UNIVERSITÀ DEGLI STUDI DI PADOVA

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Final dissertation

Scalar-tensor mode mixing in higher-order cosmological perturbations

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Contents

Notation

In this whole thesis project some notation will be used and it is explained in this small section.

- In the Einstein summation convention, Greek letters range from 0 to 3 and Latin indices are from 1 to 3

$$T^{\mu}_{\mu} = T^{0}_{0} + T^{1}_{1} + T^{2}_{2} + T^{3}_{3} \tag{1}$$

$$T^{i}_{i} = T^{1}_{1} + T^{2}_{2} + T^{3}_{3}, \tag{2}$$

  so the delta tensor

$$\delta^{\mu}_{\mu} = 4 \tag{3}$$

- Mostly positive signature in the metric $(-, +, +, +)$

- The derivative of a tensor will be often written as a comma in the indices

$$\partial_{\mu} T^{\nu\rho}_{\lambda} = T^{\nu\rho}_{\lambda,\mu} \tag{4}$$

- Derivative with respect to time t will be denoted with an overdot $\partial_t \bullet = \dot{\bullet}$; derivative with respect to conformal time η will be denoted with a prime $\partial_\eta \bullet = \bullet'$

1

Introduction

Cosmology ... restrains the aberrations of the mere undisciplined imagination.

Whitehead, A. N., *The Function of Reason*, 1929

The "standard model" of Cosmology is based on the assumption that the Universe is on average spatially homogeneous and isotropic. Thus it is based on a spatially homogeneous and isotropic Friedman-Robertson-Walker (FRW) model. Starting from this flat background it is possible to recover the expansion of the Universe on large scales, according to Einstein's theory of General Relativity.

If we consider stars, galaxies and clusters, forming a complex, inhomogeneous and anisotropic universe, we need to go beyond the FRW model: we should, in principle find anisotropic and inhomogeneous solutions of Einstein's equations. This is done by introducing cosmological perturbation theory, which is a perturbative approach adding, order by order, small perturbations to the background solution.

The theory of cosmological perturbations has become a very important subject of modern Cosmology because it allows to link the models of the very early Universe, such as the inflationary scenario, with the massive high-precision data on the Cosmic Microwave Background radiation, on large-scale structures and future data on the primordial Gravitational-Wave stochastic background [4, 21].

When working with perturbations within General Relativity a difficulty arises: we have to deal with perturbed fields in a given geometry, but also the geometry itself is perturbed. We then have to assign a mapping between points in the inhomogeneous Universe and points in the given known homogeneous background. The mapping is dubbed "gauge" and the freedom in this choice is called gauge freedom, leading to different expressions of the same phenomenon simply due to different gauge choices [13, 26].

In my thesis project I start recalling the basis of the FRW background, recovering the expansion and evolution equations. Then, I move forward to perturbation theory, describing the perturbed metric at first and second order. In the general FRW perturbed metric it is possible to characterize scalars, vector and tensor perturbation according to how they transform under coordinate transformation of the background. We have scalars, vector and tensor perturbations. Then, we can decompose those perturbation using Helmholtz theorem: vectors can be decomposed into scalars and true (zero-divergence) vector components; tensors can be decomposed into scalar, vector (zero-divergence) and tensor (transverse and traceless) components. Thus, ending with 4 scalars, 2 vectors and one tensor (GW).

Then, a part will be dedicated to the gauge issue, finding the first and second-order gauge transformation for all the scalar, vector and tensor quantities appearing in the perturbed metric. As done in literature, one can find that the first-order tensor perturbation are gauge-invariant but when one goes to second order, gravitational waves are no longer gauge invariant and the specific form of these waves is gauge-dependent. A description of the main gauge choices is done, focusing on the Poisson or longitudinal gauge, which is the one used in the calculations.

After this introductory section, I start recovering the first-order results: scalar and tensor perturbations evolve independently, thus I can easily study and write Einstein's equations for scalars and then for tensors. First order vectors are neglected due to the fact that, if generated, they are fast redshifted away with the expansion of the Universe. So, the equations governing the evolution of these quantities are obtained, recovering the results in the literature. When one goes to second order, computations start to be more complex, revealing the underlying non-linearity of Einstein's equations. For the first time, the second-order perturbed metric is obtained directly in the Poisson gauge, with scalars at first and second order, vectors at second order, tensors at first and second order. With this choice, I show that we have second-order mixed terms which source scalar and tensor perturbations. Namely, second-order scalar modes are sourced by first-order scalars coupled with first-order tensors, by two coupled first-order scalar and two first-order tensors. The same problem has been discussed in [8] and in [7] but in a different gauge. Now, Einstein's equations start to be more complicated and finding solutions is not so easy.

Starting from computing the second-order perturbed quantities such as the Ricci tensor and the Ricci scalar, the second order energy-momentum tensor, from the traceless $i - j$ component of the Einstein field equation I can find the difference between the second order scalar perturbations, which as a first result is non zero even for a perfect fluid in absence of an anisotropic stress tensor, in contrast with the first order result, which gives $\psi = \phi$ for a perfect fluid. With this equation in hand, I derived the other Einstein equations and the continuity equation to close the system for the scalar quantities ψ_2, ϕ_2, $\delta^2\rho$ and $\delta^2 v$.

The following part is focused on writing the Boltzmann equations at second order in order to study the evolution of particle species, such as photons, baryons and cold dark matter. In this section I write the Boltzmann equation accounting for first and second order scalar perturbation, second order vector perturbation, first and second order tensor perturbation. The main difference w.r.t. the literature here is that first-order tensor modes had not been included when going to second order [31, 30], but it is interesting to study the coupling of these tensor modes with first-order scalars or other first order tensors. One can find the solution of the second-order collisional Boltzmann equation for photons, which can lead to second-order anisotropies in the CMB, e.g. the level of non-Gaussianity, to help in the hard task of discriminating among different mechanisms for the generation of the primordial perturbations. The same formalism can be later applied to derive the Boltzmann equation for other particle species, such as baryons and cold dark matter (CDM). Studying the evolution of cold dark matter is a very important topic because it plays a fundamental role in structure formation.

In conclusion the main goal of this project is to add an original contribution to the second-order evolution of scalar quantities, in Poisson gauge, such as the gravitational potentials, the density contrast and the velocity of baryons and CDM, starting from the perturbed expression of the metric through the Einstein and Boltzmann equations, considering as non-negligible the contributions from first-order tensor modes which can be coupled to themselves and to other first-order scalar modes.

2
The standard cosmological lore

From my point of view one cannot arrive, by way of theory, at any at least somewhat reliable results in the field of cosmology, if one makes no use of the principle of general relativity.

Albert Einstein, *Reply to criticisms*, 1949

On large scales, observations reveal that our observable Universe is on average homogeneous and isotropic, but on scales much shorter we can see clumps of matter forming stars, galaxies and clusters of galaxies. But on a large-scale viewpoint there is no difference on average if we look to a portion of the universe of order larger than ~ 100 Mpc centered on the Earth and other volumes with the same size centered elsewhere. The fact that our observable universe is on average homogeneous and isotropic by no means implies that the *entire* universe is also smooth, but we can assume it to be spatially homogeneous and isotropic in order to describe our local Hubble volume.

The metric for a space with those characteristics of homogeneity and isotropy is the maximally-symmetric Friedmann-Robertson-Walker (FRW) metric, which can be described by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{2.1}$$

where t is the cosmic time, namely the time coordinate of an observer comoving with the cosmic fluid; $a(t)$ is the scale factor which spans the expansion of the universe; $d\Omega^2$ is the infinitesimal solid angle and the curvature k can be either positive, null or negative

$$k = \begin{cases} -1 & \text{open} \\ 0 & \text{flat} \\ +1 & \text{closed} \end{cases} \tag{2.2}$$

2.1 Friedmann equations

The dynamic of the smooth Universe is described by the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{2.3}$$

where on the left-hand side $R_\mu\nu$ is the Ricci tensor, $g_{\mu\nu}$ is the metric and R is the Ricci scalar, on the right-hand side we have the Newton constant G and the energy-momentum tensor $T_{\mu\nu}$. The Ricci tensor is a contraction of the first and third indices of the Riemann tensor which is

$$R^{\lambda}_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\sigma\nu} - \partial_\nu \Gamma^\lambda_{\sigma\mu} + \Gamma^\rho_{\nu\sigma}\Gamma^\lambda_{\mu\rho} - \Gamma^\rho_{\mu\sigma}\Gamma^\lambda_{\nu\rho}, \tag{2.4}$$

so the Ricci tensor is

$$R_{\sigma\nu} = \partial_\lambda \Gamma^\lambda_{\sigma\nu} - \partial_\nu \Gamma^\lambda_{\sigma\lambda} + \Gamma^\rho_{\nu\sigma}\Gamma^\lambda_{\lambda\rho} - \Gamma^\rho_{\lambda\sigma}\Gamma^\lambda_{\nu\rho}. \tag{2.5}$$

The Ricci scalar R is defined as the contraction of the two indices of the Ricci tensor

$$R = R^\alpha_\alpha, \tag{2.6}$$

and in conclusion the Christoffel symbols are defined as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}\left(g_{\mu\lambda,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}\right) \tag{2.7}$$

The Einstein equations (2.3) can be derived from an action principle through the action

$$S = \frac{1}{16\pi G}\int R\sqrt{-g}\,d^4x + S_{\text{matter}}, \tag{2.8}$$

where g is the determinant of the metric tensor.

The RHS of (2.3) is proportional to the energy momentum tensor, which is for a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{2.9}$$

where ρ and p are the energy density and pressure of the fluid and u^μ is the 4-velocity, such that $u^\mu u_\mu = -1$.

Since current observations are consistent with a flat ($k = 0$) universe, which is also in agreement with inflation, we adopt this choice and so write the line element as

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{2.10}$$

We need to quantify the expansion rate of the universe so we introduce the Hubble parameter, $H(t)$ defined as

$$H(t) = \frac{da/dt}{a} = \frac{\dot{a}}{a}. \tag{2.11}$$

It is more convenient to introduce a new quantity which is the conformal time coordinate η defined as

$$d\eta = \frac{dt}{a} \tag{2.12}$$

and write the new conformal metric as

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \delta_{ij}dx^i dx^j\right]. \tag{2.13}$$

The new conformal Hubble parameter is

$$\mathcal{H} = \frac{da/d\eta}{a} = \frac{a'}{a}. \tag{2.14}$$

In the case of interest of the FRW universe, we can compute the Christoffel symbols, and the only non vanishing ones are

$$\Gamma^0_{00} = \mathcal{H}, \quad \Gamma^0_{ij} = \mathcal{H}\delta_{ij}, \quad \Gamma^i_{0j} = \mathcal{H}\delta^i_j. \tag{2.15}$$

With these in hand we can compute the Ricci tensor and the Ricci scalar,

$$R_{00} = -3\mathcal{H}' \tag{2.16}$$

$$R_{ij} = \left[2\mathcal{H}^2 + \mathcal{H}'\right]\delta_{ij} \tag{2.17}$$

$$R = 6a^{-2}\left[\mathcal{H}^2 + \mathcal{H}'\right], \tag{2.18}$$

we plug everything into Einstein equations (2.3) and we have

$$\mathcal{H}^2 = \frac{8\pi G}{3}\rho a^2, \tag{2.19}$$

$$\mathcal{H}' = -\frac{4\pi G}{3}(\rho + p)a^2. \tag{2.20}$$

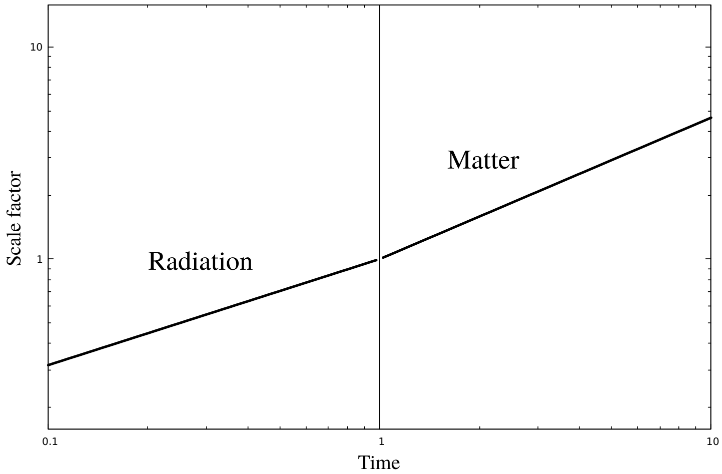


Figure 2.1: Evolution of the scale factor a as a function of time during radiation and matter domination epochs. The unity on time axis is the equivalence epoch.

From the energy conservation equation

$$\nabla_\nu T^{0\nu} = 0 \tag{2.21}$$

we find

$$\rho' = -3\mathcal{H}(\rho + p). \tag{2.22}$$

To close the system of these three equations, one needs an extra relation linking p and ρ: the equation of state $P(\rho)$. The simplest possibility is to have

$$p = \omega\rho, \tag{2.23}$$

where ω can be parametrized as

$$\omega = \begin{cases} 0 & \text{dust} \\ \frac{1}{3} & \text{radiation} \\ -1 & \Lambda \end{cases} \tag{2.24}$$

Now we can solve the system and we find the usual relations (plot in Figure 2.1)

$$a(t) = a_*(t)\left(\frac{t}{t*}\right)^{2/3(1+\omega)} \Longrightarrow a(t) \propto \begin{cases} t^{1/2} & \text{radiation} \\ t^{2/3} & \text{matter} \end{cases} \tag{2.25}$$

The same can be done for the energy density ρ, resulting in

$$\rho = \rho_* \left(\frac{a}{a_*}\right)^{-3(1+\omega)} \implies \rho \propto \begin{cases} a^{-3} & \text{radiation} \\ a^{-4} & \text{matter} \\ \text{constant} & \Lambda \end{cases} \tag{2.26}$$

2.2 Inflation

Before conluding a small and trivial introduction on standard cosmology, one should not forget about a short but necessary discussion on the inflationary paradigm. Following [1] we can say that the standard cosmology, as happens in particle physics, has its own shortcomings, which are inconsistent with the hot big bang model, invoking questions that the model allows to ask but lacks of answers.

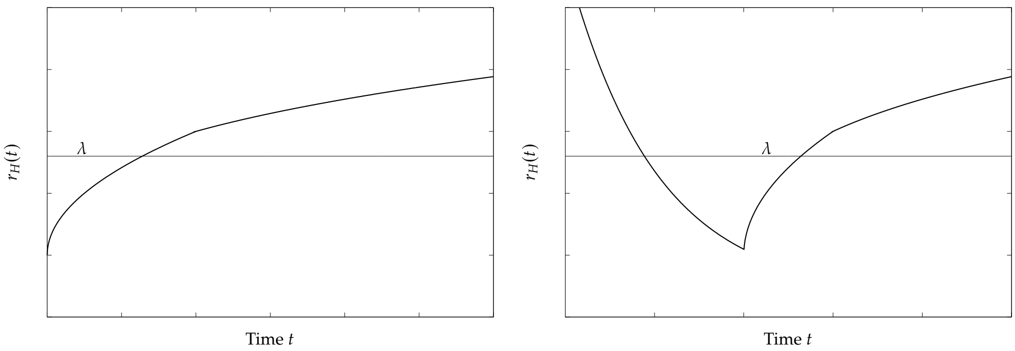
The first "problem" is the so-called *Horizon problem*, connected to the Cosmic Microwave Background (CMB) radiation. Firstly we define the comoving Hubble radius as

$$r_H = \frac{1}{\dot{a}(t)} \tag{2.27}$$

which is always growing, since $\dot{r}_H > 0$ because the acceleration of the expansion is negative for a fluid with $\omega > -1/3$.

Here a problem arises: when we look at the CMB we can observe regions that have the same statistical properties, such as the temperature T up to small fluctuations, without having been in causal contact before. Experimentally we see that two patches of the CMB shares the same temperature of $T_{CMB} \approx 2.7$ K with only small deviations $\delta T/T \approx 10^{-5}$. So, how can this be possible? There can be two solutions: the first one is to assume that our universe was born homogeneous and isotopic, but this solution is not satisfactory, because we want to explain the evolution of the universe, starting from the most general initial conditions.

The second solution is to modify our beloved hot Big Bang model. We need to push our singularity to $-\infty$ and call this period *inflation*, during which the comoving Hubble radius decreased causing a lot of scales to exit the horizon and to re-enter when r_H started to grow again. Now, all the scales that exited and re-entered thereafter, have been in causal contact before inflation, hence they can share similar statistical properties. Since the largest scale we can probe is the CMB, we can safely say that all the patches were before causally connected and we have no problems with causality.



(a) Without inflation (b) With inflation

Figure 2.2: Comoving Hubble radius as a function of time and comoving wavelength λ. In the left panel we see the evolution in the case without inflation. In the right panel we have inflation, in which r_H decreases. The perturbation λ was inside the horizon, then it crossed for the first time exiting and re-entering a second time.

A second shortcoming of the "old" Big Bang model is the *flatness problem*. This is a fine-tuning problem and it can be easily understood considering the value of the density of matter and energy in the universe. We have that the quantity $\Omega - 1$ is very close to zero with small deviations: at 95% CL we know that $|\Omega - 1| < 0.4\%$. At Planck epoch we have that

$$|\Omega(t_{pl}) - 1| < 10^{-62} \tag{2.28}$$

and we can see that we have a big problem of fine-tuning of this specific parameter. How does inflation solve this problem? We can see that

$$\frac{|\Omega - 1|_{tf}}{|\Omega - 1|_{t_i}} \sim \left(\frac{a_i}{a_f}\right)^2 \sim \exp\{-2N\}, \tag{2.29}$$

where N is the number of e-folds. So, for some big enough values of N we can see that the quantity $|\Omega - 1|$ at the end of the inflation will be close to zero even is at the beginning of the inflation it wasn't zero. Thus we say that the inflationary solution is an *attractor solution*.

In the context of the modern unified gauge theories we find an interesting variety of superheavy, stable particle that should have been generated during the early Universe. These particle species, surviving annihilation processes, would contribute quite generously to the present energy density, i.e. $\Omega_{0X} > 1$, overclosing the Universe. For example, using the Kibble mechanism, the expected relic abundance of monopoles is

$$\Omega_{0,\text{mono}} \sim 10^{11} \left(\frac{T_{GUT}}{10^{14}\text{GeV}} \right)^3 \left(\frac{m_{\text{mono}}}{10^{16}\text{GeV}} \right), \tag{2.30}$$

far too big. These unwanted relics can be massive particles or other topological defects, such as cosmic strings, which are mono-dimensional defects, arising from the spontaneous breaking of the U(1) symmetry; two-dimensional defects are the so-called domain walls generated during phase-transitions. The energy density of these *unwanted* relics must be, in some way, redshifted away during the expansion of the Universe and the inflationary solution has the power of solve this problem. However, inflation is not discerning, and in doing so any trace of radiation or dust-like matter is similarly redshifted away to nothing [14].

2.2.1 Inflationary solution

In order to have a positive acceleration of the expansion, we need to have

$$w < -\frac{1}{3} \tag{2.31}$$

and a proper candidate to drive inflation with this requirement is a simple real scalar field φ as a function of time and space. The Lagrangian of a real scalar field is of the form

$$\mathcal{L}_\varphi = -\frac{1}{2} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} - V(\varphi). \tag{2.32}$$

We can compute the energy momentum tensor from the action S, defined as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}, \tag{2.33}$$

where g is the determinant of the metric and S is the action of the scalar field

$$S = \int d^4x \sqrt{-g} \mathcal{L}. \tag{2.34}$$

With our choice of the Lagrangian we have

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi + g_{\mu\nu} \mathcal{L}. \tag{2.35}$$

The scalar field can be decomposed into a background value and into a small perturbation

$$\varphi(x, t) = \varphi_0(t) + \delta\varphi(x, t) \tag{2.36}$$

and focusing on the background part we have

$$\rho = \frac{1}{2}\dot{\varphi}_0^2 + V(\varphi_0) \tag{2.37}$$

$$P = \frac{1}{2}\dot{\varphi}_0^2 - V(\varphi_0). \tag{2.38}$$

From the variational principle we can find the Klein-Gordon equation for this scalar field in an expanding universe

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\nabla^2\varphi = -\frac{\partial V}{\partial \varphi}, \tag{2.39}$$

where $3H\dot{\varphi}$ is a new term which accounts for the expansion of the Universe.

We can satisfy our condition on the equation of state for example if we impose that

$$\dot{\varphi}^2 \ll V(\varphi), \tag{2.40}$$

which is the *slow-roll* motion. The most relevant equations become

$$H^2 = \frac{8}{3}\pi G V(\varphi_0) \tag{2.41}$$

$$3H\dot{\varphi}_0 = -\frac{\partial V}{\partial \varphi_0}. \tag{2.42}$$

Now we can introduce two parameters, called slow-roll parameters ϵ and η

$$\epsilon = \frac{-\dot{H}}{H^2} \tag{2.43}$$

$$\eta = -\frac{\ddot{\varphi}}{H\varphi} \tag{2.44}$$

and in order inflation to occur we need to impose them to be much smaller than 1 and under this conditions it is called *slow-roll* inflation. Inflation ends when ϵ is almost unity.

As we said, the inflaton field φ can be decomposed into a background component to which we add a small perturbation. This perturbation will obviously satisfy the Klein-Gordon equation

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \frac{\nabla^2\delta\varphi}{a^2} = -\frac{\partial^2 V}{\partial \varphi^2}\delta\varphi. \tag{2.45}$$

We can quantize the scalar field and then we can rewrite the KG equation replacing $\delta\varphi$ with the mode function u_k from the rescaled quantized scalar field. Our new equations reads

$$u_k'' + \left[k^2 - \frac{a''}{a} + a^2\frac{\partial^2 V}{\partial\varphi^2}\right]u_k = 0 \tag{2.46}$$

and in the case of a massless scalar field $\partial^2 V/\partial\varphi^2 = 0$ and in pure de-Sitter with H constant we can solve the equation into two scenarios: sub-horizon and super-horizon. In the sub-horizon regime we can see that $k \gg H$ and we have

$$u_k'' + k^2 u_k = 0, \tag{2.47}$$

which is an harmonic oscillator with solution

$$u_k = \frac{1}{\sqrt{2k}}e^{-ik\tau}. \tag{2.48}$$

In the super-horizon regime we can neglect k^2 since $k \ll H$. Our KG equation for the mode function then is

$$u_k'' - \frac{a''}{a}u_k = 0 \tag{2.49}$$

with a growing solution proportional to a, namely

$$u_k = B(k)a, \tag{2.50}$$

which tells us that on super-horizon there are no causal connections and thus the amplitude of the perturbation to the scalar field remains constant. Matching the two solutions at horizon crossing we find that the value of the amplitude is

$$|\delta\varphi_k| = \frac{H}{\sqrt{2k^3}} \tag{2.51}$$

constant on super-horizon scales.

3

Cosmological Perturbations

[...] the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible [...]

Sachs and Wolfe, 1967

The Universe in its totality, on large scales, as we already know, is considered to be homogeneous and isotropic on average, but the presence of structures such as stars, galaxies and clusters makes the Universe not so homogeneous. We should go beyond the FRW model: either finding a new model which accounts for anisotropies and inhomogeneities or perturbing our simple FRW one. In this chapter I will firstly tackle the problem of interest, focusing on a deeper problem which arises when dealing with General Relativity and perturbations: *the gauge issue*. Later on I will introduce perturbations and compute perturbations up to second order.

3.1 The gauge problem

Let us start from an ideal universe $\bar{S}$, where all the quantities are denoted by an overbar, so for example the metric tensor will be $\bar{g}_{\mu\nu}$ and the energy density $\bar{\rho}$. Now we perturb this smooth universe to obtain a new S model in which the metric is $g_{\mu\nu}$. The perturbation $\delta g_{\mu\nu}$ is thus defined as the difference between the metric in the pertubed spacetime and the metric in the ideal smooth universe

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}. \tag{3.1}$$

We are then trying to compare two tensors which live in two different spacetime: it is a known fact of differential geometry that, in order to be meaningful at all, the tensors must be considered at the same point. We need to find a prescription, a one-to-one map to identify points of these two spacetimes: this map is called *gauge* and a change in this mapping is denoted *gauge transformation*. The problem is that one has the freedom to choose this gauge and this freedom gives rise to the arbitrariness of the value of the perturbation at any given spacetime point [13].

