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On The Axioms Of Common Meadows: Fracterm Calculus, Flattening And Incompleteness

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Common meadows are arithmetic structures with inverse or division, made total on 0 by a flag $\perp$ for ease of calculation. We examine some axiomatizations of common meadows to clarify their relationship with commutative rings and serve different theoretical agendas. A common meadow fracterm calculus is a special form of the equational axiomatization of common meadows, originally based on the use of division on the rational numbers. We study axioms that allow the basic process of simplifying complex expressions involving division. A useful axiomatic extension of the common meadow fracterm calculus imposes the requirement that the characteristic of common meadows be zero (using a simple infinite scheme of closed equations). It is known that these axioms are complete for the full equational theory of common cancellation meadows of characteristic 0. Here, we show that these axioms do *not* prove all conditional equations which hold in all common cancellation meadows of characteristic 0.

Keywords: arithmetic structures; rational numbers; division by zero; meadows; common meadows; fracterm calculus; equational specification; initial algebra semantics

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1. INTRODUCTION

The basic algebra for computing in arithmetical structures starts with rings and fields, with their operations

$$x + y,\ -x,\ x \cdot y.$$

Although a field has inverses for non-zero elements, to study division we must add an operator. The choices begin with its formulation: there is the inverse operator, with its notations

$$x^{-1},\ 1/x$$

and the division operator, with its notations

$$x/y,\ x \div y,\ \frac{x}{y}.$$

One can derive each from the other, of course, e.g. $x/y = x \cdot 1/y$. For rings and fields with inverse or division we have coined the term *meadow* in [1].

Whatever the choice, we have to deal with division by zero. There are several designs for the algebra of division by zero: the operators can simply be partial, or made total in different ways (we will mention some later). For general information on division by zero see the survey papers [2] and [3].

We will focus on division in fields in the case that division by zero is total and its value is a flag $\perp$ that propagates, suspending further computation. Technically, this means that if $\perp$ is an argument to an operation then the value is $\perp$. We call this $\perp$ an *absorbtive element*. The addition of division and $\perp$ changes the algebra of rings and fields immediately. For example, in a commutative ring $0 \cdot x = 0$ but on adding an absorbtive element $\perp$,

$$0 \cdot x \neq 0 \text{ because } 0 \cdot \perp = \perp.$$

These changes form part of the emerging theory of *common meadows*, to which this note is a contribution.

Common meadows are algebraic structures close to rings and fields that are defined by a set of equational axioms [4–6]. The axioms were designed to accommodate the changes wrought by adding $\perp$ to totalize inverse or division, i.e. setting $0^{-1} = \perp$ or $x/0 = \perp$; division is needed to make the field of rational numbers into an abstract data type.[1] Not surprisingly, there are a number of variations for the axioms and one objective here is to review the axioms of common meadows and pick out significant alternatives that are elegant and/or useful: we clarify their relationship with commutative rings and their relevance for different theoretical tasks and agendas.

It is not an accident that different notations exist for inverse and division; as well as historical reasons, they embody points of view. This is most notable in the case of 'fractions' which are everywhere in elementary teaching and essentially disappear as a concept in advanced mathematics. What exactly is a fraction? In [7], the informal term fraction is replaced with the formally defined syntactic concept of a *fracterm*. A fracterm is an expression or term over an arithmetic signature having division as its leading function symbol. In this paper, we examine the axiomatization of common meadows re-expressed as a calculus of fracterms; we call this axiomatization the *common meadow fracterm calculus*.

Looking at the axioms of common meadows in a notation for fracterms adds to the theory both in its scope and technically. An important example is as follows: central to using fractions are calculations like

$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{2}{3} \cdot \frac{5}{4} = \frac{10}{12}.$$

Here, a complex fracterm is transformed into a simple fracterm containing only one division. A fracterm is *flat* if the division operator occurs only once and it does not have an occurrence of $\perp$. Given a fracterm, the derivation of an equivalent flat fracterm is called *flattening* [7]. From [4], we know that the equational axioms of common meadow fracterm calculus enable all fracterms to be flattened. Here, we seek the most relevant equational axiomatization to deliver this fundamental property of flattening; this is a slightly weaker set than those of the common meadows.

