^{0}$ of $\operatorname{Gtl}_{q} Q$ is defined as follows: For every puncture $q \in Q_{0}$, denote by $\ell_{q}$ the sum of all full turns around $q$, summed over all arc ends at $q$. The total curvature $\mu_{q}^{0}$ of $\operatorname{Gtl}_{q} Q$ is defined as the sum over all puncture contributions:

$$
\mu_{q}^{0}:=\sum_{q \in Q_{0}} q \ell_{q} .
$$

The product $\mu_{q}^{1}$ still vanishes and the product $\mu_{q}^{2}:=\mu^{2}$ is not deformed. The higher products $\mu_{\bar{q}}^{\geq 3}$ however count discrete immersed disks which are now also allowed to cover punctures, weighting the result of every disk with the product $\in \mathbb{C} \llbracket Q_{0} \rrbracket$ of the punctures covered. The definition of $\mu_{q}^{0}$ and $\mu_{q}^{\geq 3}$ is depicted in Figure 17.1.


Figure 17.2: A zigzag path $L$


Figure 17.3: On consistency

### 17.2 Zigzag paths

In this section, we recall the notions of zigzag paths and geometric consistency for dimers. An exhaustive reference is 17 . More information is also found in Paper II
Definition 17.7. Let $Q$ be a dimer. A zigzag path $L$ is an infinite path $\ldots a_{2} a_{1} a_{0} a_{-1} a_{-2} \ldots$ of arcs in $Q$ together with an alternating choice of "left" or "right" for every $i \in \mathbb{N}$ such that

- $a_{i+1} a_{i}$ lies in a clockwise polygon if $i$ is assigned "right",
- $a_{i+1} a_{i}$ lies in a counterclockwise polygon if $i$ is assigned "left".

We also say that $L$ turns left at $a_{i}$ if $i$ is assigned "left" and turns right if $a_{i}$ is assigned "right". Two zigzag paths are identified if their paths including left/right indications differ only by integer shift.

Slightly simplified, a zigzag path $L$ is a path in $Q$ that turns alternatingly maximally right and maximally left in $Q$. The typical shape of a zigzag path is drawn in Figure 17.2 If every puncture of $Q$ has valence at least 4 , then the left/right indication is a superfluous datum. In this case, the left/right indication for zigzag paths is a superfluous part of the datum of a zigzag path. For other dimers $Q$, the left/right indication is very important. An example is the $M$-punctured sphere $Q_{M}$ which we will recall in section 18 Our definition deviates slightly from the definition of 18 .

Geometric consistency is a specific instance of various consistency conditions which can be imposed on dimers. To define it, we denote by $\tilde{Q}$ the lift of the arc system $Q$ to the universal cover of the closed surface $|Q|$. We define the auxiliary notion of zigzag rays as follows, depicted in Figure 17.3a
Definition 17.8. Let $a \in \tilde{Q}_{1}$ be an arc. Then the four zigzag rays starting at $a$ are the sequences of $\operatorname{arcs}\left(a_{i}^{1}\right)_{i \geq 0},\left(a_{i}^{2}\right)_{i \geq 0},\left(a_{i}^{3}\right)_{i \geq 0}$ and $\left(a_{i}^{4}\right)_{i \geq 0}$ in $\tilde{Q}$ determined by $a_{0}^{1}=a_{0}^{2}=a_{0}^{3}=a_{0}^{4}=a$ and the following properties:

- The sequences $\left(a_{i}^{1}\right)$ and $\left(a_{i}^{2}\right)$ satisfy $h\left(a_{i}^{1 / 2}\right)=t\left(a_{i+1}^{1 / 2}\right)$.
- The sequences $\left(a_{i}^{3}\right)$ and $\left(a_{i}^{4}\right)$ satisfy $t\left(a_{i}^{3 / 4}\right)=h\left(a_{i+1}^{3 / 4}\right)$.
- The path $a_{i+1}^{1 / 2} a_{i}^{1 / 2}$ lies in the boundary of a counterclockwise polygon when $i$ is odd/even, and clockwise when $i$ is even/odd.
- The path $a_{i}^{3 / 4} a_{i+1}^{3 / 4}$ lies in the boundary of a counterclockwise polygon when $i$ is odd/even, and clockwise when $i$ is even/odd.
A dimer is geometrically consistent if the zigzag rays starting with an arc $a$ in the universal cover intersect nowhere, except at $a$ itself. The precise definition reads as follows:

Definition 17.9. Let $Q$ be a dimer. Then $Q$ is geometrically consistent if for every $a \in \tilde{Q}_{1}$ the four zigzag rays $\left(a_{i}^{1}\right),\left(a_{i}^{2}\right),\left(a_{i}^{3}\right)$ and $\left(a_{i}^{4}\right)$ satisfy the following property: Whenever $a_{i}^{k}=a_{j}^{l}$, then $i=j$ and $k=l$, or $i=j=0$.
Remark 17.10. Many dimers are geometrically consistent. In contrast, a dimer on a sphere is never geometrically consistent. A dimer which contains the pattern sketched in Figure 17.3b is also not geometrically consistent. A geometrically consistent dimer satisfies the [NMDC] condition.

### 17.3 Matrix factorizations

In this section, we recall the notion of matrix factorizations. After recalling the definition, we focus on matrix factorization categories. Matrix factorizations serve as B-side in mirror symmetry. As such, our standard reference is 18 .

Matrix factorizations go back to 20th century work of Buchweitz and others. In a nutshell, the observation is as follows: Let $A$ be an algebra and $\ell \in Z(A)$ a central element. Such a pair $(A, \ell)$ is also called a Landau-Ginzburg model. If $\ell$ is prime in $A$, there is no nontrivial factorization $\ell=a b$ in $A$. However, there may be modules $M, N \in \operatorname{Mod} A$ with maps $f: M \rightarrow N$ and $g: N \rightarrow M$ such that both $g \circ f: M \rightarrow M$ and $f \circ g: N \rightarrow N$ are multiplication by $\ell$. In other words, for typical $\ell$ there are more factorizations on the module level than in the algebra itself. This gives rise to the following definition:

Definition 17.11. Let $A$ be an associative algebra and $\ell \in A$ a central element. A matrix factorization of $(A, \ell)$ is a pair of finitely generated projective $A$-modules $(P, Q)$ together with $A$-module morphisms $f: P \rightarrow Q$ and $g: Q \rightarrow P$ such that $f \circ g=\ell \operatorname{id}_{Q}$ and $g \circ f=\ell \operatorname{id}_{P}$.

Remark 17.12. There is an alternative definition: A matrix factorization is a $\mathbb{Z} / 2 \mathbb{Z}$-graded projective $A$-module $M$ together with an odd $A$-module map $\delta: M \rightarrow M$ such that $\delta^{2}=\ell \mathrm{id}_{M}$. This definition is equivalent to Definition 17.11 Given $(M, \delta)$, simply put $P:=M^{\text {even }}$ and $Q:=M^{\text {odd }}$. The odd map $\delta$ then automatically splits into two maps $f: P \rightarrow Q$ and $g: Q \rightarrow P$. Conversely given $f: P \rightleftarrows Q: g$, put

$$
M=P \oplus Q[1], \quad \delta=\left(\begin{array}{ll}
0 & g \\
f & 0
\end{array}\right)
$$

The set of matrix factorizations of $(A, \ell)$ can be turned into a $\mathbb{Z} / 2 \mathbb{Z}$-graded dg category $\operatorname{MF}(A, \ell)$. The intuition behind the dg structure is to interpret every matrix factorization $(M, \delta)$ as an almost chain complex, more precisely a twisted complex over the curved $A_{\infty}$-category $(A, \ell)$. We use the notation $\tilde{\delta}$ to denote the tweaked version $\tilde{\delta}(m)=(-1)^{|m|} \delta(m)$ of $\delta$.

Definition 17.13. Let $(A, \ell)$ be a Landau-Ginzburg model. The category of matrix factorizations $\operatorname{MF}(A, \ell)$ is defined as follows:

- Objects are the matrix factorizations $(M, \delta)$ of $(A, \ell)$.
- Hom spaces are given by $\operatorname{Hom}\left(\left(M, \delta_{M}\right),\left(N, \delta_{N}\right)\right)=\operatorname{Hom}_{A}(M, N)$, naturally $\mathbb{Z} / 2 \mathbb{Z}$-graded.
- The differential is given by $\mu^{1}(f)=\tilde{\delta}_{N} \circ f-(-1)^{|f|} f \circ \tilde{\delta}_{M}$ for $f \in \operatorname{Hom}\left(\left(M, \delta_{M}\right),\left(N, \delta_{N}\right)\right)$.
- The product is given by $\mu^{2}(f, g)=(-1)^{\|f\||g|} f \circ g$.

More explicitly, regard two matrix factorizations $f: P \rightleftarrows Q: g$ and $f^{\prime}: P^{\prime} \rightleftarrows Q^{\prime}: g^{\prime}$. Then their hom space is

$$
\underbrace{\operatorname{Hom}\left(P, P^{\prime}\right) \oplus \operatorname{Hom}\left(Q, Q^{\prime}\right)}_{\text {even }} \oplus \underbrace{\operatorname{Hom}\left(P, Q^{\prime}\right) \oplus \operatorname{Hom}\left(Q, P^{\prime}\right)}_{\text {odd }}
$$

A morphism can be presented as a 2-by-2 matrix $\left(\begin{array}{ll}A & B \\ C & D\end{array}\right)$, where $A: P \rightarrow P^{\prime}$ and $B: Q \rightarrow P^{\prime}$ and so on. In these terms, we can write

$$
\mu_{\mathrm{MF}(A, \ell)}^{1}\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right)=\left(\begin{array}{cc}
-g^{\prime} C+B f & -g^{\prime} D+A g \\
f^{\prime} A-D f & f^{\prime} B-C g
\end{array}\right)
$$

The product $\mu_{\mathrm{MF}(A, \ell)}^{2}$ is simply given by signed matrix multiplication

$$
\mu_{\mathrm{MF}(A, \ell)}^{2}\left(\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right),\left(\begin{array}{ll}
A^{\prime} & B^{\prime} \\
C^{\prime} & D^{\prime}
\end{array}\right)\right)=\left(\begin{array}{cc}
A A^{\prime}+B C^{\prime} & -A B^{\prime}+B D^{\prime} \\
C A^{\prime}-D C^{\prime} & C B^{\prime}+D D^{\prime}
\end{array}\right)
$$



Figure 17.4: F-term equivalence

### 17.4 Jacobi algebras of dimers

In this section, we recollect the Jacobi algebras of dimers, together with their associated categories of matrix factorizations. We also fix some notation in this section. For example, the Jacobi algebra of a dimer will be denoted $\operatorname{Jac} Q$, its special central element will be denoted $\ell \in \operatorname{Jac} Q$. We would like to remind the reader that the full category of matrix factorizations is denoted $\operatorname{MF}(\operatorname{Jac} Q, \ell)$. Meanwhile, we recall in this section a specific small subcategory, denoted by lowercase letters $\operatorname{mf}(\operatorname{Jac} Q, \ell) \subseteq \operatorname{MF}(\operatorname{Jac} Q, \ell)$.

Let us start by fixing terminology and notation for cyclicity:
Definition 17.14. Let $Q$ be a quiver. If $p$ is a path in $Q$, we denote by $p_{\text {cyc }} \in \mathbb{C} Q$ the sum of its cyclic permutations. We extend this assignment linearly to $\mathbb{C} Q$ and denote it by $p \mapsto p_{\text {cyc }}$ as well. An element $W \in \mathbb{C} Q$ is cyclic if it is a linear combination of cyclic paths in $Q$ whose coefficients are invariant under cyclic permutation:

$$
W=\sum_{\text {cycles } a_{k} \ldots a_{1}} \lambda_{a_{k} \ldots a_{1}} a_{k} \ldots a_{1}, \quad \text { with } \quad \forall i=1, \ldots, k: \quad \lambda_{a_{k} \ldots a_{1}}=\lambda_{a_{i-1} \ldots a_{i+1} a_{i}} .
$$

We recall that superpotentials are defined as cyclic elements of length at least two:
Definition 17.15. A superpotential is a cyclic element $W \in \mathbb{C} Q_{\geq 2}$. Its relations are the elements

$$
\partial_{a} W=\sum_{\substack{\text { paths } k_{k} \ldots a_{1} \\ \text { with } a_{k}=a}} \lambda_{a_{k} \ldots a_{1}} a_{k-1} \ldots a_{1}, \quad a \in Q_{1}
$$

Its Jacobi algebra is given by

$$
\operatorname{Jac}(Q, W)=\frac{\mathbb{C} Q}{\left(\partial_{a} W\right)}
$$

Here $\left(\partial_{a} W\right)$ denotes the two-sided ideal generated by the partial derivatives $\partial_{a} W$ for $a \in Q_{1}$.
A dimer $Q$ is nothing else than a specific type of quiver embedded in a surface. In particular it comes with an associated path algebra $\mathbb{C} Q$. The dimer structure of $Q$ provides us with an additional central element $W \in \mathbb{C} Q$, given by the difference of the clockwise polygons of $Q$ and the counterclockwise polygons, cyclically permuted:

$$
W=\sum_{\substack{a_{1}, \ldots, a_{k} \\ \text { clockwise }}}\left(a_{1} \ldots a_{k}\right)_{\mathrm{cyc}}-\sum_{\substack{a_{1}, \ldots, a_{k} \\ \text { counterclockwise }}}\left(a_{1} \ldots a_{k}\right)_{\mathrm{cyc}} .
$$

Definition 17.16. Let $Q$ be a dimer. Then its Jacobi algebra is the associative algebra Jac $Q=$ $\mathbb{C} Q /\left(\partial_{a} W\right)$.

The relations $\partial_{a} W$ equate two neighboring polygons: Flipping a path over an arc $a$ is possible if the path follows all arcs of a neighboring polygon apart from $a$. These flip moves are known as $\mathbf{F}$-term moves and the equivalence relation on the set of paths in $Q$ is known as F-term equivalence. The terminology is depicted in Figure 17.4 A good reference is 28.

Regard the set of paths in $Q$ modulo F-term equivalence. The set contains a special element $\ell_{v}$ for each vertex $v \in Q_{0}$, given by the boundary of a chosen polygon incident at $v$. All boundaries of polygons incident at $v$ are F-term equivalent, hence $\ell_{v}$ does not depend on the choice. In other words, it can be rotated around $v$. We may drop the subscript from $\ell_{v}$ if it is clear from the context. The element $\ell$ commutes with all paths, that is, $u \ell \sim \ell u$. Davison 28 introduced the following consistency condition for dimers:


Figure 17.5: Three-punctured sphere and its mirror dimer

Definition $17.17(\boxed{28})$. A dimer $Q$ is cancellation consistent if it has the following cancellation property:

$$
p \ell \sim q \ell \Longrightarrow p \sim q .
$$

The Jacobi algebra contains a special element which we denote by $\ell$ as well and which is called the potential. It is given by the sum of the elements $\ell_{v}$ over $v \in Q_{0}$ :

$$
\ell=\sum_{v \in Q_{0}} \ell_{v} \in \operatorname{Jac} Q
$$

The relations of Jac $Q$ ensure that $\ell$ is central and as an element of $\operatorname{Jac} Q$ is independent of the choices of incident polygons.

We are ready to discuss matrix factorizations of the Landau-Ginzburg model (Jac $Q, \ell$ ). For every vertex $v \in Q_{0}$, the module $(\operatorname{Jac} Q) v$ is projective. There are many further projectives, for instance given by taking direct sums of these elementary projectives. The hom space $\operatorname{Hom}_{\mathrm{Jac}} Q((\operatorname{Jac} Q) v,(\operatorname{Jac} Q) w)$ between two standard projectives is naturally identified with $v(\operatorname{Jac} Q) w$, the subspace of paths in Jac $Q$ starting at $w$ and ending at $v$. A matrix factorization between two such projectives can be visualized as a bipartite graph consisting of vertices in $Q$, connected by paths in $Q$, such that all products sum up to $\ell$.

There is a special subcategory $\operatorname{mf}(\operatorname{Jac} Q, \ell) \subseteq \operatorname{MF}(\operatorname{Jac} Q, \ell)$. The idea is that every polygon boundary can be factorized as the product of a single boundary arrow and all the other boundary arrows. More precisely, let $a \in Q_{1}$ be an arrow. There are precisely two polygons neighboring $a$. The complements of their boundary are paths $r_{a}^{+}$and $r_{a}^{-}$. Within $\operatorname{Jac} Q$, these two paths are identified and we simply denote them by $\bar{a}=r_{a}^{+}=r_{a}^{-} \in \operatorname{Jac} Q$. Since $a \bar{a}=\ell_{h(a)}$ and $\bar{a} a=\ell_{t(a)}$, we can build matrix factorizations from $a$ and $\bar{a}$ :

Definition 17.18. Let $Q$ be a dimer and $a \in Q_{1}$. Then $\operatorname{mf}(\operatorname{Jac} Q, \ell) \subseteq \operatorname{MF}(\operatorname{Jac} Q, \ell)$ is the subcategory given by the matrix factorizations

$$
M_{a}=(\operatorname{Jac} Q) h(a) \underset{\stackrel{a}{a}}{\stackrel{a}{\rightleftarrows}}(\operatorname{Jac} Q) t(a), \quad a \in Q_{1}
$$

Remark 17.19. These matrix factorizations $M_{a}$ are no factorizations of $\ell$ as element of Jac $Q$. Instead their behavior is "local". We expect that the collection $\left\{M_{a}\right\}_{a \in Q_{1}}$ already generates $\mathrm{H} \operatorname{Tw} \operatorname{MF}(\operatorname{Jac} Q, \ell)$ under shifts and cones. Such a result might be obtained by a local analysis or re-interpretation in terms of the commutative model of 36 or 58 .

### 17.5 Mirror symmetry for punctured surfaces

In this section, we recall mirror symmetry for punctured surfaces from 18 . We start with an overview of the ingredients and the original proof. In particular, we exhibit A- and B-side of this mirror equivalence, explain the equivalence on object level and on hom spaces. Regarding terminology, we recall here the construction of the dual dimer $\breve{Q}$ attached to $Q$. To prepare the reader for the rest of the paper, we explain why the original setup makes it so hard to deform mirror symmetry and how the more recent Cho-Hong-Lau construction solves this issue.

The basic ingredient for mirror symmetry for punctured surfaces is a dimer $Q$ and its dual dimer $\check{Q}$, defined as follows:

Definition 17.20. Let $Q$ be a dimer. Then its dual dimer $\mathscr{Q}$ is the dimer obtained by cutting $Q$ into its polygons, flipping over the counterclockwise polygons and inverting their arrows, and gluing everything together again along the arrows.


Figure 17.6: Opposite paths for even and odd $k$

Example 17.21. In Figure 17.5 we have depicted the example of $Q$ being the three-punctured sphere. Its mirror dimer $\check{Q}$ is a one-punctured torus. In this example, we have $\operatorname{Jac} \check{Q}=\mathbb{C}[a, b, c]$ and $\ell=a b c$.
Remark 17.22. A basic observation is that punctures in $Q$ correspond to zigzag paths in $\check{Q}$. Moreover, let $\alpha: a \rightarrow b$ be an angle in $Q$. Denote by $k \geq 0$ its "length", or the number of indecomposable angles contained in $\alpha$. Then $\alpha$ can be reinterpreted in the mirror as a zigzag segment $Z_{\alpha}$ given by $a=z_{0}, z_{1}, \ldots, z_{k}=b$ between the arcs $a$ and $b$ in $\check{Q}$, with the property that $h\left(z_{i}\right)=t\left(z_{i+1}\right)$.

In the remainder of this section, we explain the following mirror symmetry of punctured surfaces:
Theorem $17.23(\underline{18})$. Let $Q$ be a dimer such that the dual dimer $\check{Q}$ is cancellation consistent. Then there exists an isomorphism of $\mathbb{Z} / 2 \mathbb{Z}$-graded $A_{\infty}$-categories

$$
F: \operatorname{Gtl} Q \rightarrow \operatorname{Hmf}(\operatorname{Jac} \check{Q}, \ell)
$$

Let us recollect this functor $F$ on object level. The objects of $\mathrm{Gtl} Q$ are the arcs of $Q$. The $\operatorname{arcs}$ of $Q$ are in bijection with those of $\check{Q}$. The category $\operatorname{Hmf}(\operatorname{Jac} \check{Q}, \ell)$ is the minimal model of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ and as such has the same objects as $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$. Combining these three observations, it is easy to grasp the functor $F$ on object level: It simply maps $a \in Q_{1}$ to the matrix factorization $M_{a} \in \operatorname{Hmf}(\operatorname{Jac} \mathscr{Q}, \ell)$.

We can also grasp $F$ on the level of morphisms. Recall that the hom space $\operatorname{Hom}_{\operatorname{Gt1} Q}(a, b)$ is spanned by angles starting on $a$ and ending on $b$, rotating around a common puncture. The corresponding basis for $\operatorname{Hom}_{\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)}\left(M_{a}, M_{b}\right)$ was determined in 18, Lemma 8.3]. Let $\alpha: a \rightarrow b$ be an angle in $Q$. By Remark 17.22 the angle $\alpha$ comes with an associated zigzag segment $Z_{\alpha}$ in $\check{Q}$. Define opp ${ }_{1}$ to be the path from $t(b)$ to $t(a)$ and $\mathrm{opp}_{2}$ the path from $h(b)$ to $h(a)$ along $Z_{\alpha}$. This definition is depicted in Figure 17.6. Denote by $\zeta\left(Z_{\alpha}\right)$ the morphism of matrix factorizations $\zeta\left(Z_{\alpha}\right): M_{a} \rightarrow M_{b}$ given by

$$
\zeta\left(Z_{\alpha}\right)=\left(\begin{array}{cc}
\mathrm{opp}_{2} & 0 \\
0 & \mathrm{opp}_{1}
\end{array}\right)
$$

In case $k$ is odd, let $\mathrm{opp}_{1}$ be the path from $h(b)$ to $t(a)$ and opp ${ }_{2}$ the path from $t(b)$ to $h(a)$. Denote by $\zeta\left(Z_{\alpha}\right)$ the morphism of matrix factorizations $\zeta(Z): M_{a} \rightarrow M_{b}$ given by

$$
\zeta\left(Z_{\alpha}\right)=\left(\begin{array}{cc}
0 & \mathrm{opp}_{1} \\
\mathrm{opp}_{2} & 0
\end{array}\right)
$$

It is an easy check that for any parity of $k$ the morphism $\zeta\left(Z_{\alpha}\right)$ is a closed morphism in $\operatorname{mf}(\operatorname{Jac} Q, \ell)$ in the sense that $\mu_{\operatorname{mf}(\operatorname{Jac} \mathscr{Q}, \ell)}^{1}(\zeta(Z))=0$. In terms of $\zeta\left(Z_{\alpha}\right)$, the functor $F$ on level of objects and on the level $F^{1}$ on hom spaces is given by

$$
\begin{aligned}
F(a) & =M_{a}, \quad \forall a \in Q_{1} \\
F^{1}(\alpha) & =\zeta\left(Z_{\alpha}\right), \quad \forall \alpha: a \rightarrow b
\end{aligned}
$$

Remark 17.24. This minimal model $\operatorname{Hmf}(\operatorname{Jac} Q, \ell)$ is by no means canonical. It can be calculated by the Kadeishvili construction, which however depends on the a choice of a so-called homological splitting for all hom spaces $\inf \operatorname{mac} \check{Q}, \ell)$. In 18, it was observed that whatever splitting is chosen, the map $F^{1}$ alone is never an $A_{\infty}$-functor. Instead, it is necessary to include higher components $F \geq 2$.

The higher products $F^{\geq 2}$ have been constructed in 18 , Appendix A]. The idea is to construct $F^{k+1}$ inductively from $F^{1}, \ldots, F^{k}$. A first important ingredient is knowledge of the products

$$
\mu_{\mathrm{H} \operatorname{mf}(\mathrm{Jac} \check{Q}, \ell)}^{1}\left(F^{1}\left(\alpha_{k}\right), \ldots, F^{1}\left(\alpha_{1}\right)\right)
$$

at least for sequences $\alpha_{1}, \ldots, \alpha_{k}$ which are consecutive interior angles of some polygon. The second ingredient is a temporary restriction to the case that $Q$ is large enough to force certain unknown terms to vanish. With these two premises the component $F^{k+1}$ can be constructed inductively.

Remark 17.25. The construction of $F^{k+1}$ consists of solving a Hochschild cocycle equation, which of course has no unique solution. Correspondingly, the functor $F$ cannot be computed explicitly from 18.

The lack of explicit functor $F$ seemed to make it very difficult to deform mirror symmetry: Once we deform $\operatorname{Gtl} Q$ to $\operatorname{Gtl}_{q} Q$, there must be a deformation $\operatorname{mf}_{q}(\operatorname{Jac} \check{Q}, \ell)$ of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ such that $\operatorname{Gtl}_{q} Q$ and $\operatorname{Hmf}_{q}(\operatorname{Jac} \check{Q}, \ell)$ are still isomorphic. Actually finding this mirror deformation is a nontrivial task. The most basic approach is to write down the $A_{\infty}$-functor equations and find manually a collection of deformed $A_{\infty}$-products on $\operatorname{Hmf}(\operatorname{Jac} \check{Q}, \ell)$ which still keep $F$ a functor. This is impossible if $F$ itself is unknown.

Fortunately, a modern explicit construction of a mirror functor $F: \operatorname{Gtl} Q \rightarrow \operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ is available due to Cho, Hong and Lau [26. We shall explain here what makes this Cho-Hong-Lau construction so suited for deformations. The idea can be formulated in more generality: Let $\mathcal{C}$ be an $A_{\infty}$-category and $\mathbb{L} \subseteq \mathcal{C}$ an subcategory. Then the Cho-Hong-Lau construction produces from the structure of the endomorphism algebra of $\mathbb{L}$ a mirror Landau-Ginzburg model $(J, \ell)$ and a functor $F: \mathcal{C} \rightarrow \operatorname{MF}(J, \ell)$ :

$$
\begin{array}{ccc}
\mathbb{L} \subseteq \mathcal{C} \\
A_{\infty} \text {-category with subcategory } & \sim & \begin{array}{c}
\mathcal{C} \rightarrow \operatorname{mf}(J, \ell) \\
\text { mirror functor }
\end{array}
\end{array}
$$

Remark 17.26. Let $\mathcal{C}=\mathrm{HTw} \operatorname{Gtl} Q$ and $\mathbb{L} \subseteq \mathrm{HTw} \operatorname{Gtl} Q$ be the category of zigzag paths in $Q$, which we recall in section 18.1. Then the mirror $(J, \ell)$ is precisely $(\operatorname{Jac} \check{Q}, \ell)$ and the construction gives an explicit functor $\bar{F}: \operatorname{Gtl} Q \rightarrow \mathrm{MF}(\operatorname{Jac} \check{Q}, \ell)$. We recall this fact in more detail in section 21.1

In the context of deformations, the Cho-Hong-Lau construction is extraordinarily useful. Simply speaking, applying the construction to the deformed category $\mathrm{Gtl}_{q} Q$ yields a deformed Landau-Ginzburg $\operatorname{model}\left(\mathrm{Jac}_{q} \check{Q}, \ell_{q}\right)$ and a deformed functor $F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{MF}\left(\mathrm{Jac}_{q} \check{Q}, \ell_{q}\right)$. In the rest of this paper, we make this rigorous. One bottleneck consists of proving that $\mathrm{Jac}_{q} \check{Q}$ is actually a deformation of $\operatorname{Jac} \check{Q}$.

## 18 Preliminaries on the category of zigzag paths

In this section, we recollect the description of the deformed category of zigzag paths from Paper II. The aim is to translate the material in such a way that it becomes directly usable in section 21 .

In section 18.1, we recall the three ways of thinking about zigzag paths: as zigzag paths in the dimer $Q$, as zigzag curves in the surface $|Q|$, and as twisted complexes lying in Tw Gtl $Q$. In particular, we recall the interpretation of intersection points between two zigzag curves as basis of cohomology hom space of the corresponding twisted complexes. We recall the definition of the $A_{\infty}$-category $\mathbb{L}$ of zigzag paths. In section 18.2 we recall the deformed counterpart $\mathbb{L}_{q}$ of $\mathbb{L}$. In section 18.3 , we review the deformed $A_{\infty}$-structure on $\mathrm{H}_{q}$ in terms of CR, ID, DS and DW disks. Overall, we try to introduce the reader to the translation presented in Table 18.1. In section 18.4 we review another class of $A_{\infty}$-products on the minimal model $\mathrm{HTw} \mathrm{Gtl}_{q} Q$.

Throughout this section, $Q$ denotes a fixed geometrically consistent dimer or a standard sphere dimer $Q_{M}$ with $M \geq 3$, depicted in Figure 18.2a and 18.2b For section 18.3 and 18.4 , we assume additionally Convention 18.1.

### 18.1 Category of zigzag paths

In this section, we review the category $\mathbb{L}$ of zigzag paths fromPaper II Throughout, $Q$ is a geometrically consistent dimer or one of the standard sphere dimers $Q_{M}$ with $M \geq 3$.

There is a triad correspondence between zigzag paths, zigzag curves and corresponding twisted complexes. We have depicted this in Figure 18.3 Let us explain their relation as follows:

- Zigzag paths are combinatorial gadgets in $Q$.
- Zigzag curves are defined as their smoothed analogs in the surface $|Q|$.
- A zigzag path $L$ comes with a canonical twisted complex presentation also denoted $L \in \operatorname{Tw} \mathrm{Gtl} Q$. The datum of $L \in \mathrm{Tw} \operatorname{Gtl} Q$ includes a $\delta$-matrix consisting of angles between the arcs lying on the zigzag path.

| Gadget | Discrete | Smooth |
| :---: | :---: | :---: |
| Input datum | $Q$ | $\|Q\| \backslash Q_{0}$ |
| (deformed) | $Q$ | $\left(\|Q\|, Q_{0}\right)$ |
| Wrapped category | $\mathrm{Gtl} Q$ | wFuk $\left(\|Q\| \backslash Q_{0}\right)$ |
| (deformed) | $\operatorname{Gtl}_{q} Q$ | nonexistent |
| Zigzag object | $L$ | $\tilde{L} \subseteq\|Q\| \backslash Q_{0}$ |
| (deformed) | $L$ | $\tilde{L} \subseteq\|Q\|$ |
| Zigzag category | $\mathrm{H} \mathbb{L}$ | " $\widetilde{\mathbb{L}}$ |
| (deformed) | $\mathrm{H} \mathbb{L}_{q}$ | " $\mathbb{L}_{q} "$ |
| Basis morphisms | $h \in \operatorname{Hom}_{\mathrm{H} \mathbb{L}}\left(L_{1}, L_{2}\right)$ | $p \in \tilde{L}_{1} \cap \tilde{L}_{2}$ |

$A_{\infty}$-products $\quad \mathrm{CR} / \mathrm{ID} / \mathrm{DS} / \mathrm{DW}$ disks Morse-Bott disks
Table 18.1: Translation between discrete and smooth


Figure 18.2: The sphere dimers and their zigzag curves

In our setting, we allow additional signs in the entries of the $\delta$-matrix of the twisted complex. Once a specific choice of signs has been selected for every zigzag path, the sets of zigzag paths, zigzag curves and their twisted complexes are in one-to-one correspondence. We therefore allow ourselves to switch liberally between the three corresponding objects. For additional clarity, we may denote zigzag paths or their twisted complexes by letters $L, L_{1}, \ldots$ and associated zigzag curves by $\tilde{L}, \tilde{L}_{1}, \ldots$.

Interpreting a zigzag curve as object in the wrapped Fukaya category or as a twisted complex in $\mathrm{Tw} \operatorname{Gtl} Q$ requires two more pieces of data. The first datum is the choice of spin structure. Choosing simultaneous spin structures for all zigzag paths in $Q$ is equivalent to associating a sign $(-1)^{\# \alpha}$ to every internal angle of every polygon in $Q$. Once the spin structure is chosen, we can form the twisted complex $L \in \mathrm{Tw} \operatorname{Gtl} Q$ for every zigzag path $L$, depicted with signs in Figure 18.3 For the purpose of the present paper, we choose the $\#$ signs in a very specific way, described in Convention 18.1 .

The second piece of data required is the choice of location for the identity and co-identity on every zigzag path. More precisely, this choice entails a choice of one of one indexed arc $a_{0}$ on $L$ and the selection of one single angle $\alpha_{0}$ out of all angles present in the $\delta$-matrix of $L$. In the twisted complex presentation, the choice of identity and co-identity location are not visible. They are however needed as a choice to compute the products in the minimal model of $\operatorname{Tw~}_{\mathrm{Gtl}_{q}} Q$, just like the products in the wrapped Fukaya category also depend on these choices.

We codify the convention on these choices as follows:
Convention 18.1. The dimer $Q$ is a geometrically consistent dimer or standard sphere dimer $Q_{M}$ with $M \geq 3$. Each zigzag path is supposed to come with a choice of an identity location $a_{0}$ and a co-identity location $\alpha_{0}$. The co-identity $\alpha_{0}$ shall be chosen to lie in a counterclockwise polygon. The spin structures


Figure 18.3: Three descriptions of a zigzag path


Figure 18.4: Intersection degree
is given by assigning to every interior angle $\alpha$ of a clockwise polygon the $\# \operatorname{sign} \# \alpha=0$ and to every interior angle $\alpha$ of a counterclockwise polygon the $\#$ sign $\# \alpha=1$.

Definition 18.2. The category of zigzag paths is the category $\mathbb{L} \subseteq \operatorname{Tw} \operatorname{Gtl} Q$ given by the twisted complexes associated with all zigzag paths of $Q$, each with its single associated choice of spin structure.

In Paper II we have investigated the minimal model $\mathrm{H} \mathrm{Tw} \mathrm{Gtl} Q$. The objects of this minimal model are the same as those of $\mathrm{Tw} \operatorname{Gtl} Q$. In particular, this category contains all twisted complexes associated with zigzag paths. However, the hom spaces are compressed in comparison to $\mathrm{Tw} \mathrm{Gtl} Q$. We have provided explicit basis elements for these hom spaces $\operatorname{Hom}_{\mathrm{HTw} \operatorname{Gt1} Q}\left(L_{1}, L_{2}\right)$ in Paper II The basis elements can be identified with intersection points of $\tilde{L}_{1}$ and $\tilde{L}_{2}$ :

Lemma 18.3. Let $L_{1}$ and $L_{2}$ be two zigzag paths. Then the intersection points between $\tilde{L}_{1}$ and $\tilde{L}_{2}$ naturally provide a basis for the hom space $\operatorname{Hom}_{\mathbb{H} \mathbb{L}}\left(L_{1}, L_{2}\right)$. In case $L_{1}=L_{2}$, this concerns only the transversal self-intersection points, which count double, plus the identity and co-identity points.

Remark 18.4. The description of the hom spaces by means of intersection points between $\tilde{L}_{1}$ and $\tilde{L}_{2}$ is expected from the derived equivalence of $\operatorname{Gtl} Q$ and the wrapped Fukaya category 18 . Under this equivalence, the zigzag path $L$ corresponds to the zigzag curve $\tilde{L}$. As such, the hom space $\operatorname{Hom}_{\mathrm{wFuk}\left(|Q|, Q_{0}\right)}\left(L_{1}, L_{2}\right)$ has basis given by intersections between $\tilde{L}_{1}$ and $\tilde{L}_{2}$. Since zigzag curves bound no digons in the punctured surface $|Q| \backslash Q_{0}$, the differential on $\operatorname{Hom}_{w F u k}\left(|Q|, Q_{0}\right)\left(\tilde{L}_{1}, \tilde{L}_{2}\right)$ vanishes and consequently intersection points also provide a basis for the cohomology $\operatorname{Hom}_{\text {wFuk }\left(|Q|, Q_{0}\right)}\left(\tilde{L}_{1}, \tilde{L}_{2}\right)$. In summary, this identifies intersections between $\tilde{L}_{1}$ and $\tilde{L}_{2}$ as hom space $\operatorname{Hom}_{\mathrm{H} \operatorname{Tw~Gtl} Q}\left(L_{1}, L_{2}\right)$.

Every basis morphism $p: L_{1} \rightarrow L_{2}$ comes with a degree $|p| \in \mathbb{Z} / 2 \mathbb{Z}$ assigned. The degree depends on the orientation of the intersection between $\tilde{L}_{1}$ and $\tilde{L}_{2}$. The precise convention is depicted in Figure 18.4

Remark 18.5. The basis of the hom space $\operatorname{Hom}_{\mathrm{HTw}_{\mathrm{Gtl}}( }\left(L_{1}, L_{2}\right)$ is special in case $L_{1}=L_{2}$. Its basis is then given by transversal intersections plus two special morphisms, namely the identity and co-identity. The transversal intersections are self-intersections, and they give in fact two morphisms in H Tw Gtl $Q$, of which one is odd and the other is even. Whenever we refer to self-intersections of a zigzag curve, it is understood that the datum of a self-intersection shall include the choice of whether we mean the odd or the even morphism. For more details we refer to Paper II or 16. Chapter 9].

### 18.2 Deformed category of zigzag paths

In this section, we recall the deformation $\mathbb{L}_{q}$ of $\mathbb{L}$ from Paper II First, we recall the use of the deformed gentle algebra $\mathrm{Gtl}_{q} Q$. We then explain that the zigzag curves survive upon deforming $\mathrm{Gtl} Q$ to $\mathrm{Gtl}_{q} Q$. Finally, we recall how the process of "uncurving" produces the deformed, yet curvature-free category $\mathbb{L}_{q}$.

We are interested in the minimal model of $\mathrm{Tw} \mathrm{Gtl}_{q} Q$. For our purposes, it suffices in fact to look at the subcategory of $\mathrm{Tw} \mathrm{Gtl}_{q} Q$ given by zigzag paths:

Definition 18.6. The category $\mathbb{L}_{q}^{\text {pre }} \subseteq \operatorname{Tw~}_{\operatorname{Gtl}}^{q}$ $Q$ is the subcategory consisting of the twisted complexes of all zigzag paths of $Q$, each with their chosen spin structure.

In Paper II we showed how to compute a minimal model of a deformed $A_{\infty}$-category $\mathcal{C}_{q}$. The first step in the procedure consists of optimizing curvature according to a well-defined prescription. The result of the curvature optimization procedure is an $A_{\infty}$-deformation that is gauge equivalent to $\mathcal{C}_{q}$.

In the specific case of $\mathbb{L}_{q}^{\text {pre }}$, we can explicitly describe the result $\mathbb{L}_{q}$ of the curvature optimization procedure:

Definition 18.7. Let $L$ be a zigzag path of $Q$, with associated twisted complex

$$
L=\left(a_{1} \oplus a_{3} \oplus \ldots \oplus a_{k} \oplus a_{2} \oplus \ldots \oplus a_{2 k}, \delta\right)
$$

as in Figure 18.3 Then the corresponding deformed zigzag path is the following object of $\mathrm{Tw}^{\prime} \mathrm{Gtl}_{q}$, still denoted $L$ :

$$
\begin{aligned}
L & =\left(a_{1} \oplus a_{3} \oplus \ldots \oplus a_{k} \oplus a_{2} \oplus \ldots \oplus a_{2 k}, \delta\right), \\
\delta & =\left[\begin{array}{ccccc|c} 
\\
\hline(-1)^{\# \alpha_{1}} q_{1} \alpha_{1}^{\prime} & (-1)^{\# \alpha_{2}} q_{2} \alpha_{2}^{\prime} & 0 & \ldots & 0 & \text { ditto } \\
0 & (-1)^{\# \alpha_{3}} q_{3} \alpha_{3}^{\prime} & (-1)^{\# \alpha_{4}} q_{4} \alpha_{4}^{\prime} & \cdots & 0 & \\
\ldots & \ldots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & (-1)^{\# \alpha_{2 k-2} q_{2 k-2} \alpha_{2 k-2}^{\prime}} & \\
(-1)^{\# \alpha_{2 k}} q_{2 k} \alpha_{2 k}^{\prime} & 0 & 0 & \cdots & (-1)^{\# \alpha_{2 k-1} q_{2 k-1} \alpha_{2 k-1}^{\prime}} &
\end{array}\right] .
\end{aligned}
$$

Here "ditto" denotes the same matrix entries as in Figure 18.3. The letter $q_{i}$ denotes the puncture around which $\alpha_{i}$ winds. The angle $\alpha_{i}^{\prime}$ is defined as the complementary angle to $\alpha_{i}$. In other words, the angle $\alpha_{i}^{\prime}$ is such that $\alpha_{i}^{\prime} \alpha_{i}$ comprises a single full turn around a puncture.

The category of deformed zigzag paths is the subcategory $\mathbb{L}_{q} \subseteq \mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} Q$ consisting of the deformed zigzag paths.

Remark 18.8. The definition of the deformed zigzag path is a specific application of the "complementary angle trick" laid out in Paper II On the level of twisted complexes, the complementary angle trick infinitesimally changes the $\delta$-matrices of the zigzag paths. The resulting $\delta$-matrices are not upper triangular anymore, whence the notation $\mathrm{Tw}^{\prime} \mathrm{Gtl}_{q} Q$. On the level of the category $\mathbb{L}_{q}^{\text {pre }}$ itself, the complementary angle trick consists of a specific gauge transformation.

Remark 18.9. The category $\mathbb{L}_{q}$ of deformed zigzag paths is a deformation of the category $\mathbb{L}$ of zigzag paths. In Paper II, we proved that $\mathbb{L}_{q}$ has "optimal curvature" in the sense of the deformed Kadeishvili theorem. In fact, if $Q$ is geometrically consistent, then $\mathbb{L}_{q}$ is curvature-free. If $Q=Q_{M}$ for $M$ odd, then $\mathbb{L}_{q}$ is curvature-free as well (due to choice of spin structure). If $Q=Q_{M}$ for $M$ even, then the curvature of the two zigzag paths in $\mathbb{L}_{q}$ is an infinitesimal multiple of their identity morphisms.

### 18.3 Minimal model structure

In this section, we recall the deformed $A_{\infty}$-products on $\mathrm{H} \mathbb{L}_{q}$ from Paper II As it turns out, these products have striking similarity with the products of the relative Fukaya category. Although we do not need Fukaya categories here, it is helpful to recall that their products are enumerated in terms of what we may call smooth immersed disks. In the present section, we recall that also the products of $\mathrm{H} \mathbb{L}_{q}$ can be enumerated in terms of certain types of smooth immersed disks. We recollect their precise rule.

The category $\mathbb{L}_{q}$ is a deformed $A_{\infty}$-category and as such has no classical minimal model. In Paper II. we define a notion of minimal models for deformed $A_{\infty}$-categories and show that every deformed $A_{\infty}$ has a minimal model. Moreover, we show how to compute minimal models by means of a deformed Kadeishvili construction. In Paper II, we compute the minimal model $H \mathbb{L}_{q}$ of $\mathbb{L}_{q}$ in its entirety.


Figure 18.5: This pictures depicts a typical CR disk. The disk has twelve inputs of which eight are co-identities. There are two groups of co-identities, each consisting of four co-identities stacked together. By definition, the zigzag curves on which the two stacks of co-identities lie are required to run clockwise, as indicated by the arrows.

Remark 18.10. The aim of our minimal model computation in Paper II was to identify the higher products of $\mathrm{H} \mathbb{L}_{q}$ as higher products of the relative Fukaya category. Here is a sketch of this computation: We provided an explicit homological splitting as well as corresponding deformed codifferential $h_{q}$ and deformed projection $\pi_{q}$ in terms of "tails of morphisms". It remained to evaluate all Kadeishvili trees. The essential step was to analyze Kadeishvili trees by what we called "result components". It turned out that every result component can be matched with a disk between the zigzag curves. Finally, we classified the disks that appeared this way into the types CR, ID, DS and DW. In other words, the minimal model $\mathrm{H} \mathbb{L}_{q}$ is given by intersection points of the zigzag curves, together with a deformed $A_{\infty}$-structure which counts disks.

The minimal model $\mathrm{H} \mathbb{L}_{q}$ can be expressed by means of counting CR, ID, DS and DW disks. The first step in this section is to recall these four types of disks. We give a verbal and visual characterization of these disks. As auxiliary disk type we recall the SL disks (shapeless disks), with a slightly abridged definition.

For the definition of these disk types, let $L_{1}, \ldots, L_{N+1}$ be a fixed sequence of $N+1 \geq 1$ zigzag paths and $h_{i}: L_{i} \rightarrow L_{i+1}$ for $i=1, \ldots, N$ some basis morphisms. Since identities among the inputs $h_{i}$ yield well-known products $\mu_{\mathrm{H} \mathbb{L}_{q}}\left(h_{N}, \ldots, h_{1}\right)$ by the unitality property of $\mathrm{H} \mathbb{L}_{q}$, we assume that none of the inputs $h_{i}$ is an identity. Note that co-identities are however allowed. Let $t: L_{1} \rightarrow L_{N+1}$ be another intersection point, the letter standing for "target candidate".

Definition 18.11. An SL disk (shapeless disk) with inputs $h_{1}, \ldots, h_{N}$ and output $t$ consists of an oriented immersion $D: P_{N+1} \rightarrow|Q|$ of the standard $(N+1)$-gon $P_{N+1}$ such that

- the $i$-th edge of $D$ is mapped to a segment of $\tilde{L}_{i}$, for $i=1, \ldots, N+1$,
- the $i$-th corner of $D$ is mapped to the intersection point corresponding with $h_{i}$, for $i=1, \ldots, N$,
- the $N+1$-th corner of $D$ is mapped to the intersection point corresponding with $t$,
- all corners of $D$ are convex.

The immersion $D$ itself is taken up to reparametrization.
Remark 18.12. SL disks are allowed to be monogons $(N=0)$ or digons $(N=1)$. All the $\tilde{L}_{i}$-segments are allowed to be empty. We can also imagine these empty segments as being infinitesimally short.

The standard polygon $P_{N+1}$ together with its numbering of corners and edges is depicted in Figure 17.1a. Note that disk inputs are numbered in opposite direction as they would be in the standard definition 11. The difference is necessary in order to match with the convention for gentle algebras 18. With the definition of SL disks in mind, we are ready to recall the CR, ID, DS and DW disk types. Among these four disk types, only CR and ID disks are relevant for this paper and we have depicted their schematic in Figure 18.5 and 18.6. For DS and DW disks we given an abridged definition and explain what makes them irrelevant.

Definition 18.13. A CR disk (co-identity rule disk) is an SL disk all of whose segments are of nonempty, with the exception that multiple stacked co-identity inputs with empty segments in between are allowed, as long as their zigzag curve is oriented clockwise with the disk. We denote by Disk ${ }_{\mathrm{CR}}$ the set of all CR disks, taking the union over arbitrary input sequence $h_{1}, \ldots, h_{N}$ and output $t$.


Figure 18.6: These pictures depict typical ID disks, categorized according to the type of their degenerate input. All depicted ID disks have nine inputs, of which four consist of a stack of co-identities and one is the degenerate input. The degenerate input is the input located directly next to the output mark. The orientations of the zigzag curves near the output mark are enforced by the specific rules of ID disks. The orientation of the zigzag curve carrying the co-identities is enforced by the requirement that the disk becomes CR upon excision of the output mark.

Definition 18.14. A ID disk (identity degenerate disk) is an SL disk satisfying the following conditions:

- The output is an identity,
- Precisely one input, the degenerate input, is infinitesimally close to the output,
- The degenerate input is an odd or even transversal intersection,
- The disk becomes CR upon excision of the output and substitution of the output mark by the degenerate input,
- In case the degenerate input is odd, it precedes respectively succeeds the output mark if $\tilde{L}_{1}$ is oriented clockwise respectively counterclockwise with the disk,
- In case the degenerate input is even, then the source zigzag curve of the degenerate input is counterclockwise and the target zigzag curve of the degenerate input is clockwise.
We denote by Disk ${ }_{\text {ID }}$ the set of ID disks with arbitrary inputs and output.
DS and DW disks (degenerate strip disks, degenerate wedge disks) are immersed strips fitting into one of the two digons bounded by a zigzag curve $\tilde{L}$ and its Hamiltonian deformation. It is possible to make this more precise. However, every DS and DW disk necessarily includes at least one even input. This already renders DS and DW disks irrelevant for the present paper. Even without recalling the precise definition, we denote by Disk ${ }_{D S}$ and Disk $_{\text {DW }}$ the set of DS and DW disks, respectively.

In order to give the description of the products $\mu_{\mathrm{H} \mathbb{L}_{q}}$ in terms of disks, we have to introduce two pieces of notation here: the Abouzaid sign and the deformation parameter attached to a disk. We have chosen to name this sign rule after Abouzaid for the reason that it is the same as in 1 .

Definition 18.15. Let $D$ be an SL disk. Then its Abouzaid sign $\operatorname{Abou}(D) \in \mathbb{Z} / 2 \mathbb{Z}$ is the sum of all \# signs on the boundary of $D$, plus the number of odd inputs $h_{i}: L_{i} \rightarrow L_{i+1}$ where $\tilde{L}_{i+1}$ is oriented counterclockwise relative to $D$, plus one if the output $t: L_{1} \rightarrow L_{N+1}$ is odd and $\tilde{L}_{N+1}$ is oriented counterclockwise. The deformation parameter $\operatorname{Punc}(D) \in \mathbb{C} \llbracket Q_{0} \rrbracket$ is the total product of all punctures covered by $D$, counted with multiplicity.

We will now make the description of the product structure on $\mathrm{H} \mathbb{L}_{q}$ precise. The procedure is familiar: Let $h_{1}, \ldots, h_{N}$ be a sequence of inputs. The product $\mu_{\mathrm{H}_{q}}\left(h_{N}, \ldots, h_{1}\right)$ is given by enumerating all disks with inputs $h_{1}, \ldots, h_{N}$ and arbitrary output $t: L_{1} \rightarrow L_{N+1}$. In our specific case, the types of disks involved are the CR, ID, DS and DW disks. The contribution from a disk $D$ carries the Abouzaid sign $(-1)^{\operatorname{Abou}(D)}$ and is weighted by the deformation parameter $\operatorname{Punc}(D) \in \mathbb{C} \llbracket Q_{0} \rrbracket$. For a disk $D$, denote its output $t: L_{1} \rightarrow L_{N+1}$ by $\mathrm{t}(D)$.

Theorem 18.16 Paper II). Let $Q$ be a geometrically consistent dimer or standard sphere dimer $Q_{M}$ with $M \geq 3$. The $A_{\infty}$-product $\mu_{\mathrm{H} \mathbb{L}_{q}}$ is strictly unital. Let $h_{1}, \ldots, h_{N}$ be a sequence of $N \geq 0$ non-identity basis morphisms with $h_{i}: L_{i} \rightarrow L_{i+1}$. Then their product is given by

$$
\mu_{\mathrm{H} \mathbb{L}_{q}}^{N}\left(h_{N}, \ldots, h_{1}\right)=\sum_{\substack{D \in \text { Disk }_{\mathrm{CR}} \dot{\operatorname{USisk}} \\ D \text { has inputs }_{\mathrm{ID}} \dot{\text { ind}}_{1}, \ldots, h_{N}}}(-1)^{\operatorname{Abou}(D)} \operatorname{Punc}(D) \mathrm{t}(D)
$$

Remark 18.17. The description of Theorem 18.16 also describes curvature $\mu_{\mathrm{H} \mathbb{L}_{q}}^{0}$ and differential $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}$ of the minimal model accurately. Let us exprain this as follows:

In case $Q$ is geometrically consistent, the curvature $\mu_{\mathrm{H} \mathbb{L}_{q}}^{0}$ and the differential $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}$ vanish. This is witnessed by the fact that there are no monogons or digons in $Q$ bounded by zigzag curves.

In case $Q=Q_{M}$ for odd $M$, the curvature $\mu_{\mathrm{H} \mathbb{L}_{q}}^{0}$ vanishes, but the differential $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}$ is nonzero. This vanishing of curvature is witnessed by the fact that there are monogons in $Q_{M}$ bounded by zigzag curves, but they cancel each other due to the spin structure. The differential is witnessed by the fact that there are digons in $Q_{M}$ bounded by zigzag curves.

In case $Q=Q_{M}$ for even $M$, both curvatue $\mu_{\mathrm{H} \mathbb{L}_{q}}^{0}$ and $\mu_{\mathrm{H} \mathbb{L}_{q}}^{1}$ are nonzero. This is witnessed by the fact that there are both monogons and digons in $Q_{M}$ bounded by zigzag paths.

### 18.4 Preparation for mirror objects

In this section we review further selected products in the category $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ from Paper II Namely, we regard products of the form $\mu\left(m, h_{N}, \ldots, h_{1}\right)$, where $h_{1}, \ldots, h_{N}$ is a sequence of odd morphisms $h_{i}: L_{i} \rightarrow L_{i+1}$ not containing co-identities and $m \in \operatorname{Hom}_{H T_{w} \operatorname{Gtl}_{q} Q}\left(L_{N+1}, a\right)$ is a morphism to an arc $a \in \operatorname{Gtl}_{q} Q$. Ultimately, knowledge of these products serves the calculation of mirror objects in section 21.6

We start with reviewing the hom space $\operatorname{Hom}_{\tilde{H} \operatorname{Tw} \operatorname{Gtl} Q}(L, a)$. Here $L \in \mathrm{HTw} \operatorname{Gtl} Q$ is a zigzag path in $Q$ and $a \in \operatorname{Gtl} Q$ is an arc. The zigzag curve $\tilde{L}$ and the $\operatorname{arc} a$ are both curves in the surface $|Q|$. While $\tilde{L}$ is a closed curve, the arc $a$ is an interval. The curves $\tilde{L}$ and $a$ may be disjoint or intersect. If they intersect, they intersect in a single point, namely the midpoint of $a$ :


The picture on the left depicts a single odd intersection $L \rightarrow a$. The picture in the middle depicts an even intersection $L \rightarrow a$. It is also possible that $L$ intersects the arc $a$ twice, depicted on the right. Given that HTw Gtl $Q$ and the wrapped Fukaya category are equivalent, we expect that intersections $L \rightarrow a$ provide a natural basis for the hom space $\operatorname{Hom}_{\operatorname{HTw}_{\operatorname{GtI} Q}(L, a) \text {. We confirmed this in Paper II }}^{\text {Pa }}$
Lemma 18.18. Let $L$ be a zigzag path and $a$ an arc in $Q$. Then a natural basis for the hom space $\operatorname{Hom}_{\mathrm{HTw} \operatorname{Gtl} Q}(L, a)$ is given by the intersections between $a$ and $\tilde{L}$.

Every intersection point $m: L_{1} \rightarrow a$ comes with a partner $m^{*}: L_{2} \rightarrow a$. To see this, let $L_{2}$ be the zigzag path departing from $a$ on the opposite side of $L_{1}$. Then also $\tilde{L}_{2}$ intersects $a$ at its midpoint. When $m$ is even, its partner $m^{*}$ is odd, and vice versa. It is possible that $L_{1}=L_{2}$, namely in case the two zigzag paths departing from $a$ are equal. Apart from the zigzag paths $L_{1}$ and $L_{2}$, there is not a single other zigzag path in $Q$ that intersects $a$. Let us review an additional type of disk, depicted in Figure 18.7

Definition 18.19. An MD disk (mirror disk) is a CR disk whose

- inputs $h_{1}, \ldots, h_{N}$ are all odd and do not contain co-identities,
- output is even and not an identity,
- zigzag segments all run clockwise,
which has undergone the following surgery: The output mark, located at a certain arc $a$, has been cut off. The odd morphism at $a$ is added as final input, and the even morphism at $a$ is indicated as new output.

The Abouzaid sign $\operatorname{Abou}(D)$ and the deformation parameter Punc $(D)$ of an MD disk are defined in analogy to Abouzaid signs and deformation parameters for SL disks. Given an arc $a \in Q_{1}$, there are two zigzag paths departing from $a$. In particular, there is one single odd basis morphism between these two zigzag paths located at $a$. With this in mind, we are ready to recall the description of some products of the form $\mu_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}\left(m, h_{N}, \ldots, h_{1}\right)$ from Paper II.

Lemma 18.20. Let $h_{1}, \ldots, h_{N}$ be a sequence of $N \geq 0$ odd cohomology basis elements $h_{i}: L_{i} \rightarrow L_{i+1}$ such that none of them is the co-identity. Let $a$ be an arc. Let $m \in \operatorname{Hom}_{\mathrm{H}_{\mathrm{Tw}} \operatorname{Gtl}_{q} Q}\left(L_{N+1}, a\right)$ be an odd intersection. Then we have

$$
\mu_{\mathrm{HTw}_{\operatorname{Gtl}}^{q} Q}\left(m, h_{N}, \ldots, h_{1}\right)=\sum_{\substack{\text { MD disk } D \\ \text { with inputs } h_{1}, \ldots, h_{N}, m}}(-1)^{\operatorname{Abou}(D)} \operatorname{Punc}(D) m^{*} .
$$



Figure 18.7: Disks contributing to products $\mu(m, b, \ldots, b)$

Let $m \in \operatorname{Hom}_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}\left(L_{N+1}, a\right)$ be an even intersection. Then the product $\mu\left(m, h_{N}, \ldots, h_{1}\right)$ vanishes, except if $N=1$ and $h_{1}$ is the odd intersection at $a$ between the two zigzag paths departing from $a$. In this case, we have

$$
\mu_{\mathrm{HTw} \mathrm{Gtl}}^{q}, ~\left(m, h_{1}\right)=-m^{*} .
$$

Remark 18.21. In case $Q$ is geometrically consistent, the differential $\mu_{\mathrm{HTw} \mathrm{Gtl}}^{q} \boldsymbol{Q} Q(m)$ vanishes. This is also the case if $Q=Q_{M}$ for even $M$. In case of $Q=Q_{M}$ with $M$ odd, the differential is instead given by counting digons which lie on the clockwise face of $Q_{M}$. By definition of MD disks, this, corner case is included in Lemma 18.20

## 19 Flatness of superpotential deformations

In this section, we show that a superpotential deformation of a CY3 Jacobi algebra is flat under assumption of a certain boundedness condition:


# Deformed Jacobi algebra <br> $\operatorname{Jac}\left(Q, W_{q}\right)=\frac{B \widehat{\otimes} \mathbb{C} Q}{\left(\partial_{a} W_{q}\right)}$ 

Deformation theory for algebras has historically followed the question when a deformation of an ideal gives rise to a deformation of the algebra. The core indicators of this development are the type of algebra studied, the degree $|R|$ of the relations in the algebra, and the degrees $\left|R^{\prime}\right|$ admitted for the deformation of the relations. Past development shows a continuous improvement on these two degrees:

| Year | Authors | Type of algebra | Deformation | $\|R\|$ | $\left\|R^{\prime}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1994 | Braverman and Gaitsgory 20 | Koszul | PBW | 2 | 2 |
| 2001 | Berger 10 | $N$-Koszul | - | $N$ | - |
| 2006 | Berger and Ginzburg 11 | $N$-Koszul | PBW | $N$ | $<N$ |
| 2006 | Berger and Taillefer 12 | CY3 | PBW | $N$ | $<N$ |
| 2023 | this paper | CY3 | formal | bounded | any |

In the present section, we examine this "flatness question" for the case of formal deformations of CY3 Jacobi algebras. We build on the work of Ginzburg-Berger, which shows that CY3-ness and homogeneity of the superpotential $W$ with respect to a positive integer grading on $Q$ give rise to an inductive argument that ensures flatness. The proof can however be continued without homogeneity requirement and we land in the completed path algebra $\widehat{\mathbb{C Q}}$. In this section, we construct a boundedness condition which allows us to land back in $\mathbb{C} Q$ by means of a posteriori estimates. The result is a flatness theorem that goes beyond the work of Berger and Ginzburg. Finally, we show that Jacobi algebras of most dimers satisfy our boundedness condition.

The general context of our work is the theory of deformations of associative algebras. The main question can be sketched as follows: Let $A$ be an algebra and $I \subseteq A$ an ideal, then we have the quotient algebra $A / I$. Let now $B$ be a deformation base, for instance $B=\mathbb{C} \llbracket q \rrbracket$. Let now $I_{q} \subseteq B \widehat{\otimes} A$ be an ideal, for instance $I_{q} \subseteq A \llbracket q \rrbracket$. We can then form the quotient algebra $(B \widehat{\otimes} A) / I_{q}$, for instance $A \llbracket q \rrbracket / I_{q}$. The main question is:

How need $I$ and $I_{q}$ be related so that $(B \widehat{\otimes} A) / I_{q}$ is a deformation of $A / I$ ?

Our results are presented in section 19.11 and can be summarized as follows: Let $A=\operatorname{Jac}(Q, W)$ be a CY3 algebra given by a quiver $Q$ with superpotential $W$. The relations of this algebra are the derivatives $\partial_{a} W$ of the superpotential. Let $W^{\prime} \in \mathfrak{m} \mathbb{C} Q$ be a deformation of the superpotential, that is, any additional cyclic terms lying in order $\mathfrak{m}$. Then we can consider the deformed relations $\partial_{a}\left(W+W^{\prime}\right) \in B \widehat{\otimes} \mathbb{C} Q$. We show that if the relations $\partial_{a} W$ satisfy a certain boundedness condition, then $\operatorname{Jac}\left(Q, W+W^{\prime}\right)$ is automatically a deformation of $\operatorname{Jac}(Q, W)$.

The boundedness condition for the relations $\partial_{a} W$ is best described as follows: Each relation $\partial_{a} W$ is a linear combination of paths in $Q$. We define an equivalence relation on the set of paths in $Q$ as follows:

- If $p$ and $q$ are paths appearing in the same relation $\partial_{a} W$, then $p \sim q$.
- If $p \sim q$, then $a p b \sim a q b$ for arbitrary paths $a, b$.
- Take the transitive hull.

This equivalence relation partitions the set of paths in $Q$ into equivalence classes. Our boundedness condition demands that the path length in every equivalence class is bounded. In other words, we demand that path length cannot increase to infinity if we apply (parts of) relations.

We have structured this section as follows: In section 19.1, we provide some intuition and terminology regarding flatness. In section 19.2 , we recollect the tools of Berger and Ginzburg 11. In section 19.3 we set up notation and formulate a stronger versions of the tools of Berger and Ginzburg. In section 19.4 , we introduce our boundedness argument and demonstrate its strength, most of which can be read independent of the deformations context. In section 19.5, we introduce more notation for ideal-like sets which allows us to apply the boundedness argument to the flatness question. In section 19.6 we prove a bounded version of the tools of Berger and Ginzburg. In section 19.7 we derive our first flatness result which concerns quasi-flatness of a certain auxiliary ideal in $B \widehat{\otimes} \widehat{C Q}$. In section 19.8 , we prove our second flatness result by tracing our way back to the non-completed path algebra. In section 19.9, we provide an a posteriori interpretation of the auxiliary ideal, which promotes our first flatness result to a flatness result for completed ideals in $\widehat{\mathbb{C} Q}$. In section 19.10 we provide criteria for the Jacobi algebra of a dimer to satisfy our boundedness condition. In section 19.11 we streamline our flatness theorems.

### 19.1 Flatness and quasi-flatness

In this section, we recapitulate flatness of algebra deformations and quasi-flatness of ideal deformations. The core connection is that a quasi-flat ideal deformation typically makes the quotient algebra a flat algebra deformation:

## Quasi-flat ideal deformation

Flat algebra deformation
$(B \widehat{\otimes} A) / I_{q}$ of $A / I$

It is our aim to study deformations of ideals. We deploy notation and terminology from section 15.3 in particular the distinct notation $\mathfrak{m}^{k} Y \neq \mathfrak{m}^{k} \cdot Y$. Let $A$ be an algebra and $I \subseteq A$ an ideal. Regard a deformation base $B$ and let $I_{q} \subseteq B \widehat{\otimes} A$ be an ideal. The question is in which cases $(B \widehat{\otimes} A) / I_{q}$ can be identified with $B \widehat{\otimes}(A / I)$. A first observation is that $I_{q}$ needs to be close enough to $I$. More precisely, we should have $\pi\left(I_{q}\right)=I$, or in other words $I_{q}+\mathfrak{m} A=I+\mathfrak{m} A$. This condition is not strong enough however. The basic problem is that $I_{q}$ may be too large in higher orders of $\mathfrak{m}$. Elements of $A$ may get unexpectedly annihilated in the quotient $(B \widehat{\otimes} A) / I_{q}$ when multiplied by elements of $\mathfrak{m}$. The further condition that $I_{q}$ therefore needs to satisfy is the quasi-flatness property $I_{q} \cap \mathfrak{m} A \subseteq \mathfrak{m} I_{q}$.

We start by fixing terminology for deformations of algebras and ideals as follows:
Definition 19.1. Let $A$ be an algebra and $B$ a deformation base. Then a $B$-algebra $A_{q}$ is a deformation of $A$ if there is a $B$-linear algebra isomorphism $\varphi: A_{q} \xrightarrow{\sim}\left(B \widehat{\otimes} A, \mu_{q}\right)$ where $\mu_{q}$ is a deformation of the product $\mu: A \otimes A \rightarrow A$.

Definition 19.2. Let $A$ be an algebra and $I \subseteq A$ an ideal. Let $B$ be a deformation base and $I_{q} \subseteq B \widehat{\otimes} A$ an ideal. Then $I_{q}$ is a deformation of $I$ if $I+\mathfrak{m} A=I_{q}+\mathfrak{m} A$.
Remark 19.3. Our terminology is slightly confusing: We have decided to call $I_{q}$ a deformation of $I$ already if it is loosely related to $I$. In contrast, our notion for deformations of algebras is very strict.

Recall from section 15.3 that $I_{q}$ is quasi-flat if $I_{q} \cap \mathfrak{m} A \subseteq \mathfrak{m} I_{q}$. The ideal $I_{q}$ is pseudoclosed if $B I_{q} \subseteq I_{q}$. In the following, we present two sample deformations of the ideal $I=(X) \subseteq A=\mathbb{C}[X]$ over $B=\mathbb{C} \llbracket q \rrbracket$. One of the deformations is quasi-flat and the other is not:

| Object | Quasi-flat | Not quasi-flat |
| :---: | :---: | :---: |
| Algebra $A$ | $\mathbb{C}[X]$ | $\mathbb{C}[X]$ |
| Ideal $I$ | $(X)$ | $(X)$ |
| Quotient $A / I$ | $\mathbb{C}$ | $\mathbb{C}$ |
| Deformed ideal $I_{q}$ | $(X+q)$ | $(X)+(q)$ |
| $I_{q} \cap q A$ | $q I_{q}$ | $q A$ |
| Quotient $A \llbracket q \rrbracket / I_{q}$ | $\frac{\mathbb{C}[X] \llbracket q \rrbracket}{(X+q)}=\mathbb{C} \llbracket q \rrbracket$ | $\frac{\mathbb{C}[X] \llbracket q \rrbracket}{(X)+(q)}=\mathbb{C}$ |

Not every deformation of an algebra $A / I$ can be described through a deformation of its ideal $I$. Conversely, not every deformation $I$ gives rise to a deformation of $A / I$. We recall here a classification of those deformations $I_{q}$ that give rise to deformations of $A / I$ :

Proposition 19.4. Let $A$ be an algebra and $I \subseteq A$ an ideal. Let $B$ be a deformation basis and $I_{q} \subseteq B \widehat{\otimes} A$ a deformation of $I$. Put $A_{q}=(B \widehat{\otimes} A) / I_{q}$. Then the following are equivalent:

- $I_{q}$ is quasi-flat and pseudoclosed.
- $A_{q}$ is a deformation of $A / I$.

In this case, the $\mathfrak{m}$-adic topology on $A_{q}$ agrees with the quotient topology and any $B$-linear isomorphism $A_{q} \xrightarrow{\sim} B \widehat{\otimes}(A / I)$ is automatically continuous.

Proof. We divide the proof into three parts: We first show that $I_{q}$ being quasi-flat and pseudoclosed implies that $A_{q}$ is a deformation. Second we show the converse, and third we draw the topological conclusions.

For the first part, assume $I_{q}$ is quasi-flat and pseudoclosed. It is our task to find an isomorphism of $B$-algebras $\varphi:(B \widehat{\otimes} A) / I_{q} \xrightarrow{\sim}\left(B \widehat{\otimes}(A / I), \mu_{q}\right)$, where $\mu_{q}$ is a deformation of the algebra structure of $A / I$. Pick a complement $V \subseteq A$ of $I$ such that $A=I \oplus V$. Both $I_{q} \subseteq B \widehat{\otimes} A$ and $B V \subseteq B \widehat{\otimes} A$ are quasi-flat and pseudoclosed. By Proposition 15.29 we conclude that $B \bar{\otimes} A=I_{q} \oplus B V$. We obtain a $B$-linear isomorphism

$$
\varphi: \frac{B \widehat{\otimes} A}{I_{q}}=\frac{I_{q} \oplus B V}{I_{q}} \xrightarrow{\sim} B V \xrightarrow{\sim} \frac{B I \oplus B V}{B I} \xrightarrow{\sim} B \widehat{\otimes} \frac{A}{I} .
$$

This already provides $\varphi$ as $B$-linear map. It remains to check that the algebra structure $\mu_{q}$ induced on $B \widehat{\otimes}(A / I)$ from $(B \widehat{\otimes} A) / I_{q}$ via $\varphi$ is a deformation of the natural algebra structure $\mu$ of $A / I$. Pick $a, b \in V \subseteq A$. Write $a b=x+m+v$ with $x+m \in I_{q}, x \in I, m \in \mathfrak{m} A, v \in B V$. Projecting $a b$ to $B V$ along $I_{q} \oplus B V$ gives $v$. Projecting $a b$ to $B V$ along $B I \oplus B V$ instead gives $v+\mathcal{O}(\mathfrak{m})$. This shows that $\mu_{q}$ is a deformation of $\mu$.

For the second part, assume there is an isomorphism of $B$-modules $\varphi:(B \widehat{\otimes} A) / I_{q} \rightarrow B \widehat{\otimes}(A / I)$. Regard the $B$-linear projection map $\pi: B \widehat{\otimes} A \rightarrow(B \widehat{\otimes} A) / I_{q}$. The composition $\varphi \pi: B \widehat{\otimes} A \rightarrow B \widehat{\otimes}(A / I)$ is $B$-linear and surjective. Pick a linear section $A / I \rightarrow B \widehat{\otimes} A$ of $\varphi \pi$ and extend to a $B$-linear and continuous map $\psi: B \widehat{\otimes}(A / I) \rightarrow B \widehat{\otimes} A$ following Remark 15.11 Thanks to continuity, $\psi$ is a section of $\varphi \pi$ in the sense that $\varphi \pi \psi=$ id. In other words, we have the $B$-linear map $\psi \varphi:(B \widehat{\otimes} A) / I_{q} \rightarrow B \widehat{\otimes} A$ with $\pi(\psi \varphi)=$ id. This shows that the projection $\pi$ has a $B$-linear section, hence $I_{q}$ is a direct summand of $B \widehat{\otimes} A$. By Proposition 15.29, we conclude that $I_{q}$ is quasi-flat and pseudoclosed.

For the third part, let $\varphi: A_{q} \rightarrow B \widehat{\otimes}(A / I)$ be any $B$-linear isomorphism. By Lemma 15.34 this map is automatically continuous when $A_{q}$ is equipped with the $\mathfrak{m}$-adic topology. By Remark 15.11 the composition $B \widehat{\otimes} A \rightarrow A_{q} \xrightarrow{\sim} B \widehat{\otimes}(A / I)$ is automatically continuous as well, and the universal property of the quotient topology renders $\varphi$ continuous when $A_{q}$ is equipped with the quotient topology. This settles all claims.

Remark 19.5. When $A$ is unital, then the $B$-algebra isomorphism $\varphi:(B \widehat{\otimes} A) / I_{q} \rightarrow\left(B \widehat{\otimes} A, \mu_{q}\right)$ can be chosen to preserve the unit. Indeed, one simply chooses the complement $V$ to contain the unit.

Remark 19.6. The statement of Proposition 19.4 still holds true when $A$ is an algebra over a semisimple algebra $\Lambda$, for instance $\Lambda=\mathbb{C} Q_{0}$ with $Q$ a quiver. The ideals $I$ and $I_{q}$ are automatically a $\Lambda$-submodule as well, and the complement $V \subseteq A$ can be chosen as a $\Lambda$-submodule due to semisimplicity. The $B$-algebra isomorphism $\varphi:(B \widehat{\otimes} A) / I_{q} \rightarrow\left(B \widehat{\otimes}(A / I), \mu_{q}\right)$ then becomes $\Lambda$-linear.

Let us illustrate flatness in the case of deformations of a deformation of a superpotential $W$ on a quiver $Q$. Our typical starting point is a cyclic deformation $W_{q} \in B \widehat{\otimes} \mathbb{C} Q$ of $W$. This deformation gives rise to deformed relations $\partial_{a} W_{q}$ for $a \in Q_{1}$. The question we try to settle is when the ideal $\left(\partial_{a} W_{q}\right)$ or its closure in $B \widehat{\otimes} \mathbb{C} Q$ is quasi-flat. On an intuitive level this means the following: We start from any element $x \in \mathbb{C} Q$ and continue adding up deformed relations $\partial_{a} W_{q}$ :

$$
x+p_{1} \partial_{a_{1}} W_{q} q_{1}+p_{2} \partial_{a_{2}} W_{q} q_{2}+\ldots
$$

Here $p_{i}, q_{i}$ are arbitrary paths in $Q$. If we arrive at some point at an element of the form $x+\mathcal{O}(\mathfrak{m})$, does the infinitesimal part necessarily vanish or have we incurred additional terms in higher order? If the infinitesimal part always vanishes, then the deformed ideal ( $\partial_{a} W_{q}$ ) contains no unexpected new relations. If the infinitesimal part consists of higher-order multiples of relations, then the deformed ideal $\left(\partial_{a} W_{q}\right)$ still contains no new relations. If the infinitesimal part consists of strictly more than $\mathfrak{m}\left(\partial_{a} W_{q}\right)$, then the deformed ideal $\left(\partial_{a} W_{q}\right)$ contains new relations in higher order which cannot be made by combining existing relations in lower order.

Relations in higher order which cannot be made from combining existing relations in lower order indicate an ideal that is not quasi-flat. More precisely, these relations lie in $\left(\left(\partial_{a} W_{q}\right) \cap \mathfrak{m} \mathbb{C} Q\right) \backslash \mathfrak{m}\left(\partial_{a} W_{q}\right)$. They kill more than expected in the higher-order part of the quotient $(B \widehat{\otimes} \mathbb{C} Q) /\left(\partial_{a} W_{q}\right)$ and prevent the quotient from being isomorphic to $B \widehat{\otimes} \mathbb{C} Q /\left(\partial_{a} W\right)$.

### 19.2 Berger-Ginzburg inclusion

In this section we recall superpotentials, the CY3 property and the Berger-Ginzburg inclusion. More precisely, we start by recalling superpotentials and their associated Jacobi algebras. In terms of a bimodule resolution, we recall what it means for an algebra to have the CY3 property. Jacobi algebras are sometimes CY3. If so, their superpotentials satisfies a certain condition which appeared in 11. The reformulation of the CY3 property in terms of this condition is essential for us, and we shall refer to the this condition as the Berger-Ginzburg inclusion.
Remark 19.7. This section is meant as a motivational section. The reader who wishes to take the origin of the Berger-Ginzburg inclusion for granted is advised to skip to section 19.3 For instance, the original Berger-Ginzburg inclusion presented here is in fact not sufficient for our purposes, so we will build a stronger Berger-Ginzburg inclusion from scratch in section 19.3

To start with, let $Q$ be a quiver. Recall from section 16.6 that a superpotential on $Q$ is a cyclic element of $\mathbb{C} Q_{\geq 2}$. Recall also that we may denote by $W$ the $\mathbb{C} Q_{0}$-bimodule generated by $W$ in $\mathbb{C} Q$ :

$$
W=\mathbb{C} Q_{0} W Q_{0}=\bigoplus_{v \in Q_{0}} \mathbb{C} v W v \subseteq \mathbb{C} Q
$$

Recall that we denote the relations space by

$$
R=\operatorname{span}\left\{\partial_{a} W \mid a \in Q_{1}\right\}
$$

Remark 19.8. By abuse of notation, we have used the same notation for $W \in \mathbb{C} Q_{\geq 2}$ as for its associated bimodule in the above definition. We shall stick to this notation. The distinction should always be clear. In section 19.3, the terminology will be changed and improved.

Recall that the Jacobi algebra associated with $(Q, W)$ is the algebra $A=\operatorname{Jac}(Q, W)=\mathbb{C} Q /\left(\partial_{a} W\right)$. Let us temporarily keep the shorthand $A$ for the Jacobi algebra. Recall from section 16.6 the candidate resolution of $A$ as $A$-bimodule:

$$
\begin{equation*}
0 \rightarrow A \underset{\mathbb{C} Q_{0}}{\otimes} W \underset{\mathbb{C} Q_{0}}{\otimes} A \xrightarrow{g_{1}} A \underset{\mathbb{C} Q_{0}}{\otimes} R \underset{\mathbb{C} Q_{0}}{\otimes} A \xrightarrow{g_{2}} A \underset{\mathbb{C} Q_{0}}{\otimes} \underset{\mathbb{C} Q_{1}}{\underset{C}{ } Q_{0}} \underset{\rightarrow}{\otimes} A \xrightarrow{g_{Q_{0}}} A \underset{\underset{\sim}{Q_{0}}}{\otimes} A \rightarrow A \rightarrow 0 . \tag{19.1}
\end{equation*}
$$

We have described this sequence in Remark 16.51. As we recall in section 16.6 the Jacobi algebra $A=\operatorname{Jac}(Q, W)$ is CY3 if and only if the sequence is exact:
Theorem 19.9 ( 14 , Theorem 4.3]). Let $Q$ be a quiver and $W \in \mathbb{C} Q_{\geq 3}$ a superpotential. Then the algebra $\operatorname{Jac}(Q, W)$ is CY3 if and only if the sequence 19.1) is exact.

Recall also from Lemma 16.42 that the exact sequence can be used to produce projective resolutions for all left and right modules of $A$. When tensoring the sequence on the left with $\mathbb{C} Q_{0}$ over $A$, then we obtain a resolution of $\mathbb{C} Q_{0}$ as right $A$-modules:

$$
\begin{equation*}
0 \rightarrow W \underset{\mathbb{C} Q_{0}}{\otimes} A \xrightarrow{f_{1}} R \underset{\mathbb{C} Q_{0}}{\otimes} A \xrightarrow{f_{2}} \mathbb{C} Q_{1} \underset{\mathbb{C} Q_{0}}{\otimes} A \xrightarrow{f_{3}} A \rightarrow \mathbb{C} Q_{0} \rightarrow 0 . \tag{19.2}
\end{equation*}
$$

Remark 19.10. The maps $f_{1}, f_{2}, f_{3}$ are described as follows: For $f_{1}$, write an element $w=\sum_{i \in I} r_{i} a_{i}$ in terms of relations $r_{i} \in R$ and arrows $a_{i} \in Q_{1}$, then map $f_{1}(w)=\sum_{i \in I} r_{i} \otimes a_{i}$. For $f_{2}$, write an element $r=\sum_{i \in I} a_{i} p_{i}$ with arrows $a_{i}$, then map $f_{2}(r)=\sum_{i \in I} a_{i} \otimes p_{i}$. The map $f_{3}$ is the multiplication map and the fourth map is the projection onto the quotient of $A$ by all arrows.

