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UNIQUENESS OF STEADY STATE POSITIVE SOLUTIONS TO A GENERAL ELLIPTIC SYSTEM WITH DIRICHLET BOUNDARY CONDITIONS

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Abstract The purpose of this paper is to give conditions for the uniqueness of positive solution to a rather general type of elliptic system of the Dirichlet problem on a bounded domain Ω in R^n. Also considered are the effects of perturbations on the coexistence state and uniqueness.

Keywords Coexistence, cooperating species of animals.

MSC(2010) 35J66, 35J67.

1. Introduction

One of the prominent subjects of study and analysis in mathematical biology concerns the competition, predator-prey or cooperation of two or more species of animals in the same environment. Especially pertinent areas of investigation include the conditions under which the species can coexist, as well as the conditions under which any one of the species becomes extinct, that is, one of the species is excluded by the others. In this paper, we focus on the general cooperation model to better understand the cooperative interactions between two species. Specifically, we investigate the conditions needed for the coexistence of two species when the factors affecting them are fixed or perturbed.

2. Literature Review

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the simple cooperation model, commonly known as the Lotka-Volterra cooperation model. This system describes the cooperative interaction of two species residing in the same environment in the following manner:

Suppose two species of animals, rabbits and squirrels for instance, are cooperating in a bounded domain Ω. Let $u(x,t)$ and $v(x,t)$ be densities of the two habitats in the place x of Ω at time t. Then we have the dynamic cooperation model

$$\begin{aligned}
&u_t(x,t) = \Delta u(x,t) + au(x,t) - bu^2(x,t) + cu(x,t)v(x,t),\\
&v_t(x,t) = \Delta v(x,t) + dv(x,t) - fv^2(x,t) + eu(x,t)v(x,t) \ \text{ in } \ \Omega \times [0,\infty),\\
&u(x,t) = v(x,t) = 0 \ \text{ for } \ x \in \partial\Omega,
\end{aligned}$$

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where $a, d > 0$ are growth rates, $b, f > 0$ are self-limitation rates, and $c, e > 0$ are cooperation rates. Here we are interested in the time independent, positive solutions, i.e. the positive solutions $u(x), v(x)$ of

$$\begin{aligned} &\Delta u(x) + u(x)(a - bu(x) + cv(x)) = 0, \\ &\Delta v(x) + v(x)(d - fv(x) + eu(x)) = 0 \text{ in } \Omega, \\ &u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{aligned} \tag{2.1}$$

which are called the coexistence state or the steady state. The coexistence state is the positive density solution depending only on the spatial variable x, not on the time variable t, and so, its existence means that the two species of animals can live peacefully and forever.

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to (2.1).([9, 11, 13, 14, 22])

One of the initial important results for the time-independent Lotka-Volterra model was obtained by Korman and Leung. In 1987, they published the following equivalent condition for the existence of a positive steady state solution to (2.1):

Theorem 2.1 ([9]). *Let $b = f = 1$ and $a > \lambda_1, d > \lambda_1$. Then for existence of a positive solution to* (2.1)*, it is necessary and sufficient that $ce < 1$.*

Biologically, the conditions in Theorem 2.1 implies that if the cooperation rates are too large, in other words, if members of each species interact strongly with members of the other species, then there is no positive steady state solution to (2.1), that is, the two species within the same domain can not coexist.

In 1992, Zhengyuan and Mottoni classified the region of reproduction rates (a, d) for the coexistence of species of animals.

Their primary result is given below:

Theorem 2.2 ([22]). *If $ce < bf$, then there exists a function $\gamma_0(a)$ with*

$$\begin{aligned} \gamma_0(a) &< \lambda_1, a > \lambda_1, \\ &= \lambda_1, a = \lambda_1, \\ &> \lambda_1, a < \lambda_1, \end{aligned}$$

which is decreasing for $a \in R$ and satisfies

$$\lim_{a \to -\infty} \gamma_0(a) = \infty,$$
$$\lim_{a \to \infty} \gamma_0(a) = -\infty$$

such that the set S^+ of nonnegative solutions to (2.1) *is characterized as follows:*
(1) If $a \le \lambda_1, d \le \lambda_1$, then $S^+ = \{(0,0)\}$.
(2) If $a \le \lambda_1, \lambda_1 < d < \gamma_0(a)$, then $S^+ = \{(0,0), (0, \frac{1}{f}\theta_d)\}$.
(3) If $a > \lambda_1, d < \gamma_0(a)$, then $S^+ = \{(0,0), (\frac{1}{b}\theta_a, 0)\}$.
(4) If $a \le \lambda_1, d > \gamma_0(a)$, then $S^+ = \{(0,0), (0, \frac{1}{f}\theta_d), (u^+, v^+)\}$.
(5) If $a > \lambda_1, \gamma_0(a) < d \le \lambda_1$, then $S^+ = \{(0,0), (\frac{1}{b}\theta_a, 0), (u^+, v^+)\}$.
(6) If $a > \lambda_1, d > \lambda_1$, then $S^+ = \{(0,0), (\frac{1}{b}\theta_a, 0), (0, \frac{1}{f}\theta_d), (u^+, v^+)\}$.