A second task is to gauge the strength of the set E_{cmfc} of equational laws of the common meadow fracterm calculus. How much can be proved using them? A useful extension of common meadow fracterm calculus imposes the requirement that the characteristic is zero, by means of adding an infinite set

$$\chi_0 = \left\{ \frac{\underline{n}}{\underline{n}} = 1 | \in \mathbb{N}, n > 0 \right\}$$

[1] Data types are minimal algebras: this condition requires that all rational numbers can be generated by applying the operations to the constants 0, 1.

of closed equations expressing $n/n = 1$ for non-zero natural numbers n. From [6], we know that $E_{cmfc} \cup \chi_0$ axiomatizes all the equations true in all common cancellation meadows with characteristic 0. While many conditional equations can be derived, here we will show that $E_{cmfc} \cup \chi_0$ does *not* prove all conditional equations which hold in all common cancellation meadows of characteristic 0.

In summary, our new contributions are to shed new light on the following:

(i) totalized arithmetic division through several structured axiomatizations of common meadows and their relationships with some key algebraic and logical properties;
(ii) known and unknown completeness properties of the axiomatizations for equations and conditional equations, notably with a new incompleteness theorem; and
(iii) the literature on these matters, showing how earlier sets of axioms may be adapted and reinterpreted.

The structure of the note is as follows. In Section 2, we study axiomatizations of common meadows and the fracterm calculus. In Section 3, we prove the incompleteness result for conditional equations. In Section 4, we comment on the context of these investigations.

We assume the reader is versed in basic algebra and equational logic; familiarity with the theory of abstract data types is desirable but not necessary.

2. COMMON MEADOW FRACTERM CALCULUS

We will build up the axioms for common meadows and the fracterm calculus in stages, starting with commutative rings.

2.1. Commutative rings

First, let us simply add the flag $\perp$ to a ring. Let Σ_{cr} be the signature of commutative rings and fields. Let $\Sigma_{cr,\perp}$ be Σ_{cr} with a constant $\perp$ added.

As we noted in the Introduction, the absorbtion property $\perp$ changes the algebra (recall $0 \cdot x \neq 0$). Table 1 contains equations for commutative rings that are weakened so as to be compatible with the presence of an absorptive element $\perp$. Let this set of equations be denoted $E_{wcr,\perp}$.

The axioms are mostly those of commutative rings, only $0 \cdot x = 0 \cdot (x \cdot x)$ is unfamiliar but is true of $\perp$. The equation $x + \perp = \perp$ is an absorption property. We note that the other absorbtion properties are derivable:

$$E_{wcr,\perp} \vdash x \cdot \perp = \perp \text{ and } E_{wcr,\perp} \vdash -\perp = \perp,$$

following [4].

Consider the semantics. Following [8], given any commutative ring R, we define the transformation $\mathsf{Enl}_{\perp}(R)$ to be the algebra that results from R by extending the signature with a

TABLE 1. $E_{wcr,\bot}$: Weakened equations for commutative rings.

$$(x+y)+z = x+(y+z)$$
$$x+y = y+x$$
$$x+0 = x$$
$$x+(-x) = 0\cdot x$$
$$x\cdot(y\cdot z) = (x\cdot y)\cdot z$$
$$x\cdot y = y\cdot x$$
$$1\cdot x = x$$
$$x\cdot(y+z) = x\cdot y + x\cdot z$$
$$-(-x) = x$$
$$0\cdot x = 0\cdot(x\cdot x)$$
$$x+\bot = \bot$$

new element $\bot$ as a constant and then extending all operations such that $\bot$ is an absorptive element of the new algebra.

The following proposition confirms the soundness of the axiomatization $E_{wcr,\bot}$ and is easy to prove:

PROPOSITION 2.1. *Let R be a commutative ring then* $\mathsf{Enl}_\bot(R) \models E_{wcr,\bot}$.

We assume that $x_1,\ldots,x_n$ is a fixed listing of n variables and we consider multivariate polynomials with no other variables than these n variables. A monomial is a product of an integer coefficient and a product of non-negative powers of these variables, where the powers of variables are given by $x^0 = 1, x^{n+1} = x^n \cdot x$. A monomial is determined by the coefficient and the vector of its powers. A polynomial is a sum of monomials written in lexicographical ordering of the power vectors with (implicit) bracketing to the left.