Requiring an algebra to be CY3 is an algebro-geometric condition. Berger and Ginzburg 11 translate this condition for $A=\operatorname{Jac}(Q, W)$ to a more tractable condition in terms of the superpotential and relations. More precisely, their approach is to evaluate exactness of the sequence 19.2 at the module second from the left. The result is an inclusion in terms of $R$ and $W$ :
Lemma 19.11 (11, Theorem 2.6]). If $\operatorname{Jac}(Q, W)$ is CY3, then the Berger-Ginzburg inclusion holds:

$$
\begin{equation*}
R \mathbb{C} Q \cap \mathbb{C} Q_{1} R \subseteq W \mathbb{C} Q+R I(R) \tag{19.3}
\end{equation*}
$$

Remark 19.12. The inclusion 19.3 can be roughly interpreted as follows: Let $p \in \mathbb{C} Q$ be a path. Add any amount of ideal elements $r y \in R \mathbb{C} Q$ and $a r \in \mathbb{C} Q_{1} R$ to $p$. If we land at $p$ again after the addition, the additions of the two types have been equal and therefore lie in the intersection $R \mathbb{C} Q \cap \mathbb{C} Q_{1} R$. By the inclusion 19.3 , the relations we have added are equivalent to applying the entire set of relations $\partial_{a} W$ around some vertices and terms that "vanish quadratically" in $A / I(R)$.

Berger and Taillefer 12 use the intermediate results of Berger and Ginzburg 11 to deduce flatness of PBW deformations. Apart from the CY3 condition in the form of the Berger-Ginzburg inclusion, their core assumption is that the superpotential $W \in \mathbb{C} Q_{\geq 2}$ is homogeneous in path length. Translated to the setting of formal deformations, their statement would read as follows:
Proposition 19.13. Let $Q$ be a quiver and $W \in \mathbb{C} Q_{\geq 2}$ a homogeneous superpotential. If $(B, \mathfrak{m})$ is a deformation base and $W^{\prime} \in \mathfrak{m} \mathbb{C} Q_{\geq 2}$ is a cyclic element, then the deformed ideal $B\left(\partial_{a} W+\partial_{a} W^{\prime}\right)$ is quasi-flat. In other words, the "deformed Jacobi algebra"

$$
\operatorname{Jac}\left(Q, W+W^{\prime}\right):=\frac{B \widehat{\otimes} \mathbb{C} Q}{B\left(\partial_{a} W+\partial_{a} W^{\prime}\right)}
$$

is a deformation of $\operatorname{Jac}(Q, W)$.
We shall not prove Lemma 19.11 and Proposition 19.13 here. Namely, the core assumption that $W$ is homogeneous is too restrictive for our purposes. We will prove stronger versions of both.

### 19.3 Notation and conventions

In this section, we define the setup in which we prove flatness properties. As a first step, we start from relation spaces instead of superpotentials. Second, we introduce a stronger version of the BergerGinzburg inclusion. We fix both items in a notational framework, which does not make explicit reference to superpotentials. Finally, we show that relations coming from CY3 superpotentials fall within the framework.

It may happen that one is interested in checking flatness of Calabi-Yau like algebras where the relation space $R$ is given, without the relations coming from a single superpotential $W$. In this case, one may form the space $W=\mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1}$ which serves as a replacement for the superpotential:

$$
\begin{array}{cccc}
\text { CY3 viewpoint } & \begin{array}{c}
W \in \mathbb{C} Q_{\geq 2} \\
\text { superpotential }
\end{array} & \rightsquigarrow & \begin{array}{c}
R=\operatorname{span}\left(\partial_{a} W\right) \\
\text { relation space }
\end{array} \\
\text { Deformation viewpoint } & R \subseteq \mathbb{C} Q & \rightsquigarrow & \begin{array}{c}
W=\mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1} \\
\text { superpotential space }
\end{array}
\end{array}
$$

The approach of starting with a relation space $R$ is also the context used in 11. We will codify the setup in Convention 19.14 During the remainder of section 19. we stick to this convention.
Convention 19.14. The letter $Q$ denotes a quiver. The space $R \subseteq \mathbb{C} Q_{\geq 1}$ is a finite-dimensional $\mathbb{C} Q_{0^{-}}$ bimodule of relations. We write $W:=\mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1}$. When regarding deformations, $(B, \mathfrak{m})$ is a deformation base and $\psi: R \rightarrow \mathfrak{m} \widehat{\mathbb{C Q}}$ is a $\mathbb{C} Q_{0}$-bimodule map. The ideal generated by $R$ is denoted $I(R)=\mathbb{C} Q R \mathbb{C} Q \subseteq \mathbb{C} Q$. The space of deformed relations is denoted $P:=(\operatorname{Id}+\psi)(R) \subseteq B \widehat{\otimes} \widehat{\mathbb{C} Q}$. Two additional properties may be assumed:
[BG] The strong Berger-Ginzburg inclusion

$$
\begin{equation*}
R \otimes \mathbb{C} Q \cap c^{-1}\left(\mathbb{C} Q_{1} I(R)\right) \subseteq f(W \otimes \mathbb{C} Q)+R \otimes I(R) \tag{19.4}
\end{equation*}
$$

Here, $c: R \otimes \mathbb{C} Q \rightarrow R \mathbb{C} Q$ denotes the contraction map and $f: W \otimes \mathbb{C} Q \rightarrow R \otimes \mathbb{C} Q_{1} \mathbb{C} Q$ denotes the map that splits $W$ into relations and arrows.
$[\mathrm{CP}]$ The property $\left(\mathbb{C} Q_{1} P+P \mathbb{C} Q_{1}\right) \cap \mathfrak{m} \widehat{\mathbb{C} Q}=0$.
Remark 19.15. We drop subscripts from the tensor product $\otimes$. When the two tensorands come from quiver algebras, the tensor product always refers to the tensor product over $\mathbb{C} Q_{0}$ :

$$
R \otimes \mathbb{C} Q=R \otimes \mathbb{C} Q_{0} \mathbb{C} Q, \quad R \otimes I(R)=R \otimes \mathbb{C} Q_{0} I(R), \quad W \otimes \mathbb{C} Q=W \otimes \mathbb{C} Q_{0} \mathbb{C} Q, \quad \ldots
$$

In the remainder of the section, we illustrate which algebras fall under the framework of Convention 19.14 In particular, we show that Jacobi algebras with the CY3 property satisfy the conditions [BG] and [CP].

Example 19.16. The algebra $\mathbb{C}\langle A, B, C\rangle /(B C, C A, A B)$ falls into the framework of Convention 19.14 Its relations are given by $R=\operatorname{span}(B C, C A, A B)$ and by definition we have

$$
W=\mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1}=\operatorname{span}(A B C, B C A, C A B)
$$

This space is three-dimensional and not spanned by a single superpotential on $\mathbb{C} Q$. The algebra is not CY3. However the strong Berger-Ginzburg inclusion is satisfied:

$$
R \otimes \mathbb{C} Q \cap c^{-1}\left(\mathbb{C} Q_{1} I(R)\right)=B C \otimes A \mathbb{C} Q+C A \otimes B \mathbb{C} Q+A B \otimes C \mathbb{C} Q=f(W \otimes \mathbb{C} Q)
$$

The algebra therefore falls within our framework Convention 19.14 and can therefore be treated with our approach.

For this specific algebra another approach to deformations is possible: In 8, Barmeier and Wang study quiver algebras whose relations are defined by so-called reduction systems. The deformations of these algebras are in combinatorial correspondence with deformations of their reduction system. The specific relations $B C=C A=A B=0$ constitute a reduction system, therefore this specific algebra falls under the framework of 8 .

Next, we show that the case of superpotentials falls within the framework of Convention 19.14. As a preparation, we need the following lemma, modeled after 11 .
Lemma 19.17. Assume [CP]. Then the two replacement maps

$$
\begin{aligned}
\psi^{0}: W \rightarrow \mathfrak{m} \mathbb{C} Q_{1} \widehat{\mathbb{C} Q}, & \sum_{i \in I} a_{i} r_{i} \mapsto \sum_{i \in I} a_{i} \psi\left(r_{i}\right), \\
\psi^{1}: W \rightarrow \mathfrak{m} \widehat{\mathbb{C} Q} \mathbb{C} Q_{1}, & \sum_{i \in I} r_{i} a_{i} \mapsto \sum_{i \in I} \psi\left(r_{i}\right) a_{i}
\end{aligned}
$$

are well-defined and equal.
Proof. For well-definedness, it suffices to note that $\psi^{0}$ and $\psi^{1}$ are nothing else than the splitting maps $W \subseteq R \mathbb{C} Q_{1} \rightarrow R \otimes \mathbb{C} Q_{1}$ and $W \subseteq \mathbb{C} Q_{1} R \rightarrow \mathbb{C} Q_{1} \otimes R$, composed with $\psi$ acting either on the left or right factor. Now let us explain why $\overline{\psi^{0}}=\psi^{1}$. Let $w \in W$, then

$$
\psi^{0}(w)-\psi^{1}(w)=\left(w+\psi^{0}(w)\right)-\left(w+\psi^{1}(w)\right) \in P \mathbb{C} Q_{1}+\mathbb{C} Q_{1} P
$$

The left-hand side simultaneously lies in $\mathfrak{m} \widehat{\mathbb{C Q}}$. By $[\mathrm{CP}]$, the difference $\psi^{0}(w)-\psi^{1}(w)$ vanishes. This proves $\psi^{0}=\psi^{1}$.

We are now ready to see how the case of relations coming from a superpotential fits into the framework of Convention 19.14 Let $W$ be a superpotential and $W^{\prime} \in \mathfrak{m} \mathbb{C} Q$ a deformation. We simply put $R=$ $\operatorname{span}\left\{\partial_{a} W\right\}$ and $\psi\left(\partial_{a} W\right):=\partial_{a} W^{\prime}$. To make these definitions work, one needs to check that the relations $\partial_{a} W$ are linearly independent and that $\mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1}$ is indeed the $\mathbb{C} Q_{0}$-bimodule generated by the superpotential $W \in \mathbb{C} Q$. We verify this in the following lemma. Note that $W$ is required to consist of paths of length $\geq 3$ this time:

Lemma 19.18. Let $Q$ be a quiver with superpotential $W \in \mathbb{C} Q_{\geq 3}$. Assume $\operatorname{Jac}(Q, W)$ is CY3. Then the relations $\partial_{a} W$ for $a \in Q_{1}$ are linearly independent. Put $R=\operatorname{span}\left\{\partial_{a} W\right\}$. Then $\mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1}$ is equal to the $\mathbb{C} Q_{0}$-bimodule generated by $W$. The property $[\mathrm{BG}]$ holds.

Moreover, let $W^{\prime} \in \mathfrak{m} \widehat{\mathbb{C Q}}$ be a (cyclic) deformation. Define $\psi: R \rightarrow \mathfrak{m} \widehat{\mathbb{C Q}}$ by $\psi\left(\partial_{a} W\right)=\partial_{a} W^{\prime}$. Then [CP] holds.
Proof. We prove all four statements one after another. For sake of simplicity, we use the notation $W$ to mean both the superpotential and the $\mathbb{C} Q_{0}$-bimodule $\mathbb{C} Q_{0} W \mathbb{C} Q_{0}$ generated by the superpotential. In an expression like $W \subseteq \mathbb{C} Q_{1} R$ or $w \in W$, the letter $W$ is to be interpreted as the $\mathbb{C} Q_{0}$-bimodule $\mathbb{C} Q_{0} W \mathbb{C} Q_{0}$. This way, no confusion should arise.

The first statement on linear independence of the relations $\partial_{a} W$ is now widely known, due to Ginzburg, Bocklandt, Berger, Taillefer and others. For example, the CY3 property implies by [14. Section 4.2] that the sequence 19.1 is a self-dual resolution of $\operatorname{Jac}(Q, W)$. Hence $R$ and $\mathbb{C} Q_{1}$ are equal in dimension and the relations are linearly independent.

For the second part of the proof, we show that $\mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1}$ is the $\mathbb{C} Q_{0}$-bimodule generated by $W$. The inclusion $W \subseteq \mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1}$ is easy and follows from cyclicity: Indeed for any vertex $v \in Q_{0}$ we have

$$
v W v=\sum_{h(a)=v} a \partial_{a} W=\sum_{t(a)=v} \partial_{a} W a .
$$

The first sum lies in $\mathbb{C} Q_{1} R$ and the second sum lies in $R \mathbb{C} Q_{1}$, hence $v W v$ lies in their intersection.
The converse inclusion $\mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1} \subseteq W$ follows from inspection of the exact sequence 19.2. To see this, pick an element $\sum r_{i} a_{i}$ with $r_{i} \in R$ and $a_{i} \in Q_{1}$ that simultaneously lies in $\mathbb{C} Q_{1} R$. We claim that $\sum r_{i} \otimes a_{i}$ goes to zero under $f_{2}: R \otimes \operatorname{Jac}(Q, W) \rightarrow \mathbb{C} Q_{1} \otimes \operatorname{Jac}(Q, W)$. Indeed, the assumption that $\sum r_{i} a_{i} \in \mathbb{C} Q_{1} R$ implies

$$
f_{2}\left(\sum r_{i} \otimes a_{i}\right)=\sum r_{i} a_{i}=0 \in \frac{\mathbb{C} Q_{1} \mathbb{C} Q}{\mathbb{C} Q_{1} I(R)} \cong \mathbb{C} Q_{1} \otimes \operatorname{Jac}(Q, W)
$$

By exactness of 19.2 , we deduce that $\sum r_{i} \otimes a_{i}$ lies in the image of $f_{1}$. Therefore within $R \otimes \operatorname{Jac}(Q, W)$ we can write $\sum r_{i} \otimes a_{i}=f_{1}\left(\sum w_{i} \otimes p_{i}\right)$, where $w_{i} \in W$ and $p_{i}$ are paths. We shall now try to lift this equality to $R \otimes \mathbb{C} Q$. As a first step, recall that

$$
R \otimes \operatorname{Jac}(Q, W)=\frac{R \otimes \mathbb{C} Q}{R \otimes I(R)}
$$

In consequence there exists a $z \in R \otimes I(R)$ such that within $R \otimes \mathbb{C} Q$ we have

$$
\sum r_{i} \otimes a_{i}=f\left(\sum w_{i} \otimes p_{i}\right)+z
$$

Here $f: W \otimes \mathbb{C} Q \rightarrow R \otimes \mathbb{C} Q_{1} \mathbb{C} Q$ denotes the map that splits $W$ into relations and arrows. The terms $w_{i} \otimes p_{i}$ and $z \in R \otimes I(R)$ look wild, but in reality we can vastly reduce the complexity: Collect in an index set $S$ all indices $i$ where the path $p_{i}$ is not a vertex. For such indices $i \in S$, we have that all terms in $f\left(w_{i} \otimes p_{i}\right)$ have right tensor component of length at least 2 . Note that all terms in $R \otimes I(R)$ also have right tensor length at least 2 . Nevertheless, the left-hand side $\sum r_{i} \otimes a_{i}$ has terms only with right tensor length 1 , since $a_{i}$ are arrows. We deduce that $f\left(\sum_{i \notin S} w_{i} \otimes p_{i}\right)+z=0$. Finally, we conclude

$$
\sum r_{i} \otimes a_{i}=f\left(\sum_{i \in S} w_{i} \otimes 1\right)
$$

Contracting the tensors on both sides gives $\sum r_{i} a_{i} \in W$ as desired. This proves the second statement.
For the third statement, it is our task to prove that the strong Berger-Ginzburg inclusion holds. We note that

$$
W \otimes \frac{\mathbb{C} Q}{I(R)} \cong \frac{W \otimes \mathbb{C} Q}{W \otimes I(R)}, \quad R \otimes \frac{\mathbb{C} Q}{I(R)} \cong \frac{R \otimes \mathbb{C} Q}{R \otimes I(R)} \quad \mathbb{C} Q_{1} \otimes \frac{\mathbb{C} Q}{I(R)} \cong \frac{\mathbb{C} Q_{1} \mathbb{C} Q}{\mathbb{C} Q_{1} I(R)}
$$

Inserting this into the exact sequence 19.2 , we get the exact sequence

$$
\frac{W \otimes \mathbb{C} Q}{W \otimes I(R)} \stackrel{[f]}{\rightarrow} \frac{R \otimes \mathbb{C} Q}{R \otimes I(R)} \stackrel{[c]}{\longrightarrow} \frac{\mathbb{C} Q_{1} \mathbb{C} Q}{\mathbb{C} Q_{1} I(R)}
$$

Evaluating Ker $\subseteq \operatorname{Im}$ on this sequence yields the strong Berger-Ginzburg inclusion.

The fourth statement follows again from cyclicity. Any element in $\mathbb{C} Q_{1} P+P \mathbb{C} Q_{1}$ can be written as $x+\psi^{0}(x)+y+\psi^{1}(y)$, where $x \in R \mathbb{C} Q_{1}$ and $y \in \mathbb{C} Q_{1} R$. If such an element additionally lies in $\mathfrak{m} \mathbb{C} Q$, we get $x+y=0$, since $\psi^{0}$ and $\psi^{1}$ only give terms in $\mathfrak{m} \mathbb{C} Q$. Hence $x=-y \in \mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1}=W$. Without loss of generality, we can assume $x$ and $y$ are in the form

$$
x=\sum_{t(a)=v} \partial_{a} W a=\sum_{h(a)=v} a \partial_{a} W=-y .
$$

We deduce

$$
\psi^{0}(x)=\sum_{t(a)=v} \partial_{a} W^{\prime} a=\sum_{h(a)=v} a \partial_{a} W^{\prime}=-\psi^{1}(y) .
$$

In total, $x+\psi^{0}(x)+y+\psi^{1}(y)$ vanishes. This demonstrates the fourth statement and finishes the proof.
We have seen that the case of a CY3 superpotential is covered by Convention 19.14 From here on, we proceed in the context of Convention 19.14. We finish the present section with remarks on the difference between our strong Berger-Ginzburg inclusion and the original Berger-Ginzburg inclusion, as well as an explanation on how Berger and Ginzburg proceed in case of a homogeneous superpotential.

Remark 19.19. Let us compare the standard Berger-Ginzburg inclusion 19.3 and strong BergerGinzburg inclusion 19.4). We claim that the strong inclusion implies the standard inclusion and the converse holds if $W \in \mathbb{C} Q$ is homogeneous. To show the strong-to-weak implication, lift any element $x \in R \mathbb{C} Q \cap \mathbb{C} Q_{1} R$ to the tensor product $R \otimes \mathbb{C} Q$, apply the strong inclusion and contract again. To prove the weak-to-strong implication, assume $W$ is homogeneous. Let $x \in R \otimes \mathbb{C} Q \cap c^{-1}\left(\mathbb{C} Q_{1} I(R)\right)$, then

$$
c(x) \in R \mathbb{C} Q \cap \mathbb{C} Q_{1} I(R) \subseteq W \mathbb{C} Q+R I(R)=c(f(W \otimes \mathbb{C} Q)+R \otimes I(R))
$$

Homogeneity makes the contraction map $c$ injective and hence $x \in f(W \otimes \mathbb{C} Q)+R \otimes I(R)$. This proves the weak-to-strong implication.

The work of Berger and Ginzburg 11 takes place in the context of PBW deformations. We have translated their result to the case of formal deformations in Proposition 19.13 Now that we have introduced better terminology, we are ready to explain how Berger and Ginzburg prove their result. Translated again to formal deformations, their core lemma is the following:

Lemma 19.20 (Berger-Ginzburg). Assume the standard Berger-Ginzburg inclusion and [CP]. Then

$$
\begin{equation*}
I(P)_{\mathbb{C} Q} \cap \mathfrak{m} \mathbb{C} Q \subseteq \mathbb{C} Q_{1}\left(I(P)_{\mathbb{C} Q} \cap \mathfrak{m} \mathbb{C} Q\right)+\mathfrak{m} I(P)_{\mathbb{C} Q} \tag{19.5}
\end{equation*}
$$

After proving this lemma, Berger and Ginzburg assume that $Q$ has a grading in which every arrow is positive and $W$ is homogeneous. They continue as follows: Pick an element $x$ on the left-hand side of homogeneous degree in zeroth order. By the lemma we can write $x=y+z$, where $y \in \mathbb{C} Q_{1}\left(I(P)_{\mathbb{C} Q} \cap \mathfrak{m} \mathbb{C} Q\right)$ and $z$ already lies in $\mathfrak{m} I(P)_{\mathbb{C} Q}$. In $y$, we can split off arrows on the left. This results in reducing the degree and we can assume by induction that the part with an arrow less already lies in $\mathfrak{m} I(P)_{\mathbb{C} Q}$, hence $y \in \mathbb{C}_{1} \mathfrak{m} I(P)_{\mathbb{C} Q} \subseteq \mathfrak{m} I(P)_{\mathbb{C} Q}$. It follows that $I(P)_{\mathbb{C} Q} \cap \mathfrak{m} \mathbb{C} Q \subseteq \mathfrak{m} I(P)_{\mathbb{C} Q}$. This proves flatness in the setting of Berger and Ginzburg where $W$ is homogeneous.

### 19.4 Relations of bounded type

In this section, we show how to circumvent the homogeneity requirement of Berger and Ginzburg. Our substitution for the homogeneity requirement is a simple yet powerful boundedness condition. In the present section, we first introduce the boundedness argument in high generality. Then we demonstrate its strength in a sequence of applications. In section 19.6, we tailor it specifically to the case of deformed relations and derive a bounded version of the strong Berger-Ginzburg inclusion. The boundedness condition allows us in section 19.7 to continue the proof of flatness without homogeneity assumption.

In the most general way, our boundedness argument is stated as follows:
Lemma 19.21. Let $V$ be a vector space with basis $V_{0}$ and an equivalence relation $\sim$ on $V_{0}$. Let $X \subseteq V$ be a subspace with spanning set $X_{0}$ such that the $V_{0}$-constituents of every $x_{0} \in X_{0}$ are $\sim$-related. Then for every set $C \subseteq V_{0}$ closed under $\sim$ we have

$$
X \cap \operatorname{span}(C) \subseteq \operatorname{span}\left(X_{0} \cap \operatorname{span}(C)\right)
$$

Proof. The strategy is to write an element on the left-hand side in terms of the spanning set $X_{0}$ and then realize that the terms not related to $C$ vanish collectively. Indeed, let

$$
x=\sum_{i \in I} \lambda_{i} x_{i} \in X \cap \operatorname{span}(C), \quad x_{i} \in X_{0},|I|<\infty
$$

Now let $S \subseteq I$ be the set of indices $i \in I$ where the constituents of $x_{i}$ lie in $C$. Regard the decomposition

$$
x=\sum_{i \in S} \lambda_{i} x_{i}+\sum_{i \notin S} \lambda_{i} x_{i} .
$$

The $V_{0}$-constituents of the first summand all lie in $C$, while the constituents of the second summand all lie outside of $C$. Since the left-hand side $x$ lies in $\operatorname{span}(C)$, the second summand necessarily vanishes. Since $x_{i} \in X_{0} \cap \operatorname{span}(C)$ for $i \in S$, this finishes the proof.

We have a specific application to quiver algebras with relations $R$ in mind: Declare paths $p c q$ and $p d q$ related if $c$ and $d$ appear in a relation of $R$ together. Let us make this precise.
Definition 19.22. Let $Q$ be a quiver and $R \subseteq \mathbb{C} Q$ a finite-dimensional $\mathbb{C} Q_{0}$-bimodule. Assume a basis $F$ for $R$ is given. For each $c \in F$, decompose $c=\sum \lambda_{i} c_{i}$ as linear combination of paths ( $\lambda_{i} \neq 0$ ), then set $p c_{i} q \sim p c_{j} q$ for any paths $p, q$ and indices $i, j$. Denote by $\sim$ the transitive hull of this relation. Two paths $p$ and $q$ are $F$-related if $p \sim q$. For $N \in \mathbb{N}$ denote the supremum of all lengths of paths related to paths of lengths $\leq N$ by

$$
h(N)=\sup \{|q||\exists p:|p| \leq N, p \sim q\} \in \mathbb{N} \cup\{\infty\} .
$$

The basis $F$ is of bounded type if $h(N)<\infty$ for all $N \in \mathbb{N}$.
Remark 19.23. The basis $F$ is always assumed to be a basis for $R$ as $\mathbb{C} Q_{0}$-bimodule. Simply speaking, for every $c \in F$ there shall be vertices $v, w \in Q_{0}$ such that every path in $c$ runs from $v$ to $w$.

During the present section, we provide applications of the bounded type condition. Typically, we will work with the basis $F$ explicitly. In later sections it is only relevant that there exists a basis of bounded type. We therefore set up the following terminology:

Definition 19.24. Let $Q$ be a quiver and $R \subseteq \mathbb{C} Q$ a finite-dimensional $\mathbb{C} Q_{0}$-bimodule of relations. Then $R$ is of bounded type if it has a basis $F$ of bounded type. A superpotential $W$ is of bounded type if its relation space $R=\operatorname{span}\left\{\partial_{a} W\right\}$ is of bounded type.
Example 19.25. If $R \subseteq \mathbb{C} Q$ is graded space with respect to path length or any other grading that is positive on the arrows, then any homogeneous basis for $R$ is of bounded type: Degree is then an invariant under $F$-relatedness and length becomes bounded by a multiple of the degree. Here are three easy instances for the algebra $\mathbb{C} Q=\mathbb{C}\langle A, B, C\rangle$ :

- Regard $R=\operatorname{span}(A B-B A, B C-C B, C A-A C)$ with basis $F=\{A B-B A, B C-C B, C A-A C\}$. We claim that $F$ is of bounded type. Indeed, two paths $p, q$ in $Q$ are $F$-related if and only if they differ by reordering of $A, B$ and $C$. For instance, $A B A C \sim A A C B$. In particular, any two $F$-related paths are of equal length and $h(N)=N$. This shows that $F$ is of bounded type.
- Regard $R=\operatorname{span}\left(A B-C^{4}, B A-C^{2} B\right)$ and $F=\left\{A B-C^{4}, B A-C^{2} B\right\}$. This gives for instance $B C^{4} \sim B A B \sim C^{2} B^{2}$. To see that $F$ is of bounded type, give $A, B$ degree 2 and $C$ degree 1. Then $F$-related paths are equal in degree. We have $h(0)=0$ and $h(1)=1$ and $h(N)=2 N$ for $N \geq 2$.
- Regard $R=\operatorname{span}(A B, A B C)$. The basis $F=\{A B, A B C\}$ is of bounded type and $h(N)=N$. The basis $F=\{A B+A B C, A B-A B C\}$ is however not of bounded type, because $A B \sim A B C \sim A B C C$ etc.

Remark 19.26. Given a basis $F \subseteq R$ and an integer $N \in \mathbb{N}$, denote by $l(N)$ the minimal length of paths $F$-related to paths of length $\geq N$ :

$$
l(N)=\min \{|q||\exists p:|p| \geq N, p \sim q\} .
$$

Then $h(l(N)) \geq N$. Indeed, pick a path of length $l(N)$. Then it is $F$-related to a path of length $\geq N$ and hence the supremum $h(l(N))$ is at least $N$.

Assume $F$ is of bounded type. Then $h$ is finite on every integer. The inequality $h(l(N)) \geq N$ ensures that $l(N)$ converges to infinity as $N \rightarrow \infty$. Simply speaking, if $l(N)$ stays small, this means there are short paths equivalent to longer and longer paths, precisely the opposite of the assumption. In total, we conclude that the interval $[l(M), h(N)]$ goes to infinity as $[M, N]$ goes to infinity. Moreover, we can say

$$
p \sim q,|q| \in[M, N] \quad \Longrightarrow \quad|p| \in[l(M), h(N)] .
$$



Figure 19.1: Decomposition into related and unrelated paths

To demonstrate the strength of the boundedness assumption, we apply Lemma 19.21 to relations of bounded type. For instance, the next lemma brings elements $x \in X$ known to satisfy length bounds back to a bounded part of the spanning set. We write ${ }_{M \leq} \mathbb{C} Q_{\leq N}$ for the subspace of $\mathbb{C} Q$ spanned by paths of length between $M$ and $N$.

Lemma 19.27. Let $Q$ be a quiver and $R \subseteq \mathbb{C} Q$ a $\mathbb{C} Q_{0}$-bimodule with basis $F$. Let $X \subseteq \mathbb{C} Q$ be a subspace with spanning set $X_{0}$ such that for every $x_{0} \in X_{0}$ all paths appearing in $x_{0}$ are $F$-related. Then

$$
\begin{equation*}
\forall M, N \in \mathbb{N}: \quad X \cap \cap_{M \leq} \mathbb{C} Q_{\leq N} \subseteq \operatorname{span}\left(X_{0} \cap_{l(M) \leq} \mathbb{C} Q_{\leq h(N)}\right) \tag{19.6}
\end{equation*}
$$

Proof. We apply Lemma 19.21 As ambient space use $V=\mathbb{C} Q$, as spanning set $V_{0}$ use the set of paths in $Q$, as relation $\sim$ use $F$-relatedness, and as restriction $C$ use the set of paths in $Q$ that are $F$-related to paths of length between $M$ and $N$. Lemma 19.21 now yields $X \cap \operatorname{span}(C) \subseteq \operatorname{span}\left(X_{0} \cap \operatorname{span}(C)\right)$.

The desired inclusion 19.6 is slightly weaker than this. Let us check. The left side of 19.6 is contained in $X \cap \operatorname{span}(C)$, since $C$ includes in particular all paths of length between $M$ and $N$. Finally $\operatorname{span}\left(X_{0} \cap \operatorname{span}(C)\right)$ is contained in the right side of 19.6 , since paths in $C$ always have length at least $l(M)$ and at most $h(N)$.

We provide a further example of path length analysis, a toy version of our later flatness results.
Lemma 19.28. Let $Q$ be a quiver and $R \subseteq \mathbb{C} Q$ a $\mathbb{C} Q_{0}$-bimodule with basis $F$. Let $(B, \mathfrak{m})$ be a deformation base. Let $X \subseteq B \widehat{\otimes} \mathbb{C} Q$ be a subspace with spanning set $X_{0}$ such that for every $x_{0} \in X_{0}$ all zeroth-order paths in $x_{0}$ are $F$-related. Then

$$
X \cap\left(\mathbb{C} Q_{\leq N}+\mathfrak{m} \mathbb{C} Q\right) \subseteq \operatorname{span}\left(X_{0} \cap\left(\mathbb{C} Q_{\leq h(N)}+\mathfrak{m} \mathbb{C} Q\right)\right)+X \cap \mathfrak{m} \mathbb{C} Q
$$

Proof. It is possible to build on Lemma 19.27, but we deploy Lemma 19.21 instead. Denote by $\pi$ : $X \rightarrow \mathbb{C} Q$ the projection to zeroth order. Define $Y_{0}:=\pi\left(X_{0}\right)$. It is a spanning set for $\pi(X)$. Now let $x \in X \cap\left(\mathbb{C} Q_{\leq N}+\mathfrak{m} \mathbb{C} Q\right)$. Then

$$
\pi(x) \in \pi(X) \cap \mathbb{C} Q_{\leq N} \subseteq \operatorname{span}\left(Y_{0} \cap \mathbb{C} Q_{\leq h(N)}\right) \subseteq \operatorname{span}\left(\pi\left(X_{0} \cap\left(\mathbb{C} Q_{\leq h(N)}+\mathfrak{m} \mathbb{C} Q\right)\right)\right)
$$

Hence we can choose $x^{\prime} \in \operatorname{span}\left(X_{0} \cap\left(\mathbb{C} Q_{\leq h(N)}+\mathfrak{m} \mathbb{C} Q\right)\right)$ such that $\pi(x)=\pi\left(x^{\prime}\right)$. In particular, we have $x-x^{\prime} \in X \cap \mathfrak{m} \mathbb{C} Q$. We finish the proof with the observation that $x=x^{\prime}+\left(x-x^{\prime}\right)$.

Lemma 19.28 is best summarized as follows: When forming a large sum $x$ over $X_{0}$ elements, once their zeroth-order length surpasses $h(N)$ they cannot contribute anymore to $x$, up to higher order terms. In other words, summands of zeroth-order length $>h(N)$ contribute only to higher order terms.

Remark 19.29. The original idea behind our boundedness condition is best explained as follows: We are given an element $x \in \mathbb{C} Q_{1}\left(I(P)_{\mathbb{C} Q} \cap \mathfrak{m} \mathbb{C} Q\right)$ and are supposed to improve on this. Berger and Ginzburg proceed by splitting off arrows on the left and thereby reducing degree. Our idea is to iterate the inclusion instead which increases degree. The resulting elements $x_{i}$ lie in higher an higher path length. If we can show that the terms in very high path length are not related at all with the low length terms we started with, then the $x_{i}$ must vanish. This insight led to the boundedness condition presented in this section.

### 19.5 Ideals tailored to boundedness

In this section, we collect and introduce notation for many ideal-like sets. For instance, we have already used the notation $I(R)$ earlier to denote ideals generated by the relation space $R$. In the present section, we define new ideal-like sets that are tailored to our boundedness argument of section 19.4

The setting of this section is Convention 19.14 The ideal-like sets we define here are depicted in Table 19.2 In Definition 19.30, we start with simple subspaces of $\mathbb{C} Q$ where the lengths are bounded. In Definition 19.31 we define ideal-like spaces in which length bounds are imposed on the paths embracing a relation:

| Notation |  | Definition | Typical element | Requirement |
| :---: | :---: | :---: | :---: | :---: |
| $I(R)$ | $=$ | $\mathbb{C} Q R \mathbb{C} Q$ | $\sum_{i=0}^{K} p_{i} r_{i} q_{i}$ | finite sum |
| $I(R)_{\widehat{\mathbb{C Q}}}$ | $=$ | $\begin{gathered} \operatorname{Im}(\widehat{\mathbb{C Q}} \widehat{\otimes} R \widehat{\otimes} \widehat{\mathbb{C} Q} \\ \rightarrow \widehat{\mathbb{C Q}}) \end{gathered}$ | $\sum_{i=0}^{\infty} p_{i} r_{i} q_{i}$ | $\left\|p_{i}\right\|+\left\|q_{i}\right\| \rightarrow \infty$ |
| $\widehat{\mathbb{C Q}} R \widehat{R \mathbb{C} Q}$ | $=$ | $\begin{gathered} \operatorname{Im}(\widehat{\mathbb{C Q}} \otimes R \otimes \widehat{\mathbb{C Q}} \\ \rightarrow \widehat{\mathbb{C Q}}) \end{gathered}$ | $\sum_{i=0}^{K}\left(\sum_{j=0}^{\infty} \lambda_{j} p_{j}\right) r_{i}\left(\sum_{j=0}^{\infty} \eta_{j} q_{j}\right)$ | $\left\|p_{i}\right\|,\left\|q_{i}\right\| \rightarrow \infty$ |
| (P) | $=$ | $\begin{gathered} \operatorname{Im}(\widehat{\mathbb{C Q}} \widehat{\otimes} P \widehat{\otimes} \widehat{\mathbb{C Q}} \\ \rightarrow B \widehat{\otimes} \widehat{\mathbb{C} Q}) \end{gathered}$ | $\sum_{i=0}^{\infty} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i}$ | $\left\|p_{i}\right\|+\left\|q_{i}\right\| \rightarrow \infty$ |
| $I(P)$ | $=$ | $B(P)$ | $\sum_{j=0}^{\infty} m_{j} \sum_{i=0}^{\infty} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i}$ | $\begin{gathered} m_{j} \in \mathfrak{m} \rightarrow \infty \\ \left\|p_{i}\right\|+\left\|q_{i}\right\| \rightarrow \infty \end{gathered}$ |
| $I(P)_{\mathbb{C} Q}$ | $=$ | $B \mathbb{C} Q P \mathbb{C} Q$ | $\sum_{i=0}^{\infty} m_{i} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i}$ | $m_{i} \in \mathfrak{m}^{\rightarrow \infty}$ |
| $M \leq \mathbb{C} Q_{\leq N}$ | $=$ | $\operatorname{span}_{M \leq\|p\| \leq N} p$ | $\sum_{i=0}^{K} p_{i}$ | $M \leq\left\|p_{i}\right\| \leq N$ |
| $M \leq(\mathbb{C} Q R \mathbb{C} Q)_{\leq N}$ | $=$ | $\operatorname{span}_{M \leq\|p r q\| \leq N} p r q$ | $\sum_{i=0}^{K} p_{i} r_{i} q_{i}$ | $M \leq\left\|p_{i} r_{i} q_{i}\right\| \leq N$ |
| $M \leq(\mathbb{C} Q P \mathbb{C} Q)_{\leq N}$ | $=$ | $\operatorname{span}_{M \leq\|p r q\| \leq N} p(r+\psi(r)) q$ | $\sum_{i=0}^{K} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i}$ | $M \leq\left\|p_{i} r_{i} q_{i}\right\| \leq N$ |

Table 19.2: Ideal-like sets with their definitions, typical elements and requirements. For sake of legibility, we have omitted double indices in the description of the typical elements. For instance $p_{j}$ in double sums should read $p_{i, j}$.

Definition 19.30. We define the following four subspaces of $\mathbb{C} Q$ and $\mathbb{C} Q \otimes \mathbb{C} Q_{0} \mathbb{C} Q$ :

- The spaces

$$
M \leq \mathbb{C} Q_{\leq N}, \quad \mathbb{C} Q_{\geq N}, \quad \mathbb{C} Q_{\leq N}
$$

are the subspaces of $\mathbb{C} Q$ spanned by paths of length in $[M, N],[N, \infty)$ or $[0, N]$, respectively.

- The spaces

$$
M \leq(\mathbb{C} Q \otimes \mathbb{C} Q)_{\leq N}, \quad(\mathbb{C} Q \otimes \mathbb{C} Q)_{\geq N}, \quad(\mathbb{C} Q \otimes \mathbb{C} Q)_{\leq N}
$$

are the subspaces of $\mathbb{C} Q \otimes \mathbb{C} Q_{0} \mathbb{C} Q$ spanned by pure tensors $p \otimes p^{\prime}$ where $p$ and $p^{\prime}$ are paths with length bound $\left|p p^{\prime}\right| \in[M, N],\left|p p^{\prime}\right| \geq N$ or $\left|p p^{\prime}\right| \leq N$, respectively.

Definition 19.31. We define the following six subspaces of $\mathbb{C} Q$ and $B \widehat{\otimes} \widehat{\mathbb{C} Q}$ :

- The spaces

$$
M \leq(\mathbb{C} Q R \mathbb{C} Q)_{\leq N}, \quad(\mathbb{C} Q R \mathbb{C} Q)_{\geq M}, \quad(\mathbb{C} Q R \mathbb{C} Q)_{\leq N}
$$

are the subspaces of $\mathbb{C} Q$ consisting of elements that can be written in the form

$$
x=\sum_{\text {finite }} p_{i} r_{i} q_{i},
$$

where $p_{i}, q_{i}$ are paths in $Q$ and $r_{i} \in R$ are relations such that for all $i$ every path contained in $p_{i} r_{i} q_{i} \in \mathbb{C} Q$ has length in $[M, N]$ or $[M, \infty)$ or $[0, N]$, respectively.

- The spaces

$$
M \leq(\mathbb{C} Q P \mathbb{C} Q)_{\leq N}, \quad(\mathbb{C} Q P \mathbb{C} Q)_{\geq M}, \quad(\mathbb{C} Q P \mathbb{C} Q)_{\leq N}
$$

are the subspaces of $B \widehat{\otimes} \widehat{\mathbb{C} Q}$ consisting of elements that can be written in the form

$$
x=\sum_{\text {finite }} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i},
$$

where $p_{i}, q_{i}$ are paths in $Q$ and $r_{i} \in R$ are relations such that for all $i$ every path contained in $p_{i} r_{i} q_{i} \in \mathbb{C} Q$ has length in $[M, N]$ or $[M, \infty)$ or $[0, N]$, respectively.

We finally define several ideal-like spaces which are finely tuned to the purpose of bounding path lengths:

Definition 19.32. We define the spaces $I(R), I(R)_{\widehat{\mathbb{C Q}}}, \widehat{\mathbb{C Q}} R \widehat{\mathbb{C Q}},(P), I(P)$ and $I(P)_{\mathbb{C} Q}$ as follows:

- The space $I(R)=\mathbb{C} Q R \mathbb{C} Q \subseteq \mathbb{C} Q$ is the ideal generated by $R$. In other words, it contains elements of the form

$$
x=\sum_{i=0}^{K} p_{i} r_{i} q_{i} .
$$

Here $r_{i} \in R$ and $p_{i}, q_{i}$ are paths in $Q$.

- The space $I(R)_{\widehat{\mathbb{C Q}}} \subseteq \widehat{\mathbb{C Q}}$ is defined as the image of the multiplication map $\widehat{\mathbb{C Q}} \widehat{\otimes} R \widehat{\otimes} \widehat{\mathbb{C Q}} \rightarrow B \widehat{\otimes} \widehat{\mathbb{C Q}}$. In other words, its elements are of the form

$$
x=\sum_{i=0}^{\infty} p_{i} r_{i} q_{i}
$$

Here $r_{i} \in R$ and $p_{i}, q_{i}$ are paths with $\left|p_{i}\right|+\left|q_{i}\right| \rightarrow \infty$.

- The space $\widehat{\mathbb{C Q}} R \widehat{\mathbb{C Q}}$ is the ideal generated by $R$ in $\widehat{\mathbb{C Q}}$. In other words, its elements are of the form

$$
x=\sum_{j=0}^{K}\left(\sum_{i=0}^{\infty} \lambda_{i, j} p_{i, j}\right) r_{i, j}\left(\sum_{i=0}^{\infty} \eta_{i, j} q_{i, j}\right) .
$$

Here $r_{i} \in R$ and $p_{i, j}, q_{i, j}$ are paths in $Q$ such that for every $j \in \mathbb{N}$ the lengths $\left|p_{i, j}\right|$ and $\left|q_{i, j}\right|$ converge to $\infty$ as $i \rightarrow \infty$.

- The space $(P) \subseteq B \widehat{\otimes} \widehat{\mathbb{C Q}}$ is image of the multiplication map $\widehat{\mathbb{C Q}} \widehat{\otimes} P \widehat{\otimes} \widehat{\mathbb{C Q}} \rightarrow B \widehat{\otimes} \widehat{\mathbb{C Q}}$. In other words, it contains elements of the form

$$
x=\sum_{i=0}^{\infty} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i} .
$$

Here $r_{i} \in R$, and $p_{i}, q_{i}$ are paths in $Q$ with combined length $\left|p_{i}\right|+\left|q_{i}\right| \rightarrow \infty$.

- The space $I(P) \subseteq B \widehat{\otimes} \widehat{\mathbb{C Q}}$ is defined as $I(P)=B(P)$. In other words, it contains elements of the form

$$
x=\sum_{j=0}^{\infty} m_{j} \sum_{i=0}^{\infty} p_{i, j}\left(r_{i, j}+\psi\left(r_{i, j}\right)\right) q_{i, j} .
$$

Here $m_{j} \in \mathfrak{m}^{\rightarrow \infty}$ and for every $j \in \mathbb{N}$ the combined length $\left|p_{i, j}\right|+\left|q_{i, j}\right|$ of $p_{i, j}$ and $q_{i, j}$ is supposed to converge to $\infty$ as $i \rightarrow \infty$.

- If $\operatorname{Im}(\psi) \subseteq \mathfrak{m} \mathbb{C} Q$, then the space $I(P)_{\mathbb{C} Q} \subseteq B \widehat{\otimes} \mathbb{C} Q$ is defined as $B(\mathbb{C} Q P \mathbb{C} Q) \subseteq B \widehat{\otimes} \mathbb{C} Q$. In other words, it contains elements of the form

$$
x=\sum_{i=0}^{\infty} m_{i} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i}
$$

Here $m_{i} \in \mathfrak{m}^{\rightarrow \infty}$ and $r_{i} \in R$. The elements $p_{i}, q_{i}$ are paths in $Q$.
Remark 19.33. In Definition 19.32, we have done our best to give both an abstract and a practical definition of all ideal-like sets. The abstract definitions come with the following two peculiarities: First, the notation $B(P)$ in the definition of $I(P)$ makes use of the shorthand notation of Definition 15.18 Second, the given abstract definition of $(P)$ makes use of the completed tensor product $\widehat{\mathbb{C} Q} \widehat{\otimes} \widehat{\otimes} \mathbb{C} Q$. The completion here is taken with respect to the Krull topology, which we actually only explain in section 19.9 Just like $B \widehat{\otimes} X$ consists of formal power series in elements of $X$, the space $\widehat{\mathbb{C Q}} \widehat{\otimes} P \widehat{\otimes} \widehat{\mathbb{C} Q}$ consists of formal two-sided power series of paths embracing elements of $P$. For the present context, it suffices to accept the explicit description of the elements of $(P)$ in terms of series.

Remark 19.34. The individual definitions are tedious to memorize, but the hidden structure becomes apparent once we compare the definitions:

|  | $R$ | $P$ |
| :--- | :--- | :--- |
| formed in $\mathbb{C} Q$ | $I(R)$ | $I(P)_{\mathbb{C} Q}$ |
| formed in $\widehat{\mathbb{C} Q}$ | $I(R)_{\widehat{\mathbb{C}}}$ | $I(P)$ |

The default objects are $I(R)$ and $I(P)$. The ideals $I(R)_{\widehat{\mathbb{C}}}$ and $I(P)_{\mathbb{C} Q}$ only appear in section 19.8 and 19.9

Remark 19.35. We list here a few warnings:

- $I(R)_{\widehat{\mathbb{C} Q}}$ is an ideal in $\widehat{\mathbb{C} Q}$, but not the ideal generated by $R$ and not its closure either.
- $I(P)$ is an ideal in $B \widehat{\otimes} \widehat{\mathbb{C} Q}$, but not the ideal generated by $P$ and not its closure either.
- ${ }_{M \leq}(\mathbb{C} Q P \mathbb{C} Q)_{\leq N}$ is not the same as $\mathbb{C} Q P \mathbb{C} Q \cap_{M \leq} \mathbb{C} Q_{\leq N}$.

For instance, closedness of $I(P)$ in the $\mathfrak{m}$-adic topology would entail roughly the following: Whenever a sequence $\left(x_{n}\right) \subseteq I(P)$ becomes concentrated in high powers of $\mathfrak{m}^{k}$, the differences $x_{n}-x_{n+1}$ are not only concentrated in $\mathfrak{m}^{k}$ as a whole, but can be written as a sum over $m_{i} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i}$ where every single coefficient $m_{i}$ lies in $\mathfrak{m}^{k}$. This observation visualizes that $I(P)$ is not necessarily closed. Once we prove $I(P)$ quasi-flat however, it is also closed in the $\mathfrak{m}$-adic topology according to Proposition 15.43

We have depicted abbreviated descriptions of the various sets in Table 19.2

### 19.6 Bounded strong Berger-Ginzburg inclusion

In this section, we prove a bounded version of the strong Berger-Ginzburg inclusion. The necessity of this bounded version is illustrated by the large amount of path length estimates we need to deploy in section 19.7. Indeed, the lack of homogeneity for $W$ makes it necessary to estimate lengths of virtually every vector involved in the flatness argument. One of the sources of vectors in the flatness argument is the strong Berger-Ginzburg inclusion. As such, we need a version of the strong Berger-Ginzburg inclusion that is suited for the bounded world. The present section is meant to provide this bounded version. It is surprising that the bounded version follows directly from the ordinary strong Berger-Ginzburg inclusion. It is a kind of a posteriori estimate and works without additional assumptions on $Q$ or $W$ :


Since the strong Berger-Ginzburg inclusion works with tensors instead of paths, our first step in this section is to introduce a notion of $F$-relatedness for tensors:

Definition 19.36. Let $F$ be a basis for $R$. Then two pure tensors of paths $p \otimes p^{\prime}$ and $q \otimes q^{\prime}$ are $F$-related if $p p^{\prime}$ and $q q^{\prime}$ are $F$-related.

We shall now construct a spanning set $W_{0}$ for $W$ with the property that for every $w \in W_{0}$ all pure tensors of paths appearing in $f(w)$ are $F$-related. Recall from Convention 19.14 that $f: W \otimes \mathbb{C} Q \rightarrow R \otimes$ $\mathbb{C} Q_{1} \mathbb{C} Q$ is the map which splits $W$ into relations and arrows. When $w \in W$, then we have $f(w) \in R \otimes \mathbb{C} Q_{1}$.

Lemma 19.37. Let $F$ be a basis for $R$. Then $W=\mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1}$ has a spanning set $W_{0}$ such that for every $w \in W_{0}$ the constituents of $f(w) \in R \otimes \mathbb{C} Q_{1}$ are $F$-related.

Proof. The strategy is to decompose an arbitrary $w \in W$ into a sum $\sum w_{p}$ such that the constituents of $f\left(w_{p}\right)$ are $F$-related. Pick $w \in W$. Since $W=\mathbb{C} Q_{1} R \cap R \mathbb{C} Q_{1}$, we can write $w$ in two ways as finite sums

$$
w=\sum_{i \in I} a_{i} r_{i}=\sum_{j \in J} r_{j}^{\prime} b_{j} .
$$

Here $a_{i}$ and $b_{j}$ are scalar multiples of arrows and $r_{i}$ and $r_{j}^{\prime}$ lie in $F$. Note that in both sums the constituents of every individual summand are $F$-related. Denote by $P$ the finite set of paths appearing anywhere in these sums, modulo $F$-relatedness. Now $P$ splits both $I$ and $J$ into classes. Namely for $p \in P$, let $I_{p} \subseteq I$ be the set of indices $i$ where the constituents of $a_{i} r_{i}$ are $F$-related to $p$. Similarly, let $J_{p} \subseteq J$ be the set of indices where the constituents of $r_{j}^{\prime} b_{j}$ are $F$-related to $p$. For every $p \in P$, both sums

$$
w_{p}=\sum_{i \in I_{p}} a_{i} r_{i}, \quad w_{p}^{\prime}=\sum_{j \in J_{p}} r_{j}^{\prime} b_{j}
$$

have path support lying in the equivalence class of $p$. Since the index sets $\left(I_{p}\right)_{p \in P}$ and $\left(J_{p}\right)_{p \in P}$ both exhaust disjointly $I$ and $J$, we conclude $w_{p}=w_{p}^{\prime}$ for every $p \in P$. Furthermore $w_{p} \in \mathbb{C} Q_{1} R$ and $w_{p}^{\prime} \in R \mathbb{C} Q_{1}$, hence $w_{p}=w_{p}^{\prime} \in W$. By construction, all constituents in

$$
f\left(w_{p}\right)=\sum_{j \in J_{p}} r_{j}^{\prime} \otimes b_{j} .
$$

contract to paths $F$-related to $p$. In particular, all constituents of $f\left(w_{p}\right)$ are related to each other. We have decomposed $w \in W$ into summands $w_{p} \in W$ such that all constituents in $f\left(w_{p}\right)$ are $F$-related to each other. Running this algorithm for every $w \in W$ provides a spanning set $W_{0}$ with the desired property. Naturally one can extract a finite one from it.

Remark 19.38. The statement of Lemma 19.37 is obvious in case $R$ is the space of relations $R=$ $\operatorname{span}\left(\partial_{a} W\right)$ coming from a CY3 superpotential. Indeed, simply use the spanning set

$$
\begin{equation*}
W_{0}=\left\{\sum_{t(a)=v} \partial_{a} W a \mid v \in Q_{0}\right\} . \tag{19.7}
\end{equation*}
$$

Let us check that the spanning set $W_{0}$ satisfies the claimed condition. For $w=\sum_{t(a)=v} \partial_{a} W a$, we simply have $f(w)=\sum_{t(a)=v} \partial_{a} W \otimes a$. By cyclicity of $W$, all constituents of this sum are related. This makes that $W_{0}$ satisfies the requirements of Lemma 19.37

Before we devote ourselves to the bounded version of the strong Berger-Ginzburg inclusion, we need length bounds for the paths in $R$ and $W$. While $R$ and $W$ are not homogeneous, any element $r \in R$ or $w \in W$ can still be decomposed as a linear combination of paths in $Q$ :

$$
r=\lambda_{1} p_{1}+\ldots \lambda_{k} p_{k} \quad \text { or } \quad w=\varepsilon_{1} q_{1}+\ldots+\varepsilon_{l} q_{l} .
$$

The paths $p_{1}, \ldots, p_{k}$ or $q_{1}, \ldots, q_{1}$ are typically not of the same length. However, $R$ and $W$ are finitedimensional by Convention 19.14 This implies that the path lengths encountered in $R$ and $W$ are bounded. We therefore fix the following notation:

Definition 19.39. The maximum path length encountered in $R$ and $W$ is denoted by $|R| \in \mathbb{N}$ and $|W| \in \mathbb{N}$.

We are ready to prove our bounded version of the strong Berger-Ginzburg inclusion:
Lemma 19.40. Assume $R$ is of bounded type. If the strong Berger-Ginzburg inclusion [BG] holds, then it also holds in the bounded form

$$
\begin{align*}
R \otimes \mathbb{C} Q \cap c^{-1}\left(\mathbb{C} Q_{1} I(R)\right) & \cap{ }_{M \leq}(\mathbb{C} Q \otimes \mathbb{C} Q)_{\leq N} \\
& \subseteq f\left(W \otimes_{l(M)-|W| \leq} \mathbb{C} Q_{\leq h(N)}\right)+R \otimes_{l(M)-|R| \leq}(\mathbb{C} Q R \mathbb{C} Q)_{\leq h(N)} \tag{19.8}
\end{align*}
$$

Here $N, M \in \mathbb{N}$ are arbitrary integers. The inclusion also holds when two-sided bounds are replaced by one-sided bounds from above or below.

Proof. The proof boils down to applying Lemma 19.27 to the right side of the Berger-Ginzburg inclusion. Pick a spanning set $W_{0} \subseteq W$ as in Lemma 19.37. Put $X=f(W \otimes \mathbb{C} Q)+R \otimes I(R)$ and use splits $f(w \otimes p)$ and products $r \otimes p r^{\prime} q$ as spanning set $X_{0} \subseteq X$, where $w \in W_{0}, r, r^{\prime} \in F$, and $p, q$ are paths. Regard the ambient space $V=\mathbb{C} Q \otimes \mathbb{C} Q$ and its basis $V_{0}$ consisting of pure tensors of paths.

By construction, all constituents in a split $f(w \otimes p)$ are $F$-related. Namely they consist of $F$-related tensors, with additional $p$ on the right side. Also all constituents in a product $r \otimes p r^{\prime} q$ are $F$-related. Indeed, decomposing $r=\sum \lambda_{i} c_{i}$ and $r^{\prime}=\sum \lambda_{i}^{\prime} c_{i}^{\prime}$ into scalar multiples of paths, we have

$$
\forall i_{0}, j_{0}, i_{1}, j_{1}: \quad c_{i_{0}} \otimes p c_{j_{0}}^{\prime} q \sim c_{i_{1}} \otimes p c_{j_{0}}^{\prime} q \sim c_{i_{1}} \otimes p c_{j_{1}}^{\prime} q
$$

Further choose $C \subseteq V_{0}$ as the set of path tensors $p \otimes p^{\prime}$ whose contraction $p p^{\prime}$ is related to a path of length at least $M$ and at most $N$. This set is closed under $\sim$. Finally, all assumptions of Lemma 19.21 are satisfied and we obtain

$$
\begin{equation*}
(f(W \otimes \mathbb{C} Q)+R \otimes I(R)) \cap \operatorname{span}(C) \subseteq \operatorname{span}\left(X_{0} \cap \operatorname{span}(C)\right) \tag{19.9}
\end{equation*}
$$

It remains to interpret both sides of this inclusion. Let us start with the right side. Tensors in span $(C)$ have length between $l(M)$ and $h(N)$. If an element $f(w \otimes p)$ lies in $\operatorname{span}(C)$, then $p$ has length $l(M)-|W| \leq$ $|p| \leq h(N)$. If an element $r \otimes p r^{\prime} q$ lies in span $(C)$, then all paths $c$ in $p r^{\prime} q$ have length $l(M)-|R| \leq|c| \leq$ $h(N)$. We conclude that the right side of 19.9 is contained in the right side of 19.8 .

We finish the proof with the remark that the left side of $\sqrt{19.8}$ is contained in the left side of $\sqrt[19.9]{ }$, since the strong Berger-Ginzburg inclusion holds by assumption and tensors of length between $M$ and $N$ lie in $\operatorname{span}(C)$.

### 19.7 Quasi-flatness in the completed path algebra

In this section, we prove our first quasi-flatness result. The idea is to use our boundedness condition from section 19.4 to continue the line of Berger and Ginzburg without homogeneity assumption. The flatness result in this section deals with the ideals $I(P) \subseteq B \widehat{\otimes} \widehat{\mathbb{C} Q}$ in the completed quiver algebra. In section 19.8 we will reduce the result of the present section to the case of the non-completed quiver algebra.

For convenience of the reader, we have sketched in Lemma 19.20 the core idea of Berger and Ginzburg. It is however not necessary to be aware of the statement. Rather, we prove from scratch a bounded version of their statement. Before we proceed, recall fromDefinition 19.39 that $|R|$ and $|W|$ denote the maximum path length encountered in $R$ and $W$.

Lemma 19.41. Assume $R$ is of bounded type and [BG] and [CP] hold. Then

$$
\begin{equation*}
(\mathbb{C} Q P \mathbb{C} Q)_{\geq N} \cap \mathfrak{m} \widehat{\mathbb{C} Q} \subseteq \mathbb{C} Q_{1}(\mathbb{C} Q P \mathbb{C} Q)_{\geq l(N)-|R|-1}+\mathfrak{m} I(P)_{\geq l(N)-2|R|} \tag{19.10}
\end{equation*}
$$

Proof. By assumption, $R$ has a basis $F$ of bounded type. Now pick an element

$$
\tilde{x}=\sum_{i \in I} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i} \in(\mathbb{C} Q P \mathbb{C} Q)_{\geq N} \cap \mathfrak{m} \widehat{\mathbb{C} Q}
$$

By definition of $(\mathbb{C} Q P \mathbb{C} Q)_{\geq N}$, we can assume $p_{i}$ and $q_{i}$ are scalar multiples of paths, $r_{i}$ lies in $F$ and all paths in $p_{i} r_{i} q_{i}$ have length at least $N$. Let us inspect the sum. The terms where $p_{i} \in \mathbb{C} Q_{\geq 1}$ already lie on the right-hand side of 19.10 and do not need further treatment. The terms where $p_{i}$ is a vertex are nasty. Denote by $I_{0} \subseteq I$ the set of these nasty indices. Put

$$
x:=\sum_{i \in I_{0}} p_{i} r_{i} \otimes q_{i} \in R \otimes \mathbb{C} Q .
$$

Since $\tilde{x}$ is supposed to vanish on zeroth order, we have

$$
c(x)=\sum_{i \in I_{0}} p_{i} r_{i} q_{i}=-\sum_{i \in I \backslash I_{0}} p_{i} r_{i} q_{i} \in \mathbb{C} Q_{1} I(R) .
$$

At the same time, $x$ lies in $(\mathbb{C} Q \otimes \mathbb{C} Q)_{\geq N}$ and hence on the left-hand side of the bounded strong BergerGinzburg inclusion 19.8, using $N$ for the lower bound and dropping the upper bound. We conclude

$$
x \in f\left(W \otimes \mathbb{C} Q_{\geq l(N)-|W|}\right)+R \otimes(\mathbb{C} Q R \mathbb{C} Q)_{\geq l(N)-|R|}
$$

Split $x=y+z$ according to this decomposition. We want to determine $\psi(x)=\psi(y)+\psi(z)$, where $\psi$ here acts on the left tensor factor of $R \otimes \mathbb{C} Q$. Let us regard $y$ first. We can write $y$ as a finite sum

$$
y=\sum_{i \in K} f\left(w_{i} \otimes p_{i}^{\prime}\right)
$$

with $w_{i}$ lying in $W$ and $p_{i}^{\prime}$ scalar multiples of paths with $\left|p_{i}^{\prime}\right| \geq l(N)-|W|$. By Lemma 19.17. we have $\psi^{0}=\psi^{1}$ on $W$. Therefore we can swap $\psi^{0}\left(w_{i}\right)$ over:

$$
c\left(y+\psi^{0}(y)\right)=\sum_{i \in K}\left(w_{i}+\psi^{0}\left(w_{i}\right)\right) p_{i}^{\prime}=\sum_{i \in K}\left(w_{i}+\psi^{1}\left(w_{i}\right)\right) p_{i}^{\prime} .
$$

Note that $w_{i}+\psi^{1}\left(w_{i}\right) \in \mathbb{C} Q_{1} P$ by nature. We obtain the first intermediate result

$$
c\left(y+\psi^{0}(y)\right) \in \mathbb{C} Q_{1} P \mathbb{C} Q_{\geq l(N)-|W|}
$$

Now regard $z$. We can write

$$
z=\sum_{i \in L} r_{i}^{\prime} \otimes s_{i} r_{i}^{\prime \prime} t_{i}
$$

where $r_{i}^{\prime}, r_{i}^{\prime \prime} \in F$ and $s_{i}, t_{i}$ are scalar multiples of paths, and all paths in $s_{i} r_{i}^{\prime \prime} t_{i}$ have length at least $l(N)-|R|$. We get

$$
\begin{aligned}
c(z+\psi(z)) & =\sum_{i \in L}\left(r_{i}^{\prime}+\psi\left(r_{i}^{\prime}\right)\right) s_{i} r_{i}^{\prime \prime} t_{i} \\
& =\sum_{i \in L} r_{i}^{\prime} s_{i}\left(r_{i}^{\prime \prime}+\psi\left(r_{i}^{\prime \prime}\right)\right) t_{i}+\sum_{i \in L} \psi\left(r_{i}^{\prime}\right) s_{i}\left(r_{i}^{\prime \prime}+\psi\left(r_{i}^{\prime \prime}\right)\right) t_{i}-\sum_{i \in L}\left(r_{i}^{\prime}+\psi\left(r_{i}^{\prime}\right)\right) s_{i} \psi\left(r_{i}^{\prime \prime}\right) t_{i} \\
& \in \mathbb{C} Q_{1}(\mathbb{C} Q P \mathbb{C} Q)_{\geq l(N)-|R|}+\mathfrak{m} I(P)_{\geq l(N)-2|R|} .
\end{aligned}
$$

In the last row, we have used that $r_{i}^{\prime} \in R \subseteq \mathbb{C} Q_{\geq 1}$. In total, we get

$$
\begin{aligned}
\tilde{x}=\sum_{i \in I} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i} & =c(y+\psi(y))+c(z+\psi(z))+\sum_{i \in I \backslash I_{0}} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i} \\
& \in \mathbb{C} Q_{1}(\mathbb{C} Q P \mathbb{C} Q)_{\geq l(N)-|R|-1}+\mathfrak{m} I(P)_{\geq l(N)-2|R|} .
\end{aligned}
$$

We have used that $|W| \leq|R|+1$.
The following is a continuation of the line of Berger and Ginzburg.
Lemma 19.42. Assume $R$ is of bounded type and $[\mathrm{BG}]$ and $[\mathrm{CP}]$ holds. Then

$$
I(P) \cap \mathfrak{m} \widehat{\mathbb{C} Q} \subseteq \mathbb{C} Q_{1}(I(P) \cap \mathfrak{m} \widehat{\mathbb{C} Q})+\mathfrak{m} I(P)
$$

Proof. The strategy is to divide an element $x$ on the left-hand side into chunks $x_{N}$ that individually lie in $(\mathbb{C} Q P \mathbb{C} Q)_{\geq N} \cap \mathfrak{m}$. We estimate that the lengths of the ideal paths used in $x_{N}$ converge to $\infty$, so that summing up the chunks again lands on the right-hand side and not in completion of $I(P)$.

Since $\mathfrak{m} I(P)$ already lies on the right side, it suffices to regard elements of $(P)$ on the left-hand side. Regard such an element

$$
x=\sum_{i=0}^{\infty} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i} \in(P) \cap \mathfrak{m} \widehat{\mathbb{C} Q}
$$

For $N \geq 0$, let $S_{N} \subseteq \mathbb{N}$ be the set of indices $i$ where all paths in $p_{i} r_{i} q_{i}$ are related to a path of length $\leq N$. All sets $S_{N}$ are finite and together exhaust $\mathbb{N}$. In order to rewrite $x$, let us make these sets disjoint by setting $S_{N}^{\prime}:=S_{N} \backslash S_{N-1}$ for $N \geq 1$ and $S_{0}^{\prime}:=S_{0}$. Put

$$
x_{N}=\sum_{i \in S_{N}^{\prime}} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i}
$$

Then we get $x=\sum_{N=0}^{\infty} x_{N}$. We show that each chunk $x_{N}$ lies in $\widehat{\mathfrak{m} \mathbb{C Q}}$. Let $M<N$ and $i \in S_{M}^{\prime}, j \in S_{N}^{\prime}$. Then paths in $p_{i} r_{i} q_{i}$ are related to paths of length $\leq M$, while paths in $p_{j} r_{j} q_{j}$ are not related to paths of length $\leq M$, since $j \notin S_{M}$. This implies that the zeroth order path supports of all chunks $x_{N}$ are pairwise disjoint. However we know that $x \in \mathfrak{m} \widehat{\mathbb{C} Q}$, hence $x$ vanishes on zeroth order. We conclude $x_{N} \in \mathbb{C} Q P \mathbb{C} Q \cap \mathfrak{m} \widehat{\mathbb{C} Q}$.

Now note that for $i \in S_{N}^{\prime}$ the term $p_{i} r_{i} q_{i}$ has length at least $N$, for otherwise $i$ would be contained in $S_{N-1}$. We conclude that

$$
x_{N} \in(\mathbb{C} Q P \mathbb{C} Q)_{\geq N} \cap \mathfrak{m} \widehat{\mathbb{C} Q}
$$

Using Lemma 19.41, we deduce

$$
x_{N} \in \mathbb{C} Q_{1}(\mathbb{C} Q P \mathbb{C} Q)_{\geq l(N)-|R|-1}+\mathfrak{m} I(P)_{\geq l(N)-2|R|}
$$

Since $R$ is of bounded type, we have $l(N) \rightarrow \infty$ as $N \rightarrow \infty$. When summing up $x_{N}$ over $N \in \mathbb{N}$, this estimate on path lengths embracing $P$ ensures that we get two well-defined sums in $I(P)$ and not e.g. in the completion of $I(P)$ :

$$
x=\sum_{N=0}^{\infty} x_{N} \in \mathbb{C} Q_{1} I(P)+\mathfrak{m} I(P)
$$

Since $x$ lies in $\mathfrak{m} \widehat{\mathbb{C Q}}$ by assumption, we obtain

$$
x \in\left(\mathbb{C} Q_{1} I(P) \cap \mathfrak{m} \widehat{\mathbb{C Q}}\right)+\mathfrak{m} I(P)=\mathbb{C} Q_{1}(I(P) \cap \mathfrak{m} \widehat{\mathbb{C} Q})+\mathfrak{m} I(P)
$$

This finishes the proof.

We now arrive at our first flatness result. Recall that quasi-flatness for $I(P)$ refers to the inclusion $I(P) \cap \mathfrak{m} \widehat{\mathbb{C Q}} \subseteq \mathfrak{m} I(P)$.

Proposition 19.43 (Quasi-flatness I). Assume $R$ is of bounded type and [BG] and [CP] hold. Then $I(P)$ is quasi-flat.

Proof. Iterating the statement of Lemma 19.42, we get

$$
\begin{aligned}
I(P) \cap \mathfrak{m} \widehat{\mathbb{C Q}} & \subseteq \mathbb{C} Q_{1}(I(P) \cap \mathfrak{m} \widehat{\mathbb{C Q}})+\mathfrak{m} I(P) \\
& \subseteq \mathbb{C} Q_{1}\left(\mathbb{C} Q_{1}(I(P) \cap \mathfrak{m} \widehat{\mathbb{C} Q})+\mathfrak{m} I(P)\right)+\mathfrak{m} I(P) \subseteq \ldots
\end{aligned}
$$

In other words, pick $x \in I(P) \cap \mathfrak{m} \widehat{\mathbb{C Q}}$. Then we get a sequence of elements $x_{N} \in \mathbb{C}_{N}(I(P) \cap \mathfrak{m} \widehat{\mathbb{C Q}})$ and $y_{N} \in \mathfrak{m} \mathbb{C} Q_{N-1} I(P)$ with

$$
\forall N \in \mathbb{N}: \quad x=x_{N}+y_{1}+\ldots+y_{N}
$$

Now the series $\sum_{N=1}^{\infty} y_{N}$ defines an element in $\mathfrak{m} I(P)$, since its summands $y_{N}$ contain only paths of higher and higher length. We claim that this element is precisely $x$. Indeed, disregard all paths longer than an arbitrary number. Then the series stabilizes to $x$ after finitely many summands and does not change anymore thereafter. Finally, we conclude $x$ is the sum of the series, which we already know lies in $\mathfrak{m} I(P)$.

### 19.8 Quasi-flatness in the path algebra

In this section, we build our second quasi-flatness result. Namely, we show that $I(P)_{\mathbb{C} Q}$ is quasi-flat if $R$ is of bounded type and the conditions $[\mathrm{BG}]$ and $[\mathrm{CP}]$ hold. In section 19.7 we have already seen that $I(P)$ is quasi-flat. The idea of the present section is to use the boundedness argument to bring quasi-flatness from $I(P)$ to $I(P)_{\mathbb{C} Q}$.

The first step is to refine the Berger-Ginzburg lemma Lemma 19.20 We deploy the bounded version of the Berger-Ginzburg inclusion, sticking to the lower bounds and forgetting the upper bounds.

Lemma 19.44. Let $R$ be of bounded type. Then for any natural numbers $M \leq N$ we have

$$
I(P) \cap\left(M \leq \widehat{\mathbb{C Q}}_{\leq N}+\mathfrak{m} \widehat{\mathbb{C Q}}\right) \subseteq l_{l(M) \leq}(\mathbb{C} Q P \mathbb{C} Q)_{\leq h(N)}+\mathfrak{m} \widehat{\mathbb{C Q}} \cap I(P)
$$

The inclusion also holds once the upper bound or lower bound is dropped.
Proof. Since $\mathfrak{m} I(P)$ already lies on the right-hand side, it suffices to prove the inclusion of

$$
(P) \cap\left(\widehat{M \leq}^{\widehat{\mathbb{C}}}{ }_{\leq N}+\mathfrak{m} \widehat{\mathbb{C Q}}\right)
$$

Pick an element

$$
x=\sum_{i=0}^{\infty} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i} \in(P) \cap\left({ }_{M \leq} \widehat{\mathbb{C} Q}_{\leq N}+\mathfrak{m} \widehat{\mathbb{C} Q}\right) .
$$

We can assume $\left|p_{i}\right|+\left|q_{i}\right| \rightarrow \infty$. Our strategy is to decompose $x$ into two parts. Let $S \subseteq \mathbb{N}$ be the set of indices $i$ where the constituents of $p_{i} r_{i} q_{i}$ are related to paths of length in the interval $[M, N] \subseteq \mathbb{N}$. Then for $i \in S$ all paths in $p_{i} r_{i} q_{i}$ are related to paths of length in $[M, N]$ and are hence of length in $[l(M), h(N)]$ themselves. In particular $S$ is finite. Meanwhile for $i \notin S$ no paths in $p_{i} r_{i} q_{i}$ are related to paths of length in $[M, N]$ and we conclude that

$$
\sum_{i \in \mathbb{N} \backslash S} p_{i} r_{i} q_{i}=0 .
$$

In other words, we have

$$
x_{\mathbb{N} \backslash S}:=\sum_{i \in \mathbb{N} \backslash S} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i} \in(P) \cap \mathfrak{m} \mathbb{C} Q,
$$

while

$$
\left.x_{S}:=\sum_{i \in S} p_{i}\left(r_{i}+\psi\left(r_{i}\right)\right) q_{i} \in l(M) \leq \mathbb{C} Q P \mathbb{C} Q\right)_{\leq h(N)}
$$

Recalling $x=x_{S}+x_{\mathbb{N} \backslash S}$ gives that $x$ lies in the right-hand side of the desired inclusion.

We seek to apply Lemma 19.44 iteratively. As we pull out more and more powers of the maximal ideal, we need to ensure that the remainder is still bounded in length at zeroth order. We can achieve this if we require from the very beginning that an element $x \in I(P)$ has bounded length at order of $\mathfrak{m}$. For $B=\mathbb{C} \llbracket q \rrbracket$ we would require that the path lengths at every $q$ power $q^{k}$ are bounded. For general deformation basis $B$, we shall require that the path lengths in $x$ shall be bounded if we project to $B / \mathfrak{m}^{k}$. This gives rise to a subset of $B \widehat{\otimes} \mathbb{C} Q$, namely the space of elements with length at most $N_{k}$ at $\mathfrak{m}$-order $\leq k$. Let us record this in the following definition:

Definition 19.45. Let $N_{0}, N_{1}, \ldots$ be an increasing sequence of integers. Then we set

$$
\begin{aligned}
\mathbb{C} Q_{\leq\left(N_{0}, N_{1}, \ldots\right)} & =\left\{x \in B \widehat{\otimes} \mathbb{C} Q \mid \pi_{k}(x) \in\left(B / \mathfrak{m}^{k}\right) \otimes \mathbb{C} Q_{\leq N_{k}} \forall k \in \mathbb{N}\right\} \\
& =\mathbb{C} Q_{\leq N_{0}}+\mathfrak{m}^{1} \mathbb{C} Q_{\leq N_{1}}+\mathfrak{m}^{2} \mathbb{C} Q_{\leq N_{2}}+\ldots
\end{aligned}
$$

Here $\pi_{k}: B \widehat{\otimes} \mathbb{C} Q \rightarrow\left(B / \mathfrak{m}^{k}\right) \otimes \mathbb{C} Q$ denotes the standard projection.
Example 19.46. For $B=\mathbb{C} \llbracket q \rrbracket$ the space $\mathbb{C} Q_{\leq\left(N_{0}, N_{1}, \ldots\right)}$ is simply the space of $\mathbb{C} Q$-valued power series in $q$ where the paths at order $q^{k}$ are of length $\leq N_{k}$.

We are now getting closer to proving quasi-flatness of $I(P)_{\mathbb{C} Q}$. As we shall see, elements of $I(P)_{\mathbb{C} Q}$ are namely distinguished elements of $I(P)$ in the sense that they simultaneously lie in $\mathbb{C} Q_{\leq\left(N_{0}, \ldots\right)}$ for some sequence $N_{0} \leq N_{1} \leq \ldots$. To exploit this property, let us prove the following lemma.

Lemma 19.47. Assume $R$ is of bounded type and $I(P)$ is quasi-flat. Let $N_{1} \leq N_{2} \leq \ldots$ be a sequence. Then there exists a sequence $N_{1}^{\prime} \leq N_{2}^{\prime} \leq \ldots$ such that

$$
\begin{aligned}
I(P) \cap \mathbb{C} Q_{\leq\left(0, N_{1}, \ldots\right)} & \subseteq \sum_{k=1}^{\infty} \mathfrak{m}^{k}(\mathbb{C} Q P \mathbb{C} Q)_{\leq N_{k}^{\prime}} \\
& =\mathfrak{m}^{1}(\mathbb{C} Q P \mathbb{C} Q)_{\leq N_{1}^{\prime}}+\mathfrak{m}^{2}(\mathbb{C} Q P \mathbb{C} Q)_{\leq N_{2}^{\prime}}+\ldots
\end{aligned}
$$

Proof. The idea is to iterate Proposition 15.32 in combination with Lemma 19.44 and quasi-flatness of $I(P)$. In the first part of the proof, we construct the desired sequence $\left(N_{k}^{\prime}\right)$. In the second part, we construct the desired decomposition for individual elements $x \in I(P) \cap \mathbb{C} Q_{\leq\left(0, N_{1}, \ldots\right)}$. In the third part, we wrap up and prove the desired inclusion.

For the first part of the proof, let us construct the sequence $\left(N_{k}^{\prime}\right)$. Given arbitrary $k \in \mathbb{N}$, let $M_{k}$ be the maximum path length encountered in the image of $\pi \circ \psi: R \rightarrow B / \mathfrak{m}^{k} \otimes \mathbb{C} Q$. We construct an auxiliary sequence as

$$
\begin{aligned}
& N_{1}^{\prime \prime}=N_{1}, \\
& N_{2}^{\prime \prime}=\max \left(M_{1}+h\left(N_{1}^{\prime \prime}\right), N_{2}\right), \\
& N_{3}^{\prime \prime}=\max \left(M_{2}+h\left(N_{2}^{\prime \prime}\right), N_{3}\right),
\end{aligned}
$$

The sequence $\left(N_{k}^{\prime \prime}\right)$ is increasing, since $h\left(N_{i}^{\prime \prime}\right) \geq N_{i}^{\prime \prime}$. We construct the final desired sequence as

$$
N_{k}^{\prime}=h\left(N_{k}^{\prime \prime}\right)
$$

For the second part of the proof, we prove a finite version of the desired inclusion by iterating Lemma 19.44 Pick an arbitrary element $x \in I(P) \cap \mathbb{C} Q_{\leq\left(0, N_{1}, \ldots\right)}$. We shall inductively construct sequences $\left(y_{k}\right)_{k \geq 1},\left(x_{k}\right)_{k \geq 0}$ such that

$$
\begin{array}{ll}
\forall k \geq 0: & x=y_{1}+\ldots+y_{k}+x_{k} \\
\forall k \geq 0: & x_{k} \in \mathfrak{m}^{k+1} I(P)  \tag{19.11}\\
\forall k \geq 1: & y_{k} \in \mathfrak{m}^{k}(\mathbb{C} Q P \mathbb{C} Q)_{\leq h\left(N_{k}^{\prime \prime}\right)}
\end{array}
$$

Our induction starts at $k=0$. Note that we are not required to construct an element $y_{0}$. For the induction base $k=0$, note that quasi-flatness gives

$$
x \in I(P) \cap \mathbb{C} Q_{\leq\left(0, N_{1}, \ldots\right)} \subseteq \mathfrak{m} I(P)
$$

We simply put $x_{0}=x$. This satisfies the requirement 19.11 for the induction base $k=0$.

Towards the induction step, let $k \geq 0$ and assume that $x_{0}, \ldots, x_{k}$ and $y_{1}, \ldots, y_{k}$ have been constructed with property 19.11. Regard the element $x_{k}=x-y_{1}-\ldots-y_{k}$. Since $x \in \mathbb{C} Q_{\leq\left(0, N_{1}, \ldots\right)}$ and $y_{i} \in$ $\mathfrak{m}^{i}(\mathbb{C} Q P \mathbb{C} Q)_{\leq h\left(N_{k}^{\prime \prime}\right)}$, the total length in $x_{k}$ at level $\mathfrak{m}^{k+1}$ is less or equal to

$$
\max \left(N_{k+1}, M_{k}+h\left(N_{1}^{\prime \prime}\right), M_{k-1}+h\left(N_{2}^{\prime \prime}\right), \ldots, M_{1}+h\left(N_{k}^{\prime \prime}\right)\right) \leq \max \left(N_{k+1}, M_{k}+h\left(N_{k}^{\prime \prime}\right)\right)=N_{k+1}^{\prime \prime}
$$

In the above inequality, the term $M_{k}+h\left(N_{k}^{\prime \prime}\right)$ in the second maximum only appears when $k \geq 1$. In either case, the bound by $N_{k+1}$ is valid. In combination with Proposition 15.32 Lemma 19.44 and quasi-flatness we conclude

$$
\begin{aligned}
x_{k} & \in \mathfrak{m}^{k+1} I(P) \cap \mathfrak{m}^{k+1}\left(\mathbb{C} Q_{\leq\left(N_{k+1}^{\prime \prime}\right)}+\mathfrak{m} \widehat{\mathbb{C} Q}\right) \\
& \subseteq \mathfrak{m}^{k+1}\left(I(P) \cap\left(\mathbb{C} Q_{\leq N_{k+1}^{\prime \prime}}+\mathfrak{m} \widehat{\mathbb{C Q}}\right)\right)+\mathfrak{m}^{k+2} I(P) \\
& \subseteq \mathfrak{m}^{k+1}\left((\mathbb{C} Q P \mathbb{C} Q)_{\leq h\left(N_{k+1}^{\prime \prime}\right)}+\mathfrak{m} I(P)\right)+\mathfrak{m}^{k+2} I(P) \\
& \subseteq \mathfrak{m}^{k+1}(\mathbb{C} Q P \mathbb{C} Q)_{\leq h\left(N_{k+1}^{\prime \prime}\right)}+\mathfrak{m}^{k+2} I(P)
\end{aligned}
$$

According to this sum decomposition, write $x_{k}=y_{k+1}+x_{k+1}$. This finishes the induction step. Ultimately, we have constructed the sequences $\left(x_{k}\right)_{k \geq 0}$ and $\left(y_{k}\right)_{k \geq 1}$ with property 19.11.