The work of Korman, Leung, Zhengyuan and Mottoni provides insight into the cooperative interactions of two species operating under the conditions described in the Lotka-Volterra model. However, their results are somewhat limited by a few key assumptions. In the Lotka-Volterra model that they studied, the rates of change of densities largely depend on constant rates of reproduction, self-limitation, and cooperation. The model also assumes a linear relationship of the terms affecting the rate of change for both population densities.

However, in reality, the rates of change of population densities may vary in a more complicated and irregular manner than can be described by the simple cooperation model. Therefore, in the last decade, significant research has been focused on the existence and uniqueness of the positive steady state solution of the general cooperation model for two species,

$$\begin{aligned}
&u_t(x,t) = \Delta u(x,t) + u(x,t)g(u(x,t), v(x,t))\\
&v_t(x,t) = \Delta v(x,t) + v(x,t)h(u(x,t), v(x,t)) \text{ in } \Omega \times R^+,\\
&u(x,t)|_{\partial\Omega} = v(x,t)|_{\partial\Omega} = 0,
\end{aligned}$$

or, equivalently, the positive solution to

$$\begin{aligned}
&\Delta u(x) + u(x)g(u(x), v(x)) = 0\\
&\Delta v(x) + v(x)h(u(x), v(x)) = 0 \text{ in } \Omega,\\
&u|_{\partial\Omega} = v|_{\partial\Omega} = 0,
\end{aligned} \tag{2.2}$$

where $g, h \in C^1$ are such that $g_u < 0, g_v > 0, h_u > 0, h_v < 0$ and designate reproduction, self-limitation and cooperation rates that satisfy certain growth conditions.

Because of its broader applicability, the general cooperation model has become a more popular subject of research within the mathematical community over the past few years.

The functions g, h describe how species 1 (u) and 2 (v) interact among themselves and with each other.

The followings are questions raised in the general model with nonlinear growth rates.

Problem 1 : What are the conditions for existence or nonexistence of positive solutions?
Problem 2 : What are the sufficient conditions for uniqueness of positive solutions?
Problem 3 : What is the effect of perturbation for existence and uniqueness?

In our analysis we focus on the conditions required for the maintenance of the coexistence state of the model when the growth rate functions (g, h) are slightly perturbed. Biologically, our conclusion implies that two species may slightly relax ecologically and yet continue to coexist at unique densities.

In [7], we established the following existence result:

Theorem 2.3. *If $g(0,0) > \lambda_1, h(0,0) > \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ with homogeneous boundary condition, and*

$$\sup(g_v)\sup(h_u) < \sup(g_u)\sup(h_v),$$

then (2.2) *has a positive solution.*

We achieve solution estimates in the section 4 to prove the uniqueness and the invertibility of linearization in sections 5, 6, 7 and 8, where we investigate the effect of perturbation for existence and uniqueness.

An especially significant aspect of the global uniqueness result is the stability of the positive steady state solution, which has become an important subject of mathematical study. Indeed, researchers have obtained several stability results for the Lotka-Volterra model with constant rates(see [2,3,6,10]). However, the stability of the steady state solution for the general model remains open to investigation. The research presented in this paper therefore begins the mathematical community's discussion on the stability of the steady state solution for the general cooperation model.

3. Preliminaries

Lemma 3.1 ([4]). *We consider the system*

$$\begin{aligned} &-\Delta u + q(x)u = \lambda u \ \text{ in } \ \Omega, \\ &u|_{\partial\Omega} = 0, \end{aligned} \tag{3.1}$$

where $q(x)$ is a smooth function from Ω to R and Ω is a bounded domain in R^n.

(i) The first eigenvalue $\lambda_1(q)$, denoted by simply λ_1 with the corresponding eigenfunction ϕ_1 when $q \equiv 0$, is simple with a positive eigenfunction.

(ii) If $q_1(x) < q_2(x)$ for all $x \in \Omega$, then $\lambda_1(q_1) < \lambda_1(q_2)$.

(iii) (Variational Characterization of the first eigenvalue)

$$\lambda_1(q) = \min_{\phi \in W_0^1(\Omega), \phi \neq 0} \frac{\int_\Omega (|\nabla\phi|^2 + q\phi^2)dx}{\int_\Omega \phi^2 dx}.$$

Lemma 3.2 ([12]). *Consider*

$$\begin{aligned} &\Delta u + uf(u) = 0 \ \text{ in } \ \Omega, \\ &u|_{\partial\Omega} = 0, u > 0, \end{aligned}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \leq 0$ for $u \geq c_0$ and Ω is a bounded domain in R^n.

If $f(0) > \lambda_1$, then the above equation has a unique positive solution. We denote this unique positive solution as θ_f.

The most important property of this positive solution is that θ_f is increasing as f is increasing.

We specifically note that for $a > \lambda_1$, the unique positive solution of

$$\begin{aligned} &\Delta u + u(a - u) = 0 \ \text{ in } \ \Omega, \\ &u|_{\partial\Omega} = 0, u > 0, \end{aligned}$$

is denoted by θ_a. Hence, θ_a is increasing as $a > 0$ is increasing.