Another way to state the same problem is to try to answer to this question: if we don't know how the model $\bar{S}$ was used to make the construction, *can we uniquely recover $\bar{S}$ from S?* [28]. The answer is no because we can't uniquely determine the background quantities from the realistic universe. We need to define, as before, a map ϕ from $\bar{S}$ to S and the perturbations depend on the choice of this not unique map.

Let us suppose that we have a family of manifolds $\mathcal{M}_\lambda$ where λ identifies the order of the perturbation: $\lambda = 0$ is the background, $\lambda = 1$ the first order and so on. On $\mathcal{M}_\lambda$ live tensors, such as the metric $g_{\mu\nu(\lambda)}$, and we need to confront these tensorial quantities with the ones on the background $\mathcal{M}_0$: the one-to-one mapping are denoted as ψ_λ or ϕ_λ. The choice of this function is the gauge choice and a change in this correspondence, keeping the background space-time fixed, is a gauge transformation [28]. Following the Bardeen definition [26]

> A change in the correspondence, keeping the background coordinates fixed, is called a *gauge transformation*, to be distinguished from a coordinate transformation which changes the labeling of points in the background and physical spacetime together. [...] Thus, even if a quantity is a scalar under coordinate transformations, the value of the *perturbation* in the quantity will *not* be invariant under gauge transformations if the quantity is nonzero and position dependent in the background.

Following [13], we consider a point p in the background $\mathcal{M}_0$, with coordinates $x^\mu(p)$ and choose a gauge defined by a function ψ_λ. In this way we have a new point o which is the correspondence of the point p through the mapping induced by ψ_λ

$$o = \psi_\lambda(p). \tag{3.2}$$

We could also have used a different gauge, say φ_λ and consider o as the as the point of $\mathcal{M}_\lambda$ which corresponds to a new point q of the background

$$o = \varphi_\lambda(q). \tag{3.3}$$

We start from the point p on $\mathcal{M}_0$, reach the point o on $\mathcal{M}_\lambda$ through ψ_λ and eventually we go back to the background to the point q with φ_λ^{-1}. The overall gauge transformation can be seen as a one-to-one correspondence between different points in the background since

$$q = \Phi_\lambda(p) := \varphi_\lambda^{-1}(\psi_\lambda(p)), \tag{3.4}$$

such transformation is called *active coordinate transformation*. So we can see that the point q is at a parametric distance ξ from p where ξ is defined as

$$\frac{dx^\mu}{d\lambda} = \xi^\mu, \tag{3.5}$$

having the transformation

$$x^\mu(q) = x^\mu(p) + \lambda\xi^\mu + \dots, \tag{3.6}$$

which is the usual infinitesimal point transformation. We can now define a new system of coordinates y^μ such that $y^\mu(q) := x^\mu(p)$, from (3.6) we have easily

$$\begin{aligned} y^\mu(q) := x^\mu(p) &= x^\mu(q) - \lambda\xi^\mu(x(p)) + \dots \\ &\simeq x^\mu(q) - \lambda\xi^\mu(x(q)) + \dots, \end{aligned} \tag{3.7}$$

where in the last equality we Taylor expanded to first order in ξ. Equation (3.7) is the so called *passive approach*.

To be more precise, consider now a vector field Z with components Z^μ in the x-coordinate system. Define a new vector field $\tilde{Z}$, with components $\tilde{Z}$ in the x-coordinates, such that these components evaluated at the coordinate point $x^\mu(p)$ are equal to the components Z'^μ the "old" vector field Z had in the y-coordinates at the coordinate point $y^\mu(q)$. So we have the relation

$$\tilde{Z}^\mu(x(p)) = Z'^\mu(y(q)) = \left.\frac{\partial y^\mu}{\partial x^\nu}\right|_{x(q)} Z^\nu(x(q)). \tag{3.8}$$

Looking at the first and last term in the equation is the "active approach" looking at the last two terms is the "passive approach".

The Jacobian is

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu - \lambda\xi^\mu_{,\nu} \tag{3.9}$$

and Taylor expanding $Z(x(q))$ at first order

$$\tilde{Z}(x(p)) \simeq Z^\mu(x(p)) + Z^\mu_{,\nu}\lambda\xi^\nu(x(p)) - \lambda\xi^\mu_{,\nu}Z^\nu(x(p)), \tag{3.10}$$

where we can recognize the Lie derivative

$$\mathcal{L}_\xi Z^\mu = Z^\mu_{,\nu}\xi^\nu - \xi^\mu_{,\nu}Z^\nu \tag{3.11}$$

so our gauge transformation up to first order reads

$$\tilde{Z} = Z + \mathcal{L}_\xi Z. \tag{3.12}$$

Our goal, however, is to go beyond first order and we can expand everything up to second order, starting from (3.6)

$$x^\mu(q) = x^\mu(p) + \lambda\xi^\mu(x(p)) + \frac{\lambda^2}{2}\xi^\mu_{,\nu}\xi^\nu(x(p)) + \ldots, \tag{3.13}$$

where

$$\frac{d^2x^\mu}{d\lambda^2} = \frac{d}{d\lambda}\xi^\mu = \frac{dx^\nu}{d\lambda}\xi^\mu_{,\nu} = \xi^\nu\xi^\mu_{,\nu}. \tag{3.14}$$

The passive "approach" relation (3.7) becomes

$$y^\mu(q) = x^\mu(q) - \lambda\xi^\mu(x(p)) - \frac{\lambda^2}{2}\xi^\mu_{,\nu}\xi^\nu(x(p)) + \ldots, \tag{3.15}$$

expanding everything

$$y^\mu(q) = x^\mu(q) - \lambda\xi^\mu(x(q)) + \frac{\lambda^2}{2}\xi^\mu_{,\nu}\xi^\nu(x(q)) + \ldots. \tag{3.16}$$

Since ξ^μ can be expanded as [18]

$$\xi = \sum_{h=0}^{\infty}\frac{\lambda^h}{h!}\xi_{(h+1)} \tag{3.17}$$

we have easily

$$y^\mu(q) = x^\mu(q) - \lambda\xi^\mu_1(x(q)) + \frac{\lambda^2}{2}\left[\xi^\mu_{1,\nu}\xi^\nu_1(x(q)) - \xi^\mu_2(x(q))\right] + \ldots. \tag{3.18}$$

Now we need to plug this equation into (3.8) to get the general rule for a tensor. The second order Jacobian reads

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu - \lambda\xi^\mu_{1,\nu} + \frac{\lambda^2}{2}\left[\xi^\mu_{1,\nu\lambda}\xi^\lambda_1 + \xi^\mu_{1,\lambda}\xi^\lambda_{1,\nu} - \xi^\mu_{2,\nu}\right] + \ldots, \tag{3.19}$$

Taylor expanding everything up to second order as before we have

$$\tilde{Z}^\mu = Z^\mu + \lambda\mathcal{L}_{\xi_1}Z^\mu + \frac{\lambda^2}{2}\left(\mathcal{L}^2_{\xi_1} + \mathcal{L}_{\xi_2}\right)Z^\mu, \tag{3.20}$$

where we used the fact that the second order Lie derivative is

$$\mathcal{L}_\xi^2 Z^\mu = \xi^\beta \xi^\delta_{,\beta} Z^\mu_{,\delta} - 2\xi^\beta Z^\delta_{,\beta}\xi^\mu_{,\delta} + Z^\beta \xi^\delta_{,\beta}\xi^\mu_{,\delta} + \xi^\beta\xi^\delta Z^\mu_{,\beta\delta} - Z^\beta\xi^\delta\xi^\mu_{,\beta\delta}. \quad (3.21)$$

For a generic tensor T we have thus the gauge transformation up to second order

$$\tilde{T} = T + \lambda\mathcal{L}_{\xi_1}T + \frac{\lambda^2}{2}\left(\mathcal{L}^2_{\xi_1} + \mathcal{L}_{\xi_2}\right)T + \ldots. \quad (3.22)$$

Let us split the tensor T into zero, first and second order

$$T = \bar{T} + T_1 + \frac{1}{2}T_2, \quad \tilde{T} = \bar{T} + \tilde{T}_1 + \frac{1}{2}\tilde{T}_2, \quad (3.23)$$

so we have

$$\tilde{T}_1 = T_1 + \lambda\mathcal{L}_{\xi_1}\bar{T} \quad (3.24)$$

$$\tilde{T}_2 = T_2 + 2\lambda\mathcal{L}_{\xi_1}T_1 + \lambda^2\left(\mathcal{L}^2_{\xi_1} + \mathcal{L}_{\xi_2}\right)\bar{T} \quad (3.25)$$

where T_r is the perturbation at the r-th order: sometimes it will be denoted as $T_{(r)} = \delta^r T$, i.e. $T_1 = \delta T$, $T_2 = \delta^2 T$.

Lie derivatives

Our transformation law requires to compute Lie derivatives of quantities which could be scalars, vectors or tensors and here I recall the main results which will be useful later. First order Lie derivatives along the vector field ξ^μ are

- $\mathcal{L}_\xi\phi = \phi_{,\mu}\xi^\mu$
- $\mathcal{L}_\xi v^\mu = v^\mu_{,\alpha}\xi^\alpha - v^\alpha\xi^\mu_{,\alpha}$
- $\mathcal{L}_\xi\omega_{\mu\nu} = \omega_{\mu\nu,\alpha}\xi^\alpha + \omega_{\mu\alpha}\xi^\alpha_{,\nu} + \omega_{\alpha\nu}\xi^\alpha_{,\nu}$

Second order Lie derivatives can be computed easily from the definition, i.e. $\mathcal{L}^2_\xi T = \mathcal{L}_\xi[\mathcal{L}_\xi T]$, having

- $\mathcal{L}^2_\xi\phi = \xi^\beta\xi^\delta_{,\beta}\phi_{,\delta} + \xi^\beta\xi^\delta\phi_{,\beta\delta}$
- $\mathcal{L}^2_\xi v^\mu = \xi^\beta\xi^\delta_{,\beta}v^\mu_{,\delta} - 2\xi^\beta v^\delta_{,\beta}\xi^\mu_{,\delta} + v^\beta\xi^\delta_{,\beta}\xi^\mu_{,\delta} + \xi^\beta\xi^\delta v^\mu_{,\beta\delta} - v^\beta\xi^\delta\xi^\mu_{,\beta\delta}$
- $\mathcal{L}^2_\xi\omega_{\mu\nu} = \xi^\beta\omega_{\delta\nu}\xi^\delta_{,\mu\beta} + \xi^\beta\omega_{\mu\delta}\xi^\delta_{,\nu\beta} + \xi^\beta\xi^\delta_{,\beta}\omega_{\mu\nu,\delta} + \xi^\beta\xi^\delta\omega_{\mu\nu,\beta\delta} + 2\xi^\beta\omega_{\delta\nu,\beta}\xi^\delta_{,\mu} + \omega_{\beta\nu}\xi^\beta_{,\delta}\xi^\delta_{,\mu} + 2\xi^\beta\omega_{\mu\delta,\beta}\xi^\delta_{,\nu} + \omega_{\mu\beta}\xi^\beta_{,\delta}\xi^\delta_{,\nu} + 2\omega_{\beta\delta}\xi^\beta_{,\mu}\xi^\delta_{,\nu}$

3.2 A perturbed universe

Our smooth background is the so-called FRW model described by a metric

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \delta_{ij}dx^i dx^j\right] \tag{3.26}$$

and we can introduce some quantities to perturb this metric up to any order r. We can start introducing a scalar quantity ψ called the *lapse function*, such that the 00 component of the metric will be described as

$$g_{00} = -a^2(\eta)\left(1 + 2\sum_{r=1}^{\infty}\frac{1}{r!}\psi_{(r)}(x^i,\eta)\right). \tag{3.27}$$

Then, the $0i$ component of the metric can be perturbed using a new function ω_i which is called *shift* perturbation

$$g_{0i} = a^2(\eta)\sum_{r=1}^{\infty}\frac{1}{r!}\omega_{(r)i}(x^i,\eta). \tag{3.28}$$

Lastly, the spatial part of the metric ij can be perturbed using two functions

$$g_{ij} = a^2(\eta)\left[\left(1 - 2\sum_{r=1}^{\infty}\frac{1}{r!}\phi_{(r)}(x^i,\eta)\right)\delta_{ij} + \sum_{r=1}^{\infty}\frac{1}{r!}\chi_{(r)ij}(x^i,\eta)\right]. \tag{3.29}$$

If we stop to second order we will have components

$$\begin{aligned}
g_{00} &= -a^2(1 + 2\psi_1 + \psi_2)\\
g_{0i} &= a^2(\omega_{1i} + \frac{1}{2}\omega_{2i})\\
g_{ij} &= a^2\left[(1 - 2\phi_1 - \phi_2)\delta_{ij} + \chi_{1ij} + \frac{1}{2}\chi_{2ij}\right]
\end{aligned}$$

where the dependence on η and x^i is hidden because of brevity: remember that zero order quantities are only time dependent and perturbations depend on both time and space coordinates.

It is a known fact that the quantities appearing in the metric, i.e. ω_i and χ_{ij}, can be further decomposed according to the *Helmholtz decomposition theorem* at any order in perturbation theory. This states that the quantity ω_i can be decomposed in

$$\omega_i = \omega^{\parallel}_{,i} + \omega^{\perp}_i, \tag{3.30}$$

where $\omega_i^{\perp}$ is a solenoidal (divergence-free o transverse) vector, namely $\omega_i^{\perp,i} = 0$. Similarly, the traceless part of the spatial perturbed metric χ_{ij} can be splitted in scalar, vector and tensor part as (cfr. [15] with different notation)

$$\chi_{ij} = 2D_{ij}\chi^{\parallel} + \chi^{\perp}_{i,j} + \chi^{\perp}_{j,i} + \chi^{T}_{ij}. \tag{3.31}$$

Here D_{ij} is a traceless operator defined as

$$D_{ij} = \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2, \tag{3.32}$$

the vector quantity $\chi_i^{\perp}$ is transverse, namely $\chi_i^{\perp,i} = 0$, and χ^T_{ij} is a tensor perturbation which is by definition transverse and traceless, i.e. $\chi_{ij}^{T,i} =$ and $\chi_i^{Ti} = 0$.

We can summarise saying that the most general perturbations of the FRW metric can be decomposed *at each point in space* into four scalar parts with one degree of freedom each, two vectors each having two d.o.f. and one tensor having 2 d.o.f.. So in total we have 10 degrees of freedom. The question is easily served: why do we bother with this "fancy" mathematical decomposition? Obviously one can say that different perturbations are linked to distinct physical phenomena: Newtonian gravity is a scalar phenomenon, the presence of magnetic fields in the universe is linked to some vector quantities and gravitational waves are only tensor perturbations. Another explanation is that we can thus decompose Einstein equations and in some specific gauge choices, up to first order, these perturbations evolve independently.

The first order vector perturbations are neglected because our models of inflation predict that no vector components are generated during the inflationary epoch, but, even if they are created, they are quickly diluted away as we will see later.

This was about the metric perturbations but we are not satisfied enough yet. We have to account for matter perturbations: we have to perturb the 4-velocity and the matter-energy density. Let us start with the 4-velocity of the matter, defined as $u^\mu = dx^\mu/d\eta$. We set it to be

$$u^\mu = \frac{1}{a}\left(\delta^\mu_0 + v^\mu\right), \tag{3.33}$$

where v^μ is the peculiar velocity, expanded up to second order as $v^\mu = v^\mu_1 + v^\mu_2/2$. We know from General Relativity that the 4-velocity must staisfy the normalization condition $u^\mu u_\mu = -1$, so we can link the time component v^0 at any order with the lapse perturbation ψ. Easily we start at first order and we find that

$$v_1^0 = -\psi_1. \tag{3.34}$$

Now we do the same calculations, keeping only terms up to second order, from the condition

$$\begin{aligned} u^\mu u_\mu &= -\left(1 + v_1^0 + \frac{1}{2}v_2^0\right)^2 (1 + 2\psi_1 + \psi_2) + v_1^i v_1^j \delta_{ij} \\ &= -\left(1 + (v_1^0)^2 + 2v_1^0 + v_2^0\right)(1 + 2\psi_1 + \psi_2) + v_1^k v_{1k} \\ &= -\left(1 + \psi_2 + \psi_1^2 - 4\psi_1^2 + v_2^0\right) + v_1^k v_{1k}, \end{aligned}$$

since it must be equal to -1 we have

$$v_2^0 = 3\psi_1^2 - \psi_2 + v_{1k} v_1^k. \tag{3.35}$$

Now, as we did before, the spatial perturbation of the velocity can be decomposed into a scalar and a true transverse vector

$$v^i = v_{\|}^{,i} + v_\perp^i. \tag{3.36}$$

The energy density can be expanded as

$$\rho = \bar{\rho} + \sum_{r=1}^{\infty} \frac{1}{r!} \delta^r \rho. \tag{3.37}$$

One last quantity which needs to be decomposed is the vector ξ along which the Lie dragging is performed, we can write (at any order)

$$\xi^0 = \alpha \tag{3.38}$$

and

$$\xi^i = \beta^{,i} + d^i, \tag{3.39}$$

with d^i transverse, i.e. $d^i_{,i} = 0$.

3.3 Choosing a gauge

In this section we will see two of the most "popular" gauges used in cosmological perturbation theory [4, 21]. These gauges can be specified by assigning a particular set of variables to zero, either in the gravitational sector or the matter sector. As we have seen before, our gauge freedom in the metric part is completely contained in the tensor ξ^μ, which is made of two scalars (α, β) and one vector (d^i): thus two scalars and one vector to zero in our metric is a perfect way to fix a gauge and this is what is done in the two following examples.

3.3.1 Synchronous gauge

Introduced by Lifshitz in 1946, it is characterised by the conditions

$$\psi = \omega_i = 0 \tag{3.40}$$

which remove two scalars (ψ and $\omega^{||}$) and a vector ($\omega_i^{\perp}$). In this gauge it can be easily seen that the time coordinate corresponds to the proper time of comoving observers at fixed spatial coordinates. The only problem is that this gauge does not eliminate all the possible gauge freedom. At first order we can write

$$\tilde{\psi} = \psi + \alpha' + \mathcal{H}\alpha = \psi + \frac{(a\alpha)'}{a} = 0 \tag{3.41}$$

so we have

$$a\alpha = -\int a\psi d\eta + C(x^i) \tag{3.42}$$

where we have a constant $C(x^i)$ which needs to be fixed from initial conditions.

This gauge is very useful in lots of cosmological calculations, and it is the gauge employed for CMBFAST [10] and CAMB [9].

3.3.2 Poisson gauge

Introduced by Bertschinger [21] it is defined by eliminating one scalar from g_{0i}, another scalar and a vector from g_{ij}, namely

$$\omega^{||} = 0 \tag{3.43}$$

$$\chi^{||} = 0 \tag{3.44}$$

$$\chi_i^{\perp} = 0 \tag{3.45}$$

A previous version of this gauge is the so-called longitudinal or conformal Newtonian gauge [11], defined by

$$\omega_i = \chi_{ij} = 0 \tag{3.46}$$

valid only if the stress-energy tensor contains no vector or tensor parts and there are only scalar metric perturbations (no gravitational waves). Since the scalar shear is given by $\sigma = \chi^{\|\prime} - \omega^{\|}$, this gauge is also known as the *zero-shear* gauge because σ vanishes in this gauge.

Bardeen in his work [26] proposed two gauge-invariant scalar potentials, defined as

$$2\Psi_A = 2\psi + 2\omega^{\|\prime} + 2\mathcal{H}\omega^{\|} - \left(\chi^{\|\prime\prime} + \mathcal{H}\chi^{\|\prime}\right) \tag{3.47}$$

$$2\Phi_H = -2\phi - \frac{1}{3}\nabla^2\chi^{\|} + 2\mathcal{H}\omega^{\|} - \mathcal{H}\chi^{\|\prime}, \tag{3.48}$$

and we can easily see that in Poisson gauge the two scalar potential are also gauge invariant, since $\psi = \Psi_A$ and $\phi = -\Phi_H$. Thus, the metric perturbations in the Poisson gauge correspond exactly with several of the gauge-invariant variables introduced by Bardeen.

As we will see later Poisson gauge gives the relativistic cosmological generalization of Newtonian gravity.

3.4 First order

Now that we have the essential resources, we start sneaking into the lumpy world of perturbation theory, recalling the very basic and well estabilished results in the linear theory.

3.4.1 Gauge transformations

As we have seen before, we need to specify a gauge choice and we can have the possibility to choose the one we want. We saw how to transform from a gauge to another, but how do really the perturbations transform at first order?

Since these results are easily found on textbooks and reviews (see [13, 4, 15, 3]), we will walk through this computations quite quickly.We can see now how the perturbations appering in the metric transform, starting with the lapse perturbation ψ_1, from δg_{00}

$$\delta\tilde{g}_{00} = \delta g_{00} + \mathcal{L}_{\xi_1}\overline{g}_{00}, \tag{3.49}$$

which is

$$
\begin{aligned}
-2a^2\tilde{\psi}_1 &= -2a^2\psi_1 + \left(\overline{g}_{00,\mu}\xi_1{}^{\mu} + \xi_1{}^{\mu}_{,0}\overline{g}_{0\mu} + \xi_1{}^{\mu}_{,0}\overline{g}_{0\mu}\right) \\
&= -2a^2\psi_1 + \left(\overline{g}_{00,0}\xi_1{}^{0} + \overline{g}_{00,i}\xi_1{}^{i} + \xi_1{}^{0}_{,0}\overline{g}_{00} + \xi_1{}^{i}_{,0}\overline{g}_{0i} + \xi_1{}^{0}_{,0}\overline{g}_{00} + \xi_1{}^{i}_{,0}\overline{g}_{0i}\right) \\
&= -2a^2\psi_1 + \left(-2aa'\alpha_1 - 2a^2\alpha_1'\right),
\end{aligned}
$$

so

$$
\tilde{\psi}_1 = \psi_1 + \mathcal{H}\alpha_1 + \alpha_1'. \tag{3.50}
$$

For $\omega_1^{\|}$ we have

$$
\begin{aligned}
a^2\tilde{\omega}^{\|}_{1,i} &= a^2\omega^{\|}_{1,i} + \mathcal{L}_{\xi_1}\overline{g}_{0i} \\
&= a^2\omega^{\|}_{1,i} + \overline{g}_{0i,0}\xi_1{}^{0} + \overline{g}_{0i,j}\xi_1{}^{j} + \xi_1{}^{0}_{,0}\overline{g}_{i0} + \xi_1{}^{j}_{,0}\overline{g}_{ij} + \xi_1{}^{0}_{,i}\overline{g}_{00} + \xi_1{}^{j}_{,i}\overline{g}_{ij} \\
&= a^2\omega^{\|}_{1,i} - a^2\alpha_{1,i} + a^2\beta'_{,j}\delta_{ij} \\
&= a^2\omega^{\|}_{1,i} - a^2\alpha_{1,i} + a^2\beta'_{,i},
\end{aligned}
$$

thus we have

$$
\tilde{\omega}_1^{\|} = \omega_1^{\|} - \alpha_1 + \beta'. \tag{3.51}
$$

From g_{ij} we can see how ϕ_1 transforms

$$
\begin{aligned}
-2\tilde{\phi}_1\delta_{ij} &= -2\phi_1\delta_{ij} + \frac{1}{a^2}\left(2aa'\alpha_1\delta_{ij} + \beta_{1,ik}a^2\delta_{kj} + \beta_{1,ik}a^2\delta_{kj}\right) \\
&= -2\phi_1\delta_{ij} + 2\mathcal{H}\alpha_1'\delta_{ij} + 2\beta_{1,ij},
\end{aligned}
$$

multiplying by δ^{ij} we have

$$
-6\tilde{\phi}_1 = -6\phi_1 + 6\mathcal{H}\alpha_1 + 2\nabla^2\beta_1 \tag{3.52}
$$

and in conclusion

$$
\tilde{\phi}_1 = \phi_1 - \mathcal{H}\alpha_1 - \frac{1}{3}\nabla^2\beta_1. \tag{3.53}
$$