A *CM-polynomial* over $x_1,\ldots,x_n$ is a sum

$$t + 0\cdot t',$$

where t is a polynomial and t' is either 0 or a sum $x_{i_1}+\ldots+x_{i_k}$ of different variables (chosen from $x_1,\ldots,x_n$) none of which have occurrences with a positive power in t; CM is for common meadow.

We will now prove the converse with the help of these lemmas.

PROPOSITION 2.2. *For each term r with free variables* $x_1,\ldots,x_n$,

$$E_{wcr,\bot} \vdash 0\cdot r = 0\cdot(x_1+\ldots+x_n).$$

Proof. Straightforward with induction on the structure of r, making use of the fact that $E_{wcr,\bot} \vdash 0\cdot(x\cdot y) = (0\cdot x)+(0\cdot y)$, which is an immediate corollary of the results of [4]. ■

PROPOSITION 2.3. *For each term r with free variables among* $x_1,\ldots,x_n$ *there is a CM-polynomial* $t+0\cdot t'$, *with variables in* $x_1,\ldots,x_n$, *such that*

$$E_{wcr,\bot} \vdash r = t + 0\cdot t'.$$

Proof. Straightforward with induction on the structure of t. ■

THEOREM 2.1. *Suppose for an equation* $t = r$ *that for every ring* $R, \mathsf{Enl}_\bot(R) \models t = r$. *Then* $E_{wcr,\bot} \vdash t = r$ *and so combined with soundness we have*

$$E_{wcr,\bot} \vdash t = r \iff \mathsf{Enl}_\bot(R) \models t = r.$$

Proof. Soundness of the equations for all $\bot$-enlarged rings is immediate. For completeness, assume that $\mathsf{Enl}_\bot(R) \models t = r$, and let the variables of t and r be among $x_1,\ldots,x_n$. Now using Proposition 2.3 we find that there is a polynomial t' and a term t'' such that

$$E_{wcr,\bot} \vdash t = t' + (0\cdot t'')$$

and such that no variable occurring in t' also occurs in t''. Using Proposition 2.1, t'' can be chosen as a sum over zero (in which case it is 0) or more variables. Similarly, we can find r' and a term r'' such that $E_{wcr,\bot} \vdash r = r' + 0\cdot r''$.

Now choose $R = \mathbb{Z}$, then $\mathsf{Enl}_\bot(\mathbb{Z}) \models t = r$. Then t and r are equal as multivariate functions on $\mathbb{Z}$, and so t' and r' are the same polynomials, i.e. have the same coefficients and so are syntactically identical as terms. Further, it must be the case that on both sides precisely the same variables occur. The latter is the case because of the presence of $\bot$. To see this, notice that, e.g. $0\cdot(x+y) = 0\cdot x$ fails in any structure $\mathsf{Enl}_\bot(\mathbb{Z})$ on a valuation $(x = 0, y = \bot)$. It follows that t'' and r'' are syntactically identical too so that

$$E_{wcr},\bot \vdash t = t' + (0\cdot t'') = r' + (0\cdot r'') = r.$$

■

2.2. Common meadows

To define the common meadows, we add division to commutative rings. As we have noted in the Introduction, there are various ways to do this. We add $\frac{x}{y}$ in preparation for the fracterm calculus. Let Σ_{cm} be $\Sigma_{cr,\bot}$ with this division operator added.

Next, we will add six equations to the axioms $E_{wcr,\bot}$ for rings with $\bot$ to make the set E_{wcm} that almost defines the common meadows. These equations are given in Table 2.

Commentary on the axioms. The axiom system E_{wcm} is an adaptation of the axiom system $\mathrm{Md}_{\mathbf{a}}$ that was proposed in [4]: it is written in division notation rather than in inverse notation, and **a** is renamed into $\bot$; further the axiom $x\cdot\bot = \bot$ has been left out as it is derivable from the axioms in Table 1, an

TABLE 2. E_{wcm}: Weak equations for common meadows.

$$\texttt{import:}\ E_{wcr,\bot}$$
$$\frac{x}{y} = x \cdot \frac{1}{y}$$
$$\frac{x}{x} = 1 + \frac{0}{x}$$
$$\frac{1}{(\frac{1}{x})} = x + \frac{0}{x}$$
$$\frac{1}{x \cdot y} = \frac{1}{x} \cdot \frac{1}{y}$$
$$\frac{1}{1} = 1$$
$$\frac{1}{0} = \bot$$