Consider the system

$$\begin{aligned} &\Delta u + f(x, u) = 0 \ \text{ in } \ \Omega, \\ &u = 0 \ \text{ on } \ \partial\Omega, \end{aligned} \tag{3.2}$$

where $u = (u_1, ..., u_m)$ and $f = (f_1, ..., f_m)$ is quasimonotone increasing, i.e. $f_i(x, u)$ is increasing in u_j for all $j \neq i$.

Lemma 3.3 ([9]). *Let w_λ be a family of subsolutions[supersolutions]$(\alpha \le \lambda \le \beta)$ to* (3.2)*, increasing in λ such that*

$$\begin{aligned}&\Delta w_\lambda + f(x, w_\lambda) \ge [\le] 0 \ \ in \ \ \Omega,\\ &w_\lambda = 0 \ \ on \ \ \partial\Omega.\end{aligned}$$

Assume also $u \ge w_\alpha[u \le w_\beta]$, w_λ does not satisfy (3.2) *for any λ, and $\frac{\partial w_\lambda}{\partial n}$ changes continuously in λ on $\partial\Omega$. Then $u \ge \sup w_\lambda [u \le \inf w_\lambda]$.*

Lemma 3.4 ([17]). *Assume that $f(x, z)$ is quasimonotone increasing in z and that there exist v, w satisfying*

$$\begin{aligned}&v \le w, \Delta v + f(x, v) \ge 0, \Delta w + f(x, w) \le 0 \ \ in \ \ \Omega,\\ &v \le 0 \le w \ \ on \ \ \partial\Omega.\end{aligned}$$

Then there exist solutions u^*, u_* of (3.2) *satisfying $v \le u_* \le u^* \le w$ and with the property that any solution u with $v \le u \le w$ satisfies $u_* \le u \le u^*$. The solutions u^*, u_* are called maximum and minimum solutions, respectively.*

4. Solution estimate

In this section, we build up upper and lower bounds of solutions under certain conditions to establish uniqueness results in the next sections.

We consider

$$\begin{aligned}&\Delta u + ug(u, v) = 0 \ \text{ in } \ \Omega,\\ &\Delta v + vh(u, v) = 0 \ \text{ in } \ \Omega,\\ &u = v = 0 \ \text{ on } \ \partial\Omega,\end{aligned} \tag{4.1}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$ and $g, h \in C^1$ are such that

(SE1) $g_u < 0, g_v > 0, h_u > 0, h_v < 0$,
(SE2) $\sup(g_u) \le -1, \sup(h_v) \le -1$,
(SE3) $\sup(g_v)\sup(h_u) < 1$,
(SE4) there is $c > 0$ such that $g(u, 0) \le 0, h(0, v) \le 0$ for all $u, v \ge c$.

We have the following solution estimate.

Theorem 4.1. *If $g(0, 0) \ge h(0, 0) > \lambda_1$, then for any positive solution (u, v) to* (4.1)*, we have*

$$\begin{aligned}&\theta_{g(\cdot,0)} \le u \le \alpha\theta_{g(0,0)},\\ &\theta_{h(0,\cdot)} \le v \le \beta\theta_{g(0,0)},\end{aligned}$$

where $\alpha = \frac{1+\sup(g_v)}{1-\sup(g_v)\sup(h_u)}$ and $\beta = \frac{1+\sup(h_u)}{1-\sup(g_v)\sup(h_u)}$.

Proof. Suppose (u, v) is a positive solution to (4.1). Then

$$\begin{aligned}\Delta u + ug(u, 0) &= \Delta u + u[g(u, v) + g(u, 0) - g(u, v)]\\ &= u[g(u, 0) - g(u, v)]\\ &< 0\end{aligned}$$

by the monotonicity of g, and so, u is a supersolution to

$$\begin{aligned}&\Delta Z + Zg(Z,0) = 0 \ \text{ in } \ \Omega,\\ &Z|_{\partial\Omega} = 0.\end{aligned}$$

Furthermore, for sufficiently large $n \in N$, since $g(0,0) > \lambda_1$,

$$\begin{aligned}\Delta\frac{\phi_1}{n} + \frac{\phi_1}{n}g(\frac{\phi_1}{n},0) &= \frac{1}{n}[\Delta\phi_1 + \phi_1 g(\frac{\phi_1}{n},0)]\\ &= \frac{1}{n}[-\lambda_1\phi_1 + \phi_1 g(\frac{\phi_1}{n},0)]\\ &= \frac{\phi_1}{n}[-\lambda_1 + g(\frac{\phi_1}{n},0)]\\ &> 0\end{aligned}$$

by the continuity of g, and so, $\frac{\phi_1}{n}$ is a subsolution to

$$\begin{aligned}&\Delta Z + Zg(Z,0) = 0 \ \text{ in } \ \Omega,\\ &Z|_{\partial\Omega} = 0.\end{aligned}$$

Hence, by the super-subsolution method and Lemma 3.2, we conclude that

$$\theta_{g(\cdot,0)} \leq u. \tag{4.2}$$

Similarly, we can establish that

$$\theta_{h(0,\cdot)} \leq v. \tag{4.3}$$

Let $u_\lambda = \alpha\lambda\theta_{g(0,0)}, v_\lambda = \beta\lambda\theta_{g(0,0)}, \lambda \geq 1$, where

$$\begin{aligned}\alpha &= \frac{1+\sup(g_v)}{1-\sup(g_v)\sup(h_u)},\\ \beta &= \frac{1+\sup(h_u)}{1-\sup(g_v)\sup(h_u)}.\end{aligned}$$