For the third part of the proof, we wrap up and prove the desired inclusion. Regard an element $x \in I(P) \cap \mathbb{C} Q_{\leq\left(0, N_{1}, \ldots\right)}$ together with its sequences $\left(x_{k}\right)$ and $\left(y_{k}\right)$. With respect to the $\mathfrak{m}$-adic topology on $B \widehat{\otimes} A$, we have the converging series

$$
x=\sum_{k=1}^{\infty} y_{k} .
$$

Indeed, the summands lie in increasingly high powers of $\mathfrak{m}$. The limit is $x$, since the difference between $y_{1}+\ldots+y_{k}$ and $x$ is $x_{k}$ which lies in increasingly high powers of $\mathfrak{m}$. Even better, we conclude

$$
x \in \sum_{k=1}^{\infty} \mathfrak{m}^{k}(\mathbb{C} Q P \mathbb{C} Q)_{\leq h\left(N_{k}^{\prime \prime}\right)} .
$$

Since $x$ was arbitrary and $N_{k}^{\prime}$ is defined as $h\left(N_{k}^{\prime \prime}\right)$, this proves the claim.
We are ready to prove quasi-flatness of $I(P)_{\mathbb{C} Q}$. The requirement is that $I(P)$ is already quasi-flat. By Proposition 19.43 this happens for example $R$ is of finite type and [BG] and [CP] hold.
Proposition 19.48 (Quasi-flatness II). Assume $R$ is of bounded type and $\psi$ maps to $\mathfrak{m} \mathbb{C} Q$. If $I(P)$ is quasi-flat, then $I(P)_{\mathbb{C} Q}$ is quasi-flat.
Proof. It is our task to show $I(P)_{\mathbb{C} Q} \cap \mathfrak{m} \mathbb{C} Q \subseteq \mathfrak{m} I(P)_{\mathbb{C} Q}$. Pick an element $x \in I(P)_{\mathbb{C} Q} \cap \mathfrak{m} \mathbb{C} Q$. Start with the observation that the relation space $R$ is finite-dimensional, the image of $\psi$ lies in $\mathfrak{m} \mathbb{C} Q$ instead of $\mathfrak{m} \widehat{\mathbb{C} Q}$ and in $I(P)_{\mathbb{C} Q}$ only finitely many paths are multiplied to $P$ at order $\leq k$. We see that $x$ has bounded path length at order $\leq k$ for every $k \in \mathbb{N}$. In other words, there is a sequence $N=\left(N_{0}, N_{1}, \ldots\right)$ such that $x \in \mathbb{C} Q_{\leq\left(0, N_{1}, \ldots\right)}$. In total, we have $x \in I(P) \cap \mathbb{C} Q_{\leq\left(0, N_{1}, \ldots\right)}$. According to Lemma 19.47, we get

$$
x \in \sum_{k=1}^{\infty} \mathfrak{m}^{k}(\mathbb{C} Q P \mathbb{C} Q)_{\leq N_{k}^{\prime}}
$$

This shows $x \in I(P)_{\mathbb{C} Q}$ and finishes the proof.

### 19.9 Closedness results

In this section, we prove additional closedness results. The idea is that a quotient of an algebra by an ideal can always be taken. However, if the algebra enjoys topological properties, they can be lost in the quotient if the ideal is not good enough. In the present section we devote ourselves to the study of this problem in the case of the deformed ideals $I(P) \subseteq \widehat{\mathbb{C} Q}$. Essentially, we show that $I(P)$ is good enough for the "Krull topology" on $\widehat{\mathbb{C Q}}$ if $R$ is of bounded type.

Let us start by recalling the classical topology on $\widehat{\mathbb{C Q}}$.
Definition 19.49. The Krull topology on $\mathbb{C} Q$ (or $\widehat{\mathbb{C} Q}$ ) is the topology generated by the neighborhood basis

$$
x+\mathbb{C} Q_{\geq N} \quad \text { or } \quad x+\widehat{\mathbb{C}}_{\geq N}, \quad \text { for } x \in \mathbb{C} Q, N \in \mathbb{N} .
$$

Remark 19.50. With the Krull topology, the space $\mathbb{C} Q$ is first countable and sequential. A sequence $x_{n} \in \mathbb{C} Q$ converges to some $x \in \mathbb{C} Q$ if $x_{n}-x_{n+1}$ is concentrated in higher and higher path length. The space $\mathbb{C} Q$ is not complete: The space $\widehat{\mathbb{C Q}}$ is in fact its completion.
Remark 19.51. A standard setup in algebra is as follows: Given is a path algebra $\mathbb{C} Q$ of a quiver and one is interested in dividing out an ideal $I \subseteq \mathbb{C} Q$. One accepts the quotient algebra $\mathbb{C} Q / I$ without questioning its topological properties. If $I_{q}$ is a quasi-flat deformation of $I$, then $(B \widehat{\otimes} \mathbb{C} Q) / I_{q}$ is a deformation of $\mathbb{C} Q / I$. For path algebras of quivers, this is all one typically desires.

In contrast, consider an ideal $I \subseteq \widehat{\mathbb{C} Q}$ in the completed path algebra. Then one can still form the quotient $\widehat{\mathbb{C Q}} / I$, but one is interested in the Krull topology on the quotient. Therefore one typically requires that $I \subseteq \widehat{\mathbb{C Q}}$ is closed with respect to the Krull topology.
Remark 19.52. Let us use the standard notation $\widehat{\mathbb{C Q}} R \widehat{\mathbb{C Q}}$ for the ideal generated by $R$ in $\widehat{\mathbb{C Q}}$. We claim that

$$
\widehat{\mathbb{C} Q} R \widehat{\mathbb{C} Q} \subseteq I(R)_{\widehat{\mathbb{C} Q}} \subseteq \overline{I(R)}
$$

Indeed, the left-hand side is the finite span of elements of the form $p r q$ where $r \in R$ and $p, q \in \widehat{\mathbb{C Q}}$. A reordering, or counting argument, for the constituents of $p$ and $q$ shows that $p r q \in I(R)_{\widehat{\mathbb{C Q}}}$. On the other hand, pick an element $x \in I(R)_{\widehat{\mathbb{C}}}$, presented as a series Definition 19.32. Truncating the series at high indices immediately shows that $x$ lies in the closure of $I(R)$.

We arrive at the following inclusion:

$$
R \subseteq I(R) \subseteq \widehat{\mathbb{C Q}} R \widehat{\mathbb{C} Q} \subseteq I(R)_{\widehat{\mathbb{C Q}}} \subseteq \overline{I(R)}
$$

In case $R$ is of bounded type, we can improve on these inclusions: We shall see that $I(R)_{\widehat{\mathbb{C Q}}}$ is the closure of $I(R)$ and thereby also the closure of $\widehat{\mathbb{C Q}} R \widehat{\mathbb{C Q}}$.
Remark 19.53. At this point, we shall comment on the stark difference with the commutative case. In fact, $\widehat{\mathbb{C Q}} R \widehat{\mathbb{C} Q}$ is not necessarily closed with respect to the Krull topology, while this would hold if $A=\widehat{\mathbb{C} Q}$ were commutative. Indeed, let $A$ be a commutative local ring with maximal ideal $\mathfrak{m}_{A}$. Then $\left(A, \mathfrak{m}_{A}\right)$ is a Zariski ring and hence any ideal $I \subseteq A$ is automatically closed with respect to the $\mathfrak{m}_{A}$-adic topology. If $\widehat{\mathbb{C Q}}$ were commutative, this would imply that $\widehat{\mathbb{C Q}} R \widehat{\mathbb{C Q}}$ is closed with respect to the Krull topology.

We shall here give an example of a quiver $Q$ and a space $R$ where $\widehat{\mathbb{C Q}} R \widehat{\mathbb{C Q}}$ is clearly not closed:


We note that the series $\sum_{i=0}^{\infty} A^{i} B C^{i}$ lies in the closure of $\mathbb{C} Q R \mathbb{C} Q$, and claim that it does not lie in $\widehat{\mathbb{C Q}} B \widehat{\mathbb{C} Q}$. The clue is to analyze the structure of infinite sums of paths. For any element $\sum_{i, j=0}^{\infty} \lambda_{i, j} A^{i} B C^{j}$ let us call $\left(\lambda_{i, j}\right)$ its coefficient matrix. An element of the form

$$
\left(\sum_{i=0}^{\infty} \lambda_{i} A^{i}\right) B\left(\sum_{j=0}^{\infty} \eta_{j} C^{j}\right)
$$

has a coefficient matrix of column rank at most one, since all columns are multiples of each other. Any element in the ideal $\widehat{\mathbb{C Q}} B \widehat{\mathbb{C Q}}$ is a finite sum of elements of this form and therefore has coefficient matrix with finite column rank. The coefficient matrix of the element $x=\sum_{i=0}^{\infty} A^{i} B C^{i}$ is however an infinite diagonal matrix which has infinite rank. We conclude $x \notin \widehat{\mathbb{C Q}} B \widehat{\mathbb{C} Q}$. This illustrates a difference with the commutative case and shows how intricate closedness can be.

Lemma 19.54. Let $R$ be of bounded type. Then $I(R)_{\widehat{\mathbb{C Q}}}$ is the closure of $I(R)$ in the Krull topology.
Proof. It is clear that $I(R)_{\widehat{\mathbb{C Q}}}$ is contained in the closure of $I(R)$. Conversely, $I(R)$ is naturally a subset of $I(R)_{\widehat{\mathbb{C Q}}}$. It remains to be shown that $I(R)_{\widehat{\mathbb{C Q}}}$ is closed.

Regard a sequence $\left(x_{n}\right) \subseteq I(R)_{\widehat{\mathbb{C Q}}}$ with $x_{n} \rightarrow x \in \widehat{\mathbb{C Q}}$. It is our task to show that $x$ can be written as a series

$$
x=\sum_{i=0}^{\infty} p_{i} r_{i} q_{i} \quad \text { where } \quad r_{i} \in R,\left|p_{i}\right|+\left|q_{i}\right| \rightarrow \infty
$$

We can assume that $x_{n+1}-x_{n} \in \mathbb{C} Q_{\geq k_{n}}$ for a sequence $\left(k_{n}\right) \subseteq \mathbb{N}$ with $k_{n} \rightarrow \infty$. The standard boundedness argument shows that

$$
x_{n}-x_{n+1} \in I(R) \cap \widehat{\mathbb{C}}_{\geq k_{n}} \subseteq(\mathbb{C} Q R \mathbb{C} Q)_{\geq l\left(k_{n}\right)}
$$

Summing up over $n \geq 0$, we get that $x$ is indeed an element of $I(R)_{\widehat{C Q}}$.
Next, we shall prove that $I(R)_{\widehat{\mathbb{C Q}}}$ has a closed complement in $\widehat{\mathbb{C Q}}$.
Example 19.55. The existence of closed complements is intricate. It is not true that a complement of an ideal $I \subseteq \mathbb{C} Q$ gives rise to a closed complement of its closure $\bar{I} \subseteq \widehat{\mathbb{C} Q}$. More precisely, if $\mathbb{C} Q=I+V$, then it does not necessarily hold that $\widehat{\mathbb{C} Q}=\bar{I}+\bar{V}$. For instance, pick $\mathbb{C} Q=\mathbb{C}\langle A\rangle$ and regard

$$
\begin{aligned}
I & =(1+A)=\operatorname{span}\left(1+A, A+A^{2}, \ldots\right) \\
V & =\operatorname{span}(1)
\end{aligned}
$$

We have $\mathbb{C}\langle A\rangle=I \oplus V$. We have $1-A^{n} \in I$, hence $1 \in \bar{I} \cap \bar{V}$.
Lemma 19.56. Let $R$ be of bounded type. Then $I(R)_{\widehat{\mathbb{C Q}}}$ has a closed complement.
Proof. It is our task to find a closed subspace $V \subseteq \widehat{\mathbb{C Q}}$ such that $\widehat{\mathbb{C Q}}=I(R)_{\widehat{\mathbb{C Q}}} \oplus V$. The naive idea is to fill $V$ by picking paths of increasing length which complement $I(R)$. However, the closure of the space spanned by these paths need not have vanishing intersection with $I(R)_{\widehat{\mathbb{C}}}$ in general. The remedy is provided by the boundedness condition, which allows us to reduce the vanishing intersection property to checking it on finite lengths of paths. We divide the proof into three steps: In the first part of the proof, we construct the space $V$. In the second part, we show $I(R)_{\widehat{\mathbb{C} Q}} \cap V=0$. In the third part, we show $\widehat{\mathbb{C} Q}=I(R)_{\widehat{\mathbb{C}}}+V$.

For the first part of the proof, we construct the space $V$. Pick for every $N \in \mathbb{N}$ a set $V_{N}$ of paths of length $N$ such that all paths in $V_{n}$ for $n \leq N$ are linearly independent of $I(R)$, but span $\mathbb{C} Q_{\leq N}$ when combined with $I(R)$ :

$$
\mathbb{C} Q_{\leq N} \subseteq \operatorname{span}\left(V_{0} \cup \ldots \cup V_{N}\right) \oplus I(R)
$$

Our candidate space for the closed complement is

$$
V=\prod_{N=0}^{\infty} \operatorname{span} V_{N} \subseteq \widehat{\mathbb{C} Q}
$$

This is the subspace consisting of elements of $\widehat{\mathbb{C} Q}$ which are supported on paths lying in the union $\cup_{N \in \mathbb{N}} V_{N}$. The space $V$ is closed with respect to the Krull topology. Indeed, sticking to our sequential viewpoint, whenever $\left(x_{n}\right) \subseteq V$ converges in $\widehat{\mathbb{C} Q}$, then coefficients of $x_{n}$ on paths in low path length stabilize. From a purely topological viewpoint, if $x \in \widehat{\mathbb{C Q}} \backslash V$, then $x$ has support on some path $p \notin$ $\cup_{N \in \mathbb{N}} V_{N}$. Any element $y \in x+\mathbb{C} Q_{>|p|}$ then also has support on this path $p$ and therefore $y \notin V$. This shows $V$ is closed.

For the second part of the proof, we show $I(R)_{\widehat{\mathbb{C Q}}} \cap V=0$. Pick any element $x \in I(R)_{\widehat{\mathbb{C Q}}} \cap V$. We shall prove that $x \in \mathbb{C} Q_{>N}$ for every $N \in \mathbb{N}$. Indeed, let $N \in \mathbb{N}$. Then decompose $x=x_{1}+x_{2}$ into path support related to length $\leq N$ and path support not related to length $\leq N$. Since $x$ lies in $V$ and $V$ is built from pure paths, we have that $x_{1}, x_{2} \in V$. More specifically, since all paths related to length $\leq N$ are of length $\leq h(N)$, we have $x_{1} \in \operatorname{span}\left(V_{0} \cup \ldots \cup V_{h(N)}\right)$.

We claim that $x_{1} \in I(R)$. Indeed, write

$$
x=\sum_{i=0}^{\infty} p_{i} r_{i} q_{i}, \quad r_{i} \in R,\left|p_{i}\right|+\left|q_{i}\right| \rightarrow \infty
$$

Let $S \subseteq \mathbb{N}$ be the set of indices where all paths in $p_{i} r_{i} q_{i}$ are related to paths of length $\leq N$. Then

$$
x=\sum_{i \in S} p_{i} r_{i} q_{i}+\sum_{i \in \mathbb{N} \backslash S} p_{i} r_{i} q_{i}
$$

In the first summand, all paths are related to paths of length $\leq N$. In the second summand, no paths are related to length $\leq N$. In conclusion, we have

$$
x_{1}=\sum_{i \in S} p_{i} r_{i} q_{i}, \quad x_{2}=\sum_{i \in \mathbb{N} \backslash S} p_{i} r_{i} q_{i} .
$$

We obtain $x_{1} \in I(R)$. In conclusion $x_{1} \in I(R) \cap \operatorname{span}\left(V_{0}, \ldots, V_{h(N)}\right)$. However by construction of the sets $V_{n}$, the sum $I(R)+\operatorname{span}\left(V_{0}, \ldots, V_{h(N)}\right)$ is actually direct and hence $x_{1}=0$. We conclude $x=x_{2}$. Since all paths in the support of $x_{2}$ are not related to length $\leq N$, they are of length $>N$. We conclude $x \in \widehat{\mathbb{C} Q}_{>N}$. Since $N$ was arbitrary, we conclude $x=0$. This shows $I(R)_{\widehat{\mathbb{C}}} \cap V=0$.

For the third part of the proof, we show that $\widehat{\mathbb{C Q}}=I(R)_{\widehat{\mathbb{Q}}}+V$. The idea is to break down an element of $\widehat{\mathbb{C Q}}$ into pieces which individually lie in the two spaces and make sure the pieces sum up appropriately. Let now $x \in \widehat{\mathbb{C} Q}$. Split $x$ into pieces according to path length:

$$
x=\sum_{n=0}^{\infty} x_{n}, \quad\left|x_{n}\right|=n .
$$

We claim that for every $n \in \mathbb{N}$ we have $x_{n} \in I(R)_{\geq l(n)}+\operatorname{span}\left(V_{l(n)} \cup \ldots V_{n}\right)$. To prove this, fix $n \in \mathbb{N}$. By construction, we can write $x=y_{n}+z_{n}$ with $y_{n} \in I(R)$ and $z_{n} \in \operatorname{span}\left(V_{0} \cup \ldots \cup V_{n}\right)$. Write

$$
y_{n}=\sum_{i \in I}^{\infty} p_{i} r_{i} q_{i}, \quad r_{i} \in R,|I|<\infty
$$

Let $S \subseteq I$ be the set of indices $i \in I$ where all paths in $p_{i} r_{i} q_{i}$ are related to a path of length $n$. We have $y_{n}=y_{n}^{(1)}+y_{n}^{(2)}$ with

$$
y_{n}^{(1)}=\sum_{i \in S} p_{i} r_{i} q_{i}, \quad y_{n}^{(2)}=\sum_{i \in I \backslash S} p_{i} r_{i} q_{i} .
$$

For $i \in S$, all paths in $p_{i} r_{i} q_{i}$ are of length $\geq l(n)$. This implies $y_{n}^{(1)} \in I(R)_{\geq l(n)}$.
Next, split $z_{n}=z_{n}^{(1)}+z_{n}^{(2)}$ with $z_{n}^{(1)}, z_{n}^{(2)} \in \operatorname{span}\left(V_{0} \cup \ldots \cup V_{n}\right)$ such that $z_{n}^{(1)}$ only contains paths related to paths of length $n$ and $z_{n}^{(2)}$ only contains paths not related to paths of length $n$. Since all paths related to paths of length $n$ have length at least $l(n)$, we have $z_{n}^{(1)} \in \operatorname{span}\left(V_{l(n)} \cup \ldots \cup V_{n}\right)$.

We argue that $y_{n}^{(2)}+z_{n}^{(2)}=0$. Indeed, all paths in $y_{n}^{(1)}+z_{n}^{(1)}$ are related to paths of length $n$, while all paths in $y_{n}^{(2)}+z_{n}^{(2)}$ are not related to length $n$. Since $x_{n}$ itself has length $n$, we conclude $y_{n}^{(2)}+z_{n}^{(2)}=0$.

Finally, we get $x_{n}=y_{n}^{(1)}+z_{n}^{(1)}$. Since $y_{n}^{(1)} \in I(R)_{\geq l(N)}$ and $z_{n}^{(1)} \in \operatorname{span}\left(V_{l(n)} \cup \ldots \cup V_{n}\right)$ and $l(n) \rightarrow \infty$, we get

$$
x=\sum_{n=0}^{\infty} y_{n}^{(1)}+z_{n}^{(1)}=\sum_{n=0}^{\infty} y_{n}^{(1)}+\sum_{n=0}^{\infty} z_{n}^{(1)} \in I(R)_{\widehat{\mathbb{C Q}}}+V
$$

This proves that $\widehat{\mathbb{C Q}}=I(R)_{\widehat{\mathbb{C} q}}+V$. Together with the fact $I(R)_{\widehat{\mathbb{C Q}}} \cap V=0$ proven before, this proves the direct sum decomposition $\widehat{\mathbb{C Q}}=I(R)_{\widehat{\mathbb{C Q}}} \oplus V$ and finishes the proof.

In the remainder of this section, we devote ourselves to the study of $P$ and its associated ideal-like spaces. Let us give an indication of the additional topology we want to regard. As we recalled in Remark 19.51 for a quotient $\widehat{\mathbb{C} Q} / I$ one typically demands that $I \subseteq \widehat{\mathbb{C Q}}$ be closed with respect to the Krull topology. The quotient $\widehat{\mathbb{C} Q} / I$ then inherits a quotient topology. The tensor product $B \widehat{\otimes}(\widehat{\mathbb{C Q}} / I)$ then obtains a combination of the Krull and $\mathfrak{m}$-adic topology. We define this topology as follows:

Definition 19.57. Let $B$ be a deformation base. Then the tensor topology on $B \widehat{\otimes} \widehat{\mathbb{C} Q}$ is the limit topology induced from

$$
B \widehat{\otimes} \widehat{\mathbb{C Q}}=\lim \left(B / \mathfrak{m}^{k} \otimes \widehat{\mathbb{C} Q}\right)
$$

Here the individual spaces $B / \mathfrak{m}^{k} \otimes \widehat{\mathbb{C Q}}$ are equipped with the Krull topology. An explicit neighborhood basis for the topology is given by the subspaces $x+\mathfrak{m}^{k} \widehat{\mathbb{C Q}}+B \widehat{\mathbb{C Q}} \geq N$ for $x \in B \widehat{\otimes} \widehat{\mathbb{C Q}}$.

Remark 19.58. With respect to the tensor topology, a sequence $x_{n} \subseteq \widehat{\mathbb{C} Q}$ converges if for every $k \in \mathbb{N}$, the path lengths of the differences $x_{n}-x_{n+1}$ go to infinity once $\mathfrak{m}^{k}$ is divided out.

We have the following chain of inclusions for $P$ and its associated ideal-like spaces:

$$
P \subseteq \mathbb{C} Q P \mathbb{C} Q \subseteq \widehat{\mathbb{C Q} P \widehat{\mathbb{C} Q}} \stackrel{(P)}{\subseteq} \quad \begin{aligned}
& (B \widehat{\otimes} \widehat{\mathbb{C} Q}) P(B \widehat{\otimes} \widehat{\mathbb{C} Q})
\end{aligned} \subseteq I(P) \subseteq \overline{I(P)} \subseteq \overline{I(P)}^{\otimes}
$$

Here $(B \widehat{\otimes} \widehat{\mathbb{C Q}}) P(B \widehat{\otimes} \widehat{\mathbb{C Q}})$ denotes the ideal generated by $P$ in $B \widehat{\otimes} \widehat{\mathbb{C} Q}$. If $R$ is of bounded type and $I(P)$ is quasi-flat, the inclusions simplify: We shall see that $I(P)$ is the closure of $(B \widehat{\otimes} \widehat{\mathbb{C Q}}) P(B \widehat{\otimes} \widehat{\mathbb{C Q}})$.

Lemma 19.59. Let $R$ be of bounded type and $I(P)$ quasi-flat. Write $\tilde{l}(N)=l(N)-|R|$ for $N \in \mathbb{N}$. Then for $k \geq 1$ and $N \geq 0$ we have

$$
\begin{aligned}
& I(P) \cap\left(B \widehat{\mathbb{C}}_{\geq N}+\mathfrak{m}^{k} \widehat{\mathbb{C} Q}\right) \\
& \subseteq \sum_{i=0}^{k-1} \mathfrak{m}^{i}(\mathbb{C} Q P \mathbb{C} Q)_{\geq l(\bar{i}(N))}+\mathfrak{m}^{k} I(P) \\
& =(\mathbb{C} Q P \mathbb{C} Q)_{\geq l(N)}+\mathfrak{m}(\mathbb{C} Q P \mathbb{C} Q)_{\geq l(\tilde{l}(N))}+\mathfrak{m}^{2}(\mathbb{C} Q P \mathbb{C} Q)_{\geq l(\tilde{l}(\tilde{l}(N)))}+\ldots+\mathfrak{m}^{k} I(P) .
\end{aligned}
$$

Proof. The proof is very similar to the proof of Lemma 19.47. The idea is to iterate Proposition 15.32 in combination with Lemma 19.44 and quasi-flatness of $I(P)$.

Let $x \in I(P) \cap\left(B \widehat{\mathbb{C}}_{\geq N}+\mathfrak{m}^{k} \widehat{\mathbb{C} Q}\right)$. Then in particular

$$
x \in I(P) \cap\left(\widehat{\mathbb{C Q}}_{\geq N}+\mathfrak{m} \widehat{\mathbb{C} Q}\right) \subseteq(\mathbb{C} Q P \mathbb{C} Q)_{\geq l(N)}+\mathfrak{m} I(P)
$$

According to this sum decomposition, write $x=y_{1}+x_{1}$. We clearly have $y_{1} \in B \widehat{\mathbb{C} Q}_{\geq l(N)-|R|}$. Since $x \in B \widehat{\mathbb{C} Q}_{>N}+\mathfrak{m}^{k} \widehat{\mathbb{C} Q}$, we conclude $x_{1} \in \mathfrak{m} I(P) \cap\left(B \widehat{\mathbb{C} Q}_{\geq \tilde{l}(N)}+\mathfrak{m}^{k} \widehat{\mathbb{C Q}}\right)$. We now continue this way, using Proposition 15.32

$$
\begin{aligned}
x_{1} & \in \mathfrak{m} I(P) \cap\left(\mathfrak{m} \widehat{\mathbb{C} Q}_{\geq \tilde{l}(N)}+\mathfrak{m}^{2} \widehat{\mathbb{C} Q}\right) \\
& \subseteq \mathfrak{m}\left(I(P) \cap\left(\widehat{\mathbb{C} Q}_{\geq \tilde{l}(N)}+\mathfrak{m} \widehat{\mathbb{C} Q}\right)\right)+\mathfrak{m}^{2} I(P) \\
& \subseteq \mathfrak{m}\left((\mathbb{C} Q P \mathbb{C} Q)_{\geq l(\tilde{l}(N))}+\mathfrak{m} I(P)\right)+\mathfrak{m} I(P) \\
& \subseteq \mathfrak{m}(\mathbb{C} Q P \mathbb{C} Q)_{\geq l(\tilde{l}(N))}+\mathfrak{m}^{2} I(P) .
\end{aligned}
$$

Split $x_{1}=y_{2}+x_{2}$ according to this sum decomposition and continue. The result is immediate.
Proposition 19.60. Assume $R$ is of bounded type and $I(P)$ is quasi-flat. Then $I(P)$ is the closure of the ideal $(B \widehat{\otimes} \widehat{\mathbb{C} Q}) P(B \widehat{\otimes} \widehat{\mathbb{C} Q})$ with respect to the tensor topology.

Proof. It is clear that $(B \widehat{\otimes} \widehat{\mathbb{C Q}}) P(B \widehat{\otimes} \widehat{\mathbb{C Q}})$ is contained in $I(P)$ and that $I(P)$ is contained in the closure of $(B \widehat{\otimes} \widehat{\mathbb{C} Q}) P(B \widehat{\otimes} \widehat{\mathbb{C} Q})$. It remains to show that $I(P)$ is closed.

To prove $I(P)$ closed, regard a series of $\left(x_{n}\right) \subseteq I(P)$ converging in $B \widehat{\otimes} \widehat{\mathbb{C Q}}$ :

$$
x=\sum_{n=0}^{\infty} x_{n} .
$$

We now explain how to prove $x \in I(P)$. It entails inspecting every $x_{n}$ and dividing it into chunks in such a way it becomes evident that a reordering of the chunks sums up to an element of $I(P)$. The reordering is unproblematic, keeping $x$ intact as element of $B \widehat{\otimes} \widehat{\mathbb{C} Q}$.

In order to define the chunks, we shall build a sequence $\left(n_{k}\right)_{k \geq 1} \subseteq \mathbb{N}$. Let $k \geq 1$. Since we assume the series $\sum x_{n}$ converges in the tensor topology, there exists for every $n \in \mathbb{N}$ a (maximally chosen) $N_{n}^{(k)} \in \mathbb{N}$ such that $x_{n} \in B \mathbb{C} Q_{\geq N_{n}^{(k)}}+\mathfrak{m}^{k} \widehat{\mathbb{C} Q}$. We have that $N_{n}^{(k)} \rightarrow \infty$ as $n \rightarrow \infty$. In particular, we have $l\left(\tilde{l}^{k-1}\left(N_{n}^{(k)}\right)\right) \rightarrow \infty$ as $n \rightarrow \infty$. Choose $n_{k} \in \mathbb{N}$ so high that $l\left(\tilde{l}^{k-1}\left(N_{n}^{(k)}\right)\right) \geq k$ for all $n \geq n_{k}$. Of course, we can enforce that $n_{k}$ is an increasing sequence: $n_{k} \geq n_{k-1}$. We have now built the sequence $\left(n_{k}\right)_{k \geq 1}$ which is essential in the construction of the chunks.

Let now $n \in \mathbb{N}$ and assume $n \geq n_{1}$. Then there is a unique $k \in \mathbb{N}$ such that $n_{k} \leq n<n_{k+1}$. We get

$$
\begin{aligned}
x_{n} & \in I(P) \cap\left[B \mathbb{C} Q_{\geq N_{n}^{(k)}}+\mathfrak{m}^{k} \widehat{\mathbb{C} Q}\right] \\
& \subseteq \sum_{i=0}^{k-1} \mathfrak{m}^{i}(\mathbb{C} Q P \mathbb{C} Q)_{\geq l\left(\tilde{l}^{i}\left(N_{n}^{(k)}\right)\right)}+\mathfrak{m}^{k} I(P) \\
& \subseteq B(\mathbb{C} Q P \mathbb{C} Q)_{\geq k}+\mathfrak{m}^{k} I(P) .
\end{aligned}
$$

In the second row, we have applied Lemma 19.59. In the third row, we have used that $l\left(\tilde{l}^{i}\left(N_{n}^{(k)}\right)\right) \geq$ $l\left(\tilde{l}^{k-1}\left(N_{n}^{(k)}\right)\right) \geq k$ minding $n \geq n_{k}$. With respect to the two summands, write now

$$
x_{n}=x_{n}^{(1)}+x_{n}^{(2)}, \quad x_{n}^{(1)} \in B(\mathbb{C} Q P \mathbb{C} Q)_{\geq k}, \quad x_{n}^{(2)} \in \mathfrak{m}^{k} I(P)
$$

We can now write

$$
x=\sum_{n=0}^{n_{1}-1} x_{n}+\underbrace{\sum_{n=n_{1}}^{n_{2}-1} x_{n}^{(1)}}_{\in B(\mathbb{C} Q P \mathbb{C} Q) \geq 1}+\underbrace{\sum_{n=n_{2}}^{n_{3}-1} x_{n}^{(1)}}_{\in B(\mathbb{C} Q P \mathbb{C} Q)_{\geq 2}}+\ldots+\underbrace{\sum_{n=n_{1}}^{n_{2}-1} x_{n}^{(1)}}_{\in \mathfrak{m}^{1} I(P)}+\underbrace{\sum_{n=n_{2}}^{n_{3}-1} x_{n}^{(1)}}_{\in \mathfrak{m}^{2} I(P)}+\ldots \in I(P) .
$$

The top row is a finite sum. The second row adds up to an element of $I(P)$ since the path lengths increase. The third summand adds up to an element of $I(P)$ since the powers of $\mathfrak{m}$ increase. This finally shows $x \in I(P)$ and we conclude $I(P)$ is closed with respect to the tensor topology.

We finish this section by commenting on deformations of algebras where the Krull topology is taken into account. Recall from Proposition 19.4 that $B \widehat{\otimes} A / I_{q}$ is a deformation of $A / I$ if $I_{q}$ is a quasi-flat deformation of $I$. It would be delightful to have a version of this statement which takes into account the Krull topology in case $A=\widehat{\mathbb{C} Q}$. This requires two steps: First we shall define what it means for a deformation to be tensor continuous. Second we shall prove that $B \widehat{\otimes} \widehat{\mathbb{C Q}} / I_{q}$ is a tensor continuous deformation in case $I_{q}$ is quasi-flat and closed with respect to the tensor topology. Let us start as follows:

Definition 19.61. Let $I \subseteq \widehat{\mathbb{C} Q}$ be a closed ideal and $I_{q}$ a deformation closed with respect to the tensor topology. Then $(B \widehat{\otimes} \widehat{\mathbb{C} Q}) / I_{q}$ is a (tensor continuous) deformation of $\widehat{\mathbb{C Q}} / I$ if there exists a deformation $\mu_{q}$ of the product $\mu:(\widehat{\mathbb{C Q}} / I) \otimes(\widehat{\mathbb{C Q}} / I) \rightarrow(\widehat{\mathbb{C Q}} / I)$ together with a $B$-linear algebra isomorphism

$$
\varphi: \frac{B \widehat{\otimes} \widehat{\mathbb{C} Q}}{I_{q}} \xrightarrow{\sim}\left(B \widehat{\otimes}(\widehat{\mathbb{C Q}} / I), \mu_{q}\right)
$$

which is a homeomorphism with respect to the tensor topology.
Lemma 19.62. Let $Q$ be a quiver, $I \subseteq \widehat{\mathbb{C} Q}$ a closed ideal which has a closed complement in $\widehat{\mathbb{C} Q}$. Let $I_{q}$ be a deformation closed with respect to the tensor topology. If $I_{q}$ is quasi-flat, then $(B \widehat{\otimes} \widehat{\mathbb{C} Q}) / I_{q}$ is a (tensor continuous) deformation of $\widehat{\mathbb{C} Q} / I$.
Proof. The proof proceeds as in Proposition 19.4 with minor adaptions. The first step is to choose a complement $V \subseteq \widehat{\mathbb{C Q}}$ such that $\widehat{\mathbb{C} Q}=I \oplus V$. In comparison with Proposition 19.4 we can choose $V$ to be closed. As in Proposition 19.4 we obtain the direct sum decompositions

$$
\widehat{\mathbb{C Q}}=I \oplus V, \quad B \widehat{\otimes} \widehat{\mathbb{C} Q}=I_{q} \oplus B V
$$

The difference in our case is that the first is not only a direct sum of vector spaces and the second not only a direct sum of $\mathfrak{m}$-adically closed subspaces. Instead, the direct summands of the first sum are closed with respect to the Krull topology and the summands of the second sum are closed with respect to the tensor topology. As in Proposition 19.4, we define the map

$$
\varphi: B \widehat{\otimes} \frac{\widehat{\mathbb{C Q}}}{I}=B \widehat{\otimes} \frac{V \oplus I}{I} \xrightarrow{\sim} B \widehat{\otimes} V \xrightarrow{\sim} \frac{I_{q} \oplus B V}{I_{q}}=\frac{B \widehat{\otimes} \widehat{\mathbb{C Q}}}{I_{q}} .
$$

This map is immediately a homeomorphism with respect to the tensor topology. One then transfers the product of $(B \widehat{\otimes} \widehat{\mathbb{C} Q}) / I_{q}$ onto a deformed product $\mu_{q}$ on $B \widehat{\otimes}(\widehat{\mathbb{C Q}} / I)$. This finishes the proof.

### 19.10 Dimers of bounded type

In this section, we introduce a boundedness condition of dimers. Our motivation comes from Jacobi algebras of dimers, which are a special case of Jacobi algebras of quivers with superpotential. It is our interest to show that a deformation of the superpotential leads to a flat deformation of the Jacobi algebra. Fortunately, Jacobi algebras of most dimers are CY3. In other words, a Jacobi algebra of a dimer falls under the framework developed in the past sections. The only question remaining is whether the superpotential is of bounded type.

In the present section, we show that the superpotential of a Jacobi algebra of a dimer is of bounded type in the following cases:


Figure 19.3: Intuition on F-term equivalence

- All polygons in $Q$ have equal number of edges (obvious),
- $Q$ is cancellation consistent and sits in a torus Lemma 19.66,
- $Q$ is cancellation consistent and has no triangles Theorem 19.72.

To start our investigation, let us recall the construction of this Jacobi algebra and the meaning of consistency. Let $Q$ be a dimer. Then its superpotential $W \in \mathbb{C} Q$ is given by the difference of the clockwise polygons of $Q$ and the counterclockwise polygons, cyclically permuted:

$$
W=\sum_{\substack{a_{1}, \ldots, a_{k} \\ \text { clockwise }}}\left(a_{1} \ldots a_{k}\right)_{\mathrm{cyc}}-\sum_{\substack{a_{1}, \ldots, a_{k} \\ \text { counterclockwise }}}\left(a_{1} \ldots a_{k}\right)_{\mathrm{cyc}}
$$

The relations $\partial_{a} W$ equate two neighboring polygons: Flipping a path over an arc $a$ is possible if the path follows all arcs of a neighboring polygon apart from $a$. These flip moves are known as $\mathbf{F}$-term moves and the equivalence relation on the set of paths in $Q$ is known as F-term equivalence. The terminology is depicted in Figure 19.3. A good reference is 28.

Regard the set of paths in $Q$ modulo F-term equivalence. The set contains a special element $\ell_{v}$ for each vertex $v \in Q_{0}$, given by the boundary of a chosen polygon at $v$. All boundaries of polygons incident at $v$ are F-term equivalent, hence $\ell_{v}$ does not depend on the choice. In other words, it can be rotated around $v$. We may drop the subscript from $\ell_{v}$ if it is clear from the context. The element $\ell$ commutes with all paths, that is, $u \ell \sim \ell u$. Davison 28 introduced the following consistency condition for dimers:

Definition 19.63 ( 28$]$ ). A dimer $Q$ is cancellation consistent if it has the following cancellation property:

$$
p \ell \sim q \ell \Longrightarrow p \sim q
$$

Remark 19.64. An equivalent requirement is $p r \sim q r \Rightarrow p \sim q$ for all compatible paths $p, q, r$. Indeed assume $p r \sim q r$. Then pick an "inverse" path $r^{\prime}$ in $Q$ such that $r r^{\prime} \sim \ell^{k}$ for some $k \in \mathbb{N}$, and observe $p \ell^{k} \sim p r r^{\prime} \sim q r r^{\prime} \sim q \ell^{k}$, hence $p \sim q$.

The Jacobi algebra of $Q$ is given by $\operatorname{Jac}(Q)=\mathbb{C} Q /\left(\partial_{a} W\right)$. It has a special central element $\ell \in \operatorname{Jac}(Q)$, given by the sum of the $\ell_{v}$ at all vertices $v \in Q_{0}$ :

$$
\ell=\sum_{v \in Q_{0}} \ell_{v} \in \operatorname{Jac}(Q)
$$

The set of paths in $Q$ modulo F-term equivalence is a basis for $\operatorname{Jac}(Q)$. Davison shows in 28, Theorem 8.1] that $\operatorname{Jac}(Q)$ is CY3 if $Q$ is cancellation consistent. A complete survey of consistency conditions can be found in 17. For example, zigzag consistency implies cancellation consistency.

The previous sections of this paper establish that CY3 algebras with relations of bounded type have favorable deformation theory: Whenever we deform their superpotential in a cyclic way, the deformation is flat. We want to apply this to Jacobi algebras of dimers. Since $\operatorname{Jac}(Q)$ is already known to be CY3 if $Q$ is cancellation consistent, it remains to ask if the relations $\partial_{a} W$ are of bounded type. This is the case for some dimers, but not for all. In the present section, we provide criteria for this to happen. Let us give this notion a name.

Definition 19.65. A dimer $Q$ is of bounded type if all F-term equivalence classes are bounded in path length. Equivalently, all F-term equivalence classes are finite sets.

(a) Initial flip always has an $\ell$

(b) Zigzag paths act as cage

(c) Crossing a zigzag path

Figure 19.4: Zigzag intuition

In this new terminology, the Jacobi algebra of a dimer $Q$ has favorable deformation theory if $Q$ if cancellation consistent and of bounded type. Which dimers are of bounded type? An easy example are the dimers whose polygons are all of equal length. Indeed, in such case an F-term move preserves length. A more intricate case is the following:
Lemma 19.66. Let $Q$ be a cancellation consistent torus dimer. Then $Q$ is of bounded type.
Proof. In 17. Theorem 7.6] it is shown that a cancellation consistent torus dimer has a so-called "consistent R-grading", or "anomaly-free R-symmetry". Broomhead shows in [23, Section 2.3] that the existence of an anomaly-free R-symmetry implies that for every arc there exists a perfect matching $P: Q_{1} \rightarrow\{0,1\}$ that is positive on that arc. Summing up all the perfect matching gradings, we obtain a positive integer grading $d$ on $\mathbb{C} Q$ in which $W$ is homogeneous. The degree $d$ of a path is then preserved under F-term equivalence. Since length of a path $p$ is bounded by its total degree $d(p)$, we conclude that any cancellation consistent dimer on a torus is of bounded type.

In this section, we give another wide class of cancellation consistent dimers of bounded type: those where all polygons are of length at least 4 . The core observation is that whenever a path flips over a zigzag path for the first time, it includes a cycle $\ell$ right there, see Figure 19.4a. In short, this gives rise to the following line of proof: It suffices to regard paths equivalent to $\ell^{k}$ and proceed by induction over $k \in \mathbb{N}$. Once F-term moves bring a path $p \sim \ell^{k}$ to cross a zigzag path for the first time, it includes a cycle $\ell$ right there. Stripping away $\ell$ from the path makes it equivalent to $\ell^{k-1}$ and by induction hypothesis such a path is bounded in length. In other words, zigzag paths at sufficient distance provide a "cage" for F-term equivalence classes, see Figure 19.4b,

The problem with this "cage proof" is that construction of an effective cage requires topological arguments and geometric consistency. We derive a more refined proof, focusing on the crossings between $p$ and individual zigzag paths.
Definition 19.67. A crossing of $p$ over a zigzag path $Z$ is a sequence of $k \geq 1$ consecutive arcs in $p$ that follow $Z$, such that $p$ leaves $Z$ to the left or right before the sequence, and the right resp. left after the sequence Figure 19.4c.

Paths containing a full cycle $\ell$ around a polygon are easy to bound in length by induction. We therefore regard mainly paths that are $\ell$-free, that is, do not contain a full cycle $\ell$ around some polygon. In other words, a path is $\ell$-free if it is not of the form $p \ell q$. Note this is not the same as being a minimal path, since minimality refers to F-term equivalence: A path is minimal if it is not F-term equivalent to a path of the form $\ell q$. We are now ready to prove that crossings with zigzag paths are a partial invariant.
Lemma 19.68. Let $Q$ be a dimer without triangles and let $Z$ be a zigzag path. Let $p$ and $q$ be two closed $\ell$-free paths differing only by an F-term move. Then $p$ and $q$ have the same number of crossings with $Z$.

Proof. The strategy is to inspect the crossings of $p$ and $q$ over $Z$, and match them up. It is essential that $Q$ has no triangles, because triangles bordering $Z$ make it possible to create new crossings, see Figure 19.5

Let us inspect a crossing of $p$ over $Z$. Without loss of generality we can assume that $p$ turns right at the end of the sequence and left at the beginning. We show that the crossing is preserved when $p$ flips to $q$. We scrutinize this by a case distinction on where the flip happens. Recall that a flip always involves precisely one polygon minus an arc.

Regard the sequence $a, b_{1}, \ldots, b_{k}, c$ of arcs on $p$ crossing over $Z$. Regard the case that the arcs involved in the flip are all before $a$. Then $a, b_{1}, \ldots, b_{k}, c$ stay entirely part of the path and the crossing is preserved.

Regard the case that $k \geq 2$. Then no three consecutive $b_{i}$ arcs can be involved in the flip, because they follow a zigzag and do not circle around a polygon. Two consecutive $b_{i}$ arcs are not enough for a


Figure 19.5: Triangles ruin our invariant.

(a) Case 1

(b) Case 2

(c) Case 3

(d) Case 4

Figure 19.6: F-term moves of $\ell$-free paths preserve zigzag crossings.
flip, since all polygons are assumed to consist of at least 4 arcs. Hence arcs before $a, a$ itself, $b_{1}$ and $b_{2}$ remain as possible arcs involved in the flip. In all cases we check that a crossing at the same point still exists in $q$. The case where $\operatorname{arcs} b_{k-1}, b_{k}, c, \ldots$ on the other side of the crossing are involved follows similarly.

We distinguish 4 cases, depicted in Figure 19.6 In cases 1-3, only arcs before $a$ and $a$ itself are involved in the flipping. Case 1 depicts the situation where $a$ lies maximally left, case 2 depicts an average situation, case 3 depicts the situation where $a$ lies maximally right. Due to arrow directions, case 1 and 3 differ in appearance.

It turns out in case 1 that there is a new arc, indicated by a checkmark in the figure, that leaves $Z$ and everything between that arc and $b_{1}$ follows $Z$. In other words $q$ still has a crossing at the same location. In case 2, the arc leaving $Z$ also changes through the flip, but the crossing as a whole remains. Case 3 is actually impossible: By assumption $a$ leaves $Z$, and in order to conclude a polygon $P$ minus an arc, $p$ needs to continue turning around $P$. It ends precisely at the head of $b_{1}$, concluding an $\ell$-cycle $\ldots, a, b_{1}$.

Case 4 is the situation where arcs before $a$, the arc $a$ itself and $b_{1}$ are involved in the flipping. Since $a$ and $b_{1}$ are supposed to be part of the flipping, the polygon to be flipped is necessarily the one lying in the corner of the zigzag path at $b_{1}, b_{2}$. The first arc of $p$ involved in the flip starts at the head of $b_{2}$ (or the corresponding vertex of $Z$ in case $k=1$ ). What comes before that arc in $p$ ? The arc $b_{2}$ following $b_{1}$ on $Z$, which we also label this way by abuse if $k=1$, cannot come before it, because $p$ is $\ell$-free. By arrow directions, it can also not concern the arc $b_{3}$, similarly labeled by abuse if $k \leq 2$. Hence it must concern an arc that turns left of the zigzag path. In the figure this arc is depicted again by a checkmark. This demonstrates that also in case 4 the crossing is preserved.

Finally it is also easy to see that the crossing is preserved in case where paths before $a, a$ itself and $b_{1}$ and $b_{2}$ are involved in the flip. Moreover, no flip is possible that includes $a, b_{1}$ and $c$ if $k=1$.

Let us scrutinize the conclusion. We have associated to each crossing of $p$ over $Z$, let us call it $\chi$, a crossing $\varphi(\chi)$ of $q$ over $Z$. Is this map $\chi \mapsto \varphi(\chi)$ a one-to-one correspondence? Swapping the roles of $p$ and $q$, we also have a map $\psi$ from crossings of $q$ over $Z$ to crossings of $p$ over $Z$. Inspecting the case distinction Figure 19.6 again, it becomes apparent that $\psi$ has no other choice than associating to $\varphi(\chi)$ back $\chi$ again. For example, a crossing $\chi$ and its image $\varphi(\chi)$ always have an arc on $Z$ in common, and similarly $\chi$ and $\psi(\chi)$ do. In other words $\varphi \circ \psi=\mathrm{Id}$ and similarly $\psi \circ \varphi=\mathrm{Id}$. We conclude that $p$ and $q$ have the same number of crossings over $Z$.

Given a path $p$, recall our plan is to utilize the crossings of $p$ over arbitrary zigzag paths as a partial invariant to bound the length of $p$. As announced, we do not construct an explicit cage, but rather argue as follows: The only way to avoid crossing zigzag paths is to follow the boundary of a polygon. If we assume $p$ is $\ell$-free, then following the boundary of a polygon is possible for at most $K$ consecutive arcs, where $K$ is the maximum length of polygons in $Q$. We conclude that $p$ necessarily crosses a zigzag path at least once every $K$ arcs. Let us make this precise.

Lemma 19.69. Let $Q$ be a cancellation consistent dimer. If $p$ is an $\ell$-free path having $C$ crossings with zigzag paths, then its length is bounded by $K(C+1)$.

Proof. The strategy is to show that $p$ necessarily crosses a zigzag path at least once every $K$ arcs, and then apply Lemma 19.68

We claim that a path $c b a$ of length 3 either crosses a zigzag path at $b$ or is part of a polygon. To check this, we take on the perspective of $b$, allowing us to find words for where $a$ and $c$ turn at head and tail of $b$. The generic case is when $a$ and $c$ are neither left-most nor right-most. Then, the path $c b a$ crosses both zigzag paths starting at $b$.

Let us treat the special cases. If $c$ is the left-most (resp. right-most) at the head of $b$ and $a$ is the right-most (resp. left-most) at the tail of $a$, then $c b a$ is part of a counterclockwise (resp. clockwise) polygon. If $c$ is the left-most (resp. right-most), but $a$ is not the right-most (resp. left-most), then the zigzag path starting at $b$ and turning right (resp. left) crosses $p$ at $b$. Similarly if $a$ is the right-most (resp. left-most), but $c$ is not the left-most (resp. right-most), then the zigzag path starting at $b$ and turning left (resp. right) crosses $p$ at $b$.

Either way, we conclude a path $c b a$ of length 3 either crosses a zigzag path at $b$ or is part of a polygon. Now regard a path longer than 3 arrows. How many consecutive arcs are possible without crossing a zigzag path? By consistency, there are at least four polygons incident at every vertex. Hence if $d c b$ lies in a polygon and $c b a$ lies in a polygon, then both lie in the same polygon. We conclude that after $K$ arcs, a path has either completed a cycle around a polygon or crossed a zigzag path. If a path $p$ contains no cycle at all, then it has crossed at least $\lfloor|p| / K\rfloor$ many zigzag paths. Reading this inequality the other way around gives the desired bound.

We recall some notions, before diving into the proof. Let $p$ be a path in $Q$. By consistency, $p$ is equivalent to a composition $\ell^{k} q$ of a cycle power $\ell^{k}$ and a minimal path $q$. That is, $q$ cannot be written as a multiple of $\ell$. This decomposition $p=\ell^{k} q$ is unique up to equivalence of $q$. Let us call $k$ the "looseness" of $p$.
Remark 19.70. If $p=p_{2} \ell p_{1}$ is a path containing a cycle, then $p_{2} \ell p_{1} \sim p_{2} p_{1} \ell$. If $Q$ is cancellation consistent, it satisfies the cancellation condition: If $p$ is equivalent to $\ell^{k}$, then $p_{2} p_{1}$ is equivalent to $\ell^{k-1}$. Once established that paths equivalent to $\ell^{k-1}$ have bounded length, then $p_{2} p_{1}$ and hence $p_{2} \ell p_{1}$ are also bounded.

Lemma 19.71. Let $Q$ be a cancellation consistent dimer without triangles. For any vertex $v$ and integer $k$, the paths equivalent to $\ell^{k}$ at $v$ are of bounded length.

Proof. We have a partial invariant at hand: the number of times a given pass crosses a zigzag path. This number is not preserved under F-term equivalence in general, but becomes an invariant once we restrict to closed $\ell$-free paths.

We proceed by induction. Assume all closed paths equivalent to $\ell^{k-1}$ starting at vertex $v$ have length $\leq N$. Let $p$ be a path equivalent to $\ell^{k}$. If $p$ contains a cycle $\ell$, then we can bound $|p| \leq N+K$ by Remark 19.70 and we are done. Therefore we can assume $p$ is $\ell$-free.

Pick a sequence $\ell^{k}=p_{1}, \ldots, p_{n}=p$ of paths, each related to its successor by an F-term move. Let $m<n$ be the maximal number where $p_{m}$ still contains a cycle $\ell$. Then $\left|p_{m}\right| \leq N+K$ by the induction hypothesis. An F-move changes length by at most $K$, hence $\left|p_{m+1}\right| \leq N+2 K$. By Lemma 19.68, the path $p_{m+1}$ has the same total number of crossings with zigzag paths as $p_{m+2}, \ldots, p_{n}$ do.

Since $\left|p_{m+1}\right| \leq N+2 K$, the path $p_{m+1}$ has at most $2 N+4 K$ crossings with zigzag paths. We have seen this number stays constant and hence also $p_{n}$ has at most $2 N+4 K$ crossings with zigzag paths. By Lemma 19.69 its length is bounded by $K(2 N+4 K+1)$. This finishes the induction.

Theorem 19.72. Any cancellation consistent dimer without triangles is of bounded type.
Proof. Lemma 19.71 already establishes the claim for the paths $p=\ell^{k}$. We deduce from this the general case where $p$ is an arbitrary path. Fix some path $p^{\prime}$ from the end of $p$ to the start of $p$, such that $p^{\prime} p$ is contractible. Then $p^{\prime} p \sim \ell^{k_{0}}$ for some $k_{0}$. Now let $q$ be an arbitrary path equivalent to $p$. We get

$$
q \sim p \quad \Longrightarrow \quad p^{\prime} q \sim p^{\prime} p \sim \ell^{k_{0}}
$$

By Lemma 19.71. the length of $p^{\prime} q$ is bounded. In particular, the length of $q$ is bounded.
Remark 19.73. The bound of Lemma 19.71 is exponential in $k$ :

$$
q \sim \ell^{k} \Longrightarrow|q| \leq \mathcal{O}\left((2 K)^{k}\right)
$$

The bound is also exponential if we fix $p$ and regard paths $q \sim p \ell^{k}$. Indeed, let $p$ and $p^{\prime}$ be fixed paths with $p^{\prime} p \sim \ell^{k_{0}}$, then

$$
q \sim \ell^{k} p \Longrightarrow|q| \leq\left|p^{\prime} q\right| \leq \mathcal{O}\left((2 K)^{k}\right)
$$

since $p^{\prime} q \sim \ell^{k+k_{0}}$. These bounds are far from sharp. Regard for example a relatively straight dimer like the one in Figure 19.3b This figure convinces us that the expected bound is actually linear in $k$.

### 19.11 Main theorems on flatness

In this section, we collect our main theorems on flatness. In particular, we return to the case where the relations come from a superpotential. For the statement of our theorem, the deformed relations are supposed to come from a deformation of the superpotential and the algebra is supposed to be CY3.

Let us state our flatness result first in the most general way, taking the setup from Convention 19.14.
Remark 19.74. Recall that $I(R)$ is the ideal generated by $R$ in $\mathbb{C} Q$. The spaces $I(R)_{\widehat{\mathbb{C}}}, I(P)_{\mathbb{C} Q}$ and $I(P)$ are a bit more complicated. We defined them in an intricate way in section 19.5 Under the assumptions of Proposition 19.75, the definitions however simplify: The space $I(P) \subseteq B \widehat{\otimes} \mathbb{C} Q$ is quasi-flat by Proposition 19.43 and $I(P)_{\mathbb{C} Q} \subseteq B \widehat{\otimes} \mathbb{C} Q$ is quasi-flat if $\psi$ only maps to $\mathfrak{m} \mathbb{C} Q$ by Proposition 19.48

In simplified terms, the space $I(R)_{\widehat{\mathbb{C}}}$ is the closure of the ideal generated by $R$ in $\widehat{\mathbb{C} Q}$ by Lemma 19.54 The space $I(P)_{\mathbb{C} Q}$ is the closure of the ideal generated by $P$ in $B \widehat{\otimes} \mathbb{C} Q$ if $\psi$ only maps to $\mathfrak{m} \mathbb{C} Q$, since $I(P)_{\mathbb{C} Q}$ is quasi-flat and hence closed. The space $I(P)$ is the closure of the ideal generated by $P$ in $B \widehat{\otimes} \widehat{\mathbb{C} Q}$ with respect to the tensor topology by Proposition 19.60. Written out, we have

$$
\begin{aligned}
I(R) & =\mathbb{C} Q R \mathbb{C} Q \subseteq \mathbb{C} Q \\
I(R)_{\widehat{\mathbb{C} Q}} & =\overline{\mathbb{C} Q R \mathbb{C Q} \subseteq \widehat{\mathbb{C} Q}}, \\
I(P)_{\mathbb{C} Q} & =\overline{(B \widehat{\otimes} \mathbb{C} Q) P(B \widehat{\otimes} \mathbb{C} Q)} \subseteq B \widehat{\otimes} \mathbb{C} Q, \quad \text { if } \psi(R) \subseteq \mathfrak{m} \mathbb{C} Q, \\
I(P) & =\overline{(B \widehat{\otimes} \widehat{\mathbb{C} Q}) P(B \widehat{\otimes} \widehat{\mathbb{C} Q})} \subseteq B \widehat{\otimes} \widehat{\mathbb{C} Q} .
\end{aligned}
$$

With these preparations, we are ready to state our flatness result in the most general way.
Proposition 19.75. Under Convention 19.14 assume $R$ is of bounded type and [BG] and [CP] hold. Then we have:

- $\frac{B \widehat{\mathbb{Q}} \widehat{\mathrm{CQ}}}{I(P)}$ is a (tensor continuous) deformation of $\frac{\widehat{\mathrm{CQ}}}{I(R) \widehat{\mathrm{CQ}}}$.
- $\frac{B \widehat{\mathbb{Q}} \mathbb{C} Q}{I(P)}$ is a deformation of $\frac{\mathbb{C} Q}{I(R)}$ if $\psi$ only maps to $\mathfrak{m} \mathbb{C} Q$.

Proof. This is a culmination of what we have proved in the preceding sections. Regard the first statement. Since $R$ is of bounded type and $[\mathrm{BG}]$ and [CP] hold, Proposition 19.43 implies that $I(P)$ is quasi-flat. By Lemma 19.54 we have that $I(R)_{\widehat{\mathbb{C Q}}}$ is closed and by Lemma 19.56 it has a closed complement. Invoking Lemma 19.62 gives that $(B \widehat{\otimes} \widehat{\mathbb{C} Q}) / I(P)$ is a deformation of $\widehat{\mathbb{C} Q} / I(R)_{\widehat{\mathbb{C}}}$.

Regard the second statement. By Proposition 19.48 also $I(P)_{\mathbb{C} Q}$ is quasi-flat. Invoking Proposition 19.4 proves the second statement. This finishes the proof.

Let us restate this proposition in case the relations come from a superpotential. Recall that a superpotential $W$ gives a relation space $R=\operatorname{span}\left\{\partial_{a} W\right\}$ and a deformation $W^{\prime}$ of the superpotential gives a deformed relation space $P=\operatorname{span}\left\{\partial_{a}\left(W+W^{\prime}\right)\right\}$. All details are taken care of by Lemma 19.18 Let us use the following notation:

$$
\begin{aligned}
\operatorname{Jac}(Q, W) & =\frac{\mathbb{C} Q}{\left(\partial_{a} W\right)}, \\
\operatorname{Jac}(\widehat{Q}, W) & =\frac{\widehat{\mathbb{C} Q}}{\left(\partial_{a} W\right)}, \\
\operatorname{Jac}\left(Q, W+W^{\prime}\right) & =\frac{B \widehat{\otimes} \mathbb{C} Q}{(B \widehat{\otimes} \mathbb{C} Q)\left(\partial_{a}\left(W+W^{\prime}\right)\right)(B \widehat{\otimes} \mathbb{C} Q)}, \quad \text { if } W^{\prime} \in \mathfrak{m} \mathbb{C} Q, \\
\operatorname{Jac}\left(\widehat{Q}, W+W^{\prime}\right) & =\frac{B \widehat{\otimes} \widehat{\mathbb{C} Q}}{(\widehat{(B Q})\left(\partial_{a}\left(W+W^{\prime}\right)\right)(B \widehat{\mathbb{C} Q})}{ }^{\otimes}
\end{aligned}
$$

With these considerations, the following theorem is an immediate consequence of Proposition 19.75
Theorem 19.76. Let $Q$ be a quiver, $W \in \mathbb{C} Q_{\geq 3}$ a superpotential and $W^{\prime} \in \widehat{\mathfrak{m} \mathbb{C} Q}$ be cyclic. If $\operatorname{Jac}(Q, W)$ is CY3 and $W$ is of bounded type, then

- $\operatorname{Jac}\left(\widehat{Q}, W+W^{\prime}\right)$ is a deformation of $\operatorname{Jac}(\widehat{Q}, W)$.
- $\operatorname{Jac}\left(Q, W+W^{\prime}\right)$ is a deformation of $\operatorname{Jac}(Q, W)$ if $W^{\prime} \in \mathfrak{m} \mathbb{C} Q$.

The theorem also applies to the standard Jacobi algebra $\operatorname{Jac} Q$ of a dimer. Here cancellation consistency of $Q$ already implies that $\operatorname{Jac} Q$ is CY3 [28, Theorem 8.1]. We have investigated the specific superpotential $W=W_{\text {cyc }}^{+}-W_{\text {cyc }}^{-}$in section 19.10 It is the difference of the clockwise and the counterclockwise polygons in $Q$. If all polygons in $Q$ have equal length or $Q$ sits in a torus or has no triangles, then the superpotential $W$ is of bounded type.

Theorem 19.77. Let $Q$ be a cancellation consistent dimer of bounded type. Denote by $W$ the superpotential of $Q$ and let $W^{\prime} \in \mathfrak{m} \widehat{\mathbb{C} Q}$ be cyclic. Then

- $\operatorname{Jac}\left(\widehat{Q}, W+W^{\prime}\right)$ is a deformation of $\operatorname{Jac}(\widehat{Q}, W)$.
- $\operatorname{Jac}\left(Q, W+W^{\prime}\right)$ is a deformation of $\operatorname{Jac}(Q, W)$ if $W^{\prime} \in \mathfrak{m} \mathbb{C} Q$.


## 20 A deformed Cho-Hong-Lau construction

In this section, we recapitulate the mirror construction of Cho, Hong and Lau 26 and formulate a deformed version. Let us sketch this procedure: The construction of Cho, Hong and Lau starts from an $A_{\infty}$-category $\mathcal{C}$ together with a designated subcategory $\mathbb{L} \subseteq \mathcal{C}$ whose $A_{\infty}$-products are cyclic. From this pair $(\mathcal{C}, \mathbb{L})$ they construct a Landau-Ginzburg model: an algebra $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$ with a central element $\ell \in \operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$. They also construct a functor $F: \mathcal{C} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right)$ :

$$
\begin{array}{clc}
\text { Cyclic subcategory } \\
\mathbb{L} \subseteq \mathcal{C}
\end{array} \quad \longrightarrow \quad F: \mathcal{C} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right)
$$

The idea to deform this construction is as easy as it can get: Once we change $\mathcal{C}$ to a deformation $\mathcal{C}_{q}$, the relations of $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$ deform and the central element $\ell$ changes. As long as the subcategory $\mathbb{L}_{q} \subseteq \mathcal{C}_{q}$ is still cyclic, this gives a deformed Landau-Ginzburg model $\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right), \ell_{q}\right)$ together with a deformed functor $F_{q}: \mathcal{C}_{q} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right), \ell_{q}\right)$ :

$$
\begin{array}{clc}
\text { Cyclic deformed subcategory } & \longrightarrow & \text { Deformed mirror functor } \\
\mathbb{L}_{q} \subseteq \mathcal{C}_{q} & F: \mathcal{C}_{q} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right), \ell_{q}\right)
\end{array}
$$

In section 20.1 we motivate the Cho-Hong-Lau construction via Koszul duality. In section 20.2 we recall the construction and fix notation. Insection 20.3 , we start deforming the construction by building a deformed Landau-Ginzburg model. In section 20.4 we prepare a category of projective modules for deformed algebras. In section 20.5, we define categories of deformed matrix factorizations. In section 20.6 we construct the deformed mirror functor.

After discussing the Cho-Hong-Lau construction in Section 20.2, we will assume Convention 20.6 throughout the rest of the section. From section 20.3 onwards, we assume its deformed version Convention 20.22. In text and lemmas, we may typically omit mentioning the conventions, while for the actual results we will always remind the reader. We may refer to the non-deformed Cho-Hong-Lau construction as the "classical construction" and to the deformed version as the "deformed construction".

### 20.1 Perspective from Koszul duality

In this section, we motivate the construction of Cho, Hong and Lau from the perspective of Koszul duality. In fact, the Cho-Hong-Lau construction is a specialized variant of Koszul duality adapted to the case that $\mathcal{C}$ is not an augmented $A_{\infty}$-category. In section 16 we have already recalled Koszul duality and the connection between cyclic $A_{\infty}$-algebras and Calabi-Yau algebras. We have also provided a series of direct tweaks to Koszul duality which motivate the Cho-Hong-Lau construction. In the present section, we compile explicitly a roadmap from Koszul duality to the Cho-Hong-Lau construction:

Starting from arbitrary $\mathcal{C}$ : Koszul duality departs from an augmented finite-dimensional $A_{\infty}$-algebra $A$ and yields a functor $\operatorname{Mod}_{\text {right }}^{\mathrm{fd}} A \rightarrow \mathrm{Tw} A^{!}$. Cho, Hong and Lau instead depart from an $A_{\infty^{-}}$ category $\mathcal{C}$ and subcategory of reference objects $\mathbb{L} \subseteq \mathcal{C}$. They form the $A_{\infty}$-algebra $A=\operatorname{Hom}(\mathbb{L}, \mathbb{L})$. Their mirror functor is then roughly the composition

$$
F: \mathcal{C} \xrightarrow{\operatorname{Hom}(\mathbb{L},-)} \operatorname{Mod}_{\mathrm{right}}^{\mathrm{fd}} A \xrightarrow{\mathrm{Koszul}} \operatorname{Tw} A^{!}
$$

Multiple reference objects: The Koszul dual $A^{!}$is always an algebra of noncommutative power series. When departing from a category $\mathbb{L}$ with multiple objects, the algebra $A$ becomes structured over the semisimple ring $\mathbb{C} Q_{0}$. The Koszul dual $A^{!}$inherits the structuring and becomes a quiver algebra with vertex set $Q_{0}$. The dual element $x_{i}^{\vee}$ runs in the opposite direction of $x_{i}$.
Passing to cohomology: The Koszul dual of an $A_{\infty}$-algebra is a dg algebra. Cho, Hong and Lau forget the dg algebra and pass to cohomology. The root cause for the success of this procedure consists of an $A_{\infty}$-morphism $A^{!} \rightarrow \mathrm{H}^{0} A^{!}$. As we demonstrate in Lemma 16.61, this $A_{\infty}$-morphism exists naturally if $A$ is positively graded.
Restriction to odd morphisms: The Koszul dual of an $A_{\infty}$-algebra $A$ spanned by $x_{1}, \ldots, x_{n}$ is generated by dual variables $x_{1}^{\vee}, \ldots, x_{n}^{\vee}$ of degree $\left|x_{i}^{\vee}\right|=1-\left|x_{i}\right|$. Cho, Hong and Lau take the even part of $A$ into account, but restrict the mirror $\mathrm{H}^{0} A^{!}$to the degree zero generators only. The only relations divided out are $d_{A^{\prime}} x^{\vee}$ for $x \in A^{2}$. As we demonstrate in Lemma 16.63, this approach comes completely naturally if $A$ is positively graded.
Relaxing the grading: $\mathbb{Z}$-gradedness of $A$ is not a necessity for Koszul duality. As we demonstrate in Lemma 16.65 it is however essential for using $\mathrm{H}^{0} A^{!}$as codomain of Koszul duality. Cho, Hong and Lau admit $A$ to be $\mathbb{Z} / 2 \mathbb{Z}$-graded. To obtain a functioning Koszul duality in this case, the codomain of the functor is not based on the actual cohomology $\mathrm{H}^{0} A^{!}$but deploys a surrogate. As we demonstrate in Lemma 16.68, the functioning surrogate for $\mathrm{H}^{0} A^{!}$is a quotient of the tensor algebra generated by $x_{i}^{\vee}$ for $x_{i}$ odd. The surrogate ideal is generated by the restrictions of $d_{A^{!}} x_{i}^{\vee} \in A^{!}$to $T\left(\bar{A}^{\text {odd }}[1]\right) \subseteq T(\bar{A}[1])$ for $x_{i}$ even.
Cyclicity of relations: Cyclicity of the $A_{\infty}$-algebra $A$ typically makes its Koszul dual $A^{!}$a Calabi-Yau dg algebra. While its cohomology $\mathrm{H}^{0} A^{!}$is not necessarily a Calabi-Yau, the surrogate algebra used instead of $\mathrm{H}^{0} A^{!}$is always the Jacobi algebra $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$ of a quiver with superpotential. It is a candidate for being a Calabi-Yau algebra of dimension 3.
Non-augmented $\mathcal{C}$ : Koszul duality requires that the algebra $A$ is augmented in the sense that $A_{\infty^{-}}$ products of non-identities never yield identities. Cho, Hong and Lau solve this by accumulating all products that yield identities in an element $\ell \in A^{!}$. The element $\ell$ is central in $A^{!}$and therefore forms a curved dg algebra $\left(A^{!}, \ell\right)$.
Matrix factorizations: Together with the curvature mentioned above, the result is a Landau-Ginzburg model ( $\mathrm{Jac}_{W} Q^{\mathbb{L}}, \ell$ ) consisting of the Jacobi algebra of a quiver with superpotential, together with the additional potential $\ell$ as curvature. The suitable analog of the codomain $\operatorname{Tw} \operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$ in this curved setting is category $\operatorname{MF}\left(\operatorname{Jac}_{W} Q^{\mathbb{L}}, \ell\right)$ of matrix factorizations.
Opposite construction: Koszul duality yields a strictly defined Koszul dual $A^{!}$and a functor to $\mathrm{Tw} A^{!}$. Cho, Hong and Lau instead construct a Jacobi algebra based on the opposite algebra of $A^{!}$. In this algebra, generator $x_{i}^{\vee}$ points in the same direction as $x_{i}$. The codomain of the functor never has to be written $\operatorname{Tw}\left(A^{!}\right)^{\circ}$ op , because Tw is replaced by MF. While Tw is naturally a category of right modules, the category MF is by definition a category of left modules and therefore already takes the opposite into account.

The most important ingredients of the Cho-Hong-Lau construction are the category $\mathcal{C}$ and a choice of subcategory $\mathbb{L}=\left\{L_{1}, \ldots, L_{N}\right\} \subseteq \mathcal{C}$ which is largely cyclic with respect to a non-degenerate odd graded-symmetric pairing $\langle-,-\rangle$. We shall fix this terminology as follows:
Definition 20.1. Let $\mathcal{C}$ be an $\mathbb{Z} / 2 \mathbb{Z}$-graded $A_{\infty}$-category. An odd non-degenerate graded-symmetric pairing on $\mathcal{C}$ consists of a family of non-degenerate odd bilinear pairings $\langle-,-\rangle_{L_{1}, L_{2}}$ indexed by all pairs of objects $L_{1}, L_{2} \in \mathcal{C}$, with

$$
\langle-,-\rangle_{L_{1}, L_{2}}: \operatorname{Hom}\left(L_{1}, L_{2}\right) \times \operatorname{Hom}\left(L_{2}, L_{1}\right) \rightarrow \mathbb{C}, \quad\langle x, y\rangle=(-1)^{|x||y|}\langle y, x\rangle .
$$

Remark 20.2. We simply write $\langle-,-\rangle$ instead of $\langle-,-\rangle_{L_{1}, L_{2}}$. We set $\langle x, y\rangle=0$ whenever $x, y$ lie in incompatible hom spaces.

When choosing a basis for the odd part of the hom spaces of $\mathbb{L}$, one obtains a dual basis for the even part of the hom spaces of $\mathbb{L}$ as in Definition 16.56. We shall prepare here terminology for the precise type of basis that the Cho-Hong-Lau construction requires:

Definition 20.3. Let $\mathbb{L}=\left\{L_{1}, \ldots, L_{N}\right\}$ be a unital $A_{\infty}$-category with non-degenerate odd gradedsymmetric pairing $\langle-,-\rangle$. Let $E_{i j}$ be disjoint index sets for every $1 \leq i, j \leq N$ and let

$$
\left\{X_{e}\right\}_{e \in E_{i j}} \subseteq \operatorname{Hom}^{\text {odd }}\left(L_{i}, L_{j}\right), \quad\left\{Y_{e}\right\}_{e \in E_{j i}} \subseteq \operatorname{Hom}^{\text {even }}\left(L_{i}, L_{j}\right), \quad \operatorname{id}_{L_{i}}^{*} \in \operatorname{Hom}^{\text {odd }}\left(L_{i}, L_{i}\right)
$$

for every $1 \leq i, j \leq N$. Then the triple $\left\{X_{e}\right\},\left\{Y_{e}\right\},\left\{\operatorname{id}_{L_{i}}^{*}\right\}$ is a CHL basis for $\mathbb{L}$ if

1. These families of morphisms form a basis for the hom spaces of $\mathbb{L}$ when combined with the identities $\mathrm{id}_{L_{i}}$ :

$$
\operatorname{Hom}\left(L_{i}, L_{j}\right)=\operatorname{span}\left\{X_{e}\right\}_{e \in E_{i j}} \oplus \operatorname{span}\left\{Y_{e}\right\}_{e \in E_{j i}} \quad\left[\oplus \operatorname{span}\left\{\operatorname{id}_{L_{i}}, \mathrm{id}_{L_{i}}^{*}\right\} \text { if } i=j\right] .
$$

2. We have the pairing identities

$$
\begin{array}{ll}
\left\langle Y_{e}, X_{f}\right\rangle=\left\langle X_{f}, Y_{e}\right\rangle=\delta_{e f}, & \left\langle\mathrm{id}_{L_{i}}^{*}, \operatorname{id}_{L_{j}}\right\rangle=\left\langle\operatorname{id}_{L_{j}}, \operatorname{id}_{L_{i}}^{*}\right\rangle=\delta_{i j}, \\
\left\langle X_{e}, X_{f}\right\rangle=\left\langle Y_{e}, Y_{f}\right\rangle=0, & \left\langle\operatorname{id}_{L_{i}}, \operatorname{id}_{L_{j}}\right\rangle=\left\langle\operatorname{id}_{L_{i}}^{*}, \operatorname{id}_{L_{j}}^{*}\right\rangle=0,  \tag{20.1}\\
\left\langle X_{e}, \operatorname{id}_{L_{i}}\right\rangle=\left\langle X_{e}, \mathrm{id}_{L_{i}}^{*}\right\rangle=0, & \left\langle Y_{e}, \operatorname{id}_{L_{i}}\right\rangle=\left\langle Y_{e}, \mathrm{id}_{L_{i}}^{*}\right\rangle=0 .
\end{array}
$$

The element $\mathrm{id}_{L_{i}}^{*}$ is the co-identity of $L_{i}$ in $\mathbb{L}$.
Remark 20.4. In contrast to the case of cyclic $A_{\infty}$-algebras, the pairing pairs the opposite hom spaces $\operatorname{Hom}\left(L_{i}, L_{j}\right)$ and $\operatorname{Hom}\left(L_{j}, L_{i}\right)$. The dual basis element for $X_{e} \in \operatorname{Hom}\left(L_{i}, L_{j}\right)$ is therefore an element $Y_{e} \in \operatorname{Hom}\left(L_{j}, L_{i}\right)$. This is the reason why the basis elements $Y_{e}$ for $\operatorname{Hom}^{\text {even }}\left(L_{i}, L_{j}\right)$ are indexed by the index set $E_{j i}$ borrowed from the basis of the opposite hom space $\operatorname{Hom}^{\text {odd }}\left(L_{j}, L_{i}\right)$.

Remark 20.5. A Cho-Hong-Lau basis for $\mathbb{L}$ always exists if the hom spaces of $\mathbb{L}$ are finite-dimensional. This is a simple consequence of Definition 16.56 which we shall briefly explain. First, the identity $\mathrm{id}_{L_{i}}$ determines a dual odd element $\mathrm{id}_{L_{i}}^{*} \in \operatorname{Hom}^{\text {oda }}\left(L_{i}, L_{i}\right)$. One now freely chooses further odd elements $X_{e} \in \operatorname{Hom}^{\text {odd }}\left(L_{i}, L_{j}\right)$, where the index $e$ ranges over an arbitrary set $E_{i j}$ for every $1 \leq i, j \leq N$. Together with the co-identities, the elements $X_{e}$ are supposed to form a basis for the odd part of the hom spaces of $\mathcal{C}$ :

$$
\operatorname{Hom}^{\text {odd }}\left(L_{i}, L_{j}\right)=\operatorname{span}\left\{X_{e}\right\}_{e \in E_{i j}} \quad\left[\oplus \mathbb{C i d}_{L_{i}}^{*} \text { if } i=j\right] .
$$

According to Definition 16.56, we obtain a unique dual basis for the even hom spaces. It is of the form

$$
\operatorname{Hom}^{\operatorname{even}}\left(L_{i}, L_{j}\right)=\operatorname{span}\left\{Y_{e}\right\}_{e \in E_{j i}} \quad\left[\oplus \mathbb{C} \operatorname{id}_{L_{i}} \text { if } i=j\right]
$$

By construction, the triple $\left\{X_{e}\right\},\left\{Y_{e}\right\}$ and $\left\{\operatorname{id}_{L_{i}}^{*}\right\}$ now satisfies the pairing identities 20.1 and forms a CHL basis according to Definition 20.3 .

### 20.2 The Cho-Hong-Lau construction

In this section, we recall the construction of the noncommutative mirror functor due to Cho, Hong and Lau 26. The aim is to define the mirror functor as fast as possible. The mirror functor recalled here serves as leading term of the deformed mirror functor which we construct in the next sections. The present section also serves to fix notation and terminology as well as to fix sign conventions.

In Convention 20.6 we record the complete list of input data and assumptions for the Cho-Hong-Lau construction. The input data include a category $\mathcal{C}$ and a chosen subcategory $\mathbb{L} \subseteq \mathcal{C}$. The input data also include a choice of CHL basis for $\mathbb{L}$. The assumptions include that $\mathbb{L}$ is "cyclic on the odd augmented part" of $\mathbb{L}$. The precise list reads as follows:

Convention 20.6. The $A_{\infty}$-category $\mathcal{C}$ is $\mathbb{Z} / 2 \mathbb{Z}$-graded and unital. A subset of reference objects $\mathbb{L}=$ $\left\{L_{1}, \ldots, L_{N}\right\} \subseteq \mathcal{C}$ is provided. The category $\mathbb{L}$ is supposed to come with an odd non-degenerate gradedsymmetric pairing $\langle-,-\rangle$. A CHL basis $\left\{X_{e}\right\}_{e \in E_{i j}, 1 \leq i, j \leq N},\left\{Y_{e}\right\}_{e \in E_{i j}, 1 \leq i, j \leq N},\left\{\operatorname{id}_{L_{i}}^{*}\right\}_{1 \leq i \leq N}$ for $\mathbb{L}$ is provided. The category $\mathbb{L}$ is required to be cyclic on the odd part with respect to $\langle-,-\rangle$ :

$$
\left\langle\mu\left(X_{e_{k+1}}, \ldots, X_{e_{2}}\right), X_{e_{1}}\right\rangle=\left\langle\mu\left(X_{e_{k}}, \ldots, X_{e_{1}}\right), X_{e_{k+1}}\right\rangle
$$

The hom spaces $\operatorname{Hom}\left(L_{i}, X\right)$ are assumed to be finite-dimensional for $1 \leq i \leq N$ and $X \in \mathcal{C}$.
The construction of mirror and mirror functor proceeds by Koszul transforming the $A_{\infty}$-structure of $\mathbb{L}$. In classical Koszul duality, one transforms the $A_{\infty}$-structure of an $A_{\infty}$-algebra by looking at which sequences of basis input elements produce a given basis element as output. In the construction of Cho, Hong and Lau, the role of the input sequences is played by sequences of basis elements $X_{e_{1}}, \ldots, X_{e_{k}}$, ranging over all hom spaces in $\mathbb{L}$. To record the products on these sequences, one introduces a formal variable $x_{e}$ for every $e \in E_{i j}$ and $1 \leq i, j \leq N$. The variables are subject to constraints on composition, coming from a quiver structure:

Definition 20.7. The CHL quiver $Q^{\mathbb{L}}$ has one vertex $L_{i}$ for every reference object $L_{i}$ and an arrow $x_{e}: L_{i} \rightarrow L_{j}$ for every $e \in E_{i j}$ and $1 \leq i, j \leq N$.