Now for the last scalar $\chi_1^{\|}$ we look again at g_{ij}; as before we have

$$
\mathcal{L}_{\xi_1}\overline{g}_{ij} = 2aa'\alpha_1\delta_{ij} + 2a^2\beta_{1,ij}, \tag{3.54}
$$

so

$$
\begin{aligned}
2a^2D_{ij}\tilde{\chi}_1^{\|} &= 2a^2D_{ij}\chi_1^{\|} + 2aa'\alpha_1\delta_{ij} + 2a^2a^2\beta_{1,ij} \\
\tilde{\chi}^{\|}_{1,ij} - \frac{1}{3}\nabla^2\tilde{\chi}_1^{\|}\delta_{ij} &= \chi^{\|}_{1,ij} + \frac{1}{3}\nabla^2\chi_1^{\|}\delta_{ij} + \mathcal{H}\alpha_1\delta_{ij} + \beta_{1,ij},
\end{aligned}
$$

taking the traceless part for $i \neq j$, for which the deltas are zero

$$\tilde{\chi}^{\parallel}_{1,ij} = \chi^{\parallel}_{1,ij} + \beta_{1,ij}. \tag{3.55}$$

Now I switch to the vector part of the metric, starting from $\omega^{\perp}_{1i}$. Here the Lie derivative is (keeping only the vector part of ξ_1)

$$\mathcal{L}_{\xi_1}\overline{g}_{0i} = a^2 d'_{1i}, \tag{3.56}$$

so we have

$$a^2\tilde{\omega}^{\perp}_{1i} = a^2\omega^{\perp}_{1i} + a^2 d'_{1i},$$

which gives

$$\tilde{\omega}^{\perp}_{1i} = \omega^{\perp}_{1i} + d'_{1i}. \tag{3.57}$$

For $\chi^{\perp}_{1i}$ we take the g_{ij}. The Lie derivative is

$$\mathcal{L}_{\xi_1}\overline{g}_{ij} = a^2 d_{1j,i} + a^2 d_{1j,i}, \tag{3.58}$$

thus we have

$$\tilde{\chi}^{\perp}_{1i,j} + \tilde{\chi}^{\perp}_{1j,i} = \chi^{\perp}_{1i,j} + \chi^{\perp}_{1j,i} + d_{1j,i} + d_{1j,i},$$

which gives

$$\tilde{\chi}^{\perp}_{1i} = \chi^{\perp}_{1i} + d_{1i}. \tag{3.59}$$

The last step is now to see the tensor perturbation χ^{T}_{1ij}

$$a^2\tilde{\chi}^{T}_{1ij} = a^2\chi^{T}_{1ij} + \mathcal{L}_{\xi_1}\overline{g}_{ij}. \tag{3.60}$$

and we can easily write

$$\tilde{\chi}_{1ij} = \chi_{1ij}. \tag{3.61}$$

Now we can study how the velocity field transorms, recalling that $v^0_1 = -\psi_1$, namely

$$\begin{aligned}\tilde{v}^0_1 &= -\tilde{\psi}_1\\ &= -\psi_1 - \mathcal{H}\alpha_1 - \alpha'_1,\end{aligned} \tag{3.62}$$

so, trivially

$$\tilde{v}^0_1 = v^0_1 - \mathcal{H}\alpha_1 - \alpha'_1. \tag{3.63}$$

The spatial part of the 4-velocity can be computed from

$$\delta\tilde{u}^i = \delta u^i + \mathcal{L}_{\xi_1}\overline{u}^i,$$

recalling that only $\overline{u}^0$ is non vanishing and $\delta u^i = v_1^i/a$ and that the lie derivative of a vector field is

$$\mathcal{L}_\xi v^\alpha = v^\alpha_{,\mu}\xi^\mu - v^\mu \xi^\alpha_{,\mu}, \tag{3.64}$$

thus we have

$$\begin{aligned} \frac{1}{a}\tilde{v}_1^i &= \frac{1}{a}v_1^i + \overline{u}^i_{,\mu}\xi_1^\mu - \overline{u}^\mu \xi^i_{1,\mu} \\ &= \frac{1}{a}v_1^i - \overline{u}^0 \xi^i_{1,0} \qquad (3.65) \\ &= \frac{1}{a}v_1^i - \frac{1}{a}\left(\beta_1'^{,i} + d_1'^{,i}\right), \qquad (3.66) \end{aligned}$$

so the scalar part of the spatial components of the 4-velocity transforms as

$$\tilde{v}^{\|}1 = v^{\|}1 - \beta_1' \tag{3.67}$$

and the vector part follows easily

$$\tilde{v}_1^{\perp i} = v_1^{\perp i} - d_1'^{\,i}. \tag{3.68}$$

Another important quantity is the density ρ, this transforms as

$$\tilde{\delta\rho} = \delta\rho + \mathcal{L}_{\xi_1}\overline{\rho}, \tag{3.69}$$

where $\mathcal{L}_{\xi_1}\overline{\rho} = \overline{\rho}_{,\mu}\xi_1^\mu$. So we have

$$\tilde{\delta\rho} = \delta\rho + \overline{\rho}'\alpha_1, \tag{3.70}$$

since $\overline{\rho}$ depends only on time.

3.4.2 Einstein equations for scalar perturbations

To recover the first order Einstein equations I have to compute the components of the equations up to first order: Ricci tensor, Ricci scalar and the energy momentum tensor. Using the definition we have, up to first order

$$\begin{aligned} &\delta\Gamma^0_{00} = \psi_1', \quad \delta\Gamma^0_{0i} = \psi_{1,i}, \quad \delta\Gamma^i_{00} = \psi_1'^{\,i} \\ &\delta\Gamma^0_{ij} = -2\mathcal{H}\phi_1\delta_{ij} - 2\mathcal{H}\psi_1\delta_{ij} - \phi_1'\delta_{ij}, \quad \delta\Gamma^i_{j0} = -\delta^i_j\phi_1' \\ &\delta\Gamma^i_{jk} = -\delta^i_j\phi_{1,k} - \delta^i_k\phi_{1,j} + \delta_{jk}\phi_1'^{\,i} \end{aligned}$$

from which we can compute the first order Ricci tensor

$$\delta R_{ij} = \left(\phi_1 - \psi_1\right)_{,ij} + \delta_{ij}\left[-\mathcal{H}\left(5\phi_1' + \psi_1'\right) - \left(2\mathcal{H}' + 4\mathcal{H}^2\right)\left(\phi_1 + \psi_1\right) - \phi_1'' + \nabla^2\phi_1\right]$$

and

$$\delta R_{00} = 3\phi_1'' + \nabla\psi_1 + 3\mathcal{H}\left(\psi_1' + \phi_1'\right), \quad \delta R_{0i} = 2\left(\phi_1' + \mathcal{H}\psi_1\right)_{,i}.$$

Now, the Ricci scalar can be easily written as

$$\delta R = g^{00}R_{00} + g^{ij}R_{ij} + 2g^{0i}R_{i0} \tag{3.71}$$

where the last term is a second order quantity and can be neglected.

After some algebra one finds

$$\delta R = a^{-2}\left[-6\phi_1'' + 6\mathcal{H}(\psi_1' + 3\phi_1') - 12\psi_1(\mathcal{H}' + \mathcal{H}^2) + 2\nabla^2(2\phi_1 - \psi_1)\right]$$

We can write the Einstein tensor G^μ_ν

$$\delta G^i_j = \delta R^i_j - \frac{1}{2}\delta^i_j\delta R$$

and we have

$$\delta G^i_j = a^{-2}\delta^i_j\left[\mathcal{H}(4\phi_1' + 2\psi_1') + \psi_1(4\mathcal{H}' + 2\mathcal{H}^2) + 2\phi_1'' - \nabla^2(\phi_1 - \psi_1)\right] + \\ - a^{-2}(\phi_1 - \psi_1)_{,ij}$$

Now we need to write the first order energy momentum tensor $\delta T_{\mu\nu}$ for a perfect fluid, which gives

$$\begin{aligned} T^0_0 &= -\bar{\rho} - \delta\rho \\ T^i_0 &= -(\bar{\rho} + \bar{p})v_{1,i} = -T^0_i \\ T^i_j &= \bar{p}\delta^i_j + \delta p\delta^i_j \end{aligned}$$

Putting everything together and taking the tracefree part of the ij equation

$$\delta G^i_j - \frac{1}{3}\delta G^k_k\delta^i_j = 8\pi G\left(\delta T^i_j - \frac{1}{3}\delta T^k_k\delta^i_j\right)$$

which gives us

$$\left(\partial_i\partial_j - \frac{1}{3}\nabla^2\delta^i_j\right)(\phi_1 - \psi_1) = 0 \tag{3.72}$$

which is satisfied for

$$\psi_1 = \phi_1 = \varphi. \tag{3.73}$$

With this, we can find the evolution equation for dust inserting the previous result into the trace part of the ij Einstein equation, namely

$$a^{-2}\left[6\mathcal{H}\varphi' + \varphi(4\mathcal{H}' + 2\mathcal{H}^2) + 2\varphi''\right]\delta^i_j = 8\pi G\delta p\delta^i_j. \tag{3.74}$$

In the easy case of dust we have $\delta p = 0$, so

$$\varphi'' + 3H\varphi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\varphi = 0 \tag{3.75}$$

3.4.3 Einstein equations for tensor perturbations

For tensor perturbations we follow the previous computations but with only tensors. Namely we have

$$\delta\Gamma^i_{j0} = \frac{1}{2}\chi^{i\,\prime}_{1j}, \quad \delta\Gamma^i_{jk} = \frac{1}{2}\left(\chi^i_{1j,k} + \chi^i_{1k,j} - \chi^{,i}_{1jk}\right),$$

where $\chi_{1ij} \equiv \chi^{\perp}_{1ij}$. The non vanishing perturbations in the Ricci tensor is only in the ij components, because we have that tensor perturbations are traceless and transverse

$$\delta R^i_j = a^{-2}\left(\frac{1}{2}\chi^{i\,\prime\prime}_{1j} - \frac{1}{2}\nabla^2\chi^i_{1j} + \mathcal{H}\chi^{i\,\prime}_{1j}\right). \tag{3.76}$$

The first order pertubed Ricci scalar is zero and from the ij equation we have

$$\chi^{i\,\prime\prime}_{1j} + 2\mathcal{H}\chi^{i\,\prime}_{1j} - \nabla^2\chi^i_{1j} = 0, \tag{3.77}$$

which is the equation of a harmonic oscillator damped by the expansion of the Universe, described by the term

$$2\mathcal{H}\chi^{i\,\prime}_{1j} \neq 0 \tag{3.78}$$

3.4.4 Einstein equations for vector perturbations

In the case of first order vector perturbations, we set to zero the scalar and tensors components in our metric. Recalling that we are in Poisson gauge we can set $\omega^{\perp}_{1i} \equiv \omega_{1i}$. Up to first order we have easily

$$\delta\Gamma^0_{0i} = \omega_{1,i}\mathcal{H}, \quad \delta\Gamma^0_{ij} = -\frac{1}{2}\left(\omega_{1i,j} + \omega_{1j,i}\right), \quad \delta\Gamma^i_{jk} = -\omega^i_1\delta_{jk}\mathcal{H}$$

$$\delta\Gamma^i_{j0} = \frac{1}{2}\left(\omega^k_{1,j} - \omega^{,k}_{1j}\right), \quad \delta\Gamma^i_{00} = \mathcal{H}\omega^i_1 + \omega^{i\,\prime}_1.$$

Now we can write the $0i$ Einstein equation for the vector shear and we find

$$\nabla^2\Psi_i = 16\pi G a^2 V_{ic} \tag{3.79}$$

where $\Psi_i = \omega_i^{\perp} - \chi_i^{\perp\prime}$ is the vector shear and $V_{ic} = v_i^{\perp} + \omega_i^{\perp}$ is the gauge invariant vorticity.

From the momentum conservation equation derived from $T^{i\mu}_{;\mu} = 0$, in case of no anisotropic stress tensor (perfect fluid), we have

$$[(\bar{\rho} + \bar{p})V_{ic}]' + 4\mathcal{H}(\bar{\rho} + \bar{p})V_{ic} = 0 \tag{3.80}$$

which can be solved having

$$(\bar{\rho} + \bar{p})V_{ic} \propto a^{-4}. \tag{3.81}$$

The previous equation gives us an interesting result, namely

$$a^4(\bar{\rho} + \bar{p})V_{ic} = \text{constant} \tag{3.82}$$

which is the Kelvin circulation theorem, saying that the vorticity is conserved along fluid trajectories, in the absence of dissipative effects. The last equation can be rearranged in a more intuitive form

$$a^3(\bar{\rho} + \bar{p})V_{ic}a = \text{constant} \tag{3.83}$$

which is a "momentum" multiplied by a, representing the conservation of the intrinsic angular momentum. From this result we see that the vorticity tends to be diluted during the expansion of the universe since it is inversely proportional to the scale factor.

So, even if inflation would have generated first order vector perturbations, they would have been quickly diluted and thus it is quite reasonable to neglect those first order vector components.

3.5 Second order

After this straightforward path, leaving behind us the linear world, we go a bit higher and at the beginning of this somewhat winding and intricate path, we start to see how the trees, branches and shrubs begin to intertwine and merge with each other, making the crossing more complicated.

3.5.1 Gauge transformations

The first thing we encounter in our path is the gauge transformation, as we did in the linear world: we need to find how second order perturbations transform.

We start with the lapse perturbation ψ_2, namely from the relation

$$\delta^2 \tilde{g}_{00} = \delta^2 g_{00} + 2\mathcal{L}_{\xi_1}\delta^1 g_{00} + \mathcal{L}^2_{\xi_1}\bar{g}_{00} + \mathcal{L}_{\xi_2}\bar{g}_{00} \tag{3.84}$$

where $\delta^2 g_{00} = -2a^2\psi_2$, since $g_{00} = \bar{g}_{00} + \delta^1 g_{00} + \frac{1}{2}\delta^2 g_{00}$. The Lie derivative along ξ_1 of $\delta^1 g_{00}$ reads

$$\begin{aligned}
\mathcal{L}_{\xi_1}\delta^1 g_{00} &= \delta^1 g_{00,\mu}\xi_1^\mu + 2\xi^\mu_{1,0}\delta^1 g_{0\mu} \\
&= \delta^1 g_{00,0}\xi_1^0 + \delta^1 g_{00,i}\xi_1^i + 2\left[\xi^0_{1,0}\delta^1 g_{00} + \xi^i_{1,0}\delta^1 g_{01}\right] \\
&= -2(2aa'\psi_1 + a^2\psi_1')\alpha_1 - a^2\psi_{,i}\xi_1^i + 2(-2a^2\psi_1\alpha_1' + a^2\xi_1^{i\,\prime}\omega_{1i}).
\end{aligned}$$

The Lie derivative along ξ_2 is esily found

$$\begin{aligned}
\mathcal{L}_{\xi_2}\bar{g}_{00} &= \bar{g}_{00,\mu}\xi_2^\mu + 2\xi^\mu_{2,0}\bar{g}_{0\mu} \\
&= -2aa'\alpha_2 - 2\alpha_2' a^2.
\end{aligned}$$

Now comes the trickiest part, computing the second order Lie derivative along ξ_1

$$\begin{aligned}
\mathcal{L}^2_{\xi_1}\bar{g}_{00} = {}& \bar{g}_{00,\mu\nu}\xi_1^\mu\xi_1^\nu + \bar{g}_{00,\mu}\xi^\mu_{1,\nu}\xi_1^\nu + \xi^\nu_{1,0\mu}\bar{g}_{0\nu}\xi_1^\mu + 2\xi^\mu_{1,0}\bar{g}_{0\mu,\nu}\xi_1^\nu + \\
&+\xi^\mu_{1,0\nu}\bar{g}_{0\mu}\xi_1^\nu + 2\xi^\mu_{1,0}\bar{g}_{0\mu,\nu}\xi_1^\nu + \xi^\nu_{1,0}\xi^\mu_{1,\nu}\bar{g}_{0\mu} + 2\xi^\nu_{1,0}\xi^\mu_{1,0}\bar{g}_{\nu\mu} + \xi^\nu_{1,0}\xi^\mu_{1,\nu}\bar{g}_{0\mu},
\end{aligned}$$

expanding all the summation, we end up with

$$\begin{aligned}
\mathcal{L}^2_{\xi_1}\bar{g}_{00} = {}& -2\alpha_1^2(a'^2 + aa'') - 2\alpha_1'\alpha_1 aa' - 2aa'\alpha_{1,i}\xi_1^i - \alpha_1''\alpha_1 a^2 - a^2\alpha'_{1,i}\xi_1^i + \\
&-4\alpha_1'\alpha_1 aa' - \alpha_1''\alpha_1 a^2 - \alpha'_{1,i}\xi_1^i a^2 - 4aa'\alpha_1'\alpha_1 - a^2\alpha_1'^2 - a^2\alpha_{1,i}\xi_1^{i\,\prime} - 2a^2\alpha_1'^2 + \\
&+2a^2\xi_1^{i\,\prime}\xi'_{1i} - a^2\alpha_1'^2 - a^2\xi_1^{i\,\prime}\alpha_{1,i}.
\end{aligned}$$

Now we can put everything together and reach the desired result

$$\tilde{\psi}_2 = \psi_2 + \mathcal{H}\alpha_2 + \alpha_2' + \Pi, \tag{3.85}$$

where Π is a contribution from products of first-order transformation, defined as

$$\begin{aligned}
\Pi = {}& \alpha_1\left[\alpha_1'' + 5\mathcal{H}\alpha_1' + \left(\mathcal{H}^2 + \frac{a''}{a}\right)\alpha_1 + 4\mathcal{H}\psi_1 + 2\psi_1'\right] + 2\alpha_1'\left(\alpha_1' + 2\psi_1\right) + \\
&+\xi_1^i\left(\alpha_1' + \mathcal{H}\alpha_1 + 2\psi_1\right)_{,i} + \xi_1^{i\,\prime}\left(\alpha_{1,i} - 2\omega_{1i} - \xi'_{1i}\right)
\end{aligned}$$

Now, from the $0i$ component we derive the transformation rule for $\omega_2^{\parallel}$ and $\omega_{2i}^{\perp}$

$$\delta^2\tilde{g}_{0i} = \delta^2 g_{0i} + 2\mathcal{L}_{\xi_1}\delta^1 g_{0i} + \mathcal{L}^2_{\xi_1}\bar{g}_{0i} + \mathcal{L}_{\xi_2}\bar{g}_{0i}. \tag{3.86}$$

Again, we have

$$\begin{aligned}\mathcal{L}_{\xi_1}\delta^1 g_{01} &= \delta^1 g_{0i,\mu}\xi_1^{\mu} + \xi^{\mu}_{1,0}\delta^1 g_{\mu i} + \xi^{\mu}_{1,i}\delta^1 g_{0\mu}\\ &= 2aa'\omega_{1i}\alpha_1 + a^2\omega'_{1i}\alpha_1 + a^2\omega_{1i,k}\xi_1^k + \alpha'_1\omega_{1i} +\\ &\quad + 2\xi_1^{k\prime}C_{1ki} - 2\alpha_{1,i}a^2\psi_1 + \xi^k_{1,i}a^2\omega_{1k},\end{aligned}$$

where for brevity I introduced the quantity $C_{(r)ij}$ defined as

$$C_{(r)ij} = -\phi_{(r)}\delta_{ij} + \chi_{(r)ij}. \tag{3.87}$$

The other Lie derivative is

$$\mathcal{L}_{\xi_2}\bar{g}_{0i} = a^2\delta_{ki}\xi_2^{k\prime} - a^2\alpha_{2,i}.$$

The second order Lie derivative acting on $\bar{g}_{0i}$ is defined as before (changing the indices) and we have

$$\begin{aligned}\mathcal{L}^2_{\xi_1}\bar{g}_{0i} &= \alpha_1\xi_1^{k\prime\prime} + \xi^{k\ \prime}_{1,j}\delta_{ik}\xi_1^j - 4aa'\alpha_1\alpha_{1,i} - \alpha_{1,ij}a^2\xi_1^j + 4\xi_1^{k\prime}aa'\delta_{ik}\alpha_1 - a^2\alpha'_1\alpha_{1,i} +\\ &-a^2\alpha_{1,k}\xi^k_{1,i} - 2a^2\alpha'_1\alpha_{1,i} + 2a^2\delta_{kj}\xi_1^{j\prime}\xi^k_{1,i} + \alpha'_1\xi_1^{k\prime}a^2\delta_{ki} + a^2\delta_{ik}\xi_1^{j\prime} - a^2\alpha_{1,k}\xi^k_{1,i}.\end{aligned}$$

After some algebra, we can write

$$\tilde{\omega}_{2i} = \omega_{2i} + \xi'_{2i} - \alpha_{2,i} + \Sigma_i, \tag{3.88}$$

where Σ_i is defined as

$$\begin{aligned}\Sigma_i &= 2\left[2\mathcal{H}\omega_{1i}\alpha_1 + \omega'_{1i}\alpha_1 + \omega_{1i,k}\xi_1^k + \alpha'_1\omega_{1i} + 2\xi_1^j C_{1ij} - 2\alpha_{1,i}\psi_1 + \omega_{1k}\xi^k_{1,i}\right] +\\ &+4\mathcal{H}\alpha_1\left(\xi'_{1i} - \alpha_{1,i}\right) + \alpha'_1\left(\xi'_{1i} - 3\alpha_{1,i}\right) + \alpha_1\left(\xi''_{1i} - \alpha'_{1,i}\right) + \xi_1^{j\prime}\left(\xi_{i,j} + 2\xi_{j,i}\right) +\\ &+\xi_1^j\left(\xi'_{1i,j} - \alpha_{1,ij}\right) - \alpha_{1,j}\xi^j_{1,i}.\end{aligned}$$

Equation (3.88) is the transformation rule for the perturbation ω_{2i} and since we decomposed it into scalar and true vector, we can write the transformation for $\omega_2^{\parallel}$ and $\omega_{2i}^{\perp}$. For the scalar part we can apply the operator ∂^i to (3.88) and then take the inverse laplacian ∇^{-2}, recalling that $\omega_i^{\perp,i} = d_i^{,i} = 0$, namely

$$\tilde{\omega}_2^{\parallel} = \omega_2 + \beta'_2 - \alpha_2 + \nabla^{-2}\Sigma^i_{,i}, \tag{3.89}$$

now we insert this inside (3.88) and we solve for the solenoidal vector part

$$\tilde{\omega}^{\perp}_{2i} = \omega^{\perp}_{2i} + d'_{2i} + \Sigma_i - \nabla^{-2}\Sigma^{k}_{,ki}. \tag{3.90}$$

The last transformation rules of the perturbations are derived from the trace and traceless part of the ij components of the metric

$$\delta^2\tilde{g}_{ij} = \delta^2 g_{ij} + 2\mathcal{L}_{\xi_1}\delta^1 g_{ij} + \mathcal{L}^2_{\xi_1}\bar{g}_{ij} + \mathcal{L}_{\xi_2}\bar{g}_{ij}. \tag{3.91}$$

The first Lie derivative reads

$$\mathcal{L}_{\xi_1}\delta^1 g_{ij} = 2\left(2aa'C_{1ij} + a^2C'_{1ij}\right)\alpha_1 + a^2C_{1ij,k}\xi^k_1 + a^2\omega_{1i}\alpha_{1,j} + 2a^2C_{1ik}\xi^k_{1,j} + \\ + a^2\omega_{1j}\alpha_{1,i}2a^2C_{1kj}\xi^k_{1,i}.$$

The third one is easily

$$\mathcal{L}_{\xi_2}\bar{g}_{ij} = 2aa'\alpha_2\delta_{ij} + a^2\xi_{2i,j} + a^2\xi_{2j,i}.$$

As previously the calculation of the second order Lie derivative is a bit tedious but straightforward and the result is

$$\mathcal{L}^2_{\xi_1}\bar{g}_{ij} = 2\left[\left(a'^2 + aa''\right)\alpha_1^2 + aa'\left(\alpha_1\alpha'_1\alpha_{1,k}\xi^k_1\right)\right]\delta_{ij} + 4aa'\alpha_1\left(\xi_{1i,j} + \xi_{1j,i}\right) + \\ + a^2\alpha_1\left(\xi'_{1i,j} + \xi'_{1j,i}\right) + \xi^k_1 a^2\left(\xi_{1i,jk} + \xi_{1j,ik}\right) + a^2\xi^k_{1,j}\xi_{1j,k} + a^2\alpha_{1,j}\xi'_{1i} + \\ + a^2\alpha_{1,i}\xi'_{1j} - 2a^2\alpha_{1,j}\alpha_{1,i} + 2a^2\xi^k_{1,j}\xi_{1k,i}.$$