Then by the Mean Value Theorem

$$\begin{aligned}&\Delta u_\lambda + u_\lambda g(u_\lambda, v_\lambda)\\ =&\Delta u_\lambda + u_\lambda[g(0,0) + g(u_\lambda,v_\lambda) - g(0,v_\lambda) + g(0,v_\lambda) - g(0,0)]\\ \leq&\Delta u_\lambda + u_\lambda[g(0,0) + u_\lambda\sup(g_u) + v_\lambda\sup(g_v)]\\ \leq&\alpha\lambda[\Delta\theta_{g(0,0)} + \theta_{g(0,0)}\{g(0,0) - \alpha\lambda\theta_{g(0,0)} + \sup(g_v)\beta\lambda\theta_{g(0,0)}\}]\\ =&\alpha\lambda[\Delta\theta_{g(0,0)} + \theta_{g(0,0)}\{g(0,0) - \lambda\theta_{g(0,0)}(\alpha - \sup(g_v)\beta)\}]\\ =&\alpha\lambda[\Delta\theta_{g(0,0)} + \theta_{g(0,0)}\\ &\{g(0,0) - \lambda\theta_{g(0,0)}(\frac{-\sup(g_v) - \sup(h_u)\sup(g_v) + 1 + \sup(g_v)}{1-\sup(g_v)\sup(h_u)})\}]\\ =&\alpha\lambda[\Delta\theta_{g(0,0)} + \theta_{g(0,0)}\{g(0,0) - \lambda\theta_{g(0,0)}\}]\\ \leq&\alpha\lambda[\Delta\theta_{g(0,0)} + \theta_{g(0,0)}\{g(0,0) - \theta_{g(0,0)}\}]\\ =&0,\end{aligned}$$

and

$$
\begin{aligned}
&\Delta v_\lambda + v_\lambda h(u_\lambda, v_\lambda)\\
=&\Delta v_\lambda + v_\lambda[h(0,0) + h(u_\lambda, v_\lambda) - h(u_\lambda, 0) + h(u_\lambda, 0) - h(0,0)]\\
\le&\Delta v_\lambda + v_\lambda[h(0,0) + v_\lambda \sup(h_v) + u_\lambda \sup(h_u)]\\
\le&\beta\lambda[\Delta\theta_{g(0,0)} + \theta_{g(0,0)}\{h(0,0) - \beta\lambda\theta_{g(0,0)} + \sup(h_u)\alpha\lambda\theta_{g(0,0)}\}]\\
\le&\beta\lambda[\Delta\theta_{g(0,0)} + \theta_{g(0,0)}\{g(0,0) - \lambda\theta_{g(0,0)}(\beta - \sup(h_u)\alpha)\}]\\
=&\beta\lambda[\Delta\theta_{g(0,0)} + \theta_{g(0,0)}\\
&\{g(0,0) - \lambda\theta_{g(0,0)}(\frac{-\sup(h_u) - \sup(h_u)\sup(g_v) + 1 + \sup(h_u)}{1 - \sup(g_v)\sup(h_u)})\}]\\
=&\beta\lambda[\Delta\theta_{g(0,0)} + \theta_{g(0,0)}\{g(0,0) - \lambda\theta_{g(0,0)}\}]\\
\le&\beta\lambda[\Delta\theta_{g(0,0)} + \theta_{g(0,0)}\{g(0,0) - \theta_{g(0,0)}\}]\\
=&0.
\end{aligned}
$$

Therefore, by the Lemma 3.3,

$$u \le \alpha\theta_{g(0,0)}, v \le \beta\theta_{g(0,0)}. \tag{4.4}$$

By (4.2), (4.3) and (4.4), we have the desired inequalities. □

We also established the following solution estimate in [7].

Lemma 4.1. *Let (u, v) be any pair of positive solution to (2.2).*

(i) If $g(0,0) \ge h(0,0), -1 \le g_u < 0, h_v \le -1$, then

$$[\sup(h_u) + 1]u \ge [\inf(g_v) + 1]v.$$

(ii) If $g(0,0) \le h(0,0), -1 \le h_v < 0, g_u \le -1$, then

$$[\inf(h_u) + 1]u \le [\sup(g_v) + 1]v.$$

5. Uniqueness 1

In this section, we prove the following uniqueness result.

Theorem 5.1. *If $g(0,0) \ge h(0,0) > \lambda_1$ and*

$$\alpha \sup\frac{\theta_{g(0,0)}}{\theta_{h(0,\cdot)}}[\sup(g_v)]^2 + \beta\sup\frac{\theta_{g(0,0)}}{\theta_{g(\cdot,0)}}[\sup(h_u)]^2 + 2\sup(g_v)\sup(h_u) < 4,$$

where $\alpha = \frac{1+\sup(g_v)}{1-\sup(g_v)\sup(h_u)}$ and $\beta = \frac{1+\sup(h_u)}{1-\sup(g_v)\sup(h_u)}$, then (4.1) has a unique positive solution.