TABLE 3. E_{ffl}: Equations for fracterm flattening.

$$x = \frac{x}{1}$$
$$-\frac{x}{y} = \frac{-x}{y}$$
$$\frac{x}{y} \cdot \frac{u}{v} = \frac{x \cdot u}{y \cdot v}$$
$$\frac{x}{y} + \frac{u}{v} = \frac{(x \cdot v) + (y \cdot u)}{y \cdot v}$$
$$\frac{x}{(\frac{u}{v})} = x \cdot \frac{v \cdot v}{u \cdot v}$$
$$\bot = \frac{1}{0}$$

observation made by Alban Ponse after [4] was published, and which has been incorporated in the updated version [5] of [4]. (Currently) we call the set E_{wcm} the weak axioms of common meadows as shortly we will add an equation that is needed for a completeness theorem. This commentary is continued in Section 4.2.

The distinction between inverse notation and division notation was discussed in detail in [9] for the case of involutive meadows, where the transformation to division is as follows: add an equation $\frac{x}{y} = x \cdot \frac{1}{y}$, which expresses division in terms of inverse, and replace all occurrences of r^{-1} by $\frac{1}{r}$, and in some cases replace $t \cdot r^{-1}$ by $\frac{t}{r}$.

An explicit construction of an initial algebra of (Σ_{cm}, E_{wcm}) has been given in [10].

In [4] the following is shown (though stated in terms of inverse notation):

Proposition 2.4. *(Fracterm flattening I). For each open expression t over Σ_{cm} there are expressions r and s without an occurrence of division such that*

$$E_{wcm} \vdash t = \frac{r}{s}.$$

2.3. Simpler axioms for fracterm flattening

Flattening is so important in arithmetic that it is wise to find out the basic laws that enable it. Table 3 contains straightforward equations that have been collected with the sole purpose to axiomatize fracterm flattening.

Using the results from [4] we can show that the equations of Table 3 are derivable from the equations E_{wcm}. In fact, it is the case that:

Proposition 2.5. *The equational specifications E_{wcm} and $E_{wcr,\bot} \cup E_{ffl}$ are logically equivalent.*

Proof. This result was obtained by Alban Ponse with the help of the system *Prover9* [11]. ■

In other words the equations of E_{wcm} can be replaced by an extension of the weakened commutative ring equations $E_{wcr,\bot}$ by E_{ffl}, being an equational inductive definition of fracterm flattening plus an explicit definition of the constant $\bot$.

We note that without the presence of the constant name $\bot$ it is still the case that $\frac{0}{0}$ is absorptive so that the weakening (by omitting $x + (-x) = 0$ and adopting $x + (-x) = 0 \cdot x$ instead) of E_{cr} to E_{wcr} is unavoidable. Indeed, w.r.t. values of closed expressions:

$$\frac{0}{0} \cdot \frac{\underline{n}}{\underline{m}} = \frac{0 \cdot \underline{n}}{0 \cdot \underline{m}} = \frac{0}{0} \text{ and } \frac{0}{0} + \frac{\underline{n}}{\underline{m}} = \frac{0 \cdot \underline{m} + \underline{n} \cdot 0}{0 \cdot \underline{m}} = \frac{0}{0}.$$

2.4. Common meadow fracterm calculus

In [6] it was noticed that in order to prove a completeness result for the equational theory of common cancellative meadows of characteristic 0, the axiom

$$\frac{1}{1 + 0 \cdot x} = 1 + 0 \cdot x$$

is needed; this cannot be derived from the equations of E_{wcm} in Table 2. Thus, adding this equation as an axiom, removing redundant axioms, and using a division notation leads to a set of equations we refer to as the *equations of common meadows* and can be denoted E_{cm}. With a focus on syntax rather than on semantics, we present the equations of common meadows of E_{cm} as the equational rules of the common meadow fracterm calculus E_{cmfc} as displayed in Table 4; for this version we use CFC as an abbreviation.