Thus we have

$$2\tilde{C}_{2ij} = 2C_{2ij} + 2\mathcal{H}\alpha_2\delta_{ij} + \xi_{1i,j} + \xi_{2j,i} + \Upsilon_{ij}, \tag{3.92}$$

where

$$\Upsilon_{ij} = 2\left[\left(\mathcal{H}^2 + \frac{a''}{a}\right)\alpha_1^2 + \mathcal{H}\left(\alpha_1\alpha'_1 + \alpha_{\,k}\xi^k_1\right)\right]\delta_{ij} + 4\Big[\alpha_1\left(C'_{1ij} + 2\mathcal{H}C_{1ij}\right) + \\ + C_{1ik}\xi^k_{1,j} + C_{1ij,k}\xi^k_1 + C_{1kj}\xi^k_{1,i}\Big] + 2\left(\omega_{1i}\alpha_{1,j} + \omega_{1j}\alpha_{1,i}\right) + \\ + 4\mathcal{H}\alpha_1\left(\xi_{1i,j} + \xi_{1j,i}\right) + \alpha_1\left(\xi'_{1i,j} + \xi'_{1j,i}\right) + \xi^k_1\left(\xi_{1i,jk} + \xi_{1j,ik}\right) + \\ + \xi^k_{1,j}\xi_{1i,k} + \xi^k_{1,i}\xi_{1j,k} + \alpha_{1,j}\xi'_{1i} + \alpha_{1,i}\xi'_{1j} - 2\alpha_{1,j}\alpha_{1,i} + \xi^k_{1,j}\xi_{1k,i}.$$

To have the transformation rule for the scalar ϕ_2 we take the trace $i = j$, thus

$$-6\tilde{\phi}_2 = -6\phi_2 + 6\mathcal{H}\alpha_2 + 2\beta^i_{2,i} + \Upsilon^i_i$$

which gives

$$\tilde{\phi}_2 = \phi_2 - \mathcal{H}\alpha_2 - \frac{1}{3}\nabla^2\beta_2 - \frac{1}{6}\Upsilon^i_i. \tag{3.93}$$

For $\chi_2^{\parallel}$ we substitute the previous relation into (3.92) and apply to it the operator $\partial^i\partial^j$ and take the double inverse laplacian $\nabla^{-2}\nabla^{-2}$ and after some algebra

$$\tilde{\chi}_2^{\parallel} = \chi_2^{\parallel} + \beta_2 - \frac{1}{4}\nabla^{-2}\Upsilon^i_i + \frac{3}{4}\nabla^{-2}\nabla^{-2}\Upsilon^{,ij}_{ij}. \tag{3.94}$$

For the vector perturbation $\chi_{2i}^{\perp}$, we plug the two previous results into (3.92) and apply to it again ∂^i, then we take the inverse laplacian. After some other tedious but straightforward math, we get

$$\tilde{\chi}_{2i}^{\perp} = \chi_{2i}^{\perp} + d_{2i} - \nabla^{-2}\nabla^{-2}\Upsilon^{,jk}_{jk,i} + \nabla^{-2}\Upsilon^{,k}_{ki}. \tag{3.95}$$

In the end, for tensor perturbation, namely gravitational waves, we can substitute everything into (3.92) and after some algebra (again) we find

$$\begin{aligned}\tilde{\chi}_{2ij}^{T} = \chi_{2ij}^{T} + \Upsilon_{ij} + \frac{1}{2}\left(\nabla^{-2}\Upsilon^{,kl}_{kl} - \Upsilon^k_k\right)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\Upsilon^{,kl}_{kl,ij} + \\ + \frac{1}{2}\nabla^{-2}\Upsilon^k_{k,ij} - \nabla^{-2}\left(\Upsilon^{,k}_{jk,i} + \Upsilon^{,k}_{ik,j}\right).\end{aligned} \tag{3.96}$$

From this last equation, despite its complexity, it's easy to see that gravitational waves at second order are not gauge invariant anymore.

Done with metric perturbations, we need to transform also the matter perturbations. We start from the 4-velocity and in particular from the time component of v_2^{μ} because we know that from (3.35)

$$v_2^0 = 3\psi_1^2 - \psi_2 + v_{1k}v_1^k, \tag{3.97}$$

so, using the transformation laws we derived before, we can write

$$\tilde{v}_2^0 = v_2^0 - \mathcal{H}\alpha_2 - \alpha_2' + \Delta, \tag{3.98}$$

where for brevity

$$\begin{aligned}\Delta = \alpha_1\left[\mathcal{H}\alpha_1' + \alpha_1\left(2\mathcal{H}^2 - \frac{a''}{a}\right) - 2\mathcal{H}v_1^0 + 2v_1^{0\prime} - \alpha_1''\right] + \\ + \xi_1^k\left(2v^0_{1,k} - \mathcal{H}\alpha_{1,k} - \alpha'_{1,k}\right) + \alpha_1'\left(\alpha_1' - 2v_1^0\right) - 2v_{1k}\alpha_1^{\prime,k} + \xi'_{1k}\alpha_1^{\prime,k}.\end{aligned}$$

For the spatial part of the velocity perturbation we have

$$\delta^2\tilde{u}^i = \delta^2 u^i + \mathcal{L}_{\xi_2}\bar{u}^i + 2\mathcal{L}_{\xi_1}\delta^1 u^i + \mathcal{L}^2_{\xi_1}\bar{u}^i, \tag{3.99}$$

now we use the definition of the Lie derivative of a vector

$$\mathcal{L}_{\xi_2}\bar{u}^i = \bar{u}^i_{,\mu}\xi_2^\mu - \bar{u}^\mu\xi^i_{2,\mu} = -\frac{1}{a}\xi_2^{i\,\prime}.$$

The second Lie derivative term reads

$$\mathcal{L}_{\xi_1}\delta^1 u_1^i = \delta u^i_{,\mu}\xi_1^\mu - \delta^1 u^\mu \xi^i_{1,\mu} = \frac{1}{a}\left(v_1^{i\,\prime}\alpha_1 + v^i_{1,k}\xi_1^k - v_1^0\xi_1^{i\,\prime} - v_1^k\xi^i_{1,k}\right) - \frac{a'}{a^2}\alpha_1 v_1^i$$

Now, the usual second order Lie derivative

$$\mathcal{L}^2_\xi\bar{u}^i = \xi^\beta\xi^\delta_{,\beta}\bar{u}^i_{,\delta} - 2\xi^\beta\bar{u}^\delta_{,\beta}\xi^i_{,\delta} + \bar{u}^\beta\xi^\delta_{,\beta}\xi^i_{,\delta} + \xi^\beta\xi^\delta\bar{u}^i_{,\beta\delta} - \bar{u}^\beta\xi^\delta\xi^i_{,\beta\delta}$$
$$= 2\frac{a'}{a^2}\alpha_1\xi_1^{i\,\prime} - \frac{1}{a}\alpha_1\xi_1^{i\,\prime\prime} - \frac{1}{a}\xi_1^k\xi^i_{1,k}{}' + \frac{1}{a}\alpha_1'\xi_1^{i\,\prime} + \frac{1}{a}\xi^i_{1,k}\xi_1^{k\,\prime},$$

putting everything together we get

$$\tilde{v}_2^i = v_2^i - \xi_2^{i\,\prime} + \Omega^i \tag{3.100}$$

where

$$\Omega^i = \alpha_1\left[2\left(v_1^{i\,\prime} - \mathcal{H}v_1^i\right) - \left(\xi_1^{i\,\prime\prime} - 2\mathcal{H}\xi_1^{i\,\prime}\right)\right] + \xi_1^{i\,\prime}\left(2\psi_1 + \alpha_1'\right) +$$
$$-\xi^i_{1,k}\left(2v_1^k + \xi_1^{k\,\prime}\right) + \xi_1^k\left(2v^i_{1,k} - \xi^i_{1,k}{}'\right).$$

The scalar part of the spatial second order velocity perturbation can be found easily taking the derivative along the i direction of (3.100) and then the inverse laplacian

$$\tilde{v}_2^{\parallel} = v_2^{\parallel} - \beta_2' + \nabla^{-2}\Omega^k_{,k} \tag{3.101}$$

and the vector part, substituting, is

$$\tilde{v}_2^{\perp i} = v_2^{\perp i} - d_2^{i\,\prime} + \Omega^i - \nabla^{-2}\Omega^{k,i}_{,k}. \tag{3.102}$$

The last matter perturbation is the energy density $\delta^2\rho$ and it transforms as

$$\delta^2\tilde{\rho}^i = \delta^2\rho + \mathcal{L}_{\xi_2}\bar{\rho} + 2\mathcal{L}_{\xi_1}\delta^1\rho + \mathcal{L}^2_{\xi_1}\bar{\rho}, \tag{3.103}$$

since ρ is a scalar quantity, the Lie derivatives up to second order are very easy to be computed and the result is

$$\delta^2\tilde{\rho} = \delta^2\rho + \bar{\rho}'\alpha_2 + \Xi, \tag{3.104}$$

where

$$\Xi = \alpha_1\left(\bar{\rho}''\alpha_1 + \bar{\rho}'\alpha_1' + 2\delta^1\rho'\right) + \xi_1^k\left(\bar{\rho}'\alpha_{1,k} + 2\delta^1\rho_{,k}\right)$$

3.5.2 Einstein equations

Before starting with the computations of the Einstein equations, I want to stress that, since the complete calculations done by hand are very long, some passages may be omitted for the sake of brevity, not to bother the reader. Moreover, since we know we are in Poisson gauge, we can shorten our notation putting

$$\omega^{\perp}_{(r)i} = \omega_{(r)i}$$
$$\chi^{T}_{(r)ij} = \chi_{(r)ij}$$

The first thing to do, in order to simplify a bit the calculations is to expand the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} + \frac{1}{2}\delta^2 g_{\mu\nu} \tag{3.105}$$

and the Christoffel symbols as

$$\Gamma^{\alpha}_{\beta\gamma} = \bar{\Gamma}^{\alpha}_{\beta\gamma} + \delta\Gamma^{\alpha}_{\beta\gamma} + \delta^2\Gamma^{\alpha}_{\beta\gamma} \tag{3.106}$$

where the interesting part is the last term, defined as

$$\delta^2\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\left\{\frac{1}{2}\bar{g}^{\alpha\lambda}\left[\delta^2 g_{\lambda\gamma,\beta} + \delta^2 g_{\lambda\beta,\gamma} - \delta^2 g_{\beta\gamma,\lambda}\right] + \delta g^{\alpha\lambda}\left[\delta g_{\lambda\gamma,\beta} + \delta g_{\lambda\beta,\gamma} + \right.\right.$$
$$\left.\left. -\delta g_{\beta\gamma,\lambda}\right] + \frac{1}{2}\delta^2 g^{\alpha\lambda}\left[\bar{g}_{\lambda\gamma,\beta} + \bar{g}_{\lambda\beta,\gamma} - \bar{g}_{\beta\gamma,\lambda}\right]\right\}. \tag{3.107}$$

Now, to proceed further we need to invert the metric up to second order, using the relation

$$g_{\mu\nu}g^{\nu\lambda} = \delta^{\lambda}_{\mu}, \tag{3.108}$$

which, up to second order in perturbation theory gives

$$g^{00} = -a^{-2}\left(1 - 2\psi_1 - \psi_2 + 4\psi_1^2\right)$$
$$g^{0i} = \frac{1}{2}a^{-2}\omega_2^i$$
$$g^{ij} = a^{-2}\left[\left(1 + 2\phi_1 + \phi_2 + 4\phi_1^2\right)\delta^{ij} - \chi_1^{ij} - \frac{1}{2}\chi_2^{ij} + \chi^i_{1l}\chi_1^{lj} - 4\phi_1\chi_1^{ij}\right].$$

Using the definition and the inverse metric we can now proceed and we can write the affine connections to second order

$$\delta^2\Gamma^0_{00} = \frac{1}{2}\psi_2' - 2\psi_1'\psi_1$$
$$\delta^2\Gamma^0_{i0} = \frac{1}{2}\psi_{2,i} - 2\psi_1\psi_{1,i} + \frac{1}{2}\mathcal{H}\omega_{2i}$$
$$\delta^2\Gamma^i_{00} = \frac{1}{2}\mathcal{H}\omega_2^i + \frac{1}{2}\omega_2^{i\,\prime} + \frac{1}{2}\psi_1^{,i} + 2\phi_1\psi_1^{,i} - \chi_1^{ik}\psi_{1,k}$$
$$\delta^2\Gamma^0_{ij} = \frac{1}{4}\chi_{2ij}' + \frac{1}{2}\mathcal{H}\chi_{2ij} - \psi_1\chi_{1ij}' - 2\mathcal{H}\psi_1\chi_{1ij} + 4\mathcal{H}\psi_1^2\delta_{ij} - \mathcal{H}\psi_2\delta_{ij} +$$
$$+ 4\mathcal{H}\psi_1\phi_1\delta_{ij} + 2\psi_1\phi_1'\delta_{ij} - \mathcal{H}\phi_2\delta_{ij} - \frac{1}{2}\phi_2'\delta_{ij} - \frac{1}{4}\omega_{2i,j} - \frac{1}{4}\omega_{2j,i}$$
$$\delta^2\Gamma^i_{0j} = -\frac{1}{2}\phi_1'\delta^{ij} + \frac{1}{4}\left(\chi_{2j}^{i\;\prime} + \omega^i_{2,j} - \omega_{2j}^{,i}\right) - 2\phi_1\phi_1'\delta^i_j + \phi_1\chi_{1j}^{i\;\prime} + \phi_1'\chi^i_{1j} - \frac{1}{2}\chi_1^{ik}\chi_{1kj}'$$
$$\delta^2\Gamma^i_{jk} = -\frac{1}{2}\mathcal{H}\omega_2^i\delta_{jk} - \frac{1}{2}\left(\phi_{2,j}\delta^i_k + \phi_{2,k}\delta^i_j - \phi_2^{,i}\delta_{jk}\right) + \frac{1}{4}\left(\chi^i_{2k,j} + \chi^i_{2j,k} - \chi^{,i}_{2jk}\right) +$$
$$- 2\phi_1\left(\phi_{1,k}\delta^i_j + \phi_{1,j}\delta^i_k - \phi_1^{,i}\delta_{jk}\right) + \phi_1\left(\chi^i_{1j,k} + \chi^i_{1k,j} - \chi^{,i}_{1jk}\right) + \phi_{1,k}\chi^i_{1j} + \phi_{1,j}\chi^i_{1k} +$$
$$- \phi_{1,l}\chi_1^{il}\delta_{kj} - \frac{1}{2}\chi_1^{il}\left(\chi_{1lj,k} + \chi_{1kl,j} - \chi_{1jk,l}\right).$$

In the same way as for the Christoffel symbols I can expand the Ricci tensor, having

$$\delta^2 R_{\mu\nu} = \delta^2\Gamma^\sigma_{\mu\nu,\sigma} - \delta^2\Gamma^\sigma_{\sigma\mu,\nu} + \delta^2\Gamma^\sigma_{\sigma\lambda}\overline{\Gamma}^\lambda_{\mu\nu} + \delta\Gamma^\sigma_{\sigma\lambda}\delta\Gamma^\lambda_{\mu\nu} + \overline{\Gamma}^\sigma_{\sigma\lambda}\delta^2\Gamma^\lambda_{\mu\nu} +$$
$$-\overline{\Gamma}^\sigma_{\nu\lambda}\delta^2\Gamma^\lambda_{\sigma\mu} - \delta\Gamma^\sigma_{\nu\lambda}\delta\Gamma^\lambda_{\sigma\mu} - \delta^2\Gamma^\sigma_{\nu\lambda}\overline{\Gamma}^\lambda_{\sigma\mu}.$$

Now we need to compute the derivative and products of Christoffel symbols at second order in perturbation theory. The algebra is straightforward but very tedious and involving. I'm going to reproduce the computations for $\delta^2 R_{00}$ and later I will report only the results for the other components, for brevity.

The first term is

$$\delta^2\Gamma^\sigma_{00,\sigma} = \delta^2\Gamma^0_{00,0} + \delta^2\Gamma^i_{00,i}$$
$$= \frac{1}{2}\psi_2'' - 2\psi_1''\psi_1 - 2\psi_1^2 + \frac{1}{2}\nabla^2\psi_2 + 2\phi_{1,i}\psi_1^{,i} + 2\phi_1\nabla^2\psi_1 - \chi_1^{ik}\psi_{1,ik}.$$

Then we have

$$\delta^2\Gamma^\sigma_{\sigma0,0} = \delta^2\Gamma^0_{00,0} + \delta^2\Gamma^i_{i0,0}$$
$$= \frac{1}{2}\psi_2'' - 2\psi_1''\psi_1 - 2\psi_1'^2 - \frac{3}{2}\phi_2'' - 6\phi_1' - 6\phi_1\phi_1'' - \frac{1}{2}\chi_1^{ik\,\prime}\chi_{1ki}' - \frac{1}{2}\chi_1^{ik}\chi_{1ki}''.$$

Now we need to compute the products of the connections

$$
\begin{aligned}
\delta^2\Gamma^\sigma_{\sigma\lambda}\bar{\Gamma}^\lambda_{00} &= \frac{1}{2}\mathcal{H}\psi_2' - 2\mathcal{H}\psi_1'\psi_1 - \frac{3}{2}\mathcal{H}\phi_2' - 6\mathcal{H}\phi_1\phi_1' - \frac{1}{2}\mathcal{H}\chi_1^{ik}\chi_{1ki}' \\
\delta\Gamma^\sigma_{\sigma\lambda}\delta\Gamma^\lambda_{00} &= \psi_1'^2 - 3\phi_1'\psi_1' - 2\psi_{1,i}\psi_1'^{,i} - 3\phi_{1,k}\psi_1'^{,k} \\
\bar{\Gamma}^\sigma_{\sigma\lambda}\delta^2\Gamma^\lambda_{00} &= 2\mathcal{H}\psi_2' - 8\mathcal{H}\psi_1'\psi_1 \\
\bar{\Gamma}^\sigma_{0\lambda}\delta^2\Gamma^\lambda_{\sigma 0} &= \frac{1}{2}\mathcal{H}\psi_2' - 2\mathcal{H}\psi_1'\psi_1 - \frac{3}{2}\mathcal{H}\phi_2' - 6\phi_1\phi_1' - \frac{1}{2}\mathcal{H}\chi_1^{il}\chi_{1li}' \\
\delta\Gamma^\sigma_{0\lambda}\delta\Gamma^\lambda_{\sigma 0} &= \psi_1'^2 + 2\psi_{1,i}\psi_1'^{,i} + 3\phi_1'^2 + \frac{1}{4}\chi^i_{1k}{}'\chi^k_{1i}{}' \\
\delta^2\Gamma^\sigma_{0\lambda}\bar{\Gamma}^\lambda_{\sigma 0} &= \frac{1}{2}\mathcal{H}\psi_2' - 2\mathcal{H}\psi_1'\psi_1 - \frac{3}{2}\phi_2'\mathcal{H} - 6\mathcal{H}\phi_1\phi_1' - \frac{1}{2}\mathcal{H}\chi_1^{il}\chi_{1il}'.
\end{aligned}
$$

Putting everything together and summing, we arrive to the desired result

$$
\begin{aligned}
\delta^2 R_{00} = &\frac{1}{2}\nabla^2\psi_2 - \phi_{1,i}\psi_1'^{,i} - \psi_{1,i}\psi_1'^{,i} + \frac{3}{2}\mathcal{H}\psi_2' + \frac{3}{2}\mathcal{H}\psi_2' + \frac{3}{2}\psi_2'' + 3\phi_1'^2 + \\
&+ 6\phi_1\phi_1'' + \frac{1}{4}\chi_1^{ik}{}'\chi_{1ki}' + \frac{1}{2}\chi_1^{ik}\chi_{1ki}'' + \frac{1}{2}\mathcal{H}\chi_1^{ik}\chi_{1ik}' - 6\mathcal{H}\psi_1'\psi_1 + 6\mathcal{H}\phi_1\phi_1' + \\
&+ 2\phi_1\nabla^2\psi_1 - \chi_1^{ik}\psi_{1,ik} - 3\phi_1'\psi_1'
\end{aligned}
\tag{3.109}
$$

For $\delta^2 R_{0i}$ we follow the same path, even if it is a bit more involving, and we have

$$
\begin{aligned}
\delta^2 R_{0i} = &\frac{1}{2}\mathcal{H}'\omega_{2i} + \psi_{2,i}' + 4\phi_1'\phi_{1,i} + 4\phi_1\phi_{1,i}' - \frac{1}{2}\phi_{1,k}\chi^k_{1i}{}' - \frac{1}{2}\chi_1^{kl}\chi_{1li,k}' + \\
&+ \frac{1}{2}\chi_1^{kl}\chi_{1lk,i}' + \frac{1}{4}\chi^{kl}_{1,i}{}' + \mathcal{H}^2\omega_{2i} - 2\phi_1'\psi_{1,i} - \frac{1}{2}\psi_{1,k}\chi^k_{1i}{}' + \\
&+ \mathcal{H}\psi_{2,i} - 4\mathcal{H}\psi_1\psi_{1,i} - \frac{1}{4}\nabla^2\omega_{2i}
\end{aligned}
\tag{3.110}
$$

The last component is the ij one and it is convenient to split it into a "diagonal" part, proportional to δ_{ij} and into a "non diagonal" one. We end up with

$$\begin{aligned}
\delta^2 R^{ND}_{ij} &= -\frac{1}{2}\psi_{2,ij} + \frac{1}{2}\phi_{2,ij} + 3\phi_{1,i}\phi_{1,j} + \psi_{1,i}\psi_{1,j} + \mathcal{H}^2\chi_{2ij} + \\
&- \frac{1}{2}(\omega_{2i,j} + \omega_{2j,i}) - \frac{1}{4}(\omega'_{2i,j} + \omega'_{2j,i}) - \frac{1}{4}\nabla^2\chi_{2ij} + \frac{1}{4}\chi^{kl}_{1,i}\chi_{1kl,j} + \\
&+ \frac{1}{2}\chi^{kl}_1\chi_{1kl,ij} - \psi'_1\chi'_{1ij} - \mathcal{H}\psi'_1\chi_{1ij} - \psi_{1,i}\phi_{1,j} - \psi_{1,j}\phi_{1,i} + \\
&+ \frac{1}{2}\psi_{1,k}\chi^k_{1i,j} + \frac{1}{2}\psi_{1,k}\chi^k_{1j,i} - \frac{1}{2}\psi_{1,k}\chi'^k_{1ij} + \frac{1}{2}\phi'_1\chi'_{1ij} - 3\mathcal{H}\phi'_1\chi_{1ij} + \\
&+ \frac{1}{2}\phi_{1,k}\chi^k_{1i,j} + \frac{1}{2}\phi_{1,k}\chi^k_{1j,i} - \frac{3}{2}\phi_{1,k}\chi'^k_{1ij} + 2\phi_1\phi_{1,ij} - \phi_1\nabla^2\chi_{1ij} + \\
&+ \phi_{1,ik}\chi^k_{1j} + \phi_{1,jk}\chi^k_{1i} - \frac{1}{2}\chi^{kl}_1\chi_{1li,jk} - \frac{1}{2}\chi^{kl}_1\chi_{1lj,ik} + \frac{1}{2}\chi^{kl}_1\chi_{1ij,kl} + \\
&+ 2\psi_1\psi_{1,ij} - 2\mathcal{H}\psi_1\chi'_{1ij} - 4\mathcal{H}^2\psi_1\chi_{1ij} + \frac{1}{4}\chi''_{2ij} + \frac{1}{2}\mathcal{H}'\chi_{2ij} + \frac{1}{2}\chi'_{2ij} + \\
&- \psi_1\chi''_{1ij} - 2\mathcal{H}'\psi_1\chi_{1ij} + \frac{1}{2}\chi^k_{1i,l}\chi'^l_{1jk} - \frac{1}{2}\chi'_{1ik}\chi^{k\,\prime}_{1j} - \frac{1}{2}\chi^k_{1i,l}\chi^l_{1j,k}
\end{aligned} \tag{3.111}$$

and

$$\begin{aligned}
\delta^2 R^{D}_{ij} &= \delta_{ij}\Bigg[4\mathcal{H}'\psi_1^2 + 4\mathcal{H}\psi_1\psi'_1 - \mathcal{H}'\psi_2 - \frac{1}{2}\mathcal{H}\psi'_2 + 4\mathcal{H}'\psi_1\phi_1 + \\
&+ 2\mathcal{H}\psi'_1\phi_1 + \phi'_1\psi'_1 + 2\psi_1\phi''_1 - \mathcal{H}'\phi_2 - \frac{1}{2}\phi''_2 + \frac{1}{2}\nabla^2\phi_2 + \phi_{1,k}\phi'^k_1 + \\
&+ 2\phi_1\nabla^2\phi_1 - \phi_{1,lk}\chi^{kl}_1 + \psi_{1,k}\phi'^k_1 + \phi'^2_1 - \frac{5}{2}\mathcal{H}\phi'_2 - \frac{1}{2}\mathcal{H}\chi^{kl}_1\chi'_{1lk} + \\
&+ 8\mathcal{H}^2\psi^-_1 2\mathcal{H}^2\psi_2 - 2\mathcal{H}^2\phi_2 + 8\mathcal{H}^2\psi_1\phi_1 + 10\mathcal{H}\phi'_1\psi_1\Bigg].
\end{aligned} \tag{3.112}$$

Going through these expressions we can start seeing terms like

$$\frac{1}{2}\phi_{1,k}\chi^k_{1i,j}$$

which are appearing due to the fact that we are not neglecting first order tensor perturbations and all these terms will appear in the following expressions, making the results much longer and difficult.