With the definition of $Q^{\mathbb{L}}$ in mind, we build the auxiliary formal element

$$
\begin{equation*}
b=\sum_{i, j=1}^{N} \sum_{e \in E_{i j}} x_{e} X_{e} \tag{20.2}
\end{equation*}
$$

In principle, the basis morphisms $X_{e}$ lie in different hom spaces. If we view $\mathbb{L}$ as a direct sum of its elements $L_{1}, \ldots, L_{N}$, we can interpret $b$ as an element of $\widehat{\mathbb{C} Q^{\mathbb{L}}} \otimes \operatorname{Hom}(\mathbb{L}, \mathbb{L})$. Our convention is that product of the type $\mu\left(m_{k}, \ldots, m_{1}, b, \ldots, b\right)$ are always to be understood as multlinear expansions of the product under use of the sum 20.2. More background can be found in 26, Chapter 2].

Summing up products of the type $\mu\left(m_{k}, \ldots, m_{1}, b, \ldots, b\right)$ over increasing number of $b$-insertions gives an infinite series. The summands consist of paths in $Q^{\mathbb{L}}$ multiplied by basis elements $X_{e}, Y_{e}$ or (co)identities. The coefficients series of a basis element $X_{e}$ need not converge in $\mathbb{C} Q^{\mathbb{L}}$, but generally only in the completed path algebra $\widehat{\mathbb{C} Q^{\mathbb{L}}}$. The special case where the coefficient series terminate is however relevant, as it allows one to obtain a Landau-Ginzburg model building on the quiver algebra $\mathbb{C} Q^{\mathbb{L}}$ instead of its completion. We shall give this case a name:

Definition 20.8. $\mathbb{L}$ is of bounded growth if for all morphisms $m_{1}, \ldots, m_{1}$ in $\mathcal{C}$ there is an $l_{0} \in \mathbb{N}$ such that

$$
\forall l \geq l_{0}: \quad \mu^{k+l}\left(m_{k}, \ldots, m_{1}, b, \ldots, b\right)=0
$$

With this in mind, we can define all relevant intermediates of the Cho-Hong-Lau construction as follows:
Definition 20.9. The relations $R_{e} \in \widehat{\mathbb{C} Q^{\mathbb{L}}}$ and the potential $\ell \in \widehat{\mathbb{C} Q^{\mathbb{L}}}$ are defined by

$$
\begin{equation*}
\sum_{k \geq 1} \mu^{k}(b, \ldots, b)=\ell \operatorname{id}_{\mathbb{L}}+\sum_{i, j=1}^{N} \sum_{e \in E_{i j}} R_{e} Y_{e} \tag{20.3}
\end{equation*}
$$

The superpotential is defined as

$$
W=\left\langle\sum_{k \geq 1} \mu^{k}(b, \ldots, b), b\right\rangle \in \widehat{\mathbb{C} Q^{\mathbb{L}}}
$$

The Jacobi algebra is defined as

$$
\begin{equation*}
\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)=\frac{\widehat{\mathbb{C} Q^{\mathbb{L}}}}{\left(\partial_{x_{e}} W\right)} . \tag{20.4}
\end{equation*}
$$

The Landau-Ginzburg model is the pair $\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right), \ell\right)$. If $\mathbb{L}$ is of bounded growth, then $R_{e}, \ell, W$ are regarded as elements of $\mathbb{C} Q^{\mathbb{L}}$, the Jacobi algebra is defined as $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)=\mathbb{C} Q /\left(\partial_{x_{e}} W\right)$, and the Landau-Ginzburg model is $\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right)$.

Remark 20.10. The element $W \in \widehat{\mathbb{C} Q^{\mathbb{L}}}$ is cyclic, as we recall in Lemma 20.16 Its derivative $\partial_{x_{e}} W$ is defined by stripping off $x_{e}$ from the front (or rear) side of all terms in $W$ that start (or end) with $x_{e}$.

Remark 20.11. The description of $\ell$ and $R_{e}$ in 20.3 is to be interpreted as follows: All the products $\mu(b, \ldots, b)$ are even and can hence be written as a sum of the even basis elements. The even basis elements are by assumption of the form $Y_{e}$ and $\mathrm{id}_{L_{i}}$. The element $R_{e}$ is formed by recording the coefficient of $Y_{e}$ and the element $\ell$ is formed by recording the coefficients of the identities $\mathrm{id}_{L_{i}}$ and summing up.

Remark 20.12. We have used the notation $\bar{X}$ for the closure of a set $X \subseteq \widehat{\mathbb{C} Q^{\mathbb{L}}}$ with respect to the Krull topology on $\widehat{\mathbb{C} Q^{\mathbb{L}}}$. More information on the Krull topology can be found in section 19.9

Remark 20.13. In Definition 20.9 the two uses of the symbol $\left(\partial_{x_{e}} W\right)$ differ slightly. Namely in 20.4, the expression $\left(\partial_{x_{e}} W\right)$ denotes the ideal $\widehat{\mathbb{C} Q^{\mathbb{L}}} \operatorname{span}\left(\partial_{x_{e}}\right)_{e} \widehat{\mathbb{C} Q^{\mathbb{L}}}$ generated by the relations $\partial_{x_{e}} W$ in $\widehat{\mathbb{C} Q^{\mathbb{L}}}$, while in the case of bounded growth it denotes the ideal generated in $\mathbb{C} Q^{\mathbb{L}}$. Alternatively, the shared definition $\left(\partial_{x_{e}} W\right)=\mathbb{C} Q^{\mathbb{L}} \operatorname{span}\left(\partial_{x_{e}} W\right) \mathbb{C} Q^{\mathbb{L}}$ can be used in both cases since in 20.4 closure is taken. The notation for the two ideals also differs slightly from the notation of section 19 .

Remark 20.14. We have decided to treat the case that $\mathbb{L}$ is of bounded growth in parallel with the general case. In contrast, Cho, Hong and Lau 26 specialize to the case of bounded growth only in Chapter 10. In fact, their construction departs from the even more general situation involving the Novikov ring. We shall try to make explicit every time whether we regard the general construction which uses the completed path algebra and the closure of the ideal, or the specific construction which uses the ordinary path algebra and ideal.

Remark 20.15. Unfortunately the calculations of 26 and 25 appear to contain at least two independent sign issues. We have therefore decided to repeat the calculations here and repair the signs. We shall here trace back the signs: The first dubious sign can be found in 25, Theorem 2.19]: The expressions (2.10), (2.12), (2.14) indeed add up to zero due to the $A_{\infty}$-relation for $\mathcal{C}$, but we need to show that the difference $+(2.10)-(2.12)-(2.14)$ vanishes. This issue breaks the functor equations for $\widehat{F}$, even if the matrix factorizations under consideration have vanishing $\delta$. The second dubious sign can be found in the combination of 26, Definition 4.3] and [26, Definition 4.4]. The specific combination of sign conventions for $\mu_{\mathrm{MF}}^{1}$ and for the endomorphism $\delta$ of $\widehat{F}$ seems to break the functor equations for $\widehat{F}$. A third issue is that even without regarding the functor relations the definition of $\widehat{F}$ immediately renders $\widehat{F}$ (id) $=-\mathrm{id}$, while it would be desirable to have $\widehat{F}(\mathrm{id})=$ id. We have tried to repair all issues, even though it makes the sign convention for $\widehat{F}$ slightly unesthetic.

From now on, the element $\ell$ is typically regarded as an element in the quotient $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$. Its significance in the quotient and the relation between $R_{e}$ and $W$ is explained as follows:

Lemma 20.16. The superpotential $W \in \widehat{\mathbb{C} Q^{\mathbb{L}}}$ is cyclic and we have $R_{e}=\partial_{x_{e}} W$. The potential $\ell \in$ $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$ is central. The analogous statements hold if $\mathbb{L}$ is of bounded growth.
Proof. We divide the proof into the four obvious parts. The first part of the proof is to check cyclicity of $W$. To show this, regard the sum decomposition

$$
\begin{equation*}
\left\langle\mu^{k}(b, \ldots, b), b\right\rangle=\sum_{e_{1}, \ldots, e_{k+1}}\left\langle\mu^{k}\left(X_{e_{k+1}}, \ldots, X_{e_{2}}\right), X_{e_{1}}\right\rangle x_{e_{k+1}} \ldots x_{e_{1}} \tag{20.5}
\end{equation*}
$$

We will group the collection of all summands into cyclic orbits. Namely, regard any term $\lambda x_{e_{k+1}} \ldots x_{e_{1}}$ appearing in this sum. Recall that by the cyclicity assumption of Convention 20.6 we have

$$
\begin{equation*}
\lambda=\left\langle\mu^{k}\left(X_{e_{k+1}}, \ldots, X_{e_{2}}\right), X_{e_{1}}\right\rangle=\left\langle\mu^{k}\left(X_{e_{k}}, \ldots, X_{e_{1}}\right), X_{e_{k+1}}\right\rangle \tag{20.6}
\end{equation*}
$$

This means the permuted version $\lambda x_{e_{k}} \ldots x_{e_{1}} x_{e_{k+1}}$ also appears in the sum 20.5, with equal coefficient $\lambda$. This renders $\left\langle\mu^{k}(b, \ldots, b), b\right\rangle$ cyclic. Summing over $k \geq 1$, we derive that the entire superpotential $W$ is cyclic.

The second part of the proof is to check that $R_{e}=\partial_{x_{e}} W$. Usually, derivatives of a cyclic superpotential are written in terms of those paths starting with the given variable $x_{e}$. Since $W$ is already cyclic, it suffices to extract all paths which instead end on $x_{e}$. To avoid double indexing by $e$, we will write $x_{f}$ in the expansion of $\left\langle\mu^{k}(b, \ldots, b), b\right\rangle$. Ultimately, we calculate within $\widehat{\mathbb{C} Q^{\mathbb{L}}}$ that

$$
\begin{aligned}
\partial_{x_{e}} W & =\partial_{x_{e}}\left\langle\sum_{k \geq 1} \mu^{k}(b, \ldots, b), \sum_{f} x_{f} X_{f}\right\rangle \\
& =\partial_{x_{e}}\left\langle\ell \operatorname{id}_{\mathbb{L}}+\sum_{f} R_{f} Y_{f}, \sum_{f} x_{f} X_{f}\right\rangle=\partial_{x_{e}}\left(\sum_{f} R_{f} x_{f}\right)=R_{e}
\end{aligned}
$$

The third part of the proof is to check that $\ell \in \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$ is central. In this part of the proof, we shall regard $\ell$ interchangeably as element of $\widehat{\mathbb{C} Q^{\mathbb{L}}}$ or $\operatorname{Jac}(\widehat{Q}, W)$. We start with the observation that within $\widehat{\mathbb{C} Q^{\mathbb{L}}} \otimes \operatorname{Hom}(\mathbb{L}, \mathbb{L})$ we have

$$
0=\sum_{k \geq 1} \sum_{l \geq 1} \mu^{k}\left(b, \ldots, \mu^{l}(b, \ldots, b), \ldots, b\right)=\sum_{k \geq 1} \mu^{k}\left(b, \ldots, \sum_{i=1}^{N} \ell_{i} \operatorname{id}_{L_{i}}+\sum R_{e} Y_{e}, \ldots, b\right)
$$

We have essentially performed a reordering of the double sum which is legitimate since the path lengths regarding $Q^{\mathbb{L}}$ encountered in $\mu(b, \ldots, b)$ increase as the number of inputs increases. We now claim
that apart from $\mu^{2}\left(b, \ell_{i} \operatorname{id}_{L_{i}}\right)$ and $\mu^{2}\left(\ell_{i} \operatorname{id}_{L_{i}}, b\right)$, the entire sum on the right-hand side of the equation lies in $\overline{\left(\partial_{x_{e}} W\right)} \otimes \operatorname{Hom}(\mathbb{L}, \mathbb{L})$. Indeed, all terms with identities vanish for $k \geq 3$ and $k=1$ and the products involving $Y_{e}$ all embrace relations $R_{e}$. We caution that the expression does in general not lie in $\widehat{\mathbb{C Q}^{\mathbb{L}}} \operatorname{span}\left(\partial_{x_{e}} W\right) \widehat{\mathbb{C} Q^{\mathbb{L}}}$, see Remark 19.53 for an illustration. Ultimately, we conclude within $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) \otimes$ $\operatorname{Hom}(\mathbb{L}, \mathbb{L})$ that

$$
0=\mu^{2}\left(b, \ell \operatorname{id}_{\mathbb{L}}\right)+\mu^{2}\left(\ell \operatorname{id}_{\mathbb{L}}, b\right)
$$

Let $1 \leq i, j \leq N$ and $e \in E_{i j}$. Then extracting the $X_{e}$-component gives

$$
0=x_{e} \ell-\ell x_{e} \text { within } \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) .
$$

We conclude that $\ell$ commutes with all arrows in $Q^{\mathbb{L}}$ and hence with all finite paths. Let now $x \in \widehat{\mathbb{C Q}^{\mathbb{L}}}$ be an arbitrary element. If one wants to take Krull-continuity of the projection $\pi: \widehat{\mathbb{C Q}^{\mathbb{L}}} \rightarrow \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$ for granted, simply choose a sequence $\left(x_{n}\right) \in \mathbb{C} Q^{\mathbb{L}}$ with $x_{n} \rightarrow x$, then $0=\pi\left(x_{n}\right) \ell-\ell \pi\left(x_{n}\right) \rightarrow \pi(x) \ell-\ell \pi(x)$, hence $x \ell=\ell x$. If not, write $x=\sum_{k \geq 0} x_{k}$ where $x_{k}$ is homogeneous of length $k$. Then

$$
\left(\sum_{k \geq 0} \pi\left(x_{k}\right)\right) \ell=\pi\left(\left(\sum_{k \geq 0} x_{k}\right) \ell\right)=\pi\left(\sum_{k \geq 0} x_{k} \ell\right)=\pi\left(\sum_{k \geq 0} \ell x_{k}+z_{k}\right)=\ell\left(\sum_{k \geq 0} \pi\left(x_{k}\right)\right)
$$

Here $z_{k} \in \overline{\left(\partial_{x_{e}} W\right)}$ denotes the difference of $x_{k} \ell$ and $\ell \underline{x_{k}}$ when regarded as elements of $\widehat{\mathbb{C} Q^{\mathbb{L}}}$. Note that $z_{k}$ has length at least $k$ since $x_{k}$ does, hence $\sum z_{k} \in \overline{\left(\partial_{x_{e}} W\right)}$. We conclude that $\ell$ commutes with any element $x \in \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$ and hence $\ell \in Z\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)\right)$. The fourth part of the proof consists of observing that the calculations still hold, even simplify, in case $\mathbb{L}$ is of bounded growth. This finishes the proof.

It is time to demonstrate how one calculates in the Jacobi algebra. For instance, within $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) \otimes$ $\operatorname{Hom}(\mathbb{L}, \mathbb{L})$ the expression (20.3) simplifies to

$$
\sum_{k \geq 1} \mu^{k}(b, \ldots, b)=\sum_{i=1}^{N} \ell_{i} \operatorname{id}_{L_{i}} \in \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) \otimes \operatorname{Hom}(\mathbb{L}, \mathbb{L}) .
$$

We are ready to recall the construction of the mirror functor of Cho, Hong and Lau. The idea is to tweak the Koszul duality functor for the $A_{\infty}$-algebra $A=\operatorname{End}(\mathbb{L})$. We shall construct two functors $\widehat{F}$ and $F$, where $\widehat{F}$ serves the general case and $F$ the case where $\mathbb{L}$ is of bounded growth. The domain of both functors is $\mathcal{C}$ and the codomains are the matrix factorization categories $\operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right), \ell\right)$ and $\operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right)$, respectively. We start with the explicit descriptions on the level of objects:

$$
\begin{align*}
& \widehat{F}(X)=\left(\begin{array}{ll}
\bigoplus_{i=1}^{N} \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) L_{i} \otimes \operatorname{Hom}\left(L_{i}, X\right), & \delta(m)=\sum_{k \geq 1}(-1)^{\|m\|} \mu^{k}(m, b, \ldots, b)
\end{array}\right),  \tag{20.7}\\
& F(X)=\left(\bigoplus_{i=1}^{N} \operatorname{Jac}\left(Q^{\mathbb{L}}, W\right) L_{i} \otimes \operatorname{Hom}\left(L_{i}, X\right), \quad \delta(m)=\sum_{k \geq 1}(-1)^{\|m\|} \mu^{k}(m, b, \ldots, b)\right) . \tag{20.8}
\end{align*}
$$

Let us explain how to interpret these expressions as matrix factorizations. Recall from Remark 17.12 that a matrix factorization can be defined as a $\mathbb{Z} / 2 \mathbb{Z}$-graded module which together with an odd endomorphism which squares to the desired central element. In our case, the $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$-module $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) L_{i} \otimes$ $\operatorname{Hom}\left(L_{i}, X\right)$ shall have $\mathbb{Z} / 2 \mathbb{Z}$-grading inherited from $\operatorname{Hom}\left(L_{i}, X\right)$. Since $b$ is odd, the map $\delta$ itself becomes odd. The module is projective and finitely generated since $\operatorname{Hom}\left(L_{i}, X\right)$ is finite-dimensional by assumption. If we check that $\delta$ squares to $\ell$, then $\widehat{F}(X)$ is indeed a matrix factorization.

Lemma 20.17. For $X \in \mathcal{C}$ the object $\widehat{F}(X)$ is indeed a matrix factorization of $\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right), \ell\right)$. If $\mathbb{L}$ is of bounded growth, then $F(X)$ is a matrix factorization of $\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right)$.

Proof. We merely check the case of $\widehat{F}(X)$. It is our task to show that $\delta^{2}$ equates to multiplying by $\ell$.

Calculating in $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) \otimes \operatorname{Hom}(\mathbb{L}, X)$, we get

$$
\begin{aligned}
\delta(\delta(m)) & =\delta\left(\sum_{k \geq 1}(-1)^{\|m\|} \mu^{k}(m, b, \ldots, b)\right) \\
& =\sum_{l \geq 1} \sum_{k \geq 1}(-1)^{\|m\|+\|m\|-1} \mu^{l}\left(\mu^{k}(m, b, \ldots, b), b, \ldots, b\right) \\
& =-\sum_{n \geq 1}(\mu \cdot \mu)^{n}(m, b, \ldots, b)+\sum_{k \geq 2} \sum_{l \geq 1} \mu^{k}\left(m, b, \ldots, \mu^{l}(b, \ldots, b), \ldots, b\right) \\
& =\mu^{2}\left(m, \ell \operatorname{id}_{\mathbb{L}}\right)=\ell m \in \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) \otimes \operatorname{Hom}(\mathbb{L}, X) .
\end{aligned}
$$

Here we have used the $A_{\infty}$-relation $\mu \cdot \mu=0$ and $\sum_{l \geq 1} \mu^{l}(b, \ldots, b)=\ell \operatorname{id}_{\mathbb{L}}$. We conclude that $\delta^{2}=\ell$. This shows that $\widehat{F}(X)$ is a matrix factorization. The analogous calculations show that $F(X)$ is a matrix factorization when $\mathbb{L}$ is of bounded growth.

We are finally ready to write down the functors $\widehat{F}$ and $F$. On objects, these functors map $X \in \mathcal{C}$ to their associated matrix factorizations $\widehat{F}(X)$ and $F(X)$ defined in 20.7. On morphisms, the functors are intuitively constructed through viewing $\operatorname{Hom}\left(L_{i}, X\right)$ as a module over $\operatorname{End}(\mathbb{L})$ and composing with the Koszul duality functors. We explain the origin of the functors in more detail in Remark 20.21
Definition 20.18. The CHL functor $\widehat{F}$ is the mapping

$$
\begin{align*}
& \widehat{F}: \mathcal{C} \longrightarrow \operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right), \ell\right),  \tag{20.9}\\
& X \longmapsto \widehat{F}(X),  \tag{20.10}\\
& \widehat{F}\left(m_{k}, \ldots, m_{1}\right)(m)=(-1)^{\left(\left\|m_{1}\right\|+\ldots+\left\|m_{k}\right\|\right)\|m\|+1} \sum_{l \geq 0} \mu^{k+l+1}\left(m_{k}, \ldots, m_{1}, m, b, \ldots, b\right)  \tag{20.11}\\
& \text { for } m_{i}: X_{i} \rightarrow X_{i+1}, \quad m \in \widehat{F}\left(X_{1}\right) . \tag{20.12}
\end{align*}
$$

In case $\mathbb{L}$ is of bounded growth, the functor $F: \mathcal{C} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right)$ is defined analogously.
Remark 20.19. Let us elaborate and make sense of the definition of $\widehat{F}\left(m_{k}, \ldots, m_{1}\right)$. To start with, the sequence $m_{1}, \ldots, m_{k}$ is an arbitrary sequence of morphisms $m_{i}: X_{i} \rightarrow X_{i+1}$ in $\mathcal{C}$. The morphism $\widehat{F}\left(m_{k}, \ldots, m_{1}\right)$ is supposed to be a $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$-module map $\widehat{F}\left(X_{1}\right) \rightarrow \widehat{F}\left(X_{k+1}\right)$. Let us make sense of its definition: The element $m$ used to define this map lies in the direct sum

$$
\widehat{F}\left(X_{1}\right)=\bigoplus_{i \in Q_{0}^{\mathbb{L}}} \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) L_{i} \otimes \operatorname{Hom}\left(L_{i}, X_{1}\right)
$$

Its image $\widehat{F}\left(m_{k}, \ldots, m_{1}\right)(m)$ is defined by the products $\mu\left(m_{k}, \ldots, m_{1}, m, b, \ldots, b\right)$, with an arbitrary amount of $b$-insertions. The result of each of these products lives in

$$
\widehat{F}\left(X_{k+1}\right)=\bigoplus_{j \in Q_{0}^{\mathbb{L}}} \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) L_{j} \otimes \operatorname{Hom}\left(L_{j}, X_{k+1}\right)
$$

To see this, recall that the formal parameters $x_{e}$ in $b$ simply get multiplied up as we evaluate the product. Simply speaking, a product where the right-most $b$-summand $x_{e} X_{e}$ is consumed lands in $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) t(e) \otimes$ $\operatorname{Hom}\left(L_{t(e)}, X_{k+1}\right)$. Finally, note that $\widehat{F}\left(m_{k}, \ldots, m_{1}\right)$ is indeed a module map: If $a \in \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$, then $F\left(m_{k}, \ldots, m_{1}\right)(a m)=a F\left(m_{k}, \ldots, m_{1}\right)(m)$ since the factor $a$ can be pulled to the front.

The main algebraic result of Cho, Hong and Lau entails that $\widehat{F}$ indeed defines an $A_{\infty}$-functor:
Lemma 20.20. The CHL functor is a unital $A_{\infty}$-functor $\widehat{F}: \mathcal{C} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right), \ell\right)$. In case $\mathbb{L}$ is of bounded growth, then $F$ is a unital $A_{\infty}$-functor as well.
Proof. We have that $\widehat{F}(\mathrm{id})(m)=(-1)^{\|m\|+1} \mu^{2}(\mathrm{id}, m)=m$, therefore $\widehat{F}(\mathrm{id})=$ id. Similarly, we have $\widehat{F}\left(m_{k}, \ldots, m_{1}\right)=0$ whenever $k \geq 1$ and one $m_{i}$ is an identity. For the functor relations, we need to check

$$
\begin{aligned}
\sum_{i, j}(-1)^{\left\|m_{1}\right\|+\ldots+\left\|m_{j}\right\|} \widehat{F}\left(m_{k}, \ldots, \mu\right. & \left.\left(m_{i}, \ldots, m_{j+1}\right), \ldots, m_{1}\right) \\
& =\mu_{\mathrm{MF}}^{2}\left(\widehat{F}\left(m_{k}, \ldots, m_{i+1}\right), \widehat{F}\left(m_{i}, \ldots, m_{1}\right)\right)+\mu_{\mathrm{MF}}^{1}\left(\widehat{F}\left(m_{k}, \ldots, m_{1}\right)\right)
\end{aligned}
$$

Both sides are homomorphisms $\widehat{F}(X) \rightarrow \widehat{F}(Y)$ of certain $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$-modules. In order to equate both sides, we plug in an arbitrary element $m \in \widehat{F}(X)$ and start evaluating from the left-hand side. Minding the notation $\tilde{\delta}(m)=(-1)^{|m|} \delta(m)$, we get:

$$
\begin{aligned}
& \sum_{0 \leq i \leq j \leq k}(-1)^{\left\|m_{1}\right\|+\ldots+\left\|m_{j}\right\| \widehat{F}\left(m_{k}, \ldots, \mu\left(m_{i}, \ldots, m_{j+1}\right), \ldots, m_{1}\right)(m)} \\
= & \sum_{0 \leq i \leq j \leq k}(-1)^{\left\|m_{1}\right\|+\ldots+\left\|m_{j}\right\|+\left(\left\|m_{1}\right\|+\ldots+\left\|m_{k}\right\|+1\right)\|m\|+1} \mu^{k+l}\left(m_{k}, \ldots, \mu\left(m_{i}, \ldots, m_{j+1}\right), \ldots, m_{1}, m, b, \ldots, b\right) \\
= & \sum_{1 \leq i \leq k-1}(-1)^{\|, l \geq 0} \\
& +\sum_{i, j \geq 0}(-1)^{\|m\|+\left(\left\|m_{1}\right\|+\ldots+\left\|m_{k}\right\|+1\right)\|m\| \mu^{k-i+l+1}\left(m_{k}, \ldots, \mu^{i+1+l}\left(m_{i}, \ldots, m_{1}, m, b, \ldots, b\right), b, \ldots, b\right)} \\
& +\sum_{i, j \geq 0}(-1)^{\|m\|+\left(\left\|m_{1}\right\|+\ldots+\left\|m_{k}\right\|+1\right)\|m\|} \mu^{k+i+1}\left(m_{k}, \ldots, m_{1}, \mu^{j+1}(m, b, \ldots, b), b, \ldots, b\right) \\
= & \sum_{1 \leq i \leq k-1}(-1)^{\text {I }} \widehat{F}\left(m_{k}, \ldots, m_{i+1}\right)\left(\widehat{F}\left(m_{i}, \ldots, m_{1}\right)(m)\right) \\
& +(-1)^{\|m\|+\left(\left\|m_{1}\right\|+\ldots+\left\|m_{k}\right\|+1\right)\|m\|+\left(\left\|m_{1}\right\|+\ldots+\left\|m_{k}\right\|\right)\|m\|+1+1} \tilde{\delta}\left(\widehat{F}\left(m_{k}, \ldots, m_{1}\right)(m)\right) \\
& +(-1)^{\|m\|+\left(\left\|m_{1}\right\|+\ldots+\left\|m_{k}\right\|+1\right)\|m\|+1+\left(\left\|m_{1}\right\|+\ldots+\left\|m_{k}\right\|\right)\|\tilde{\delta}(m)\|+1} \widehat{F}\left(m_{k}, \ldots, m_{1}\right)(\tilde{\delta}(m)) \\
= & \sum_{1 \leq i \leq 1}\left(\mu^{k+j+1}\left(m_{k}, \ldots, m_{1}, m_{1}, b, \ldots, b\right), b, \ldots, b\right) \\
& (-1)^{\left(\left\|m_{i+1}\right\|+\ldots+\left\|m_{k}\right\|\right)\left(\left\|m_{1}\right\|+\ldots+\left\|m_{i}\right\|+1\right) \widehat{F}\left(m_{k}, \ldots, m_{i+1}\right)\left(\widehat{F}\left(m_{i}, \ldots, m_{1}\right)(m)\right)} \\
& +\tilde{\delta}\left(\widehat{F}\left(m_{k}, \ldots, m_{1}\right)(m)\right)-(-1)^{\left|\widehat{F}\left(m_{k}, \ldots, m_{1}\right)\right| \widehat{F}\left(m_{k}, \ldots, m_{1}\right)(\tilde{\delta}(m))} \\
= & \mu_{\mathrm{MF}}^{2}\left(\widehat{F}\left(m_{k}, \ldots, m_{i+1}\right), \widehat{F}\left(m_{i}, \ldots, m_{1}\right)\right)(m)+\mu_{\mathrm{MF}}^{1}\left(\widehat{F}\left(m_{k}, \ldots, m_{1}\right)\right)(m) .
\end{aligned}
$$

This calculation deserves a few comments. In the first equality, we have unraveled the definition of $\widehat{F}$. In the second equality, we have used the $A_{\infty}$-relations for $\mathcal{C}$, which adds an absolute flip to the sign. In the third equality, we have reinterpreted the inner and outer $\mu$ applications as $\widehat{F}$ or $\tilde{\delta}$, using that $\tilde{\delta}(m)=-\sum_{l \geq 1} \mu^{l}(m, b, \ldots, b)$. In the fourth equality, we have rewritten the expressions in terms of the products $\mu_{\mathrm{MF}}$. Since $m$ was arbitrary, we conclude that the $A_{\infty}$-functor relations hold. The very same calculations apply for $F$ in case $\mathbb{L}$ is of bounded growth.

Remark 20.21. The approach of Cho, Hong and Lau seems to originate from Yoneda functors instead of Koszul duality. We sketch here their line of reasoning 25, 26, focusing on the case of a single reference object, $\mathbb{L}=\{L\}$. Let Ch denote the dg category of chain complexes.

A typical Yoneda functor in the $A_{\infty}$-world takes the shape $F=\operatorname{Hom}(L,-): \mathcal{C} \rightarrow \mathrm{Ch}$ and sends $X \in \mathcal{C}$ to the chain complex $\left(\operatorname{Hom}(L, X), \mu^{1}\right)$. The functor $F$ is not strict, but has higher components $F^{\geq 1}$ given by $F\left(m_{k}, \ldots, m_{1}\right)(m)= \pm \mu\left(m_{k}, \ldots, m_{1}, m\right)$.

Twisting an $A_{\infty}$-category $\mathcal{C}$ by any element $b$ gives a new possibly curved $A_{\infty}$-category. Cho, Hong and Lau simply decided to enlarge the category $\mathcal{C}$ by introducing formal variables $x_{e}$ for every odd basis element $X_{e}$ of the hom space $\operatorname{Hom}(L, L)$. Their twist is given by the element $b=\sum x_{e} X_{e}$. This gives an enlarged and twisted category $\mathcal{C}_{b}=\left(\mathcal{C}\left\langle\left\langle x_{e}\right\rangle\right\rangle, b\right)$. Its products $\mu_{b}^{k}$ are given by twisting $\mu$ with $b$. This category in principle has a Yoneda functor $F_{b}: \mathcal{C}_{b} \rightarrow \mathrm{Ch}$ itself, given by

$$
\begin{align*}
F_{b}(X) & =\left(\operatorname{Hom}(L, X), \mu_{b}^{1}\right), \\
F_{b}\left(m_{k}, \ldots, m_{1}\right)(m) & =\mu_{b}\left(m_{k}, \ldots, m_{1}, m\right) . \tag{20.13}
\end{align*}
$$

At this point, adaptations have to be made. For instance the map $\mu_{b}^{1}$, explicitly $\mu_{b}^{1}(m)=\mu(b, \ldots, m, \ldots, b)$, is not a differential anymore because $\mathcal{C}_{b}$ has curvature $\mu(b, \ldots, b)$. Before we proceed, simplify $F_{b}$ slightly by inserting $b$ only at the right-most positions in 20.13).

To make $\mu_{b}^{1}$ a differential and restore the functor $F_{b}$, the crucial idea is to enforce relations among the variables $x_{e}$ to at least render the curvature a multiple of the identity. This precisely explains the choice of the relations $R_{e}$. It be noted that $\mu_{b}^{1}$ is still not a differential since $\mu(b, \ldots, b)$ may contain an identity term $\ell \mathrm{id}_{L}$. This in mind, $\mu_{b}^{1}$ however squares to $\ell$ and we are in the realm of matrix factorizations. The functor $F_{q}$ can be adapted accordingly. This explains the original motivation of Cho, Hong and Lau.

| Gadget | Classical | Deformed |
| :---: | :---: | :---: |
| Completed quiver algebra | $\widehat{\mathbb{C} Q^{\mathbb{L}}}$ | $B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$ |
| (slow growth) | $\mathbb{C} Q^{\mathbb{L}}$ | $B \widehat{\otimes} \mathbb{C} Q^{\mathbb{L}}$ |
| Superpotential | $W \in \widehat{\mathbb{C} Q^{\mathbb{L}}}$ | $W_{q} \in B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$ |
| (slow growth) | $W \in \mathbb{C} Q^{\mathbb{L}}$ | $W_{q} \in B \widehat{\otimes} \mathbb{C} Q^{\mathbb{L}}$ |
| Relations | $R_{e} \in \widehat{\mathbb{C} Q^{\mathbb{L}}}$ | $R_{q, e} \in B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$ |
| (slow growth) | $R_{e} \in \mathbb{C} Q^{\mathbb{L}}$ | $R_{q, e} \in B \widehat{\otimes} \mathbb{C} Q^{\mathbb{L}}$ |
| Jacobi algebra | $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)=\frac{\widehat{\mathbb{C} Q^{\mathbb{L}}}}{\left(\partial_{x_{e} W}\right)}$ | $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)=\frac{B \widehat{\otimes} \widehat{Q^{\mathbb{L}}}}{\left(\partial_{x_{e} W}\right)}$ |
| (slow growth) | $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)=\frac{\mathbb{C} Q^{\mathbb{L}}}{\left(\partial_{x_{e} W} W\right)}$ | $\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right)=\frac{B \widehat{\mathbb{Q}} Q^{\mathbb{L}}}{\left(\partial_{x_{e}} W\right)}$ |
| Potential | $\ell \in \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$ | $\ell_{q} \in \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$ |
| (slow growth) | $\ell \in \operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$ | $\ell_{q} \in \operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right)$ |

Table 20.1: Terminology of deformed Cho-Hong-Lau construction

### 20.3 Deformed Landau-Ginzburg model

In this section, we start deforming the construction of Cho, Hong and Lau. The idea is as easy as it can get: We apply the Cho-Hong-Lau construction formally to the whole formal family of categories $\left(\mathcal{C}_{q}\right)$. Intuitively, we obtain a family of matrix factorization categories. More precisely, we get a new algebra $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$, a new central element $\ell_{q} \in Z\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)\right)$, a deformation $\operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right)$ of $\operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) \ell\right)$ and a functor $\widehat{F}_{q}$ running from $\mathcal{C}_{q}$ to $\operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right)$. In other words: Thanks to the generality the Cho-Hong-Lau construction, the mirror gets deformed when the input category $\mathcal{C}$ gets deformed.

In the present section, we devote ourselves to the construction of $W_{q}, \ell_{q}$ and the Jacobi algebra $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$. In section 20.5, we define the deformed matrix factorizations category and in section 20.6 we construct our deformed mirror functor. We document our setup in the following convention:
Convention 20.22. The category $\mathcal{C}$, the reference objects $\mathbb{L}=\left\{L_{1}, \ldots, L_{N}\right\}$, the basis elements $X_{e}, Y_{e}$, $\mathrm{id}_{L_{i}}^{*}$ and the pairing $\langle-,-\rangle$ are as in Convention 20.6 Let $\mathcal{C}_{q}$ be a deformation of $\mathcal{C}$ over a deformation base $(B, \mathfrak{m})$. We assume $\mathcal{C}_{q}$ is (deformed) cyclic on odd elements:

$$
\left\langle\mu_{q}\left(X_{e_{k+1}}, \ldots, X_{e_{2}}\right), X_{e_{1}}\right\rangle=\left\langle\mu_{q}\left(X_{e_{k}}, \ldots, X_{e_{1}}\right), X_{e_{k+1}}\right\rangle .
$$

We assume that $\mathcal{C}_{q}$ is strictly unital with the same identities $\mathrm{id}_{X}$ as in $\mathcal{C}$ :

$$
\mu_{q}^{1}\left(\mathrm{id}_{X}\right)=0, \quad \mu_{q}^{2}\left(a, \mathrm{id}_{X}\right)=a, \quad \mu_{q}^{2}\left(\operatorname{id}_{X}, a\right)=(-1)^{|a|} a, \quad \mu_{q}^{\geq 3}\left(\ldots, \mathrm{id}_{X}, \ldots\right)=0 .
$$

The letter $\mathbb{L}_{q}$ denotes the subcategory of $\mathcal{C}_{q}$ given by the objects $L_{1}, \ldots, L_{N} \in \mathbb{L}$.
In overview, our construction of the deformed Landau-Ginzburg model $\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right)$ proceeds as follows: We model the algebra $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$ still via the CHL quiver $Q^{\mathbb{L}}$ and relations. More precisely, we regard the enlarged version $B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$ instead of $\widehat{\mathbb{C} Q^{\mathbb{L}}}$. The deformed superpotential $W_{q}$ lives in this enlarged algebra instead of $\widehat{\mathbb{C} Q^{\mathbb{L}}}$. The deformed Jacobi algebra $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$ is defined as quotient of $B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$ instead of $\widehat{\mathbb{C} Q^{\mathbb{L}}}$. The deformed potential $\ell_{q}$ lives in the center of $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$. We develop in parallel a variant in case $\mathbb{L}$ is of bounded type. For the variant in case of bounded type, we need an additional condition on the deformed products $\mu_{q}$. We refer to this condition as being of slow growth, and the condition already implies that $\mathbb{L}$ is of bounded growth. All terminology is collected in Table 20.1
Definition 20.23. $\mathbb{L}_{q}$ is of slow growth if for all morphisms $m_{1}, \ldots, m_{k}$ in $\mathcal{C}$ and for every $n \in \mathbb{N}$ there exists an $l_{0} \in \mathbb{N}$ such that

$$
\forall l \geq l_{0}: \quad \mu_{q}^{k+l}\left(m_{k}, \ldots, m_{1}, b, \ldots, b\right) \in \mathfrak{m}^{n} \operatorname{Hom}\left(\mathbb{L}, X_{k+1}\right) .
$$

Remark 20.24. If $\mathbb{L}_{q}$ is of slow growth, then $\mathbb{L}$ is automatically of bounded growth.
We are now ready to define the deformed relations $R_{q, e}$ and potential $\ell_{q}$. As the reader may expect, these are simply read off from $\mu_{q}(b, \ldots, b)$.

Definition 20.25. The deformed relations $R_{q, e} \in B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$ and the deformed potential are defined by

$$
\begin{equation*}
\sum_{k \geq 0} \mu_{q}^{k}(b, \ldots, b)=\ell_{q} \mathrm{id}+\sum_{i, j=1}^{N} \sum_{e \in E_{i j}} R_{q, e} Y_{e} \tag{20.14}
\end{equation*}
$$

The deformed superpotential is defined as

$$
W_{q}=\left\langle\sum_{k \geq 0} \mu_{q}^{k}(b, \ldots, b), b\right\rangle \in B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}} .
$$

The deformed Jacobi algebra is defined as

$$
\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)=\frac{B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}}{\left(\partial_{x_{e}} W\right)^{\otimes}} .
$$

The deformed Landau-Ginzburg model is the pair $\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right)$. If $\mathbb{L}_{q}$ is of slow growth, then $R_{q, e}, \ell_{q}, W_{q}$ are regarded as elements of $B \widehat{\otimes} \mathbb{C} Q^{\mathbb{L}}$, the deformed Jacobi algebra is defined as $\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right)=$ $\left(B \widehat{\otimes} \mathbb{C} Q^{\mathbb{L}}\right) / \overline{\left(\partial_{x_{e}} W\right)}$, and the deformed Landau-Ginzburg model is $\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right)$.

Remark 20.26. The element $W_{q} \in B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$ is cyclic, as we shall see in Lemma 20.30. Cyclicity for $W_{q}$ is to be understood in the sense that $W_{q} \in B \widehat{\otimes}{\widehat{\mathbb{C}} Q^{\mathbb{L}}}^{\text {cyc }}$, where ${\widehat{\mathbb{C}}{ }^{\mathbb{L}}}^{\text {cyc }} \subseteq \subseteq{\widehat{\mathbb{C}} Q^{\mathbb{L}}}$ is the subspace of elements invariant under cyclic permutation. More explicitly, $W_{q}$ is of the form $\sum m_{i} p_{i}$ with $m_{i} \in \mathfrak{m} \rightarrow \infty$ and $p_{i} \in \widehat{\mathbb{C} Q^{\mathbb{L}}}$ cyc . This explains the sense in which $W_{q}$ is cyclic.

The derivatives $\partial_{x_{e}} W_{q}$ are elements of $B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$. More precisely, the derivative $\partial_{x_{e}} W_{q}$ is defined as $\lim _{k} \partial_{x_{e}} \pi_{k}\left(W_{q}\right)$ where $\pi_{k}$ denotes the projection $\pi_{k}: B \widehat{\otimes}{\widehat{\mathbb{C}}{ }^{\mathbb{L}}}^{\text {cyc }}$. $\rightarrow B / \mathfrak{m}^{k} \otimes \widehat{\mathbb{C} Q^{\mathbb{L}}}{ }_{\text {cyc }}$. More explicitly, if $W_{q}=\sum m_{k} p_{k}$ with $m_{k} \in \mathfrak{m}^{\rightarrow \infty}$ and $p_{k} \in \widehat{\mathbb{C Q}^{\mathbb{L}}}{ }_{\text {cyc }}$, then $\partial_{x_{e}} W_{q}=\sum m_{k} \partial_{x_{e}} p_{k}$. This explains how to understand $\partial_{x_{e}} W_{q}$.

Remark 20.27. The definition of the relations $R_{q, e}$ is understood as follows: The chunk of $R_{q, e}$ read off from $\mu_{q}^{k}(b, \ldots, b)$ for a certain $k \geq 0$ consists of paths in $Q^{\mathbb{L}}$ of length $k$, weighted with deformation parameters. More precisely, this chunk lies in $B \widehat{\otimes} \mathbb{C} Q_{k}^{\mathbb{L}}$. The total relation $R_{q, e}$ is the sum of these chunks over $k \geq 0$. As such, the deformed relations $R_{q, e}$ all lie in $B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$. As in the classical case, the relation $R_{q, e}$ only contain paths running from $h(e)$ to $t(e)$ for every $e \in E_{i j}$.

Similarly, the potential $\ell_{q}$ lies in $B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$. We shall from now on typically denote by $\ell_{q}$ its projection to $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$. As in the classical case, the only paths contained in $\ell_{q}$ are loops of $Q^{\mathbb{L}}$.

Remark 20.28. In contrast to the classical case, the category $\mathcal{C}_{q}$ may also have infinitesimal curvature. This is not a problem. We simply start counting from $k=0$ instead of $k=1$ in 20.14.

Remark 20.29. Within the present section 20 we have decided to stick to the following closure notations: If $X \subseteq \widehat{\mathbb{C} Q^{\mathbb{L}}}$, then $\bar{X}$ is the closure with respect to the Krull topology. If $X \subseteq B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$ or $X \subseteq B \widehat{\otimes} \mathbb{C} Q^{\mathbb{L}}$, then $\bar{X}$ denotes the closure with respect to the m-adic topology. If $X \subseteq B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$, then $\bar{X}^{\otimes}$ denotes the closure with respect to the tensor topology. For more information on the tensor topology, we refer to section 19.9

In the rest of this section, we check properties of the deformed Landau-Ginzburg model. Our first milestone is a deformed version of Lemma 20.16
Lemma 20.30. Assume Convention 20.22 Then the superpotential $W_{q} \in B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$ is cyclic and we have $R_{q, e}=\partial_{x_{e}} W_{q}$. The potential $\ell_{q} \in \operatorname{Jac}\left(\widetilde{Q^{\mathbb{L}}}, W_{q}\right)$ is central. The analogous statements hold if $\mathbb{L}_{q}$ is of slow growth.

Proof. The proof is analogous to the proof of the classical version Lemma 20.16. Cautious is due when working with the completed ideals. We shall therefore spell out a few details. By comparison, the fact that $\mathcal{C}_{q}$ is allowed to have curvature is technically unproblematic.

Cyclicity of $W_{q}$ and the property that $R_{q, e}=\partial_{x_{e}} W_{q}$ follow immediately from the cyclicity assumption in Convention 20.22 and are unproblematic in the sense that these properties hold in $B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$.

We shall rather comment in detail on the centrality of the deformed potential $\ell_{q}$ : Within $B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}} \otimes$ $\operatorname{Hom}(\mathbb{L}, \mathbb{L})$, we have

$$
\begin{align*}
0 & =\sum_{k \geq 0} \sum_{0 \leq l \leq k} \mu_{q}^{k-l+1}\left(b, \ldots, \mu_{q}^{l}(b, \ldots, b), b, \ldots, b\right) \\
& =\sum_{k \geq 1} \sum_{l \geq 0} \mu_{q}^{k}\left(b, \ldots, \mu_{q}^{l}(b, \ldots, b), \ldots, b\right)  \tag{20.15}\\
& =\sum_{k \geq 1} \mu_{q}^{k}\left(b, \ldots, \sum_{i=1}^{N} \ell_{q, i} i_{L_{i}}+\sum R_{e} Y_{e}, \ldots, b\right) .
\end{align*}
$$

In the first row, we have used the curved $A_{\infty}$-relations. In the second row, we have applied a reordering of the double sum with increasing path length. In the third row, we have reproduced 20.14. We claim that up to the two terms $\mu_{q}^{2}\left(b, \ell_{q, i} \operatorname{id}_{L_{i}}\right)$ and $\mu_{q}^{2}\left(\ell_{q, i} \operatorname{id}_{L_{i}}, b\right)$, the entire sum lies in $\overline{\operatorname{span}\left(\partial_{x_{e}} W_{q}\right)} \otimes \otimes \operatorname{Hom}(\mathbb{L}, \mathbb{L})$. Indeed, for every $k \geq 3$ and for $k=1$ the summand lies in the intersection of $B \widehat{\otimes} \mathbb{C} Q^{\mathbb{L}} \operatorname{span}\left(R_{e}\right) \mathbb{C} Q^{\mathbb{L}}$ and $B \widehat{\otimes} \mathbb{C} Q_{\geq k-1}^{\mathbb{L}}$. Therefore the series converges, apart from the two special summands, to an element of $\overline{\operatorname{span}\left(\partial_{x_{e}} W_{q}\right)}{ }^{\otimes} \otimes \operatorname{Hom}(\mathbb{L}, \mathbb{L})$. Within $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right) \otimes \operatorname{Hom}(\mathbb{L}, \mathbb{L})$, we conclude that

$$
0=\mu_{q}^{2}\left(b, \ell_{q} \mathrm{id}\right)+\mu_{q}^{2}\left(\ell_{q} \mathrm{id}, b\right)
$$

Let $1 \leq i, j \leq N$ and $e \in E_{i j}$. Then extracting the $X_{e}$-component gives

$$
0=x_{e} \ell_{q}-\ell_{q} x_{e} \text { within } \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)
$$

We conclude that $\ell$ commutes with all arrows in $Q^{\mathbb{L}}$, hence with all finite paths and the image of $B \otimes \mathbb{C} Q^{\mathbb{L}}$. Let now $x \in B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}}$ be an arbitrary element. We shall prove that $\pi(x) \ell=\ell \pi(x)$, where $\pi: B \widehat{\otimes} \widehat{\mathbb{C} Q^{\mathbb{L}}} \rightarrow \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$ is the projection. The easiest way to achieve this is by approximating $x$ by finite paths via the tensor topology. Pick any sequence $\left(x_{n}\right) \subseteq B \otimes \mathbb{C} Q^{\mathbb{L}}$ such that $x_{n} \rightarrow x$ in the tensor topology. Since $\overline{\left(\partial_{x_{e}} W_{q}\right)}{ }^{\otimes}$ is closed with respect to the tensor topology, the projection $\pi$ is continuous. We obtain

$$
0=\pi\left(x_{n}\right) \ell-\ell \pi\left(x_{n}\right) \rightarrow \pi(x) \ell-\ell \pi(x) \text { within } \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right) .
$$

We conclude that $\ell$ commutes with any element $x \in \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$ and hence $\ell \in Z\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)\right)$. This finishes the third part.

The fourth part of the proof consists of observing that the calculations still hold in case $\mathbb{L}_{q}$ is of slow growth. Indeed, in this case the double sums in the calculation 20.15 all become finite and apart from the two special terms lie in $\overline{\operatorname{span}\left(\partial_{x_{e}} W_{q}\right)}$, noting that the closure is taken this time only with respect to the $\mathfrak{m}$-adic topology. The approximation of $x \in B \widehat{\otimes} \mathbb{C} Q^{\mathbb{L}}$ happens by a sequence $x_{n} \in B \otimes \mathbb{C} Q^{\mathbb{L}}$ which converges to $x$ in the $\mathfrak{m}$-adic topology. This proves the fourth step and finishes the proof.

### 20.4 Projectives of deformed algebras

In this section, we show that projectives of an algebra and any deformation are in one-to-one correspondence. The purpose of this section is to build a rigorous foundation for our deformed category of matrix factorizations.

$$
\text { Projectives of } A_{q} \quad \longleftrightarrow \quad \text { Projectives of } A
$$

The core idea of matching projectives is as follows: Projectives of $A$ are direct $A$-module summands of a free $A$-module. Given such a decomposition $A^{\oplus n}=P_{1} \oplus \ldots P_{k}$ into projectives, intuition says that the modules $P_{i}$ can be extended to $A_{q}$-modules $P_{q, i}$ such that $A_{q}^{\oplus n}=P_{q, 1} \oplus \ldots \oplus P_{q, k}$. The extensions $P_{q, i}$ are direct summands of $A_{q}^{\oplus n}$ and therefore automatically projective. This procedure allows us to turn projectives of $A$ into projectives of $A_{q}$. Conversely, our expectation is that a projective module $P_{q}$ of $A_{q}$ gives rise to a projective $P$ of $A$ by merely setting $P=P_{q} / \mathfrak{m} P_{q}$. This allows us to match projectives of $A_{q}$ with projectives of $A$.

If $P$ is a projective of $A$, then $B \widehat{\otimes} P$ is not a projective of $A_{q}$. Instead, the module needs to be tweaked a little. This is best accomplished by realizing $P$ as image of an $A$-linear idempotent $p: A^{\oplus n} \rightarrow A^{\oplus n}$ and lifting $p$ to an $A_{q}$-linear idempotent $p_{q}: A_{q}^{\oplus n} \rightarrow A_{q}^{\oplus n}$. We achieve this by an adaption of the classical idempotent lifting argument to the case of formal deformations:

Lemma 20.31. Let $A$ be an algebra and $A_{q}=\left(B \widehat{\otimes} A, \mu_{q}\right)$ a deformation. Let $p: A^{\oplus n} \rightarrow A^{\oplus n}$ be an $A$-linear idempotent in the sense that $p^{2}=p$. Then there exists an $A_{q}$-linear idempotent $p_{q}: A_{q}^{\oplus n} \rightarrow A_{q}^{\oplus n}$ with leading term $p$.

Proof. The idea is to lift $p$ to an arbitrary $A_{q}$-module map and then turn it iteratively into a projection. As a first step, let $p_{0}: A_{q}^{\oplus n} \rightarrow A_{q}^{\oplus n}$ be any $A_{q}$-linear continuous lift of $p$, for instance given by setting $p_{0}\left(e_{i}\right)=p\left(e_{i}\right) \in A^{\oplus n} \subseteq A_{q}^{\oplus n}$ where $e_{1}, \ldots, e_{n}$ are the basis vectors of $A_{q}^{\oplus n}$.

As a second step, we construct a sequence of $A_{q}$-module maps $p_{k}: A_{q}^{\oplus n} \rightarrow A_{q}^{\oplus n}$ such that $p_{k}^{2}-p_{k} \in$ $\mathfrak{m}^{2^{k}} \operatorname{End}\left(A_{q}^{\oplus n}\right)$ and $p_{k+1}-p_{k} \in \mathfrak{m}^{2^{k}} \operatorname{End}\left(A_{q}^{\oplus n}\right)$. The first item in the sequence is the already constructed map $p_{0}$. Assume for induction that the sequence has already been constructed up to $p_{k}$. Then put $\varepsilon=p_{k}^{2}-p_{k}$. By induction hypothesis, $\varepsilon: A_{q}^{\oplus n} \rightarrow A_{q}^{\oplus n}$ is an $A_{q}$-module map which lies in $\mathfrak{m}^{2^{k}} \operatorname{End}\left(A_{q}^{\oplus n}\right)$.

The trick is to set $p_{k+1}=p_{k} \pm \varepsilon$ in order to render $p_{k+1}^{2}-p_{k+1}$ of lesser order than $p_{k}^{2}-p_{k}=\varepsilon$. The correct sign is neither plus or minus in general, but differs for two parts of $\varepsilon$. More precisely, we split $\varepsilon$ into a part mapping to the image of $p_{k}$ and a part almost mapping to the kernel of $p_{k}$ :

$$
\varepsilon=p_{k} \circ \varepsilon+\left(p_{k}-\mathrm{id}\right) \circ \varepsilon
$$

The right sign for the first part is -1 , while the sign for the second part is +1 . In consequence, we put

$$
p_{k+1}=p_{k}-p_{k} \circ \varepsilon+\left(\mathrm{id}-p_{k}\right) \circ \varepsilon
$$

Since $p_{k}$ is a $A_{q}$-linear, so is $p_{n+1}$. Noting that $\varepsilon$ commutes with $p_{k}$ by definition, we get

$$
\begin{aligned}
p_{k+1}^{2}-p_{k+1} & =\left(p_{k}-p_{k} \varepsilon+\left(\mathrm{id}-p_{k}\right) \varepsilon\right)^{2}-\left(p_{k}-p_{k} \varepsilon+\left(\mathrm{id}-p_{k}\right) \varepsilon\right) \\
& =\left(p_{k}^{2}-p_{k}\right)-2 p_{k}^{2} \varepsilon+2\left(p_{k}-p_{k}^{2}\right) \varepsilon+p_{k} \varepsilon-\left(\mathrm{id}-p_{k}\right) \varepsilon+\mathcal{O}\left(\varepsilon^{2}\right) \\
& =p_{k} \varepsilon+\left(\mathrm{id}-p_{k}\right) \varepsilon-2 p_{k}^{2} \varepsilon+2\left(p_{k}-p_{k}^{2}\right) \varepsilon+p_{k} \varepsilon-\left(\mathrm{id}-p_{k}\right) \varepsilon+\mathcal{O}\left(\varepsilon^{2}\right) \\
& =4\left(p_{k}-p_{k}^{2}\right) \varepsilon+\mathcal{O}\left(\varepsilon^{2}\right)=\mathcal{O}\left(\varepsilon^{2}\right)
\end{aligned}
$$

In the calculation, we have written $\mathcal{O}\left(\varepsilon^{2}\right)$ for any expressions containing an $\varepsilon^{2}$ factor. We conclude that $p_{k+1} \in\left(\mathfrak{m}^{2^{k}}\right)^{2} \operatorname{End}\left(A_{q}^{\oplus n}\right)=\mathfrak{m}^{2^{k+1}} \operatorname{End}\left(A_{q}^{\oplus n}\right)$. This finishes the construction of $p_{k+1}$ and thereby the inductive construction of the sequence $\left(p_{k}\right)$.

Finally, the sequence $\left(p_{k}\right)$ converges to a limit map $p_{q}: A_{q}^{\oplus n} \rightarrow A_{q}^{\oplus n}$. It can be explicitly described as limit of $p_{k}$ in every matrix entry, where matrix representation is taken with respect to the direct sum description $A_{q}^{\oplus n}$. We conclude that $p_{q}$ is an $A_{q}$-linear map with $p_{q}^{2}=p_{q}$. Since $x_{k+1}-x_{k} \in \mathfrak{m} \operatorname{End}\left(A_{q}^{\oplus n}\right)$ for $k \geq 0$, the map $p_{q}$ reduces to $p$ modulo $\mathfrak{m}$. This finishes the proof.

Let $P \subseteq A_{q}^{\oplus n}$ be a submodule, not necessarily projective. Regard the projection map $\pi: B \widehat{\otimes} A \rightarrow A$. Then $\pi(P) \subseteq A^{\oplus n}$ is an $A$-submodule of $A^{\oplus n}$. To avoid confusion, let us write $\mu_{q}$ for the deformed product of $\bar{A}_{q}$ and $\mu$ for the product of $A$. In these terms, we have for $a \in A$ and $x \in B \widehat{\otimes} A^{\oplus n}$ that $\mu(a, \pi(x))=\pi\left(\mu_{q}(a, x)\right)$. Now if $x \in P$, then $\mu_{q}(a, x) \in P$ and we get $\mu(a, \pi(x))=\pi\left(\mu_{q}(a, x)\right) \in \pi(P)$. This shows that $\pi(P)$ is an $A$-submodule. With this observation in mind, we can make the following statement:

Lemma 20.32. Let $P \subseteq A_{q}^{\oplus n}$ be a projective module. Then the $A$-module $\pi(P)$ is projective.
Proof. Write $A_{q}^{\oplus n}=P \oplus Q$ for some $A_{q}$-module $Q$. As kernels of projections, both are automatically closed with respect to the $\mathfrak{m}$-adic topology, in particular pseudoclosed. According to Proposition 15.29, we get that $A^{\oplus n}=\pi(P) \oplus \pi(Q)$ as vector spaces. As observed before, both $\pi(P)$ and $\pi(Q)$ are actually $A$-submodules of $A^{\oplus n}$. We conclude that $\pi(P)$ is a projective $A$-module. This finishes the proof.

The endomorphism space $\operatorname{Hom}_{A_{q}}\left(A_{q}^{\oplus n}, A_{q}^{\oplus n}\right)$ is the same as $A_{q}^{\oplus n}=B \widehat{\otimes} A^{\oplus n}$ as $B$-module. If $A_{q}^{\oplus n}=$ $P_{1} \oplus \ldots \oplus P_{k}$ is a decomposition into $A_{q}$-submodules, then every hom space $\operatorname{Hom}_{A_{q}}\left(P_{i}, P_{j}\right)$ can be naturally interpreted as $B$-linear subspace of $B \widehat{\otimes} A^{\oplus n}$. In fact, $B \widehat{\otimes} A^{\oplus n}$ is their direct sum.

Lemma 20.33. Let $A_{q}^{\oplus n}=P_{1} \oplus \ldots \oplus P_{k}$ be a decomposition as $A_{q}$-module. Then $\operatorname{Hom}_{A_{q}}\left(P_{i}, P_{j}\right) \subseteq$ $B \widehat{\otimes} A^{\oplus n}$ is pseudoclosed and quasi-flat, and we have $\pi\left(\operatorname{Hom}_{A_{q}}\left(P_{i}, P_{j}\right)\right)=\operatorname{Hom}_{A}\left(\pi\left(P_{i}\right), \pi\left(P_{j}\right)\right)$.

Proof. We have $A_{q}^{\oplus n^{2}}=\operatorname{Hom}_{A_{q}}\left(A_{q}^{\oplus n}, A_{q}^{\oplus n}\right)=\bigoplus_{i, j} \operatorname{Hom}_{A_{q}}\left(P_{i}, P_{j}\right)$. By Proposition 15.29 we conclude that every $\operatorname{Hom}_{A_{q}}\left(P_{i}, P_{j}\right)$ is quasi-flat and pseudoclosed.

It remains to show that $\pi\left(\operatorname{Hom}_{A_{q}}\left(P_{i}, P_{j}\right)\right)=\operatorname{Hom}_{A}\left(\pi\left(P_{i}\right), \pi\left(P_{j}\right)\right) \subseteq A_{q}^{\oplus n^{2}}$. Pick any $A_{q}$-linear morphism $\varphi: P_{i} \rightarrow P_{j}$. We claim that $\pi(\varphi): \pi\left(P_{i}\right) \rightarrow \pi\left(P_{j}\right)$ is an $A$-module map. Indeed, for $a \in A$ and $x \in P_{i}$ we have

$$
\pi(\varphi)(\mu(a, \pi(x)))=\pi\left(\varphi\left(\mu_{q}(a, x)\right)\right)=\pi\left(\mu_{q}(a, \varphi(x))\right)=\mu(a, \pi(\varphi(x)))=\mu(a, \pi(\varphi)(x))
$$

Conversely, let $\varphi: \pi\left(P_{i}\right) \rightarrow \pi\left(P_{j}\right)$ be an $A$-module morphism. We shall construct an $A_{q}$-module map $\varphi_{q}: P_{i} \rightarrow P_{j}$ such that $\pi\left(\varphi_{q}\right)=\varphi$. The idea is to view the composition $\tilde{\varphi}: A^{\oplus n} \rightarrow \pi\left(P_{i}\right) \rightarrow \pi\left(P_{j}\right) \hookrightarrow A^{\oplus n}$. We can write $\tilde{\varphi}$ as an element of $A^{\oplus n^{2}}$. In particular, we obtain a map of $A_{q}$-modules $\varphi_{q}: P_{i} \hookrightarrow A_{q}^{\oplus n^{2}} \rightarrow$ $A_{q}^{\oplus n^{2}} \rightarrow P_{j}$ with $\pi\left(\varphi_{q}\right)=\varphi$. This shows $\pi\left(\operatorname{Hom}_{A_{q}}\left(P_{i}, P_{j}\right)\right)=\operatorname{Hom}_{A}\left(\pi\left(P_{i}\right), \pi\left(P_{j}\right)\right)$ and finishes the proof.

Let us set up more terminology. We denote by Proj $A$ the category of finitely generated projective $A$-modules and by Proj $A_{q}$ the category of finitely generated $A_{q}$-modules. Both are ordinary unital $\mathbb{C}$ linear categories. We use the terminology of $A_{\infty}$-deformations for these two categories, even though the only product of both is an ordinary associative composition with different sign rule. In terms of Definition 15.41 we shall prove that $\operatorname{Proj} A_{q}$ is a loose object-cloning deformation of $\operatorname{Proj} A$.

The first step is to define the mapping $F:$ Ob Proj $A_{q} \rightarrow$ Ob Proj $A$ as given by $F(P)=P /(\mathfrak{m} \cdot P)$. Obviously, the quotient $P /(\mathfrak{m} \cdot P)$ carries a natural action of $A=A_{q} / \mathfrak{m} A_{q}$, rendering it an $A$-module. This mapping $F$ is the generalization of the mapping $P \mapsto \pi(P)$ to projectives not presented as submodules of free modules. We shall now prove that $F(P)$ is actually projective:

Lemma 20.34. Let $P \in \operatorname{Proj} A_{q}$. Then the $A$-module $F(P)=P /(\mathfrak{m} \cdot P)$ is projective.
Proof. It suffices to regard the case of a submodule $P \subseteq A_{q}^{\oplus n}$. We claim that $P /(\mathfrak{m} \cdot P) \cong \pi(P)$ as $A$-modules. Regard the $A$-linear projection map $\pi: P \rightarrow \pi(P)$. The map is surjective. Its kernel is $P \cap \mathfrak{m} A^{\oplus n}$. Since $P$ is a direct summand of $B \widehat{\otimes} A^{\oplus n}$ as $B$-module, $P \subseteq B \widehat{\otimes} A^{\oplus n}$ is quasi-flat and pseudoclosed. In particular we have $P \cap \mathfrak{m} A^{\oplus n}=\mathfrak{m} P=\mathfrak{m} \cdot P$. We obtain an isomorphism of $A$-modules $P /(\mathfrak{m} \cdot P) \xrightarrow{\sim} \pi(P)$ By Lemma 20.32, $\pi(P)$ is a projective $A$-module and we finish the proof by concluding that $P /(\mathfrak{m} \cdot P) \in \operatorname{Proj} A$.

Proposition 20.35. The category $\operatorname{Proj} A_{q}$ is an essentially surjective loose object-cloning deformation of $\operatorname{Proj} A$ over $B$ via the mapping $F: \operatorname{Ob} \operatorname{Proj} A_{q} \rightarrow \operatorname{Ob} \operatorname{Proj} A$.

Proof. The proof consists of two steps: First we show that $F$ is essentially surjective. Second, we show that $\operatorname{Proj} A_{q}$ becomes an object cloning deformation of $\operatorname{Proj} A$.

To see that $F$ is essentially surjective, let $P \in \operatorname{Proj} A$ be any finitely generated projective $A$ module. There exists an isomorphic module $P^{\prime}$ and a decomposition $A^{\oplus n}=P^{\prime} \oplus Q$ for some $n \in \mathbb{N}$ and some module $Q$. Pick a lift $A_{q}^{\oplus n}=P_{q}^{\prime} \oplus Q_{q}$. Then $P_{q}^{\prime} /\left(\mathfrak{m} \cdot P_{q}^{\prime}\right) \cong P$, since $\mathfrak{m} P_{q}^{\prime}=\mathfrak{m} \cdot P_{q}^{\prime}$. This shows that $P$ is reached by $F$ up to isomorphism. In other words, $F$ is essentially surjective.

To see that $\operatorname{Proj} A_{q}$ is a loose object-cloning deformation of $\operatorname{Proj} A$, we have to provide linear isomorphisms

$$
\psi_{P, Q}: \operatorname{Hom}_{A_{q}}(P, Q) /\left(\mathfrak{m} \cdot \operatorname{Hom}_{A_{q}}(P, Q)\right) \xrightarrow{\sim} \operatorname{Hom}_{A}(F(P), F(Q)) \quad \text { for } P, Q \in \operatorname{Proj} A_{q} .
$$

Let $\varphi \in \operatorname{Hom}_{A_{q}}(P, Q)$. Then $\varphi$ descends to a map $F(P) \rightarrow F(Q)$. We define $\psi_{P, Q}(\varphi)$ as this descended map. We need to check that $\psi_{P, Q}$ is an isomorphism. For this, we note that $\psi_{P, Q}$ is defined purely in terms of the module structure of $P$ and $Q$. It therefore suffices to show bijectivity of $\psi_{P, Q}$ only in case $P$ and $Q$ are embedded as submodules of a free module $A_{q}^{\oplus n}$. In this case, the hom space $\operatorname{Hom}_{A_{q}}(P, Q)$ has an interpretation as pseudoclosed quasi-flat $B$-submodule of $B \widehat{\otimes} A^{\oplus n^{2}}$. The map $\psi_{P, Q}$ is in this case the map $\operatorname{Hom}_{A_{q}}(P, Q) /\left(\mathfrak{m} \cdot \operatorname{Hom}_{A_{q}}(P, Q)\right) \rightarrow \operatorname{Hom}_{A}(\pi(P), \pi(Q))$ simply induced from the projection to zeroth order $\operatorname{Hom}_{A_{q}}(P, Q) \rightarrow \operatorname{Hom}_{A}(\pi(P), \pi(Q))$. The map $\psi_{P, Q}$ is surjective since $\operatorname{Hom}_{A}(\pi(P), \pi(Q))=\pi\left(\operatorname{Hom}_{A_{q}}(P, Q)\right)$ by Lemma 20.33 and injective since $\operatorname{Hom}_{A_{q}}(P, Q)$ is quasi-flat. This shows that $\psi_{P, Q}$ is an isomorphism.

Finally, we have to check that composition in the category Proj $A_{q}$ reduces to composition in Proj $A$ via $\left\{\psi_{P, Q}\right\}_{P, Q}$ once $\mathfrak{m}$ is divided out. This is however immediate, since composition commutes with descending to the quotient by $\mathfrak{m}$. This finishes the proof.

Remark 20.36. The correspondence between projectives of $A_{q}$ and $A$ can typically be made more explicit. Assume for instance, an element $v \in A$ is an idempotent in the sense that $v^{2}=v$. Then $A v$ is a projective of $A$ since $A v$ is the image of the $A$-linear idempotent map $(-) v: A \rightarrow A$. Moreover, if $v^{2}=v$ within $A_{q}$, then $A_{q} v$ is a projective of $A_{q}$, since it is the image of the idempotent $A_{q}$-linear map $(-) v: A_{q} \rightarrow A_{q}$.

The $A$-projective $F\left(A_{q} v\right)=A_{q} v /\left(\mathfrak{m} \cdot A_{q} v\right)$ can be naturally identified with $A v$ via the map $\varphi$ : $A_{q} v /\left(\mathfrak{m} \cdot A_{q} v\right) \xrightarrow{\sim} A v$ given by $\varphi(a)=\pi(a) v$ for $a \in A_{q} v \subseteq B \widehat{\otimes} A$, where $\pi: B \widehat{\otimes} A \rightarrow A$ is the standard projection. Since $\mu_{q}$ is a deformation of $\mu$, we have $\varphi\left(\mu_{q}(a, b)\right)=\pi\left(\mu_{q}(a, b)\right) v=\mu(\pi(a), \pi(b)) v$ and conclude that $\varphi$ is $A$-linear. The map $\varphi$ is surjective since for $a \in A$ we have $\varphi(a)=a v$. The map $\varphi$ is injective, since $\pi(a) v=0$ for $a \in A_{q} v$ implies $\pi\left(\mu_{q}(a, v)\right)=0$, hence $\mu_{q}(a, v) \in \mathfrak{m} \cap A_{q} v \subseteq \mathfrak{m} \cdot A_{q} v$, since $A_{q} v \subseteq B \widehat{\otimes} A$ is quasi-flat.

Hom spaces between projectives can also be identified. Let $v, w$ be two idempotents of $A$ still idempotent in $A_{q}$. Then we have the natural identification

$$
\frac{\operatorname{Hom}_{A_{q}}\left(A_{q} v, A_{q} w\right)}{m \cdot \operatorname{Hom}_{A_{q}}\left(A_{q} v, A_{q} w\right)} \xrightarrow{\sim} \operatorname{Hom}_{A}\left(F\left(A_{q} v\right), F\left(A_{q} w\right)\right) \xrightarrow{\sim} \operatorname{Hom}_{A}(A v, A w) .
$$

### 20.5 Deformed matrix factorizations

In this section, we define deformed categories of matrix factorizations. There are several issues on the road to their definition. In order to satisfy our demand to serve as mirror model in the deformed Cho-Hong-Lau construction, the definition furthermore needs to deviate from what one expects. The first step in this section is to explain these issues. We then provide a successful construction of the deformed category of matrix factorizations for any deformed Landau-Ginzburg model. We finish this section with an explanation how this category becomes an deformation of the ordinary category of matrix factorizations.

Our starting point is a pair $(A, \ell)$ consisting of an associative algebra $A$ with a central element $\ell \in A$. Recall from section 17.4 that the category of matrix factorizations $\operatorname{MF}(A, \ell)$ is a dg category which has as objects pairs of finitely generated projective $A$-modules $M, N$ together with $A$-module morphisms $f: M \rightarrow N$ and $g: N \rightarrow M$ such that $f \circ g=\ell \operatorname{id}_{N}$ and $g \circ f=\ell \mathrm{id}_{M}$.

The starting point for the deformed setup is a pair $\left(A_{q}, \ell_{q}\right)$ of a deformation $A_{q}$ of $A$ and a deformed central element $\ell_{q} \in Z\left(A_{q}\right)$. More precisely, $A_{q}=\left(B \widehat{\otimes} A, \mu_{q}\right)$ shall be an associative deformation of the algebra $A$ over a deformation base $B$, in the sense that $\mu_{q}:(B \widehat{\otimes} A) \otimes(B \widehat{\otimes} A) \rightarrow B \widehat{\otimes} A$ is a $B$-linear associative product that reduces to the product $\mu: A \otimes A \rightarrow A$ once $\mathfrak{m}$ is divided out. The element $\ell_{q} \in Z\left(A_{q}\right)$ is required to be a deformation of $\ell$ in the sense that $\ell_{q}-\ell \in \mathfrak{m} A$. We fix this terminology as follows:

Definition 20.37. A Landau-Ginzburg model $(A, \ell)$ is a pair of an associative algebra and a central element. A deformation $\left(A_{q}, \ell_{q}\right)$ of $(A, \ell)$ consists of an algebra deformation $A_{q}=\left(B \widehat{\otimes} A, \mu_{q}\right)$ of $A$ together with a central element $\ell_{q} \in Z\left(A_{q}\right)$ which is a deformation of $\ell$. Disregarding the reference to $(A, \ell)$, we may call $\left(A_{q}, \ell_{q}\right)$ a deformed Landau-Ginzburg model.

The following is a naive candidate for a deformed matrix factorization category: Objects are projective $A_{q}$-modules $M, N$ together with $A_{q}$-module maps $f: M \rightarrow N$ and $g: N \rightarrow M$ such that $f \circ g=$ $\ell_{q} \mathrm{id}_{N}$ and $g \circ f=\ell_{q} \mathrm{id}_{M}$. The hom space of two such matrix factorizations $f: M \rightleftarrows N: g$ and $f^{\prime}: M^{\prime} \rightleftarrows N^{\prime}: g^{\prime}$ would be defined as $\operatorname{Hom}_{A_{q}}\left(M, M^{\prime}\right) \oplus \ldots$ similar to the classical case. The differential $\mu^{1}$ on this category would be given by commuting a morphism with $f, g$ and $f^{\prime}, g^{\prime}$ as in the classical case. The product $\mu^{2}$ would be given by standard matrix composition. This defines a dg category and a naive candidate for a deformed category of matrix factorizations.

Let us now sketch the issues associated with this naive definition. The leading question is how to interpret this category as a deformation of $\operatorname{MF}(A, \ell)$, both on object and morphism level.

The first issue consists of matching projective modules of $A_{q}$ with projective modules of $A$. Even more, we also need that the hom spaces between projective modules of the two kinds match. We have resolved this in section 20.4

The second issue consists of matching the factorization morphisms $f, g$ for $\ell_{q}$ with factorization morphisms for $\ell$. In fact, multiple matrix factorizations of $\left(A_{q}, \ell_{q}\right)$ reduce to the same matrix factorization of $(A, \ell)$. A simple example is to multiply $f$ by an element $1+\varepsilon$ with $\varepsilon \in \mathfrak{m}$ and multiply $g$ by $(1+\varepsilon)^{-1}=\sum(-\varepsilon)^{i}$. Conversely, it is unclear whether every matrix factorization of $(A, \ell)$ extends to a matrix factorization of $\left(A_{q}, \ell_{q}\right)$. This makes a precise correspondence between matrix factorizations of $(A, \ell)$ and of $\left(A_{q}, \ell_{q}\right)$ impossible. We resolve this by building $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ as an object-cloning deformation of $\operatorname{MF}(A, \ell)$.

The third issue consists of liberalizing the category $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ enough so that it can serve as codomain of the deformed mirror functor. As we shall see, our deformed mirror functor does not map to objects $(M, N, f, g)$ where all compositions $f \circ g$ and $g \circ f$ are equal to each other. Instead, the compositions will only agree with the predefined central element $\ell_{q}$ on zeroth order and differ per object. This phenomenon is inevitable when starting form a curved $A_{\infty}$-deformation. It requires us to admit objects very liberally into the category $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$.

We resolve the third issue as follows: The objects of $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ are pairs $(M, N, f, g)$ of projective modules and module maps, but $f \circ g$ and $g \circ f$ need only equal $\ell_{q}$ up to zeroth order. The difference of $f \circ g$ and $\ell_{q}$, and of $g \circ f$ and $\ell_{q}$, serves as curvature of $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$. Since our deformed mirror functor typically requires the codomain to carry curvature as well, this resolves the third issue sufficiently.

At this point, we stress the crucial importance of $A_{q}$ being a deformation of $A$ in the sense of Definition 19.1 Intuitively, if $A_{q}$ is smaller than $B \widehat{\otimes} A$, then its modules have smaller hom spaces as well, which breaks all chances to make the category of matrix factorizations of $\left(A_{q}, \ell_{q}\right)$ a deformation of $\operatorname{MF}(A, \ell)$.

Definition 20.38. Let $\left(A_{q}, \ell_{q}\right)$ be a deformed Landau-Ginzburg model. A deformed matrix factorization of $\left(A_{q}, \ell_{q}\right)$ consists of two finitely generated projective $A_{q}$-modules $P, Q$ together with $A_{q}$-module maps $f: P \rightarrow Q$ and $g: Q \rightarrow P$ such that $f \circ g-\ell_{q} \operatorname{id}_{Q} \in \mathfrak{m} \cdot \operatorname{Hom}_{A_{q}}(Q, Q)$ and $g \circ f-\ell_{q} \operatorname{id}_{P} \in$ $\mathfrak{m} \cdot \operatorname{Hom}_{A_{q}}(P, P)$.

Example 20.39. Let $A=\mathbb{C}[X, Y]$ and regard the trivial deformation $A_{q}=(A \llbracket q \rrbracket, \mu)$. Regard the central element $\ell=X Y$ and the deformed central element $\ell_{q}=X Y+q$. The object $(A, A, X, Y)$ is a matrix factorization of $(A, \ell)$. Both $\left(A_{q}, A_{q}, X, Y\right)$ and $\left(A_{q}, A_{q}, X+5 q, Y+q X\right)$ are deformed matrix factorizations of $\left(A_{q}, \ell_{q}\right)$. Both however do not factor to $\ell_{q}$ precisely, but only on zeroth order. In fact, there are no single elements $f, g \in A_{q}$ such that $f g=X Y+q$ and $f-X \in(q)$ and $g-Y \in(q)$. To see this, assume $f=X+q z$ and $g=Y+q w$, then $f g=X Y+q(z Y+X w)+q^{2} z w$. In order for this to be equal to $\ell_{q}=X Y+q$, we need that $z Y+X w$ is 1 on first order, which is impossible. This shows that matrix factorizations need not have strict lifts, but deformed matrix factorizations are rather abundant.

Remark 20.40. As in the classical case, the pair of modules $(P, Q)$ can also be described as a $\mathbb{Z} / 2 \mathbb{Z}$ graded module $M=P \oplus Q[1]$ (where both graded parts are projective). The pair of morphisms $(f, g)$ can be described as an odd morphism $\delta: P \oplus Q[1] \rightarrow P \oplus Q[1]$. We shall liberally switch between the two kinds of notation, identifying

$$
f: P \rightleftarrows Q: g \quad \longleftrightarrow \quad\left(P \oplus Q[1],\left(\begin{array}{ll}
0 & g \\
f & 0
\end{array}\right)\right)
$$

We are now ready to define $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$. As in the classical case, if $(M, \delta)$ is a deformed matrix factorization, we denote by $\tilde{\delta}$ the tweaked differential given by $\tilde{\delta}(m)=(-1)^{|m|} \delta(m)$.