According to [6], the axioms of Table 4, as given in that paper in inverse notation, are logically independent (with the

TABLE 4. E_{cmfc}: Equations of common meadow fracterm calculus.

$$\texttt{import:}\ E_{wcr,\perp}$$
$$\frac{x}{y} = x \cdot \frac{1}{y}$$
$$\frac{x}{x} = 1 + \frac{0}{x}$$
$$\frac{1}{x \cdot y} = \frac{1}{x} \cdot \frac{1}{y}$$
$$\frac{1}{1 + 0 \cdot x} = 1 + 0 \cdot x$$
$$\frac{1}{0} = \perp$$

exception of $0{\cdot}x = 0{\cdot}(x{\cdot}x)$ that comes with $E_{wcr,\perp}$ while it is not mentioned in the version of the axioms $\mathrm{Md}_{\mathbf{a}}$ from [6]). Indeed, although Table 4 makes use of an attractive modularization of the equational axioms, the preference for having $E_{wcr,\perp}$ as a modular component for which a meaningful completeness result (i.e. Proposition 2.1) can be shown leads to the inclusion of $0 \cdot x = 0 \cdot (x \cdot x)$ in $E_{wcr,\perp}$, while, remarkably, by including further equations about division, that same equation becomes derivable.

A version of these axioms in division notation is given in Table 4, the equations of which correspond to the equations of common meadows as given in [6] (see Table 2 of [6]) though written in division notation. Following [9] the step from inverse notation (as in Table 2 of [6]) to division notation is made by introducing division defined by $\frac{x}{y} =_{def} x \cdot y^{-1}$, and writing $x^{-1} = \frac{1}{x}$.

PROPOSITION 2.6. *In E_{cmfc}, the equation $0 \cdot x = 0 \cdot (x \cdot x)$ is derivable from the other equations, and the other equations of E_{cmfc} are logically independent.*

Proof. This was established by Alban Ponse with the *Mace4* tool (see [11]) as reported in [6] for the table of equations in inverse notation and subsequently for the equations of Table 4. ■

Finally, we repeat the flattening property that is shown in [6]:

PROPOSITION 2.7. *(Fracterm flattening II). For each CFC expression t there are CR expressions (i.e. CFC expressions without occurrences of the division operator) r and s such that*

$$E_{cmfc} \vdash t = \frac{r}{s}.$$

In any case, allowing fracterm flattening has been the main design requirement for the common meadow fracterm calculus. Another design requirement for the common meadow fracterm calculus has been that its equations axiomatize the equational theory of a meaningful class of arithmetical datatypes, which we discuss next.

3. AN INCOMPLETENESS RESULT FOR CFC

The term meadow reflects the concept's intimate relationship with fields (see [1]). Although meadows are a wider class of algebras, the following property brings them closer: A meadow M is cancellative if for all $x, y \in M$,

$$x \neq 0 \wedge x \neq \perp \implies \frac{x}{x} = 1.$$

A common cancellation meadow is a common meadow which is the enlargement of a field with division and $\perp$.

In [6] it was shown that E_{cmfc} together with the equations

$$\chi_0 = \left\{ \frac{\underline{n}}{\underline{n}} = 1 | \in \mathbb{N}, n > 0 \right\}$$

axiomatizes all the equations true in all common cancellation meadows with characteristic 0.

Many conditional equations can be derived from these equations, however we will show that $E_{cmfc} \cup \chi_0$ does not completely axiomatize the conditional equational theory of common cancellation meadows.

We will focus on the following conditional equation ϕ:

$$\phi \equiv \frac{1}{x} = \perp \rightarrow 0 \cdot x = x$$

which is trivially valid in all common cancellation meadows, because only 0 and $\perp$ will meet the premise of the conditional equation in such structures.

THEOREM 3.1. *The conditional equation ϕ is not derivable from $E_{cmfc} \cup \chi_0$; hence $E_{cmfc} \cup \chi_0$ does not axiomatize the conditional equational theory of common cancellation meadows of characteristic* 0.

Proof. Suppose for a contradiction that $E_{cmfc} \cup \chi_0 \vdash \frac{1}{x} = \perp \rightarrow 0 \cdot x = x$. Then for some $K \in \mathbb{N}$,

$$E_{cmfc} \cup \left\{ \frac{\underline{n}}{\underline{n}} = 1 | \in \mathbb{N}, 0 < n < K \right\} \vdash \frac{1}{x} = \perp \rightarrow 0 \cdot x = x.$$

We will show that such a number K does not exist.