Our goal in this section is to write down the Einstein equations in the mixed form

$$G^i_j = R^i_j - \frac{1}{2}\delta^i_j R = 8\pi G T^i_j, \tag{3.113}$$

so we need to raise an index, by doing

$$R^\mu_\nu = R_{\rho\nu} g^{\mu\rho}. \tag{3.114}$$

At second order in perturbation theory our previous expression can be written as

$$\delta^2 R^\mu_\nu = \bar{g}^{\mu\rho}\delta^2 R_{\rho\nu} + \delta g^{\mu\rho}\delta R_{\rho\nu} + \frac{1}{2}\delta^2 g^{\mu\rho}\bar{R}_{\rho\nu}. \tag{3.115}$$

Thus we have

$$\begin{aligned}\delta^2 R^0_0 = -a^{-2}\Bigg[&\frac{1}{2}\nabla^2\psi_2 - \phi_{1,i}\psi_1^{,i} - \psi_{1,i}\psi_1^{,i} + \frac{3}{2}\mathcal{H}\psi_2' + \frac{3}{2}\phi_2'' + 3\phi_1'^2 + 6\phi_1\phi_1''+\\ &+ \frac{1}{4}\chi_1^{ik'}\chi_{1ki}' + \frac{1}{2}\chi_1^{ik}\chi_{1ki}'' + \frac{1}{2}\mathcal{H}\chi_1^{ik}\chi_{1ki}' - 12\mathcal{H}\psi_1\psi_1' + 6\mathcal{H}\phi_1\phi_1' + 2\phi_1\nabla^2\psi_1+\\ &- \chi_1^{ik}\psi_{1,ik} - 3\phi_1'\psi_1' - 6H\psi_1\phi_1' - 6\psi_1\phi_1'' - 1\psi_1\nabla^2\psi_1 + 3\mathcal{H}'\psi_1 - 12\mathcal{H}'\psi_1^2+\\ &+ \frac{3}{2}\mathcal{H}\phi_2'\Bigg]\end{aligned} \tag{3.116}$$

$$\begin{aligned}\delta^2 R^i_0 = a^{-2}\Bigg[&-\mathcal{H}'\omega_2' + \phi_2'^{,i} + 4\phi_1'\phi_1^{,i} - \frac{1}{2}\phi_{1,k}\chi_1^{ki'} - \phi_{1,k}'\chi_1^{ki} - \frac{1}{2}\chi_1^{ml}\chi_{1l,m}'{}^{i}+\\ &+ \frac{1}{2}\chi_1^{ml}\chi'^{,i}_{1lm} + \frac{1}{4}\chi_1^{ml,i}\chi_1 ml' + \mathcal{H}^2\omega_2^i - 2\phi_1'\psi_1^{,i} - \frac{1}{2}\psi_{1,k}\chi_1^{ki'} + \mathcal{H}\psi_2^{,i}+\\ &- 4\mathcal{H}\psi_1\psi_1^{,i} - \frac{1}{4}\nabla^2\omega_2^i + 4\mathcal{H}\phi_1\psi_1^{,i} - 2\mathcal{H}\psi_{1,k}\chi_1^{ik}\Bigg]\end{aligned} \tag{3.117}$$

$$\begin{aligned}\delta^2 R^{i\,D}_j = a^{-2}\Bigg[&4\mathcal{H}'\psi_1^2 + 4\mathcal{H}\psi_1'\psi_1 - \mathcal{H}'\psi_2 - \frac{1}{2}\mathcal{H}\psi_2' + \phi_1'\psi_1' + 2\psi_1\phi_1''+\\ &- \frac{1}{2}\phi_2'' + \frac{1}{2}\nabla^2\phi_2 + \phi_{1,k}\phi_1^{,k} + 4\phi_1\nabla^2\phi_1 - \phi_{1,lk}\chi_1^{kl} + \psi_{1,k}\phi_1^{,k}+\\ &+ \phi_1'^2 - \frac{5}{2}\mathcal{H}\phi_2' - \frac{1}{2}\mathcal{H}\chi_1^{kl}\chi_{1kl}' + 8\mathcal{H}^2\psi_1^2 - 2\mathcal{H}^2\psi_2 + 10\mathcal{H}\phi_1'\psi_1+\\ &- 10\mathcal{H}\phi_1'\phi_1 - 2\phi_1\phi_1''\Bigg]\delta^i_j\end{aligned} \tag{3.118}$$

$$\delta^2 R^{i\,ND}_{j} = a^{-2}\Bigg[-\frac{1}{2}\psi'^{,i}_{2,j}+\frac{1}{2}\phi'^{,i}_{2,j}+3\phi'^{,i}_{1}\phi_{1,j}+\psi'^{,i}_{1}\psi_{1,j}-\frac{1}{2}\mathcal{H}\left(\omega^{i}_{2,j}+\omega'^{,i}_{2j}\right)+$$
$$-\frac{1}{4}\left(\omega^{i}_{1,j}{}'+\omega'^{,i}_{2j}{}'\right)-\frac{1}{4}\nabla^2\chi^{i}_{2j}+4\phi_1\phi'^{,i}_{1,j}+\frac{1}{4}\chi^{lk,i}_{1}{}'\chi_{1kl,j}+\frac{1}{2}\chi^{kl}_{1}\chi'^{,i}_{1lk,j}+$$
$$-\psi'_1\chi^{i}_{1j}{}'-\psi'^{,i}_{1}\phi_{1,j}-\psi_{1,j}\phi'^{,i}_{1}+\frac{1}{2}\psi_{1,k}\chi^{ki}_{1,j}+\frac{1}{2}\psi_{1,k}\chi^{k,i}_{1j}+$$
$$-\frac{1}{2}\psi_{1,k}\chi^{i,k}_{1j}+\frac{1}{2}\phi'_1\chi^{i}_{1j}{}'+2\mathcal{H}\phi'_1\chi^{i}_{1j}+\frac{1}{2}\phi_{1,k}\chi^{ki}_{1,j}+\frac{1}{2}\phi_{1,k}\chi^{k,i}_{1j}+$$
$$-\frac{3}{2}\phi_{1,k}\chi^{i,k}_{1j}-2\phi_1\nabla^2\chi^{i}_{1j}+\phi'^{,i}_{1,k}\chi^{k}_{1j}-\frac{1}{2}\chi^{kl}_{1}\chi^{i}_{1l,jk}-\frac{1}{2}\chi^{kl}_{1}\chi'^{,i}_{1lj,k}+$$
$$+\frac{1}{2}\chi^{kl}_{1}\chi^{i}_{1j,kl}+2\psi_1\psi'^{,i}_{1,j}-2\mathcal{H}\psi_1\chi^{i}_{1j}{}'+\frac{1}{4}\chi^{i}_{2j}{}''+\frac{1}{2}\mathcal{H}\chi^{i}_{2j}{}'-\psi_1\chi^{i}_{1j}{}''+$$
$$+\frac{1}{2}\chi^{ik}_{1,l}\chi'^{,l}_{1jk}-\frac{1}{2}\chi^{i}_{1k}{}'-\frac{1}{2}\chi^{ki}_{1,l}\chi^{l}_{1j,k}+\phi_1\chi^{i}_{1j}{}''+2\mathcal{H}\phi_1\chi^{i}_{1j}{}'+$$
$$-2\phi_1\psi^{i}_{1,j}-\frac{1}{2}\chi^{ik}_{1}\chi''_{1kj}-\mathcal{H}\chi^{ik}_{1}\chi'_{1kj}+\phi''_1\chi^{i}_{1j}-\nabla^2\phi_1\chi^{i}_{1j}+$$
$$+\frac{1}{2}\chi^{ik}_{1}\nabla^2\chi_{1kj}+\chi^{ik}_{1}\psi_{1,kj} \tag{3.119}$$

The Ricci scalar can be computed and it is

$$\delta^2 R = a^{-2}\Bigg[24\mathcal{H}'\psi_1^2+24\mathcal{H}\psi_1\psi'_1-6\mathcal{H}\psi_2-3\mathcal{H}\psi'_2+6\phi'_1\psi'_1+12\psi_1\phi''_1+$$
$$-3\phi''_2+2\nabla^2\phi_2+6\phi_{1,k}\phi'^{,k}_{1}+16\phi_1\nabla^2\phi_1-2\phi_{1,lk}\chi^{kl}_{1}-9\mathcal{H}\phi'_2+$$
$$-3\mathcal{H}\chi^{kl}_{1}\chi'_{1kl}+24\mathcal{H}^2\psi_1^2-6\mathcal{H}^2\psi_2+36\mathcal{H}\psi_1\phi_1-\nabla^2\psi_2+$$
$$+2\psi_{1,k}\psi'^{,k}_{1}+\frac{3}{4}\chi^{ml}_{1,k}\chi'^{,k}_{1lm}+\chi^{ml}_{1}\nabla^2\chi_{1ml}+4\psi_1\nabla^2\psi_1+$$
$$-\frac{3}{4}\chi^{kl\prime}_{1}-\frac{1}{2}\chi^{m}_{1k,l}\chi^{lk}_{1,m}-36\mathcal{H}\phi_1\phi'_1-4\phi_1\nabla^2\psi_1-\chi^{kl}_{1}\chi''_{1lk}+$$
$$+2\chi^{kl}_{1}\psi_{1,lk}-12\phi_1\phi''_1+2\psi_{,k}\phi'^{,k}_{1}\Bigg] \tag{3.120}$$

Now we move to the matter perturbations computing the second order energy momentum tensor from the definition

$$T^{\mu}_{\nu} = (\rho+p)u^{\mu}u_{\nu}+p\delta^{\mu}_{\nu}+\Sigma^{\mu}_{\nu} \tag{3.121}$$

where, from now on I set the anisotropic stress tensor Σ^{μ}_{ν} to be zero because, treating the baryons and the cold dark matter, being massive particles, a perfect fluid approximation is good. We recall that

$$u^{\mu} = \frac{1}{a}\left(\delta^{\mu}_{0}+v^{\mu}_{1}+\frac{1}{2}v^{\mu}_{2}\right) \tag{3.122}$$

from which we can compute u_μ. We have

$$u_0 = g_{0\rho}u^\rho = a\left(-1 - \psi_1 - \frac{1}{2}\psi_2 + \frac{1}{2}\psi_1^2 - \frac{1}{2}v_{1k}v_1^k\right) \tag{3.123}$$

$$u_i = \frac{1}{2}a\omega_{2i} + a\left(v_{1i} + \frac{1}{2}v_{2i} - 2\phi_1 v_{1i} + v_1^j\chi_{1ij}\right) \tag{3.124}$$

For the 00 component at second order we find

$$\delta^2 T_0^0 = -\frac{1}{2}\delta^2\rho - (1+w)\bar{\rho}v_{1k}v_1^k \tag{3.125}$$

where w is the equation of state. The other components of T_ν^μ are given by

$$\delta^2 T_0^i = -(1+w)\bar{\rho}\left[\frac{1}{2}v_2^i + \left(\psi_1 + \frac{\delta\rho}{\bar{\rho}}\right)v_1^i\right] \tag{3.126}$$

$$\delta^2 T_j^i = (1+w)\,\bar{\rho}v_1^i v_{ij} + \frac{1}{2}w\delta^2\rho\delta_j^i. \tag{3.127}$$

With all these expressions, we can finally write the mixed Einstein equations. We can start from the 00 equation

$$\delta^2 G_0^0 = \delta^2 R_0^0 - \frac{1}{2}R = 8\pi G\delta^2 T_0^0 \tag{3.128}$$

which reads

$$\frac{1}{8}\chi_1^{ik\prime}\chi_{1ki}' + \mathcal{H}\chi_1^{ik}\chi_{1ki}' - 12\mathcal{H}^2\phi^2 - 3\phi'^2 + 3\mathcal{H}\Phi_2' + 3\mathcal{H}^2\Psi_2 - \nabla^2\Phi_2 - 3\phi_{,k}\phi'^{,k} +$$
$$+\phi_{,ik}\chi_1^{ik} - \frac{1}{2}\chi_1^{mk}\nabla^2\chi_{1mk} - \frac{3}{8}\chi_{1,k}^{ml}\chi_{ml}^{1,k} + \frac{1}{4}\chi_{1k,l}^{m}\chi_{1,m}^{lk} - 8\phi\nabla^2\phi = 8\pi Ga^2\delta^2 T_0^0$$

where ϕ is the first order scalar potential (remember that at first order, for a perfect fluid we had $\phi_1 = \psi_1 \equiv \phi$).

The second equation is derived from the traceless ij component

$$\delta^2 G_j^i = 8\pi G\delta^2 T_j^i \tag{3.129}$$

and we can write that the traceless part is

$$\delta^2 G_j^{iTL} = \delta^2 R_j^{iND} + \delta^2 R_j^{iD} - \frac{1}{2}\delta_j^i\delta^2 R - \frac{1}{3}\delta_j^i\left(\delta^2 R_k^{kND} + \delta^2 R_k^{kD} - \frac{1}{2}\delta_k^k\delta^2 R\right)$$
$$= \delta^2 R_j^{iND} + \delta^2 R_j^{iD} - \frac{1}{3}\delta_j^i\delta^2 R_k^{kND} - \frac{1}{3}\delta_j^i\delta^2 R_k^{kD}.$$

We can easily see that

$$\delta^i_j \delta^2 R^{kD}_k = 3\delta^2 R^{iD}_j, \tag{3.130}$$

thus we have

$$\delta^2 G^{iTL}_j = \delta^2 R^{iND}_j - \frac{1}{3}\delta^i_j \delta^2 R^{kND}_k. \tag{3.131}$$

It is convenient to split the $\delta^2 G^i_j$ into 3 parts, respectively proportional to terms at second order in ϕ or Φ_2 and Ψ_2, quadratic in χ and mixed like "$\phi\chi$"

$$\delta^2 G^{iTL}_j = a^{-2}\left(S^i_j + I^i_j + M^i_j\right), \tag{3.132}$$

where

$$S^i_j = 2\partial^i\phi\partial_j\phi + 4\phi\partial^i\partial_j\phi + \frac{1}{2}\left(\Phi_2 - \Psi_2\right)^{,i}_{,j} - \frac{1}{3}\delta^i_j\left[\frac{1}{2}\nabla^2\left(\Phi_1 - \Psi_2\right) + \right.$$
$$\left. + 2\partial^k\phi\partial_k\phi + 4\phi\partial^k\partial_k\phi\right] \tag{3.133}$$

$$I^i_j = 2 - \frac{1}{4}\nabla^2\chi^i_{2j} + \frac{1}{4}\chi^{lk,i}_1\chi_1 kl,i + \frac{1}{2}\chi^{kl}_1\chi'^{i}_{1lk,j} - \frac{1}{2}\chi^{kl}_1\chi^i_{1l,jk} - \frac{1}{2}\chi^{kl}_1\chi^i_{1lj,k} +$$
$$+ \frac{1}{2}\chi^{kl}_1\chi^i_{1j,kl} + \frac{1}{4}\chi^{i}_{2j}{}'' + \frac{1}{2}\mathcal{H}\chi^{i}_{2j}{}' + \frac{1}{2}\chi^{ik}_{1,l}\chi'^{l}_{1jk} - \frac{1}{2}\chi^{i}_{1k}{}'\chi^{k}_{1j}{}' - \frac{1}{2}\chi^{ki}_{1,l}\chi^l_{1j,k} +$$
$$- \frac{1}{2}\chi^{ik}_1\chi''_{1kj} - \mathcal{H}\chi^{ik}_1\chi'_{1kj} + \frac{1}{2}\chi^{ik}_1\nabla^2\chi_{1kj} - \frac{1}{3}\delta^i_j\left[\frac{1}{4}\chi^{lk,m}_1\chi_{1kl,m} + \frac{1}{2}\chi^{kl}_1\nabla^2\chi_{1lk} + \right.$$
$$\left. + \frac{1}{2}\chi^{mk}_{1,l}\chi'^{l}_{1mk} - \frac{1}{2}\chi^{m}_{1k}{}'\chi^{k}_{1m}{}' - \frac{1}{2}\chi^{km}_{1,l}\chi^l_{1m,k} - \frac{1}{2}\chi^{mk}_1\left(\chi''_{1km} + 2\mathcal{H}\chi'_{1km} - \nabla^2\chi_{1km}\right)\right] \tag{3.134}$$

and

$$M^i_j = -\frac{1}{2}\phi'\chi^{i}_{1j}{}' + \phi_{,k}\chi^{ki}_{1,j} + 2\mathcal{H}\phi'\chi^i_{1j} + \phi_{,k}\chi^{k,i}_{1j} - 2\phi_{,k}\chi^{i,k}_{1j} + \phi'^{,i}_{,k}\chi^k_{1j} +$$
$$\phi''\chi^i_{1j} - \nabla^2\phi\chi^i_{1j} + \phi_{,kj}\chi^{ik}_1 - \frac{2}{3}\delta^i_j\phi_{,km}\chi^{km}_1. \tag{3.135}$$

Since we are interested in the second order scalar part of this equation, we can remove second order tensors and vectors just by applying the operator $\partial_i\partial^j$ to the previous equation. We need to remember here the properties of the tensors and vectors: the first are transverse and traceless and the second are transverse. Thus we can compute the second derivative applied to S^i_j and we find

$$\partial_i\partial^j S^i_j = \frac{1}{3}\nabla^2\nabla^2\left(\Phi_2 - \Psi_2\right) + \frac{4}{3}\nabla^2\nabla^2\phi^2 - \partial_i\partial^j\left(2\partial^i\phi\partial_j\phi\right) + \frac{1}{3}\nabla^2\left(2\partial^i\phi\partial_i\phi\right), \tag{3.136}$$

where we can see the interesting quantity $\Phi_2 - \Psi_2$. For the term I^i_j we need to use the first order equation (3.77) with some tricks and we have

$$\begin{aligned}\partial_i\partial^j I^i_j = &\frac{1}{6}\nabla^2\nabla^2\left(\chi_1^{kl}\chi_{lk}\right) - \frac{1}{4}\partial_i\partial^j\Big[\partial^i\chi_1^{kl}\partial_j\chi_{kl} + 2\big(\chi_1^{kl}\partial_j\partial_k\chi^i_{1l} + \chi^{kl}\partial^i\partial_k\chi_{1lj} + \\ &- \chi_1^{kl}\partial_k\partial_l\chi^i_j - \partial_l\chi_1^{ik}\partial^l\chi_{1jk} + \chi^i_{1k}{}'\chi^k_{1j}{}' + \partial_l\chi_1^{ki}\partial_k\chi^l_{1j}\big)\Big] + \\ &+ \frac{1}{12}\nabla^2\Big[\partial^m\chi_1^{lk}\partial_m\chi_{1kl} + 2\big(\chi^m_{1k}{}'\chi^k_{1m}{}' + \partial_l\chi_1^{km}\partial_k\chi^l_{1m} - \partial_l\chi_1^{mk}\partial^l\chi_{1mk}\big)\Big].\end{aligned} \tag{3.137}$$

In the end, we can do the same for the term M^i_j

$$\begin{aligned}\partial_i\partial^j M^i_j = &-\frac{2}{3}\nabla^2\left(\phi_{,km}\chi_1^{km}\right) + \partial_i\partial^j\Big[2\mathcal{H}\phi'\chi^i_{1j} - \frac{1}{2}\phi'\chi^i_{1j}{}' + \phi_{,k}\big(\chi^{ki}_{1,j} + \chi^{k,i}_{1j} + \\ &- 2\chi^{i,k}_{1j}\big) + \phi''\chi^i_{1j} - \nabla^2\phi\chi^i_{1j} + \phi'^{,i}_{,k}\chi^k_{1j} + \phi_{,kj}\chi_1^{ik}\Big].\end{aligned} \tag{3.138}$$

Everything is done for the left-hand-side of the equation, now we need to apply the operator $\partial_i\partial^j$ also to the traceless part of the second order energy momentum tensor $\delta^2 T^i_j$. Namely, we have

$$\partial_i\partial^j\delta^2 T^{iTL}_j = \partial_i\partial^j\left[(1+w)\bar{\rho}v^i_1 v_{1j}\right] - \frac{1}{3}\nabla^2\left[(1+w)\bar{\rho}v^k_1 v_{1k}\right]. \tag{3.139}$$

Inserting all the previous expressions into the Einstein equation and solving for the quantity $\Phi_2 - \Psi_2$ we can write

$$\frac{1}{3}\nabla^2\nabla^2\left(\Phi_2 - \Psi_2\right) = -\frac{4}{3}\nabla^2\nabla^2\phi^2 - \frac{1}{6}\nabla^2\nabla^2\left(\chi_1^{lk}\chi_{1lk}\right) + \frac{1}{4}\partial_i\partial^j\Big[8\partial^i\phi\partial_j\phi + \\ + \partial^i\chi_1^{kl}\partial_j\chi_{1kl} + 2\big(\chi_1^{kl}\partial_j\partial_k\chi^i_{1l} + \chi^{kl}\partial^i\partial_k\chi_{1lj} - \chi_1^{kl}\partial_k\partial_l\chi^i_j + \\ - \partial_l\chi_1^{ik}\partial^l\chi_{1jk} + \chi^{i\ \prime}_{1k}\chi^{k\ \prime}_{1j} + \partial_l\chi_1^{ki}\partial_k\chi^l_{1j}\big) - 8\mathcal{H}\phi'\chi^i_{1j} + \\ + 2\phi'\chi^{i\ \prime}_{1j} - 4\phi_{,k}\big(\chi^{ki}_{1,j} + \chi^{k,i}_{1j} - 2\chi^{i,k}_{1j}\big) - 4\phi''\chi^i_{1j} + 4\nabla^2\phi\chi^i_{1j} + \\ - 4\phi'^{i}_{,k}\chi^k_{1j} - 4\phi_{,kj}\chi_1^{ik} + 12(1+w)\mathcal{H}^2 v_1^i v_{1j}\Big] + \\ - \frac{1}{12}\nabla^2\Big[8\partial^i\phi\partial_i\phi + \partial^m\chi_1^{lk}\partial_m\chi_{1kl} + 2\big(\chi^{m\ \prime}_{1k}\chi^{k\ \prime}_{1m} + \\ + \partial_l\chi_1^{km}\partial_k\chi^l_{1m} - \partial_l\chi_1^{mk}\partial^l\chi_{1mk}\big) - 8\phi_{,km}\chi_1^{km} + \\ + 12(1+w)\mathcal{H}^2 v_1^i v_{1i}\Big], \tag{3.140}$$

where I used the background Friedmann equation

$$8\pi G\bar{\rho}a^2 = 3\mathcal{H}^2. \tag{3.141}$$

In conclusion we have

$$\Phi_2 - \Psi_2 = -4\phi^2 - \frac{1}{2}\chi_1^{kl}\chi_{1kl} + \frac{3}{4}\nabla^{-2}\nabla^{-2}\partial_i\partial^j A^i_j - \frac{1}{4}\nabla^{-2}B, \tag{3.142}$$

where I introduced A^i_j and B for brevity: they are defined respectively as the first and the second term in the square bracket. This equation is the second order equivalent of the linear constraint $\phi_1 = \psi_1$ and it reveals the underlying effects of terms quadratic in the first-order scalar and tensor perturbations which can create a source of difference between the two second order scalar potentials in Poisson gauge, even if we are considering a perfect fluid with zero anisotropic stress tensor. Inside A^i_j and B we have contributions from "$\phi\phi$", "$\chi\chi$" and "$\chi\phi$" and these last two contributions are one of the original results of this thesis project (cfr. [2] and [22]).