Proof. Let $(u_1, v_1), (u_2, v_2)$ be positive solutions to (4.1), and let $p = u_1 - u_2, q = v_1 - v_2$. We want to show that $p \equiv q \equiv 0$. Since $(u_1, v_1), (u_2, v_2)$ are solutions to (4.1),

$$
\begin{aligned}
\Delta(p) + g(u_1, v_1)p + [g(u_1, v_1) - g(u_2, v_2)]u_2 = 0,\\
\Delta(q) + h(u_2, v_2)q + [h(u_1, v_1) - h(u_2, v_2)]v_1 = 0.
\end{aligned}
$$

So,

$$\Delta(p) + g(u_1, v_1)p + [g(u_1, v_1) - g(u_2, v_1) + g(u_2, v_1) - g(u_2, v_2)]u_2 = 0,$$
$$\Delta(q) + h(u_2, v_2)q + [h(u_1, v_1) - h(u_2, v_1) + h(u_2, v_1) - h(u_2, v_2)]v_1 = 0.$$

By the Mean Value Theorem, there are $\bar{u}, \bar{v}$,and $\tilde{u}, \tilde{v}$ such that $\bar{u}$ and $\tilde{u}$ are between u_1 and u_2, $\bar{v}$ and $\tilde{v}$ are between v_1 and v_2, and

$$g(u_1, v_1) - g(u_2, v_1) = g_u(\bar{u}, v_1)p,$$
$$g(u_2, v_1) - g(u_2, v_2) = g_v(u_2, \bar{v})q,$$
$$h(u_1, v_1) - h(u_2, v_1) = h_u(\tilde{u}, v_1)p,$$
$$h(u_2, v_1) - h(u_2, v_2) = h_v(u_2, \tilde{v})q.$$

Therefore,

$$\Delta(p) + g(u_1, v_1)p + [g_u(\bar{u}, v_1)p + g_v(u_2, \bar{v})q]u_2 = 0,$$
$$\Delta(q) + h(u_2, v_2)q + [h_u(\tilde{u}, v_1)p + h_v(u_2, \tilde{v})q]v_1 = 0.$$

Since

$$\Delta(u_1) + g(u_1, v_1)u_1 = 0,$$
$$\Delta(v_2) + h(u_2, v_2)v_2 = 0,$$

by the Lemma 3.1, we have

$$\int_\Omega -p\Delta(p) - g(u_1, v_1)p^2 dx \geq 0,$$
$$\int_\Omega -q\Delta(q) - h(u_2, v_2)q^2 dx \geq 0,$$

and so, $\int_\Omega -g_u(\bar{u}, v_1)u_2 p^2 - [g_v(u_2, \bar{v})u_2 + h_u(\tilde{u}, v_1)v_1]pq - h_v(u_2, \tilde{v})v_1 q^2 dx \leq 0$. Hence, $p \equiv q \equiv 0$ if the integrand is positive definite, in other words,

$$[g_v(u_2, \bar{v})u_2 + h_u(\tilde{u}, v_1)v_1]^2 < 4g_u(\bar{u}, v_1)h_v(u_2, \tilde{v})u_2 v_1,$$

which is true if

$$[\sup(g_v)]^2 u_2^2 + [\sup(h_u)]^2 v_1^2 + 2u_2 v_1 \sup(g_v)\sup(h_u) < 4u_2 v_1,$$

which is true if

$$[\sup(g_v)]^2 \frac{u_2}{v_1} + [\sup(h_u)]^2 \frac{v_1}{u_2} + 2\sup(g_v)\sup(h_u) < 4,$$

which is true if the condition is satisfied by the solution estimates in the Theorem 4.1. □

6. Uniqueness 2

In this section, we derive another uniqueness result.

We consider the model

$$\begin{aligned}
&\Delta u(x) + u(x)g(u(x), v(x)) = 0,\\
&\Delta v(x) + v(x)h(u(x), v(x)) = 0 \ \text{ in } \ \Omega,\\
&u|_{\partial\Omega} = v|_{\partial\Omega} = 0,
\end{aligned} \tag{6.1}$$

where $g, h \in C^1$ are such that $g_u < 0, g_v > 0, h_u > 0, h_v < 0$, $g(\cdot, 0) \geq h(0, \cdot)$, $\sup(g_u) \leq -1, \sup(h_v) \leq -1$, $\sup(g_v)\sup(h_u) < 1$, and there is $c > 0$ such that $g(u, 0) \leq 0, h(0, v) \leq 0$ for all $u, v \geq c$.

Define $\delta = \inf_\Omega \frac{\theta_{h(0,\cdot)}}{\theta_{g(0,0)}}$.