Definition 20.41. Let $\left(A_{q}, \ell_{q}\right)$ be a deformed Landau-Ginzburg model. The deformed category of matrix factorizations $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ is defined as follows:

- Objects are the deformed matrix factorizations $(M, \delta)$ of $\left(A_{q}, \ell_{q}\right)$.
- Hom spaces are given by $\operatorname{Hom}\left(\left(M, \delta_{M}\right),\left(N, \delta_{N}\right)\right)=\operatorname{Hom}_{A_{q}}(M, N)$, naturally $\mathbb{Z} / 2 \mathbb{Z}$-graded.
- The curvature of an object $(M, \delta)$ is given by $\mu_{(M, \delta)}^{0}=\ell_{q} \operatorname{id}_{M}-\delta^{2}$.
- The differential is given by $\mu^{1}(f)=\tilde{\delta}_{N} \circ f-(-1)^{|f|} f \circ \tilde{\delta}_{M}$ for $f \in \operatorname{Hom}\left(\left(M, \delta_{M}\right),\left(N, \delta_{N}\right)\right)$.
- The product is given by $\mu^{2}(f, g)=(-1)^{\|f\|| | g \mid} f \circ g$.

Remark 20.42. In writing $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$, we have abused notation: The category $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ is not the same as (classical) matrix factorizations of the pair $\left(A_{q}, \ell_{q}\right)$.

We aim at showing that $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ is an object-cloning deformation of $\operatorname{MF}(A, \ell)$. To make this true, we provide a map $\operatorname{ObMF}\left(A_{q}, \ell_{q}\right) \rightarrow \operatorname{ObMF}(A, \ell)$. The construction of this map is easy and consists of taking the leading term of any matrix factorization:

Definition 20.43. Let $\left(A_{q}, \ell_{q}\right)$ be a deformed Landau-Ginzburg model. Let $(M, \delta)$ be a deformed matrix factorization of $(A, \ell)$. Then the leading term of $(M, \delta)$ is the matrix factorization $O(M, \delta)$ of $(A, \ell)$ given by

$$
O(M, \delta)=(M /(\mathfrak{m} \cdot M), \pi(\delta))
$$

In this definition, $\pi(\delta)$ denotes the induced map $M /(\mathfrak{m} \cdot M) \rightarrow M /(\mathfrak{m} \cdot M)$ and is automatically an $A$ module map. The quotient $M /(\mathfrak{m} \cdot M)$ is understood to be performed in even and odd degree separately. We are now ready to show that $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ is an object-cloning deformation of $\operatorname{MF}(A, \ell)$.
Proposition 20.44. The category $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ is a loose object-cloning deformation of $\operatorname{MF}(A, \ell)$ along the map $O: \operatorname{Ob} \operatorname{MF}\left(A_{q}, \ell_{q}\right) \rightarrow \operatorname{ObMF}(A, \ell)$.

Proof. We divide the proof into three steps: The first step is to show that $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ satisfies the (curved) $A_{\infty}$-axioms, merely regarding its hom spaces as $\mathbb{Z} / 2 \mathbb{Z}$-graded vector spaces instead of $B$-modules. The second step is to investigate the $B$-linear structure on its hom spaces and the shape of the products. The third step is to draw the conclusion that $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ is indeed an object-cloning deformation of $\operatorname{MF}(A, \ell)$.

For the first step, we check all curved $A_{\infty}$-relations for $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ one after another. The first relation reads

$$
\mu^{1}\left(\mu_{(M, \delta)}^{0}\right)=\tilde{\delta} \circ\left(\ell_{q} \operatorname{id}_{M}-\delta^{2}\right)-\left(\ell_{q} \operatorname{id}_{M}-\delta^{2}\right) \circ \tilde{\delta}=0 .
$$

Note we have used that $\delta^{2}=-\tilde{\delta}^{2}$ and $\ell_{q}$ is central. The second relation reads

$$
\begin{aligned}
\mu^{1}\left(\mu^{1}(f)\right)+(-1)^{\|f\|} \mu^{2}\left(\mu^{0}, f\right)+\mu^{2}\left(f, \mu^{0}\right)= & \tilde{\delta} \circ\left(\tilde{\delta} \circ f-(-1)^{|f|} f \circ \tilde{\delta}\right) \\
& -(-1)^{|f|+1}\left(\tilde{\delta} \circ f-(-1)^{|f|} f \circ \tilde{\delta}\right) \circ \tilde{\delta} \\
& +(-1)^{\|f\|+|f|}\left(\ell_{q} \operatorname{id}_{M}-\delta^{2}\right) \circ f+f \circ\left(\ell_{q} \operatorname{id}_{M}-\delta^{2}\right) \\
= & -\delta^{2} \circ f+f \circ \delta^{2}-\ell_{q} f+f \ell_{q}+\delta^{2} \circ f-f \circ \delta^{2}=0 .
\end{aligned}
$$

We have used again that $\ell_{q}$ is central. The third relation is analogous to the classical case and reads

$$
\begin{aligned}
\mu^{1}\left(\mu^{2}(f, g)\right)+(-1)^{\|g\|} \mu^{2}\left(\mu^{1}(f), g\right)+\mu^{2}\left(f, \mu^{1}(g)\right)= & (-1)^{\|f\|| | g \mid}\left(\tilde{\delta} \circ f \circ g-(-1)^{|f g|} f \circ g \circ \tilde{\delta}\right) \\
& +(-1)^{|f||g|+\|g\|}\left(\tilde{\delta} \circ f-(-1)^{|f|} f \circ \tilde{\delta}\right) \circ g \\
& +(-1)^{\|f\|\|g\|} f \circ\left(\tilde{\delta} \circ g-(-1)^{|g|} g \circ \tilde{\delta}\right)=0 .
\end{aligned}
$$

The fourth relation is associativity and all other relations vanish.
For the second part of the proof, we give $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ the structure of loose object-cloning deformation. This entails providing for every two deformed matrix factorizations $\left(M, \delta_{M}\right),\left(N, \delta_{N}\right)$ a linear $\mathbb{Z} / 2 \mathbb{Z}$-graded isomorphism

$$
\psi_{\left(M, \delta_{M}\right),\left(N, \delta_{N}\right)}: \frac{\operatorname{Hom}_{\mathrm{MF}\left(A_{q}, \ell_{q}\right)}\left(\left(M, \delta_{M}\right),\left(N, \delta_{N}\right)\right)}{\mathfrak{m} \cdot \operatorname{Hom}_{\mathrm{MF}\left(A_{q}, \ell_{q}\right)}\left(\left(M, \delta_{M}\right),\left(N, \delta_{N}\right)\right)} \stackrel{\sim}{\rightarrow} \operatorname{Hom}_{\mathrm{MF}(A, \ell)}\left(O\left(M, \delta_{M}\right), O\left(N, \delta_{N}\right)\right) .
$$

The hom space $\operatorname{Hom}_{\operatorname{MF}\left(A_{q}, \ell_{q}\right)}\left(\left(M, \delta_{M}\right),\left(N, \delta_{N}\right)\right)$ merely consists of the direct sum of four hom spaces from the category Proj $A_{q}$. Thanks to Proposition 20.35 every of these four hom spaces, quotiented by $\mathfrak{m}$, comes with a natural isomorphism to the corresponding hom space from the category Proj $A$. We now construct the isomorphism $\psi_{\left(M, \delta_{M}\right),\left(N, \delta_{N}\right)}$ by simply combining these four isomorphisms on their respective domains.

For the third part of the proof, we explain why $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ is a loose object-cloning deformation of $\operatorname{MF}(A, \ell)$ via $O$. Indeed, composition in $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ is merely given by matrix composition and we have seen in Proposition 20.35 that it is compatible via $\psi$ with composition in Proj $A$. Similarly, the differential in $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ is defined via commuting with $\delta$, which reduces via $\psi$ to commuting with the induced $\delta$ in $\operatorname{Proj} A$ and hence to the differential of $\operatorname{MF}(A, \ell)$. Finally, the curvature in $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ is infinitesimal and reduces to zero after dividing out $\mathfrak{m}$. This proves $\operatorname{MF}\left(A_{q}, \ell_{q}\right)$ a loose object-cloning deformation of $\operatorname{MF}(A, \ell)$ via $O$ and finishes the proof.

### 20.6 Deformed mirror functor

In this section, we construct our deformed mirror functor. The idea is to simply repeat the Cho-Hong-Lau construction using the deformed category $\mathcal{C}_{q}$ as input instead of $\mathcal{C}$. As we have seen in section 20.3, we obtain a deformed Landau-Ginzburg model $\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right)$. The target of the mirror functor is then the deformed category of matrix factorizations $\operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right)$. We deploy notation and assumptions from Convention 20.22 As in the classical case, we construct functors in the general case and the case of slow growth in parallel.

The first step of this section is to define the deformed functors $\widehat{F}_{q}: \mathcal{C}_{q} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right)$ and $F_{q}: \mathcal{C}_{q} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right), \ell_{q}\right)$ on object level. The second step is to define them on morphism level.

Finally we check that $\widehat{F}_{q}$ and $F_{q}$ satisfy the deformed $A_{\infty}$-functor relations and their leading terms are the classical Cho-Hong-Lau functors $\widehat{F}$ and $F$.

An important assumption in this section is that $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$ be a deformation of $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$, or $\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right)$ be a deformation of $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$, in the sense of Definition 19.1. As such, these algebras come with $B$-linear algebra isomorphisms

$$
\begin{aligned}
& \widehat{\varphi}_{\mathrm{Jac}}: \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right) \xrightarrow{\sim}\left(B \widehat{\otimes} \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right), \mu_{\mathrm{Jac}, q}\right), \\
& \varphi_{\mathrm{Jac}}: \operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right) \xrightarrow{\sim}\left(B \widehat{\otimes} \operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \mu_{\mathrm{Jac}, q}\right) .
\end{aligned}
$$

We fix these isomorphisms and will always view $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$ and $\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right)$ as deformations of $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$ and $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$.
Remark 20.45. By Remark 19.6, the isomorphisms $\widehat{\varphi}_{\mathrm{Jac}}$ and $\varphi_{\mathrm{Jac}}$ can be assumed to be unital and be $\mathbb{C} Q_{0}^{\mathbb{L}}$-bimodule morphisms. For instance, any vertex element $L_{i} \in \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$ is mapped to the vertex $L_{i} \in \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$, without getting deformed. We also have $L_{i}^{2}=L_{i}$ in $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$ and $\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right)$, since the same holds in $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$ and $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$. The two module maps $(-) L_{i}: \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right) \rightarrow$ $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$ and $(-) L_{i}: \operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right) \rightarrow \operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right)$ are therefore idempotents. We conclude that $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right) L_{i}$ and $\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right) L_{i}$ are projectives.

We start with the explicit description of $\widehat{F}_{q}$ and $F_{q}$ on object level.

$$
\begin{aligned}
& \widehat{F}_{q}(X)=\left(\bigoplus_{i=1}^{N} \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right) L_{i} \otimes \operatorname{Hom}_{\mathcal{C}}\left(L_{i}, X\right), \quad \delta(m)=\sum_{k \geq 1}(-1)^{\|m\|} \mu_{q}^{k}(m, b, \ldots, b)\right), \\
& F_{q}(X)=\left(\bigoplus_{i=1}^{N} \operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right) L_{i} \otimes \operatorname{Hom}_{\mathcal{C}}\left(L_{i}, X\right), \quad \delta(m)=\sum_{k \geq 1}(-1)^{\|m\|} \mu_{q}^{k}(m, b, \ldots, b)\right) .
\end{aligned}
$$

Note that each map $\delta$ is well-defined since $\mathbb{L}_{q}$ is of slow growth.
Lemma 20.46. For $X \in \mathcal{C}$ the object $\widehat{F}_{q}(X)$ is indeed a deformed matrix factorization of $\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right)$. If $\mathbb{L}_{q}$ is of slow growth, then $F_{q}(X)$ is a deformed matrix factorization of $\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right), \ell_{q}\right)$.

Proof. We merely check the case of $\widehat{F}_{q}(X)$. It is our task to show that $\ell_{q} \mathrm{id}_{\widehat{F}_{q}(X)}-\delta^{2}$ is infinitesimal. Calculating in $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right) \otimes \operatorname{Hom}(\mathbb{L}, X)$, we find

$$
\begin{aligned}
\ell_{q} m-\delta^{2}(m) & =\ell_{q} m-\delta^{2}(m) \\
& =\ell_{q} m+\sum_{i, j \geq 0} \mu_{q}^{i+1}\left(\mu_{q}^{j+1}(m, b \ldots, b), b, \ldots, b\right) \\
& =\ell_{q} m-\sum_{i, j, k \geq 0} \mu_{q}(m, \underbrace{b, \ldots, b}_{i}, \mu_{q}(\underbrace{b, \ldots, b}_{j}), \underbrace{b, \ldots, b}_{k})-(-1)^{\|m\|} \sum_{i \geq 0} \mu_{q}^{i+2}\left(\mu_{q}^{0}, m, b, \ldots, b\right) \\
& =(-1)^{|m|} \sum_{i \geq 0} \mu_{q}^{i+2}\left(\mu_{q}^{0}, m, b, \ldots, b\right) \\
& \in \mathfrak{m} \cdot\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right) \otimes \operatorname{Hom}(\mathbb{L}, X)\right) .
\end{aligned}
$$

Here we have used the curved $A_{\infty}$-relation for $\mu_{q}$ and $\sum_{l \geq 0} \mu_{q}^{l}(b, \ldots, b)=\ell_{q} \mathrm{id}_{\mathbb{L}_{q}}$. We conclude that $\ell_{q} \operatorname{id}_{\widehat{F}_{q}(X)}-\delta^{2}$ is infinitesimal and $\widehat{F}_{q}(X)$ is a deformed matrix factorization. The analogous calculations show that $F_{q}(X)$ is a matrix factorization when $\mathbb{L}_{q}$ is of bounded growth.

Remark 20.47. Explicitly, the curvature of the object $\widehat{F}_{q}(X)$ is the endomorphism of $\widehat{F}_{q}(X)$ given by

$$
\mu_{\mathrm{MF}, \widehat{F}_{q}(X)}^{0}(m)=\ell_{q} m-\delta^{2}(m)=(-1)^{|m|} \sum_{l \geq 0} \mu_{q}^{l+2}\left(\mu_{q}^{0}, m, b, \ldots, b\right)
$$

We now define the deformed CHL functor in analogy to the classical CHL functor:

Definition 20.48. The deformed CHL functor $\widehat{F}_{q}$ is the mapping

$$
\begin{aligned}
& \widehat{F}_{q}: \mathcal{C}_{q} \longrightarrow \operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right), \\
& X \longmapsto \widehat{F}_{q}(X), \\
& \widehat{F}_{q}\left(m_{k}, \ldots, m_{1}\right)(m)=(-1)^{\left(\left\|m_{1}\right\|+\ldots+\left\|m_{k}\right\|\right)\|m\|+1} \sum_{l \geq 0} \mu_{q}^{k+l+1}\left(m_{k}, \ldots, m_{1}, m, b, \ldots, b\right) \\
& \text { for } m_{i}: X_{i} \rightarrow X_{i+1}, \quad m \in \widehat{F}_{q}\left(X_{1}\right) .
\end{aligned}
$$

In case $\mathbb{L}_{q}$ is of slow growth, the functor $F_{q}: \mathcal{C}_{q} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right), \ell_{q}\right)$ is defined analogously.
Remark 20.49. Note that we put $F_{q}^{0}=0$, as opposed to $F_{q}^{0}:=\sum_{l \geq 0} \mu_{q}^{l+1}(-, b, \ldots, b)$.
We shall now prove that $\widehat{F}_{q}$ and $F_{q}$ are actually functors. We shall also show that their leading terms are the classical functors $\widehat{F}$ and $F$. The categories $\operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right)$ and $\operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right), \ell_{q}\right)$ are only object-cloning deformations of $\operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right), \ell\right)$ and $\operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right)$, so making the statement on leading terms rigorous we need to provide an identification map on object level. In Definition 20.43, we have already provided such identification maps

$$
\begin{aligned}
& O: \operatorname{ObMF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right), \ell_{q}\right) \rightarrow \operatorname{ObMF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right), \ell\right), \\
& O: \operatorname{Ob} \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right), \ell_{q}\right) \rightarrow \operatorname{ObMF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right) .
\end{aligned}
$$

The map $O$ sends $\widehat{F}_{q}(X)$ and $F_{q}(X)$ to

$$
\begin{aligned}
& O\left(\widehat{F}_{q}(X)\right)=\left(\bigoplus_{i=1}^{N} \frac{\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right) L_{i}}{\mathfrak{m} \cdot \operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right) L_{i}} \otimes \operatorname{Hom}_{\mathcal{C}}\left(L_{i}, X\right), \quad \delta(m)=\sum_{k \geq 1}(-1)^{\|m\|}(\pi \otimes \mathrm{id})\left(\mu_{q}^{k}(m, b, \ldots, b)\right)\right), \\
& O\left(F_{q}(X)\right)=\left(\bigoplus_{i=1}^{N} \frac{\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right) L_{i}}{\mathfrak{m} \cdot \operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right) L_{i}} \otimes \operatorname{Hom}_{\mathcal{C}}\left(L_{i}, X\right), \quad \delta(m)=\sum_{k \geq 1}(-1)^{\|m\|}(\pi \otimes \mathrm{id})\left(\mu_{q}^{k}(m, b, \ldots, b)\right)\right)
\end{aligned}
$$

Here $\pi$ denotes the projection from the Jacobi algebra to its quotient by $\mathfrak{m}$. Meanwhile, the object $\widehat{F}(X)$ is given by a sum of projectives of the form $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right) L_{i}$. As discussed in Remark 20.36 the difference is entirely cosmetic and we shall naturally identify

$$
\begin{align*}
& O\left(\widehat{F}_{q}(X)\right) \xrightarrow{\sim} \widehat{F}(X),  \tag{20.16}\\
& O\left(F_{q}(X)\right) \xrightarrow{\sim} F(X) .
\end{align*}
$$

A similar statement holds for the hom spaces. Regard a hom space between two image objects $\widehat{F}_{q}(X)$ and $\widehat{F}_{q}(Y)$, divide out $\mathfrak{m}$, and identify the quotient via the map $\psi$ from section 20.4 According to Remark 20.36 the result can further be identified naturally with the hom space in $\operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right), \ell\right)$ or $\operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathrm{L}}, W\right), \ell\right)$ :

$$
\begin{align*}
& \frac{\operatorname{Hom}\left(\widehat{F}_{q}(X), \widehat{F}_{q}(Y)\right)}{\mathfrak{m} \cdot \operatorname{Hom}\left(\widehat{F}_{q}(X), \widehat{F}_{q}(Y)\right)} \xrightarrow[\sim]{\sim} \operatorname{Hom}\left(\frac{\widehat{F}_{q}(X)}{\mathfrak{m} \cdot \widehat{F}_{q}(X)}, \frac{\widehat{F}_{q}(Y)}{\mathfrak{m} \cdot \widehat{F}_{q}(Y)}\right) \xrightarrow{\sim} \operatorname{Hom}(\widehat{F}(X), \widehat{F}(Y)),  \tag{20.17}\\
& \frac{\operatorname{Hom}\left(F_{q}(X), F_{q}(Y)\right)}{\mathfrak{m} \cdot \operatorname{Hom}\left(F_{q}(X), F_{q}(Y)\right)} \stackrel{\psi}{\sim} \operatorname{Hom}\left(\frac{F_{q}(X)}{\mathfrak{m} \cdot F_{q}(X)}, \frac{F_{q}(Y)}{\mathfrak{m} \cdot F_{q}(Y)}\right) \xrightarrow{\sim} \operatorname{Hom}(F(X), F(Y)) .
\end{align*}
$$

We claim that with respect to this identification, the leading terms of the two functors $\widehat{F}_{q}$ and $F_{q}$ are the classical Cho-Hong-Lau functors $\widehat{F}$ and $F$ :
Theorem 20.50. Assume Convention 20.22 and that $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$ is a deformation of $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W\right)$. Then the mapping $\widehat{F}_{q}$ defines a functor of loose object-cloning $A_{\infty}$-deformations

$$
\widehat{F}_{q}: \mathcal{C}_{q} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(\widehat{Q^{\mathrm{L}}}, W_{q}\right), \ell_{q}\right) .
$$

The leading term of $\widehat{F}_{q}$ via the identifications 20.16 and 20.17 is the classical Cho-Hong-Lau functor $\widehat{F}$.

Assume instead Convention 20.22 that $\mathbb{L}_{q}$ is of slow growth and that $\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right)$ is a deformation of $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$. Then the mapping $F_{q}$ defines a functor of loose object-cloning $A_{\infty}$-deformations

$$
F_{q}: \mathcal{C}_{q} \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W_{q}\right), \ell_{q}\right)
$$

The leading term of $F_{q}$ via the identifications 20.16 and 20.17 is the classical Cho-Hong-Lau functor $F$.

Proof. First we check the $A_{\infty}$-functor relations, then we comment on the leading term. To start with, for the functor relations it is our task to check that

$$
\begin{aligned}
\sum_{0 \leq j \leq i \leq k}(-1)^{\left\|m_{1}\right\|+\ldots+\left\|m_{j}\right\|} \widehat{F}_{q}\left(m_{k}\right. & \left., \ldots, \mu_{q}\left(m_{i}, \ldots, m_{j+1}\right), \ldots, m_{1}\right) \\
& =\mu_{\mathrm{MF}}^{2}\left(\widehat{F}_{q}\left(m_{k}, \ldots, m_{i+1}\right), \widehat{F}_{q}\left(m_{i}, \ldots, m_{1}\right)\right)+\mu_{\mathrm{MF}}^{1}\left(\widehat{F}_{q}\left(m_{k}, \ldots, m_{1}\right)\right) .
\end{aligned}
$$

Checking these relations is similar to the classical case Lemma 20.20 The calculation does not use explicitly that $\delta^{2}$ vanishes, therefore remains intact. However, due to the possible curvature of $\mathcal{C}_{q}$ a few new terms appear on one side of the $A_{\infty}$-functor equation. We shall check these terms in more detail.

Assume there are $k \geq 1$ input morphisms $m_{1}, \ldots, m_{k}$. Comparing with the calculation in Lemma 20.20 the new terms on the left-hand side of the functor relation are

$$
\begin{aligned}
& \sum_{0 \leq n \leq k}(-1)^{\left\|m_{1}\right\|+\ldots+\left\|m_{n}\right\|} \widehat{F}_{q}\left(m_{k}, \ldots, m_{n+1}, \mu_{q}^{0}, m_{n}, \ldots, m_{1}\right)(m) \\
= & \sum_{\substack{0 \leq n \leq k \\
0 \leq l}}(-1)^{\left\|m_{1}\right\|+\ldots+\left\|m_{n}\right\|+\|m\|\left(\left\|m_{1}\right\|+\ldots+\left\|m_{k}\right\|+1\right)+1} \mu_{q}^{k+l+1}\left(m_{k}, \ldots, m_{n+1}, \mu_{q}^{0}, m_{n}, \ldots, m_{1}, m, b, \ldots, b\right) .
\end{aligned}
$$

There are no new terms on the right-hand side of the functor relation. When applying the curved $A_{\infty}$-relation as in the proof of Lemma 20.20, the terms on the left-hand side disappear and the terms $\mu_{q}\left(m_{k}, \ldots, m_{1}, m, b, \ldots, \mu_{q}^{\geq 0}(b, \ldots, b), \ldots, b\right)$ come in, which still vanish since $\sum \mu_{q}(b, \ldots, b)=\ell_{q} \mathrm{id}_{\mathbb{L}_{q}}$. This proves the functor relations for $k \geq 1$.

Also for $k=0$ the functor relation is still satisfied:

$$
\widehat{F}_{q}^{1}\left(\mu_{X}^{0}\right)(m)=(-1)^{\|m\|+1} \sum_{l \geq 0} \mu_{q}^{l+2}\left(\mu_{q}^{0}, m, b, \ldots, b\right)=\mu_{\widehat{F}_{q}(X)}^{0}
$$

The computations for $F_{q}$ are analogous. This finishes the checks of the functor relations.
Finally, let us now comment on the leading terms of $\widehat{F}_{q}$ and $F_{q}$. Indeed, every functor component $\widehat{F}_{q}^{k}$ or $F_{q}^{k}$ is constructed via the deformed products of $\mathcal{C}_{q}$.

We need to check the $\operatorname{Jac}\left(\widehat{Q^{\mathbb{L}}}, W_{q}\right)$-linear map $\widehat{F}_{q}^{k}\left(m_{k}, \ldots, m_{1}\right): \widehat{F}_{q}\left(X_{1}\right) \rightarrow \widehat{F}_{q}\left(X_{k+1}\right)$. Up to terms of $\mathfrak{m} \cdot \widehat{F}_{q}\left(X_{k+1}\right)$, this map is equal to

$$
\sum_{l \geq 0} \mu^{k+l+1}\left(m_{k}, \ldots, m_{1},-, b, \ldots, b\right): \widehat{F}_{q}\left(X_{1}\right) \rightarrow \widehat{F}_{q}\left(X_{k+1}\right)
$$

Passing along $\psi$ gives

$$
\sum_{l \geq 0} \mu^{k+l+1}\left(m_{k}, \ldots, m_{1},-, b, \ldots, b\right): \widehat{F}_{q}\left(X_{1}\right) /\left(\mathfrak{m} \cdot \widehat{F}_{q}\left(X_{1}\right)\right) \rightarrow \widehat{F}_{q}\left(X_{k+1}\right) /\left(\mathfrak{m} \cdot \widehat{F}_{q}\left(X_{k+1}\right)\right)
$$

This map is not the same as $\widehat{F}^{k}\left(m_{k}, \ldots, m_{1}\right)$ yet. But now identify $\widehat{F}_{q}\left(X_{1}\right) /\left(\mathfrak{m} \cdot \widehat{F}_{q}\left(X_{1}\right)\right) \xrightarrow{\sim} \widehat{F}\left(X_{1}\right)$ and similarly for $X_{k+1}$. Recalling Remark 20.36, the induced map $\widehat{F}\left(X_{1}\right) \rightarrow \widehat{F}\left(X_{k+1}\right)$ is merely given by the composition $\widehat{F}\left(X_{1}\right) \rightarrow \widehat{F}_{q}\left(X_{1}\right) /\left(\mathfrak{m} \cdot \widehat{F}_{q}\left(X_{1}\right)\right) \rightarrow \widehat{F}_{q}\left(X_{k+1}\right) /\left(\mathfrak{m} \cdot \widehat{F}_{q}\left(X_{k+1}\right)\right) \rightarrow \widehat{F}\left(X_{k+1}\right)$, yielding the map

$$
\sum_{l \geq 0} \mu^{k+l+1}\left(m_{k}, \ldots, m_{k},-, b, \ldots, b\right): \widehat{F}\left(X_{1}\right) \rightarrow \widehat{F}\left(X_{k+1}\right)
$$

This is precisely $\widehat{F}^{k}\left(m_{k}, \ldots, m_{1}\right)$. We have shown that the leading term of $\widehat{F}_{q}$ is $\widehat{F}$. The same argument holds for $F_{q}$. This finishes the proof.

## 21 Deformed mirror symmetry

In this section, we collect all preliminary results and prove deformed mirror symmetry, our main theorem. The procedure is as follows: We start with a geometrically consistent dimer $Q$ under Convention 18.1. By section 18 the deformed category of zigzag curves $\mathrm{H}_{q} \subseteq \mathrm{HTw} \mathrm{Gtl}_{q} Q$ is deformed cyclic. By section 20. we obtain an algebra $\mathrm{Jac}_{q} \check{Q}$ plus central element $\ell_{q} \in \mathrm{Jac}_{q} \check{Q}$ together with a functor

$$
F_{q}: \quad \operatorname{Gtl}_{q} Q \rightarrow \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)
$$

By section 19 this algebra $\mathrm{Jac}_{q} \check{Q}$ is a deformation of $\operatorname{Jac} \check{Q}$. The necessary requirement for this last step is that $Q$ is cancellation consistent and of bounded type. Finally, we interpret the category $m f\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ as a deformation of the classical mirror $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$, and the morphism $F_{q}$ as a deformation of the classical $A_{\infty}$ mirror quasi-isomorphism

$$
F: \quad \operatorname{Gtl} Q \rightarrow \operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)
$$

In particular, we conclude that $F_{q}$ itself is a quasi-isomorphism.

### 21.1 Mirror symmetry by Cho-Hong-Lau

In this section, we recollect how a gentle algebra as specific instance of the Cho-Hong-Lau construction yields mirror symmetry for punctured surfaces. The first step in this section is to provide some details on this specific instance. Second, we read off the specific properties for the Cho-Hong-Lau construction. Third, we realize that the resulting mirror functor indeed recovers mirror symmetry for punctured surfaces. This section is an integrated summary of 26, Chapter 10].

Remark 21.1. Cho, Hong and Lau depart from a folklore version of wrapped Fukaya category, where the complete set of products is unclear. We have opted in the present paper to use the very rigorous description of $\mathrm{HTw} \operatorname{Gtl} Q$ and $\mathrm{HTw}_{\mathrm{Gtl}}^{q}$ $Q$ from Paper II. In particular, we restrict to the case that $Q$ is geometrically consistent or a standard sphere dimer.

Our starting point is a dimer $Q$ whose zigzag paths are equipped with a specific choice of spin structure, which we have codified in Convention 18.1. We shall start describing the specific instance of the Cho-Hong-Lau construction needed for mirror symmetry. The first step is to choose $\mathcal{C}=\mathrm{HTw} \mathrm{Gtl} Q$. The subcategory of reference objects, denoted $\mathbb{L}$ in section 20 is the category of zigzag paths $H \mathbb{L} \subseteq$ H Tw Gtl $Q$.

As a second step, we describe the CHL basis that needs to be chosen. The right choice of basis elements $\left\{X_{e}\right\}$ for $\operatorname{Hom}_{\mathrm{H} \mathbb{L}}\left(L_{1}, L_{2}\right)$ is the collection of all transversal intersection points of $L_{1}$ and $L_{2}$ which are odd as morphisms $L_{1} \rightarrow L_{2}$. In other words, every single transversal intersection points between two arbitrary zigzag curves appears as basis element for precisely one hom space, namely the one in which it is odd. The set of transversal intersection points between zigzag curves in $Q$ is precisely the same as the set of arcs $Q_{1}$. We shall therefore fix notation as follows:

Definition 21.2. For every arc $a \in Q_{1}$, we denote by $X_{a}$ the odd morphism in $\mathbb{L}$ which is located at the midpoint of $a$. Similarly, we denote by $Y_{a}$ the even morphism in $\mathbb{L}$ which is located at the midpoint of $a$.

Remark 21.3. Visually speaking, we have the correspondence

$$
\begin{gathered}
\underset{\text { odd morphisms }}{\text { (transversal only) }} \quad \longleftrightarrow \quad \text { arrows } a \in Q_{1} \quad \longleftrightarrow \quad \begin{array}{c}
\text { even morphisms } \\
\text { (transversal only) }
\end{array}
\end{gathered}
$$

With this in mind, the specific instance of the Cho-Hong-Lau construction which yields mirror symmetry for punctured surfaces is described as follows:

- The category $\mathcal{C}$ is the derived category $\mathrm{HTw} \operatorname{Gtl} Q$ of the gentle algebra $\operatorname{Gtl} Q$.
- The subcategory $\mathbb{L} \subseteq \mathcal{C}$ is the subcategory $\mathrm{H} \mathbb{L} \subseteq H \operatorname{Tw} \operatorname{Gtl} Q$ of zigzag paths.
- The CHL basis consists of all odd cohomology basis elements $X_{e}$ between zigzag paths, all even cohomology basis elements $Y_{e}$ between zigzag paths, and the co-identity elements $\mathrm{id}_{L_{i}}^{*}$.
- The odd pairing $\langle-,-\rangle$ on HL is defined by enforcing the pairing identities 20.1. Explicitly, one sets $\langle-,-\rangle$ to zero on all pairs of basis elements except

$$
\left\langle X_{e}, Y_{f}\right\rangle=\delta_{e f}, \quad\left\langle\mathrm{id}_{L_{i}}^{*}, \mathrm{id}_{L_{j}}\right\rangle=\delta_{i j}
$$

| Gadget | General | Specific |
| :---: | :---: | :---: |
| Category | $\mathcal{C}$ | $\mathrm{HTwGtl} Q$ |
| Reference objects | $\mathbb{L}$ | $\mathrm{H} \mathbb{L}=\left\{L_{1}, \ldots, L_{N}\right\}$ |
| Cohomology basis | $\left\{X_{e}, Y_{e}, \mathrm{id}_{L}, \mathrm{id}_{L}^{*}\right\}$ | intersection points |
| Quiver | $Q^{\mathbb{L}}$ | $\check{Q}$ |
| Superpotential $W$ | $\langle\mu(b, \ldots, b), b\rangle$ | $W=\check{Q}_{\text {cyc }}^{+}-\check{Q}_{\text {cyc }}^{-}$ |
| Relations $R_{e}$ | $\left\langle\mu(b, \ldots, b), X_{e}\right\rangle$ | $r_{a}^{+}-r_{a}^{-}$ |
| Landau-Ginzburg model | $\left(\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right), \ell\right)$ | $(\operatorname{Jac} \check{Q}, \ell)$ |
| Matrix factorization $F(a)$ | $(M, \delta)$ | $(\operatorname{Jac} \check{Q}) h(a) \underset{\bar{a}}{\stackrel{a}{\rightleftarrows}}(\operatorname{Jac} \check{Q}) t(a)$ |

Table 21.1: For every gadget involved in the Cho-Hong-Lau construction, this overview exhibits the general definition and its specific shape in the case of $\mathrm{HL} \subseteq \mathrm{HTw} \mathrm{Gtl} Q$.


Figure 21.2: This picture depicts the correspondence between odd intersections in $Q$ and arrows in $\check{Q}$. The odd basis element $X_{a}: L_{1} \rightarrow L_{2}$, given by the intersection of $\tilde{L}_{1}$ and $\tilde{L}_{2}$ located at the midpoint of the arc $a$, corresponds to an arrow $L_{1} \rightarrow L_{2}$ in the dual dimer $\check{Q}$.

Application of the Cho-Hong-Lau construction to $\mathrm{H} \mathbb{L} \subseteq \mathrm{HTw} \mathrm{Gtl} Q$ yields precisely the dual dimer $\check{Q}$, which we have recalled in section 17.5

Lemma 21.4. The CHL quiver of $\mathrm{HL} \subseteq \mathrm{HTw} \operatorname{Gtl} Q$ is the quiver $Q^{\mathbb{L}}=\check{Q}$.
Proof. By general definition, the vertices of the quiver $Q^{\mathbb{L}}$ are the reference objects $L_{i} \in \mathbb{L}$ and the arrows from $L_{i}$ to $L_{j}$ are given by the index set $E_{i j}$. For the specific case of $\mathbb{L} \subseteq \mathrm{HTw} \mathrm{Gtl} Q$, the reference objects are the zigzag paths of $Q$. Since zigzag paths of $Q$ are in correspondence with vertices of $\check{Q}$, this identifies the vertices of $Q^{\mathbb{L}}$ and $\check{Q}$. For given vertices $i, j$, the set of arrows $E_{i j}$ from $i$ to $j$ is equal to the set of odd transversal intersection points $L_{i} \rightarrow L_{j}$. Every odd transversal intersection is located at the midpoint of an arc $a \in Q_{1}$. The corresponding $\operatorname{arc} a \in \mathscr{Q}_{1}$ runs from $i$ to $j$ as well, as illustrated in Figure 21.2. We conclude that $Q^{\mathbb{L}}=\check{Q}$.

Lemma 21.5 ([26, Lemma 10.13]). Application of the Cho-Hong-Lau construction to $\mathbb{L} \subseteq$ H Tw Gtl $Q$ yields the familiar superpotential

$$
W=\sum_{\substack{a_{k} \ldots a_{1} \\
\text { clockwise }}}\left(a_{k} \ldots a_{1}\right)_{\mathrm{cyc}}-\sum_{\begin{array}{c}
a_{k} \ldots a_{1} \\
\text { counterclockwise }
\end{array}}\left(a_{k} \ldots a_{1}\right)_{\mathrm{cyc}} \in(\mathbb{C} \check{Q})_{\geq 3}
$$

In particular, application yields the familiar relations $R_{a}=r_{a}^{+}-r_{a}^{-}$and the familiar Jacobi algebra $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)=\operatorname{Jac} \check{Q}$.

Proof. It is our task to determine the superpotential $W$. After that, the conclusion on the relations $R_{a}$ and the Jacobi algebra is immediate.

The general superpotential is given by $W=\langle\mu(b, \ldots, b), b\rangle$. Specifically for $\mathrm{H} \mathbb{L} \subseteq \mathrm{H} \operatorname{Tw} \mathrm{Gtl} Q$ this description boils down to calculating the products

$$
\mu_{\mathrm{HTwGtl} Q}\left(X_{e_{k}}, \ldots, X_{e_{1}}\right)
$$



Figure 21.3: One can put a checkerboard coloring on the area of the dimer $Q$. The checkerboard coloring makes it easy to check that there are only two disks which contribute to $\partial_{e} W$.

The result of such a product is a linear combination of the even basis elements $Y_{e}$ and possibly an identity. It is our task to extract the coefficient of every $Y_{e}$ in the product. We claim that this coefficient is precisely

$$
\left\langle\mu_{\mathrm{HTw} \operatorname{Gtl} Q}\left(X_{e_{k}}, \ldots, X_{e_{1}}\right), X_{e}\right\rangle= \begin{cases}+1 & \text { if } e_{k} \ldots e_{1} e \text { is a clockwise polygon in } \check{Q}  \tag{21.1}\\ -1 & \text { if } e_{k} \ldots e_{1} e \text { is a counterclockwise polygon in } \check{Q} . \\ 0 & \text { else }\end{cases}
$$

To compute the product, we use our explicit description of the minimal model $\mathrm{H} \mathbb{L}$, see section 18.3 . Recall that section 18.3 actually describes the deformed version $\mathrm{H} \mathbb{L}_{q}$ of $\mathrm{H} \mathbb{L}$. The $A_{\infty}$-structure $\mathrm{H} \mathbb{L}$ is obtained from the deformed $A_{\infty}$-structure of $\mathrm{H}_{q}$ by extracting the leading terms of every product. Spelling this out, the higher products in HL are simply determined by only those CR, ID, DS and DW disks that do not cover any punctures.

With this in mind, we are ready to calculate the products 21.1. Let $D$ be a (CR, ID, DS or DW) disk contributing to the product. Note that all input morphisms $X_{e_{i}}$ are odd and the output is not an identity, so $D$ is necessarily a CR disk.

We claim that the disk $D$ precisely bounds the interior of an (elementary) polygon, as depicted in Figure 21.3a Indeed, the zigzag curves of $Q$ split $D$ into pieces of two types: the first type comprises an interior of a polygon of $Q$, and the second type comprises a polygonal neighborhood of a puncture bounded by neighboring zigzag curves. Two pieces of $D$ bounding each other are of opposite type. An example of this checkerboard coloring is depicted in Figure 21.3b Since the specific disk $D$ is supposed to cover no puncture, it only consists of pieces of the first type. In fact, there it consists of only one such piece, since any second bordering piece would be of the second type. This shows that $D$ precisely fits a polygon of $Q$.

Let us conclude the product formula 21.1. If $e_{k} \ldots e_{1} e$ is a clockwise polygon in $\check{Q}$, then the same path forms a clockwise polygon in $Q$. As we have just seen, there is a single disk $D$ contributing the product, and we conclude

$$
\mu\left(X_{e_{k}}, \ldots, X_{e_{1}}\right)=Y_{e}
$$

The sign is positive because all inputs of $D$ are odd and run clockwise with the disk, the output is odd, and no \# signs appear on the zigzag curve segments. If $e_{k} \ldots e_{1} e$ is a counterclockwise polygon in $\check{Q}$, then the path $e_{1} \ldots e_{k} e$ is a counterclockwise polygon in $Q$. As we have just seen, there is a single disk $D$ contributing to the product, and we conclude

$$
\mu\left(X_{e_{k}}, \ldots, X_{e_{1}}\right)=-Y_{e}
$$

The sign is negative because $D$ has $k$ odd inputs running counterclockwise with $D$, while there is also a \#-sign in every of the $k+1$ corners of the polygon.

If $e_{k} \ldots e_{1} e$ is a path in $\check{Q}$ that is not a polygon, there is no disk $D$ with inputs $X_{e_{1}}, \ldots, X_{e_{k}}$ and output $Y_{e}$ at all. In summary, we have shown 21.1. Ultimately, the superpotential is given by

$$
\begin{aligned}
W & =\left\langle\mu_{\mathrm{HTw} \mathrm{Ttl} Q}(b, \ldots, b), b\right\rangle \\
& =\sum_{e_{k}, \ldots, e_{1}, e} e_{k} \ldots e_{1} e\left\langle\mu\left(X_{e_{k}}, \ldots, X_{e_{1}}\right), Y_{e}\right\rangle \\
& =\sum_{\substack{a_{k} \ldots a_{1} \\
\text { clockwise }}}\left(a_{k} \ldots a_{1}\right)_{\mathrm{cyc}}-\sum_{\substack{a_{k} \ldots a_{1} \\
\text { counterclockwise }}}\left(a_{k} \ldots a_{1}\right)_{\mathrm{cyc}} .
\end{aligned}
$$

This finishes the calculation of the superpotential $W$. The final statement on $R_{a}$ and $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)$ is now immediate: By Lemma 20.16 we have $R_{a}=\partial_{a} W$ which simplifies to $R_{a}=r_{a}^{+}-r_{a}^{-}$. In particular, we have $\operatorname{Jac}\left(Q^{\mathbb{L}}, W\right)=\operatorname{Jac} Q$. This finishes the proof.

Remark 21.6. As preparation for the proof of Lemma 21.7, let us fix some terminology regarding the identity location of a zigzag path. Let $L_{i} \in \mathbb{L}$ be one of the zigzag paths of $Q$. The identity location of $L_{i}$ is a certain arc $a_{0}$. This arc borders precisely two polygons in $Q$. One of the two is clockwise, the other counterclockwise. The special polygon $P_{i}$ of $L_{i}$ is the one lying on the left or right side of the arc $a_{0}$, depending on whether where $L_{i}$ turns left or right at the head of $a_{0}$, respectively. If $Q$ has no punctures of valence 2 , this condition can also be expressed as follows: $P$ is the polygon which has the identity arc $a_{0}$ and its successor within $L_{i}$ as part of the boundary.

In case $P_{i}$ is clockwise, we denote by $p_{1}, \ldots, p_{l}$ be the boundary arcs of $P_{i}$, in clockwise order and ending with the identity arc $p_{l}=a_{0}$. In case $P_{i}$ is counterclockwise, let $p_{1}, \ldots, p_{l}$ be the boundary arcs of $P_{i}$, in clockwise order and starting with $p_{1}=a_{0}$. We may call $p_{1}, \ldots, p_{l}$ the special polygon sequence of $L_{i}$. Note that $p_{l} \ldots p_{1}$ is only a path in $Q$ if $P_{i}$ is clockwise, while it is always a path in $\check{Q}$.

Lemma 21.7 (26, Lemma 10.21]). Application of the Cho-Hong-Lau construction to $\mathbb{L} \subseteq$ H Tw Gtl $Q$ yields the familiar potential $\ell=\sum \ell_{v} \in \operatorname{Jac} \check{Q}$. Altogether, application yields the familiar LandauGinzburg model ( $\operatorname{Jac} \check{Q}, \ell)$.

Proof. The general potential $\ell=\sum_{i \in Q_{0}^{\mathbb{L}}} \ell_{i}$ is given by

$$
\ell_{i}=\left\langle\mu(b, \ldots, b), \mathrm{id}_{L_{i}}\right\rangle
$$

Specifically for $\mathbb{L} \subseteq \mathrm{H} \operatorname{Tw} \operatorname{Gtl} Q$ this description boils down to extracting the identities from the products

$$
\mu_{\mathrm{HTw} \mathrm{Gtl} Q}\left(X_{e_{k}}, \ldots, X_{e_{1}}\right)
$$

Denote by $p_{1}, \ldots, p_{l}$ the special polygon sequence of $L_{i}$, defined in Remark 21.6. We claim the coefficient of $\operatorname{id}_{L_{i}}$ in the product $\mu\left(X_{e_{k}}, \ldots, X_{e_{1}}\right)$ is precisely

$$
\left\langle\mu_{\mathrm{HTw} \mathrm{Gtl} Q}\left(X_{e_{k}}, \ldots, X_{e_{1}}\right), \mathrm{id}_{L_{i}}^{*}\right\rangle= \begin{cases}+1 & \text { if } e_{k} \ldots e_{1}=p_{l} \ldots p_{1}  \tag{21.2}\\ 0 & \text { else. }\end{cases}
$$

As in the proof of Lemma 21.5. we use our explicit description of the minimal model HL . Again, the only disks that count are those not covering any punctures. Let $D$ be a (CR, ID, DS or DW) disk contributing to the product 21.2 . Note that all input morphisms $X_{e_{i}}$ are odd and the output is an identity, so $D$ is necessarily a CR or ID disk. Similar to the case of the products presented in the proof of Lemma 21.5 we conclude that $D$ precisely bounds a certain polygon $P$. More precisely, the length $l$ of the polygon is equal to the number $k$ of inputs of $D$, and the boundary arcs of $P$ are equal to $e_{1}, \ldots, e_{k}$ in this order.

Let us analyze the properties of $D$. By assumption, the output of $D$ is the identity of $L_{i}$. This identity is therefore located on one of the boundary arcs of $P$. Since every single odd intersection located at arcs of the polygon is used as input of $D$, the identity location necessarily of $L_{i}$ necessarily lies infinitesimally close to one of the inputs of $D$. We conclude that $D$ is necessarily an ID disk.

We claim that $P$ is the special polygon $P_{i}$ of $L_{i}$. To show this, assume $P$ is clockwise. By the definition of ID disks, the input lying close to the output precedes the output. In other words, the identity arc is $e_{k}$ and the zigzag path $L_{i}$ is the one turning right at the head of $e_{k}$. Assume now $P$ is counterclockwise. By definition of ID disks, the input lying close to the output succeeds the output. In other words, the identity arc is $e_{1}$ and the zigzag path $L_{i}$ is the one turning left at the head of $e_{1}$. In both cases we conclude that $P$ is precisely the special polygon $P_{i}$.

We are ready to conclude the product formula 21.2). In case $e_{1}, \ldots, e_{k}=p_{1}, \ldots, p_{k}$ within $Q$ is the boundary of the special polygon $P_{i}$, then there is a single disk contributing to the product as we have just seen. We conclude

$$
\left\langle\mu\left(X_{e_{k}}, \ldots X_{e_{1}}\right), \operatorname{id}_{L_{i}}^{*}\right\rangle=+1
$$

The sign is always positive. Indeed, if $D$ lies in a clockwise polygon, then all of its $k$ inputs are odd but clockwise with $D$ and no \# signs appear on the boundary of $D$. If $D$ lies in a counterclockwise polygon, then all of its $k$ inputs are odd and counterclockwise, while there is also a \#-sign in every of the $k$ corners of the polygon

In case $e_{1}, \ldots, e_{k}$ is not $p_{1}, \ldots, p_{k}$, then there is no disk contributing to the product as we have just seen. We conclude

$$
\left\langle\mu\left(X_{e_{k}}, \ldots, X_{e_{1}}\right), \mathrm{id}_{L_{i}}^{*}\right\rangle=0
$$

Ultimately, the potential reads

$$
\begin{aligned}
\ell & =\sum_{i \in \check{Q}_{0}}\left\langle\mu_{\mathrm{HTw} \operatorname{Gtl} Q}(b, \ldots, b), \mathrm{id}_{L_{i}}^{*}\right\rangle \\
& =\sum_{i \in \check{Q}_{0}} \ell_{i} \in \operatorname{Jac} \check{Q} .
\end{aligned}
$$

Here for every $i$ the letter $\ell_{i}$ denotes an arbitrary boundary cycle of a polygon starting at vertex $i \in \check{Q}_{0}$. This finishes the proof.

Lemma 21.8 ([26, Proposition 10.30]). Application of the Cho-Hong-Lau construction to $\mathbb{L} \subseteq \mathrm{H}$ Tw Gtl $Q$ yields the familiar matrix factorizations

$$
F(a)=a:(\operatorname{Jac} \check{Q}) h(a) \rightleftarrows(\operatorname{Jac} \check{Q}) t(a): \bar{a} \text { for } a \in \operatorname{ObGtl} Q=Q_{1}
$$

Proof. By definition, the matrix factorization $F(a)$ is given by $(M, \delta)$ with module given by

$$
M=\bigoplus_{L_{i} \in \check{Q}_{0}}(\operatorname{Jac} \check{Q}) L_{i} \otimes \operatorname{Hom}_{\mathrm{H} \operatorname{Tw} \operatorname{Gtl} Q}\left(L_{i}, a\right)
$$

and differential given by

$$
\delta(m)=(-1)^{\|m\|} \sum \mu_{\mathrm{HTw} \mathrm{Gtl} Q} \mu(m, b, \ldots, b)
$$

Let us evaluate the module. As seen earlier, there are precisely two indices $i, j$ whose hom space $\operatorname{Hom}_{\mathrm{HTw} \operatorname{Gt1} Q}\left(L_{i}, a\right)$ is non-empty. These indices correspond to the two zigzag paths that cross the arc $a$. Of course, the two zigzag paths leaving $a$ might actually be equal, in which case we set $i=j$. If we set $i$ to be the zigzag path with the even intersection and $j$ the zigzag path with the odd intersection, we have $i=h(a)$ and $j=t(a)$. This already gives the shape

$$
(M, \delta)=*:(\operatorname{Jac} \check{Q}) h(a) \rightleftarrows(\operatorname{Jac} \check{Q}) t(a): *
$$

It remains to evaluate the differential $\delta$. Denote the even intersection point by $p \in \operatorname{Hom}_{\mathrm{HTw}}^{\mathrm{Gtl} Q}\left(L_{h(a)}, a\right)$ and the odd intersection point by $q \in \operatorname{Hom}_{\operatorname{HTw}_{\mathrm{GtI}}( }\left(L_{t(a)}, a\right)$. As we have seen earlier, the two products $\mu_{\mathrm{HTw} \mathrm{Gtl} Q}(p, b, \ldots, b)$ and $\mu_{\mathrm{HTwGtI} Q}(q, b, \ldots, b)$ are computed by MT and MD disks, respectively. For the present non-deformed case, this description boils down to the equations

$$
\begin{array}{r}
\delta(h(a) \otimes p)=(-1)^{\|p\|} \mu_{\mathrm{HTw} \mathrm{Gtl} Q}(p, b, \ldots, b)=(-1)^{1+1} a \otimes q \\
\delta(t(a) \otimes q)=(-1)^{\|q\|} \mu_{\mathrm{HTw} \mathrm{Gtl} Q}(q, b, \ldots, b)=+\bar{a} \otimes p
\end{array}
$$

This shows that the module $\operatorname{map}(\operatorname{Jac} \check{Q}) h(a) \rightarrow(\operatorname{Jac} \check{Q}) t(a)$ is given by left multiplication with $a$ and the module $\operatorname{map}(\operatorname{Jac} \check{Q}) t(a) \rightarrow(\operatorname{Jac} \check{Q}) h(a)$ is given by left multiplication with $\bar{a}$.

We are now ready to grasp the mirror functor associated with the specific case of $\mathrm{H} \mathbb{L} \subseteq H \mathrm{Tw}_{\mathrm{w}} \mathrm{Gtl} Q$. Indeed, we have computed the specific Landau-Ginzburg model and the mirror objects. The functor itself is not a quasi-equivalence, but it becomes one if we restrict to domain $\mathrm{Gtl} Q \subseteq \mathrm{HTw} \operatorname{Gtl} Q$ and codomain $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell) \subseteq \operatorname{MF}(\operatorname{Jac} \check{Q}, \ell)$.
Corollary 21.9 (26, Proposition 10.32]). If $\check{Q}$ is zigzag consistent, the CHL functor $F: \operatorname{Gtl} Q \rightarrow$ $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ is a quasi-isomorphism.

Proof. On the level of objects, $F$ maps an arc $a \in Q_{1}$ to the matrix factorization $F(a)$ which is bijective on the level of objects. On the level of hom spaces, we are forced to cheat a little and believe the following fact: Let $L_{1}, \ldots, L_{N+1}$ be a sequence of $N+1 \geq 1$ zigzag paths and let $a, b \in Q_{1}$ be arcs. Let $h_{1}, \ldots, h_{N}$ be odd morphisms with $h_{i} \in \operatorname{Hom}_{\mathrm{H} \mathbb{L}}\left(L_{i}, L_{i+1}\right)$ not containing co-identities, let $m \in \operatorname{Hom}_{\mathrm{HTw} \operatorname{Gtl} Q}\left(L_{N+1}, a\right)$ and $\alpha \in \operatorname{Hom}_{\mathrm{HTw} \operatorname{GtI} Q}(a, b)$ be further morphisms. Then we shall assume without proof that the product

$$
\mu_{\mathrm{HTw} \mathrm{Gtl} Q}\left(\alpha, m, h_{N}, \ldots, h_{1}\right)
$$

is computed by smooth immersed disks with Abouzaid sign. A good illustration of these disks can be found in 26, Figure 20]. This illustration makes it easy to check that $F^{1}$ sends an angle $\alpha \in \operatorname{Hom}_{\text {Gtl } Q}(a, b)$ to its associated morphism of matrix factorizations

$$
\pm\left(\begin{array}{cc}
0 & -\mathrm{opp}_{1} \\
\mathrm{opp}_{2} & 0
\end{array}\right) \text { or } \pm\left(\begin{array}{cc}
\mathrm{opp}_{2} & 0 \\
0 & \mathrm{opp}_{1}
\end{array}\right)
$$

depending on whether $\alpha$ is odd or even. It was shown in 18 that these morphisms of matrix factorizations


The original mirror functor for punctured surfaces 18 was defined in a nonconstructive and nonunique way. Its properties known precisely are its values on objects and the first component $F^{1}$. All higher components $F^{\geq 2}$ are defined nonconstructively. In contrast, the CHL functor $F: \operatorname{Gtl} Q \rightarrow \operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ has explicitly defined higher components, at least if one counts the involved higher products of H Tw Gtl $Q$ as explicit. The CHL functor should therefore be viewed as a modern constructive incarnation of the original mirror functor.
Remark 21.10. Cho, Hong and Lau use a different convention for Fukaya categories: In their convention, a disk $D$ contributing to a product $\mu\left(h_{N}, \ldots, h_{1}\right)$ is supposed to hit the intersection points $h_{1}, \ldots, h_{N}$ in counterclockwise order. We have decided to stick with the original definition of gentle algebras and let products be given by counting disks hitting the intersection points in clockwise order.

Correspondingly, the conventions for the dual dimer $\check{Q}$ also differ: In their convention, the dual dimer $\check{Q}$ is obtained from $Q$ by flipping over the clockwise faces. In our convention, the dual dimer $\check{Q}$ is obtained by flipping over the counterclockwise faces instead. The results differ by a flip of orientation and an inversion of the arrows.

### 21.2 Midpoint polygons

In this section, we introduce an auxiliary tool for the description of the deformed superpotential and deformed potential in section 21.4 and 21.5 The deformed Cho-Hong-Lau construction namely expresses superpotential and potential in term of the products on $H \mathbb{L}_{q}$ which are in turn enumerated in terms of CR, ID, DS and DW disks. In the present section, we overhaul this description and provide one single type of disks, which we call midpoint polygons.
Definition 21.11. A midpoint polygon is an immersion of the standard polygon $D: P_{N} \rightarrow|Q|$ with $N \geq 1$, such that

- The boundary of $P_{N}$ is mapped to (nonempty) zigzag curve segments.
- The corners are convex and lie on intersections points of zigzag curves.
- The corners point into the interior of a polygon.

The map $D$ itself is taken only up to reparametrization. The corners $h_{1}, \ldots, h_{N}$ of $D$ lie on intersection points of zigzag curves, in other words on the midpoints of arcs $a_{1}, \ldots, a_{N} \in Q_{1}$. We refer to the sequence of arcs as the arc sequence of $D$. The midpoint polygon starts at arc $a_{1}$ and ends at arc $a_{N}$. It starts at the left/right side of $a_{1}$ if the interior of $D$ at $a_{1}$ lies at the left/right of the arc $a_{1}$ in the arc's natural orientation. It ends at the left/right side of $a_{N}$ if the interior of $D$ lies at the left/right of the arc $a_{N}$ in the arc's natural orientation. A arc crossing of $D$ is the datum of a single (indexed) arc crossed by one of the zigzag segments of $D$, not counting the corner arcs. Arc crossings are always supposed to come with the datum of their location on the boundary of $D$.

Midpoint polygons are our most general container format for enumerating products in $\mathrm{H} \mathbb{L}_{q}$. Midpoint polygons with specific properties or specific additional data can be used to enumerate specific products. By requiring that the corners point into the interior of a polygon, we have incarnated the fact that we only regard disks with odd inputs.

Definition 21.12. Let $D$ be a midpoint polygon with sides lengths $n_{1}, \ldots, n_{k}$. Then the sign of $D$ is $|D|=\sum\left(n_{i}-1\right) / 2 \in \mathbb{Z} / 2 \mathbb{Z}$. The deformation parameter $\operatorname{Punc}(D) \in \mathbb{C} \llbracket Q_{0} \rrbracket$ is the product of the punctures covered by $D$, counting multiplicities. Let $e_{1}, \ldots, e_{k}$ be the arc sequence of $D$. Then the path recording of $D$ is the path $\operatorname{Arcs}(D)=e_{k} \ldots e_{1} \in \mathbb{C} Q$.

A sample midpoint polygon is depicted in Figure 21.4a.

### 21.3 Cyclicity and slow growth

In this section, we check that $H \mathbb{L}_{q}$ is deformed cyclic and of slow growth. These two properties are technical requirements for the deformed Cho-Hong-Lau construction according to Theorem 20.50 We check the cyclicity property directly in terms of midpoint polygons. By contrast, the slow growth property makes reference also to products between arcs and zigzag paths in $\mathrm{HTw} \mathrm{Gtl}_{q} Q$ and must therefore be dealt with from scratch. We do not prove slow growth with respect to the entire category, but restrict to arcs and zigzag paths.

(a) Midpoint polygon

(b) Lengths and signs of zigzag segments

Figure 21.4: These pictures illustrate midpoint polygons and how we measure the lengths of their boundary segments. In the midpoint polygon on the left, the intersection points are located at arcs $e_{1}, e_{2}$, $e_{3}$ of $Q$. To this midpoint polygon $D$, we assign the path recording $\operatorname{Arcs}(D)=e_{3} e_{2} e_{1}$ and the sign $|D|=3 \cdot(7-1) / 2=1 \in \mathbb{Z} / 2 \mathbb{Z}$.

Lemma 21.13. The $A_{\infty}$-deformation $\mathrm{H}_{\mathbb{L}_{q}}$ is deformed cyclic on odd sequences containing no coidentities.

Proof. Rotating a midpoint polygon contributing to $\left\langle\mu_{\mathrm{H}_{\mathbb{L}_{q}}}\left(X_{e_{k-1}}, \ldots, X_{e_{1}}\right), Y_{e_{k}}\right\rangle$ gives a midpoint polygon contributing to $\left\langle\mu_{\mathrm{H} \mathbb{L}_{q}}\left(X_{e_{k}}, \ldots, X_{e_{2}}\right), X_{e_{1}}\right\rangle$. Their Abouzaid signs and deformation parameters agree. We conclude that both pairings agree, which finishes the proof.

Let us now check the slow growth requirement property. The requirement is slightly problematic in that we are supposed to evaluate products of the form $\mu_{\mathrm{HTw}_{\mathrm{Gt1}}^{q}}\left(m_{k}, \ldots, m_{1}, b, \ldots, b\right)$. Here $m_{1}, \ldots, m_{k}$ are morphisms between arbitrary objects of $\mathrm{HTw}_{\mathrm{Gtl}}^{q}$ $Q$ and we do not have these products under control. However, for our purposes it suffices to construct the mirror functor merely on the subcategory $\mathrm{Gtl}_{q} Q \subseteq \mathrm{HTw} \mathrm{Gtl}_{q} Q$. Inspecting the construction of the functor $F_{q}$ and the proof of Lemma 20.46 and Theorem 20.50, we conclude that for constructing the functor $F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{MF}\left(\operatorname{Jac}\left(Q^{\mathrm{L}}, W_{q}\right), \ell_{q}\right)$ it suffices to check the slow growth property of Definition 20.23 only for morphisms with $m_{1}: L \rightarrow a_{1}$ and $m_{i}: a_{i-1} \rightarrow a_{i}$ with $L \in \mathbb{L}_{q}$ and $a_{1}, \ldots, a_{k} \in \operatorname{Gtl}_{q} Q$. We record this as follows:

Lemma 21.14. Let $k \geq 0$ and $a_{1}, \ldots, a_{k} \in \operatorname{Gtl}_{q} Q$. Let $L \in \mathbb{L}_{q}$ and $m_{1}: L \rightarrow a_{1}$ and $m_{i}: a_{i-1} \rightarrow a_{i}$ for $i=2, \ldots, k$. Then for every $n \in \mathbb{N}$ there exists an $l_{0} \in \mathbb{N}$ such that

$$
\forall l \geq l_{0}: \quad \mu_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}^{k+l}\left(m_{k}, \ldots, m_{1}, b, \ldots, b\right) \in \mathfrak{m}^{n} \operatorname{Hom}\left(\mathbb{L}, a_{k}\right)
$$

Proof. The intuition is that the number of $b$ insertions is a lower bound for the size of a disk contributing to the product. In other words, the number of punctures covered by a disk is at least as large as the number of $b$ insertions, up to a multiplicative constant. If one is willing to assume that the product $\mu_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}\left(m_{k}, \ldots, m_{1}, b, \ldots, b\right)$ is computed by counting disks (which we have not shown), this argument should suffice.

Otherwise, the property can be checked rigorously as follows: According to the minimal model calculation procedure detailed in Paper II, the product can be computed by summing over Kadeishvili trees. In Paper II we also provide a description of the applicable codifferential $h_{q}$ for hom spaces between zigzag paths and for hom spaces from zigzag paths to arcs.

The idea is to measure subtrees containing purely $b$ inputs separately. Thanks to the dedicated subdisk construction in Paper II we see that result components of $h$-trees only consisting of $b$ inputs immediately lie in order $\geq n$ as soon as they have a certain amount of $K \in \mathbb{N}$ inputs. The $h$-trees consisting only of $b$ inputs only yield $\beta(\mathrm{A})$ result components of order $\geq 1$. Apart from the fact that this already establishes the claim in case $k=0$, this observation is important for what follows.

If $m$ is any morphism between zigzag paths or arcs, then we call the maximum length of angles contained in $m$ simply the length of $m$. Let $I$ be the sum of the lengths of $m_{1}, \ldots, m_{k}$. Let $F$ the maximum length of a full turn around a puncture in $Q$.

Pick a result component of the product $\mu\left(m_{k}, \ldots, m_{1}, b, \ldots, b\right)$ of order $<n$. Let $T$ be the Kadeishvili $\pi$-tree which the result component is derived from. We call a node of $T$ pure if its subtree purely consumes $b$ inputs (instead of $m_{i}$ ) and if it is maximal with this property $T$ (in that its parent does not have this property). We view a subtree at a pure node of $T$ as an indecomposable unit. We call all nodes of $T$ typical that are non-leaf and not contained in a pure subtree.

Regard a typical node $N \in T$. We claim the result at $N$ has length at most one less than its total inputs including outer $\delta$ insertions and excluding inner $\delta$ inputs and direct $b$ inputs. Indeed, the result
component at the node is a morphism from a zigzag path to an arc by construction. When $m$ is an $H$ or $R$ basis morphism from a zigzag path to an arc, it can be checked easily that the following products vanish:

$$
h_{q} \mu_{\operatorname{Add~Gtl}_{q} Q}^{2}\left(m, \alpha_{3} / \alpha_{4}\right), h_{q} \mu_{\operatorname{Add~Gtl}_{q} Q}^{2}(m, \beta(\mathrm{~A})), \pi_{q} \mu_{{\operatorname{Add~} \operatorname{Gtl}_{q} Q}_{2}^{2}}\left(m, \alpha_{3} / \alpha_{4}\right), \pi_{q} \mu_{\operatorname{Add~Gtl}_{q} Q}^{2}(m, \beta(\mathrm{~A}))
$$

Therefore the node $N$ is necessarily decorated with $h_{q} \mu_{\operatorname{Add~Gtl}_{q} Q}^{\geq 3}$ or $\pi_{q} \mu_{\text {Add }^{\geq} \operatorname{Gt1}_{q} Q}$. A first-out or final-out disk definitely reduces the length of the inputs. This shows the claim.

According to the claim just proven, the total length of the output of a typical node $N \in T$ is at most $I+n F-s$, where $s$ is the number of typical nodes in the subtree at $N$. This bounds the number of typical nodes in $T$ by $I+n F$. The subtree at every pure node has at most $K$ nodes by assumption. Since there are at most $n$ pure nodes, the tree $T$ in total has at most $I+n F+n K$ nodes. This bounds the total number of nodes in $T$.

Regard a typical node $N$. Among the child nodes of $N$, there are at most $k$ typical children and at most $n$ pure children. Moreover, $N$ may have at most $n$ direct outer $\delta$ insertions. This gives rise to already $2 n+k$ inputs of the $h_{q} \mu_{\operatorname{Add~Gtl}_{q} Q}$ at $N$. Every string of direct $b$ and $\delta$ insertions between these $2 n+k$ inputs is limited to at most $F$ items, since otherwise no discrete immersed disk can be made. This means that $N$ consumes at most $(2 n+k+1) F$ many direct $b$ inputs. This number is bounded.

The total number of direct $b$ inputs at typical nodes is therefore bounded by $(2 n+k+1) F(I+n F)$. The total number of $b$ inputs used for pure trees is at most $n K$. Therefore the total number of $b$ inputs in the tree $T$ is bounded by $(2 n+k+1) F(I+n F)+n K$.

We have shown that the product $\mu_{\mathrm{HTw}_{\mathrm{Gt1}}^{q}} Q\left(m_{k}, \ldots, m_{1}, b, \ldots, b\right)$ can only have nonzero result components of order $<n$ if the number of $b$ insertions is bounded. In consequence, once the number of $b$ insertions exceeds this bound, all result components are necessarily of order $\geq n$. This finishes the proof.

### 21.4 Deformed superpotential

In this section, we start applying the deformed Cho-Hong-Lau construction to $\mathrm{H}_{q} \subseteq \mathrm{HTw} \mathrm{Gtl}_{q} Q$. The result is a deformed superpotential $W_{q} \in \mathbb{C} Q \check{Q} Q_{0} \rrbracket$, which we describe explicitly in terms of midpoint polygons. After that, we describe the resulting deformed Jacobi algebra. In order to obtain a mirror functor according to Theorem 20.50 we are required to show that the deformed Jacobi algebra is a deformation of the classical Jacobi algebra. In the present section, we invoke the flatness result Theorem 19.77 to show that this is the case. Recall that we work under Convention 18.1

Lemma 21.15. The deformed superpotential $W_{q} \in \mathbb{C} \mathscr{Q} \llbracket Q_{0} \rrbracket$ can be expressed as

$$
W_{q}=\sum_{\substack{D \text { clockwise } \\ \text { midpoint polygon }}}(-1)^{|D|} \operatorname{Punc}(D) \operatorname{Arcs}(D)-\sum_{\substack{D \text { counterclockwise } \\ \text { midpoint polygon }}}(-1)^{|D|} \operatorname{Punc}(D) \operatorname{Arcs}(D)
$$

Proof. Let us digest the statement. It is our task to evaluate the products $\mu_{\mathrm{HL}_{q}}\left(X_{e_{k-1}}, \ldots, X_{e_{1}}\right)$ and extract the coefficient of $Y_{e_{k}}$. By our explicit description of the products from section 18.3 these products boil down to counting CR, ID, DS and DW disks. Since ID disks have identity outputs and DS and DW disks are already irrelevant, we are left with the task of counting CR disks.

Let $D$ be a CR disk with inputs $X_{e_{1}}, \ldots, X_{e_{k-1}}$ and output $Y_{e_{k}}$. Since $D$ is CR and its inputs contain no co-identities, all of the zigzag segments of $D$ are actually non-empty. This immediately renders $D$ a midpoint polygon. More precisely, we associate with $D$ the midpoint polygon given by the one with the same shape as $D$ and arc sequence $e_{1}, \ldots, e_{k}$. Note that this sets up a bijection
$\left\{\mathrm{CR}\right.$ disks with inputs $X_{e_{1}}, \ldots, X_{e_{k-1}}$ and output $\left.Y_{e_{k}}\right\}$
$\quad \longleftrightarrow \quad\left\{\right.$ midpoint polygons with arc sequence $\left.e_{1}, \ldots, e_{k}\right\}$

We can therefore write

$$
\begin{aligned}
\left\langle\mu\left(X_{e_{k-1}}, \ldots, X_{e_{1}}\right), Y_{e_{k}}\right\rangle & =\sum_{\substack{\operatorname{CR} \text { disks } D \\
\text { with inputs } X_{e_{1}}, \ldots, X_{e_{k-1}} \\
\text { and output } Y_{e_{k}}}}(-1)^{\operatorname{Abou}(D) \operatorname{Punc}(D)} \underset{\substack{\text { midpoint polygons } D \\
\text { with arc sequence } e_{1}, \ldots, e_{k}}}{ }(-1)^{\operatorname{Abou}(D) \operatorname{Punc}(D)} .
\end{aligned}
$$

Here we have already anticipated that the Abouzaid sign of a CR disk $D$ with inputs $X_{e_{1}}, \ldots, X_{e_{k-1}}$ and output $Y_{e_{k}}$ is equal to the Abouzaid sign attached to its associated midpoint polygon by Definition 21.12.

In the remainder of the proof, we check that these two signs are indeed equal. Regard one of the CR disks. Its boundary cuts a number of angles of clockwise and counterclockwise polygons of $Q$. Let $n_{1}, \ldots, n_{k}$ denote the lengths of the zigzag segments, as depicted in Figure 21.4b. The \# signs alternate along the zigzag segments and the individual lengths $n_{i}$ are all odd. At every corner $X_{e_{i}}$, the two neighboring signs are equal and in fact determined by whether $D$ is clockwise or counterclockwise.

If $D$ is clockwise, the sign at every corner is $0 \in \mathbb{Z} / 2 \mathbb{Z}$ and it alternates $n_{i}$ times until the next corner. This gives a sign contribution of $\left(n_{i}-1\right) / 2$. The total sign becomes

$$
\begin{equation*}
\sum_{i=1}^{k} \frac{n_{i}-1}{2} \in \mathbb{Z} / 2 \mathbb{Z} \tag{21.3}
\end{equation*}
$$

Since all zigzag curves lie clockwise with $D$, this is already the Abouzaid sign of $D$. If $D$ is counterclockwise, the sign at every corner is $1 \in \mathbb{Z} / 2 \mathbb{Z}$. In comparison with 21.3, we incur a sign flip for all $n_{1}+\ldots+n_{k}$ angles that $L$ runs through. Since every $n_{i}$ is odd, this changes the sign by $k \in \mathbb{Z} / 2 \mathbb{Z}$. Since all zigzag paths are counterclockwise relative to $D$, the Abouzaid sign also incurs an increase by $k-1 \in \mathbb{Z} / 2 \mathbb{Z}$. This makes the sign precisely opposite to 21.3 and finishes the proof.

The deformed Cho-Hong-Lau construction proceeds by defining the deformed Jacobi algebra. The construction comes in two variants, depending on whether the category of reference objects $\mathbb{L}$ is of slow growth with respect to $\mathcal{C}$ or not. In the case of $\mathrm{H}_{q} \subseteq \mathrm{HTw} \mathrm{Gtl}_{q} Q$, we have seen in Lemma 21.14 that $\mathrm{H} \mathbb{L}_{q}$ is of slow growth, at least in a way sufficient for the Cho-Hong-Lau construction. We can therefore apply the slow growth variant of the deformed Cho-Hong-Lau construction and obtain a deformed Jacobi algebra $\operatorname{Jac}\left(\check{Q}, W_{q}\right)$ according to Definition 20.25 . We shall abbreviate this algebra by $\mathrm{Jac}_{q} \check{Q}$ :

Definition 21.16. We denote the deformed Jacobi algebra by

$$
\operatorname{Jac}_{q} \check{Q}=\operatorname{Jac}\left(\check{Q}, W_{q}\right)=\frac{\mathbb{C} \check{Q} \llbracket Q_{0} \rrbracket}{\left(\partial_{e} W_{q}\right)_{e \in \check{Q}_{1}}}
$$

Remark 21.17. Recall that the closure refers to the $\mathfrak{m}$-adic topology on $\mathbb{C} \check{Q} \llbracket Q_{0} \rrbracket$.
Let us now explain that the deformed Jacobi algebra $\mathrm{Jac}_{q} \check{Q}$ is a deformation of the classical Jacobi algebra Jac $\check{Q}$ in the sense of Definition 19.1 We have investigated this question in detail during section 19 in the general context of superpotential deformations of CY3 Jacobi algebras. We record the specific case of $\mathrm{Jac}_{q} \check{Q}$ as follows:

Corollary 21.18. If $\check{Q}$ is cancellation consistent and of bounded type, then $\mathrm{Jac}_{q} \check{Q}$ is a deformation of $\operatorname{Jac} \mathscr{Q}$.
Proof. The goal is to invoke Theorem 19.77 We have to check two conditions. First, $W_{q}$ and therefore also $W^{\prime}=W_{q}-W$ is indeed cyclic as shown in Lemma 21.13 or equivalently 21.15 Second, $W_{q}$ indeed lies in $\mathbb{C} \check{Q} \llbracket Q_{0} \rrbracket$ instead of only $\widehat{\mathbb{C} Q} \llbracket Q_{0} \rrbracket$, since $\mathrm{H} \mathbb{L}_{q}$ is of slow growth by Lemma 21.14 Finally, we conclude that Theorem 19.77 applies and $\mathrm{Jac}_{q} \check{Q}$ is a deformation of $\operatorname{Jac} \check{Q}$. This finishes the proof.
Remark 21.19. The dual dimer $\check{Q}_{M}$ of the standard sphere dimer $Q_{M}$ is cancellation consistent. The dimer $\check{Q}_{M}$ is also of bounded type because its two polygons have the same length, namely $M$. Therefore Corollary 21.18 particularly applies to the case $Q=Q_{M}$.

### 21.5 Deformed potential

In this section, we compute the deformed potential. More precisely, we evaluate the definition of the deformed potential from the deformed Cho-Hong-Lau construction in the case of $\mathrm{H} \mathbb{L}_{q}$. The result is an element $\ell_{q} \in \check{Q} \llbracket Q_{0} \rrbracket$. We describe this element in terms of midpoint polygons. Recall that we work under Convention 18.1

Recall that the deformed Cho-Hong-Lau construction in general requires us to compute a deformed potential $\ell_{q} \in B \widehat{\otimes} Q^{\mathbb{L}}$, given by counting identity outputs of products of the form $\mu\left(X_{e_{k}}, \ldots, X_{e_{1}}\right)$. More precisely, the definition reads

$$
\ell_{q}=\sum_{i \in Q_{0}^{\mathrm{L}}}\left\langle\mu(b, \ldots, b), \mathrm{id}_{i}^{*}\right\rangle .
$$


(a) First type

(b) Second type

(c) Third type

Figure 21.5: These pictures illustrate $L$-polygons of all three types. According to the definition, the midpoint polygon underlying an $L$-polygon of the first type starts on the left side of $a_{0}$. In the first picture, we have depicted this by writing $e_{1}=a_{0}$. The midpoint polygon underlying an $L$-polygon of the second type ends on the right side of $a_{0}$. We have depicted the ending in the second picture as $e_{3}=a_{0}$. The midpoint polygon underlying an $L$-polygon of the third type requires that the zigzag segment between the first and the last arc is $L$ and comes with the datum of a crossing with $a_{0}$. In the third picture, we have indicated this crossing by a thick cross.

It is our task to evaluate this deformed potential for the specific CHL pair $\mathrm{H}_{q} \subseteq \mathrm{HTw} \mathrm{Gtl}_{q} Q$. In this specific case, the products $\mu_{\mathrm{H} \mathbb{L}_{q}}(b, \ldots, b)$ are given by an enumeration of CR, ID, DS and DW disks. As announced, DS and DW disks are irrelevant. In fact, the enumeration boils down to counting midpoint polygons of specific type.

Regard a midpoint polygon. Its boundary crosses many arcs. Indeed, it consists of segments of zigzag curves, which run between midpoints of arcs. The longer the segments, the more arcs are crossed. A midpoint polygon may or may not cross arcs that are identity arcs $a_{0}$ of zigzag paths. If a midpoint polygon starts at, ends at or crosses an arc that is the identity arc $a_{0}$ of a zigzag path, then the midpoint polygon together with the datum of this crossing determines an identity contribution to the product of the intersection points associated with the polygon's corners. To make this precise, we set up the following terminology:
Definition 21.20. Let $L$ be a zigzag path in $Q$ with identity location $a_{0}$. An $L$-polygon is one of the following:

- If $L$ turns left at the head of $a_{0}$ : a midpoint polygon starting at the left side of $a_{0}$.
- If $L$ turns right at the head of $a_{0}$ : a midpoint polygon ending on the right side of $a_{0}$.
- A midpoint polygon $D$ whose segment between the last and first arc is an $L$ segment, together with the datum of a crossing of this $L$-segment with the arc $a_{0} \in L$.

We have illustrated $L$-polygons in Figure 21.5 With this in mind, we can express the potential $\ell_{q} \in \mathbb{C} \check{Q} \llbracket Q_{0} \rrbracket$ in terms of $L$-polygons:

Lemma 21.21. The deformed potential $\ell_{q}=\sum_{i \in \check{Q}_{0}} \ell_{q, i} \in \mathbb{C} \mathscr{Q} \llbracket Q_{0} \rrbracket$ can be expressed as

$$
\ell_{q, i}=\sum_{L_{i} \text {-polygons } D}(-1)^{|D|} \operatorname{Punc}(D) \operatorname{Arcs}(D)
$$

Proof. It is our task to evaluate identity terms in the products $\mu_{\mathrm{H} \mathbb{L}_{q}}(b, \ldots, b)$ and match them with $L$-polygons of the three types.

According to the description recapitulated in section 18.3 the products $\mu_{\mathrm{H} \mathbb{L}_{q}}(b, \ldots, b)$ are computed by enumerating CR and ID disks. More precisely, a counterclockwise CR disk with inputs $X_{e_{1}}, \ldots, X_{e_{k}}$ and output $\mathrm{id}_{L_{i}}$ corresponds precisely to an $L_{i}$-polygon with arc sequence $e_{1}, \ldots, e_{k}$ of the first type. A clockwise CR disk with inputs $X_{e_{1}}, \ldots, X_{e_{k}}$ and output $\mathrm{id}_{L_{i}}$ corresponds precisely to an $L_{i}$-polygon with arc sequence $e_{1}, \ldots, e_{k}$ of the second type. An ID disk with inputs $X_{e_{1}}, \ldots, X_{e_{k}}$ and output id $L_{L_{i}}$ corresponds precisely to an $L_{i}$-polygon with arc sequence $e_{1}, \ldots, e_{k}$ of the third type. This already matches all identity terms in the products $\mu_{\mathrm{H} \mathbb{L}_{q}}(b, \ldots, b)$ with $L$-polygons.