3.5.3 Energy and momentum conservation equations

In order to close the system we need two more equations and these two equations can be derived from the two second order energy and momentum conservation equations. We can start from the divergence of the energy-momentum tensor for which we know that from the Bianchi identity we must have

$$\nabla_\mu T^\mu_\nu = 0 \tag{3.143}$$

where ∇_μ is the covariant derivative. We have

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\rho} T^\rho_\nu - \Gamma^\rho_{\mu\nu} T^\mu_\rho , \tag{3.144}$$

which at second order is

$$\begin{aligned}\nabla_\mu T^\mu_\nu\big|_{\text{II order}} = {} & \partial_\mu \delta^2 T^\mu_\nu + \bar{\Gamma}^\mu_{\mu\rho}\delta^2 T^\rho_\nu + \delta\Gamma^\mu_{\mu\rho}\delta T^\rho_\nu + \delta^2\Gamma^\mu_{\mu\rho}\bar{T}^\rho_\nu + \\ & - \bar{\Gamma}^\rho_{\mu\nu}\delta^2 T^\mu_\rho - \delta\Gamma^\rho_{\mu\nu}\delta T^\mu_\rho - \delta^2\Gamma^\rho_{\mu\nu}\bar{T}^\mu_\rho .\end{aligned} \tag{3.145}$$

The energy conservation equation at second order is described by

$$\nabla_\mu T^\mu_0\big|_{\text{II order}} = 0 \tag{3.146}$$

having set $\nu = 0$. Thus, after some algebra we have

$$\begin{aligned}&\delta^2\rho' + 3\mathcal{H}(1+w)\delta^2\rho - 3(1+w)\phi_2'\bar{\rho} + 2(1+w)\bar{\rho}' v_{1i} v_1^i + 2(1+w)\bar{\rho}(v_{1i}v_1^i)' + \\ &+ 2(1+w)\bar{\rho}\left[\frac{1}{2}v^i_{2,i} + v_1^i\left(\psi_{1,i} + \frac{\delta\rho_{,i}}{\bar{\rho}}\right) + \left(\psi_1 + \frac{\delta\rho}{\bar{\rho}}\right)v^i_{1,i}\right] - 4\phi_{,i}v_1^i\bar{\rho}(1+w) + \\ &- 6(1+w)\phi'(\delta\rho + 2\bar{\rho}\phi) + 8\mathcal{H}(1+w)\bar{\rho}v_1^i v_{1i} - (1+w)\bar{\rho}\chi_1^{ik}\chi'_{1ki} = 0\end{aligned} \tag{3.147}$$

In a very analogous way, setting $\nu = i$ in (3.143), we can easily retrieve the momentum conservation equation

$$\begin{aligned}&(1+w)(\bar{\rho}v_{2i})' + (1+w)\bar{\rho}(\psi_{2,i} + 4\mathcal{H}v_{2i}) + \frac{1}{2}\delta^2 P_{,i} + \frac{1}{2}(1+w)(\bar{\rho}\omega_{2i})' + \\ &+ 2\mathcal{H}(1+w)\bar{\rho}\omega_{2i} + (1+w)(v_{1i}\delta\rho)' + (1+w)\bar{\rho}\Big[\left(v_1^k v_{1i}\right)_{,k} - 2\phi_1\phi_{1,i} + \\ &- 12\mathcal{H}\phi_1 v_{1i} + 4\mathcal{H}v_1^k\chi_{1ki} - 5v_{1i}\phi_1' - 3\phi_1 v_1'\Big] + (1+w)\delta\rho\left[\phi_{1,i} + 4\mathcal{H}v_{1i}\right] + \\ &+ (1+w)\left[\bar{\rho}v_1^k\chi_{1ki}\right]' - 3\phi_1 v_1(1+w)\delta\rho' = 0\end{aligned} \tag{3.148}$$

4

The Boltzmann equations

Who can see without admiration how the eternal stars slavishly obey the laws that the human spirit has not indeed given to but learnt from them.

Ludwig Boltzmann, *Theoretical Physics and Philosophical Problems: Selected Writings*

If one is interested in the anisotropies in the CMB and in the inhomogeneities of the matter distribution, he or she needs to study how photons, baryons and dark matter are coupled to themselves and to gravity. Recalling the book by Dodelson [32], we can see in a very simple picture (Figure 4.1) how these fields are intricate. The metric, i.e. gravity, affects everything in the Universe: starting from photons up to neutrinos and dark matter. Photons are affected by the gravity and also by the Compton scattering with free electrons: in general photons are scattered also by protons, but since Compton scattering scales as the inverse mass squared and protons are way more massive than electrons, scattering processes between photons and protons can be neglected. Electrons are coupled to protons through Coulomb scattering and are also affected by the metric. Moreover, in our picture we also have neutrinos which, after decoupling at T about 1 MeV, interact only with the metric, transferring their anisotropy to the geometry and the geometry transfers this information to the photons. Then we have the so called *dark sector*, in which we collect dark matter and dark energy. Dark matter dominates the right hand side in the Einstein equation (at later times) so the perturbations in the dark matter are the most important perturbations that can change the geometry. We also have dark energy: the cosmological constant, that can't fluctuate neither in space

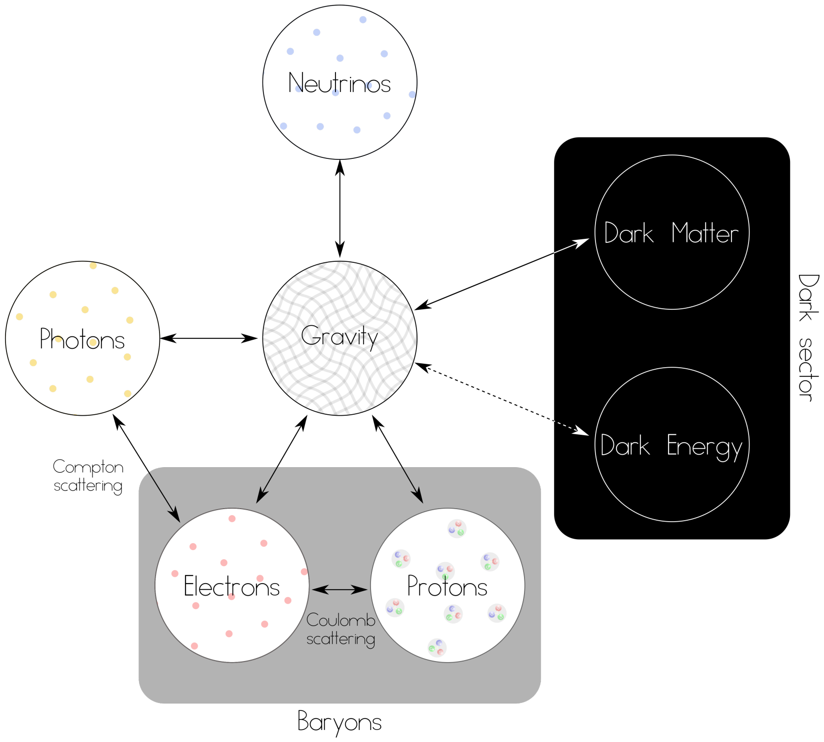


Figure 4.1: How different species interact with themselves and with the metric. The arrows are encoded in the coupled Boltzmann–Einstein equations. Since electrons and protons are tight coupled through Coulomb scattering, we can consider them together to be baryons. We added also the presence of Dark Energy for which the arrow is dashed: we do not consider perturbations to the dark energy if it is a cosmological constant.

nor in time, which appears only in the background evolution. There is a hidden arrow which connect dark matter and dark energy, because there are some models in which DM and DE can be coupled together, i.e. the *quintessence models* involve a scalar field for DE which could couple to DM.

The way to study how these species evolve is to write down and solve the Boltzmann equations. Schematically, the differential Boltzmann equation is

$$\frac{df}{d\eta} = a\mathbb{C}[f] \tag{4.1}$$

where on the left we have the total derivative of the distribution function of our species of interest and on the right-hand side we find the collision part, which is a complicate functional of the distributions functions accounting for various collision terms. In the easy case of no interactions we have that

$$\frac{df}{d\eta} = 0 \tag{4.2}$$

which tells us that the number of particles in a given element of phase space does not change in time.

4.1 The second order Boltzmann equation for photons

The first thing to do is to study the left-hand side of the Boltzmann equation. Here we have the total derivative of the distribution function of a specific particle species and this is in general a function of the conformal time, position and momentum of the particle $P^\mu = dx^\mu/d\lambda$, where λ parametrizes the particle path

$$f = f(x^i, P^\mu, \eta). \tag{4.3}$$

From the theory of General Relativity we know that the normalization condition gives us

$$P^2 = g_{\mu\nu}P^\mu P^\nu = -m^2, \tag{4.4}$$

where m is the mass of the particle. From this constraint we can eliminate the time component of the 4-momentum P^0 and the remaining phase space distribution $f(x^i, P^j, \eta)$ tells us the number of particle in a differential volume $dx^1 dx^2 dx^3 dP^1 dP^2 dP^3$ in phase space.

For photons we have that the condition (4.4) simplifies to

$$P^2 = g_{\mu\nu}P^\mu P^\nu = 0, \tag{4.5}$$

from which, using the conformal metric at second order

$$\begin{aligned} P^2 &= g_{00}P^0P^0 + 2g_{0i}P^0P^i + g_{ij}P^iP^j \\ &= \left(P^0\right)^2 (1 + 2\psi_1 + \psi_2)\, a^2 + \omega_{2i}a^2P^0P^i + p^2, \end{aligned} \tag{4.6}$$

where we defined $p^2 = g_{ij}P^iP^j$. Now we can solve for P^0 and we have

$$P^0 = \left(\omega_{2i}P^0P^i + \frac{p^2}{a^2}\right)^{1/2} (1 + 2\psi_1 + \psi_2)^{-1/2}, \tag{4.7}$$

from this expression we can easily recover the zero- and the first-order results

$$P^0\Big|_0 = \frac{p}{a}, \quad P^0\Big|_{1\text{st}} = \frac{p}{a}(1 - \psi_1). \tag{4.8}$$

To write the final expression at second order we also need the spatial component of the 4-momentum P^i and we can assume that it can be decomposed into a constant C which multiplies the direction vector n^i, defined such that $\delta_{ij}n^in^j = 1$. So we can write $P^i = Cn^i$ and we only need to find the form of C from

$$g_{ij}P^iP^j = C^2a^2P^iP^j\left[\left(1-2\phi_1-\phi_2\right)\delta_{ij}+\chi_{1ij}+\frac{1}{2}\chi_{2ij}\right]=p^2, \tag{4.9}$$

using $P^i = Cn^i$, we find

$$p^2 = C^2a^2\left(1-2\phi_1-\phi_2+\chi_{1ij}n^in^j+\frac{1}{2}\chi_{2ij}n^in^j\right). \tag{4.10}$$

Now we can invert the previous expression solving for C. We have to stress that we are doing computations up to second order, so we need to Taylor expand everything to second order, namely

$$(1-x)^{-1/2} = 1+\frac{1}{2}x+\frac{3}{8}x^2+O(x^3). \tag{4.11}$$

So we have

$$C = \frac{p}{a}\Big[1+\phi_1+\frac{1}{2}\phi_2-\frac{1}{2}\chi_{1ij}n^in^j-\frac{1}{4}\chi_{2ij}n^in^j+\frac{3}{8}\chi_{1ij}\chi_{1km}n^in^jn^kn^m+ \\ +\frac{3}{2}\phi_1^2-\frac{3}{2}\phi_1\chi_{1ij}n^in^j\Big] \tag{4.12}$$

and spatial part of the 4-momentum P^i is

$$P^i = \frac{p}{a}n^i\Big[1+\phi_1+\frac{1}{2}\phi_2-\frac{1}{2}\chi_{1kj}n^kn^j-\frac{1}{4}\chi_{2kj}n^kn^j+\frac{3}{8}\chi_{1kj}\chi_{1lm}n^kn^jn^ln^m+ \\ +\frac{3}{2}\phi_1^2-\frac{3}{2}\phi_1\chi_{1kj}n^kn^j\Big]. \tag{4.13}$$

From this we can recover again the zero- and first-order results

$$P^i\Big|_0 = \frac{p}{a}n^i, \quad P^i\Big|_{1\text{st}} = \frac{p}{a}n^i\left[1+\phi_1-\frac{1}{2}\chi_{1kj}n^kn^j\right] \tag{4.14}$$

and these expression can be confronted with [31] adding the first order tensor perturbations and with a different notation for the scalar part and the second order vector and tensor perturbations.

Now we can write the expression of P^0 as a function of p substituting in (4.7) the zero order P^0 and P^i since they multiply ω_{2i} which is already a second order quantity

$$\begin{aligned} P^0 &= \left(\omega_{2i}\frac{p^2}{a^2}n^i + \frac{p^2}{a^2}\right)^{1/2} (1+2\psi_1+\psi_2)^{-1/2} \\ &= \frac{p}{a}\left(\omega_{2i}n^i+1\right)^{1/2}\left(1-\psi_1-\frac{1}{2}\psi_2+\frac{3}{2}\psi_1^2\right) \\ &= \frac{p}{a}\left(1+\frac{1}{2}\omega_{2i}n^2\right)\left(1-\psi_1-\frac{1}{2}\psi_2+\frac{3}{2}\psi_1^2\right) \\ &= \frac{p}{a}\left(1-\psi_1-\frac{1}{2}\psi_2+\frac{3}{2}\psi_1^2+\frac{1}{2}\omega_{2i}n^i\right). \end{aligned} \tag{4.15}$$

Since we have the expressions for the momentum as functions of the momentum magnitude p and the direction n^i, we can write the distribution function as

$$f = f(x^i, n^i, p, \eta). \tag{4.16}$$

The left-hand-side of the differential Boltzmann equation (4.1) is the total derivative of the phase space distribution function f and as we know the dependencies we can split it into partial derivatives using the chain rule

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i}\frac{dx^i}{d\eta} + \frac{\partial f}{\partial p}\frac{dp}{d\eta} + \frac{\partial f}{\partial n^i}\frac{dn^i}{d\eta}. \tag{4.17}$$

We need now to compute the three time derivatives acting on x^i, p and n^i.

The first term can be computed just by noticing that

$$\frac{dx^i}{d\lambda} = \frac{dx^i}{d\eta}\frac{d\eta}{d\lambda} \tag{4.18}$$

thus

$$P^i = \frac{dx^i}{d\eta}P^0. \tag{4.19}$$

In conclusion we have that $dx^i/d\eta$ is just the ratio between P^i and P^0

$$\begin{aligned} \frac{dx^i}{d\eta} = n^i\Big[1 &+ \phi_1 + \frac{1}{2}\phi_2 - \frac{1}{2}\chi_{1kj}n^k n^j - \frac{1}{4}\chi_{2kj}n^k n^j + \frac{3}{8}\chi_{1kj}\chi_{1lm}n^k n^j n^l n^m + \\ &+ \frac{3}{2}\phi_1^2 - \frac{3}{2}\phi_1\chi_{1kj}n^k n^j\Big]\Big[1+\psi_1+\frac{1}{2}\psi_1-\frac{1}{2}\psi_1^2-\frac{1}{2}\omega_{2k}n^k\Big], \end{aligned} \tag{4.20}$$

which up to second order gives

$$\frac{dx^i}{d\eta} = n^i\Big[1 + \phi_1 + \frac{1}{2}\phi_2 - \frac{1}{2}\chi_{1kj}n^k n^j - \frac{1}{4}\chi_{2kj}n^k n^j + \frac{3}{8}\chi_{1kj}\chi_{1lm}n^k n^j n^l n^m + \\ + \frac{3}{2}\phi_1^2 - \frac{3}{2}\phi_1\chi_{1kj}n^k n^j + \psi_1 + \phi_1\psi_1 - \frac{1}{2}\psi_1\chi_{1km}n^k n^m + \frac{1}{2}\psi_2 - \frac{1}{2}\psi_1^2 + \\ - \frac{1}{2}\omega_{2k}n^k\Big]. \tag{4.21}$$

The second term in (4.17) is the time derivative of the magnitude of the momentum p and in can be calculated from the time component of the geodesic equation

$$\frac{dP^0}{d\eta} = -\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{P^0}, \tag{4.22}$$

where we used the fact that $d\eta/d\lambda = P^0$. Now, the derivative on the left-hand side can be computed, expanding P^0 to second order

$$\frac{dP^0}{d\eta} = \frac{d(p/a)}{d\eta}\left(1 - \psi_1 - \frac{1}{2}\psi_2 + \frac{3}{2}\psi_1^2 + \frac{1}{2}\omega_{2i}n^i\right) + \\ + \frac{p}{a}\frac{d}{d\eta}\left(1 - \psi_1 - \frac{1}{2}\psi_2 + \frac{3}{2}\psi_1^2 + \frac{1}{2}\omega_{2i}n^i\right), \tag{4.23}$$

where

$$\frac{d(p/a)}{d\eta} = \frac{1}{a}\frac{dp}{d\eta} - \mathcal{H}\frac{p}{a}, \tag{4.24}$$

the derivative of the scalar perturbation ψ_1 yields

$$\frac{d\psi_1}{d\eta} = \frac{\partial\psi_1}{\partial\eta} + \frac{\partial\psi_1}{\partial x^i}\frac{dx^i}{d\eta} \\ = \psi_1' + \psi_{1,i}n^i\left[1 + \phi_1 + \psi_1 - \frac{1}{2}\chi_{1km}n^k n^m\right], \tag{4.25}$$

where we used the expression of $dx^i/d\eta$ up to first order since $\partial\psi_1/\partial x^i$ is already a first order quantity. For the derivative of the second order ψ_2 we have following the previous reasoning trivially

$$\frac{d\psi_2}{d\eta} = \psi_2' + \psi_{2,i}n^i. \tag{4.26}$$

The derivative of the square of the first order scalar perturbation gives

$$\frac{d\psi_1^2}{d\eta} = 2\psi_1\psi_1' + 2\psi_1\psi_{1,i}n^i \tag{4.27}$$

and finally the derivative of the last term in (4.23) is

$$\frac{d(\omega_{2i}n^i)}{d\eta} = n^i\omega'_{2i} + n^i n^j \omega_{2i,j}, \tag{4.28}$$

where we have neglected the derivative of n^i since it is a first order quantity and ω_{2i} is only at second order.

Now we focus on the right-hand-side of (4.22), where we have, apart from the minus sign and the division by P^0

$$\Gamma^0_{\alpha\beta}P^\alpha P^\beta = \frac{g^{0\nu}}{2}\left[2g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}\right]P^\alpha P^\beta, \tag{4.29}$$

taken into account that $P^\alpha P^\beta$ is symmetric in α and β, thus the two metric derivatives in the definition of $\Gamma^\alpha_{\beta\gamma}$ contribute equally, hence the factor 2. Now we need to compute the right-hand-side of this last expression up to second order in perturbation theory for $\nu = 0, i$, $\alpha = 0, k$ and $\beta = 0, j$. Let us start from $\nu = 0$, $\alpha = 0$ and $\beta = 0$

$$\frac{g^{00}}{2}\left[2g_{00,0} - g_{00,0}\right]P^0P^0 = -\frac{1}{2}a^{-2}\left(1 - 2\psi_1 - \psi_2 + 4\psi_1^2\right)\Big[-2aa'(1+2\psi_1+\psi_2)+ \\ - a^2(2\psi'_1 + \psi'_2)\Big]\left(P^0\right)^2, \tag{4.30}$$

which gives, after multiplying and summing

$$\frac{g^{00}}{2}\left[2g_{00,0} - g_{00,0}\right]P^0P^0 = \left(\mathcal{H} + \psi'_1 + \frac{1}{2}\psi'_2 - 2\psi_1\psi'_1\right)\left(P^0\right)^2. \tag{4.31}$$

For $\nu = 0$, $\alpha = i$ and $\beta = 0$ we have

$$\frac{g^{00}}{2}\left[2g_{0i,0} - g_{i0,0}\right]P^0P^i = -P^0P^i\left(\frac{1}{2}\mathcal{H}\omega_{2i} + \frac{1}{4}\omega'_{2i}\right). \tag{4.32}$$

For $\nu = 0$, $\alpha = 0$ and $\beta = i$

$$\frac{g^{00}}{2}\left[2g_{00,i} - g_{0i,0}\right]P^0P^i = -P^0P^i\left(-2\psi_{1,i} - \psi_{2,i} - \frac{1}{2}\mathcal{H}\omega_{2i} - \frac{1}{4}\omega'_{2i} + 4\psi_1\psi_{1,i}\right). \tag{4.33}$$

In the case of $\nu = 0$, $\alpha = i$ and $\beta = j$, the expression is a bit longer

$$\frac{g^{00}}{2}\left[2g_{0i,j} - g_{ij,0}\right]P^iP^j = -\frac{1}{2}\Big[\omega_{2i} - 2\mathcal{H}\left(1 - 2\phi_1 - \phi_2\right)\delta_{ij} - 2\mathcal{H}\left(\chi_{1ij} + \chi_{2ij}\right) + \\ + \left(2\phi'_1 + \phi'_2\right)\delta_{ij} - \left(\chi'_{1ij} + \chi'_{2ij}\right) + 4\mathcal{H}\psi_1\left(1 - 2\phi_1\right)\delta_{ij} + \\ + 4\mathcal{H}\chi_{1ij}\psi_1 - 4\phi'_1\psi_1\delta_{ij} + 2\psi_1\chi'_{1ij} + 2\mathcal{H}\psi_2\delta_{ij} - 8\mathcal{H}\psi_1^2\delta_{ij}\Big]P^iP^j. \tag{4.34}$$

For $\nu = i$, the only non zero term is the one with $\alpha = j$ and $\beta = 0$

$$\frac{g^{0i}}{2}\left[2g_{ij,i} - g_{j0,i}\right] P^j P^0 = \mathcal{H}\omega_{2i}\delta_{ij}P^jP^0. \tag{4.35}$$