Theorem 6.1. *Assume* $g(0, 0) \geq h(0, 0) > \lambda_1$. *If* $\alpha\beta \sup(g_v)\sup(h_u) < \inf(1, \delta^2)$, *where*

$$\begin{aligned}
\alpha &= \frac{1 + \sup(g_v)}{1 - \sup(g_v)\sup(h_u)},\\
\beta &= \frac{1 + \sup(h_u)}{1 - \sup(g_v)\sup(h_u)},
\end{aligned}$$

then (6.1) *has a unique positive solution.*

Proof. Since $\sup(g_u) \leq -1, \sup(h_v) \leq -1$ and $\sup(g_v)\sup(h_u) < 1$, by the Theorem 2.3, the existence part was established. We focus on the uniqueness. Assume there is more than one solution. Then the Lemma 3.4 guarantees existence of the maximal solution $(\bar{u}, \bar{v})$, i.e. $u \leq \bar{u}, v \leq \bar{v}$ for any other solution (u, v) of (6.1). Consider a family of supersolutions $w_\lambda = (u + \lambda u, v + \lambda\gamma v)$ with any $\lambda > 0$ and $\gamma > 0$ to be specified. $\bar{u} \leq u + \lambda u, \bar{v} \leq v + \lambda\gamma v$ for λ sufficiently large. In order for w_λ to be a family of supersolutions, it suffices that

$$\begin{aligned}
&\Delta(u + \lambda u) + (u + \lambda u)g(u + \lambda u, v + \lambda\gamma v) \leq 0,\\
&\Delta(v + \lambda\gamma v) + (v + \lambda\gamma v)h(u + \lambda u, v + \lambda\gamma v) \leq 0,
\end{aligned} \tag{6.2}$$

which will be satisfied if

$$\begin{aligned}
&\Delta u + ug(u + \lambda u, v + \lambda\gamma v) + \lambda[\Delta u + ug(u + \lambda u, v + \lambda\gamma v)] \leq 0,\\
&\Delta v + vh(u + \lambda u, v + \lambda\gamma v) + \lambda\gamma[\Delta v + vh(u + \lambda u, v + \lambda\gamma v)] \leq 0,
\end{aligned}$$

which will be satisfied if

$$\begin{aligned}
&u[g(u + \lambda u, v + \lambda\gamma v) - g(u, v)] + \lambda u[g(u + \lambda u, v + \lambda\gamma v) - g(u, v)] \leq 0,\\
&v[h(u + \lambda u, v + \lambda\gamma v) - h(u, v)] + \lambda\gamma v[h(u + \lambda u, v + \lambda\gamma v) - h(u, v)] \leq 0,
\end{aligned}$$

which will be satisfied if

$$\begin{aligned}
&(1 + \lambda)u[g(u + \lambda u, v + \lambda\gamma v) - g(u, v)] \leq 0,\\
&(1 + \lambda\gamma)v[h(u + \lambda u, v + \lambda\gamma v) - h(u, v)] \leq 0,
\end{aligned}$$

which will be satisfied if

$$\begin{aligned}
&g(u + \lambda u, v + \lambda\gamma v) - g(u, v) \leq 0,\\
&h(u + \lambda u, v + \lambda\gamma v) - h(u, v) \leq 0,
\end{aligned}$$

which will be satisfied if

$$g(u+\lambda u, v+\lambda\gamma v) - g(u, v+\lambda\gamma v) + g(u, v+\lambda\gamma v) - g(u,v) \le 0,$$
$$h(u+\lambda u, v+\lambda\gamma v) - h(u+\lambda u, v) + h(u+\lambda u, v) - h(u,v) \le 0.$$

But, by the Mean Value Theorem there are $\bar{u}, \bar{v}, \tilde{u}, \tilde{v}$ such that

$$u \le \bar{u} \le u + \lambda u,$$
$$v \le \bar{v} \le v + \lambda\gamma v,$$
$$u \le \tilde{u} \le u + \lambda u,$$
$$v \le \tilde{v} \le v + \lambda\gamma v,$$

and

$$g(u+\lambda u, v+\lambda\gamma v) - g(u, v+\lambda\gamma v) = \lambda u g_u(\bar{u}, v+\lambda\gamma v),$$
$$g(u, v+\lambda\gamma v) - g(u,v) = \lambda\gamma v g_v(u, \bar{v}),$$
$$h(u+\lambda u, v) - h(u,v) = \lambda u h_u(\tilde{u}, v),$$
$$h(u+\lambda u, v+\lambda\gamma v) - h(u+\lambda u, v) = \lambda\gamma v h_v(u+\lambda u, \tilde{v}),$$

and so, (6.2) will be satisfied if

$$\lambda u g_u(\bar{u}, v+\lambda\gamma v) + \lambda\gamma v g_v(u, \bar{v}) \le 0,$$
$$\lambda\gamma v h_v(u+\lambda u, \tilde{v}) + \lambda u h_u(\tilde{u}, v) \le 0,$$

which will be satisfied if

$$\lambda u \sup(g_u) + \lambda\gamma v \sup(g_v) \le 0,$$
$$\lambda\gamma v \sup(h_v) + \lambda u \sup(h_u) \le 0,$$

which will be satisfied if

$$\sup(g_v) \le -\frac{u \sup(g_u)}{\gamma v},$$
$$\sup(h_u) \le -\frac{\gamma v \sup(h_v)}{u}.$$