It remains to check the signs. More precisely, we have to check that the Abouzaid sign of a CR or ID disk contributing to $\mu_{\mathrm{H}_{q}}(b, \ldots, b)$ is equal to the sign $|D| \in \mathbb{Z} / 2 \mathbb{Z}$ of its associated midpoint polygon $D$. To see that both are equal, we distinguish whether $D$ is clockwise or counterclockwise. If the disk is clockwise, its Abouzaid sign is simply the sum of the \# signs along $D$, which by the proof of Lemma 21.15 is equal to $|D|$. If $D$ is counterclockwise and has $k$ corners, its Abouzaid sign is the sum of the \# signs


Figure 21.6: The difference between two choices of $\ell_{q}$ is a relation.
plus $k$, which by the proof of Lemma 21.15 is still $|D|$. This shows that the Abouzaid sign of a disk is equal to the sign $|D|$ of the associated midpoint polygon $D$. This proves the desired formula for $\ell_{q, i}$.

The deformed potential $\ell_{q} \in \mathbb{C} \check{Q} \llbracket Q_{0} \rrbracket$ depends on the choice of identity locations. However, we claim that it becomes independent of this choice when we pass to the deformed Jacobi algebra $\mathrm{Jac}_{q} \check{Q}$ :

Lemma 21.22. The deformed potential $\ell_{q} \in \operatorname{Jac}_{q} \check{Q}=\mathbb{C} \check{Q} \llbracket Q_{0} \rrbracket / \overline{\left(\partial_{a} W_{q}\right)}$ is independent of the choice of identity locations.

Proof. The idea is to describe the difference of two possible choices explicitly in terms of the relations $\partial_{a} W_{q}$. Denote by $L_{1}, \ldots, L_{N}$ the zigzag paths of $Q$. Recall that the element $\ell_{q}$ is the sum of elements $\ell_{q, i}$ over all $1 \leq i \leq N$. The identity location on a certain zigzag path $L_{i}$ only impacts the element $\ell_{q, i}$ and no other summands. It therefore suffices to study only a single element $\ell_{q, i}$. It also suffices to assume that the two choices of identity locations to be studied are arcs which are neighbors along $L_{i}$. The strategy is to enumerate $L_{i}$-polygons with respect to the two choices.

Let us denote the two neighboring arcs are $b_{1}$ and $b_{2}$, such that $h\left(b_{1}\right)=t\left(b_{2}\right)$ and $L_{i}$ turns left at the head of $b_{1}$ and right at the head of $b_{2}$. The situation is depicted in Figure 21.6. We claim that $L_{i}$-polygons with respect to $a_{0}=b_{1}$ and $L_{i}$-polygons with respect to $a_{0}=b_{2}$ are almost the same. Indeed, the typical $L_{i}$-polygon with respect to $a_{0}=b_{1}$ has a very long $\tilde{L}_{i}$-segment and many crossings with $b_{1}$. Since $b_{2}$ is the neighbor of $b_{1}$, all these $L_{i}$-polygons can be interpreted alternatively as $L_{i}$-polygons with respect to $a_{0}=b_{2}$. The only difference lies in the corner cases.

To be very precise, the only $L_{i}$-polygons with respect to $a_{0}=b_{1}$ which cannot be interpreted as $L_{i^{-}}$ polygons with respect to $a_{0}=b_{2}$ are those which start at $b_{1}$ and end at $b_{2}$, with just the shortest possible $\tilde{L}_{i}$-segment in between, consisting of a single angle. The only $L_{i}$-polygons with respect to $a_{0}=b_{2}$ which cannot be interpreted as $L_{i}$-polygons with respect to $a_{0}=b_{1}$ are those which end at $b_{2}$. Let us denote by $\ell_{q, i}^{\left(b_{1}\right)}$ and $\ell_{q, i}^{\left(b_{2}\right)}$ the two potentials in $\mathbb{C} \mathscr{Q} \llbracket Q_{0} \rrbracket$ obtained from the two choices of identity locations. In these terms, we conclude

$$
\begin{aligned}
\ell_{q, i}^{\left(b_{2}\right)}-\ell_{q, i}^{\left(b_{1}\right)} & =\sum_{\substack{\text { clockwise } \\
\text { midpoint polygons } D \\
\text { ending at } b_{2}}} \operatorname{Punc}(D) \operatorname{Arcs}(D)-\sum_{\substack{\text { counterclockwise } \\
\text { midpoint polygons } D \\
\text { ending at } b_{2}}} \operatorname{Punc}(D) \operatorname{Arcs}(D) \\
& =b_{2} \partial_{b_{2}} W_{q} .
\end{aligned}
$$

This shows that the difference of $\ell_{q, i}^{\left(b_{1}\right)}$ and $\ell_{q, i}^{\left(b_{2}\right)}$ is a relation in the deformed Jacobi algebra. In conclusion, the potential is independent of the choice of identity locations.

### 21.6 Deformed mirror objects

In this section, we compute the deformed mirror objects. More precisely, we evaluate the definition of the deformed mirror functor for the arc objects $a \in \mathrm{Gtl}_{q} Q \subseteq \mathrm{HTw}_{\mathrm{Gtl}}^{q}$ $Q$. We describe these objects $F_{q}(a) \in \operatorname{MF}\left(\mathrm{Jac}_{\sigma} \check{Q}, \ell_{a}\right)$ in terms of midpoint polygons. Recall that we work under Convention 18.1 As we have seen in section 21.1 the mirror functor $F: \operatorname{HTw} \operatorname{Gtl} Q \rightarrow \operatorname{MF}(\operatorname{Jac} \check{Q}, \ell)$ sends the $\operatorname{arcs} a \in Q_{1}$ to very explicit matrix factorizations $F(a)$. We provide in this section an explicit computation of the deformed matrix factorizations $F_{q}(a)$, which is in fact likewise explicit:

$$
\begin{aligned}
& \text { arc } a \in \operatorname{ObGtl} Q=Q_{1} \quad \stackrel{F}{\longleftrightarrow} \quad(\operatorname{Jac} \check{Q}) h(a) \stackrel{a}{\stackrel{a}{\rightleftarrows}}(\operatorname{Jac} \check{Q}) t(a) \\
& \operatorname{arc} a \in \mathrm{Ob} \mathrm{Gtl}_{q} Q=Q_{1} \quad \stackrel{F_{q}}{\longmapsto} \quad\left(\operatorname{Jac}_{q} \check{Q}\right) h(a) \underset{\bar{a}_{q}}{\stackrel{a}{\rightleftarrows}}(\operatorname{Jac} \check{Q}) t(a)
\end{aligned}
$$

To compute the object $F_{q}(X)$ for $X \in \mathcal{C}_{q}$, we are in general required to find for every $L_{i} \in \mathbb{L}$ the hom space $\operatorname{Hom}_{\mathcal{C}_{q}}\left(L_{i}, X\right)$. We also need to compute the products $\mu_{q}(m, b, \ldots, b)$ where $m \in \operatorname{Hom}_{\mathcal{C}_{q}}\left(L_{i}, X\right)$ is a morphism from one of the reference objects to $X$.

For the specific instance $\mathrm{H} \mathbb{L}_{q} \subseteq \mathrm{HTw}_{\operatorname{Gtl}}^{q}$ $Q$ of the deformed Cho-Hong-Lau construction, we have described the hom spaces and the relevant products already section 18.4 Computing the mirror objects $F_{q}(a)$ for $a \in \operatorname{Gtl}_{q} Q$ now boils down to evaluating the products $\mu_{\mathrm{HTw}_{\mathrm{Gtl}}^{q}} Q(m, b, \ldots, b)$, where $m \in$ $\operatorname{Hom}_{\mathrm{HTw}_{\operatorname{Ttl}}^{q}} Q(L, a)$ is an even or odd intersection point between $L$ and $a$. Recall that the products for even $m$ are easy to express, while the products for odd $m$ are computed by MD disks. We shall simplify the description of these products here in terms of midpoint polygons.
Definition 21.23. Let $a \in Q_{1}$ be an arc. Define the deformed complement of $a$ as

$$
\bar{a}_{q}=\sum_{\substack{\text { clockwise } \\ \text { midpoint polygons } \\ \text { ending at } a}} \operatorname{Punc}(D) \partial_{a} \operatorname{Arcs}(D) \in \operatorname{Jac}_{q} \check{Q} .
$$

Corollary 21.24. The deformed mirror objects are the deformed matrix factorizations given by

$$
F_{q}(a)=\left(\operatorname{Jac}_{q} \check{Q}\right) h(a) \underset{\bar{a}_{q}}{\stackrel{a}{\leftrightarrows}}\left(\operatorname{Jac}_{q} \check{Q}\right) t(a)
$$

Proof. It is our task to evaluate the products $\mu_{\mathrm{HTw}^{2} \operatorname{Gtl}_{q} Q}(m, b, \ldots, b)$. Here $m$ is an even or odd intersection point between a zigzag path and an arc. We shall regard the two cases separately and explain the relevant products.

First of all, fix an arc $a \in Q_{1}$. Let $L$ be the zigzag path turning left at the head of $a$ and $L^{\prime}$ the zigzag path turning right at the head of $a$. Let $m: L \rightarrow a$ be the odd intersection point and $m^{*}: L^{\prime} \rightarrow a$ the even intersection point. Regard the odd morphism $X_{a}: L \rightarrow L^{\prime}$ of zigzag paths. According to Lemma 18.20, the product $\mu_{\mathrm{HTw}_{\mathrm{Tw}}^{q}} Q\left(m^{*}, X_{a}\right)$ is equal to $-m$ and it is the only nonvanishing term in $\mu\left(m^{*}, b, \ldots, b\right)$. We conclude

$$
\delta\left(m^{*}\right)=(-1)^{\left\|m^{*}\right\|} \sum_{l \geq 0} \mu_{\mathrm{HTw} \mathrm{Gtl}}^{q} \text { } Q ~\left(m^{*}, b, \ldots, b\right)=a m
$$

Regard now the odd intersection point $m: L \rightarrow a$ and regard a sequence $X_{e_{1}}, \ldots, X_{e_{N}}$ of odd basis morphisms with $X_{e_{i}}: L_{i} \rightarrow L_{i+1}$ and $L_{N+1}=L$. According to Lemma 18.20 we can describe the product $\mu_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}\left(m, X_{e_{k}}, \ldots, X_{e_{1}}\right)$ by means of MD disks with output $m^{*}$ :

$$
\mu_{\mathrm{HTw} \mathrm{Gtl}}^{q} \text { Q }\left(m, X_{e_{k}}, \ldots, X_{e_{1}}\right)=\sum_{\substack{D \text { MD disk } \\ \text { with inputs } X_{e_{1}}, \ldots, X_{e_{k}}, m}} \operatorname{Punc}(D) m^{*}
$$

Unraveling the definition, an MD disk with inputs $X_{e_{1}}, \ldots, X_{e_{k}}, m$ is the same as a midpoint polygon with arc sequence $e_{1}, \ldots, e_{k}, a$. We can therefore write

$$
\delta(m)=(-1)^{\|m\|} \sum_{l \geq 0} \mu_{\mathrm{HTw}_{\operatorname{Gtg}} Q}^{l+1}(m, b, \ldots, b)=\sum_{\substack{\text { clockwise } \\ \text { midpoint polygons } D \\ \text { ending at } a}} \operatorname{Punc}(D) \partial_{a} \operatorname{Arcs}(D) m^{*}=\bar{a}_{q} m^{*} .
$$

In summary, we have computed the map $\delta$ both on the odd hom space $\operatorname{Hom}(L, a)$ and the even hom space $\operatorname{Hom}\left(L^{\prime}, a\right)$. This finishes the proof.

Remark 21.25. The mirror objects $F_{q}(a) \in \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ have curvature. Explicitly, their curvature reads

$$
\mu_{\mathrm{MF}, F_{q}(a)}^{0}(m)=\ell_{q} m-\delta^{2}(m)=(-1)^{|m|} \sum_{l \geq 0} \mu_{\mathrm{HTw} \mathrm{Gtl}_{q} Q}^{l+2}\left(\mu_{\mathrm{HTw} \mathrm{Gtl}}^{q} Q, ~ m, b, \ldots, b\right)
$$

In Figure 21.7. we have depicted how to interpret this curvature geometrically.

### 21.7 Main result

In this section, we state our main result. It consists of a wide range of deformed mirror equivalences for punctured surfaces. The A -side is the deformed gentle algebra $\mathrm{Gtl}_{q} Q$ and the B -side is the deformed category of matrix factorizations $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$. When dividing out the maximal ideal, one recovers the classical mirror symmetry originally described in 18. More specifically, one recovers the classical mirror functor of Cho, Hong and Lau defined in [26, Chapter 10].


Figure 21.7: The picture depicts the products $\mu\left(\mu_{q}^{0}, m, b, \ldots, b\right)$, which determine the curvature of $F_{q}(a)$. The products can be interpreted as smooth immersed disks between odd intersections of zigzag curves, with a small embayment given by intermediately tracing an arc. The longer a disk, the more arcs can be embayed and the more it contributes to curvature of mirror objects. The less punctures a disk is supposed to cover, the shorter it is, which shows nicely why the classical mirror objects $F(a)$ are curvature-free.

Remark 21.26. For sake of completeness, we provide here a brief overview of the setting. The starting point is a dimer $Q$ which is geometrically consistent or a standard sphere dimer $Q_{M}$ with $M \geq 3$. We assume that the dual dimer $Q$ is zigzag consistent and of bounded type. The bounded type requirement for dimers is detailed in section 19.10 For instance, a cancellation consistent dimer on a torus or without triangles is automatically of bounded type.

Our mirror functor is a specific instance of the deformed Cho-Hong-Lau construction of section 20. The input datum for this specific instance is the deformed category of zigzag paths $\mathbb{L}_{q}$ and its minimal model $\mathrm{H}_{q} \subseteq \mathrm{HTw} \mathrm{Gtl}_{q} Q$. In section 21.4 we have started evaluating the deformed Cho-Hong-Lau construction for this specific instance and obtained an explicit description of the deformed superpotential $W_{q}$ and deformed Jacobi algebra $\mathrm{Jac}_{q} \check{Q}$. In section 21.5. we have obtained an explicit description of the deformed potential $\ell_{q}$. The general deformed Cho-Hong-Lau construction gives rise to a functor

$$
F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{MF}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)
$$

In section 21.6, we have computed explicitly the mirror objects $F_{q}(a)$ for $a \in \check{Q}_{1}$. We denote by $\operatorname{mf}\left(\operatorname{Jac}_{q} Q, \ell_{q}\right)$ the subcategory consisting of these objects:

$$
\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)=\left\{F_{q}(a)\right\}_{a \in \check{Q}_{1} \subseteq \operatorname{MF}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right) . . . . ~}
$$

Remark 21.27. In our main result, we wish to present a mirror functor of $A_{\infty}$-deformations. At the present point, both the Jacobi algebra $\mathrm{Jac}_{q} \check{Q}$ and the category $\operatorname{MF}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ however do not come with a canonical identification as completed tensor product of $\operatorname{Jac} \check{Q}$ and $\operatorname{MF}(\operatorname{Jac} \mathscr{Q}, \ell)$ with $B$. We amend this by the following procedure:

- The algebra $\mathrm{Jac}_{q} \check{Q}$ is a deformation of $\operatorname{Jac} \check{Q}$ in the sense that there exists a deformation $\mu_{\mathrm{Jac}, q}$ of the product of $\operatorname{Jac} \check{Q}$ and a $\mathbb{C} \llbracket Q_{0} \rrbracket$-linear algebra isomorphism

$$
\varphi_{\mathrm{Jac}, q}: \mathrm{Jac}_{q} \check{Q} \xrightarrow{\sim}\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} \mathrm{Jac} \check{Q}, \mu_{\mathrm{Jac}, q}\right) .
$$

This identification is not canonical and depends on choice. For sake of the construction, we shall fix one identification and view $\mathrm{Jac}_{q} \check{Q}$ as $\left(\mathbb{C} \llbracket Q_{0} \rrbracket \widehat{\otimes} \mathrm{Jac} \check{Q}, \mu_{\mathrm{Jac}, q}\right)$.

- The category $\operatorname{MF}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ is by construction only a loose object-cloning deformation of the category $\operatorname{MF}(\operatorname{Jac} \check{Q}, \ell)$. This means that its objects are not the same as the objects of MF(Jac $\check{Q}, \ell)$ and its hom spaces cannot be naturally identified with the completed tensor product of the hom spaces of $\operatorname{MF}(\operatorname{Jac} \mathscr{Q}, \ell)$. Instead, such identification depends on choice. We have elaborated on this phenomenon in section 20.6 We amend this by regarding the subcategory $\mathrm{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ instead of $\operatorname{MF}\left(\mathrm{Jac}_{q} \check{Q}, \ell_{q}\right)$.
- The category $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ is by construction only a loose deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$. This entails that the objects $F_{q}(a)$ are in correspondence with the objects $F(a)$, but the hom spaces have not been identified. We amend this by noting that the specific projective modules contained in $F_{q}(a)$ are of the form $\left(\mathrm{Jac}_{q} \check{Q}\right) L_{i}$. These projective modules come with natural isomorphisms

$$
\operatorname{Hom}_{\mathrm{Jac}_{q} \check{Q}}\left(\left(\mathrm{Jac}_{q} \check{Q}\right) L_{i},\left(\mathrm{Jac}_{q} \check{Q}\right) L_{j}\right) \xrightarrow{\sim} L_{i}\left(\mathrm{Jac}_{q} \check{Q}\right) L_{j} \xrightarrow{\sim} B \widehat{\otimes} L_{i}(\operatorname{Jac} \check{Q}) L_{j} .
$$

We can therefore identify the hom spaces of $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ as

$$
\begin{equation*}
\operatorname{Hom}_{\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)}\left(F_{q}(a), F_{q}(b)\right) \xrightarrow{\sim} B \widehat{\otimes} \operatorname{Hom}_{\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)}(F(a), F(b)) . \tag{21.4}
\end{equation*}
$$

This way, we can view $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ as an actual $A_{\infty}$-deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$.

- The mirror functor $F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{MF}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ is by construction only a functor of loose $A_{\infty^{-}}$ deformations. This means that it does not have a naturally defined leading term. However, once we interpret $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ as an actual $A_{\infty}$-deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$, the functor $F_{q}: \mathrm{Gtl}_{q} Q \rightarrow$ $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ becomes a functor of actual $A_{\infty}$-deformations and does have a well-defined leading term.

In Theorem 21.28 we present the main result. It is a specific instance of Theorem 20.50 and yields a wide range of deformed mirror equivalences. In the statement of the result, we have already applied the reinterpretation of $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ as actual $A_{\infty}$-deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$, according to Remark 21.27
Theorem 21.28. Let $Q$ be a geometrically consistent dimer or standard sphere dimer $Q_{M}$ with $M \geq 3$. Assume that the dual dimer $\check{Q}$ is zigzag consistent and of bounded type. Denote by $\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ the deformed Landau-Ginzburg model associated with $Q$. Then:

1. The algebra $\operatorname{Jac}_{q} \check{Q}$ is a deformation of $\operatorname{Jac} \check{Q}$.
2. The category $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ is an $A_{\infty}$-deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$.
3. The deformed Cho-Hong-Lau functor provides a quasi-isomorphism of deformed $A_{\infty}$-categories

$$
F_{q}: \operatorname{Gtl}_{q} Q \xrightarrow{\sim} \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)
$$

4. The leading term of $F_{q}$ is the classical Cho-Hong-Lau functor $F: \operatorname{Gtl} Q \rightarrow \operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$.

Proof. This is essentially a restatement of Theorem 20.50 applied to the case of $\mathrm{H} \mathbb{L}_{q} \subseteq \mathrm{HTw} \mathrm{Gtl}_{q} Q$. Although we have already worked with the functor $F_{q}$ in section 21.6 we shall recapitulate here why the conditions of Theorem 20.50 are satisfied. After that, we comment on the first, second, fourth and third claimed statement, in this order.

Let us explain that all requirements of Theorem 20.50 are satisfied. To start with, we briefly trace the requirements according to Convention 20.22 The category $\mathrm{H}_{q}$ is strictly unital with the same identities as $H \mathbb{L}$. It comes with a non-degenerate odd pairing and a choice of CHL basis. As shown in Lemma 21.13, the deformed $A_{\infty}$-structure is cyclic on the odd part. As we explained in section 21.3, we restrict the construction of the deformed Cho-Hong-Lau functor to the domain $\mathrm{Gtl}_{q} Q$. We have seen in section 18 that the hom spaces between zigzag paths and arcs are finite-dimensional. This establishes all requirements of Convention 20.22 adapted to the restriction of the functor to $\mathrm{Gtl}_{q} Q$.

The other requirements of Theorem 20.50 are that $\mathrm{H}_{q}$ is of slow growth and that $\mathrm{Jac}_{q} \check{Q}$ is a deformation of $\operatorname{Jac} \check{Q}$. In Lemma 21.14 we have indeed verified that $H \mathbb{L}_{q}$ is of slow growth, at least to an extent sufficient for application of the deformed Cho-Hong-Lau construction when restricting the functor to $\operatorname{Gtl}_{q} Q$. In Corollary 21.18, we have verified that $\mathrm{Jac}_{q} \check{Q}$ is a deformation of $\mathrm{Jac} \check{Q}$. This already proves the first claimed statement.

We see that all requirements of Theorem 20.50 are satisfied, adapted to the restriction of the functor to $\operatorname{Gtl}_{q} Q$. Invoke the slow growth version of Theorem 20.50 and conclude that the large category $\operatorname{MF}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ is a loose object-cloning deformation of $\operatorname{MF}(\operatorname{Jac} \check{Q}, \ell)$. Furthermore, the deformed Cho-Hong-Lau functor defines a functor of loose object-cloning deformations

$$
F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{MF}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)
$$

Note that the functor is restricted to $\operatorname{Gtl}_{q} Q \subseteq \operatorname{HTw}_{\mathrm{Gtl}_{q} Q}$ because we have only partially verified that $\mathrm{H} \mathbb{L}_{q}$ is of slow growth. We now explain the second, third and fourth claimed statements of the theorem.

For the second statement of the theorem, we explain why $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ is a deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$. Indeed, the subcategory $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ consists of the objects $F_{q}(a)$. As such, it is a loose object-cloning deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ via the object-cloning map

$$
\begin{aligned}
O: \mathrm{Ob}\left(\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)\right) & \longrightarrow \mathrm{Ob}(\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)), \\
F_{q}(a) & \longmapsto F(a) .
\end{aligned}
$$

Since the object-cloning map $O$ is in fact a bijection, we can call $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ simply a loose deformation of $\operatorname{mf}(\operatorname{Jac} Q, \ell)$ instead of a loose object-cloning deformation. Upon the further identification $\sqrt{21.4}$, we can naturally view $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ as an actual deformation $\operatorname{off}(\operatorname{Jac} \check{Q}, \ell)$.


Figure H.1: The 3-punctured sphere and its mirror

For the fourth statement, regard the deformed Cho-Hong-Lau functor $F_{q}$ provided to us by Theorem 20.50. We merely restrict the codomain of $F_{q}$ to the subcategory $\operatorname{mf}\left(\operatorname{Jac}_{q} Q, \ell_{q}\right) \subseteq \operatorname{MF}\left(\operatorname{Jac}_{q} Q, \ell_{q}\right)$ :

$$
F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right) .
$$

This restriction is a bijection on object level. In Theorem 20.50, we have investigated its leading term. By definition, the leading term entails dividing out $\mathfrak{m}$ on hom spaces on both sides. We have shown the leading term is $F$ when the following subsequent identification is applied:

$$
\begin{equation*}
\frac{\operatorname{Hom}_{\mathrm{MF}\left(\mathrm{Jac}_{q} \check{Q}, \ell_{q}\right)}\left(F_{q}(a), F_{q}(b)\right)}{\left(Q_{0}\right) \cdot \operatorname{Hom}_{\mathrm{MF}\left(\mathrm{Jac}_{q} \check{Q}, \ell_{q}\right)}\left(F_{q}(a), F_{q}(b)\right)} \stackrel{\sim}{\rightarrow} \operatorname{Hom}_{\mathrm{MF}(\mathrm{Jac} \check{Q}, \ell)}(F(a), F(b)), \tag{21.5}
\end{equation*}
$$

Under the present interpretation of $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ as actual deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$, we can go a step further. Indeed, passing $F_{q}$ to the quotient by $\mathfrak{m}$ and subsequently applying (21.5) is equivalent to interpreting $\operatorname{mf}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ as actual deformation of $\operatorname{mf}(\operatorname{Jac} \check{Q}, \ell)$ by means of 21.4) and taking the leading term of the functor of actual $A_{\infty}$-deformations $F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{mf}\left(\operatorname{Jac}_{q} \mathscr{Q}, \ell_{q}\right)$. This shows that $F$ is the leading term of the functor $F_{q}$ of actual deformed $A_{\infty}$-categories. This proves the fourth statement.

For the third statement, recall from Corollary 21.9 that the classical Cho-Hong-Lau functor $F$ : $\operatorname{Gtl} Q \rightarrow \operatorname{mf}(\operatorname{Jac} Q, \ell)$ is a quasi-isomorphism. Since $F$ is the leading term of $F_{q}$, the functor $F_{q}$ is then also a quasi-isomorphism in the sense of Paper II. This finishes the proof.

Remark 21.29. The classical mirror functor $F: \operatorname{HTw} \operatorname{Gtl} Q \rightarrow \operatorname{MF}(\operatorname{Jac} \check{Q}, \ell)$ is not quasi-fully-faithful on the entire category $\mathrm{HTw} \operatorname{Gtl} Q$. For instance, a narrow loop around a puncture is mapped to the zero object, because the loop does not intersect any zigzag curves. While we have formally not verified the slow growth condition for $\mathrm{H} \mathbb{L}_{q}$ with respect to the entire category $\mathrm{HTw} \mathrm{Gtl}_{q} Q$, suppose we obtain a deformed Cho-Hong-Lau functor $F_{q}: \mathrm{HTw}_{\mathrm{Gtl}}^{q} \boldsymbol{Q} Q \rightarrow \operatorname{MF}\left(\mathrm{Jac}_{q} \check{Q}, \ell_{q}\right)$. This functor would then not be quasi-fully-faithful either, in the sense of Paper II
Remark 21.30. We can build another functor $\tilde{F}_{q}: \operatorname{HTw}_{\mathrm{Gtl}_{q}} Q \rightarrow \operatorname{HTw} \operatorname{MF}\left(\operatorname{Jac}_{q} \check{Q}, \ell_{q}\right)$ by extending the deformed Cho-Hong-Lau functor $F_{q}: \operatorname{Gtl}_{q} Q \rightarrow \operatorname{MF}\left(\mathrm{Jac}_{q} Q, \ell_{q}\right)$ to the deformed twisted completion and passing to the deformed minimal model, in the sense of Paper II In contrast to $F_{q}$, the functor $\tilde{F}_{q}$ is quasi-fully-faithful.

## H Examples

In this section, we present two examples of deformed mirror symmetry. We consider the 3-punctured sphere and a 4-punctured torus. In both cases we determine the deformed Jacobi algebra $\mathrm{Jac}_{q} \check{Q}$ and the deformed potential $\ell_{q}$ explicitly.

## H. 1 3-punctured sphere

In this section, we exhibit mirror symmetry and deformed mirror symmetry for the 3-punctured sphere. We start by explaining the dimer involved together with its zigzag curve. Then we turn to the Jacobi algebra and the mirror objects. We explain all midpoint polygons and compute the deformed LandauGinzburg model and the deformed matrix factorizations.


Figure H.2: Six midpoint polygons contributing to $W_{q}$

The dimer and its mirror The sphere we regard is depicted in Figure H.1a The figures depicts the three punctures $q_{a}, q_{b}, q_{c}$, the three arcs $a, b, c$ and the zigzag curve. The zigzag curve, as an object of the Fukaya category, is also known as the Seidel Lagrangian.

The mirror dimer $\check{Q}$ is a 1 -punctured torus depicted in Figure H.1c. The superpotential $W \in \mathbb{C} \check{Q}$ equals $W=(a b c)_{\text {cyc }}-(c b a)_{\text {cyc }}$. The Jacobi algebra is $\operatorname{Jac} \check{Q}=\mathbb{C}\langle a, b, c\rangle /(a b-b a, c a-a c, b c-c b)=\mathbb{C}[a, b, c]$. The potential is $\ell=a b c \in \operatorname{Jac} \check{Q}$. The mirror objects are the matrix factorizations

$$
M_{a}=\mathrm{Jac} \check{Q} \underset{b c}{\stackrel{a}{\rightleftarrows}} \mathrm{Jac} \check{Q}, \quad M_{b}=\mathrm{Jac} \check{Q} \underset{c a}{\stackrel{b}{\rightleftarrows}} \mathrm{Jac} \check{Q}, \quad M_{c}=\mathrm{Jac} \check{Q} \underset{a b}{\stackrel{c}{\rightleftarrows}} \mathrm{Jac} \check{Q}
$$

The deformed superpotential The deformed gentle algebra $\mathrm{Gtl}_{q} Q$ has three deformation parameters $q_{a}, q_{b}$ and $q_{c}$. To compute the deformed superpotential, we have to enumerate all midpoint polygons in $Q$. There are in total 12 midpoint polygons contributing to $W_{q}$, depicted in Figure H. 2 and H.3. They fall into the following four groups: Three of them are 3-gons on the front side and have arc sequences $a b c, b c a$ and $c a b(\operatorname{sign}+1)$. They are depicted in Figure H.2a. Three of them are 3-gons on the rear side and have arc sequences $c b a, b a c$ and $a c b(\operatorname{sign}-1)$. They are depicted in Figure H.2b Three of them are monogons mainly lying on the front side with arc sequences $a, b, c$ (sign $+1, q$-parameters $q_{a}, q_{b}, q_{c}$, respectively). They are depicted in Figure H.3a H.3b and H.3c Three of them are monogons mainly lying on the rear side with arc sequences $a, b, c$ (sign $-1, q$-parameters $q_{a}, q_{b}, q_{c}$, respectively). They are depicted in Figure H.3d, H.3e and H.3f We conclude that the deformed terms cancel each other and we are left with

$$
W_{q}=W=(a b c)_{\mathrm{cyc}}-(c b a)_{\mathrm{cyc}} \in \mathbb{C}\langle a, b, c\rangle .
$$

The deformed Jacobi algebra is $\mathrm{Jac}_{q} \check{Q}=\mathbb{C}[a, b, c] \llbracket q_{a}, q_{b}, q_{c} \rrbracket$.
The deformed potential The expression for the deformed potential depends on the choice of identity location for the zigzag path $L$. Let us choose the identity to lie on the copy of $a$ which turns right at head and tail of $a$. This choice is depicted in Figure H.4 The deformed potential $\ell_{q}$ is determined by enumerating $L$-polygons. The specific choice of $a_{0}$ gives the following four $L$-polygons. We list these polygons here and recapitulate in technical terms from which part of the minimal model $\mathrm{H} \mathbb{L}_{q}$ they come from:

- One $L$-polygon is a 3 -gon lying on the front side and ending at the right side of $a$. The underlying midpoint polygon is the same as the one depicted in Figure H.2a. Its contribution to $\ell_{q}$ is $a b c$. In terms of $\mathrm{H} \mathbb{L}_{q}$, it concerns an ID disk.
- One $L$-polygon is a monogon mainly lying on the front side and ending at the right side of $a$. The underlying midpoint polygon is the same as the one depicted in Figure H.3a Its contribution to $\ell_{q}$ is $-q_{a} a$. In terms of $\mathrm{H} \mathbb{L}_{q}$, it concerns an ID disk.
- One $L$-polygon is a monogon mainly lying on the rear side and having a crossing with $a_{0}$. The underlying midpoint polygon is the same as the one depicted in Figure H.3e Its contribution to $\ell_{q}$ is $-q_{b} b$. In terms of $\mathrm{H} \mathbb{L}_{q}$, it concerns a CR disk.
- One $L$-polygon is a monogon mainly lying on the front side and having a crossing with $a_{0}$. The underlying midpoint polygon is the same as the one depicted in Figure H.3c Its contribution to $\ell_{q}$ is $-q_{c} c$. In terms of $\mathrm{H} \mathbb{L}_{q}$, it concerns a CR disk.


Figure H.3: Six canceling midpoint polygons contributing to $W_{q}$


Figure H.4: Identity location $a_{0}$

The total deformed potential amounts to

$$
\ell_{q}=a b c-q_{a} a-q_{b} b-q_{c} c \in \mathrm{Jac}_{q} \check{Q}
$$

We immediately see that $\ell_{q} \in \mathrm{Jac}_{q} \check{Q}=\mathbb{C}[a, b, c] \llbracket q_{a}, q_{b}, q_{c} \rrbracket$ is still central.

The deformed mirror objects The deformed mirror objects $F_{q}(a), F_{q}(b), F_{q}(c)$ can be determined by enumerating clockwise midpoint polygons ending at $a, b$ and $c$, respectively. For the arc $a$, there are two such midpoint polygons. The first is a 3 -gon and has $\partial_{a} \operatorname{Arcs}(D)=b c$, $\operatorname{sign}+1$ and no $q$-parameters. The second is a monogon and has $\partial_{a} \operatorname{Arcs}(D)=1$, sign -1 and $q$-parameter $q_{a}$. Both add up to a contribution of $b c-q_{a}$. The deformed mirror object is therefore

$$
F_{q}(a)=\operatorname{Jac}_{q} \underset{b c-q_{a}}{\stackrel{a}{\rightleftarrows} \operatorname{Jac}_{q} \check{Q}}
$$

Similar considerations hold for $F_{q}(b)$ and $F_{q}(c)$. The deformed mirror objects are listed in Table H.5. Note that none of the "factors" actually factor to $\ell_{q}$. Instead, the failure to factor $\ell_{q}$ is infinitesimal and serves as curvature of the deformed matrix factorizations.

## H. 2 4-punctured torus

In this section, we exhibit mirror symmetry and deformed mirror symmetry for a 4 -punctured torus dimer. We start by explaining the dimer involved together with its zigzag curve. Then we turn to the Jacobi algebra and the mirror objects. We explain all midpoint polygons and compute the deformed LandauGinzburg model and the deformed matrix factorizations. Centrality of the deformed Landau-Ginzburg potential obtained is not obvious, and we provide a manual check.

| Object | Description | Curvature |
| :---: | :---: | :---: |
| $F_{q}(a)$ | $\mathrm{Jac}_{q} \check{Q} \underset{b c-q_{a}}{\stackrel{a}{\rightleftarrows}} \mathrm{Jac}_{q} \check{Q}$ | $-q_{b} b-q_{c} c$ |
| $F_{q}(b)$ | $\mathrm{Jac}_{q} \check{Q} \underset{c a-q_{b}}{\stackrel{b}{\rightleftarrows}} \mathrm{Jac}_{q} \check{Q}$ | $-q_{a} a-q_{c} c$ |
| $F_{q}(c)$ | $\mathrm{Jac}_{q} \check{Q} \underset{a b-q_{c}}{\stackrel{c}{\rightleftarrows}} \mathrm{Jac}_{q} \check{Q}$ | $-q_{a} a-q_{b} b$ |

Table H.5: The deformed mirror objects

The dimer and its mirror The torus dimer we regard is depicted in Figure H.6a. The dimer has four punctures $q_{1}, q_{2}, q_{3}, q_{4}$, eight arcs $a_{1}, \ldots, a_{4}$ and $b_{1}, \ldots, b_{4}$ and four elementary polygons. It has four zigzag paths and is geometrically consistent. The associated zigzag curves are depicted in Figure H.6c The figure also depicts a few sample midpoint polygons of different sizes (without indication which is the first and the last midpoint).

The mirror dimer $\check{Q}$ is a 4-punctured torus as well, depicted in Figure H.6b In the numbering of the figure, the correspondence between the punctures $1,2,3,4$ of $\check{Q}$ and zigzag paths of $Q$ is as follows:

$$
\begin{aligned}
& 1 \in \check{Q}_{0} \longleftrightarrow L_{1}=\ldots, b_{1}, a_{2}, b_{4}, a_{3}, \ldots \\
& 2 \in \check{Q}_{0} \longleftrightarrow L_{2}=\ldots, b_{2}, a_{3}, b_{3}, a_{2}, \ldots \\
& 3 \in \check{Q}_{0} \longleftrightarrow L_{3}=\ldots, a_{1}, b_{2}, a_{4}, b_{3}, \ldots \\
& 4 \in \check{Q}_{0} \longleftrightarrow L_{4}=\ldots, a_{1}, b_{4}, a_{4}, b_{1}, \ldots
\end{aligned}
$$

For instance, the two associated zigzag curves $\tilde{L}_{1}$ and $\tilde{L}_{2}$ are depicted in Figure H.8a The superpotential $W \in \mathbb{C} Q$ equals

$$
W=\left(b_{1} a_{4} b_{2} a_{2}\right)_{\mathrm{cyc}}+\left(b_{4} a_{1} b_{3} a_{3}\right)_{\mathrm{cyc}}-\left(b_{2} a_{3} b_{1} a_{1}\right)_{\mathrm{cyc}}-\left(b_{3} a_{2} b_{4} a_{4}\right)_{\mathrm{cyc}} .
$$

The Jacobi algebra $\operatorname{Jac} \check{Q}=\mathbb{C} \check{Q} /\left(\partial_{a} W\right)_{a \in Q_{1}}$ is a noncommutative quiver algebra with relations. The dimer $\check{Q}$ is zigzag consistent and therefore $\mathrm{Jac} \check{Q}$ is CY3. Its 12 relations read

$$
\begin{aligned}
a_{4} b_{2} a_{2} & =a_{1} b_{2} a_{3}, \\
a_{2} b_{1} a_{4} & =a_{3} b_{1} a_{1}, \\
\ldots & =\ldots
\end{aligned}
$$

The potential $\ell \in \operatorname{Jac} \check{Q}$ reads

$$
\ell=\left(b_{1} a_{4} b_{2} a_{2}\right)_{\mathrm{cyc}} .
$$

The mirror objects are the eight matrix factorizations

$$
\begin{aligned}
& M_{a_{1}}=(\operatorname{Jac} \check{Q}) L_{3} \underset{b_{3} a_{3} b_{4}}{\stackrel{a_{1}}{\rightleftarrows}}(\operatorname{Jac} \check{Q}) L_{4}, \quad M_{a_{2}}=(\operatorname{Jac} \check{Q}) L_{2} \underset{b_{1} a_{4} b_{2}}{\stackrel{a_{2}}{\rightleftarrows}}(\operatorname{Jac} \check{Q}) L_{1}, \\
& M_{a_{3}}=(\operatorname{Jac} \check{Q}) L_{2} \underset{b_{4} a_{1} b_{3}}{\stackrel{a_{1}}{\rightleftarrows}}(\mathrm{Jac} \check{Q}) L_{1}, \quad M_{a_{4}}=(\operatorname{Jac} \check{Q}) L_{3} \underset{b_{2} a_{2} b_{1}}{\stackrel{a_{4}}{\rightleftarrows}}(\mathrm{Jac} \check{Q}) L_{4}, \\
& M_{b_{1}}=(\operatorname{Jac} \check{Q}) L_{1} \underset{a_{4} b_{2} a_{2}}{\stackrel{b_{1}}{\rightleftarrows}}(\operatorname{Jac} \check{Q}) L_{3}, \quad M_{b_{2}}=(\operatorname{Jac} \check{Q}) L_{4} \underset{a_{2} b_{1} a_{4}}{\stackrel{b_{2}}{\rightleftarrows}}(\operatorname{Jac} \check{Q}) L_{2}, \\
& M_{b_{3}}=(\operatorname{Jac} \check{Q}) L_{4} \underset{a_{3} b_{4} a_{1}}{\stackrel{b_{3}}{\rightleftarrows}}(\operatorname{Jac} \check{Q}) L_{2}, \quad M_{b_{4}}=(\operatorname{Jac} \check{Q}) L_{1} \underset{a_{1} b_{3} a_{3}}{\stackrel{b_{4}}{\rightleftarrows}}(\operatorname{Jac} \check{Q}) L_{3} .
\end{aligned}
$$

The deformed superpotential The deformed gentle algebra $\operatorname{Gtl}_{q} Q$ has four deformation parameters $q_{1}, q_{2}, q_{3}, q_{4}$. To compute the deformed superpotential $W_{q}$, we have to enumerate all midpoint polygons in $Q$. As can be seen from Figure H.6c there are infinitely many midpoint polygons. All midpoint polygons


Figure H.6: Illustration of 4-punctured torus
are rectangular, in particular for the sign $|D|$ we have $|D|=0 \in \mathbb{Z} / 2 \mathbb{Z}$ for every midpoint polygon $D$. The midpoint polygons can be classified into 64 different types according to their arc sequence, or 16 if one takes arc sequences up to cyclic permutation. All 16 types are listed in Table H. 7

The midpoint polygons of any type are a family indexed by natural numbers $k, l \in \mathbb{N}$, standing for the width and height of the polygon. More precisely, define side lengths of a midpoint polygon to be the number of angles cut by the zigzag segments. In these terms, every family comes with a minimally small midpoint polygon and all larger midpoint polygons in the family are derived from this minimal version by extending side lengths by multiples of 4 . For every of the 16 types, we have indicated in Table H.7 the $q$-parameters of the midpoint polygon of size $(k, l)$. This way, we have efficiently enumerated all midpoint polygons and their properties in 16 types and two parameters.

We use the following abbreviations:

$$
q=q_{1} q_{2} q_{3} q_{4}, q_{14}=q_{1} q_{4}, q_{23}=q_{2} q_{3} .
$$

With this in mind, the deformed superpotential can be expressed as a sum over sixteen terms, eight of which actually cancel out due to symmetry of the 4-punctured torus:

$$
\begin{aligned}
W_{q}= & \sum_{\substack{8 \text { clockwise } \\
\text { types }}} \sum_{k, l \geq 0} \operatorname{Punc}(D) \operatorname{Arcs}(D)_{\text {cyc }}-\sum_{\substack{\text { counterclockwise } \\
\text { types }}} \sum_{k, l \geq 0} \operatorname{Punc}(D) \operatorname{Arcs}(D)_{\text {cyc }} \\
= & \left(\sum_{k, l \geq 0} q_{23}^{l} q_{14}^{k} q^{2 k l}-\sum_{k, l \geq 0} q\left(q_{14} q\right)^{l}\left(q_{23} q\right)^{k} q^{2 k l}\right)\left(b_{2} a_{2} b_{1} a_{4}\right)_{\text {cyc }} \\
& +\left(\sum_{k, l \geq 0} q\left(q_{23} q\right)^{l}\left(q_{14} q\right)^{k} q^{2 k l}-\sum_{k, l \geq 0} q_{14}^{l} q_{23}^{k} q^{2 k l}\right)\left(b_{3} a_{2} b_{4} a_{4}\right)_{\text {cyc }} \\
& +\left(\sum_{k, l \geq 0} q_{23}^{l} q_{14}^{k} q^{2 k l}-\sum_{k, l \geq 0} q\left(q_{14} q\right)^{l}\left(q_{23} q\right)^{k} q^{2 k l}\right)\left(b_{3} a_{3} b_{4} a_{1}\right)_{\text {cyc }} \\
& +\left(\sum_{k, l \geq 0} q\left(q_{23} q\right)^{l}\left(q_{14} q\right)^{k} q^{2 k l}-\sum_{k, l \geq 0} q_{14}^{l} q_{23}^{k} q^{2 k l}\right)\left(b_{2} a_{3} b_{1} a_{1}\right)_{\text {cyc }} .
\end{aligned}
$$

The deformed Jacobi algebra is $\mathrm{Jac}_{q} \check{Q}=\mathbb{C} \check{Q} \llbracket q_{1}, q_{2}, q_{3}, q_{4} \rrbracket / \overline{\left(\partial_{a} W_{q}\right)_{a \in \check{Q}_{1}}}$. Here $\overline{\left(\partial_{a} W_{q}\right)_{a \in \mathscr{Q}_{1}}}$ denotes the closure of the ideal generated by the derivatives $\partial_{a} W_{q} \in \mathbb{C} Q \llbracket q_{1}, q_{2}, q_{3}, q_{4} \rrbracket$ with respect to the $\left(q_{1}, q_{2}, q_{3}, q_{4}\right)$-adic topology.

The deformed potential The deformed potential $\ell_{q}$ as element of $\mathbb{C} Q \check{\llbracket} q_{1}, q_{2}, q_{3}, q_{4} \rrbracket$ is the sum of four potentials, one for each of the four zigzag paths:

$$
\ell_{q}=\ell_{q, 1}+\ell_{q, 2}+\ell_{q, 3}+\ell_{q, 4} .
$$

The expression for the deformed potential $\ell_{q, i}$ as element of $\mathbb{C} \check{Q} \llbracket q_{1}, q_{2}, q_{3}, q_{4} \rrbracket$ depends on the choice of identity location for the zigzag path $L_{i}$. For instance, let us choose the identity of the zigzag path

| Orientation | $q$-parameter Punc $(D)$ | arc sequence $\operatorname{Arcs}(D)$ |
| :---: | :---: | :---: |
| clockwise | $q_{23}^{l} q_{14}^{k} q^{2 k l}$ | $b_{1} a_{4} b_{2} a_{2}$ |
| clockwise | $q_{3} q_{23}^{l}\left(q_{14} q\right)^{k} q^{2 k l}$ | $b_{4} a_{1} b_{2} a_{2}$ |
| clockwise | $q_{4}\left(q_{23} q\right)^{l} q_{14}^{k} q^{2 k l}$ | $b_{1} a_{1} b_{3} a_{2}$ |
| clockwise | $q\left(q_{23} q\right)^{l}\left(q_{14} q\right)^{k} q^{2 k l}$ | $b_{4} a_{4} b_{3} a_{2}$ |
| clockwise | $q_{23}^{l} q_{14}^{k} q^{2 k l}$ | $b_{4} a_{1} b_{3} a_{3}$ |
| clockwise | $q_{2} q_{23}^{l}\left(q_{14} q\right)^{k} q^{2 k l}$ | $b_{1} a_{4} b_{3} a_{3}$ |
| clockwise | $q_{1}\left(q_{23} q\right)^{l} q_{14}^{k} q^{2 k l}$ | $b_{4} a_{4} b_{2} a_{3}$ |
| clockwise | $q\left(q_{23} q\right)^{l}\left(q_{14} q\right)^{k} q^{2 k l}$ | $b_{1} a_{1} b_{2} a_{3}$ |
| counterclockwise | $q_{14}^{l} q_{23}^{k} q^{2 k l}$ | $b_{2} a_{3} b_{1} a_{1}$ |
| counterclockwise | $q_{4} q_{14}^{l}\left(q_{23} q\right)^{k} q^{2 k l}$ | $b_{3} a_{2} b_{1} a_{1}$ |
| counterclockwise | $q_{3}\left(q_{14} q\right)^{l} q_{23}^{k} q^{2 k l}$ | $b_{2} a_{2} b_{4} a_{1}$ |
| counterclockwise | $q\left(q_{14} q\right)^{l}\left(q_{23} q\right)^{k} q^{2 k l}$ | $b_{3} a_{3} b_{4} a_{1}$ |
| counterclockwise | $q_{14}^{l} q_{23}^{k} q^{2 k l}$ | $b_{3} a_{2} b_{4} a_{4}$ |
| counterclockwise | $q_{1} q_{14}^{l}\left(q_{23} q\right)^{k} q^{2 k l}$ | $b_{2} a_{3} b_{4} a_{4}$ |
| counterclockwise | $q_{2}\left(q_{14} q\right)^{l} q_{23}^{k} q^{2 k l}$ | $b_{3} a_{3} b_{1} a_{4}$ |
| counterclockwise | $q\left(q_{14} q\right)^{l}\left(q_{23} q\right)^{k} q^{2 k l}$ | $b_{2} a_{2} b_{1} a_{4}$ |

Table H.7: Enumeration of midpoint polygons
$L_{1}=\ldots, a_{2}, b_{1}, a_{3}, b_{4}, \ldots$ to be the arc $a_{2}$. Then the summands $\ell_{q, 1}$ and $\ell_{q, 2}$ read

$$
\begin{aligned}
\ell_{q, 1}= & \left(\sum_{k, l \geq 0} l q_{23}^{l} q_{14}^{k} q^{2 k l} b_{1} a_{4} b_{2} a_{2}+\sum_{k, l \geq 0}(l+1) q\left(q_{14} q\right)^{l}\left(q_{23} q\right)^{k} q^{2 k l}\right) b_{1} a_{4} b_{2} a_{2} \\
& +\left(\sum_{k, l \geq 0} l q_{4}\left(q_{23} q\right)^{l} q_{14}^{k} q^{2 k l}+\sum_{k, l \geq 0}(l+1) q_{4} q_{14}^{l}\left(q_{23} q\right)^{k} q^{2 k l}\right) b_{1} a_{1} b_{3} a_{2} \\
& +\left(\sum_{k, l \geq 0} l q_{3} q_{23}^{l}\left(q_{14} q\right)^{k} q^{2 k l}+\sum_{k, l \geq 0}(l+1) q_{3}\left(q_{14} q\right)^{l} q_{23}^{k} q^{2 k l}\right) b_{4} a_{1} b_{2} a_{2} \\
& +\left(\sum_{k, l \geq 0} l q\left(q_{23} q\right)^{l}\left(q_{14} q\right)^{k} q^{2 k l}+\sum_{k, l \geq 0}(l+1) q_{14}^{l} q_{23}^{k} q^{2 k l}\right) b_{4} a_{4} b_{3} a_{2} \\
& +\left(\sum_{k, l \geq 0}(l+1) q_{2} q_{23}^{l}\left(q_{14} q\right)^{k} q^{2 k l}+\sum_{k, l \geq 0} l q_{2}\left(q_{14} q\right)^{l} q_{23}^{k} q^{2 k l}\right) b_{1} a_{4} b_{3} a_{3} \\
& +\left(\sum_{k, l \geq 0}(l+1) q\left(q_{23} q\right)^{l}\left(q_{14} q\right)^{k} q^{2 k l}+\sum_{k, l \geq 0} l q_{14}^{l} q_{23}^{k} q^{2 k l}\right) b_{1} a_{1} b_{2} a_{3} \\
& +\left(\sum_{k, l \geq 0} l q_{23}^{l} q_{14}^{k} q^{2 k l}+\sum_{k, l \geq 0}(l+1) q\left(q_{14} q\right)^{l}\left(q_{23} q\right)^{k} q^{2 k l}\right) b_{4} a_{1} b_{3} a_{3} \\
& +\left(\sum_{k, l \geq 0} l q_{1}\left(q_{23} q\right)^{l} q_{14}^{k} q^{2 k l}+\sum_{k, l \geq 0}(l+1) q_{1} q_{14}^{l}\left(q_{23} q\right)^{k} q^{2 k l}\right) b_{4} a_{4} b_{2} a_{3}
\end{aligned}
$$

and

$$
\begin{aligned}
\ell_{q, 2}= & \left(\sum_{k, l \geq 0}(k+1) q_{23}^{l} q_{14}^{k} q^{2 k l}+\sum_{k, l \geq 0} k q\left(q_{14} q\right)^{l}\left(q_{23} q\right)^{k} q^{2 k l}\right) a_{2} b_{1} a_{4} b_{2} \\
& +\left(\sum_{k, l \geq 0}(k+1) q_{4}\left(q_{23} q\right)^{l} q_{14}^{k} q^{2 k l}+\sum_{k, l \geq 0} k q_{4} q_{14}^{l}\left(q_{23} q\right)^{k} q^{2 k l}\right) a_{2} b_{1} a_{1} b_{3} \\
& +\left(\sum_{k, l \geq 0}(k+1) q_{3}\left(q_{23}\right)^{l}\left(q_{14} q\right)^{k} q^{2 k l}+\sum_{k, l \geq 0} k q_{3}\left(q_{14} q\right)^{l} q_{23}^{k} q^{2 k l}\right) a_{2} b_{4} a_{1} b_{2} \\
& +\left(\sum_{k, l \geq 0}(k+1) q\left(q_{23} q\right)^{l}\left(q_{14} q\right)^{k} q^{2 k l}+\sum_{k, l \geq 0} k q_{14}^{l} q_{23}^{k} q^{2 k l}\right) a_{2} b_{4} a_{4} b_{3} \\
& +\left(\sum_{k, l \geq 0} k\left(q_{23}\right)^{l}\left(q_{14}\right)^{k} q^{2 k l}+\sum_{k, l \geq 0}(k+1) q\left(q_{14} q\right)^{l}\left(q_{23} q\right)^{k} q^{2 k l}\right) a_{3} b_{4} a_{1} b_{3} \\
& +\left(\sum_{k, l \geq 0}(k+1) q_{1}\left(q_{23} q\right)^{l} q_{14}^{k} q^{2 k l}+\sum_{k, l \geq 0} k q_{1} q_{14}^{l}\left(q_{23} q\right)^{k} q^{2 k l}\right) a_{3} b_{4} a_{4} b_{2} \\
& +\left(\sum_{k, l \geq 0} k q_{2} q_{23}^{l}\left(q_{14} q\right)^{k} q^{2 k l}+\sum_{k, l \geq 0}(k+1) q_{2}\left(q_{14} q\right)^{l} q_{23}^{k} q^{2 k l}\right) a_{3} b_{1} a_{4} b_{3} \\
& +\left(\sum_{k, l \geq 0}(k+1) q\left(q_{23} q\right)^{l}\left(q_{14} q\right)^{k} q^{2 k l}+\sum_{k, l \geq 0} k q_{14}^{l} q_{23}^{k} q^{2 k l}\right) a_{3} b_{1} a_{1} b_{2}
\end{aligned}
$$

Centrality The general mirror construction guarantees that the deformed potential $\ell_{q}$ is central in the deformed Jacobi algebra $\mathrm{Jac}_{q} \check{Q}$. In what follows, we verify centrality manually in the case of the 4 -punctured torus. Our starting point is the explicit description of $W_{q}$ and $\ell_{q}$ in terms of midpoint polygons. It turns out that centrality is not as obvious as in the classical case of $\ell \in \mathrm{Jac} \check{Q}$. The checks we present incarnate particular cases of the $A_{\infty}$-relations for $\mathrm{H} \mathbb{L}_{q}$, in fact going beyond the transversal case. Checking centrality therefore presents evidence that the description of $\mathrm{H} \mathbb{L}_{q}$ in terms of CR, ID, DS and DW disks is accurate and is therefore evidence for the correctness of Paper II

To get started, recall that proving $\ell_{q}$ central entails finding for every arc $a \in \dot{Q}_{1}$ an element $x \in \overline{\left(\partial_{a} W_{q}\right)}$ such that $a \ell_{q}+x=\ell_{q} a$ within $\mathbb{C} \check{Q}$. In the classical case of $\ell \in \operatorname{Jac} \check{Q}$, this element $x$ can be described as a sequence of F-term flips of the path $\ell a \in \mathbb{C} \check{Q}$, see section 19.10 In the deformed case of $\ell_{q} \in \mathrm{Jac}_{q} \check{Q}$, the potential $\ell_{q}$ is much more complicated and the element $x$ can only be obtained by inspecting and Koszul dualizing the $A_{\infty}$-relations of $\mathrm{H} \mathbb{L}_{q}$.

For symmetry reasons, it suffices to prove $\ell_{q} a-a \ell_{q} \in \overline{\left(\partial_{a} W_{q}\right)}$ for a single arc $a \in \check{Q}_{1}$, which we choose to be the arc $a=a_{2}$. Recall that $t\left(a_{2}\right)=L_{1}$ and $h\left(a_{2}\right)=L_{2}$. The arrow set $\check{Q}_{1}$ identifies with odd transversal intersections of zigzag paths in $Q$, and we write $X_{a}$ for the odd basis morphism whose corresponding intersection point is located at the midpoint of the arc $a$. For instance, the two zigzag paths $L_{1}$ and $L_{2}$ and the morphism $X_{a_{2}}: L_{1} \rightarrow L_{2}$ are depicted in Figure H.8a Recall that the element $b$ is a formal sum of the odd transversal intersections $X_{a}$, weighted by their formal parameter $a \in \check{Q}_{1}$. In our case, we have explicitly

$$
b=\sum_{i=1}^{4} a_{i} X_{a_{i}}+\sum_{i=1}^{4} b_{i} X_{b_{i}} .
$$

The first step to centrality is to obtain a candidate expression for $x=\ell_{q, 2} a_{2}-a_{2} \ell_{q, 1}$ in terms of the relations $\partial_{a} W_{q}$. The idea is to inspect those $A_{\infty}$-relations which lead to centrality in the proof of Lemma 20.30

$$
\begin{align*}
& \mu_{\mathrm{H} \mathbb{L}_{q}}\left(\mu^{\geq 0}(b, \ldots, b)\right)+\mu_{\mathrm{H} \mathbb{L}_{q}}\left(b, \mu_{\mathrm{H} \mathbb{L}_{q}}^{\geq 0}(b, \ldots, b)\right) \\
+ & \mu_{\mathrm{H} \mathbb{L}_{q}}\left(\mu^{\geq 0}(b, \ldots, b), b\right)+\mu_{\mathrm{H} \mathbb{L}_{q}}^{\geq 3}\left(b, \ldots, \mu^{\geq 0}(b, \ldots, b), \ldots, b\right)=0 . \tag{H.1}
\end{align*}
$$

The expression on the left-hand side of (H.1) is a sum of multiple odd morphisms weighted by paths in $\check{Q}$. To guess $x$, we extract the coefficient of the odd morphism $X_{a_{2}}$.

(a) The zigzag paths $L_{1}$ and $L_{2}$ and the morphism $X_{a_{2}}: L_{1} \rightarrow L_{2}$

Let us inspect all four terms of H.1. The first term vanishes in the case of the 4 -punctured torus, since it is geometrically consistent. For the second and third term, recall that $\mu^{\geq 0}(b, \ldots, b)$ is the sum of the idententities of the zigzag paths $L_{i}$ weighted by $\ell_{q, i}$, and the even intersections $Y_{a}$ weighted by the relations $\partial_{a} W_{q}$ :

$$
\mu^{\geq 0}(b, \ldots, b)=\sum_{i=1}^{4} \ell_{q, i} \operatorname{id}_{L_{i}}+\sum_{i=1}^{4} \partial_{a_{i}} W_{q} Y_{a_{i}}+\sum_{i=1}^{4} \partial_{b_{i}} W_{q} Y_{b_{i}}
$$

We insert this expression into the second and third terms of H.1 and extract the coefficient of $X_{a_{2}}$. Since there are no 3 -gons among zigzag curves in $Q$, the $X_{a_{2}}$ coefficient of the second term is $a_{2} \ell_{q, 1}$ and the $X_{a_{2}}$ coefficient of the third term is $-\ell_{q, 2} a_{2}$.

For the fourth term in H.1, we have to count CR, ID, DS and DW disks with output $X_{a_{2}}$ and input sequence consisting of odd elements of the form $X_{a_{i}}$ or $X_{b_{i}}$, mixed with one single even input of the form $Y_{a_{i}}$ or $Y_{b_{i}}$. The $X_{a_{2}}$ coefficient from every CR, ID, DS or DW disk is then the path in $\check{Q}$ given by the concatenation of the $a_{i}$ or $b_{i}$ symbols, inserting the relation $\partial_{a_{i}} W_{q}$ or $\partial_{b_{i}} W_{q}$ instead at the even input.

It remains to enumerate all relevant CR, ID, DS and DW disks explicitly in the case of the 4 -punctured torus:

- CR disks are 4-gons, in fact rectangles due to the shape of $Q$. CR disks contribute both to the inner and to the outer $\mu_{\mathrm{H} \mathbb{L}_{q}}^{3}$ in H.1.
- ID disks are 5-gons, in fact rectangles of which one side has an output marking on the identity location $a_{2}$. They are only relevant for the inner $\mu_{\mathrm{H} \mathbb{L}_{q}}$ since their output is an identity and not $X_{a_{2}}$.
- DS disks are only relevant for the outer $\mu_{\mathrm{H} \mathbb{L}_{q}}$ since they require at least one even input. According to the intricate rules for DS disks, the only DS disk with output $X_{a_{2}}$ is the DS disk with infinitesimally small side lengths contributing to $\mu_{\mathrm{H} \mathbb{L}_{q}}^{3}\left(X_{a_{2}}, Y_{a_{2}}, X_{a_{2}}\right)$. The existence of this DS disk is independent of the choice of co-identity location. In terms of the Kadeishvili construction from Paper I this DS disk corresponds to the $\pi$-tree with associated DS result component depicted in Figure H. 9 .
- DW disks are irrelevant. In fact, a DW disk has at least one even input and has a co-identity output and can therefore neither contribute to the inner $\mu_{\mathrm{H}_{q}}$ nor contribute $X_{a_{2}}$ to the outer $\mu_{\mathrm{H} \mathbb{L}_{q}}$.
This finishes our evaluation of H.1. All in all, the $A_{\infty}$-relations for $\mathrm{H} \mathbb{L}_{q}$ claim that the $X_{a_{2}}$ coefficients of H.1 add up to zero. This provides us with a candidate expression for $x=\ell_{q, 2} a_{2}-a_{2} \ell_{q, 1}$ in terms of the relations $\partial_{a} W_{q}$. We record and verify this guess as follows:

Lemma H.1. Regard the 4 -punctured torus $Q$ with its zigzag paths $L_{1}, L_{2}, L_{3}, L_{4}$. Denote by $\ell_{q, 1}$ and $\ell_{q, 2}$ the potentials in $\mathbb{C} \mathscr{Q} \llbracket q_{1}, q_{2}, q_{3}, q_{4} \rrbracket$ associated with $L_{1}$ and $L_{2}$ under the choice of $a_{2}$ as identity


Figure H.9: DS result component contributing to $\mu^{3}\left(X_{a_{2}}, Y_{a_{2}}, X_{a_{2}}\right)$
location. Then within $\mathbb{C} \check{Q} \llbracket q_{1}, q_{2}, q_{3}, q_{4} \rrbracket$ we have

$$
\begin{align*}
\ell_{q, 2} a_{2}-a_{2} \ell_{q, 1}= & a_{2}\left(\partial_{a_{2}} W_{q}\right) a_{2}  \tag{H.2}\\
& +\left(\sum_{k, l \geq 0} q_{1} q^{l} q_{14}^{k} q^{2 k l}-\sum_{k, l \geq 0} q_{1} q_{23} q^{l}\left(q_{23} q\right)^{k} q^{2 k l}\right) a_{3} b_{4}\left(\partial_{b_{1}} W_{q}\right) \\
& +\left(\sum_{k, l \geq 0} q_{14} q^{2 l} q_{14}^{k} q^{2 k l}-\sum_{k, l \geq 0} q_{23} q q^{2 l}\left(q_{23} q\right)^{k} q^{2 k l}\right) a_{2} b_{1}\left(\partial_{b_{1}} W_{q}\right) \\
& +\left(\sum_{k, l \geq 0} q_{1} q_{2} q_{4} q^{l}\left(q_{14} q\right)^{k} q^{2 k l}-\sum_{k, l \geq 0} q_{2} q^{l} q_{23}^{k} q^{2 k l}\right) a_{3} b_{1}\left(\partial_{b_{4}} W_{q}\right) \\
& +\left(\sum_{k, l \geq 0} q_{14} q q^{2 l}\left(q_{14} q\right)^{k} q^{2 k l}-\sum_{k, l \geq 0} q_{23} q^{2 l} q_{23}^{k} q^{2 k l}\right) a_{2} b_{4}\left(\partial_{b_{4}} W_{q}\right) \\
& +\left(\sum_{k, l \geq 0} q_{1} q_{2} q^{l} q^{k} q^{2 k l}-\sum_{k, l \geq 0} q_{1} q_{2} q^{l} q^{k} q^{2 k l}\right) a_{3}\left(\partial_{a_{2}} W_{q}\right) a_{3} \\
& +\left(\sum_{k, l \geq 0} q^{2} q^{2 l} q^{2 k} q^{2 k l}-\sum_{k, l \geq 0} q^{2} q^{2 l} q^{2 k} q^{2 k l}\right) a_{2}\left(\partial_{a_{2}} W_{q}\right) a_{2} \\
& +\left(\sum_{k, l \geq 0} q q^{l} q^{2 k} q^{2 k l}-\sum_{k, l \geq 0} q q^{l} q^{2 k} q^{2 k l}\right) a_{3}\left(\partial_{a_{3}} W_{q}\right) a_{2} \\
& +\left(\sum_{k, l \geq 0} q q^{2 l} q^{k} q^{2 k l}-\sum_{k, l \geq 0} q q^{2 l} q^{k} q^{2 k l}\right) a_{2}\left(\partial_{a_{3}} W_{q}\right) a_{3} \\
& +\left(\sum_{k, l \geq 0} q_{2} q_{14}\left(q_{14} q\right)^{l} q^{k} q^{2 k l}-\sum_{k, l \geq 0} q_{2} q_{23}^{l} q^{k} q^{2 k l}\right)\left(\partial_{b_{2}} W_{q}\right) b_{3} a_{3} \\
& +\left(\sum_{k, l \geq 0} q_{14} q\left(q_{14} q\right)^{l} q^{2 k} q^{2 k l}-\sum_{k, l \geq 0} q_{23} q_{23}^{l} q^{2 k} q^{2 k l}\right)\left(\partial_{b_{2}} W_{q}\right){b_{2} a_{2}} \\
& +\left(\sum_{k, l \geq 0} q_{1} q_{14}^{l} q^{k} q^{2 k l}-\sum_{k, l \geq 0} q_{1} q_{23}\left(q_{23} q\right)^{l} q^{k} q^{2 k l}\right)\left(\partial_{b_{3}} W_{q}\right){b_{2} a_{3}} \\
& +\left(\sum_{k, l \geq 0} q_{14} q_{14}^{l} q^{2 k} q^{2 k l}-\sum_{k, l \geq 0} q_{23} q\left(q_{23} q\right)^{l} q^{2 k} q^{2 k l}\right)\left(\partial_{b_{3}} W_{q}\right){b_{3} a_{2}}^{2}
\end{align*}
$$

In particular, we have $\ell_{q} a_{2}=a_{2} \ell_{q}$ within $\operatorname{Jac}_{q} \check{Q}$.
Proof. The strategy is to cancel terms on the right-hand side against each other. The correct means of finding cancellation partners is by tracing back all terms to their respective geometric origins. More precisely, recall that the claimed identity (H.2) is supposed to reflect the $A_{\infty}$-relations for $\mathrm{H} \mathbb{L}_{q}$. The $A_{\infty}$-relations again reflect the geometric property that nonconvex disks can be divided into convex disks in two different ways. Once we trace back every term on the right-hand side of $H .2$ to its nonconvex disk origin, we find the correct cancellation partner. A few corner cases remain in which the disk is actually convex, and these correspond to the left-hand side.

It might be tempting to cancel the eight middle terms on the right-hand side of (H.2) first. However, the cancellation of these eight terms is specific to the case of the 4 -punctured torus and we in fact need to preserve these terms in order to make the other cancellations transparent.

To get started, note that the identity $\quad \mathrm{H} .2$ consists of $\mathbb{C} \llbracket q_{1}, q_{2}, q_{3}, q_{4} \rrbracket$-linear combinations of paths of length five. Checking the identity entails comparing coefficients of every of the possible paths. In the remainder of the proof, we merely focus on two example paths: $a_{2} b_{1} a_{1} b_{3} a_{2}$ and $a_{2} b_{1} a_{1} b_{2} a_{3}$. Out of these two, the case of the first path is easily settled because the right-hand side contains no such paths. The case of the second path is much harder and truly illustrates the geometric reason for equality. The authors also checked manually the coefficients of all 14 other paths and used the results to eradicate all errors in H.2. For the present demonstration, we merely restrict to the two paths mentioned.

Let us comment on the first path $a_{2} b_{1} a_{1} b_{3} a_{2}$. We determine the coefficient of this path in all four entities of H.2. The coefficient coming from the term $\ell_{q, 2} a_{2}$ is

$$
\sum_{k, l \geq 0}(k+1) q_{4}\left(q_{23} q\right)^{l} q_{14}^{k} q^{2 k l}+\sum_{k, l \geq 0} k q_{4} q_{14}^{l}\left(q_{23} q\right)^{k} q^{2 k l}
$$

The coefficient coming from the term $a_{2} \ell_{q, 1}$ is

$$
\sum_{k, l \geq 0} l q_{4}\left(q_{23} q\right)^{l} q_{14}^{k} q^{2 k l}+\sum_{k, l \geq 0}(l+1) q_{4} q_{14}^{l}\left(q_{23} q\right)^{k} q^{2 k l}
$$

The coefficient coming from the term $a_{2}\left(\partial_{a_{2}} W_{q}\right) a_{2}$ is

$$
\sum_{k, l \geq 0} q_{4} q_{14}^{k}\left(q_{23} q\right)^{l} q^{2 k l}-\sum_{k, l \geq 0} q_{4} q_{14}^{l}\left(q_{23} q\right)^{k} q^{2 k l}=0
$$

Note that this vanishing is specific to the 4 -punctured torus. The coefficient in the rest of the right-hand side of H.2 vanishes because $\partial_{b_{1}} W_{q}$ has vanishing $a_{1} b_{3} a_{2}$ coefficient, because $\partial_{a_{2}} W_{q}$ has vanishing $b_{1} a_{1} b_{3}$ coefficient, and because $\partial_{b_{3}} W_{q}$ has vanishing $a_{2} b_{1} a_{1}$ coefficient. All in all, both the left-hand side and the right-hand side of H.2 cancel out. This proves the identity H.2 on the level of $a_{2} b_{1} a_{1} b_{3} a_{2}$ terms.

We now focus on the second path $a_{2} b_{1} a_{1} b_{2} a_{3} \in \mathbb{C} \check{Q}$ and prove that its coefficient in $H$ vanishes. Regarding the left-hand side, the coefficient in $\ell_{q, 2} a_{2}$ vanishes and the coefficient in $-a_{2} \ell_{q, 1}$ consists of the two sums

$$
\begin{equation*}
-\sum_{k, l \geq 0}(l+1) q\left(q_{23} q\right)^{l}\left(q_{14} q\right)^{k} q^{2 k l}-\sum_{k, l \geq 0} l q_{14}^{l} q_{23}^{k} q^{2 k l} \tag{H.3}
\end{equation*}
$$

Regarding the right-hand side, the coefficient in $a_{2}\left(\partial_{a_{2}} W_{q}\right) a_{2}$ vanishes. The relevant contribution of the 24 remaining relation terms is

$$
\begin{align*}
& +\sum_{k, l \geq 0} q_{1} q_{4} q^{2 l} q_{14}^{k} q^{2 k l} a_{2} b_{1}\left(\partial_{b_{1}} W_{q}\right) \\
& -\sum_{k, l \geq 0} q_{2} q_{3} q q^{2 l}\left(q_{23} q\right)^{k} q^{2 k l} a_{2} b_{1}\left(\partial_{b_{1}} W_{q}\right) \\
& -\sum_{k, l \geq 0} q q^{2 l} q^{k} q^{2 k l} a_{2}\left(\partial_{a_{3}} W_{q}\right) a_{3}  \tag{H.4}\\
& +\sum_{k, l \geq 0} q q^{2 l} q^{k} q^{2 k l} a_{2}\left(\partial_{a_{3}} W_{q}\right) a_{3} \\
& +\sum_{k, l \geq 0} q_{1} q_{14}^{l} q^{k} q^{2 k l}\left(\partial_{b_{3}} W_{q}\right) b_{2} a_{3} \\
& -\sum_{k, l \geq 0} q_{1} q_{2} q_{3}\left(q_{23} q\right)^{l} q^{k} q^{2 k l}\left(\partial_{b_{3}} W_{q}\right) b_{2} a_{3}
\end{align*}
$$

We count the $a_{1} b_{2} a_{3}$ coefficients in $\partial_{b_{1}} W_{q}, \partial_{a_{3}} W_{q}$ and $\partial_{b_{3}} W_{q}$ as follows:

$$
\begin{aligned}
\partial_{b_{1}} W_{q} \mid a_{1} b_{2} a_{3}= & +\sum_{k, l \geq 0} q_{14}^{2 k+1+l+2 k l} q_{23}^{2 l+1+k+2 k l} \\
& -\sum_{k, l \geq 0} q_{14}^{l+2 k l} q_{23}^{k+2 k l} \\
\partial_{a_{3}} W_{q} \mid b_{1} a_{1} b_{2}= & +\sum_{k, l \geq 0} q_{14}^{2 k+l+1+2 k l} q_{23}^{2 l+k+1+2 k l} \\
& -\sum_{k, l \geq 0} q_{14}^{l+2 k l} q_{23}^{k+2 k l} \\
\partial_{b_{3}} \mid a_{2} b_{1} a_{1}= & +\sum_{k, l \geq 0} q_{4} q_{14}^{k+l+2 k l} q_{23}^{2 l+2 k l} \\
& -\sum_{k, l \geq 0} q_{4} q_{14}^{k+l+2 k l} q_{23}^{2 k+2 k l}
\end{aligned}
$$

Plugging these expressions into H.4 , we get twelve terms which we name as follows:

$$
\begin{aligned}
& 1 \mathrm{~A}:+\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(k+1)(2 l+1)+\left(2 k^{\prime}+1\right)\left(l^{\prime}+1\right)} q_{23}^{2(k+1) l+\left(k^{\prime}+1\right)\left(2 l^{\prime}+1\right)} \\
& 1 \mathrm{~B}:-\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(k+1)(2 l+1)+l^{\prime}\left(2 k^{\prime}+1\right)} q_{23}^{2(k+1) l+k^{\prime}\left(2 l^{\prime}+1\right)}
\end{aligned}
$$

2A: $-\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(k+1)(2 l+1)+\left(2 k^{\prime}+1\right)\left(l^{\prime}+1\right)} q_{23}^{2(k+1)(l+1)+\left(k^{\prime}+1\right)\left(2 l^{\prime}+1\right)}$
$2 \mathrm{~B}:+\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(k+1)(2 l+1)+\left(2 k^{\prime}+1\right) l^{\prime}} q_{23}^{2(k+1)(l+1)+k^{\prime}\left(2 l^{\prime}+1\right)}$

3A: $-\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(k+1)(2 l+1)+\left(2 k^{\prime}+1\right)\left(l^{\prime}+1\right)} q_{23}^{(k+1)(2 l+1)+\left(k^{\prime}+1\right)\left(2 l^{\prime}+1\right)}$
$3 \mathrm{~B}:+\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(k+1)(2 l+1)+\left(2 k^{\prime}+1\right) l^{\prime}} q_{23}^{(k+1)(2 l+1)+k^{\prime}\left(2 l^{\prime}+1\right)}$
$4 \mathrm{~A}: \quad+\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(k+1)(2 l+1)+\left(2 k^{\prime}+1\right)\left(l^{\prime}+1\right)} q_{23}^{(k+1)(2 l+1)+\left(k^{\prime}+1\right)\left(2 l^{\prime}+1\right)}$
4B: $\quad-\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(k+1)(2 l+1)+\left(2 k^{\prime}+1\right) l^{\prime}} q_{23}^{(k+1)(2 l+1)+k^{\prime}\left(2 l^{\prime}+1\right)}$
$5 \mathrm{~A}: \quad+\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(1+l+k+2 k l)+\left(k^{\prime}+l^{\prime}+2 k^{\prime} l^{\prime}\right)} q_{23}^{k(2 l+1)+2 l^{\prime}\left(k^{\prime}+1\right)}$
$5 \mathrm{~B}: \quad-\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(1+l+k+2 k l)+\left(k^{\prime}+l^{\prime}+2 k^{\prime} l^{\prime}\right)} q_{23}^{k(2 l+1)+2 k^{\prime}\left(l^{\prime}+1\right)}$
$6 \mathrm{~A}: \quad-\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(1+l+k+2 k l)+\left(k^{\prime}+l^{\prime}+2 k^{\prime} l^{\prime}\right)} q_{23}^{(k+1)(2 l+1)+2\left(k^{\prime}+1\right) l^{\prime}}$
$6 \mathrm{~B}:+\sum_{k, l, k^{\prime}, l^{\prime} \geq 0} q_{14}^{(1+l+k+2 k l)+\left(k^{\prime}+l^{\prime}+2 k^{\prime} l^{\prime}\right)} q_{23}^{(k+1)(2 l+1)+2 k^{\prime}\left(l^{\prime}+1\right)}$
It is our task to sum up these twelve terms $(1 \mathrm{~A}-6 \mathrm{~B})$ and prove the sum equal to H.3). Again, note that 3 A and 3 B cancel with 4 A and 4 B . However, performing this easy cancellation would obfuscate all other cancellations. Instead, we shall obtain the correct cancellations by inspecting the geometric origin of every of these twelve terms.

We give a catalog of all cancellations below. To understand the reasons behind the cancellations, we shall demonstrate here how one finds the correct cancellation for the term 1 A in case $l \geq l^{\prime}+1$. Its geometric origin is depicted in Figure H.10a. In short, this figure depicts a pair of disks contributing to

$$
\begin{equation*}
\mu_{\mathrm{H} \mathbb{L}_{q}}\left(X_{a_{2}}, X_{b_{1}}, \mu_{\mathrm{H} \mathbb{L}_{q}}\left(X_{a_{1}}, X_{b_{2}}, X_{a_{3}}\right)\right) \tag{H.5}
\end{equation*}
$$

More precisely, the disk labeled $\partial_{b_{1}} W_{q}$ contributes to the inner $\mu_{\mathrm{H} \mathbb{L}_{q}}$, while the disk labeled "coeff" contributes to the outer $\mu_{\mathrm{H} \mathbb{L}_{q}}$. The side lengths of the disk labeled $\partial_{b_{1}} W_{q}$ are $4 k^{\prime}+3$ and $4 l^{\prime}+3$. Correspondingly, the example depicted has $k^{\prime}=l^{\prime}=0$. The side lengths of the disk labeled "coeff" are $4 k+4$ and $4 l+1$. Correspondingly, the example depicted has $k=0$ and $l=1$. The nonconvex shape can be split into two other pieces, depicted in Figure H.10b. That figure depicts a pair of disks contributing to

$$
\begin{equation*}
\mu_{\mathrm{H} \mathbb{L}_{q}}\left(X_{a_{2}}, \mu_{\mathrm{H} \mathbb{L}_{q}}\left(X_{b_{1}}, X_{a_{1}}, X_{b_{2}}\right), X_{a_{3}}\right) . \tag{H.6}
\end{equation*}
$$

The two individual contributions to H.5 and H.6 cover the same area and therefore have the same deformation parameters. A quick check shows that they have equal sign. Noting that 1A and 3A come with opposite overall signs, this makes 3 A our candidate for cancellation with 1 A in case $l \geq l^{\prime}+1$.

The cancellation of 1 A and 3 A involves a change of indices. More precisely, the term 1 A with indices $\left(k, l, k^{\prime}, l^{\prime}\right)$ cancels with 3 A with indices $\left(k, l-l^{\prime}-1, k+k^{\prime}+1, l^{\prime}\right)$. To obtain this correspondence between indices, let us understand the disks in Figure H.10b The disk labeled $\partial_{a_{3}} W_{q}$ contributes to the inner $\mu_{\mathrm{H} \mathbb{L}_{q}}$, while the disk labeled "coeff" contributes to the outer $\mathrm{H} \mathbb{L}_{q}$. The side lengths of the disk labeled $\partial_{a_{3}} W_{q}$ are $4\left(k+k^{\prime}+1\right)+3$ and $4 l^{\prime}+3$. The side lengths of the disk labeled "coeff" are $4 k+4$ and $4\left(l-l^{\prime}-1\right)+2$. This means we expect to find the partner of 1 A with indices $\left(k, l, k^{\prime}, l^{\prime}\right)$ in 3 A with indices $\left(k, l-l^{\prime}-1, k+k^{\prime}+1, l^{\prime}\right)$.