Putting everything together we find

$$\begin{aligned}\frac{dP^0}{d\eta} =& -\frac{1}{P^0}\Bigg\{\left(\mathcal{H} + \psi_1' + \frac{1}{2}\psi_2' - 2\psi_1\psi_1'\right)\left(P^0\right)^2 - P^0P^i\Big(\frac{1}{2}\mathcal{H}\omega_{2i} + \frac{1}{4}\omega_{2i}' + \\ & - 2\psi_{1,i} - \psi_{2,i} - \frac{1}{2}\mathcal{H}\omega_{2i} - \frac{1}{4}\omega_{2i}' + 4\psi_1\psi_{1,i} - \mathcal{H}\omega_{2i}\Big) - P^iP^j\Bigg[\frac{1}{2}\omega_{2i} + \\ & - \mathcal{H}\left(1 - 2\phi_1 - \phi_2\right)\delta_{ij} - \mathcal{H}\left(\chi_{1ij} + \frac{1}{2}\chi_{2ij}\right) + \frac{1}{2}\left(2\phi_1' + \phi_2'\right)\delta_{ij} + \\ & - \frac{1}{2}\left(\chi_{1ij}' + \frac{1}{2}\chi_{2ij}'\right) + 2\mathcal{H}\psi_1\left(1 - 2\phi_1\right)\delta_{ij} + 2\mathcal{H}\chi_{1ij}\psi_1 - 2\phi_1'\psi_1\delta_{ij} + \\ & + \psi_1\chi_{1ij}' + \mathcal{H}\psi_2\delta_{ij} - 4\mathcal{H}\psi_1^2\delta_{ij}\Bigg]\Bigg\},\end{aligned} \tag{4.36}$$

dividing by P^0 and keeping only terms up to second order we can write

$$\begin{aligned}\frac{dP^0}{d\eta} =& -\Big\{\frac{p}{a}\Big(\mathcal{H} + \psi_1' + \frac{1}{2}\psi_1' - 2\psi_1\psi_1' - \mathcal{H}\psi_1 - \psi_1\psi_1' - \frac{1}{2}\mathcal{H}\psi_2 + \frac{3}{2}\mathcal{H}\psi_1^2 + \\ & + \frac{1}{2}\mathcal{H}\omega_{2i}n^i\Big) - \frac{p}{a}n^i\Big(- 2\psi_{1,i} - \psi_{2,i} + 4\psi_{1,i}\psi_1 - 2\phi_1\psi_{1,i} - \mathcal{H}\omega_{2i} + \\ & - \chi_{1km}n^kn^m\psi_{1,i}\Big) - \frac{1}{2}\frac{p}{a}n^in^j\Big[\omega_{2i} - 2\mathcal{H}\left(1 - 2\phi_1 - \phi_2\right)\delta_{ij} - 2\mathcal{H}\big(\chi_{1ij} + \\ & + \chi_{2ij}\big) + \left(2\phi_1' + \phi_2'\right)\delta_{ij} - \left(\chi_{1ij}' + \chi_{2ij}'\right) + 4\mathcal{H}\psi_1\left(1 - 2\phi_1\right)\delta_{ij} + \\ & + 4\mathcal{H}\chi_{1ij}\psi_1 - 4\phi_1'\psi_1\delta_{ij} + 2\psi_1\chi_{1ij}' + 2\mathcal{H}\psi_2\delta_{ij} - 8\mathcal{H}\psi_1^2\delta_{ij} + \\ & - 4\mathcal{H}\phi_1\delta_{ij}(1 - 2\phi_1) - 4\mathcal{H}\phi_1\chi_{1ij} + 4\phi_1\phi_1'\delta_{ij} - 2\phi_1\chi_{1ij}' + 8\psi_1\phi_1\mathcal{H}\delta_{ij} + \\ & - 2\mathcal{H}\phi_2\delta_{ij} + 2\mathcal{H}(1 - 2\phi_1)\chi_{1km}n^kn^m\delta_{ij} + 2\mathcal{H}\chi_{1ij}\chi_{1km}n^kn^m + \\ & - 2\phi_1'\chi_{1km}n^kn^m\delta_{ij} + \chi_{1ij}'\chi_{1km}n^kn^m - 4\mathcal{H}\psi_1\chi_{1km}n^kn^m\delta_{ij} + \\ & + \mathcal{H}\chi_{2km}n^kn^m\delta_{ij} - 8\phi_1^2\mathcal{H}\delta_{ij} - 2\mathcal{H}\chi_{1ln}\chi_{1km}n^kn^mn^ln^n\delta_{ij} + \\ & + 8\mathcal{H}\phi_1\chi_{1km}n^kn^m\delta_{ij} - 2\mathcal{H}\delta_{ij}\psi_1(1 - 2\phi_1) - 2\mathcal{H}\psi_1\chi_{1ij} + 2\phi_1'\psi_1\delta_{ij} + \\ & - \psi_1\chi_{1ij}' + 4\mathcal{H}\delta_{ij}\psi_1^2 - 4\mathcal{H}\phi_1\psi_1\delta_{ij} + 2\mathcal{H}\psi_1\chi_{1km}n^kn^m\delta_{ij} + \\ & - 2\mathcal{H}\delta_{ij}\Big(\frac{1}{2}\psi_2 - \frac{1}{2}\psi_1^2 - \frac{1}{2}\omega_{2k}n^k\Big)\Big]\Big\}.\end{aligned} \tag{4.37}$$

This long and tedious expression can be further simplified recalling that $\delta_{ij}n^i n^j = 1$ and, skipping some algebra, we end up with

$$\frac{dP^0}{d\eta} = -\frac{p}{a}\Big[2\mathcal{H} + \psi_1' - \phi_1' - 2\mathcal{H}\psi_1 + 2\psi_{1,i}n^i + \frac{1}{2}\chi_{1ij}' n^i n^j - \mathcal{H}\psi_2 + \frac{1}{2}\psi_2' + \\ - \frac{1}{2}\phi_2' + \frac{1}{4}\chi_{2ij}' n^i n^j - 3\psi_1\psi_1' + n^i\big(\psi_{2,i} + 3\mathcal{H}\psi_1^2 - 4\psi_{1,i}\psi_1 + 2\phi_1\psi_{1,i} + \\ + \chi_{1km}n^k n^m \psi_{1,i}\big) + \phi_1'\psi_1 + \mathcal{H}\omega_{2i}n^i - \frac{1}{2}\psi_1\chi_{1ij}' n^i n^j + \phi_1\chi_{1ij}' n^i n^j + \\ - 2\phi_1\phi_1' + \phi_1'\chi_{1ij}n^i n^j - \frac{1}{2}\chi_{1ij}'\chi_{1km}n^k n^m n^i n^j - \frac{1}{2}\omega_{2i,j}n^i n^j\Big]. \tag{4.38}$$

Now we can study the left-hand-side of the geodesic equation (4.22), using the expressions of the derivatives we found before

$$\frac{dP^0}{d\eta} = \left(\frac{1}{a}\frac{dp}{d\eta} - \mathcal{H}\frac{p}{a}\right)\left(1 - \psi_1 - \frac{1}{2}\psi_2 + \frac{3}{2}\psi_1^2 + \frac{1}{2}\omega_{2i}n^i\right) + \\ + \frac{p}{a}\Big[- \psi_1' - \psi_{1,i}n^i\Big(1 + \phi_1 + \psi_1 - \frac{1}{2}\chi_{1km}n^k n^m\Big) - \frac{1}{2}\psi_2' - \frac{1}{2}\psi_{2,i}n^i + \\ + 3\psi_1\psi_1' + 3\psi_1\psi_{1,i}n^i + \frac{1}{2}\omega_{2i}' n^i + \frac{1}{2}\omega_{2i,j}n^i n^j\Big]. \tag{4.39}$$

The last step is now to equate the right part of the previous (4.39) with (4.38) and solve for the quantity $dp/pd\eta$

$$\frac{1}{p}\frac{dp}{d\eta} = -\mathcal{H} + \phi_1' + \frac{1}{2}\phi_1' + 2\phi_1\phi_1' - \psi_{1,i}n^i - \frac{1}{2}\psi_{1,i}n^i + \psi_{1,i}\psi_1 n^i - \phi_1\psi_{1,i}n^i + \\ - \frac{1}{2}\omega_{2i}' n^i - \frac{1}{4}\chi_{2ij}' n^i n^j - \frac{1}{2}\chi_{1ij}' n^i n^j - \frac{3}{2}\psi_{1,i}\chi_{1km}n^k n^m n^i - \phi_1\chi_{1ij}' n^i n^j + \\ - \phi_1'\chi_{1ij}n^i n^j, \tag{4.40}$$

which at first order exactly reproduces the result in [32], in conformal time and for only scalar perturbations

$$\frac{1}{p}\frac{dp}{d\eta} = -\mathcal{H} + \phi_1' - \psi_{1,i}n^i. \tag{4.41}$$

The last term in (4.17) to be computed is the derivative of the direction vector n^i with respect to the conformal time

$$\frac{dn^i}{d\eta}, \tag{4.42}$$

but since it is multiplying $\partial f/\partial n^i$, which is at least a first order quantity, we just need to compute (4.42) up to first order. As before we start from the geodesic equation but in this case from the spatial part of the equation

$$\frac{dP^i}{d\eta} = -\Gamma^i_{\alpha\beta}\frac{P^\alpha P^\beta}{P^0}. \tag{4.43}$$

The steps are quite similar to the one we did before for the time component, so I will skip those passages, reporting the result for the right-hand-side

$$\frac{dP^i}{d\eta} = -\frac{p}{a}\Big[\psi_1^{,i} + 2n^i\left(\mathcal{H} + \mathcal{H}\phi_1 - \frac{1}{2}\mathcal{H}\chi_{1km}n^k n^m - \phi_1'\right) + \chi^{i\ \prime}_{1j} - 2n^i n^k \phi_{1,k} + \\ + \phi_1^{,i} n^j n^k\left(\frac{1}{2}\chi^i_{1k,j} + \frac{1}{2}\chi^i_{1j,k} - \frac{1}{2}\chi^{,i}_{1jk}\right)\Big]. \tag{4.44}$$

The left-hand-side, as we did before, reads

$$\frac{dP^i}{d\eta} = \frac{d(p/a)}{d\eta}n^i\left(1 + \phi_1 - \frac{1}{2}\chi_{1km}n^k n^m\right) + \frac{p}{a}\frac{dn^i}{d\eta}\left(1 + \phi_1 - \frac{1}{2}\chi_{1km}n^k n^m\right) + \\ + \frac{p}{a}n^i\left(\frac{d\phi_1}{d\eta} - \frac{1}{2}\frac{d(\chi_{1km}n^k n^m)}{d\eta}\right), \tag{4.45}$$

using the fact that

$$\frac{d}{d\eta} = \frac{\partial}{\partial\eta} + \frac{\partial}{\partial x^i}\frac{dx^i}{d\eta}$$

and that $dn^i/d\eta$ is a first order quantity, we have

$$\frac{dP^i}{d\eta} = \left[\frac{1}{a}\frac{dp}{d\eta} - \mathcal{H}\frac{p}{a}\right]n^i\left(1 + \phi_1 - \frac{1}{2}\chi_{1km}n^k n^m\right) + \frac{p}{a}\frac{dn^i}{d\eta}\Big(1 + \phi_1 + \\ - \frac{1}{2}\chi_{1km}n^k n^m\Big) + \frac{p}{a}n^i\left[\phi_1' + \phi_{1,k}n^k - \frac{1}{2}\chi'_{1km}n^k n^m - \frac{1}{2}\chi_{1km,l}n^k n^m n^l\right]. \tag{4.46}$$

We can substitute the expression for $dp/d\eta$ from (4.40), we can equate the resulting expression with the right-hand-side of the spatial geodesic equation (4.44) and solve for $dn^i/d\eta$. At first order we have

$$\frac{dn^i}{d\eta} = \left(\psi_{1,k} + \phi_{1,k}\right)n^k n^i - \psi_1^{,i} - \psi_1^{,i} + \chi'_{1km}n^k n^m n^i + \frac{1}{2}\chi_{1km,l}n^k n^m n^l n^i + \\ - \chi^{i\ \prime}_{1j} - \chi^i_{1j,k}n^j n^k + \frac{1}{2}\chi^{,i}_{1jk}n^j n^k. \tag{4.47}$$

After all this math, one should care about the distribution function f for photons. It is assumed to be a Bose-Einstein distribution

$$\bar{f}(p,\eta) = 2\frac{1}{e^{p/T(\eta)}-1}, \tag{4.48}$$

where the overline tells us that this is only the background non-perturbed distribution function, $T(\eta)$ is the average (zero-order) temperature and the factor 2 is there because of the two spin degrees of photons. The perturbed distribution function will depend also on x^i and n^i, so to account for anisotropies and inhomogeneities we have

$$f(x^i,p,n^i,\eta) = \bar{f}(p,\eta) + f^{(1)}(x^i,p,n^i,\eta) + \frac{1}{2}f^{(2)}(x^i,p,n^i,\eta) \tag{4.49}$$

where we split the perturbation of the distribution function into a first and a second-order part.

In conclusion, we can easily see now that the left-hand side of the Boltzmann equation (4.1), up to second order reads

$$\begin{aligned}\frac{df}{d\eta} &= \frac{df^{(1)}}{d\eta} + \frac{1}{2}\frac{df^{(2)}}{d\eta} + p\frac{\partial\bar{f}}{\partial p}\left[\phi_1' + \frac{1}{2}\phi_2' + 2\phi_1\phi_1'\right] + \\ &- p\frac{\partial\bar{f}}{\partial\eta}\left[-\psi_{1,i}n^i - \frac{1}{2}\psi_{1,i}n^i + \psi_{1,i}\psi_1 n^i - \phi_1\psi_{1,i}n^i - \frac{1}{2}\omega_{2i}'n^i - \frac{1}{4}\chi_{2ij}'n^in^j + \right.\\ &\left. - \frac{1}{2}\chi_{1ij}'n^in^j - \frac{3}{2}\psi_{1,i}\chi_{1km}n^kn^mn^i - \phi_1\chi_{1ij}'n^in^j - \phi_1'\chi_{1ij}n^in^j\right]\end{aligned} \tag{4.50}$$

4.1.1 Collision Terms

In this section we will follow the reference [31], giving the main results the authors found in the article.

The idea is now to focus on the collision term related to Compton scattering in the form

$$e(q)\gamma(p) \longleftrightarrow e(q')\gamma(p') \tag{4.51}$$

The collision term $\mathbb{C}$ is thus

$$\begin{aligned}\mathbb{C}(p) = \frac{1}{E(p)}\int &\frac{dq}{(2\pi)^3 2E(q)}\frac{dq'}{(2\pi)^3 2E(q')}\frac{dp'}{(2\pi)^3 2E(p)}(2\pi)^4\delta^4(q+p-q'-p')\times \\ &\times|\mathcal{M}|^2\left\{g(q')f(p')[1+f(p)] - g(q)f(p)[1+f(p')]\right\}\end{aligned} \tag{4.52}$$

where $E(q)$ is the energy defined as $E(q) = \sqrt{q^2 + m_e^2}$, $\mathcal{M}$ is the amplitude scattering, the delta ensures energy and momentum conservation, g and f are the electron and photons distribution functions respectively. Since electrons are in thermal equilibrium with photons and are non relativistic, we can take g to be a Boltzmann distribution with velocity v

$$g(q) = n_e \left(\frac{2\pi}{m_e T_e}\right)^{3/2} \exp\left\{-\frac{(q - m_e v)^2}{2 m_e T_e}\right\} \tag{4.53}$$

The scattering amplitude $\mathcal{M}$ is

$$|\mathcal{M}|^2 = 6\pi\sigma_T m_e^2 \left[(1 + \cos^2\theta) - 2\cos\theta(1 - \cos\theta) q \cdot (\hat{p} + \hat{p}')/m_e\right] \tag{4.54}$$

where $\cos\theta = n \cdot n'$ is the scattering angle and σ_T is the Thomson cross-section.

The resulting collision term, up to second order is given by

$$\mathbb{C}(p) = \frac{3 n_e \sigma_T}{4p} \int dp' p' \frac{d\Omega'}{4\pi} \Big[c^{(1)}(p, p') + c^{(2)}_{\Delta}(p, p') + c^{(2)}_{v}(p, p') + c^{(2)}_{\Delta v}(p, p') + \\ + c^{(2)}_{vv}(p, p') + c^{(2)}_{K}(p, p') \Big], \tag{4.55}$$

where the different contributions in the square bracket can be found in [31]. The first one is the first order collision term and the others at second order are arranged in 4 components: one is the so-called *anisotropy suppression* term; the second one depends on the second-order velocity perturbation; a set of terms with the velocity coupling to the photon perturbation and a set of terms quadratic in the velocity v; the last contribution contains the Kompaneets terms describing spectral distortions to the CMB.

Now, following [31], one can integrate everything and reach the result, using the notation

$$\mathbb{C}(p) = C^{(1)}(p) + \frac{1}{2} C^{(2)}(p). \tag{4.56}$$

Here, $C^{(1)}(p)$ is defined as

$$C^{(1)}(p) = n_e \sigma_T \left[f_0^{(1)}(p) + \frac{1}{2} f_2^{(1)} P_2(\hat{v} \cdot n) - f^{(1)} - p \frac{\partial f^{(0)}}{\partial p} v \cdot n \right] \tag{4.57}$$

having used the decomposition in Legendre polynomials

$$f^{(1)}(x, p, n) = \sum_l (2l + 1) f_l^{(1)}(p) P_l(\cos\theta) \tag{4.58}$$

At second order, we have to remember that the second-order velocity term includes a vector part, breaking the azimutal symmetry, leading to a generic angular decomposition of the distribution function, such that

$$f_{lm} = (-i)^{-l}\sqrt{\frac{2l+1}{4\pi}}\int d\Omega f Y^{*}_{lm}(n). \tag{4.59}$$

Integrating everything, see [31] for the detailed calculations, we can find that the second order contribution to the collision term is

$$\begin{aligned}\frac{1}{2}C^{(2)}(\mathbf{p}) = n_e\sigma_T\Bigg\{&\frac{1}{2}f^{(2)}_{00}(p) - \frac{1}{4}\sum_{m=-2}^{2}\frac{\sqrt{4\pi}}{5^{3/2}}f^{(2)}_{2m}(p)Y_{2m}(\mathbf{n}) - \frac{1}{2}f^{(2)}(\mathbf{p})+\\
&+\delta^{(1)}_e\left[f^{(1)}_0(p) + \frac{1}{2}f^{(1)}_2P_2(\hat{\mathbf{v}}\cdot\mathbf{n}) - f^{(1)} - p\frac{\partial f^{(0)}}{\partial p}\mathbf{v}\cdot\mathbf{n}\right] - \frac{1}{2}p\frac{\partial f^{(0)}}{\partial p}\mathbf{v}^{(2)}\cdot\mathbf{n}+\\
&+\mathbf{v}\cdot\mathbf{n}\Bigg[f^{(1)}(\mathbf{p}) - f^{(1)}_0(p) - p\frac{\partial f^{(1)}_0(p)}{\partial p} - f^{(1)}_2(p) + \frac{1}{2}P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\Bigg(f^{(1)}_2(p)+\\
&-p\frac{\partial f^{(1)}_2(p)}{\partial p}\Bigg)\Bigg] + v\Bigg[2f^{(1)}_1(p) + p\frac{\partial f^{(1)}_1(p)}{\partial p} + \frac{1}{5}P_2(\hat{\mathbf{v}}\cdot\mathbf{n})\Bigg(-f^{(1)}_1(p)+\\
&+p\frac{\partial f^{(1)}_1(p)}{\partial p} + 6f^{(1)}_3(p) + \frac{3}{2}p\frac{\partial f^{(1)}_3(p)}{\partial p}\Bigg)\Bigg] + (\mathbf{v}\cdot\mathbf{n})^2\left[p\frac{\partial f^{(0)}}{\partial p} + \frac{11}{20}p^2\frac{\partial^2 f^{(0)}}{\partial p^2}\right]+\\
&+v^2\left[p\frac{\partial f^{(0)}}{\partial p} + \frac{3}{20}p^2\frac{\partial^2 f^{(0)}}{\partial p^2}\right] + \frac{1}{m_e^2}\frac{\partial}{\partial p}\left[p^4\left(T_e\frac{\partial f^{(0)}}{\partial p} + f^{(0)}\left(1+f^{(0)}\right)\right)\right]\Bigg\}\end{aligned} \tag{4.60}$$

4.2 The second-order Boltzmann equation for dark matter and baryons

In this section we will derive the second order collisionless equation for cold dark matter. Following the previous reasoning, we can apply the same formalism developed before to derive the Boltzmann equation for any other constituent in the universe. It is very important to study the the evolution of the dark matter because in almost all popular models of structure formation, it plays an important role in structure formation and in determining the gravitational field in the universe [32].

As the term *dark* suggests and as we already know, dark matter does not interact with any of the other species in the universe, thus we don't need to deal with any collision terms. A difference from the photons case is that dark matter is considered to be *cold* dark matter, namely it is non relativistic, and the constraint on the 4-momentum gives

$$g_{\mu\nu}Q^{\mu}Q^{\nu} = -m^2, \tag{4.61}$$

where m is the mass of the dark matter particle. Moreover, the energy is defined as

$$E = \sqrt{q^2 + m^2}, \tag{4.62}$$

where as before q is defined as $q^2 = g_{ij}Q^iQ^j$. We need to split the total derivative on the left-hand side of the Boltzmann equation as

$$\frac{dg}{d\eta} = \frac{\partial g}{\partial \eta} + \frac{\partial g}{\partial x^i}\frac{dx^i}{d\eta} + \frac{\partial g}{\partial p^i}\frac{dp^i}{d\eta}, \tag{4.63}$$

keeping implicit the direction of the momentum n^i into $q^i = qn^i$.

Following the same path we followed for the photons case, remembering the difference between the two cases, we have (skipping all the passages for the sake of brevity)

$$Q^0 = \frac{E}{a}\left(1 - \psi_1 - \frac{1}{2}\psi_2 + \frac{3}{2}\psi_1^2 + \frac{1}{2}\omega_{2i}\frac{q^i}{E}\right), \tag{4.64}$$

which reduces to the massless case because $p = E$ and $p^i/E = n^i$.

For the spatial component of the momentum Q^i we reach the same expression we had for the massless case

$$Q^i = \frac{q^i}{a}\Big[1 + \phi_1 + \frac{1}{2}\phi_2 - \frac{1}{2}\chi_{1kj}n^kn^j - \frac{1}{4}\chi_{2kj}n^kn^j + \frac{3}{8}\chi_{1kj}\chi_{1lm}n^kn^jn^ln^m + \\ + \frac{3}{2}\phi_1^2 - \frac{3}{2}\phi_1\chi_{1kj}n^kn^j\Big]. \tag{4.65}$$

Now we need to compute the term $dx^i/d\eta$ and as we did for photons we have

$$\frac{dx^i}{d\eta} = \frac{q^i}{E}\Big[1 + \phi_1 + \frac{1}{2}\phi_2 - \frac{1}{2}\chi_{1kj}n^kn^j - \frac{1}{4}\chi_{2kj}n^kn^j + \frac{3}{8}\chi_{1kj}\chi_{1lm}n^kn^jn^ln^m + \\ + \frac{3}{2}\phi_1^2 - \frac{3}{2}\phi_1\chi_{1kj}n^kn^j + \psi_1 + \phi_1\psi_1 - \frac{1}{2}\psi_1\chi_{1km}n^kn^m + \frac{1}{2}\psi_2 - \frac{1}{2}\psi_1^2 + \\ - \frac{1}{2}\omega_{2k}\frac{q^k}{E}\Big]. \tag{4.66}$$

For the term $dq^i/d\eta$ we have to write the geodesic equation

$$\frac{dQ^i}{d\eta} = -\Gamma^i_{\alpha\beta}\frac{Q^\alpha Q^\beta}{Q^0}, \tag{4.67}$$

where the right-hand side, apart from the minus sign, can be written as

$$\Gamma^i_{\alpha\beta}\frac{Q^\alpha Q^\beta}{Q^0} = A^i Q^0 + B^i_j Q^j + C^i_{jk}\frac{Q^j Q^k}{Q^0}, \tag{4.68}$$

where I introduced

$$A^i = \frac{1}{2}\omega'_{2i} + 2\phi_1\psi_1^{,i} - \psi_{1,k}\chi_1^{ik} + \psi_1^{,i} + \frac{1}{2}\psi_2^{,i} + \frac{1}{2}\mathcal{H}\omega_{2i} \tag{4.69}$$

$$\begin{aligned}B^i_j =& 2\mathcal{H}\delta_{ij} - 2\phi'_1\delta^i_j - \phi'_1\delta^i_j + \chi^i_{1j}{}' + \frac{1}{2}\chi^i_{1j}{}' - \frac{1}{2}\omega^{,i}_{2j} + 2\chi^i_{1j}{}' + 2\chi^i_{1j}\phi'_1 + \\ & - 4\phi'_1\phi_1\delta^i_j - \chi_1^{ik}\chi'_{1kj} + \frac{1}{2}\omega^i_{2,j}\end{aligned} \tag{4.70}$$

$$\begin{aligned}C^i_{jk} =& \frac{1}{2}\Big[\left(-4\phi_{1,j} - 2\phi_{2,j}\right)\delta^i_k + 2\chi^i_{1k,j} + \chi^i_{2k,m} + \left(2\phi_1^{,i} + \phi_2^{,i}\right)\delta_{kj} - \chi^{,i}_{1kj} + \\ & - \frac{1}{2}\chi^{,i}_{2kj} - 8\phi_1\phi_{1,j}\delta^i_k + 4\phi_1\chi^i_{1k,j} + 4\phi_1\phi_1^{,i}\delta_{kj} - 2\phi_1\chi^{,i}_{1kj} + 4\phi_{1,j}\chi^i_{1k} + \\ & - 2\chi_1^{il}\chi_{1lk,j} - 2\phi_{1,l}\chi_1^{il}\delta_{kj} + \chi_1^{il}\chi_{1kj,l} - \mathcal{H}\omega_2^i\delta_{kj}\Big].\end{aligned} \tag{4.71}$$