To this end choose a prime number p such that $p > K$. Let H be the ring $\mathbb{Z}/p^2\mathbb{Z}$. Upon extending the signature with a new constant $\perp$, the interpretation of which will be denoted with $\perp$ as well, H is enlarged to $\mathsf{Enl}_\perp(H)$. We notice that $\mathsf{Enl}_\perp(H)$ satisfies the first nine equations of Table 4. Next, a division function is added to the signature and an expansion of $\mathsf{Enl}_\perp(H)$ to $\mathsf{Enl}_\perp(H)(-)$ is obtained.

For that purpose the interpretation of the division function is given by $\frac{x}{y} = x \cdot \frac{1}{y}$ while choosing inverses $\frac{1}{x}$ as follows: $\frac{1}{0} = \frac{1}{\perp} = \perp$; if n is a multiple of p then $\frac{1}{n} = \perp$; otherwise if $\gcd(n, p) = 1$ then find $m < p^2$ such that $n{\cdot}m = 1 \mod p^2$ and let $\frac{1}{n} = m$. We notice that m is unique: suppose that for some $u < p^2, u \neq m, n \cdot u = 1 \mod p^2$. W.l.o.g. we may assume that $m > u$. Then $n \cdot m - n \cdot u = n \cdot (m-u) = 0 \mod p^2$ and it follows that n must have a factor p because $m - u < p^2$ and $m - u$ cannot be a multiple of p^2.

We will check that $\mathsf{Enl}_\bot(H)(–)$ satisfies the other equations of $E_{cmfc} \cup \{\frac{n}{\underline{n}} = 1 | \in \mathbb{N}, 0 < n < K\}$. For $0 < n < K, 0 < n < p$ and thus $\gcd(n,p) = 1$ so that $\mathsf{Enl}_\bot(H)(–) \models \frac{n}{\underline{n}} = 1$. For the equations about division of E_{cmfc} all cases are immediate except for $\frac{1}{x \cdot y} = \frac{1}{x} \cdot \frac{1}{y}$. By distinguishing four cases ($\gcd(n,p) = 1$ or not and $\gcd(m,p) = 1$ or not) it is easily checked that for all n, m with $0 < n, m < p^2$, $\mathsf{Enl}_\bot(H)(–) \models \frac{1}{\underline{n} \cdot \underline{m}} = \frac{1}{\underline{n}} \cdot \frac{1}{\underline{n}}$. ■

Question. Does the conditional theory of common meadows of characteristic 0 have an attractive axiomatization?

4. CONCLUDING REMARKS

Partial data types may be bearable in system design and specification but total data types are needed in implementations. Totalizing operators that are partial is complicated semantically but any method must allow algebraic and logical analyses to be useable and safe. The apparently simple method of using an absorbtive element $\bot$ to make division total leads to subtle semantical consequences for other arithmetic operations; it also leads to a forensic interest in designing new equational specifications to capture and control them.

4.1. Summary

We have re-examined the axioms for common meadows and tried to provide a modular construction that allows the theory to be re-focussed or customized for various agendas.

We started the modularization with the equational theory $E_{wcr,\bot}$ of commutative rings with an absorbtive flag $\bot$ added. With this base, we focused on additional axioms designed to accommodate division. To illuminate elementary arithmetic, the set E_{wcm} of axioms and its equivalents were shown to be enough to enable flattening. Adding one more axiom produced E_{cm} and its form as the common meadow fracterm calculus E_{cmfc}. This last axiomatization enables a completeness theorem that axiomatizes the whole equational theory of common meadows of characteristic 0. However, we showed that E_{cmfc} does not axiomatize the conditional equational theory.

4.2. On axiomatizing the common meadows

This paper is about axiomatizations. All established classes of algebras have several axiomatizations and closely related weaker formulations. These are acquired as their theories develop, often over many years – one thinks of the theory of groups, rings and fields, and their many derived classes of algebras. Indeed, they are still developing as our work on meadows of various kinds demonstrates.

Since their proposal in [4], the common meadows have found several applications and with each application come new opportunities for developing their theory, which includes refinements of their axiomatizations. Criteria that shape our development so far have been

(i) equational axiomatizations,
(ii) initial algebra models of the rationals,
(iii) classical properties and practices with calculating with fractions and
(iv) logical issues of independence, completeness, and incompleteness.