Similar considerations for all other pairs of terms give rise to a list of cancellation candidates. In what follows, we compile this list and check for every entry individually that it indeed cancels out with its partner:

- 1 A in case $l \geq l^{\prime}+1$ cancels with 3 A with indices $\left(k, l-l^{\prime}-1, k+k^{\prime}+1, l^{\prime}\right)$ :
$\left(q_{14}\right):(k+1)(2 l+1)+\left(2 k^{\prime}+1\right)\left(l^{\prime}+1\right)=(k+1)\left(2\left(l-l^{\prime}-1\right)+1\right)+\left(2\left(k+k^{\prime}+1\right)+1\right)\left(l^{\prime}+1\right)$,
$\left(q_{23}\right): \quad 2(k+1) l+\left(k^{\prime}+1\right)\left(2 l^{\prime}+1\right)=(k+1)\left(2\left(l-l^{\prime}-1\right)+1\right)+\left(\left(k+k^{\prime}+1\right)+1\right)\left(2 l^{\prime}+1\right)$.
- 1 A in case $l \leq l^{\prime}$ cancels with 6 A with indices $\left(k^{\prime}, l^{\prime}-l, k+k^{\prime}+1, l\right)$ :

$$
\begin{aligned}
\left(q_{14}\right): & (k+1)(2 l+1)+\left(2 k^{\prime}+1\right)\left(l^{\prime}+1\right) \\
& =\left(1+\left(l^{\prime}-l\right)+k^{\prime}+2 k^{\prime}\left(l^{\prime}-l\right)\right)+\left(\left(k+k^{\prime}+1\right)+l+2\left(k+k^{\prime}+1\right) l\right), \\
\left(q_{23}\right): & 2(k+1) l+\left(k^{\prime}+1\right)\left(2 l^{\prime}+1\right)=\left(k^{\prime}+1\right)\left(2\left(l^{\prime}-l\right)+1\right)+2\left(\left(k+k^{\prime}+1\right)+1\right) l .
\end{aligned}
$$

- 1B in case $k \geq k^{\prime}$ cancels with 5A with indices $\left(k^{\prime}, l+l^{\prime}, k-k^{\prime}, l\right)$ :
$\left(q_{14}\right): \quad(k+1)(2 l+1)+l^{\prime}\left(2 k^{\prime}+1\right)=\left(1+\left(l+l^{\prime}\right)+k^{\prime}+2 k^{\prime}\left(l+l^{\prime}\right)\right)+\left(\left(k-k^{\prime}\right)+l+2\left(k-k^{\prime}\right) l\right)$,
$\left(q_{23}\right): \quad 2(k+1) l+k^{\prime}\left(2 l^{\prime}+1\right)=k^{\prime}\left(2\left(l+l^{\prime}\right)+1\right)+2 l\left(\left(k-k^{\prime}\right)+1\right)$.
- 1B in case $k+1 \leq k^{\prime}$ cancels with 3B with indices $\left(k, l+l^{\prime}, k^{\prime}-k-1, l^{\prime}\right)$ :

$$
\begin{aligned}
\left(q_{14}\right): & (k+1)(2 l+1)+l^{\prime}\left(2 k^{\prime}+1\right) & =(k+1)\left(2\left(l+l^{\prime}\right)+1\right)+\left(2\left(k^{\prime}-k-1\right)+1\right) l^{\prime}, \\
\left(q_{23}\right): & 2(k+1) l+k^{\prime}\left(2 l^{\prime}+1\right) & =(k+1)\left(2\left(l+l^{\prime}\right)+1\right)+\left(k^{\prime}-k-1\right)\left(2 l^{\prime}+1\right) .
\end{aligned}
$$

- 2 A in case $k+1 \leq k^{\prime}$ cancels with 4 A with indices $\left(k, l+l^{\prime}+1, k^{\prime}-k-1, l^{\prime}\right)$ :
$\left(q_{14}\right):(k+1)(2 l+1)+\left(2 k^{\prime}+1\right)\left(l^{\prime}+1\right)=(k+1)\left(2\left(l+l^{\prime}+1\right)+1\right)+\left(2\left(k^{\prime}-k-1\right)+1\right)\left(l^{\prime}+1\right)$,
$\left(q_{23}\right): 2(k+1)(l+1)+\left(k^{\prime}+1\right)\left(2 l^{\prime}+1\right)=(k+1)\left(2\left(l+l^{\prime}+1\right)+1\right)+\left(\left(k^{\prime}-k-1\right)+1\right)\left(2 l^{\prime}+1\right)$.
- 2 A in case $k \geq k^{\prime}$ cancels with 6 B with indices $\left(k^{\prime}, l+l^{\prime}+1, k-k^{\prime}, l\right)$ :

$$
\begin{aligned}
\left(q_{14}\right): & (k+1)(2 l+1)+\left(2 k^{\prime}+1\right)\left(l^{\prime}+1\right) \\
& =\left(1+\left(l+l^{\prime}+1\right)+k^{\prime}+2 k^{\prime}\left(l+l^{\prime}+1\right)\right)+\left(\left(k-k^{\prime}\right)+l+2\left(k-k^{\prime}\right) l\right) \\
\left(q_{23}\right): & 2(k+1)(l+1)+\left(k^{\prime}+1\right)\left(2 l^{\prime}+1\right)=\left(k^{\prime}+1\right)\left(2\left(l+l^{\prime}+1\right)+1\right)+2\left(k-k^{\prime}\right)(l+1)
\end{aligned}
$$

- 2B in case $l \geq l^{\prime}$ cancels with 4B with indices $\left(k, l-l^{\prime}, k+k^{\prime}+1, l^{\prime}\right)$ :

$$
\begin{aligned}
& \left(q_{14}\right):(k+1)(2 l+1)+\left(2 k^{\prime}+1\right) l^{\prime}=(k+1)\left(2\left(l-l^{\prime}\right)+1\right)+\left(2\left(k+k^{\prime}+1\right)+1\right) l^{\prime} . \\
& \left(q_{23}\right): 2(k+1)(l+1)+k^{\prime}\left(2 l^{\prime}+1\right)=(k+1)\left(2\left(l-l^{\prime}\right)+1\right)+\left(k+k^{\prime}+1\right)\left(2 l^{\prime}+1\right) .
\end{aligned}
$$

- 2B in case $l+1 \leq l^{\prime}$ cancels with 5B with indices $\left(k^{\prime}, l^{\prime}-l-1, k+k^{\prime}+1, l\right)$ :

$$
\begin{aligned}
\left(q_{14}\right): & (k+1)(2 l+1)+\left(2 k^{\prime}+1\right) l^{\prime} \\
& =\left(1+\left(l^{\prime}-l-1\right)+k^{\prime}+2 k^{\prime}\left(l^{\prime}-l-1\right)\right)+\left(\left(k+k^{\prime}+1\right)+l+2\left(k+k^{\prime}+1\right) l\right) \\
\left(q_{23}\right): & 2(k+1)(l+1)+k^{\prime}\left(2 l^{\prime}+1\right)=k^{\prime}\left(2\left(l^{\prime}-l-1\right)+1\right)+2\left(k+k^{\prime}+1\right)(l+1)
\end{aligned}
$$

- 3A in case $k^{\prime} \leq k$ cancels with 5 A with indices $\left(k-k^{\prime}, l, k^{\prime}, l+l^{\prime}+1\right)$ :

$$
\begin{aligned}
\left(q_{14}\right): & (k+1)(2 l+1)+\left(2 k^{\prime}+1\right)\left(l^{\prime}+1\right) \\
& =\left(1+l+\left(k-k^{\prime}\right)+2\left(k-k^{\prime}\right) l\right)+\left(k^{\prime}+\left(l+l^{\prime}+1\right)+2 k^{\prime}\left(l+l^{\prime}+1\right)\right) \\
\left(q_{23}\right): & (k+1)(2 l+1)+\left(k^{\prime}+1\right)\left(2 l^{\prime}+1\right)=\left(k-k^{\prime}\right)(2 l+1)+2\left(l+l^{\prime}+1\right)\left(k^{\prime}+1\right)
\end{aligned}
$$

- 3B in case $l^{\prime} \geq l+1$ cancels with 5B with indices $\left(k+k^{\prime}+1, l, k^{\prime}, l^{\prime}-l-1\right)$ :

$$
\begin{aligned}
\left(q_{14}\right): & (k+1)(2 l+1)+\left(2 k^{\prime}+1\right) l^{\prime} \\
& =\left(1+l+\left(k+k^{\prime}+1\right)+2\left(k+k^{\prime}+1\right) l\right)+\left(k^{\prime}+\left(l^{\prime}-l-1\right)+2 k^{\prime}\left(l^{\prime}-l-1\right)\right) \\
\left(q_{23}\right): & (k+1)(2 l+1)+k^{\prime}\left(2 l^{\prime}+1\right)=\left(k+k^{\prime}+1\right)(2 l+1)+2 k^{\prime}\left(\left(l^{\prime}-l-1\right)+1\right)
\end{aligned}
$$

- 4 A in case $l^{\prime} \geq l$ cancels with 6 A with indices $\left(k+k^{\prime}+1, l, k^{\prime}, l^{\prime}-l\right)$ :

$$
\begin{aligned}
\left(q_{14}\right): & (k+1)(2 l+1)+\left(2 k^{\prime}+1\right)\left(l^{\prime}+1\right) \\
& =\left(1+l+\left(k+k^{\prime}+1\right)+2\left(k+k^{\prime}+1\right) l\right)+\left(k^{\prime}+\left(l^{\prime}-l\right)+2 k^{\prime}\left(l^{\prime}-l\right)\right) \\
\left(q_{23}\right): & (k+1)(2 l+1)+\left(k^{\prime}+1\right)\left(2 l^{\prime}+1\right)=\left(\left(k+k^{\prime}+1\right)+1\right)(2 l+1)+2\left(k^{\prime}+1\right)\left(l^{\prime}-l\right)
\end{aligned}
$$

- 4B in case $k \geq k^{\prime}$ cancels with 6 B with indices $\left(k-k^{\prime}, l, k^{\prime}, l+l^{\prime}\right)$ :
$\left(q_{14}\right): \quad(k+1)(2 l+1)+\left(2 k^{\prime}+1\right) l^{\prime}=\left(1+l+\left(k-k^{\prime}\right)+2\left(k-k^{\prime}\right) l\right)+\left(k^{\prime}+\left(l+l^{\prime}\right)+2 k^{\prime}\left(l+l^{\prime}\right)\right)$,
$\left(q_{23}\right): \quad(k+1)(2 l+1)+k^{\prime}\left(2 l^{\prime}+1\right)=\left(\left(k-k^{\prime}\right)+1\right)(2 l+1)+2 k^{\prime}\left(\left(l+l^{\prime}\right)+1\right)$.
We have shown that most terms of (H.4) cancel out. It remains to analyze the corner cases which did not cancel and match them with H.3). In fact, the only remaining corner terms are 6 A with indices $\left(k, l, k, l^{\prime}\right)$ and 5 B with indices $\left(k, l, k, l^{\prime}\right)$. We conclude that the sum of the 12 terms $(1 \mathrm{~A}-6 \mathrm{~B})$ is

$$
\begin{aligned}
& -\sum_{k, l, l^{\prime} \geq 0} q_{14}^{1+\left(l+l^{\prime}\right)+2 k+2 k\left(l+l^{\prime}\right)} q_{23}^{(k+1)\left(2\left(l+l^{\prime}\right)+1\right)}-\sum_{k, l, l^{\prime} \geq 0} q_{14}^{1+\left(l+l^{\prime}\right)+2 k+2 k\left(l+l^{\prime}\right)} q_{23}^{k\left(2\left(l+l^{\prime}\right)+3\right)} \\
& =-\sum_{k, s \geq 0}(s+1) q_{14}^{1+s+2 k+2 k s} q_{23}^{1+k+2 s+2 k s}-\sum_{k, s \geq 0} s q_{14}^{s+2 k s} q_{23}^{k+2 k s}
\end{aligned}
$$

We recognize this expression as equal to H.3). In other words, the coefficient of $a_{2} b_{1} a_{1} b_{2} a_{3}$ in the centrality identity H.2 agrees on both sides. This finishes the checks for the coefficients of $a_{2} b_{1} a_{1} b_{2} a_{3}$, and thereby finishes our selected calculations aimed at demonstrating H.2.


Figure H.10: Term 1A cancels with term 3A if $l \geq l^{\prime}+1$

## Note IV

## Explicit Hochschild Classes for Gentle Algebras

## 22 Introduction

Hochschild cohomology is a crucial invariant for understanding an object's deformation theory. Gentle algebras are discrete models for Fukaya categories of punctured surfaces. We have computed the Hochschild cohomology of gentle algebras in Paper I under a technical restriction regarding the surface. In this note, we go beyond the restriction. More precisely, we extend the computation of Paper 1 to include all gentle algebras of punctured surfaces. The idea is to write down explicit Hochschild cocycles, instead of constructing them implicitly as in Paper I

The findings from Paper I can be summarized as follows:
Generation criterion: It determines for a given collection of cocycles with certain prescribed shape in low adicity whether it concerns a basis of Hochschild cohomology or not. We recall the generation criterion in section 24.1
Odd cocycles: We provided an explicit collection of odd cocycles. This family satisfies the requirements of the generation criterion and therefore forms a basis for odd Hochschild cohomology. We recall the odd cocycles in section 24.2
Sporadic even cocycles: We provided a semi-explicit collection of even cocycles. In the present note, we refer to them as the sporadic cocycles. We recall the sporadic cocycles in section 24.3

Ordinary even cocycles: We provided an implicit collection of even cocycles under the assumption that the arc system satisfies the [NL2] condition. In the present paper, we refer to these cocycles as ordinary even cocycles. The sporadic and ordinary even cocycles together satisfy the generation criterion. Under the condition [NL2], they form a basis for even Hochschild cohomology.
The problem with Paper I is the requirement of the [NL2] condition. In fact, the ordinary even cocycles were constructed in Paper I as cup products of carefully selected sporadic and odd classes. The sporadic classes required for this construction only exist if the [NL2] condition holds. Without the [NL2] condition, the construction via the cup product fails, leaving us without proof of existence of the ordinary even cocycles. The aim of the present note is to circumvent the [NL2] condition. The idea is to provide an explicit construction for the ordinary even cocycles. The amount of explicitness makes the construction independent of the [NL2] condition. A drawback is that long calculations are required to check that the ordinary even classes actually satisfy the cocycle condition. This is the reason we cut the present note from Paper I

This note is organized as follows: In section 23, we recall $A_{\infty}$-categories, Hochschild cohomology and gentle algebras. In section 24 we recall the generation criterion, the odd cocycles and the sporadic even cocycles from Paper I In section 25, we construct the ordinary even cocycles and perform detailed checks that they indeed satisfy the Hochschild cocycle condition. We summarize the findings in section 25.4 .

## 23 Preliminaries

In this section, we recall $A_{\infty}$-categories, Hochschild cohomology and gentle algebras. A more extensive introduction can be found in Paper I or Paper II

## $23.1 A_{\infty}$-categories

In this section we briefly recall $A_{\infty}$-categories.
Definition 23.1. A ( $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded, strictly unital) $A_{\infty}$-category $\mathcal{C}$ (over e.g. $\mathbb{C}$ ) consists of a collection of objects together with $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded hom spaces $\operatorname{Hom}(X, Y)$, distinguished identity morphisms id ${ }_{X} \in \operatorname{Hom}^{0}(X, X)$ for all $X \in \mathcal{C}$, together with multilinear higher products

$$
\mu^{k}: \operatorname{Hom}\left(X_{k}, X_{k+1}\right) \otimes \ldots \otimes \operatorname{Hom}\left(X_{1}, X_{2}\right) \rightarrow \operatorname{Hom}\left(X_{1}, X_{k+1}\right), \quad k \geq 1
$$

of degree $2-k$ such that the $A_{\infty}$-relations and strict unitality axioms hold: For every compatible morphisms $a_{1}, \ldots, a_{k}$ we have

$$
\begin{aligned}
& \sum_{0 \leq n<m \leq k}(-1)^{\left\|a_{n}\right\|+\ldots+\left\|a_{1}\right\|} \mu\left(a_{k}, \ldots, \mu\left(a_{m}, \ldots, a_{n+1}\right), a_{n}, \ldots, a_{1}\right)=0, \\
& \mu^{2}\left(a, \operatorname{id}_{X}\right)=a, \mu^{2}\left(\operatorname{id}_{Y}, a\right)=(-1)^{|a|} a, \mu^{\geq 3}\left(\ldots, \operatorname{id}_{X}, \ldots\right)=0 .
\end{aligned}
$$

The symbol $\|a\|=|a|-1$ denotes the reduced degree of $a$.

### 23.2 The Hochschild DGLA

In this section, we recall Hochschild cohomology for $A_{\infty}$-categories. First, we recall DG Lie algebras (DGLAs). Second, we recall the Hochschild complex for $A_{\infty}$-categories together with its DGLA structure. Third, we comment on the cup product.
Definition 23.2. A DG Lie algebra (DGLA) is a $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded vector space $L$ together with a differential $d: L^{i} \rightarrow L^{i+1}$ and a bracket $[-,-]: L \times L \rightarrow L$ satisfying the Leibniz rule and the Jacobi identity:

$$
\begin{aligned}
{[a, b] } & =(-1)^{|a||b|+1}[b, a], \\
d(d(a)) & =0, \\
d([a, b]) & =[d a, b]+(-1)^{|a|}[a, d b], \\
0 & =(-1)^{|a| c \mid}[a,[b, c]]+(-1)^{|b||a|}[b,[c, a]]+(-1)^{|c||b|}[c,[a, b]] .
\end{aligned}
$$

Hochschild cohomology has historically been defined for ordinary associative algebras. The Hochschild complex carries a natural DGLA structure. In more modern times, Hochschild cohomology together with the DGLA structure has been extended to the case of $A_{\infty}$-categories. We recall this construction as follows:

Definition 23.3. Let $\mathcal{C}$ be a $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded $A_{\infty}$-category. Then its Hochschild complex $\operatorname{HC}(\mathcal{C})$ is given by the graded vector space

$$
\operatorname{HC}(\mathcal{C})=\prod_{\substack{X_{1}, \ldots, X_{k}, 1 \in \mathcal{C} \\ k \geq 0}} \operatorname{Hom}\left(\operatorname{Hom}_{\mathcal{C}}\left(X_{k}, X_{k+1}\right)[-1] \otimes \ldots \otimes \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, X_{2}\right)[-1], \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, X_{k+1}\right)[-1]\right)
$$

For $\eta, \omega \in \operatorname{HC}(\mathcal{C})$, temporarily denote by $\mu \cdot \omega \in \operatorname{HC}(\mathcal{C})$ the Gerstenhaber product given by

$$
(\eta \cdot \omega)\left(a_{k}, \ldots, a_{1}\right)=\sum(-1)^{\left(\left\|a_{l}\right\|+\ldots+\left\|a_{1}\right\|\right)\|\omega\|} \eta\left(a_{k}, \ldots, \omega(\ldots), a_{l}, \ldots, a_{1}\right) .
$$

Define a $\mathbb{Z}$ - or $\mathbb{Z} / 2 \mathbb{Z}$-graded DGLA structure on $\operatorname{HC}(\mathcal{C})$ as follows: Its grading $\|\cdot\|$ is the one induced from the shifted degrees of the hom spaces of $\mathcal{C}$. In other words, we have the equation

$$
\left\|\eta\left(a_{k}, \ldots, a_{1}\right)\right\|=\|\eta\|+\left\|a_{k}\right\|+\ldots+\left\|a_{1}\right\|, \quad \eta \in \operatorname{HC}(\mathcal{C}) .
$$

The bracket on $\operatorname{HC}(\mathcal{C})$ is the Gerstenhaber bracket

$$
[\eta, \omega]=\eta \cdot \omega-(-1)^{\|\omega\|\|\eta\|} \omega \cdot \eta .
$$

Its differential is given by commuting with the product $\mu_{\mathcal{C}} \in \mathrm{HC}^{1}(\mathcal{C})$ :

$$
d \nu=\left[\mu_{\mathcal{C}}, \nu\right] .
$$

Remark 23.4. Let $\nu \in \operatorname{HC}^{1}(\mathcal{C})$. Then $\mu_{\mathcal{C}}+\varepsilon \nu$ is an infinitesimal (curved $A_{\infty^{-}}$)deformation of $\mu_{\mathcal{C}}$ over the local ring $B=\mathbb{C}[\varepsilon] /\left(\varepsilon^{2}\right)$ if and only if $d \nu=0$. More precisely, Hochschild cohomology $\operatorname{HH}^{1}(\mathcal{C})$ classifies infinitesimal deformations of $\mathcal{C}$ up to gauge equivalence.
Remark 23.5. In case $\mathcal{C}$ is only $\mathbb{Z} / 2 \mathbb{Z}$-graded, the Hochschild DGLA is only a $\mathbb{Z} / 2 \mathbb{Z}$-graded DGLA and Hochschild cohomology is only a $\mathbb{Z} / 2 \mathbb{Z}$-graded vector space.

Remark 23.6. The DGLA structure on $\operatorname{HC}(\mathcal{C})$ induces (noncanonically) the structure of an $L_{\infty}$-algebra on Hochschild cohomology $\mathrm{HH}(\mathcal{C})$.

Remark 23.7. For ordinary algebras, which are concentrated in degree zero and have vanishing higher products, the Hochschild cohomology is typically defined without the shifts. This results in a grading difference of 1 from what we present here. For example, the center of the algebra is the classical zeroth Hochschild cohomology. In our $A_{\infty}$-setting, this cohomology will rather be found in degree -1 .

There is also a second product on $\operatorname{HC}(\mathcal{C})$ : the cup product.
Definition 23.8. The cup product on $\operatorname{HC}(\mathcal{C})$ is given by

$$
(\kappa \cup \nu)\left(a_{r}, \ldots a_{1}\right):=\sum_{0 \leq i \leq j \leq u \leq v \leq r}(-1)^{\text {wr }} \mu\left(a_{r}, \ldots, \kappa\left(a_{v}, \ldots, a_{u+1}\right), \ldots, \nu\left(a_{j}, \ldots, a_{i+1}\right), \ldots, a_{1}\right)
$$

with $=\left(\left\|a_{1}\right\|+\ldots+\left\|a_{u}\right\|\right)\|\kappa\|+\left(\left\|a_{1}\right\|+\ldots+\left\|a_{i}\right\|\right)\|\nu\|+\|\nu\|+1$.

### 23.3 The gentle algebra and its deformation

In this section, we recall $A_{\infty}$-gentle algebras from 18 .
Definition 23.9. A punctured surface is a closed oriented surface $S$ with a finite set of punctures $M \subseteq S$. We assume that $|M| \geq 1$, or $|M| \geq 3$ if $S$ is a sphere.

Definition 23.10. Let $(S, M)$ be a punctured surface. An $\operatorname{arc}$ in $S$ is a not necessarily closed curve $\gamma:[0,1] \rightarrow S$. A loop is an arc $\gamma$ with $\gamma(0)=\gamma(1)$. An arc system $\mathcal{A}$ on $(S, M)$ is a finite collection of arcs such that the arcs in $\mathcal{A}$ meet only at the set $M$ of punctures. Intersections and self-intersections are not allowed.

An arc system $\mathcal{A}$ is full if the arcs cut the surface into contractible pieces, which we call polygons. The arc system $\mathcal{A}$ satisfies the condition [NMD] if:

- Any two $\operatorname{arcs}$ in $\mathcal{A}$ are non-homotopic in $S \backslash M$.
- All loops in $\mathcal{A}$ are non-contractible in $S \backslash M$.

Let us now recall the construction of the gentle algebra $\mathrm{Gtl} \mathcal{A}$. It is an $A_{\infty}$-category and we shall start by describing its objects, differential and product. After that, we will recall the higher products on $\operatorname{Gtl} \mathcal{A}$.

Definition 23.11. Let $\mathcal{A}$ be a full arc system with [NMD]. Then the gentle algebra $\operatorname{Gtl} \mathcal{A}$ is the $A_{\infty}$-category defines as follows:

- Its objects are the $\operatorname{arcs} a \in \mathcal{A}$.
- A basis for the hom space $\operatorname{Hom}_{\mathrm{Gt1} \mathcal{A}}(a, b)$ is given by the set of all angles around punctures from $a$ to $b$.
- The $\mathbb{Z} / 2 \mathbb{Z}$-grading on $\operatorname{Gtl} \mathcal{A}$ is given by declaring all indecomposable angles to have odd degree.
- The differential $\mu^{1}$ is zero.
- The product $\mu^{2}$ is defined as a signed version of the ordinary product of $\mathrm{Gtl} \mathcal{A}$ :

$$
\mu^{1}:=0, \quad \mu^{2}(\alpha, \beta):=(-1)^{|\beta|} \alpha \beta
$$

The angles include empty angles, which are the identities on the arcs. A non-empty angle is indecomposable if it cannot be written as $\alpha \beta$ where $\alpha, \beta$ are non-empty angles. The higher products are defined in Definition 23.15

Remark 23.12. The hom spaces of $\operatorname{Gtl} \mathcal{A}$ are not finite-dimensional, in contrast to what is classically called a gentle algebra.

The ordinary product $\alpha \beta$ still means concatenation of angles, and we will keep this notation. We now recall the higher products $\mu^{\geq 3}$ of Gtl $\mathcal{A}$. They capture the topology of the arcs and angles. Roughly speaking, a higher product of a sequence of angles is nonzero if the sequence bounds a disk. Such a disk may either be a polygon, or a sequence of polygons stitched together, known as an immersed disk. Let us make this precise:

Definition 23.13. An immersed disk consists of an oriented immersion of a standard polygon $P_{k}$ into the surface, such that

- Every edge of $P_{k}$ is mapped to an arc.
- The immersion does not cover any punctures.

A sequence of angles $\alpha_{1}, \ldots, \alpha_{k}$ is a disk sequence if there exists an immersed disk such that $\alpha_{1}, \ldots, \alpha_{k}$ are the interior angles of the immersed disk, counting in clockwise order.

Remark 23.14. A disk sequence $\alpha_{1}, \ldots, \alpha_{k}$ always has length $k \geq 3$ because all polygons in the arc system $\mathcal{A}$ have at least three corners. Simply speaking, there are no digons.

We can now describe the higher products $\mu^{\geq 3}$ on $\operatorname{Gtl} \mathcal{A}$ as follows:
Definition 23.15. Let $\alpha_{1}, \ldots, \alpha_{k}$ be a disk sequence. Let $\beta$ be an angle composable with $\alpha_{1}$ in the sense that $\beta \alpha_{1} \neq 0$, and let $\gamma$ be an angle post-composable with $\alpha_{k}$ in the sense that $\alpha_{k} \gamma \neq 0$. Then we define higher products

$$
\mu^{k}\left(\beta \alpha_{k}, \ldots, \alpha_{1}\right):=\beta, \quad \mu^{k}\left(\alpha_{k}, \ldots, \alpha_{1} \gamma\right):=(-1)^{|\gamma|} \gamma
$$

The higher products vanish on all angle sequences other than these. If $\alpha_{1}, \ldots, \alpha_{k}$ is a disk sequence, we call the sequence $\alpha_{1}, \ldots, \beta \alpha_{k}$ final-out if $\beta \neq \mathrm{id}$ and the sequence $\alpha_{1} \gamma, \ldots, \alpha_{k}$ first-out if $\gamma \neq \mathrm{id}$. We call either of them all-in if $\beta$ and $\gamma$ are merely identities.
Remark 23.16. In Paper I we have imposed the additional condition [NL2] on arc systems. The condition entails that $\mathcal{A}$ contains no loops and no two arcs share more than one endpoint. In particular, the [NL2] condition requires that the number of punctures $|M|$ in the surface is at least two. We do not require the [NL2] condition in the present note.

## 24 Previous calculations

In this section, we summarize findings on Hochschild cohomology from Paper I We divide the section into three parts: In section 24.1, we recall the generation criterion from Paper I The criterion determines for a given collection of cocycles with certain prescribed shape in low adicity whether it concerns a basis of Hochschild cohomology or not. In section 24.2, we recall a certain collection of odd cocycles. This family satisfies the requirements of the generation criterion and therefore forms a basis for odd Hochschild cohomology. In section 24.3 , we recall a certain collection of even cocycles, the sporadic even cocycles.

### 24.1 The generation criterion

In this section, we recall the two generation criteria for odd and even Hochschild cohomology of Gtl $\mathcal{A}$ from Paper I. The two generation criteria hold for all full arc systems of punctured surfaces and are not restricted to the assumption [NL2]. In section 24.2, we use the generation criterion for odd Hochschild cohomology to explain the basis for odd Hochschild cohomology we obtained in Paper I. In section 25 we use the generation criterion for even Hochschild cohomology to construct and verify an explicit basis for even Hochschild cohomology.

We use the notation $\ell_{m}$ to denote a full turn around the puncture $m$. The meaning is depicted in Figure 24.1b

Definition 24.1. Whenever $m \in M$ is a puncture, the letter $\ell_{m}$ denotes the sum of full turns around the puncture $m$, starting from every incident arc. Every loop incident at $m$ gives rise to two contributions to $\ell_{m}$. The element $\ell_{m}$ is a formal sum of endomorphisms of the arcs incident at $m$. In other words, $\ell_{m}$ can be interpreted as a cochain $\ell_{m} \in \operatorname{HC}(\operatorname{Gtl} \mathcal{A})$ of arity 0 . When $r \geq 1$ is a natural number, the expression $\ell_{m}^{r}$ denotes the $r$-th power of $\ell_{m}$, equally consisting of endomorphisms of the arcs incident at $m$.

Proposition 24.2 Paper I]. Let $\mathcal{A}$ be a full arc system with [NMD]. Let $\nu_{\text {id }}$ and $\left\{\nu_{m, r}\right\}_{m \in M, r \geq 1}$ be odd Hochschild cocycles. Assume the following conditions:

- $\nu_{\mathrm{id}}^{0}=\sum_{a \in \mathcal{A}} \mathrm{id}_{a}$.
- $\nu_{m, r}^{0}=\ell_{m}^{r}$.

Then the collection of $\left\{\nu_{\mathrm{id}}\right\} \cup\left\{\nu_{m, r}\right\}_{m \in M, r \geq 1}$ is a basis for $\mathrm{HH}^{\text {odd }}(\mathrm{Gtl} \mathcal{A})$.
In Definition 24.3 we fix notation for a certain class $S$ of even Hochschild cocycles which merely "scale angles". The idea is that the quotient $\mathrm{S} /[\mathrm{id},-]$ is the 1-adic part of Hochschild cohomology which merely "scales angles". The precise definition is as follows:
Definition 24.3. Let $\mathcal{A}$ be a full arc system with [NMD]. Denote by $\mathrm{S} \subseteq \mathrm{HC}^{\text {even }}(\mathrm{Gtl} \mathcal{A})$ the space of all even cochains $\nu$ such that $\nu^{\neq 1}=0, d \nu=0$, and $\nu(\alpha)=\lambda_{\alpha} \alpha$ for some scalar $\lambda_{\alpha}$ for every indecomposable angle $\alpha$. Denote by [id, -$] \subseteq S$ the subspace spanned by all Gerstenhaber commutators $\left[\mathrm{id}_{a},-\right] \in \mathrm{S}$ ranging over $a \in \mathcal{A}$.
Proposition 24.4 Paper I]. Let $\mathcal{A}$ be a full arc system with [NMD]. Let $\left\{\nu_{P}\right\}_{P \in \mathbb{P}_{0}}$ be a collection of even Hochschild cocycles, indexed by some set $\mathbb{P}_{0}$. Assume the following conditions:

- $\nu_{P} \in \mathrm{~S}$.
- The collection $\left\{\nu_{P}\right\}_{P \in \mathbb{P}_{0}}$ is a basis for $S /[i d,-]$.

Let $\left\{\nu_{m, r}\right\}_{m \in M, r \geq 1}$ be another collection of even Hochschild cocycles. Assume the following conditions:

- $\nu_{m, r}^{0}=0$,
- $\nu_{m, r}^{1}(\alpha)=\ell_{m}^{r} \alpha$ for indecomposable angles $\alpha$ winding around $m$.
- $\nu_{m, r}^{1}(\alpha)=0$ for indecomposable angles $\alpha$ not winding around $m$.

Then the two collections $\left\{\nu_{P}\right\}_{P \in \mathbb{P}_{0}}$ and $\left\{\nu_{m, r}\right\}_{m \in M, r \geq 1}$ together form a basis for $\mathrm{HH}^{\text {even }}(\mathrm{Gtl} \mathcal{A})$.
The generation criteria build on the following technical lemma:
Lemma 24.5 Paper I]. Let $\mathcal{A}$ be a full arc system with [NMD]. Then:

- A cochain $\nu \in \operatorname{HC}^{\text {odd }}(\operatorname{Gtl} \mathcal{A})$ with $d \nu=0$ and $\nu^{0}=0$ satisfies $\nu \in \operatorname{Im}(d)$.
- A cochain $\nu \in \operatorname{HC}^{\text {even }}(\operatorname{Gtl} \mathcal{A})$ with $d \nu=0$ and $\nu^{1}=0$ satisfies $\nu \in \operatorname{Im}(d)$.


### 24.2 Odd Hochschild cocycles

In this section, we recall odd Hochschild cohomology of gentle algebras from Paper I The idea to find Hochschild cocycles of $\operatorname{Gtl} \mathcal{A}$ is to trace Seidel's explanation 63 on deformations of Fukaya categories. Seidel's proposal is to define the higher products relative to a divisor. In Paper I we translated this idea to gentle algebras of punctured surfaces. In particular, we describe in Paper I a basis of the odd Hochschild cohomology. In the present section, we recall this basis.

Example 24.6. We will define the odd Hochschild cocycles $\nu_{m, r}$ by their behavior on orbigons of type $(m, r)$. If $r=1$, then orbigons of type $(m, r)$ can be interpreted in a more standard way. In fact, they are the same as immersed disks covering the puncture $m$ precisely once, and no other punctures apart from $m$. More precisely, we may say the sequence $\alpha_{1}, \ldots, \alpha_{k}$ of angles is an immersed disk covering the puncture $m \in M$ if there is an immersion of the standard polygon $P_{k}$ into $S$ such that every edge is mapped to an arc and the immersion covers a single puncture, and only once, namely $m$. An example of an immersed disk $\alpha_{1}, \ldots, \alpha_{k}$ covering a puncture is depicted in Figure 24.1a

This basis is best recalled as follows:
Definition 24.7. Let $m \in M$ be a puncture and $r \geq 1$ a natural number. Then the Hochschild cocycle $\nu_{m, r}$ is defined by

- The 0 -adic component $\nu^{0}$ is $\ell_{m}^{r}$.
- The 1-adic component $\nu^{1}$ vanishes.
- The 2 -adic component $\nu^{2}$ vanishes.
- Assume that $\alpha_{1}, \ldots, \alpha_{i}^{(1)}, \ell_{m}^{r}, \alpha_{i}^{(2)}, \ldots, \alpha_{k}$ is a disk sequence. Put $\alpha_{i}=\alpha^{(2)} \alpha^{(1)}$. Let $\beta$ be an angle composable with $\alpha_{1}$ in the sense that $\beta \alpha_{1} \neq 0$, and let $\gamma$ be an angle post-composable with $\alpha_{k}$ in the sense that $\alpha_{k} \gamma \neq 0$. Then put

$$
\nu^{k}\left(\beta \alpha_{k}, \ldots, \alpha_{1}\right)=\beta, \quad \nu^{k}\left(\alpha_{k}, \ldots, \alpha_{1} \gamma\right)=(-1)^{|\gamma|} \gamma
$$

The higher products $\nu^{\geq 3}$ vanish on all angle sequences other than these.


Figure 24.1

The single Hochschild cocycle $\nu_{\mathrm{id}}$ is given by $\nu_{\mathrm{id}}^{0}=\sum_{a \in \mathcal{A}} \mathrm{id}_{a}$ and $\nu_{\mathrm{id}}^{>0}=0$.
In terms of orbigons, the assumption in Definition 24.7 reads that $\alpha_{1}, \ldots, \alpha_{k}$ is the reduced sequence of an orbigon of type $(m, r)$.

Lemma 24.8 Paper I). The cochains $\nu_{\mathrm{id}}$ and $\nu_{m, r}$ are Hochschild cocycles.
Theorem 24.9 Paper I. The collection of $\nu_{\mathrm{id}}$ and $\left(\nu_{m, r}\right)_{m \in M, r \geq 1}$ provides a basis for $H^{\text {odd }}(\mathrm{Gtl} \mathcal{A})$.
Proof. According to the generation criterion Proposition 24.2, a collection of cocycles is a basis if it has the right 0 -ary components. This is clearly the case.

### 24.3 Sporadic even cocycles

In this section, we recall a first type of even Hochschild cocycles, the sporadic classes. The idea is to select just 1 -adic cochains $\nu=\nu^{1}$ with $\nu^{1}(\alpha)$ being a multiple of $\alpha$ whenever $\alpha$ is an indecomposable angle. More precisely, we pick a collection $\left(\nu_{P}\right)_{P \in \mathbb{P}_{0}} \subseteq \mathrm{~S}$ in such a way that the requirements of Proposition 24.4 are satisfied.

Our starting point is the set S. Recall from section 24.1 that this set consists of all 1-adic cocycles which are of the form $\nu^{1}(\alpha)=\lambda_{\alpha} \alpha$ for every angle $\alpha$. Simply speaking, S is the set of cocycles among the 1 -adic cochains which merely scale angles. Our first step in this section is to make the cocycle condition explicit:

Lemma 24.10 Paper I). Let $\mathcal{A}$ be a full arc system with [NMD]. Let $\nu \in \mathrm{HC}^{\text {even }}(\mathrm{Gtl} \mathcal{A})$ be an even cochain such that $\nu^{\neq 1}=0$ and $\nu(\alpha)=\lambda_{\alpha} \alpha$ for some scalar $\lambda_{\alpha}$ for every indecomposable angle $\alpha$. Then

$$
d \nu=0 \quad \Longleftrightarrow \quad \text { for every polygon } \alpha_{1}, \ldots, \alpha_{k}: \sum_{i=1}^{k} \lambda_{\alpha_{i}}=0
$$

Whenever $\nu \in \mathrm{S}$, we also write $\# \nu(\alpha)=\lambda_{\alpha}$ for the scalar coefficient of $\nu(\alpha)$ whenever $\alpha$ is an angle in $\mathcal{A}$. For instance, we have $\# \nu\left(\alpha_{k} \ldots \alpha_{1}\right)=\# \nu\left(\alpha_{k}\right)+\ldots+\# \nu\left(\alpha_{1}\right)$.

Lemma 24.11. The quotient $\mathrm{S} /[\mathrm{id},-]$ of sporadic cocycles by sporadic inner derivations is isomorphic to $H_{1}(S, M ; \mathbb{C})$. This space has dimension $2 g-1+|M|$.

Proof. The proof consists of two steps. To compare $\mathrm{S} /[\mathrm{id},-]$ and $H_{1}(S, M ; \mathbb{C})$, we will pick a cell decomposition of the surface $S$ and show that its degree one cocycles are S , while its degree one coboundaries are [id, -]. In the second step, we read off the dimension of this relative homology space by an alternative cell decomposition.

For the first step, let us describe the cell decomposition we put on $S$. It is a dual decomposition to the punctures, arcs and polygons of $\mathcal{A}$. The zero-dimensional cells are the midpoints of the polygons plus the punctures $M$. The one-dimensional cells are arrows from the midpoints of the polygons to all corners around the polygon. The surface is split into topological disks by the one-dimensional cells, one for each $\operatorname{arc}$ of $\mathcal{A}$. The two-dimensional cells are defined to be those disks. This cell complex is depicted in Figure 24.2a

Regard the relative cellular chain complex formed by this cell decomposition, relative to the zero-cells M. Our aim is to identify its degree-one cocycles with S and its degree-one coboundaries with [id, -].

Let us regard a degree-one cocycle $\eta$. Such a cocycle can be written as a linear combination of arrows from the centers of the polygons to the corners. Note that the arrows are precisely in correspondence with the indecomposable angles of $\mathcal{A}$. Therefore let us write $\eta=\sum_{\alpha} \lambda_{\alpha} \alpha$ with coefficients $\lambda_{\alpha} \in \mathbb{C}$. The cocycle condition, relative to $M$, is equivalent to requiring that the sum of the coefficients $\lambda_{\alpha}$ vanishes along each polygon.

(a) Cell decomposition identifying $S / I$ with relative homology. Arcs of $\mathcal{A}$ are drawn dashed.

(b) Surrounding angles

(c) Easy cell decomposition

What are the coboundaries? They are spanned by the coboundaries of all two-dimensional cells. Regard one two-dimensional cell given by an $\operatorname{arc} a \in \mathcal{A}$. Its boundary consists of the signed sum of the four one-cells bounding it. In terms of the angle interpretation of the one-cells as angles of $\mathcal{A}$, this signed sum is precisely $\alpha_{1}-\alpha_{2}+\alpha_{4}-\alpha_{3}$, where the angles are numbered as in Figure 24.2b This coboundary corresponds exactly to Hochschild coboundary $d\left(\mathrm{id}_{a}\right) \in[\mathrm{id},-]$. In other words, the quotient of degree-one cocycles by coboundaries precisely computes $\mathrm{S} /[\mathrm{id},-]$.

For the second step, we are supposed to compute the dimension of $H_{1}(S, M ; \mathbb{C})$ by choosing an easier cell decomposition. Note that the relative homology does not depend on the location of the points $M$, as long as they are distinct. Next, recall that every closed surface of genus $g$ can be split into a single disk by $2 g$ non-crossing loops $a_{1}, \ldots, a_{g}$ and $b_{1}, \ldots, b_{g}$, all starting and ending at a single point $p_{1}$. The boundary of the disk is given by the sequence $a_{1}, b_{1}, a_{1}^{-1}, b_{1}^{-1}, \ldots$.

Now form the desired cell decomposition as follows. The zero-cells are $p_{1}$, plus $|M|-1$ additional points $p_{2}, \ldots, p_{|M|}$ lying on $a_{1}$. The one-cells are the $2 g-1$ arcs plus the intervals between the points on $a_{1}$. Their complement in $S$ is a single disk. Use this disk as the single two-cell. This cell decomposition is depicted in Figure 24.2C

We are now ready to compute the degree-one homology of the cell complex of this cell decomposition, relative to $p$ and the $|M|-1$ many points lying on $a_{1}$. In fact, all $2 g-1+|M|$ arcs of the cell decomposition are cocycles, since all endpoints were chosen relative. The space of coboundaries is spanned by the boundary of the single disk. Since all arcs appear precisely twice around this disk with opposite orientation, the space of degree-one coboundaries vanishes. We conclude the relative homology $H_{1}(S, M ; \mathbb{C})$ is of dimension $2 g-1+|M|$.

Definition 24.12. The sporadic classes are any choice of basis representatives $\left(\nu_{P}\right)_{P \in \mathbb{P}_{0}} \subseteq \mathrm{~S}$ for $\mathrm{S} /[\mathrm{id},-]$. The index set $\mathbb{P}_{0}$ has cardinality $2 g-1+|M|$.

## 25 Even Hochschild cocycles

In this section, we construct explicit even Hochschild cocycles. Recall that we have already constructed sporadic even Hochschild cocycles $\left(\nu_{P}\right)_{P \in \mathbb{P}_{0}} \subseteq \mathrm{~S}$ in section 24.3 In the present section, we define a second class of Hochschild cocycles which we call the ordinary even Hochschild cocycles. The sporadic and ordinary even Hochschild cocycles together will form a basis for the even Hochschild cohomology.

We proceed as follows: In section 25.1 we given an explicit description of Hochschild cocycles $\nu$. In section 25.2 , we check that the Hochschild cochain $d \nu$ vanishes on a certain type of sequences which we call parking garage sequences. In section 25.3 , we check that $d \nu$ also vanishes on all other types of sequences. In total, we obtain that $d \nu=0$. In section 25.4 we construct the Hochschild cocycles $\nu_{m, r}$. We show that together with the sporadic cocycles they provide a basis for $\mathrm{HH}^{\text {even }}(\mathrm{Gtl} \mathcal{A})$. Finally, we comment on the Gerstenhaber bracket and cup product on Hochschild cohomology.

### 25.1 Construction

In this section we construct even Hochschild cocycles explicitly from certain input data. The input data consists of a choice of puncture $m \in M$, a natural number $r \geq 1$ and a choice of "weight" for every indecomposable angle around $m$. The construction of the Hochschild cocycle associated with this data is similar to the odd case, albeit more tricky.

$$
\begin{aligned}
& \text { Input scalars } \# \nu^{1}(\alpha) \\
& \text { Splitting angles } \Longrightarrow \text { Magic angles } \Longrightarrow \\
& \text { Turning angles }
\end{aligned} \xlongequal{\Longrightarrow} \text { Construction of } \nu
$$

Figure 25.1: Logical structure of notions

The basic idea of the construction is as follows: Let $\alpha_{1}, \ldots, \alpha_{k}, \ell_{m}^{r}$ be a disk sequence. Then we define $\nu\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ as the identity on the first, equivalently last arc of the sequence. This idea is depicted in Figure 25.2a This does not suffice however to make $\nu$ a cocycle. Instead, we need to give $\nu$ nonzero values on certain other special sequences and choose the scalars of these values in a clever way. It turns out there is no canonical choice for the scalars. We therefore start from the datum of a scalar value $\# \nu(\alpha)$ for every indecomposable angle $\alpha$ winding around $m$. Defining the special sequences is rather intricate and makes use of what we call turning angles and magic angles. To define magic angles, we have to define yet another auxiliary notion, the splitting angles. The structure of the section is summarized in the logical diagram Figure 25.1

Arc system We fix a full arc system $\mathcal{A}$ which satisfies the [NMD] condition.

Input scalars We assume the choice of a puncture $m \in M$, a natural number $r \geq 1$ and the choice of a scalar $\# \nu^{1}(\alpha)$ for every indecomposable angle $\alpha$ winding around the puncture $m$. An example of input scalars is depicted in Figure 25.2b

Splitting angles We introduce here precise measurement for certain angles. In terms of orbigons, it concerns angles between different ways of viewing an orbigon as a fold. We try to break down the terminology as far as possible to the more elementary notion of disk sequences.

Let $s=\alpha_{1}, \ldots, \alpha_{k}$ be an angle sequence. We regard indices $1 \leq i \leq k$ such that $\alpha_{i}$ has a decomposition $\alpha_{i}=\alpha_{i}^{(2)} \alpha_{i}^{(1)}$ such that $\alpha_{1}, \ldots, \alpha_{i}^{(1)}, \ell_{m}^{r}, \alpha_{i+1}^{(2)}, \ldots, \alpha_{k}$ is a disk sequence. We define the splitting set $I_{s}$ of $s=\alpha_{1}, \ldots, \alpha_{k}$ to be the set of such indices and decompositions:

$$
\begin{aligned}
I_{s}=\left\{\left(i, \alpha_{i}^{(1)}, \alpha_{i}^{(2)}\right) \mid\right. & 1 \leq i \leq k, \quad \alpha_{i}=\alpha_{i}^{(2)} \alpha_{i}^{(1)} \\
& \left.\alpha_{1}, \ldots, \alpha_{i}^{(1)}, \ell_{m}^{r}, \alpha_{i+1}^{(2)}, \ldots, \alpha_{k} \text { is a disk sequence }\right\}
\end{aligned}
$$

The set $I_{s}$ may be empty. The more often the sequence $\alpha_{1}, \ldots, \alpha_{k}$ winds around $m$, the larger the set $I_{s}$. An example sequence $s=\alpha_{1}, \ldots, \alpha_{6}$ in case $r=1$ together with its splitting set $I_{s}$ is depicted in Figure 25.2 c . The elements of $I_{s}$ are totally ordered by the number $i$, or the length of $\alpha_{i}^{(1)}$ among equal indices. We capture the angle between two elements of $I_{s}$ in the following terminology:

Definition 25.1. Let $\left(i, \alpha_{i}^{(1)}, \alpha_{i}^{(2)}\right) \leq\left(j, \alpha_{j}^{(1)}, \alpha_{j}^{(2)}\right)$ be two elements of $I_{s}$. Then the splitting angle between $\left(i, \alpha_{i}^{(1)}, \alpha_{i}^{(2)}\right)$ and $\left(j, \alpha_{j}^{(1)}, \alpha_{j}^{(2)}\right)$ is

- if $i<j$, then $\alpha$ is the angle such that $\alpha_{i}^{(2)}, \alpha_{i+1}, \ldots, \alpha_{j-1}, \alpha_{j}^{(1)}, \alpha$ is a disk sequence.
- if $i=j$, then we set $\alpha=\left(\alpha_{i}^{(1)}\right)^{-1} \alpha_{j}^{(2)}$

In case $i=j$, the splitting angle is simply speaking the difference between $\alpha_{i}^{(1)}$ and $\alpha_{j}^{(1)}$. In Figure 25.2 c , we have illustrated the splitting angle in case $i<j$. In the figure, the splitting angle is drawn dashed.

Definition 25.2. If $I_{s}$ is nonempty, the splitting angle of an element $\left(i, \alpha_{i}^{(1)}, \alpha_{i}^{(2)}\right) \in I_{s}$ is the splitting angle between $\min I_{s}$ and $\left(i, \alpha_{i}^{(1)}, \alpha_{i}^{(2)}\right)$.

In terms of orbigons, all terminology is easily described as follows: The set $I_{s}$ is nonempty if $\alpha_{1}, \ldots, \alpha_{k}$ is the reduced sequence of an orbigon $X$ of type $(m, r)$. The set $I_{s}$ is then simply the set of all possible ways the orbigon $X$ can be obtained via folding. The minimum $\min I_{s}$ is the earliest possible way to obtain $X$ via folding.


Figure 25.2: Illustration of ideas and auxiliary notions

Construction of the cochain We are now ready to construct a cochain $\nu$ from given collection of input scalars $\# \nu^{1}$. The idea is to define $\nu^{1}$ as the derivation which sends an indecomposable angles $\alpha$ to $\# \nu^{1}(\alpha) \alpha \ell_{m}^{r}$. Whenever $\alpha_{1}, \ldots, \alpha_{k}$ are indecomposable angles around $m$ such that $\alpha_{k} \ldots \alpha_{1} \neq 0$, let us already now write

$$
\# \nu^{1}\left(\alpha_{k} \ldots \alpha_{1}\right)=\# \nu^{1}\left(\alpha_{k}\right)+\ldots+\# \nu^{1}\left(\alpha_{1}\right)
$$

The higher component $\nu^{\geq 2}$ will be defined on four types of distinguished sequences. To every such distinguished sequence, we define the associated turning angle and the associated magic angle. The contribution of the sequence to $\nu^{\geq 2}$ is defined in terms of these two angles. The full definition reads as follows:
Definition 25.3. Let $\mathcal{A}$ be a full arc system with [NMD]. Let $m \in M, r \geq 1$ and let $\# \nu$ be a collection of input scalars around $m$. Then the associated even Hochschild cochain $\nu$ is defined by the following rules. For every rule, we define indicate its turning angle and its magic angle.

- The 0-adic component $\nu^{0}$ vanishes.
- The 1 -adic component $\nu^{1}$ is defined by $\nu^{1}(\alpha)=\# \nu^{1}(\alpha) \alpha \ell_{m}^{r}$ for $\alpha$ winding around $m$.
- An angle sequence $\alpha_{1}, \ldots, \alpha_{k}$ is end-split with turning angle $<\ell^{r}$ if there exists an angle $\alpha$ winding around $m$ with $\alpha<\ell^{r}$ such that $\alpha_{1}, \ldots, \alpha_{k}, \alpha$ is a disk sequence. The turning angle of the sequence is the angle $\alpha$. The magic angle of the sequence is $\ell_{m}^{r}$. The contribution to $\nu$ is

$$
\nu^{k}\left(\alpha_{k}, \ldots, \alpha_{1}\right)=(-1)^{\|\alpha\|} \# \nu^{1}(\alpha) \alpha^{-1} \ell_{m}^{r}
$$

- An angle sequence $\alpha_{1} \beta, \ldots, \alpha_{k}$ with $\alpha_{1} \beta \neq 0$ is old-era end-split with turning angle $\ell^{r}$ if $\alpha_{1}, \ldots, \alpha_{k}, \ell^{r}$ is a disk sequence. The turning angle of the sequence is $\ell^{r}$. The magic angle of the sequence is $\ell_{m}^{r}$. The contribution to $\nu$ is

$$
\nu^{k}\left(\alpha_{k}, \ldots, \alpha_{1} \beta\right)=-\# \nu^{1}\left(\ell_{m}^{r}\right) \beta
$$

- An angle sequence $\alpha_{1} \beta, \ldots, \gamma \alpha_{k}$ with $\gamma \neq \mathrm{id}$ is new-era end-split with turning angle $\ell^{r}$ if $\alpha_{1}, \ldots, \alpha_{k}, \ell^{r}$ is a disk sequence. The turning angle of the sequence is $\ell^{r}$. Regard the arc (more precisely, arc incidence) which is the head of $\alpha_{k}$, equivalently the tail of $\alpha_{1}$. The magic angle of the sequence is the indecomposable angle $n$ around $m$ which follows this arc clockwise around $m$. The contribution to $\nu$ is

$$
\nu^{k}\left(\gamma \alpha_{k}, \ldots, \alpha_{1} \beta\right)=-\# \nu^{1}(n) \gamma \beta
$$

- An angle sequence $\alpha_{1}, \ldots, \gamma \alpha_{k}$ or $\alpha_{1} \beta, \ldots, \alpha_{k}$ is middle-split if there is a $0<i<k$ such that $\alpha_{1}, \ldots, \alpha_{i}, \ell^{r}, \alpha_{i+1}, \ldots, \alpha_{k}$ is a disk sequence. The turning angle of the sequence is $\ell^{r}$. The triple $\left(i, \alpha_{i}, \alpha_{i+1}\right)$ defines an element of the splitting set $I_{s}$ of the sequence $s=\alpha_{1}, \ldots, \alpha_{i+1} \alpha_{i}, \ldots, \alpha_{k}$. The magic angle of the sequence is the splitting angle $\alpha$ of $\left(i, \alpha_{i}, \alpha_{i+1}\right)$ with respect to the angle sequence $s$. The contribution to $\nu$ is

$$
\begin{aligned}
& \nu^{k}\left(\gamma \alpha_{k}, \ldots, \alpha_{1}\right)=(-1)^{\left\|\alpha_{1}\right\|+\ldots+\left\|\alpha_{i}\right\|} \# \nu^{1}(\alpha) \gamma, \\
& \nu^{k}\left(\alpha_{k}, \ldots, \alpha_{1} \beta\right)=(-1)^{\left\|\alpha_{1}\right\|+\ldots+\left\|\alpha_{i}\right\|} \# \nu^{1}(\alpha) \beta
\end{aligned}
$$

Remark 25.4. The higher components $\nu^{\geq 2}$ are well-defined: Any angle sequence falls within at most one of the four types presented in Definition 25.3 Whenever it falls within one of the types, its presentation
in terms of $\alpha_{i}, \beta$ or $\gamma$ is unique. In case the angle sequence is middle-split, the index $i$ is unique. Alternatively, it is possible to circumvent this uniqueness statement. Indeed, add up contributions to $\nu$ instead, as in the odd case of Paper I
Remark 25.5. The sign rules follow a united pattern: If $\alpha_{1}, \ldots, \alpha_{k}$ is end-split with turning angle $\alpha<\ell^{r}$, the $\operatorname{sign}(-1)^{\|\alpha\|}$ is equal to $(-1)^{\left\|\alpha_{1}\right\|+\ldots+\left\|\alpha_{k}\right\|}$ since $\alpha_{1}, \ldots, \alpha_{k}, \alpha$ is a disk sequence and reduced degrees in a disk sequence add up to even parity. If $\alpha_{1}, \ldots, \alpha_{k}$ is (old-era or new-era) end-split with turning angle $\ell^{r}$, the sign -1 is equal to $(-1)^{\left\|\alpha_{1}\right\|+\ldots+\left\|\alpha_{k}\right\|}$. More generally, the sign consumes precisely the angles between the minimum element of $I_{s}$ and the split actually taken by the sequence.
Remark 25.6. The rules for middle-split and end-split $\nu$ are analogous: They yield exactly the same result, except that the end-split rule for magic angle $<\ell^{r}$ does not allow for additional $\beta$ and $\gamma$ at the front and at the back. For example, even the signs agree, since $\left\|\ell_{m}^{r}\right\|$ is odd. Let us explain why we distinguish the two rules. The first rule yields $\alpha_{k}^{-1} \nu^{1}\left(\alpha_{k}\right) \alpha_{k}^{-1}$. This angle always winds around $m$. Evaluate

$$
0=(d \nu)\left(\gamma, \alpha_{k}, \ldots, \alpha_{1}\right)=\mu^{2}\left(\gamma, \nu\left(\alpha_{k}, \ldots, \alpha_{1}\right)\right)-\nu\left(\mu^{2}\left(\gamma, \alpha_{k}\right), \ldots, \alpha_{1}\right)
$$

This yields $\nu\left(\gamma \alpha_{k}, \ldots, \alpha_{1}\right)=\gamma \nu\left(\alpha_{k}, \ldots, \alpha_{1}\right)$. If $\alpha_{k}$ is less than $\ell_{m}^{r}$, then $\nu\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ is non-empty and winds around $m$, while $\gamma$ leaves the arc at the opposite side. Hence the product vanishes, except if $\nu\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ is the identity. This happens precisely in the borderline case that $\alpha_{k}$ consists of $r$ full turns: $\alpha_{k}=\ell_{m}^{r}$. Only in this case additional $\beta$ and $\gamma$ on the left and right make sense. This explains the distinction between the first and second rule.

The rule for middle-split sequences has appearance similar to the odd case. One may in principle add $\gamma$ and $\beta$ simultaneously on both sides. This addition is however vacuous: The angles $\gamma$ and $\beta$ are not composable and $\gamma \beta=0$.

### 25.2 Cancellation on parking garage sequences

In this section, we perform first checks for the cocycle condition. Our starting point is a cochain $\nu$ defined in Definition 25.3 from input scalars $\# \nu^{1}$. In the present section, we check that $d \nu=[\mu, \nu]$ vanishes on certain sequences, which we call parking garage sequences.

Many of the terms in $[\mu, \nu]$ are easy to cancel away in pairs. Some are harder and cancel away only as a whole. All sequences $\alpha_{1}, \ldots, \alpha_{k}$ producing hard terms in $[\mu, \nu]$ are of the same type: they have an angle $\alpha_{i}$ around $m$ such that $\mu\left(\ldots, \nu^{1}\left(\alpha_{i}\right), \ldots\right)$ is a nonzero contribution. That is, once we prolong $\alpha_{i}$ by $r$ turns around $m$, the sequence $\alpha_{1}, \ldots, \ell^{r} \alpha_{i}, \ldots, \alpha_{k}$ becomes a disk sequence. After $\alpha_{k}$ plus $\ell^{r}$ turns around $m$, the prolonged sequence compensates its turns by winding back around $m$ in clockwise direction. Let us regard the path traced by the sequence as it winds back. Regard the polygons lying around $m$ in clockwise order. Since $\alpha_{1}, \ldots, \ell^{r} \alpha_{i}, \ldots, \alpha_{k}$ is supposed to be a disk sequence, the winding back path runs around all these polygons in clockwise order. It may have additional disks stitched to it at the polygons' outside, but the basic structure is a helix consisting of the polygons around $m$. Such a sequence resembles a parking garage spiral, with optional parking space attached on the exterior of each polygon. The schematic is depicted in Figure 25.3

Let us describe in formulas how the helix is formed. Denote for a moment by $P_{1}, \ldots, P_{l}$ the polygons traced by the sequence. Let $\alpha_{1}^{(i)}, \ldots, \alpha_{s_{i}}^{(i)}$ be their internal angles, with $\alpha_{1}^{(i)}$ being the angle at $m$. The parking garage sequence then consists of the angles

$$
\alpha_{2}^{(1)}, \ldots, \alpha_{s_{1}-1}^{(1)}, \alpha_{2}^{(2)} \alpha_{s_{1}}^{(1)}, \ldots, \alpha_{s_{2}-1}^{(2)}, \alpha_{2}^{(3)} \alpha_{s_{2}}^{(2)}, \ldots, \alpha_{s_{l}-1}^{(l)}
$$

plus the long turn angle around $m$, consisting of all the polygon angles at $m$ minus $r$ full turns:

$$
\alpha_{1}^{(1)} \ldots \alpha_{l}^{(l)} \ell^{-r}
$$

Let us explain how the additional parking space is attached. Regard a polygon $\alpha_{1}^{(i)}, \ldots, \alpha_{s_{i}}^{(i)}$ in the spiral. The angle $\alpha_{1}^{(i)}$ is the angle at $q$. The angles $\alpha_{2}^{(i)}$ and $\alpha_{s_{i}}^{(i)}$ are the angles next to $q$ and are used to attach the polygons to each other, forming the parking spiral. The parking spiral has $l$ polygons and $l-1$ interior arcs. Its outer boundary consists of $\sum\left(s_{i}-1\right)-1$ many exterior arcs, a start arc and an end arc. The additional parking space in the form of disk sequences $\beta_{1}, \ldots, \beta_{m}$ may now be attached to the exterior arcs. Wherever we add parking space around the spiral, we augment the garage sequence by that additional disk sequence. When attaching to arcs not involved in the spiral gluing, the augmentation looks like

$$
\ldots, \alpha_{j}^{(i)}, \alpha_{j+1}^{(i)}, \ldots \rightsquigarrow \ldots, \beta_{1} \alpha_{j}^{(i)}, \beta_{2}, \ldots, \beta_{m-1}, \alpha_{j+1}^{(i)} \beta_{m}, \ldots
$$



Figure 25.3: Illustration of parking garage sequences

When attaching to one of the two exterior arcs of the polygon next to the gluing, the augmentation rather looks like

$$
\ldots, \alpha_{2}^{(j+1)} \alpha_{s_{j}}^{(j)}, \alpha_{3}^{(j+1)}, \ldots \rightsquigarrow \ldots, \beta_{1} \alpha_{2}^{(j+1)} \alpha_{s_{j}}^{(j)}, \beta_{2}, \ldots, \beta_{m-1}, \alpha_{3}^{(j+1)} \beta_{m}, \ldots
$$

in case of the first exterior arc of a polygon, and looks like

$$
\ldots, \alpha_{s_{j}-1}^{(j)}, \alpha_{2}^{(j+1)} \alpha_{s_{j}}^{(j)}, \ldots \rightsquigarrow \ldots, \beta_{1} \alpha_{s_{j}-1}^{(j)}, \beta_{2}, \ldots, \beta_{m-1}, \alpha_{2}^{(j+1)} \alpha_{s_{j}}^{(j)} \beta_{m}, \ldots
$$

in case of the last exterior arc of a polygon. Let us put this definition on paper.
Definition 25.7. A parking garage sequence consists of tracing consecutive polygons around $m$ in clockwise order, and compensating all these turns minus $\ell^{r}$ by a single angle around $m$. Additional disk sequences may be stitched to the exterior arcs of the sequence.

We aim at showing that $d \nu_{P}$ vanishes on its parking garage sequences. Regard such a garage sequence $\alpha_{s}, \ldots, \alpha_{1}, \alpha, \alpha_{k}, \ldots, \alpha_{s+1}$. What terms appear in $d \nu_{P}$, applied to this sequence? First, it is possible to apply $\nu^{1}$ to $\alpha$. Indeed, the garage sequence becomes a disk sequence once we prolong $\alpha$ by $r$ turns and hence $\mu\left(\ldots, \nu^{1}(\alpha), \ldots\right)$ is one of the terms in $d \nu_{P}$. Second, it is possible to apply an inner end-split $\nu$ to any part of the outer sequence that is precisely $r$ turns long. Such terms give roughly as many contributions as there are polygon sectors around the spiral, and they add up nicely. Finally, it is also possible to apply an inner $\mu$ to the top-most part of the garage or the bottom-most part of the garage.

Let us explain why no other terms appear.
To ease the calculation, we reduce a given garage sequence to its minimal version that has all additional parking space removed. This minimal version has the same individual terms in $d \nu_{P}$ as the original garage sequence: Additional parking space merely consists of (incomplete) disk sequences and creates no further options to evaluate $\mu$ or $\nu$. Moreover, the value of the individual terms stays exactly the same: For example, the signs for inner end-split $\nu$ evaluations are independent of the length of the $\beta$ and $\gamma$ angles entering and leaving the additional parking space. While the signs for the top-most inner $\mu$ do depend on the degree of the angle leaving the parking space at angle $\alpha$ below the top, this is compensated again by the sign $(-1)^{\|\mu\| \cdot \ldots}$ associated with this term $\nu\left(\ldots, \mu\left(\alpha, \alpha_{k}, \ldots\right), \ldots\right)$ in the Hochschild differential, since $\mu$ is odd.

Let us now make this rigorous, check the signs and add up all terms.
Lemma 25.8. We have $d \nu_{P}=0$ on parking garage sequences.
Proof. Let us start by listing up all terms with signs and result. Since we choose a parking garage sequence without extra angles $\beta$ or $\gamma$ at start or end, all results of contributions $\mu(\nu)$ or $\nu(\mu)$ in $d \nu_{P}$ are
scalar multiples of the identity of the first/last $\operatorname{arc} h\left(\alpha_{s}\right)=t\left(\alpha_{s+1}\right)$ of the garage sequence. In the list below, we indicate this scalar for all terms, as well as the sign due to the Hochschild differential.

We start with the generic case, where the beginning and end of the garage sequence lie somewhere one the outer spiral. That is, the final arc is not $\alpha$ or $\alpha_{k}$. In other words, the first arc is not $\alpha$ or $\alpha_{1}$.

To ease the calculation, let us define three angles top, mid and bot, all winding around $m$. These angles can best be read off from Figure 25.3a First of all, mid is the sector around $m$ that the start/end of the sequence lies in. For example, the arc $h\left(\alpha_{s}\right)=t\left(\alpha_{s+1}\right)$ lies in this sector. The angle mid now splits the spiral angle $\alpha \ell^{r}$ into two more parts: top lying above mid, and bot lying below mid.

In other words, top forms a disk sequence together with the angles $\alpha_{s+1}, \ldots, \alpha_{k}$ (minus the part of $\alpha_{s+1}$ and those successor angles that reach into the special sector mid). Similarly, bot forms a disk sequence with $\alpha_{1}, \ldots, \alpha_{s}$ (minus the part lying in mid).

With this notation we can write bot $\cdot \mathrm{mid} \cdot \mathrm{top}=\alpha \ell^{r}$. Recall that $\# \nu^{1}(\beta)$ is the scalar coefficient of $\nu^{1}(\beta)$, for any angle $\beta$. With this in mind, we have

$$
\# \nu^{1}(\text { bot })+\# \nu^{1}(\text { mid })+\# \nu^{1}(\text { top })=\# \nu^{1}(\alpha)+\# \nu^{1}\left(\ell^{r}\right)
$$

For convenience, denote by first that sector around $m$ that is bottom-most in the garage sequence. In other words, $\alpha_{1}$ lies in this sector.

1. $\mu\left(\ldots, \nu^{1}(\alpha), \ldots\right)$

This term is characteristic for the garage sequence and appears always. Its result has scalar coefficient $\# \nu^{1}(\alpha)$. The Hochschild sign is +1 .
2. $\mu\left(\ldots, \alpha, \nu\left(\alpha_{k}, \ldots\right), \ldots\right)$

This top-most inner end-split $\nu$ appears if the outer sequence from start to top is at least $r$ full turns long. Its result has scalar coefficient $-\# \nu^{1}\left(\ell^{r}\right)$. The Hochschild sign is +1 .
3. $\mu\left(\ldots, \nu\left(\ldots, \alpha_{1}\right), \alpha, \ldots\right)$

This bottom-most inner end-split $\nu$ appears if the outer sequence from bottom to stop is at least $r$ full turns long. Its result has scalar coefficient $-\# \nu^{1}$ (first), where first is the first sector around $m$ at the bottom of the sequence. The Hochschild sign is +1 .
4. $\mu(\ldots, \alpha, \underbrace{\ldots}_{\geq 1}, \nu(\ldots), \ldots)$

Such top-part inner end-split $\nu$ terms appear if the outer sequence from start to top is more than $r$ full turns long. Their individual result coefficients are $-\# \nu^{1}$ of the next sector after their end. In total, all these terms add up to $-\# \nu^{1}\left(\right.$ top $\left.\cdot \ell^{-r}\right)=-\# \nu^{1}(\mathrm{top})+\# \nu^{1}\left(\ell^{r}\right)$. The Hochschild sign is +1 .
5. $\mu(\ldots, \nu(\ldots), \underbrace{\ldots}_{\geq 1}, \alpha, \ldots)$

Such bottom-part inner end-split $\nu$ terms appear if the outer sequence from bottom to stop is more than $r$ full turns long. Their individual result coefficients are $-\# \nu^{1}$ of the next sector after their end. In total, all these terms add up to $-\# \nu^{1}(\mathrm{mid})-\# \nu^{1}\left(\right.$ bot $\left.\cdot \ell^{-r}\right)+\# \nu^{1}$ (first). The Hochschild sign is +1 .
6. $\nu\left(\ldots, \mu\left(\alpha, \alpha_{k}, \ldots, \alpha_{t}\right), \ldots\right)$

This top-most inner $\mu$ appears if the outer sequence from start to top includes $\alpha$. The first angle of the inner $\mu$ is a certain $\alpha_{t}$, and a part $\alpha_{t}^{(1)}$ of it reaches outside the disk, so write $\alpha_{t}=\alpha_{t}^{(2)} \alpha_{t}^{(1)}$. The outer $\nu$ application is middle-split and gives an extra sign, so that the result coefficient is

$$
(-1)^{\mid \alpha_{t}^{(1)}} \# \nu^{1}\left(\operatorname{top} \cdot \alpha^{-1}\right) \cdot(-1)^{\left\|\alpha_{s+1}\right\|+\ldots+\left\|\alpha_{t}^{(1)}\right\|}
$$

The Hochschild sign is $(-1)^{1+\left\|\alpha_{s+1}\right\|+\ldots+\left\|\alpha_{t-1}\right\|}$, rendering a total contribution to $d \nu_{P}$ of merely $\# \nu^{1}$ (top $\cdot \alpha^{-1}$ ).
7. $\nu\left(\ldots, \mu\left(\alpha_{t}, \ldots, \alpha_{1}, \alpha\right), \ldots\right)$

This bottom-most inner $\mu$ appears if the outer sequence from bottom to stop includes $\alpha$. The final angle of the inner $\mu$ is a certain $\alpha_{t}$, and a part $\alpha_{t}^{(2)}$ of it reaches outside the disk, so write $\alpha_{t}=$ $\alpha_{t}^{(2)} \alpha_{t}^{(1)}$. The outer $\nu$ is again middle split, yielding a result coefficient of $(-1)^{\left\|\alpha_{s+1}\right\|+\ldots+\left\|\alpha_{k}\right\|} \# \nu^{1}$ (top). The Hochschild sign is $(-1)^{1+\left\|\alpha_{s+1}\right\|+\ldots+\left\|\alpha_{k}\right\|}$, rendering a total contribution to $d \nu_{P}$ of $-\# \nu^{1}$ (top).

It remains to check that all these terms cancel out, regardless of the length of $\alpha$ and the location of the start and end index $s$. For this, we need a case distinction after the length of the angles involved. For example, by top $\geq \alpha$ we mean that the top angle of the garage sequence includes $\alpha$. We are now ready to summarize the contributions of terms 1-7 as follows:

- Term 1 always yields a contribution of $\# \nu^{1}(\mathrm{top})+\# \nu^{1}(\mathrm{mid})+\# \nu^{1}($ bot $)-\# \nu^{1}\left(\ell^{r}\right)$.
- If top $\geq \ell^{r}$, then $2+4$ yield a total contribution of $-\# \nu^{1}$ (top).
- If bot $\geq \alpha$, then 7 yields a total contribution of $-\# \nu^{1}$ (top).
- If top $\geq \alpha$, then 6 yields a total contribution of $\# \nu^{1}\left(\ell^{r}\right)-\# \nu^{1}($ mid $)-\# \nu^{1}$ (bot).
- If bot $\geq \ell^{r}$, then $3+5$ yield a total contribution of $\# \nu^{1}\left(\ell^{r}\right)-\# \nu^{1}$ (mid) $-\# \nu^{1}$ (bot).

Since $\alpha \ell^{r}=$ bot $\cdot \mathrm{mid} \cdot$ top and mid consists of precisely one sector, we have that either top $\geq \ell^{r}$ or bot $\geq \alpha$ (but not both). Similarly, either top $\geq \alpha$ or bot $\geq \ell^{r}$. We conclude that either way, all contributions to $d \nu_{P}$ add up as

$$
\# \nu^{1}(\text { top })+\# \nu^{1}(\text { mid })+\# \nu^{1}(\text { bot })-\# \nu^{1}\left(\ell^{r}\right)-\# \nu^{1}(\text { top })+\# \nu^{1}\left(\ell^{r}\right)-\# \nu^{1}(\text { mid })-\# \nu^{1}(\text { bot })=0
$$

Let us now regard the two exceptions where the start is right before $\alpha$ or right after $\alpha$. The difference with the generic case is that there is no proper mid sector. Let us regard the first exceptional case, where the sequence consists of the angles $\alpha, \alpha_{1}, \ldots, \alpha_{k}$. Then there are no 2,4 or 6 terms, since the top is basically empty. The bottom-most terms $3+5$ contribute however with $\# \nu^{1}(\alpha)$ as in the generic case, and the bottom-most term 7 is special and contributes by $\# \nu^{1}\left(\ell^{r}\right)$. A special contribution of $-\# \nu^{1}\left(\ell^{r}\right)$ comes from $\mu\left(\nu\left(\alpha_{k}, \ldots\right), \ldots, \alpha\right)$, the equivalent of the mid term in the generic case of 5 . Finally, the 1 term contributes $\# \nu^{1}(\alpha)$ as in the generic case. In total, these four terms add up to zero.

Let us regard the second exceptional case, where the sequence consists of the angles $\alpha_{1}, \ldots, \alpha_{k}, \alpha$. Then there are no 3,5 or 7 terms, since the bottom is empty. The top-most term 2 contributes $-\# \nu^{1}(\alpha)$, the other top 4 terms contribute $-\# \nu^{1}\left(\ell^{r}\right)$, and the top 6 term contributes $\# \nu^{1}\left(\ell^{r}\right)$. Together with the type 1 term $\# \nu^{1}(\alpha)$, this adds up to zero again.

Lemma 25.9. We still have $d \nu_{P}$ on parking garage sequences when a $\gamma$ is attached at the front and/or a $\beta$ attached at the back.

Proof. Let us regard a garage sequence with additional $\gamma$ at the end. Regard first the case where the final angle is one of $\alpha_{1}, \ldots, \alpha_{k-1}$. Then that final angle $\alpha_{s+1}$ changes to $\gamma \alpha_{s+1}$. We claim the effect on $d \nu_{P}$, applied to the garage sequence, is merely a multiplication by $\gamma$. Essentially, this means checking that all old-era $\nu$ contributions stay old-era.

For example, let us check the first three types of terms explicitly. The contribution of term 1 gets only multiplied by $\gamma$. For term 2, this is also true, since the final angle is assumed not to be $\alpha$, and the outer $\mu$ itself gets multiplied by $\gamma$. This holds likewise for term 3 , except in the case when there are no angles to the left of the inner $\nu$, where the term looks like $\mu\left(\nu\left(\ldots, \alpha_{1}\right), \alpha, \ldots\right)$. Since the final angle is not $\alpha_{k}$, the inner $\nu$ was of new-era type. Prolonging the final angle naturally preserves the new-era type of the $\nu$ evaluation.

Now let us regard the two special cases where the final angle is either $\alpha$ or $\alpha_{k}$. We start with the case where the final angle is $\alpha$. The garage is then the sequence $\alpha_{1}, \ldots, \alpha_{k}, \gamma \alpha$. Inspecting all terms 1-7, most terms just get multiplied by $\gamma$, but we also incur the following changes:

- $\mu\left(\nu^{1}(\gamma \alpha), \alpha_{k}, \ldots, \alpha_{1}\right)$ now contributes $\# \nu^{1}(\gamma)$ extra,
- $\nu\left(\mu\left(\gamma \alpha, \alpha_{k}, \ldots\right), \ldots, \alpha_{1}\right)$ keeps contributing as long as $\gamma<\ell^{r}$, but the rotation amount of the outer $\nu$ now decreases from $\ell^{r}$ to $\ell^{r} \gamma^{-1}$. This means it contributes $\# \nu^{1}(\gamma)$ less. When $\gamma \geq \ell^{r}$, it stops contributing entirely, meaning it contributes $\# \nu^{1}\left(\ell^{r}\right)$ less than in the case without $\gamma$.
- The new term $\nu^{1}\left(\mu\left(\gamma \alpha, \alpha_{k}, \ldots, \alpha_{1}\right)\right)$ suddenly starts contributing when $\gamma>\ell^{r}$, namely by $\# \nu^{1}\left(\gamma \ell^{-r}\right)$. The Hochschild sign is -1 .
Adding up these extra contributions, the total remains precisely the same as in the case without $\gamma$, for $\gamma<\ell^{r}$ as well as $\gamma \geq \ell^{r}$.

Now regard the case that $\alpha_{k}$ is the final angle of the garage sequence. The sequence is then $\alpha, \alpha_{1}, \ldots, \gamma \alpha_{k}$. We have the following contributions:

1. $\mu\left(\ldots, \nu^{1}(\alpha)\right)$

This simply gets multiplied by $\gamma$ and the result is $\# \nu^{1}(\alpha) \gamma$. The Hochschild sign is +1 .

2'. $\mu\left(\nu\left(\gamma \alpha_{k}, \ldots\right), \ldots, \alpha\right)$
While the generic case term 2 has inner $\nu$ of old-era type, this pendant is new-era and yields $-\# \nu^{1}$ (next) $\gamma$, where $n$ is the next sector after the top of the garage. The Hochschild sign is +1 .
3. $\mu\left(\gamma \alpha_{k}, \ldots, \nu\left(\ldots, \alpha_{1}\right), \alpha\right)$

In no case is it possible to take an inner $\nu$ that includes all angles from $\alpha_{1}$ to $\alpha_{k}$, since these angles cover strictly more than $r$ turns around $m$. Therefore $\gamma \alpha_{k}$ lies outside of the inner $\nu$, which thereby remains old-era. The outer $\mu$ simply gets multiplied by $\gamma$ and gives $-\# \nu^{1}$ (first) $\gamma$. The Hochschild sign is +1 .
4. Terms of type 4 do not appear when $\alpha_{k}$ is the final angle.
5. $\mu(\ldots, \nu(\ldots), \underbrace{\ldots}_{\geq 1}, \alpha)$

We do not count $\mu\left(\nu\left(\gamma \alpha_{k}, \ldots\right), \ldots, \alpha\right)$ among these terms, since we already attributed it to 2 '. Then, all that changes for these type 5 terms is that they get multiplied by $\gamma$. They add up to $\left(-\# \nu^{1}(\alpha)+\# \nu^{1}(\right.$ first $\left.)\right) \gamma$. The Hochschild sign is +1 .
6. The term of type 6 does not appear when $\alpha_{k}$ is the final angle.
7. $\nu\left(\ldots, \mu\left(\alpha_{t}, \ldots, \alpha_{1}, \alpha\right)\right)$

Since the angles $\alpha_{1}, \ldots, \alpha_{k}$ cover strictly more than the angle $\alpha$ does, we have $t<k$. In other words, $\nu$ was old-era and becomes new-era. Its new value is $-\# \nu^{1}$ (next) $\gamma$. The Hochschild sign is -1 .
In total, this adds up to zero again. Let us finally comment on the changes we incur once we add $\beta$ at the back, in addition to a possible $\gamma$. Both $\mu^{\geq 3}$ and $\nu^{\geq 2}$ are "equivariant" under appending $\beta$ at the back. It remains to check the cases where $\nu^{1}$ is involved, and check for longer terms appearing because $\alpha$ gets longer. If one of $\alpha_{1}, \ldots, \alpha_{k}$ is the angle in the back of the sequence, it is readily checked that all terms simply get multiplied by $\beta$. If $\alpha$ is the angle in the back, we incur the following changes:

- $\nu(\ldots, \mu(\ldots, \alpha \beta))$ still contributes as long as $\beta<\ell^{r}$, however the sequence the inner $\mu$ is applied to becomes longer and longer, and similarly the magic angle of the outer $\nu$ becomes shorter and shorter. We lose $-\# \nu^{1}(\beta)$ as coefficient. If $\beta \geq \ell^{r}$, the term does not contribute anymore at all, and we have lost $-\# \nu^{1}\left(\ell^{r}\right)$, compared to the sequence without $\beta$. The Hochschild sign is -1 . All signs together, we deduce an extra contribution of $-\# \nu^{1}(\beta)$ or $-\# \nu^{1}\left(\ell^{r}\right)$.
- $\nu^{1}\left(\mu\left(\alpha_{k}, \ldots, \alpha_{1}, \alpha \beta\right)\right)$ starts to contribute once $\beta>\ell^{r}$, namely by $\# \nu^{1}\left(\beta \ell^{-r}\right)$. The Hochschild sign is -1 .
- $\mu\left(\alpha_{k}, \ldots, \alpha_{1}, \nu^{1}(\alpha \beta)\right)$ contributes an extra $\# \nu^{1}(\beta)$. The Hochschild sign is +1 .