Since A^i is at least first order, we need Q^0 up to first order, which is

$$Q^0 = \frac{E}{a}(1 - \psi_1), \tag{4.72}$$

so

$$A^i Q^0 = \frac{E}{a}\left(\frac{1}{2}\omega_2^{i\,\prime} + 2\phi_1\psi_1^{,i} - \psi_{1,k}\chi_1^{ki} + \psi_1^{,i} + \frac{1}{2}\psi_2^{,i} - \frac{1}{2}\mathcal{H}\omega_2^i - \psi_1\psi_1^{,i}\right). \tag{4.73}$$

The term proportional to Q^j gives

$$\begin{aligned}B^i_j Q^j =& 2\mathcal{H}\frac{q^i}{a}\Big[1 + \phi_1 + \frac{1}{2}\phi_2 - \frac{1}{2}\chi_{1kj}n^k n^j - \frac{1}{4}\chi_{2kj}n^k n^j + \frac{3}{8}\chi_{1kj}\chi_{1lm}n^k n^j n^l n^m + \\ & + \frac{3}{2}\phi_1^2 - \frac{3}{2}\phi_1\chi_{1kj}n^k n^j\Big] - 2\frac{q^i}{a}\phi'_1 - 2\frac{q^i}{a}\phi'_1\phi_1 + \frac{q^j}{a}\chi^i_{1j}{}' + \frac{q^j}{a}\chi^i_{1j}{}'\phi_1 + \\ & + \frac{1}{2}\frac{q^j}{a}\left(-2\phi'_2\delta^i_j + \chi_{2j}i' - \omega^{,i}_{2j} + 4\chi_1^{i\,\prime}\phi_1 + 4\chi^i_{1j}\phi'_1 - 8\phi'_1\phi_1\delta^i_j - 2\chi_1^{ik}\chi'_{1kj} + \omega^i_{2,j}\right).\end{aligned} \tag{4.74}$$

For the product $Q^j Q^k$ we can stop at first order, since C^i_{jk} is at least first order

$$Q^j Q^k = \frac{q^j q^k}{a^2}\left(1 + 2\phi_1 - \chi_{1im} n^i n^m\right) \tag{4.75}$$

and dividing by Q^0 we have

$$\frac{Q^j Q^k}{Q^0} = \frac{q^j q^k}{Ea}\left(1 + 2\phi_1 - \chi_{1ls} n^l n^s + \psi_1\right). \tag{4.76}$$

So, the right-hand side of the geodesic equation can be written as

$$\begin{aligned}
-\Gamma^i_{\alpha\beta}\frac{Q^\alpha Q^\beta}{Q^0} &= -\frac{E}{a}\left(\frac{1}{2}\omega_2^{i\,\prime} + 2\phi_1\psi_1^{,i} - \psi_{1,k}\chi_1^{ki} + \psi_1^{,i} + \frac{1}{2}\psi_2^{,i} - \frac{1}{2}\mathcal{H}\omega_2^i - \psi_1\psi_1^{,i}\right) + \\
&- 2\mathcal{H}\frac{q^i}{a}\Big[1 + \phi_1 + \frac{1}{2}\phi_2 - \frac{1}{2}\chi_{1kj}n^k n^j - \frac{1}{4}\chi_{2kj}n^k n^j + \frac{3}{8}\chi_{1kj}\chi_{1lm}n^k n^j n^l n^m + \\
&+ \frac{3}{2}\phi_1^2 - \frac{3}{2}\phi_1\chi_{1kj}n^k n^j\Big] + 2\phi_1'\frac{q^i}{a} + 2\phi_1'\phi_1\frac{q^i}{a} - \frac{q^j}{a}\chi^{i\;\prime}_{1j} - \frac{q^j}{a}\chi^{i\;\prime}_{1j}\phi_1 + \\
&- \frac{1}{2}\frac{q^j}{a}\Big(-2\phi_2'\delta^i_j + \chi_{2ji}{}' - \omega^{,i}_{2j} + 4\chi_1^{i\,\prime}\phi_1 + 4\chi^i_{1j}\phi_1' - 8\phi_1'\phi_1\delta^i_j - 2\chi_1^{ik}\chi_{1kj}' + \omega^i_{2,j}\Big) + \\
&- \frac{1}{2}\frac{q^k q^m}{Ea}\Big[-2\phi_{2,m}\delta^i_k + \chi^i_{2k,m} + \phi_2^{,i}\delta_{km} - \frac{1}{2}\chi^{,i}_{2km} + 4\phi_{1,m}\chi^i_{1k} + \\
&- 2\chi_1^{ij}\chi_{1jk,m} - 2\phi_{1,j}\chi_1^{ij}\delta_{km} + \chi_1^{ij}\chi_{1km,j} + 4\phi_{1,m}\chi_{1ls}n^l n^s\delta^i_k - 4\phi_{1,m}\psi_1\delta^i_k + \\
&- 2\chi^i_{1k,m}\chi_{1ls}n^l n^s + 2\psi_1\chi^i_{1k,m} - 2\chi_{1ls}n^l n^s\phi_1^{,i}\delta_{km} + 2\phi_1^{,i}\psi_1\delta_{km} + \chi^{,i}_{1km}\chi_{1ls}n^l n^s + \\
&- \chi^{,i}_{1km}\psi_1 - 4\phi_{1,m}\delta^i_k + 2\chi^i_{1k,m} + 2\phi_1^{,i}\delta_{km} - \chi^{,i}_{1km} - 16\phi_1\phi_{1,m}\delta^i_k + 8\phi_1\chi^i_{1k,m} + \\
&+ 8\phi_1\phi_1^{,i}\delta_{km} - 4\phi_1\chi^{,i}_{1km} - \mathcal{H}\omega^i_2\delta_{km}\Big]
\end{aligned} \tag{4.77}$$

On the left-hand side we need to expand the total derivative of Q^i with respect to η

$$\frac{dQ^i}{d\eta} = \left(\frac{1}{a}\frac{dq^i}{d\eta} - \mathcal{H}\frac{q^i}{a}\right)F + \frac{dF}{d\eta}\frac{q^i}{a}, \tag{4.78}$$

where F is defined as

$$\begin{aligned}
F &= 1 + \phi_1 + \frac{1}{2}\phi_2 - \frac{1}{2}\chi_{1kj}n^k n^j - \frac{1}{4}\chi_{2kj}n^k n^j + \frac{3}{8}\chi_{1kj}\chi_{1lm}n^k n^j n^l n^m + \\
&+ \frac{3}{2}\phi_1^2 - \frac{3}{2}\phi_1\chi_{1kj}n^k n^j
\end{aligned} \tag{4.79}$$

and the total derivative is

$$\frac{dF}{d\eta} = \phi_1' + \phi_{1,i}\frac{q^i}{E}(1 + \phi_1 - \frac{1}{2}\chi_{1km}n^k n^m + \psi_1) + \frac{1}{2}\phi_2' + \frac{1}{2}\phi_{2,i}\frac{q^i}{E} - \frac{1}{2}\frac{d(\chi_{1ij}n^i n^j)}{d\eta} +$$
$$- \frac{1}{4}\chi_{2ij}' n^i n^j - \frac{1}{4}\chi_{2ij,k}n^i n^j \frac{q^k}{E} + 3\phi_1\phi_1' + 3\phi_1\phi_{1,i}\frac{q^i}{E} + \frac{3}{8}\frac{d(\chi_{1ij}\chi_{1lm}n^i n^j n^l n^m)}{d\eta} +$$
$$- \frac{3}{2}\frac{d(\chi_{1ij}\phi_1 n^i n^j)}{d\eta} \tag{4.80}$$

So we can write

$$\frac{dq^i}{d\eta} = \frac{a}{F}\frac{dQ^i}{d\eta} + \mathcal{H}q^i - \frac{dF}{d\eta}\frac{q^i}{F} \tag{4.81}$$

and we can compute

$$\frac{a}{F}\frac{dQ^i}{d\eta} = -E\Bigg[\frac{1}{2}\omega_2^{i\,\prime} + \phi_1\psi_1'^{\,i} - \psi_{1,k}\chi_1^{ki} + \psi_1'^{\,i} + \frac{1}{2}\psi_2'^{\,i} - \frac{1}{2}\mathcal{H}\omega_2^i - \psi_1\psi_1'^{\,i} +$$
$$+ \frac{1}{2}\chi_{1kj}n^k n^j \psi_1'^{\,i}\Bigg] - q^i\left[2\mathcal{H} - 2\phi_1' - \phi_1'\chi_{1kj}n^k n^j - \phi_2' - 4\phi_1'\phi_1\right] +$$
$$- q^j\Bigg[\chi_{1j}^{i\;\prime} + \frac{1}{2}\chi_{2j}^{i\;\prime} - \frac{1}{2}\omega_{2j}'^{\,i} + \frac{1}{2}\omega_{2,j}^i + 2\chi_{1j}^{i\;\prime}\phi_1 + 2\chi_{1j}^i\phi_1' - \chi_1^{ik}\chi_{1kj}' +$$
$$+ \frac{1}{2}\chi_{1j}^{i\;\prime}\chi_{1km}n^k n^m\Bigg] +$$
$$- \frac{1}{2}\frac{q^k q^m}{E}\Bigg[- 2\phi_{2,m}\delta_k^i + \chi_{2k,m}^i + \phi_2'^{\,i}\delta_{km} - \frac{1}{2}\chi_{2km}'^{\,i} + 4\phi_{1,m}\chi_{1k}^i +$$
$$- 2\chi_1^{ij}\chi_{1jk,m} - 2\phi_{1,j}\chi_1^{ij}\delta_{km} + \chi_1^{ij}\chi_{1km,j} + 4\phi_{1,m}\chi_{1ls}n^l n^s\delta_k^i - 4\phi_{1,m}\psi_1\delta_k^i +$$
$$- 2\chi_{1k,m}^i\chi_{1ls}n^l n^s + 2\psi_1\chi_{1k,m}^i - 2\chi_{1ls}n^l n^s\phi_1'^{\,i}\delta_{km} + 2\phi_1'^{\,i}\psi_1\delta_{km} + \chi_{1km}'^{\,i}\chi_{1ls}n^l n^s +$$
$$- \chi_{1km}'^{\,i}\psi_1 - 4\phi_{1,m}\delta_k^i + 2\chi_{1k,m}^i + 2\phi_1'^{\,i}\delta_{km} - \chi_{1km}'^{\,i} + 4\phi_{1,m}\phi_1\delta_k^i +$$
$$- 2\phi_{1,m}\chi_{1ls}n^l n^s\delta_k^i - 2\phi_1\chi_{1k,m}^i + \chi_{1k,m}^i\chi_{1ls}n^l n^s + \phi_1\chi_{1km}'^{\,i} - \frac{1}{2}\chi_{1km}'^{\,i}\chi_{1ls}n^l n^s +$$
$$- 16\phi_1\phi_{1,m}\delta_k^i + 8\phi_1\chi_{1k,m}^i + 8\phi_1\phi_1'^{\,i}\delta_{km} - 4\phi_1\chi_{1km}'^{\,i} - 2\phi_1'^{\,i}\phi_1\delta_{km} +$$
$$+ \phi_1'^{\,i}\chi_{1ls}n^l n^s - \mathcal{H}\omega_2^i\delta_{kj}\Bigg]. \tag{4.82}$$

Now we need to compute the last term in (4.81)

$$\frac{dF}{d\eta}\frac{q^i}{F} = q^i\frac{dF}{d\eta} + q^i\Bigg[-\phi_1\phi_1' - \phi_1\phi_{1,k}\frac{q^k}{E} + \frac{1}{2}\phi_1\frac{d(\chi_{1ij}n^in^j)}{d\eta} + \frac{1}{2}\chi_{1ij}n^in^j\phi_1' + \\ + \frac{1}{2}\chi_{1ij}n^in^j\phi_{1,k}\frac{q^k}{E} - \frac{1}{2}\chi_{1ij}\frac{d(\chi_{1km}n^kn^m)}{d\eta}\Bigg]. \tag{4.83}$$

So we can sum everything and we get

$$\frac{dq^i}{d\eta} = -E\Bigg[\frac{1}{2}\omega_2^{i\,\prime} + \phi_1\psi_1^{\prime i} - \psi_{1,k}\chi_1^{ki} + \psi_1^{\prime i} + \frac{1}{2}\psi_2^{\prime i} - \frac{1}{2}\mathcal{H}\omega_2^i - \psi_1\psi_1^{\prime i} + \\ + \frac{1}{2}\chi_{1kj}n^kn^j\psi_1^{\prime i}\Bigg] - q^i\Bigg[\mathcal{H} - \phi_1' - \frac{1}{2}\phi_2' - 2\phi_1\phi_1' - \frac{1}{4}\frac{d(\chi_{2kj}n^kn^j)}{d\eta} + \\ - \frac{1}{2}\phi_1'\chi_{1kj}n^kn^j - \frac{1}{2}\frac{d(\chi_{1kj}n^kn^j)}{d\eta} + \frac{3}{8}\frac{d(\chi_{1ij}\chi_{1lm}n^in^jn^ln^m)}{d\eta} + \\ - \frac{3}{2}\frac{d(\chi_{1ij}\phi_1n^in^j)}{d\eta} + \frac{1}{2}\phi_1\frac{d(\chi_{1kj}n^kn^j)}{d\eta} + \frac{1}{2}\chi_{1kj}\frac{d(\chi_{1lm}n^ln^m)}{d\eta}\Bigg] + \\ + \frac{q^iq^k}{E}\Big[\phi_{1,k} + \frac{1}{2}\phi_{2,k} + 3\phi_1\phi_{1,k} + \phi_{1,k}\psi_1 - 2\phi_{1,k}\chi_{1ls}n^ln^s\Big] + \\ - \frac{q^2}{E}\Big[\phi_1^{\prime i} + \frac{1}{2}\phi_1^{\prime i} + \phi_1^{\prime i}\psi_1 + 3\phi_1\phi_1^{\prime i} - \frac{1}{2}\phi_1^{\prime i}\chi_{1ls}n^ln^s - \phi_{1,k}\chi_1^{ik}\Big] + \\ - \frac{1}{2}\frac{q^kq^m}{E}\Big[\chi^i_{2k,m} - \frac{1}{2}\chi^{\prime i}_{2km} + 4\phi_{1,m}\chi^i_{1k} - 2\chi_1^{ij}\chi_{1jk,m} + \chi_1^{ij}\chi_{1km,j} + \\ - 2\chi^i_{1k,m}\chi_{1ls}n^ln^s + 2\psi_1\chi^i_{1k,m} + \chi^{\prime i}_{1km}\chi_{1ls}n^ln^s + 2\chi^i_{1k,m} - \chi^{\prime i}_{1km} + \\ - 2\phi_1\chi^i_{1k,m} + \chi^i_{1k,m}\chi_{1ls}n^ln^s + \phi_1\chi^{\prime i}_{1km} - \frac{1}{2}\chi^{\prime i}_{1km}\chi_{1ls}n^ln^s + 8\phi_1\chi^i_{1k,m} + \\ - 4\phi_1\chi^{\prime i}_{1km} + \phi_1^{\prime i}\chi_{1ls}n^ln^s - \mathcal{H}\omega_2^i\delta_{km}\Big] \tag{4.84}$$

In the end we can write the full Boltzmann equation for dark matter

$$\frac{dg}{d\eta} = \frac{\partial g}{\partial\eta} + \frac{dx^i}{d\eta}\frac{\partial g}{\partial x^i} + \frac{dq^i}{d\eta}\frac{\partial g}{\partial q^i} \tag{4.85}$$

using the expressions for $dx^i/d\eta$ and $dq^i/d\eta$ that we found previously and since we are considering the collisionless case (for DM), we have

$$\frac{\partial g_{\rm dm}}{\partial\eta} + \frac{dx^i}{d\eta}\frac{\partial g_{\rm dm}}{\partial x^i} + \frac{dq^i}{d\eta}\frac{\partial g_{\rm dm}}{\partial q^i} = 0 \tag{4.86}$$

These equations reduce to the equations in the collision-less massless limit as it must, but the main difference between the two is the presence of terms such as p/E, which are velocity terms arising from the bulk velocity of the massive species.

In the case of baryons we just need to recall all the previous results we found for the dark matter and we have to add the collision part. In this way we consider the Coulomb scattering processes between the electrons and protons and the Compton scatterings between photons and electrons, namely

$$\frac{dg_e}{d\eta} = \langle c_{ep} \rangle_{QQ'q'} + \langle c_{e\gamma} \rangle_{pp'q'} \tag{4.87}$$

$$\frac{dg_p}{d\eta} = \langle c_{ep} \rangle_{qq'Q'} \tag{4.88}$$

with p and p' the initial and final momenta of the photons, q and q' the corresponding quantities for the electrons and for protons we used Q and Q', as in [31, 32]. The angle brackets indicate integration over different momenta

$$\langle \ldots \rangle_{pp'q'} = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} \ldots \tag{4.89}$$

and the quantities inside, such as $c_{e\gamma}$ can be seen as the unintegrated part of the collision term

$$c_{e\gamma} = (2\pi)^4 \delta^4(p + q - p' - q') \frac{|\mathcal{M}|^2}{8E(p)E(q)E(p')E(q')} \left\{ g_e(q') f(p') - g_e(q) f(p) \right\}. \tag{4.90}$$

The next step, now, is to take the moments of these equations to find the number density and the velocity, instead of assuming a particular form of $f_{\rm dm}$ and g. This part goes beyond the main goals of this thesis project, but I will quickly go through the main passages outlining what is needed to be done.

First of all one needs to recall that

$$n = \int \frac{d^3q}{(2pi)^3} g \tag{4.91}$$

and

$$v^i = \frac{1}{n} \int \frac{d^3q}{(2\pi)^3} g \frac{qn^i}{E} \tag{4.92}$$

The idea is to integrate over $d^3q/(2\pi)^3$ the Boltzmann equation for DM and baryons to get the energy continuity equation. To get the momentum continuity equation, we can multiply Boltzmann equation by q^i/E and integrate over $d^3q/(2\pi)^3$.

We see that our integration over d^3q can be split, as usual, into

$$d^3q = q^2 dq d\Omega \tag{4.93}$$

where Ω is the solid angle. We can use some easy relations

$$\int d\Omega n^i = \int d\Omega n^i n^j n^k = 0 \tag{4.94}$$

for parity, and

$$\int d\Omega n^i n^j = \frac{4\pi}{3}\delta^{ij}. \tag{4.95}$$

In our case we can have terms which are proportional to four n^i, so we need to find also

$$\int d\Omega n^i n^j n^k n^l = \frac{4\pi}{15}(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}), \tag{4.96}$$

which comes from tensorial algebra. These results can be found considering that, since $n^i n^j$ is a symmetric second rank Cartesian tensor, the integration over $d\Omega$ must give the symmetric rank-2 tensor δ^{ij}, since it can not depend on the position. Thus

$$\int d\Omega n^i n^j = k\delta^{ij} \tag{4.97}$$

where k is a positive constant to be found. We can multiply both sides by δ_{ij} so we have

$$\int d\Omega = 3k \tag{4.98}$$

from which $k = 4\pi/3$. In a very analogous manner we can do the same for

$$\int d\Omega n^i n^j n^k n^l = a\delta^{ij}\delta^{kl} + b\delta^{ik}\delta^{jl} + c\delta^{il}\delta^{jk}. \tag{4.99}$$

Multiplying separately by δ_{ij}, δ_{ik} and δ_{il} we get three equations in the three unknowns a, b and c. Solving this system of equation we find the desired results

$$a = b = c = \frac{4\pi}{15}. \tag{4.100}$$

This tensorial digression on the integration of the solid angle is needed to get some considerations which will simplify our moments of the Boltzmann equations for dark matter and baryons. Namely, all the vector and tensors components multiplied by an odd number of n^i give zero contribution when integrated over d^3q; the terms such as

$$\chi_{ij} n^i n^j \quad \text{and} \quad \omega_{i,j} n^i n^j \tag{4.101}$$

when integrated give zero because of the transverse and traceless properties of vectors and tensors. Vectors and tensors term multiplied by four n^i, when integrated must be computed with some care, using the previous relation.

After these integrations one can find a set of two partial differential equations for the number density n and the velocity v^i for both the dark matter and baryons.

5 Conclusions

In this thesis project I tried to add my original contribution to the cosmological perturbation theory at second order, without neglecting the first order tensor perturbations in Poisson gauge.

Starting from the standard result of the FRW background, I recovered the governing equations for the smooth and not perturbed universe. Then I introduced the concept of the inflationary scenario as a model to solve some shortcomings of the old Big Bang model, such as the *horizon problem*, the *flatness problem* and the predictions on the *unwanted relics*.

After this introductory chapter, I started the discussion on the cosmological perturbation theory, which is the main focus of this thesis project, showing how the perturbations are treated: introducing the decomposition into scalar, vector and tensor components.. The first problem that we encountered is the so called *gauge issue*, arising from the covariance of the solution of the Einstein equations: I showed how to deal with this problem, recovering the usual first order gauge transformations, used to confront quantities in different gauge choices, such as the synchronous and Poisson gauge. Later, I computed the first order governing equations of these scalar, vector and tensor perturbations, recovering the standard textbook results: in the approximation of a perfect fluid the two scalar potentials are equal, vectors decay and get diluted with the expansion of the Universe and tensor modes propagate as waves, being called gravitational waves.

The first main goal of my work is now to compute the equations beyond linear order, up to second order, where, considering first and second order scalar and tensor modes and second order vectors, second order terms can be generated by the "coupling" between first order scalars and first order tensors. Thus, rolling up our sleeves, I started with the second order gauge transformations for the metric and matter perturbations. Then, the second part was all about computing the components of the Einstein equations, such as the Ricci tensor and the energy momentum tensor. Arriving to these results was quite involving but almost straightforward: I got all the second order mixed terms like "$\phi\chi$" and "$\chi\chi$" appearing into the Ricci tensor. I constructed the Einstein equations starting from the 00 and then the traceless ij from which I found the relation between the two second order scalar potentials: $\Phi_2 - \Psi_2 \neq 0$ even for a perfect fluid, due to the presence of second order terms made of 2 first order components. Then I computed the energy and momentum conservation equations from the divergence of the energy-momentum tensor: at this stage we can have 4 equations and we can close the system.

The last part of my work is spent on the writing of the second order Boltzmann equations for the main constituents of the Universe: photons, baryons and dark matter. The original contribution here is not to neglect first order tensor perturbations and account for them in our treatment. Following the first order computations in Dodelson book [32] and the second order in [31], I retrieved the Boltzmann equations for photons, baryons and dark matter, computing the terms appearing in the total derivative of the distribution function of the various particle species. As I did, we need to account for collisions for photons and for baryons as well, but for dark matter, by definition, we can study the collisionless Boltzmann equation.

To go beyond the main goals of my thesis project, one can solve the Boltzmann equations for photons, baryons and dark matter. In the case of photons it is reasonable to assume a specific form of the distribution function but in the case of DM and baryons it is possible to use the moments to solve the equations retrieving for examples an equation for the number density and velocity of DM particles and baryons.

In this thesis project I tried to show how the presence of first order tensor perturbations can lead to second order effects due to the coupling between scalar and tensor or between two tensors. I showed thus how these terms appear into the Einstein equations, needed to study the evolution of the scalar perturbation of the "metric" part of our world. These terms also appear into the Boltzmann equation which, as I said before, describe the evolution of the baryon-photon fluid, allowing to follow the evolution of the anisotropies of the CMB radiation at second order, from the early epoch to the present, through the recombination era. In order to deal with non-Gaussianty in the CMB anisotropies is of particular importance an accurate theoretical prediction of the second order effects that can generate CMB non-Gaussian anisotropies, which, in future, could help to discriminate among the mechanisms for the generation of the cosmological perturbations and thus the various models of inflation.

A future direction of my thesis project could be to proceed and to solve numerically the set of coupled Einstein-Boltzmann equations or, under some assumptions and approximations, find an analytical solution, to study the evolution of second order CMB anisotropies accounting for first order tensor perturbations in order to confront with the literature and observations.

In this context there is a very useful Boltzmann code called SONG (Second Order Non-Gaussianity), developed in [33]. This code is an open-source project written in C and it is based on CLASS: it is able to provide predictions on second order observables. When the code was implemented, no first order tensors are considered: a possible project in the future could be to modify this code, adding the terms arising in my computations considering first order tensor perturbations. In this way one can generalize the code and see how important is this contribution, confronting with the previous results.

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