But, since $\sup(g_u) \le -1$ and $\sup(h_v) \le -1$, (6.2) will be satisfied if

$$\sup(g_v) \le \frac{u}{\gamma v},$$
$$\sup(h_u) \le \frac{\gamma v}{u}.$$

But, by the Theorem 4.1 and the definition of δ,

$$\frac{u}{v} \ge \frac{\theta_{g(\cdot,0)}}{\beta\theta_{g(0,0)}} \ge \frac{\theta_{h(0,\cdot)}}{\beta\theta_{g(0,0)}} \ge \frac{\delta}{\beta},$$
$$\frac{v}{u} \ge \frac{\theta_{h(0,\cdot)}}{\alpha\theta_{g(0,0)}} \ge \frac{\delta}{\alpha},$$

and so, (6.2) will be satisfied if

$$\sup(g_v) \le \frac{\delta}{\beta\gamma},$$
$$\sup(h_u) \le \frac{\delta\gamma}{\alpha}.$$

This can clearly be accomplished by choosing γ by the condition. Letting $\lambda \to 0$, we conclude the proof by the Lemma 3.3. □

We may also use the Lemma 4.1 at the end of proof of theorem 6.1 to derive the following another uniqueness result. We consider the model

$$\begin{aligned} &\Delta u(x) + u(x) g(u(x), v(x)) = 0, \\ &\Delta v(x) + v(x) h(u(x), v(x)) = 0 \ \text{ in } \ \Omega, \\ &u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{aligned} \tag{6.3}$$

where $g, h \in C^1$ are such that $g_u < 0, g_v > 0, h_u > 0, h_v < 0$, and $\sup(g_v)\sup(h_u) < 1$.

Theorem 6.2. *If* $g(0,0) = h(0,0) > \lambda_1, \sup(g_u) = \sup(h_v) = -1$ *and* $\frac{A}{B} < 1$*, where* $A = \sup(g_v)\sup(h_u)[\sup(g_v)+1][\sup(h_u)+1]$ *and* $B = [\inf(g_v)+1][\inf(h_u)+1]$*, then* (6.3) *has a unique positive solution.*

7. Uniqueness with perturbation

Define $B = \{(g,h) \in [C^1]^2 | g_u, g_v, h_u, h_v \ \text{ are bounded}, \ \sup(g_u) \le -1, \sup(h_v) \le -1 \ \text{ and } \ g(0,0) \ge h(0,0).\}$ with $\| (g,v) \|_B = |g(0,0)| + \sup|g_u| + \sup|g_v| + |h(0,0)| + \sup|h_u| + \sup|h_v|$ for all $(g,h) \in B$.

Then by the functional analysis theory, $(B, \|\cdot\|_B)$ is a Banach space.

The following theorem is our main result about the perturbation of uniqueness.

Theorem 7.1. *Suppose*
(A) $(g,h) \in B$ *is such that* $g(0,0) \ge h(0,0) > \lambda_1$,
(B) (4.1) *has a unique coexistence state* (u,v),
(C) the Fréchet derivative of (4.1) *at* (u,v) *is invertible.*
Then there is a neighborhood V *of* (g,h) *in* B *such that if* $(\bar g, \bar h) \in V$*, then* (4.1) *with* $(\bar g, \bar h)$ *has a unique positive solution.*

Proof. Since the Fréchet derivative of (4.1) at (u,v) is invertible, by the Implicit Function Theorem, there is a neighborhood V of (g,h) in B and a neighborhood W of (u,v) in $[C_0^{2,\alpha}(\bar\Omega)]^2$ such that for all $(\bar g, \bar h) \in V$, there is a unique positive solution $(\bar u, \bar v) \in W$ of (4.1) with $(\bar g, \bar h)$. Thus, the local uniqueness of the solution is guaranteed. To prove global uniqueness, suppose that the conclusion of Theorem 7.1 is false. Then there are sequences $(g_n, h_n, u_n, v_n), (g_n, h_n, u_n^*, v_n^*)$ in $V \times [C_0^{2,\alpha}(\bar\Omega)]^2$ such that (u_n, v_n) and (u_n^*, v_n^*) are positive solutions of (4.1)) with (g_n, h_n), $(u_n, v_n) \ne (u_n^*, v_n^*)$ and $(g_n, h_n) \to (g,h)$. By Schauder's estimate in elliptic theory and the solution estimate in the Theorem 4.1, there are uniformly convergent subsequences of u_n and v_n, which again will be denoted by u_n and v_n. Thus, let

$$(u_n, v_n) \to (\bar u, \bar v),$$

$$(u_n^*, v_n^*) \to (u^*, v^*).$$

Then $(\bar{u}, \bar{v}), (u^*, v^*) \in (C^{2,\alpha})^2$ are also solutions to (4.1) with (g, h). We claim that $\bar{u} > 0, \bar{v} > 0, u^* > 0, v^* > 0$. By the Maximum Principles, it suffices to claim $\bar{u}, \bar{v}, u^*, v^*$ are not identically zero. Suppose that it is not true. Then by the Maximum Principles again, either one of the followings will hold: (1) $\bar{u} \equiv 0, \bar{v} > 0$ (2) $\bar{u} > 0, \bar{v} \equiv 0$ (3) $\bar{u} \equiv \bar{v} \equiv 0$ First, suppose $\bar{u} \equiv 0$. Let $\tilde{u}_n = \frac{u_n}{\|u_n\|_\infty}$ and $\tilde{v}_n = v_n$. Then

$$\Delta \tilde{u}_n + \tilde{u}_n g_n(u_n, \tilde{v}_n) = 0,$$
$$\Delta \tilde{v}_n + \tilde{v}_n h_n(u_n, \tilde{v}_n) = 0.$$