Whether $\beta \leq \ell^{r}$ or $\beta>\ell^{r}$, we conclude the additional contribution vanishes, compared to the case without $\beta$.

Finally, when both non-empty $\beta$ and $\gamma$ are appended, we conclude the result is a multiple of $\gamma \beta$, which vanishes. Indeed, pick two consecutive angles around the garage sequence. Then appending an angle $\gamma$ behind the first and an angle $\beta$ behind the second necessarily makes $\beta$ and $\gamma$ incomposable.

### 25.3 Cancellation on other sequences

In the section 25.2 , we checked that $d \nu$ vanishes on parking garage sequences. Here $\nu$ is an even Hochschild cochain constructed in section 25.1 from the input data $m \in M, r \geq 1$ and input scalars $\# \nu^{1}(\alpha)$. In the present section, we check that $d \nu$ also vanishes on all other sequences of angles. The procedure is as follows: Pick a sequence $\alpha_{1}, \ldots, \alpha_{k}$ of angles and evaluate $d \nu\left(\alpha_{k}, \ldots, \alpha_{1}\right)$. This gives a collection of terms of the form $\mu(\ldots, \nu(\ldots), \ldots)$ and $\nu(\ldots, \mu(\ldots), \ldots)$. We show how to partition this collection of terms such that within each partition, the terms cancel each other. In contrast to the odd case, the partitions do not always consist of two, but sometimes also of three terms.

Of course, we cannot handle each individual sequence of angles individually, but rather need to classify sequences according to their shape. Most importantly, we distinguish the shapes according to the types of $\mu$ and $\nu$ that can be applied and the number of angles before and after the inner application. This way, we can partition all possible sequences of angles $\alpha_{1}, \ldots, \alpha_{k}$ and terms appearing in $d \nu\left(\alpha_{k}, \ldots, \alpha_{1}\right)$ in bulk format: Each of the partitions we provide makes reference to a particular shape.

In total, this procedure requires considerable case-checking effort, namely

1. listing all partitions,
2. characterizing for each partition the required sequence shape,
3. proving that each partition sums up to zero,
4. mapping each term in $d \nu(\ldots)$ to one partition,
5. proving all terms in all partitions are hit at most once,
6. proving all terms in all partitions are hit at least once.

We do not conduct all steps rigorously. In fact, we concentrate on $1,2,3,4$, but without rigor. Below, we list all partitions, ordered roughly after the type of sequence involved. Typically, such a sequence winds once around a certain area and then around another, possibly the one being nested in the other. We indicate the type of these areas as "disk", "< $\ell^{r}$ " or " $\ell$ ". In the figures, the thick dot indicates the location of $m$ and the grey rings indicate the magic angles of the $\nu(\ldots)$ for every involved cancelling term.

Remark 25.10. The list indicates clearly that the given pair or triple of terms cancels. To see this, recall that $\nu(\ldots)$ is by definition weighted with the input scalar $\# \nu^{1}$ of its magic angle. In order to make the claimed pairs or triples of terms cancel each other, we need to show that a signed sum of input scalars of the magic angles is zero. Since the input scalars $\# \nu^{1}$ are additive on angles, this amounts to checking that every indecomposable angle around $m$ appears in an even number of magic angles (ignoring signs). The reader can easily convince himself that this is the case by looking at the grey rings around $m$ in every figure: Every ray away from $m$ hits an even number of grey rings.

No. Fisures,

No. Figure
Cancelling terms
Ref.
13

$\nu\left(\mu\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \alpha_{m}, \ldots\right)$
25.17 a 3
$\mu\left(\nu^{1}\left(\alpha_{k}\right), \ldots\right)$
25.22
$\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)$
$\alpha<\ell^{r}$
25.27 p

14

$\nu\left(\ldots, \alpha_{m+1}, \mu\left(\alpha_{m}, \ldots, \alpha_{n}\right), \ldots\right)$
25.17 b
$\nu\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right)$
25.28

15

$\nu^{1}\left(\mu\left(\alpha_{k}, \ldots, \alpha_{1}\right)\right)$
$\mu\left(\ldots, \nu^{1}\left(\alpha_{1}\right)\right)$
$\mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)$
$\alpha \leq \ell^{r}, \alpha_{1} \geq \alpha$
25.20 b
25.21

$\nu\left(\ldots, \mu\left(\alpha_{m}, \ldots, \alpha_{1}\right)\right)$
$\mu\left(\ldots, \nu^{1}\left(\alpha_{1}\right)\right)$
25.17
$\mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)$
25.21 b
$\alpha \leq \ell^{r}, \alpha_{1}<\alpha$
25.24
$\nu\left(\ldots, \mu\left(\alpha_{m}, \ldots, \alpha_{n+1}\right), \ldots\right)$
$\nu\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right)$
25.17 d
25.28

18

$\nu\left(\ldots, \mu\left(\alpha_{m}, \ldots, \alpha_{n}\right), \ldots\right)$
$\nu\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right)$
25.18 b
25.28

19

$\nu\left(\ldots, \mu\left(\alpha_{m}, \ldots, \alpha_{n+1}\right), \ldots\right)$
$\nu\left(\ldots, \mu^{2}\left(\alpha_{n+1}, \alpha_{n}\right), \ldots\right)$

20

$\nu\left(\ldots, \mu\left(\alpha_{m}, \ldots, \alpha_{n+1}\right), \ldots\right)$
25.19.
25.28

No. Figure
Cancelling terms
Ref.

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| $\mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)$ | 25.23. |
| :--- | ---: |
| $\mu\left(\ldots, \nu\left(\alpha_{m}, \ldots, \alpha_{1}\right)\right)$ | 25.16 |
| $\nu\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right)$ | 25.28 |
|  |  |

22


$$
\mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)
$$

$\nu\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right)$
25.28
$\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)$


23


$$
\begin{aligned}
& \mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right) \\
& \nu\left(\alpha_{k}, \ldots, \mu^{2}\left(\alpha_{1}, \alpha_{2}\right)\right)
\end{aligned}
$$



$$
\begin{aligned}
& \mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right) \\
& \nu\left(\alpha_{k}, \ldots, \mu^{2}\left(\alpha_{2}, \alpha_{1}\right)\right)
\end{aligned}
$$



$$
\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)
$$


$\nu\left(\mu\left(\alpha_{k}, \ldots, \alpha_{m}\right), \ldots, \alpha_{1}\right)$

$\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)$
$\left.\mu\left(\nu^{1}\left(\alpha_{k}\right), \alpha_{k-1}, \ldots, \alpha_{1}\right)\right)$
$\nu\left(\mu\left(\alpha_{k}, \ldots, \alpha_{m}\right), \ldots, \alpha_{1}\right)$
$\alpha=\ell^{r}$


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$\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)$

$\left.\nu\left(\mu^{2}\left(\alpha_{k}, \alpha_{k-1}\right), \ldots, \alpha_{1}\right)\right)$


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$$
\begin{aligned}
& \mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right) \\
& \left.\mu\left(\nu\left(\alpha_{k}, \ldots, \alpha_{m}\right), \ldots, \alpha_{1}\right)\right) \\
& \nu\left(\mu^{2}\left(\alpha_{k}, \alpha_{k-1}\right), \ldots, \alpha_{1}\right)
\end{aligned}
$$



| No. Figure | Cancelling terms | Ref. |
| :--- | :--- | :--- |
| 29 | $\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)$ <br> $\nu\left(\mu^{2}\left(\alpha_{k}, \alpha_{k-1}\right), \ldots, \alpha_{1}\right)$ <br> $\mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \ldots, \alpha_{1}\right)$ |  |

We now investigate all possible terms in $d \nu(\ldots)$. For every term, we provide the other terms with which it cancels. These pairs or triples can be found back in the partition list above. A few possible terms are omitted due to analogy with other terms, and we have correspondingly not listed them in the partition table either. In Proposition 25.29, we draw the conclusion that $d \nu=0$.

Lemma 25.11. A contribution $\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}(\ldots)\right)$ with $\nu$ end-split with turning angle $<\ell^{r}$ cancels.
Proof. Since we assumed the inner $\nu$ to be end-split with turning angle $<\ell^{r}$, its result is some angle $\alpha$ winding around $m$. Then $\alpha$ forms a disk sequence together with $\alpha_{m+1}, \ldots, \alpha_{k}$. In order to find the terms canceling the contribution, we distinguish the following cases: (a) The angle $\alpha$ is an ordinary interior angle for the outer $\mu$, and the outer $\mu$ is all-in. (b) The angle $\alpha$ is an ordinary interior angle for the outer $\mu$, the outer $\mu$ is final-out, and the result part of $\alpha_{k}$ is shorter than $\alpha_{1}$. (c) The angle $\alpha$ is an ordinary interior angle for the outer $\mu$, the outer $\mu$ is final-out, and the result part of $\alpha_{k}$ is longer than $\alpha_{1}$. (d) The angle $\alpha$ is a first-out angle. See Figure 25.5

Regard case (a). Then we have the triple cancellation

$$
\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}\left(\alpha_{m}, \ldots, \alpha_{1}\right)\right)+\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \alpha_{m}, \ldots, \alpha_{1}\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right) .
$$

Regard case (b). Then $\alpha_{k}$ includes a $\gamma$ at the front, but is short. It splits the input sequence of $\nu$ into a "small" disk sequence and a remaining "big" part. We get the cancellation

$$
\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}\left(\alpha_{m}, \ldots, \alpha_{1}\right)\right)+\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{t}\right), \ldots, \alpha_{1}\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right) .
$$

Regard case (c). Then the long $\alpha_{k}$ angle makes it possible to create another inner $\nu$, the third one in the following cancellation:

$$
\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}\left(\alpha_{m}, \ldots, \alpha_{1}\right)\right)+\mu^{2}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots\right), \alpha_{1}\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right) .
$$

Regard case (d). Then we can apply $\nu$ to both disks. Their sum gets compensated by combining the two disks:

$$
\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}\left(\alpha_{m}, \ldots, \alpha_{1}\right)\right)+\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \alpha_{m}, \ldots, \alpha_{1}\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right) .
$$

Lemma 25.12. A contribution $\mu^{\geq 3}(\underbrace{\ldots}_{\geq 1}, \nu^{\geq 2}(\ldots), \underbrace{\ldots}_{\geq 1})$ with $\nu$ end-split with turning angle $<\ell^{r}$ cancels.
Proof. Write $\mu^{\geq 3}\left(\alpha_{k}, \ldots, \nu^{\geq 2}\left(\alpha_{m}, \ldots\right), \alpha_{n}, \ldots, \alpha_{1}\right)$. See Figure 25.6 We have a triple cancellation

$$
\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}(\ldots), \ldots\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{n+1}, \alpha_{n}\right), \ldots\right) .
$$

Lemma 25.13. A contribution $\mu^{\geq 3}\left(\nu^{\geq 2}(\ldots), \ldots\right)$ with $\nu$ end-split with turning angle $<\ell^{r}$ cancels.


Case (a)


Case (b)


Case (c)


Case (d)

Figure 25.5: Cancellation for Lemma 25.11. Magic angles of each contribution to $d \nu$ are drawn as gray rings around $q$. From these figures one deduces cancellation: Each sector around $q$ appears exactly an even number of times, here 0 or 2 . Inspection shows that overlapping contributions indeed appear with opposite sign.


Figure 25.6: Cancellation for Lemma 25.12 and Lemma 25.13

Proof. Since we assumed the inner $\nu$ to be end-split with turning angle $<\ell^{r}$, its result is some angle $\alpha$ winding around $m$. Then $\alpha$ forms a disk sequence together with $\alpha_{1}, \ldots, \alpha_{m}$. By assumption, $\alpha$ is necessarily an ordinary or the final out angle for this disk. In order to find the terms canceling the contribution, we distinguish the following cases: (a) The angle $\alpha$ is an ordinary interior angle for the outer disk sequence, and $\alpha_{1}$ reaches outside the orbigon at its tail. (b) The angle $\alpha$ is a final-out angle. (c) The angle $\alpha$ is an ordinary interior angle, and $\alpha_{1}$ does not reach outside.

Regard case (a), where $\alpha$ is an ordinary interior angle of the outer $\mu$. Then we have the triple cancellation

$$
\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \alpha_{m}, \ldots, \alpha_{1}\right)+\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)+\nu\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right) .
$$

Regard case (b), where $\alpha$ is a final-out angle of the outer $\mu$. In particular, the first angle $\alpha_{1}$ of the outer $\mu$ has no $\beta$ appended. We have the triple cancellation

$$
\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \ldots\right)+\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}\left(\alpha_{m}, \ldots, \alpha_{1}\right)\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right)
$$

Regard case (c). Then we have the cancellation

$$
\mu\left(\nu\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \ldots\right)+\mu\left(\ldots, \nu\left(\alpha_{t}, \ldots, \alpha_{1}\right)\right)+\nu\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right)
$$

Lemma 25.14. A contribution $\mu^{\geq 3}(\ldots, \nu \geq 2(\ldots), \ldots)$ with $\nu$ old- or new-era end-split with turning angle $\ell^{r}$ cancels.

Proof. We distinguish cases: (a) The $\nu$ is old-era and is not the final angle in $\mu$. (b) The $\nu$ is old-era and is the final angle in $\mu$, and $\mu$ is all-in or first-out. (c) The $\nu$ is old-era and is the final angle in $\mu$, and $\mu$ is final-out. (d) The $\nu$ is new-era and has $\beta$ appended. (e) The $\nu$ is new-era without $\beta$.

Regard case (a). Label the angles as $\mu^{\geq 3}\left(\alpha_{k}, \ldots, \nu^{\geq 2}\left(\alpha_{m}, \ldots, \alpha_{n+1}\right), \ldots\right)$. Then the next angle $\alpha_{m+1}$ after $\nu$ winds around $m$. Prolonging it by $\ell^{r}$ gives precisely a disk sequence, in other words the sequence is a parking garage sequence.

Regard case (b). Then $\alpha_{1}$ is the first angle of $\mu$ and winds around $m$. Prolonging it by $\ell^{r}$ gives a disk sequence and we have a parking garage again.

Regard case (c). Label the angles as $\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{m}\right), \ldots, \alpha_{1}\right)$. Then we can swap the order in which $\mu$ and $\nu$ are applied:

$$
\mu^{\geq 3}\left(\nu \geq 2\left(\alpha_{k}, \ldots, \alpha_{m}\right), \ldots, \alpha_{1}\right)+\nu^{\geq 2}\left(\alpha_{k}, \ldots, \mu^{\geq 3}\left(\alpha_{m}, \ldots, \alpha_{1}\right)\right) .
$$

Regard case (d). Then the split of $\nu$ necessarily divides the outer $\mu$ disk into two. The angle $\alpha_{t}$ where the split touches the opposite boundary of the $\mu$ disk creates a contribution $\mu^{\geq 3}\left(\ldots, \nu^{1}\left(\alpha_{t}\right), \ldots\right)$. We have a garage sequence.

Case (e) is similar to the combination of (a), (b) and (c): If there is an angle before $\nu$, then we can apply $\nu^{1}$ and have a garage sequence. If there is no angle before $\nu$, we have a contribution $\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}(\ldots)\right)$. It this is an all-in or final-out $\mu$, then the final angle $\alpha_{k}$ winds around $m$ and we have a parking garage. If this is a first-out $\mu$, then we can swap the order of $\mu$ and $\nu$ again.


Case (a)


Case (b)


Case (c)


Case (d)

Figure 25.7: Cancellation for Lemma 25.14


Case (a)


Case (b)


Case (c)


Case (d)

Figure 25.8: Cancellation for Lemma 25.15

Lemma 25.15. A contribution $\mu^{\geq 3}\left(\nu^{2}(\ldots), \ldots\right)$ with middle-split first-out $\nu$ and first-out $\mu$ cancels.
Proof. Label the angles as $\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \alpha_{m}, \ldots\right)$. We distinguish cases: (a) The result part of $\alpha_{1}$ is shorter than the corresponding interior angle of $\nu$, it ends before the $m$ puncture and cuts the $\nu$ piece into two. (b) The result part of $\alpha_{1}$ is shorter than the corresponding interior angle of the $\nu$, and it ends at the $m$ puncture in $\nu$. (c) The result part of $\alpha_{1}$ is shorter than the corresponding interior angle of $\nu$, it ends after the $m$ puncture and cuts the $\nu$ piece into two at some angle $\alpha_{t}$, and the split is neither at the end of the sequence $(s+1=k)$ nor at the cut $(t=s+1)$. (d) The result part of $\alpha_{1}$ is longer than the corresponding interior angle of $\nu$. (e) As in (c), but the split being at the end or at the cut.

Regard case (a). Write $\alpha_{t}$ for the angle where the result part of $\alpha_{1}$ hits the $\nu$ sequence. We have a cancellation

$$
\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \alpha_{m}, \ldots\right)+\nu^{\geq 2}\left(\ldots, \mu^{\geq 3}\left(\alpha_{t}, \ldots, \alpha_{1}\right)\right) .
$$

Regard case (b). Write $\alpha_{s}, \alpha_{s+1}$ for the angles where the split happens. The split gives rise to an end-split $\nu\left(\alpha_{s}, \ldots, \alpha_{1}\right)$ and its result fills up the remaining angle from $\alpha_{s+1}$ to $\alpha_{k}$, giving a contribution $\mu^{\geq 3}\left(\alpha_{k}, \ldots, \alpha_{s+1}, \nu^{\geq 2}\left(\alpha_{s}, \ldots, \alpha_{1}\right)\right)$. Together with $\mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right)$ contraction, this provides a cancellation

$$
\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \alpha_{m}, \ldots\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right)+\mu^{\geq 3}\left(\alpha_{k}, \ldots, \alpha_{s+1}, \nu^{\geq 2}\left(\alpha_{s}, \ldots, \alpha_{1}\right)\right) .
$$

Regard case (c). Write $\alpha_{t}$ for the angle where the result part of $\alpha_{1}$ hits the $\nu$ sequence. We have a cancellation

$$
\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \alpha_{m}, \ldots\right)+\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}\left(\alpha_{t}, \ldots, \alpha_{1}\right)\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right) .
$$

Note in case the result part of $\alpha_{1}$ has no arc going to $m$, the last term vanishes and the first two already cancel out. If the result part of $\alpha_{1}$ however has an arc going to $m$, then the third term precisely compensates for the difference in magic angle between the first two terms.

Regard case (d). We have a cancellation

$$
\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \alpha_{m}, \ldots\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right)+\mu^{2}\left(\alpha_{k}, \nu \nu^{\geq 2}\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)
$$

Regard case (e). We have a cancellation

$$
\mu\left(\nu\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \ldots\right)+\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)+\nu\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right)
$$

Lemma 25.16. A contribution $\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}(\ldots), \ldots\right)$ with middle-split $\nu$ cancels.


Figure 25.9: Cancellation for Lemma 25.16

Proof. We carry out the inspection only in case $\nu$ is first-out. Distinguish cases: (a) The $\nu^{\geq 2}$ result is not the final angle for $\mu$. (b) The $\nu^{\geq 2}$ result is the final angle for $\mu$, and $\mu$ is all-in or final-out. The remaining case that the $\nu^{\geq 2}$ result is the final angle for $\mu$ and $\mu$ is first-out is the content of Lemma 25.15

Regard case (a). We have a cancellation

$$
\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}\left(\alpha_{m}, \ldots, \alpha_{n+1}\right), \ldots\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right)
$$

The magic angles of both $\nu$ terms are readily seen to be equal: The input sequence for the second $\nu$ is only prolonged by a disk sequence, hence has no additional sectors around $m$.

Regard case (b). We can then swap the order in which we apply $\mu$ and $\nu$ :

$$
\mu^{\geq 3}\left(\nu^{\geq 2}\left(\ldots, \alpha_{m+1}\right), \alpha_{m}, \ldots, \alpha_{1}\right)+\nu^{\geq 2}\left(\ldots, \mu\left(\alpha_{m+1}, \ldots, \alpha_{1}\right)\right) .
$$

Lemma 25.17. $d \nu$ vanishes on any sequence that has a $\nu^{\geq 2}\left(\ldots, \mu^{\geq 3}(\ldots), \ldots\right)$ contribution, with $\nu$ end-split with turning angle $<\ell^{r}$.

Proof. We distinguish cases: (a) The inner $\mu$ is first-out, and its result is used as final angle of $\nu$. (b) The inner $\mu$ is first-out, and its result is not used as final angle of $\nu$. (c) The inner $\mu$ is final-out and its result is used as first angle for $\nu$, and the turning angle of $\nu$ together with $\alpha_{1}$ is less than or equal to $\ell^{r}$. (d) The inner $\mu$ is final-out and its result is not used as first angle for $\nu$. (e) As (c), but with angle together larger than $\ell^{r}$.

Regard case (a). Then the final angle $\alpha_{k}$ of the inner $\mu$ winds around $m$. Distinguish (a1) $\alpha_{k}$ together with the turning angle of $\nu$ is bigger than $\ell^{r}$. (a2) $\alpha_{k}$ together with the turning angle is $\ell^{r}$. (a3) $\alpha_{1}$ together with the turning angle is less than $\ell^{r}$.

Regard case (a1). Then the sequence $\alpha_{1}, \ldots, \alpha_{k-1}$ winds more than $\ell^{r}$ times around $m$ and we simply have a final-out parking garage. Regard case (a2). Then we have an end-split contribution $\nu^{\geq 2}\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)$ with turning angle $\leq \ell^{r}$, and find the cancellation

$$
\nu^{\geq 2}\left(\mu^{\geq 3}\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \alpha_{m}, \ldots\right)+\mu\left(\nu^{1}\left(\alpha_{k}\right), \ldots\right)+\mu^{2}\left(\alpha_{k}, \nu \nu^{\geq 2}\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)
$$

Case (a3) has the same cancellation as (a2).
Regard case (b). Then the result of the inner $\mu$ is not the final angle of the sequence. We can simply connect the final angle $\alpha_{m}$ of the inner $\mu$ with the next angle $\alpha_{m+1}$ of the outer $\nu$, producing a cancellation

$$
\nu^{\geq 2}\left(\ldots, \alpha_{m+1}, \mu^{\geq 3}\left(\alpha_{m}, \ldots, \alpha_{n+1}\right), \ldots\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right)
$$

Case (c) is similar to (a). Indeed, $\alpha_{1}$ winds around $m$. By assumption $\alpha_{1}$ together with the turning angle of $\nu$ is less than $\ell^{r}$. We have an end-split contribution $\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right)$ with turning angle $\leq \ell^{r}$, and find the cancellation

$$
\nu^{\geq 2}\left(\ldots, \mu^{\geq 3}\left(\alpha_{m}, \ldots, \alpha_{1}\right)\right)+\mu^{\geq 3}\left(\ldots, \nu^{1}\left(\alpha_{1}\right)\right)+\mu^{2}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)
$$

Case (d) is similar to (b). Case (e) is a parking garage sequence.

Lemma 25.18. $d \nu$ vanishes on any sequence that has a $\nu^{\geq 2}\left(\ldots, \mu^{\geq 3}(\ldots), \ldots\right)$ contribution, with $\nu$ old-era or new-era end-split with turning angle $\ell^{r}$.


Figure 25.10: Cancellation for Lemma 25.17

Proof. See Figure 25.11 Let us check the old-era case first and then comment on the new-era case.
Distinguish cases: (a) The inner $\mu$ is first-out and its result is the final angle of $\nu$. (b) The inner $\mu$ is first-out and its result is an ordinary or the first angle of $\nu$. (c) The inner $\mu$ is final-out and its result is the first angle of $\nu$, and $\nu$ is all-in. (d) The inner $\mu$ is final-out and its result is the first angle of $\nu$, and $\nu$ is first-out. (e) The inner $\mu$ is final-out and its result is not the first angle of the outer $\nu$.

Regard case (a). Write the contribution as $\nu\left(\mu\left(\alpha_{k}, \ldots, \alpha_{m+1}\right), \ldots,\right)$. Then the final angle $\alpha_{k}$ of the inner $\mu$ winds around $m$. By assumption, the sequence $\alpha_{1}, \ldots, \alpha_{m+1}$ already winds at least $\ell^{r}$ around $m$, in particular does $\alpha_{1}, \ldots, \alpha_{k-1}$. This constitutes a garage sequence.

Regard case (b). We can simply connect the angle $\alpha_{m+1}$ after $\mu$ to the final angle $\alpha_{m}$ of $\mu$ :

$$
\nu^{\geq 2}\left(\ldots, \mu^{\geq 3}\left(\alpha_{m}, \ldots, \alpha_{n+1}\right), \ldots\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right)
$$

Case (c) is similar to case (a) and yields a parking garage sequence. In case (d), there is no relevant turning around $m$ and we can swap the order in which we apply $\mu$ and $\nu$ :

$$
\nu^{\geq 2}\left(\ldots, \mu^{\geq 3}\left(\alpha_{m}, \ldots, \alpha_{1}\right)\right)+\mu^{\geq 3}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{m}\right), \ldots\right) .
$$

Both $\nu$ evaluations have equal final angle. Since the left one is old-era, the right-one is old-era as well and both have equal coefficient. Case (e) is similar to (b).

We have proven the old-era case. Finally, let us comment on the case $\nu$ is new-era instead. We claim all cancellations carry over one-to-one. Indeed, in the proof until now we have only used cancellations via parking garage sequences and cancellations in pairs. Those two cancellations from parking garage sequences carry over, since at a parking garage we are free to append $\gamma$ at the front. The cancellations in pairs consist of two cancellations with a $\mu^{2}$ insertion and one cancellation by swapping the order of applying $\mu$ and $\nu$. In the new-era case, these three cancellations still exist: In all three cancellations, the coefficients of both contributions change simultaneously to $\# \nu^{1}$ of the new-era sector and hence still cancel out.

Lemma 25.19. Any contribution $\nu^{\geq 2}\left(\ldots, \mu^{\geq 3}(\ldots), \ldots\right)$ with $\nu$ mid-split cancels.
Proof. We only check this in case $\mu$ is final-out. Distinguish cases: (a) The $\mu$ result is used as first angle for $\nu$, and $\nu$ is first-out. (b) The $\mu$ is used as first angle for $\nu$, and the result is an ordinary angle for $\nu$. (c) The $\mu$ result is used as final-out angle, as first part of the split, or as ordinary but not first angle for $\nu$. (d) The $\mu$ is used as second part of the split.

Regard case (a). Then $\nu$ has no final-out angle and we simply swap the order of $\mu$ and $\nu$.
Regard case (b). Then $\nu$ may have a final-out angle, preventing us from producing a cancellation from swapping the order. This complicated case can be dealt with in a similar way as in the case distinction of Lemma 25.16.

Regard case (c). Label angles as $\nu\left(\ldots, \mu\left(\alpha_{m}, \ldots, \alpha_{n+1}\right), \ldots\right)$. We argue there is an angle $\alpha_{n}$ before $\alpha_{n+1}$ and we can produce a cancellation with the contraction $\nu\left(\ldots, \mu^{2}\left(\alpha_{n+1}, \alpha_{n}\right), \ldots\right)$. Indeed, if the $\mu$ result is used as first part of the split, then it is not the first angle in the $\nu$ sequence, since a mid-split contribution with first angle being the first part of the split vanishes already. Therefore we can assume there is an ordinary, second part of the split or first-out angle before $\alpha_{s}$. This produces a cancellation from a simple augmentation by $\mu^{2}$.

Regard case (d). This means the first angle of the inner $\mu$ goes around $m$ and we have a parking garage.


Figure 25.11: Cancellation for Lemma 25.18


Case (a)


Case (b)


Case (c)


Case (d)

Figure 25.12: Cancellation for Lemma 25.19

Lemma 25.20. Any sequence contributing $\nu^{1}\left(\mu^{\geq 3}(\ldots)\right)$ vanishes under $d \nu$.
Proof. We shall make a case distinction whether the inner $\mu$ is first-out or final-out. Both cases work similarly, so let us just assume the inner $\mu$ is first-out. Then the sequence is of the form $\alpha_{1} \beta, \ldots, \alpha_{k}$ with $\alpha_{1}, \ldots, \alpha_{k}$ a disk sequence. This means $\alpha_{1}$ is precisely as long as the total angle that $\alpha_{2}, \ldots, \alpha_{k}$ winds back. Moreover, $\nu^{1}(\beta)$ is nonzero, hence $\beta$ and also $\alpha_{1}$ wind around $m$.

To find a cancellation, our best guess is that $\alpha_{1} \beta, \ldots, \alpha_{k}$ constitutes a parking garage with inner spiral angle (part of) $\alpha_{1} \beta$. Whether this is the case or not depends on the size of the angle that $\alpha_{2}, \ldots, \alpha_{k}$ covers. Distinguish cases as follows: (a) The angle $\alpha_{1}$ is bigger than $\ell^{r}$. (b) The angle $\alpha_{1}$ is smaller than or equal to $\ell^{r}$.

In case (a) we have a parking garage sequence. Regard case (b). Then no inner $\nu^{\geq 2}$ application is possible with at most $k-2$ terms, since $\alpha_{2}, \ldots, \alpha_{k}$ is too short. However since $\alpha_{1} \leq \ell^{r}$, we have an end-split $\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right)$ and obtain a cancellation in a triple

$$
\left.\nu^{1}\left(\mu\left(\alpha_{k}, \ldots, \alpha_{1} \beta\right)\right)+\mu^{\geq 3}\left(\alpha_{k}, \ldots, \nu^{1}\left(\alpha_{1} \beta\right)\right)+\mu^{2}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1} \beta\right)\right) .
$$

See Figure 25.13

Lemma 25.21. Any contribution $\mu^{\geq 3}\left(\alpha_{k}, \ldots, \nu^{1}\left(\alpha_{1}\right)\right)$ with first-out $\mu$ cancels.
Proof. Label the angles as $\mu\left(\alpha_{k}, \ldots, \nu^{1}\left(\alpha_{1}\right)\right)$. Since this contribution is a first-out $\mu$, we can write $\nu^{1}\left(\alpha_{k}\right)=\alpha \beta$. While $\alpha_{1}, \ldots, \nu^{1}\left(\alpha_{k}\right)$ is a first-out disk, the sequence $\alpha, \alpha_{2}, \ldots, \alpha_{k}$ is an actual (all-in) disk sequence. The angle $\alpha$ is the angle that describes how much $\alpha_{2}, \ldots, \alpha_{k}$ turns. Let us distinguish cases after the length of $\alpha$ : (a) We have $\alpha>\ell^{r}$. (b1) We have $\alpha \leq \ell^{r}$ and $\alpha_{1}<\alpha$. (c) We have $\alpha \leq \ell^{r}$ and $\alpha_{1} \geq \alpha$.

In case (a), we have first-out garage sequence.


Figure 25.13: Cancellation for Lemma 25.20


Cancellation for Lemma 25.21

Regard case (b1). Then the tail arc of $\alpha_{1}$ cuts the $\alpha, \alpha_{2}, \ldots, \alpha_{k}$ disk into two. Denote by $\alpha_{t}$ the angle where the arc touches the opposite side of the disk. We reach the cancellation

$$
\mu\left(\alpha_{k}, \ldots, \nu^{1}\left(\alpha_{1}\right)\right)+\mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)+\nu\left(\ldots, \mu\left(\alpha_{t}, \ldots, \alpha_{1}\right)\right) .
$$

(b1a) has the same cancellation as (b1).
Regard case (c). Then the angles $\alpha_{1}, \ldots, \alpha_{k}$ already conclude a disk and $\alpha_{2}, \ldots, \alpha_{k}$ are short enough to produce an end-split $\nu$ and we reach the cancellation

$$
\mu\left(\alpha_{k}, \ldots, \nu^{1}\left(\alpha_{1}\right)\right)+\nu^{1}\left(\mu\left(\alpha_{k}, \ldots, \alpha_{1}\right)\right)+\mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right) .
$$

Lemma 25.22. All contributions $\mu^{\geq 3}\left(\ldots, \nu^{1}\left(\alpha_{m}\right), \ldots\right)$ cancel.
Proof. In Lemma 25.21 we already dealt with the case of $\nu^{1}$ being the first-out angle for $\mu$. The case it is the final-out angle is similar. We are left with the checking the case it is an ordinary angle, that is, neither first-out nor final-out. Label the angles as

$$
\mu\left(\alpha_{k}, \ldots, \nu^{1}\left(\alpha_{m}\right), \ldots, \alpha_{1} \beta\right) \quad \text { or } \quad \mu\left(\gamma \alpha_{k}, \ldots, \nu^{1}\left(\alpha_{m}\right), \ldots, \alpha_{1}\right) \quad \text { or } \quad \mu\left(\alpha_{k}, \ldots, \nu^{1}\left(\alpha_{m}\right), \ldots, \alpha_{1}\right),
$$

depending on whether $\mu$ is first-out, final-out or all-in. This means $\alpha_{1}, \ldots, \nu^{1}\left(\alpha_{m}\right), \ldots, \alpha_{k}$ is a disk sequence. Since $\nu^{1}\left(\alpha_{m}\right)$ has length bigger than $\ell^{r}$, the remaining angles turn more than $r$ turns around $m$ and we conclude $\alpha_{1}, \ldots, \alpha_{m}, \ldots, \alpha_{k}$ is a garage sequence. Finally the original sequence, which may have additional $\beta$ and $\gamma$, is an (all-in, first-out or final-out) garage sequence.

Lemma 25.23. Any contribution $\mu^{2}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)$ with $\nu$ middle-split final-out or all-in cancels. We assume $\alpha_{1}$ is at the same side as $\alpha_{k}$, which is always the case if $\nu$ is final-out.

Proof. Distinguish cases: (a) The angle $\alpha_{1}$ is shorter than the corresponding interior angle of $\nu$ and starts after $m$. (b) The angle $\alpha_{1}$ is shorter and starts at $m$. (c) The angle $\alpha_{1}$ is shorter and starts before $m$. (d) The angle $\alpha_{1}$ is longer. (e) The split is at $\left(\alpha_{k-1}, \alpha_{k}\right)$ and $\alpha_{1}$ is at least as long as the interior angle after the split.

Regard case (a). Then we have a cancellation

$$
\mu^{2}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)+\nu\left(\ldots, \mu\left(\alpha_{t}, \ldots, \alpha_{1}\right)\right) .
$$

It is readily checked that the two magic angles agree.

Regard case (b). Then we have a cancellation

$$
\mu^{2}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)+\mu\left(\ldots, \nu\left(\alpha_{s}, \ldots, \alpha_{1}\right)\right)+\nu\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right)
$$

Regard case (c). Then we have a cancellation

$$
\mu^{2}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)+\mu\left(\ldots, \nu\left(\alpha_{t}, \ldots, \alpha_{1}\right)\right)+\nu\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right)
$$

Note in case there is no arc reaching to $m$ within $\alpha_{1}$, then the last term vanishes, but the first two are then already equal.

Regard case (d). Then we have a cancellation

$$
\mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)+\nu\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right)+\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)
$$

Regard case (e). Then we have a similar cancellation to (d), namely

$$
\mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)+\nu\left(\mu^{2}\left(\alpha_{k}, \alpha_{k-1}\right), \ldots\right)+\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)
$$

Lemma 25.24. Any contribution $\mu^{2}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)$ with $\nu$ first-out middle split, or end-split with turning angle $<\ell^{r}$, or end-split with turning angle $\ell^{r}$ cancels.

Proof. Distinguish cases: (a) $\nu$ is first-out middle-split. (b) $\nu$ is end-split $\leq \ell^{r}$, and $\alpha_{1}$ winds around $m$, and $\alpha_{1}$ is shorter than the turning angle of $\nu$. (c) $\nu$ is end-split, and $\alpha_{1}$ winds around $m$, and $\alpha_{1}$ is at least as long as the turning angle of $\nu$. (d) $\nu$ is end-split $\ell^{r}$ and all-in, and $\alpha_{1}$ does not wind around $m$.

Regard case (a). Then $\alpha_{1}$ can be appended to $\alpha_{2}$, still forming a middle-split $\nu$. Therefore we have a simple cancellation

$$
\mu^{2}\left(\nu^{\geq 2}, \ldots\right)+\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{2}, \alpha_{1}\right)\right)
$$

In case (b), we have a cancellation

$$
\mu^{2}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)+\mu\left(\alpha_{k}, \ldots, \alpha_{2}, \nu^{1}\left(\alpha_{1}\right)\right)+\nu\left(\alpha_{k}, \ldots, \mu\left(\alpha_{t}, \ldots, \alpha_{1}\right)\right)
$$

In case (c), we have the cancellation

$$
\mu^{2}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)+\mu\left(\alpha_{k}, \ldots, \nu^{1}\left(\alpha_{1}\right)\right)+\nu^{1}\left(\mu\left(\alpha_{k}, \ldots, \alpha_{1}\right)\right)
$$

Regard case (d). Then $\alpha_{1}$ is composable with $\alpha_{2}$ and we have the cancellation

$$
\mu^{2}\left(\nu^{\geq 2}\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)+\nu^{\geq 2}\left(\alpha_{k}, \ldots, \mu^{2}\left(\alpha_{2}, \alpha_{1}\right)\right) .
$$

Since this only changes the first angle of the $\nu$, the two terms are either both old-era or both new-era, and hence have the same magic angle and produce an equal result.

Lemma 25.25. A contribution $\mu^{2}\left(\alpha_{k}, \nu^{\geq 2}\left(\alpha_{k-1}, \ldots\right)\right)$ with $\nu$ first-out or all-in middle-split cancels.
Proof. Distinguish cases: (a) The angle $\alpha_{k}$ is shorter than the corresponding interior angle of $\nu$, and stops before $m$. (b) The angle $\alpha_{k}$ is shorter and stops at $m$. (c) The angle $\alpha_{k}$ is shorter and stops after $m$. (d) The angle $\alpha_{k}$ is longer.

Regard case (a). Then the target arc of $\alpha_{k}$ hits the opposite side of $\nu$ at some angle $\alpha_{t}$. We have the cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu^{\geq 2}\left(\alpha_{k-1}, \ldots\right)\right)+\nu\left(\mu\left(\alpha_{k}, \ldots, \alpha_{t}\right), \ldots, \alpha_{1}\right)
$$

Regard case (b). Then we have the cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu^{\geq 2}\left(\alpha_{k-1}, \ldots\right)\right)+\mu\left(\nu\left(\alpha_{k}, \ldots, \alpha_{s+1}\right), \ldots, \alpha_{1}\right)+\nu\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right)
$$

Regard case (c). Then the target arc of $\alpha_{k}$ hits the opposite side of $\nu$ at some angle $\alpha_{t}$. Then we have the cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu^{\geq 2}\left(\alpha_{k-1}, \ldots\right)\right)+\nu\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right)+\mu\left(\nu\left(\alpha_{k}, \ldots, \alpha_{t}\right), \ldots, \alpha_{1}\right)
$$

Regard case (d). Then we have the cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu^{\geq 2}\left(\alpha_{k-1}, \ldots\right)\right)+\mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)+\nu\left(\ldots, \mu^{2}\left(\alpha_{s+1}, \alpha_{s}\right), \ldots\right)
$$

Lemma 25.26. A contribution $\mu^{2}\left(\alpha_{k}, \nu^{\geq 2}\left(\alpha_{k-1}, \ldots\right)\right)$, with $\nu$ end-split with turning angle $\ell^{r}$ cancels.
Proof. Distinguish cases: (a) $\nu$ is all-in and $\alpha_{k}$ starts at the opposite side of the split angle. (b) $\nu$ is final-out. (c) $\nu$ is first-out and not final-out, and $\alpha_{k}$ is shorter than angle in $\nu$ after the split. (d) $\nu$ is first-out and not final-out, and $\alpha_{k}$ is longer.

Regard case (a). Distinguish cases: (a1) $\alpha_{k}<\ell^{r}$. (a2) $\alpha_{k} \geq \ell^{r}$. In case (a1), the target of $\alpha_{k}$ touches the opposite side of the $\nu$ orbigon at some angle $\alpha_{t}$ and we have the cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)+\mu\left(\nu^{1}\left(\alpha_{k}\right), \ldots, \alpha_{1}\right)+\nu\left(\mu\left(\alpha_{k}, \ldots, \alpha_{t}\right), \ldots, \alpha_{1}\right)
$$

In case (a2), we have the cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)+\mu\left(\nu^{1}\left(\alpha_{k}\right), \ldots, \alpha_{1}\right)+\nu^{1}\left(\mu\left(\alpha_{k}, \ldots, \alpha_{1}\right)\right)
$$

Regard case (b). Then $\alpha_{k}$ is composable with $\alpha_{k-1}$ and we have a simple cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu^{\geq 2}\left(\alpha_{k-1}, \ldots\right)\right)+\nu\left(\mu^{2}\left(\alpha_{k}, \alpha_{k-1}\right), \ldots, \alpha_{1}\right)
$$

Here both $\nu$ are new-era.
Regard case (c). The target arc of $\alpha_{k}$ hits the opposite side of $\nu$ at some angle $\alpha_{t}$ and we have the cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu^{\geq 2}\left(\alpha_{k-1}, \ldots\right)\right)+\mu\left(\nu\left(\alpha_{k}, \ldots, \alpha_{t}\right), \ldots, \alpha_{1}\right)+\nu\left(\mu^{2}\left(\alpha_{k}, \alpha_{k-1}\right), \ldots, \alpha_{1}\right)
$$

Regard case (d). We have the cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu^{\geq 2}\left(\alpha_{k-1}, \ldots\right)\right)+\nu\left(\mu^{2}\left(\alpha_{k}, \alpha_{k-1}\right), \ldots, \alpha_{1}\right)+\mu^{2}\left(\nu\left(\alpha_{k}, \ldots, \alpha_{2}\right), \alpha_{1}\right)
$$

Lemma 25.27. A contribution $\mu^{2}\left(\alpha_{k}, \nu^{\geq 2}\left(\alpha_{k-1}, \ldots\right)\right)$, with $\nu$ final-out middle-split, or end-split with turning angle $<\ell^{r}$ cancels.
Proof. Distinguish cases: (a) $\nu$ is final-out middle-split. (b) $\nu$ is end-split with turning angle $<\ell^{r}$.
Regard case (a). Then we have a simple cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu^{\geq 2}\left(\alpha_{k-1}, \ldots\right)\right)+\nu\left(\mu^{2}\left(\alpha_{k}, \alpha_{k-1}\right), \ldots, \alpha_{1}\right) .
$$

Regard case (b). Distinguish cases: (b1) $\alpha_{k}$ is smaller than the turning angle of $\nu$. (b2) $\alpha_{k}$ is at least the turning angle of $\nu$. In case (b1), the target of $\alpha_{k}$ touches the opposite side of the $\nu$ orbigon at some angle $\alpha_{t}$ and we have the cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)+\mu\left(\nu^{1}\left(\alpha_{k}\right), \ldots, \alpha_{1}\right)+\nu\left(\mu\left(\alpha_{k}, \ldots, \alpha_{t}\right), \ldots, \alpha_{1}\right)
$$

In case (b2), we have the cancellation

$$
\mu^{2}\left(\alpha_{k}, \nu\left(\alpha_{k-1}, \ldots, \alpha_{1}\right)\right)+\mu\left(\nu^{1}\left(\alpha_{k}\right), \ldots, \alpha_{1}\right)+\nu^{1}\left(\mu\left(\alpha_{k}, \ldots, \alpha_{1}\right)\right)
$$

Lemma 25.28. Any contribution $\nu^{\geq 2}\left(\ldots, \mu^{2}(\ldots), \ldots\right)$ cancels.
Proof. Write this sequence as $\nu^{\geq 2}\left(\ldots, \mu^{2}\left(\alpha_{m+1}, \alpha_{m}\right), \ldots\right)$. Distinguish cases: (a) The product $\alpha_{m+1} \alpha_{m}$ is an ordinary angle of $\nu$, and $\alpha_{m}$ splits the $\nu$ orbigon into two. (b) The product $\alpha_{m+1} \alpha_{m}$ is an ordinary angle of $\nu$, and $\alpha_{m}$ and $\alpha_{m+1}$ create another split of the orbigon. (c) $\alpha_{m+1} \alpha_{m}$ is a first-out or final-out angle of end-split $\nu$. (d) $\alpha_{m+1} \alpha_{m}$ is a final-out angle of end-split $\nu$.

In case (a), we can produce a $\nu^{\geq 2}\left(\ldots, \mu^{\geq 3}(\ldots), \ldots\right)$. In case (b), we can generally produce a $\mu^{\geq 3}\left(\ldots, \nu^{\geq 2}\left(\alpha_{m}, \ldots, \alpha_{s+1}\right), \ldots\right)$. In both cases, we fall already into the regime of the previous lemmas.

Regard case (c). Then again depending on the configuration of $\alpha_{m}$ and $\alpha_{m+1}$ we can split off a disk sequence from $\nu$ and land in one of the cases already dealt with.
Proposition 25.29. The cochain $\nu \in \operatorname{HC}(\operatorname{Gtl} \mathcal{A})$ is a Hochschild cocycle: $\nu \in \operatorname{Ker}(d)$.
Proof. We have analyzed all terms $\mu(\ldots, \nu(\ldots), \ldots)$ and $\nu(\ldots, \mu(\ldots), \ldots)$. For a given sequence, the previous lemmas show that its set of contributions can be partitioned so that the contributions in each partition cancel out together. The only terms we have not checked are $\nu^{1}\left(\ldots, \mu^{2}(\ldots), \ldots\right)$ and $\mu^{2}\left(\ldots, \nu^{1}(\ldots), \ldots\right)$. Their cancellation follows precisely from $\nu^{1}$ being a derivation.

### 25.4 Summary

In this section, we summarize our findings. Our starting point is the knowledge that the cochain defined in Definition 25.3 is indeed a Hochschild cocycle. We construct the ordinary even Hochschild cochains $\nu_{m, r}^{\text {even }}$ and show that together with the sporadic classes they form a basis for $\mathrm{HH}^{\text {even }}(\mathrm{Gtl} \mathcal{A})$. Finally, we calculate the Gerstenhaber bracket and the cup product in analogy to Paper I and conclude that the classification theorem of Paper I still holds.

The idea for the even Hochschild cocycle $\nu_{m, r}^{\text {even }}$ is to choose input scalars $\# \nu^{1}(\alpha)$ for all indecomposable angles around $m$ such that their sum is 1 . The Hochschild cocycle $\nu_{m, r}^{\text {even }}$ is then defined as the cocycle we constructed from this data in section 25.1

Definition 25.30. Let $\mathcal{A}$ be a full arc system with [NMD]. Let $m \in M$ and $r \geq 1$. Choose any collection of input scalars $\# \nu^{1}$ such that their sum is 1 . The ordinary even Hochschild cocycle $\nu_{m, r}^{\text {even }}$ is the cocycle $\nu$ constructed from $\# \nu^{1}$ by Definition 25.3

This way, the cocyle $\nu_{m, r}^{\text {even }}$ is not canonical. However, different choices $\# \nu^{1}, \# \nu^{\prime 1}$ yield gauge equivalent cocyles $\nu, \nu^{\prime}$ in the sense that $\nu-\nu^{\prime} \in \operatorname{Ker}(d)$ :

Lemma 25.31. Different choices of $\# \nu^{1}$ yield gauge equivalent cocyles $\nu_{m, r}^{\text {even }}$.
Proof. The clue is to regard the cochain $\varepsilon=\varepsilon^{0}$ given by $r$ full turns around $m$, starting from an arbitrary arc incidence at $m$. The 1-adic component $(d \varepsilon)^{1}$ then reads $(d \varepsilon)^{1}(\alpha)=\alpha \varepsilon^{0}-\varepsilon^{0} \alpha$. To apply this to $\nu$ and $\nu^{\prime}$, note that the difference $\# \nu^{1}-\# \nu^{\prime 1}$ sums up to zero around $m$, therefore the component $\nu^{1}$ can be written as a sum of cochains of the type $(d \varepsilon)^{1}$. We conclude that $\nu-\nu^{\prime}$ can be gauged so that its 0 -adic and 1-adic components vanish. By Lemma 24.5 $\nu-\nu^{\prime}$ can be gauged to zero entirely. This finishes the proof.

Together with the odd and the sporadic even classes, the ordinary even Hochschild classes form a basis for $\mathrm{HH}(\mathrm{Gtl} \mathcal{A})$ :

Theorem 25.32. Let $(S, M)$ be a punctured surface and $\mathcal{A}$ a full arc system with [NMD]. Then the odd classes $\nu_{\mathrm{id}}, \nu_{m, r}^{\text {odd }}$, the sporadic even classes $\left\{\nu_{P}\right\}_{P \in \mathbb{P}_{0}}$ and ordinary even classes $\nu_{m, r}^{\text {even }}$ for a basis for $\mathrm{HH}(\mathrm{Gtl} \mathcal{A})$.
Proof. Jointly, the classes satisfy the requirements of the generation criterion Proposition 24.2 and 24.4 The statement is then immediate.

Theorem 25.33. Let $(S, M)$ be a punctured surface and $\mathcal{A}$ a full arc system with [NMD]. Let $m_{1} \neq m_{2}$ be two distinct punctures in $M$, let $i, j \geq 0$ be two indices, and $\nu_{P}, \nu_{Q}$ two sporadic classes. Then the Gerstenhaber bracket in cohomology reads as follows:

$$
\begin{aligned}
{\left[\nu_{m_{1}, i}^{\text {odd }}, \nu_{m_{2}, j}^{\text {odd }}\right] } & =0, \\
{\left[\nu_{m_{1}, i}^{\text {eve }}, \nu_{m_{2}, j}^{\text {odd }}\right] } & =\delta_{m_{1}, m_{2}} j \cdot \nu_{m_{1}, i+j}^{\text {odd }}, \\
{\left[\nu_{m_{1}, i}^{\text {eve }}, \nu_{m_{2}, j}^{\text {even }}\right] } & =\delta_{m_{1}, m_{2}}(j-i) \cdot \nu_{m_{1}, i+j}^{\text {even }}, \\
{\left[\nu_{m_{1}, i}^{\text {eve }}, \nu_{P}\right] } & =-i \# \nu_{P}\left(\ell_{q}\right) \cdot \nu_{m_{1}, i,}^{\text {even }}, \\
{\left[\nu_{P}, \nu_{m_{1}, i}^{\text {odd }}\right] } & =i \# \nu_{P}\left(\ell_{q}\right) \cdot \nu_{m_{1}, i}^{\text {odd },} \\
{\left[\nu_{P}, \nu_{Q}\right] } & =0 .
\end{aligned}
$$

Proof. Recall that a bracket $[\nu, \eta]$ on cohomology level is defined as the projection to cohomology $\pi[\nu, \eta]$ of the bracket on chain level. This makes for the following strategy: We compute just enough of the bracket at chain level in order to deduce its projection to cohomology. In fact, by Lemma 24.5 odd Hochschild classes $\nu$ are already determined by their 0 -adic component $\nu^{0}$ and even Hochschild classes are already determined by their 1 -adic component $\nu^{1}$. It therefore suffices to compute the 0 -adic component $[\nu, \eta]^{0}$ on chain level in case the bracket value is odd, and the 1-adic component $[\nu, \eta]^{1}$ in case the bracket value is even. To compute these 0 -adic and 1 -adic components, we typically only need to know the 0 -adic and 1 -adic component of $\nu$ and $\eta$ :

$$
\begin{aligned}
{[\nu, \eta]^{0} } & =\nu^{1}\left(\eta^{0}\right)-(-1)^{\|\nu\|\|\eta\|} \eta^{1}\left(\nu^{0}\right) \\
{[\nu, \eta]^{1}(\alpha) } & =\nu^{1}\left(\eta^{1}(\alpha)\right)+(-1)^{\|\eta\|\| \| \|} \nu^{2}\left(\eta^{0}, \alpha\right)+\nu^{2}\left(\alpha, \eta^{0}\right) \\
& -(-1)^{\|\nu\|\|\eta\|}\left(\eta^{1}\left(\nu^{1}(\alpha)\right)+(-1)^{\|\nu\|\|\alpha\|} \eta^{2}\left(\nu^{0}, \alpha\right)+\eta^{2}\left(\alpha, \nu^{0}\right)\right)
\end{aligned}
$$

We are now ready to check the claimed identities. For the first identity, we have

$$
\left[\nu_{m_{1}, i}^{\text {odd }}, \nu_{m_{2}, j}^{\text {odd }}\right]^{1}(\alpha)=0
$$

Indeed, the classes $\nu_{m_{1}, i}^{\text {odd }}$ and $\nu_{m_{2}, j}^{\text {odd }}$ have no 1- and 2-adic components. For the second identity, regard

$$
\left[\nu_{m_{1}, i}^{\mathrm{even}}, \nu_{m_{2}, j}^{\mathrm{odd}}\right]^{0}=\left(\nu_{m_{1}, i}^{\mathrm{even}}\right)^{1}\left(\left(\nu_{m_{2}, j}^{\mathrm{odd}}\right)^{0}\right)=\delta_{m_{1}, m_{2}} j \ell_{m_{1}}^{i+j}
$$

Indeed, the 0 -adic component of the cocycle $\nu_{m_{2}, j}^{\text {odd }}$ consists of $j$ full turns around $m_{2}$, starting at each arc incident at $m_{2}$. The derivation $\left(\nu_{m_{1}, i}^{\text {even }}\right)^{1}$ multiplies each angle around $m_{1}$ by a certain scalar, in fact such that it multiplies one full turn by precisely 1 . Given it is a derivation, it multiplies $j$ full turns around $q$ precisely by $j$. It however sends turns around $m_{2} \neq m_{1}$ to zero. For the third identity, we have

$$
\begin{aligned}
{\left[\nu_{m_{1}, i}^{\text {even }}, \nu_{m_{2}, j}^{\text {even }}\right]^{1}(\alpha) } & =\left(\nu_{m_{1}, i}^{\text {even }}\right)^{1}\left(\left(\nu_{m_{2}, j}^{\text {even }}\right)^{1}(\alpha)\right)-\left(\nu_{m_{2}, j}^{\text {even }}\right)^{1}\left(\left(\nu_{m_{1}, i}^{\text {even }}\right)^{1}(\alpha)\right) \\
& =\delta_{m_{1}, m_{2}}\left(\# \nu_{m_{2}, j}^{1}(\alpha)\left(\# \nu_{m_{1}, i}^{1}(\alpha)+j\right) \ell_{m_{1}}^{i+j} \alpha-\# \nu_{m_{1}, i}^{1}(\alpha)\left(\# \nu_{m_{2}, j}^{1}(\alpha)+i\right) \ell_{m_{1}}^{i+j}\right) \\
& =\delta_{m_{1}, m_{2}}(j-i) \ell_{m_{1}}^{i+j} \alpha .
\end{aligned}
$$

Here we have used that $\# \nu_{m_{1}, i}^{1}(\alpha)=\# \nu_{m_{1}, j}^{1}(\alpha)$. In other words, we have assumed that the ordinary even cocycles for different $i \neq j$ but equal puncture have been constructed with the same input scalars, which is legitimate by Lemma 25.31 Finally, this bracket $\left[\nu_{m_{1}, i}^{\text {even }}, \nu_{m_{2}, j}^{\text {even }}\right]$ has the same 1 -adic component as the cohomology class $\delta_{m_{1}, m_{2}}(j-\imath) \nu_{m_{1}, i+j}^{\text {even }}$ and hence projects to it. For the fourth identity, regard

$$
\begin{aligned}
{\left[\nu_{m_{1}, i}^{\mathrm{even}}, \nu_{P}\right]^{1}(\alpha) } & =\left(\left(\nu_{m_{1}, i}^{\mathrm{even}}\right)^{1}\left(\nu_{P}^{1}(\alpha)\right)-\nu_{P}^{1}\left(\left(\nu_{m_{1}, i}^{\mathrm{even}}\right)^{1}(\alpha)\right)\right. \\
& =\# \nu_{m_{1}, i}^{1}(\alpha) \# \nu_{P}(\alpha) \ell_{m_{1}}^{i} \alpha-\# \nu_{m_{1}, i}^{1}(\alpha)\left(\# \nu_{P}(\alpha)+i \# \nu_{P}\left(\ell_{m_{1}}\right)\right) \ell_{m_{1}}^{i} \alpha
\end{aligned}
$$

We conclude that this bracket has precisely the same 1 -adic component as $-i \# \nu_{P}\left(\ell_{m_{1}}\right) \nu_{m_{1}, i}^{\text {even }}$ and hence projects to it. For the fifth identity, regard

$$
\left[\nu_{P}, \nu_{m_{1}, i}^{\mathrm{odd}}\right]^{0}=\nu_{P}^{1}\left(\ell_{m_{1}}^{i}\right)=i \# \nu_{P}\left(\ell_{m_{1}}\right) \ell_{m_{1}}^{i}
$$

We conclude that the bracket has the same 0 -adic component as $i \# \nu_{P}\left(\ell_{m_{1}}\right) \nu_{m_{1}, i}^{\text {odd }}$ and hence projects to it. For the sixth identity, regard

$$
\left[\nu_{P}, \nu_{Q}\right]^{1}(\alpha)=\nu_{P}^{1}\left(\nu_{Q}^{1}(\alpha)\right)-\nu_{Q}^{1}\left(\nu_{P}^{1}(\alpha)\right)=\# \nu_{P}(\alpha) \# \nu_{Q}(\alpha) \alpha-\# \nu_{Q}(\alpha) \# \nu_{P}(\alpha) \alpha=0
$$

This means the 1-adic component of $\left[\nu_{P}, \nu_{Q}\right]$ vanishes, this bracket is therefore gauge equivalent to zero and its projection to cohomology vanishes.

Theorem 25.34. Let $(S, M)$ be a punctured surface and $\mathcal{A}$ a full arc system with [NMD]. Let $m_{1} \neq m_{2}$ be two distinct punctures and let $i, j \geq 1$ be two indices, and $\nu_{P}, \nu_{Q}$ two sporadic classes. Then the cup product in cohomology reads as follows:

$$
\begin{aligned}
\nu_{m_{1}, i}^{\text {odd }} \cup \nu_{m_{2}, j}^{\mathrm{odd}} & =\delta_{m_{1}, m_{2}} \nu_{m_{1}, i+j}^{\text {odd }} \\
\nu_{m_{1}, i}^{\text {odd }} \cup \nu_{m_{2}, j}^{\text {even }} & =\delta_{m_{1}, m_{2}} \nu_{m_{1}, i+j}^{\text {eve }} \\
\nu_{m_{1}, i}^{\text {odd }} \cup \nu_{P} & =\# \nu_{P}\left(\ell_{m_{1}}\right) \nu_{m_{1}, i}^{\text {even }} \\
\nu_{m_{1}, i}^{\text {eve }} \cup \nu_{m_{2}, j}^{\text {even }} & =0 \\
\nu_{m_{1}, i}^{\text {eve }} \cup \nu_{P} & =0 \\
\nu_{P} \cup \nu_{Q} & =0
\end{aligned}
$$

The odd class $\nu_{\mathrm{id}}$ acts as identity element: $\nu_{\mathrm{id}} \cup \kappa=\kappa \cup \nu_{\mathrm{id}}=\kappa$.
Proof. We compute the cup products of the given Hochschild cocycles first on chain level. Then we compute their projection to cohomology. In fact, for the odd products it suffices to compute the 0 -adic component $(\nu \cup \eta)^{0}$ and for the even products it suffices to compute the 1-adic component $(\nu \cup \eta)^{1}$. We are now ready to start the calculations. For the first identity, regard

$$
\left(\nu_{m_{1}, i}^{\text {odd }} \cup \nu_{m_{2}, j}^{\text {odd }}\right)^{0}=\delta_{m_{1}, m_{2}} \mu^{2}\left(\ell_{m_{1}}^{i}, \ell_{m_{1}}^{j}\right)=\ell_{m_{1}}^{i+j}
$$

This is precisely the 0 -adic component of the Hochschild cohomology class $\nu_{m_{1}, i+j}^{\text {odd }}$ and hence projects to it. For the second identity, regard

$$
\begin{aligned}
\left(\nu_{m_{1}, i}^{\text {odd }} \cup \nu_{m_{2}, j}^{\text {even }}\right)^{1}(\alpha) & =(-1)^{\|\alpha\|+1} \mu^{2}\left(\ell_{m_{1}}^{i}, \nu_{m_{2}, j}^{1}(\alpha)\right) \\
& =\delta_{m_{1}, m_{2}}(-1)^{\|\alpha\|+1+|\alpha|} \ell_{m_{1}}^{i} \# \nu_{m_{2}, j}^{1}(\alpha) \ell_{m_{1}}^{j} \alpha=\delta_{m_{1}, m_{2}} \# \nu(\alpha) \ell_{m_{1}}^{i+j} \alpha .
\end{aligned}
$$

If $m_{1}=m_{2}$, then this is precisely equal to $\left(\nu_{m_{1}, i+j}^{\text {even }}\right)^{1}(\alpha)$, hence the cup product on chain level projects to $\nu_{m_{1}, i+j}^{\text {even }}$. For the third identity, regard

$$
\left(\nu_{m_{1}, i}^{\mathrm{odd}} \cup \nu_{P}\right)^{1}(\alpha)=(-1)^{\|\alpha\|+1} \mu^{2}\left(\ell_{m_{1}}^{i}, \nu_{P}^{1}(\alpha)\right)=(-1)^{\|\alpha\|+1} \mu^{2}\left(\ell_{m_{1}}^{i}, \# \nu_{P}(\alpha) \alpha\right)=\# \nu_{P}(\alpha) \ell_{m_{1}}^{i} \alpha
$$

This product $\nu_{m_{1}, i}^{\text {odd }} \cup \nu_{P}$ sends every angle winding around a puncture different from $m_{1}$ to zero. Since we can add commutators with arbitrary turns around $m_{1}$ as gauges, the projection of $\nu_{m_{1}, i}^{\text {odd }} \cup \nu_{P}$ to cohomology can be read off from the sum of the coefficients $\# \nu_{P}(\alpha)$ over all indecomposable angles $\alpha$ around $m_{1}$. In total, the product projects to

$$
\# \nu_{P}\left(\ell_{m_{1}}\right) \nu_{m_{1}, i}^{\text {even }}
$$

To discuss the fourth, fifth and sixth identity, let us write $\nu$ and $\eta$ for the two factors. Both $\nu$ and $\eta$ are even. This means they have vanishing 0 -adic component $\nu^{0}=\eta^{0}=0$. In particular, their cup product's 0 -adic component

$$
(\nu \cup \eta)^{0}=\mu^{2}\left(\nu^{0}, \eta^{0}\right)
$$

vanishes. This shows that $\nu \cup \eta=0$ in these three cases. To see that the odd class $\nu_{\mathrm{id}}$ acts as identity element, note that

$$
\begin{aligned}
& \left(\kappa \cup \nu_{\mathrm{id}}\right)\left(\alpha_{k}, \ldots, \alpha_{1}\right)=(-1)^{\left\|\nu_{\mathrm{id}}\right\|+1} \mu^{2}\left(\kappa\left(\alpha_{k}, \ldots, \alpha_{1}\right), \mathrm{id}\right)=\kappa\left(\alpha_{k}, \ldots, \alpha_{1}\right) \\
& \left(\nu_{\mathrm{id}} \cup \kappa\right)\left(\alpha_{k}, \ldots, \alpha_{1}\right)=(-1)^{\left\|\nu_{\mathrm{id}}\right\|\left(\left\|\alpha_{1}\right\|+\ldots+\left\|\alpha_{k}\right\|\right)+\|\kappa\|+1} \mu^{2}\left(\mathrm{id}, \kappa\left(\alpha_{k}, \ldots, \alpha_{1}\right)\right)=\kappa\left(\alpha_{k}, \ldots, \alpha_{1}\right)
\end{aligned}
$$

This finishes the proof.
Remark 25.35. In Paper I, we also proved a formality theorem for $\operatorname{HH}(\operatorname{Gtl} \mathcal{A})$ and a classification theorem for formal deformations of $\operatorname{Gtl} \mathcal{A}$. Our proof of the formality theorem builds on a topological grading for $\operatorname{Gtl} \mathcal{A}$. Without the [NL2] condition, the power of the topological grading collapses. The classification theorem however stays intact since it does not make explicit reference to the even Hochschild cocycles.

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## Summary: Deformed mirror symmetry for punctured surfaces

In this thesis we devote ourselves to a phenomenon that comes from physics and is known as homological mirror symmetry. The mirror does not concern reflection of light, but is about a mysterious correspondence between two types of geometry. The one geometry is concerned with so-called symplectic structures, that is, structures to which the principle of the Hamiltonian equations of motion applies. The other geometry deals with so-called complex structures, that is, structures where for instance a surface is parameterized by means of the complex numbers $\mathbb{C}$.

The mysterious conjecture of mirror symmetry entails that for every object of symplectic geometry there should exist a corresponding object from complex geometry, such that the symplectic deformation theory of the symplectic object is equal to the complex deformation theory of the complex object. The name mirror symmetry comes from the so-called Hodge diamond, a matrix of numbers indicating the dimensions of the deformation theories. The idea is that by mirroring the Hodge diamond of a symplectic object along its diagonal one obtains exactly the Hodge diamond of the corresponding complex object.

Kontsevich put this phenomenon into a much better mathematical form in 1994. In this form we are no longer dealing with dimension numbers, but with so-called categories. A category is actually an abstraction of the symplectic or complex structure that is extremely flexible. In connection with homological mirror symmetry, one is then interested in so-called Fukaya categories on the symplectic side and so-called categories of coherent sheaves on the complex side. Whereas a symplectic object itself cannot be compared to a complex object, it can happen that the Fukaya category of a symplectic object is equal to the category of coherent sheaves of a complex object. Homological mirror symmetry today involves trying to find the corresponding symplectic object for a given symplectic object via the formulation of the appropriate categories.

In the optimal case, one can deduce the original mirror symmetry dealing with dimension numbers from Kontsevich's homological mirror symmetry. After all, the deformation theory of a Fukaya is quite similar to the symplectic deformation theory of the given symplectic object and the deformation theory of a category of coherent sheaves is similar to the complex deformation theory of the complex object. So if the two categories are equivalent, then their deformation theories and thus very roughly the dimension numbers of the symplectic or complex deformations of the geometric objects are also the same.

As nicely as the original mirror symmetry could be traced from Kontsevich's homological mirror symmetry, it is unfortunately difficult to show that the deformation theory of a Fukaya category corresponds exactly to symplectic deformations of the geometric object. Moreover, nowadays one is interested in much more than just dimension numbers. In fact, one tries to build explicitly the whole space of deformations and to compare those spaces, rather than only their dimension numbers. This means that for a given symplectic deformation, one wants to know exactly what the corresponding complex deformation on the other side looks like.

A simple example of homological mirror symmetry deals with the case where the symplectic object is a so-called punctured surface. Measured by the generality of mirror symmetry, this is really only the 1-dimensional case. For this simple case, a good understanding of what the corresponding complex object looks like already exists in the mathematical literature. In fact, this complex object is non-commutative, meaning that it does not occur in the real world and can only be modeled by its category.

In this thesis, we devote ourselves to working out the correspondence of deformation theories for this simple example. Our first goal is to determine the entire deformation theory of the Fukaya category of such a punctured surface. Our second goal is to determine all the information we might need to achieve the corresponding deformation on the complex side. Our third goal is to actually construct the corresponding deformation and show that it is indeed the correct one. Thus, we have not equated the entire deformation theories but only a small, yet representative part.

In the first part of this thesis, we determine the entire deformation theory of the Fukaya category of a given punctured surface. In doing so, we use a discrete model for the Fukaya category that actually cuts the surface into pieces. In first view, this makes the calculations very simple. In particular, we easily succeed in writing down candidate deformations explicitly. We also show that our list actually contains all so-called formal deformations. Thus we do not end up with a description of the whole space of deformations, but rather with a kind of local approximation, a Taylor series of the space.

In the second part of this thesis, we focus on one specific deformation of the Fukaya category. Since this deformation has a lot of parameters, this is actually a summary of a lot of individual deformations and incorporates a great deal of information on how the Fukaya category can be deformed. We therefore show that this deformation is equivalent to a view of Seidel. Unlike this thesis, Seidel does not start with a discrete model, but with a smooth model of the Fukaya category. Therefore, it requires a gigantic
effort to show that these two deformations are equal. We build lots of data structures that serve to take us from the discrete side to the smooth side. Among other things, the discrete structure is described by trees, which are a kind of graphs that produce a mathematical chain reaction. The smooth side, on the other hand, is described by disks, which are polygons embedded in the surface. Eventually we reach a correspondence between these trees and disks. With this, we have achieved all the information we need to determine the corresponding deformation of the complex side, and furthermore fulfilled Seidel's vision.

In the third part of this thesis, we explicitly determine the corresponding deformation of the complex side. This is normally a very difficult process because there is no natural way to pull a deformation of a category through the equivalence of categories. But fortunately there is a construction in the literature that makes such a tracing explicitly possible when in the case of punctured surfaces. To implement this properly, we need of course the results of the gigantic calculations from the second part. As soon as we feed these into the construction we immediately get a candidate deformation on the complex side. This deformation is actually guaranteed to be the correct one, only it is written down in a way that makes it not immediately obvious that it is really a deformation. In order to prove that it is still a deformation, we must show that the category has not become smaller due to the deformation, so to speak. Fortunately, here again there is help from the literature. However, we have to work hard to adapt it to our case. Indeed, the case known from the literature is about homogeneous superpotentials where it can be shown that if the category becomes smaller this can at most concern an element of a certain length. But if we remove homogeneity, the paths spin completely out of control. We therefore define an additional boundedness condition by which we regain control of these paths and show that it actually concerns a deformation. By doing so, in the case of punctured surfaces, we are guaranteed to have the right deformation on the complex side and have successfully deformed homological mirror symmetry.

## Samenvatting: Gedeformeerde spiegelsymmetrie voor geperforeerde oppervlakken

In dit proefschrift wijden we ons aan een fenomeen dat uit de natuurkunde komt en bekend staat als homologische spiegelsymmetrie. Bij deze spiegel gaat het echter niet om de reflectie van licht, maar om een mysterieuze overeenkomst tussen twee soorten meetkunde. De ene meetkunde houdt zich bezig met zogenaamde symplectische structuren, dat wil zeggen structuren waarop het principe van de Hamiltoniaanse bewegingsvergelijkingen van toepassing is. De andere meetkunde houdt zich bezig met zogenaamde complexe structuren, dat wil zeggen structuren die bijvoorbeeld een oppervlak parametriseren door middel van de complexe getallen $\mathbb{C}$.

Het mysterieuze vermoeden van spiegelsymmetrie houdt in dat er voor ieder object van de symplectische meetkunde een corresponderend object uit de complexe meetkunde bestaat, zodanig dat de symplectische deformatietheorie van het symplectische object gelijk is aan de complexe deformatietheorie van het complexe object. Het begrip spiegelsymmetrie komt van de zogenaamde Hodge-diamant, een matrix van getallen die de dimensies van de deformatietheorieën aangeeft. Het idee is dat je de Hodgediamant een complex object verkrijgt door de Hodge-diamant van het corresponderende symplectische object langs zijn diagonaal te spiegelen.

Kontsevich heeft dit fenomeen in 1994 in een veel betere wiskundige vorm gebracht. In deze vorm hebben we niet meer met dimensiegetallen te maken, maar met zogenaamde categoriën. Een categorie is eigenlijk een abstractie van de symplectische dan wel complexe structuur die buitengewoon flexibel is. In samenhang met homologische spiegelsymmetrie spreken we dan van zogenaamde Fukaya-categorieën aan de symplectische kant en categorieën van coherente schoven aan de complexe kant. Waar een symplectisch object zelf niet vergelijkbaar is met een complex object, kan het juist wel gebeuren dat de Fukayacategorie van een symplectisch object gelijk is aan de categorie van coherente schoven van een complex object. Homologische spiegelsymmetrie houdt daarom tegenwoordig in dat men via het formuleren van de passende categoriën probeert voor een gegeven symplectisch object het corresponderende symplectisch object te vinden.

In het beste geval kan men de originele spiegelsymmetrie die over dimensiegetallen gaat, herleiden uit Kontsevich's homologische spiegelsymmetrie. Immers, de deformatietheorie van een Fukaya-categorie is goed vergelijkbaar met de symplectische deformatietheorie van het gegeven symplectische object, en de deformatietheorie van een categorie van coherente schoven is vergelijkbaar met de complexe deformatietheorie van het complexe object. Dus wanneer de twee categorieën equivalent zijn, dan zijn ook hun deformatietheorieën en daarmee ook heel grof de dimensiegetallen van de symplectische dan wel complexe deformaties van de meetkundige objecten gelijk.

Zo mooi als de originele spiegelsymmetrie zou kunnen worden herleid uit Kontsevich's homologische spiegelsymmetrie, zo moeilijk is het helaas aan te tonen dat de deformatietheorie van bijvoorbeeld een Fukaya-categorie precies overeenkomt met symplectische deformaties van het meetkundig object. Bovendien is men tegenwoordig aan veel meer geïnteresseerd dan alleen aan dimensiegetallen. In feite probeert men de gehele ruimte van deformaties expliciet op te bouwen en deze ruimtes met elkaar te vergelijken, en niet alleen maar de dimensiegetallen. Dat houdt in dat men voor een gegeven symplectische deformatie precies wil weten hoe de corresponderende complexe deformatie aan de andere kant eruit ziet.

Een simpel voorbeeld van de homologische spiegelsymmetrie is het geval waarin het symplectisch object een zogenaamd geperforeerd oppervlak is. Gemeten aan de generaliteit van spiegelsymmetrie betreft dit eigenlijk alleen het 1-dimensionale geval. Voor dit eenvoudige geval bestaat er in de wiskundige literatuur al een goed begrip van hoe het corresponderende complexe object eruit ziet. In feite is dit complexe object niet-commutatief, dwz. het komt in de echte wereld niet voor en kan alleen gemodelleerd worden via zijn categorie.

In dit proefschrift werken we de overeenstemming van deformatietheorieën voor dit eenvoudige voorbeeld uit. Ons eerste doel is het bepalen van de gehele deformatietheorie van de Fukaya-categorie van zo'n geperforeerd oppervlak. Ons tweede doel is het bepalen van alle informatie die we nodig zouden kunnen hebben om de corresponderende deformatie aan de complexe kant te verkrijgen. Ons derde doel is de corresponderende deformatie daadwerkelijk te construeren en aan te tonen dat deze constructie echt de juiste is. Hiermee hebben we weliswaar niet de deformatietheorieën als geheel gelijk kunnen stellen, maar toch een representatief deel daarvan.

In het eerste deel van dit proefschrift bepalen we dus de gehele deformatietheorie van de Fukayacategorie van een gegeven geperforeerd oppervlak. We maken daarbij gebruik van een discreet model voor de Fukaya-categorie waarmee we het oppervlak min of meer in stukken knippen. Hierdoor worden onze berekeningen heel eenvoudig. In het bijzonder lukt het ons gemakkelijk om kandidaat-deformaties expliciet af te leiden. Ook tonen we aan dat onze lijst daadwerkelijk alle zogenaamde formele deformaties
bevat. We komen dus niet uit bij een beschrijving van de gehele ruimte van deformaties, maar wel bij een soort lokale benadering, een Taylor-reeks van deze ruimte.

In het tweede deel van dit proefschrift richten wij ons op één specifieke deformatie van de Fukayacategorie. Deze deformatie heeft een groot aantal parameters en is hierdoor eigenlijk een samenvatting van heel veel individuele deformaties die een goed deel van de informatie bevat hoe de Fukaya-categorie gedeformeerd kan worden. We tonen dan ook aan dat deze deformatie gelijk is aan een visie van Seidel. In tegenstelling tot dit proefschrift start Seidel niet met een discreet model, maar met een glad model van de Fukaya-categorie. Het vereist daarom een gigantische inzet om aan te tonen dat deze twee deformaties gelijk zijn. We bouwen heel veel datastructuren die ertoe dienen ons van de discrete kant naar de gladde kant te brengen. Onder meer wordt de discrete structuur beschreven door bomen, dat zijn een soort grafen die een wiskundige kettingreactie voortbrengen. De gladde kant wordt daarentegen beschreven door schijven, dat zijn veelhoeken die in het oppervlak zijn ingebed. Uiteindelijk bereiken we een correspondentie tussen deze bomen en schijven. Daarmee hebben we alle informatie bij elkaar die we nodig hebben om de corresponderende deformatie van de complexe kant te bepalen, en bovendien Seidels visie waargemaakt.

In het derde deel van dit proefschrift bepalen we expliciet de corresponderende deformatie van de complexe kant. Dit is normaliter een moeilijk proces omdat er geen natuurlijke manier is om een deformatie van een categorie te transporteren via een equivalentie van categorieën. Maar gelukkig is er een constructie in de literatuur die zo'n transport expliciet mogelijk maakt in het geval van geperforeerde oppervlakken. Om dit goed door te voeren hebben we natuurlijk de resultaten van de uitvoerige berekeningen uit het tweede deel nodig. Zodra wij deze invoeren in de constructie krijgen we direct een kandidaat-deformatie aan de complexe kant. Deze deformatie is gegarandeerd de juiste, alleen is zij opgeschreven op een manier waarop niet direct duidelijk dat het echt een deformatie is. Om aan te tonen dat het toch een deformatie is, moeten wij aantonen dat de categorie door de deformatie als het ware niet kleiner is geworden. Gelukkig komt er ook hier weer hulp vanuit de literatuur. Wel moeten we stevig werken om deze op ons geval toe te passen. Het geval dat uit de literatuur bekend is gaat namelijk over homogene superpotentialen waarbij men kan aantonen dat als de categorie kleiner wordt, dit hooguit een element van een bepaalde lengte kan betreffen. Maar als we de homogeniteit weghalen, loopt de lengte van de paden geheel uit de hand. We definieren daarom een extra begrenzingsvoorwaarde waardoor we deze paden weer onder controle krijgen en aantonen dat het daadwerkelijk om een deformatie gaat. Daarmee hebben in het geval van geperforeerde oppervlakken gegarandeerd de juiste deformatie aan de complexe kant te pakken en succesvol homologische spiegelsymmetrie gedeformeerd.