This paper brings into focus the effects of our theoretical development on finding and comparing axioms. To be concrete, let us explain the connection of our paper with three key papers [4–6], which are three versions of [4] which first appeared in 2014. Successively, the axioms for common meadows stated in [4] were first simplified thereby obtaining an ‘intermediate’ axiom system by removing a redundant axiom, and then by incorporating a new axiom leading to the system presented in [5]. Our Table 4 corresponds to the axioms of [5] though here rendered in division rather than inverse notation.

It went unnoticed in [5] that there is indeed a clear rationale for the intermediate system of axioms, which we display in Table 2, again using division notation. This rationale of the intermediate system is brought to light in Table 3, which contains an axiom system logically equivalent to Table 2 (i.e. to the intermediate axiomatization). Now, Table 3 provides precisely the most straightforward inductive definition of fracterm flattening plus a definition of $\bot$ as being 1/0. We hold that Table 3, and because of its logical equivalence Table 2 as well, captures the idea of fracterm flattening. The importance of fracterm flattening in theory and practice constitutes a significant rationale for its independent status and study. The choice of operators – division *v.* inverse – and their notations also influences investigations and hence axiomatizations.

4.3. Division by zero as a context

The starting point for the theory of common meadows in all its possible forms is the need to make division total. The idea of adding a flag is not new as can be seen in antique algebra textbooks and modern computer arithmetics, where ∞ is regularly used, or in pocket calculators, where *error* is regularly used. Absorbtive properties of flags can be found in *ad hoc* places in computing, e.g. where *error* flags are raised and $x + error = error$ etc. What has been missing are pukka algebraic theories of ideas about totalization that can at least

(i) specify a method axiomatically;
(ii) map the algebraic consequences and
(iii) support verifiable computations.

This is quite a long-standing problem for arithmetic data types and numerical computation.

There is an interesting survey of options for division by zero by Patrick Suppes as early as 1957 in [12] (see also [13]). Suppes approved of $\frac{1}{0} = 0$, though writing $\frac{1}{0} = \bot$ is not among his options.

Our pursuit of a theory of totalization through a theories of meadows started without flags [1]. We worked with $\frac{1}{0} = 0$ in order to give an equational specification of the rationals with division using initial algebra semantics; for this purpose we coined the term *involutive meadows* since inverse is an involution. Earlier algebraic work with this option occurs in [14] and [15] where the structures are called pseudo fields. The same conventions are referred to as a division by zero calculus in [16].

A corresponding fracterm calculus is called *Suppes-Ono fracterm calculus* in [17]. Now, fracterm flattening *fails* in involutive meadows. Taking $\frac{1}{0} = 0$ in the Suppes-Ono fracterm calculus allows only quasi-flattening, i.e. an expression can be written as a sum of flat fracterms, without any bound on the length of the sum; see [18, 19]. We notice that for Suppes-Ono fracterm calculus an initial algebra has been identified in [20], while the finite models of it were characterized in [21]. (A characterization of finite models of common meadow fracterm calculus has not yet been determined.) Moreover, Suppes-Ono fracterm calculus allows an attractive equational presentation of the axioms of probability calculus, see e.g. [22].

There are other fracterm calculi. In the algebraic study of a computer arithmetic the infinity flags ∞ (unsigned) and $+\infty, -\infty$ (signed) have been used to totalize division by zero, in the notions of *wheels* [23–25] and *transreals* [26–28], respectively. These semantic models also contain $\perp$ to control the side effects of the infinities. (In the theory of transreals, the absorbtive constant $\perp$ is written as Φ and called nullity.)

The fracterm calculus of transreals is relatively complicated in comparison with common meadow fracterm calculus and Suppes-Ono fracterm caclulus. For instance, the equation

$$\frac{x}{z} + \frac{y}{z} = \frac{x+y}{z},$$

referred to as the property of quasicardinality in [29], is valid both for the common meadow fracterm calculus as well as for Suppes-Ono fracterm calculus. However, this quasicardinality *fails* for the fracterm calculus of wheels and for the fracterm calculus of transreals.

From the point of view of fracterms, and our expectations born of elementary teaching, the common meadow fracterm calculus stands out.

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