By the elliptic theory again, there is $\tilde{u}$ such that $\tilde{u}_n \to \tilde{u}$, and so,

$$\Delta \tilde{u} + \tilde{u} g(0, \bar{v}) = 0,$$
$$\Delta \bar{v} + \bar{v} h(0, \bar{v}) = 0.$$

Hence, $\lambda_1[-g(0, \bar{v})] = 0$. If $\bar{v} \equiv 0$, then $-g(0,0) + \lambda_1 = \lambda_1[-g(0,0)] = 0$, and so $g(0,0) = \lambda_1$, which is a contradiction to our assumption. If $\bar{v}$ is not identically zero, then $-g(0,0) + \lambda_1 = \lambda_1[-g(0,0)] \geq \lambda_1[-g(0, \theta_{h(0,\cdot)})] = 0$, and so, $g(0,0) \leq \lambda_1$ which is also a contradiction. Next, suppose that $\bar{u} > 0$ and $\bar{v} \equiv 0$. Let

$$\tilde{u}_n = u_n,$$
$$\tilde{v}_n = \frac{v_n}{\| v_n \|_\infty}.$$

Then

$$\Delta \tilde{u}_n + \tilde{u}_n g_n(\tilde{u}_n, v_n) = 0,$$
$$\Delta \tilde{v}_n + \tilde{v}_n h_n(\tilde{u}_n, v_n) = 0.$$

By the elliptic theory again, there is $\tilde{v}$ such that $\tilde{v}_n \to \tilde{v}$, and so,

$$\Delta \bar{u} + \bar{u} g(\bar{u}, 0) = 0,$$
$$\Delta \tilde{v} + \tilde{v} h(\bar{u}, 0) = 0.$$

Therefore, $-h(0,0) + \lambda_1 = \lambda_1[-h(0,0)] \geq \lambda_1[-h(\bar{u}, 0)] = 0$, and so, $h(0,0) \leq \lambda_1$ which is a contradiction. Consequently, $(\bar{u}, \bar{v})$ and (u^*, v^*) are positive solutions to (4.1)) with (g, h), and so, $(\bar{u}, \bar{v}) = (u^*, v^*) = (u, v)$ by the uniqueness condition. But, this is a contradiction to the Implicit Function Theorem, since $(u_n, v_n) \neq (u_n^*, v_n^*)$. $\square$

Lemma 7.1. *Suppose* (u, v) *is a positive solution to* (2.2)*. If* $4 \sup(g_u) \sup(h_v) uv > [\sup(g_v)]^2 u^2 + [\sup(h_u)]^2 v^2 + 2uv \sup(g_v) \sup(h_u)$*, then the Fréchet derivative of* (2.2) *at* (u, v) *is invertible.*

Proof. The Fréchet derivative of (2.2)) at (u, v) is

$$A = \begin{pmatrix} \Delta + g(u,v) + u g_u(u,v) & u g_v(u,v) \\ v h_u(u,v) & \Delta + h(u,v) + v h_v(u,v) \end{pmatrix}.$$

We need to show that $N(A) = \{0\}$ by the Fredholm alternative, where $N(A)$ is the null space of A. If

$$\Delta\phi + [g(u,v) + ug_u(u,v)]\phi + g_v(u,v)u\psi = 0,$$
$$\Delta\psi + h_u(u,v)v\phi + [h(u,v) + vh_v(u,v)]\psi = 0,$$

then

$$\int_\Omega |\nabla\phi|^2 - [g(u,v) + ug_u(u,v)]\phi^2 - g_v(u,v)u\phi\psi dx = 0,$$
$$\int_\Omega |\nabla\psi|^2 - h_u(u,v)v\phi\psi - [h(u,v) + vh_v(u,v)]\psi^2 dx = 0.$$

Since (u,v) is a positive solution to (2.2)), by the Lemma 3.1, we have

$$\int_\Omega |\nabla\phi|^2 - g(u,v)\phi^2 dx \geq 0,$$
$$\int_\Omega |\nabla\psi|^2 - h(u,v)\psi^2 dx \geq 0.$$

Hence,

$$\int_\Omega -ug_u(u,v)\phi^2 - g_v(u,v)u\phi\psi dx \leq 0,$$
$$\int_\Omega -h_u(u,v)v\phi\psi - h_v(u,v)v\psi^2 dx \leq 0.$$

Therefore, $\int_\Omega -ug_u(u,v)\phi^2 - [g_v(u,v)u + h_u(u,v)v]\phi\psi - h_v(u,v)v\psi^2 dx \leq 0$. Hence, $(\phi,\psi) = (0,0)$ if the integrand is positive definite, which is true if the condition is satisfied. $\square$

Combining Theorem 5.1, Theorem 7.1, and Lemma 7.1, we obtain the following corollary.

Corollary 7.1. *If $(g,h) \in B$ is such that $g(0,0) \geq h(0,0) > \lambda_1$ and*

$$\alpha \sup \frac{\theta_{g(0,0)}}{\theta_{h(0,\cdot)}}[\sup(g_v)]^2 + \beta \sup \frac{\theta_{g(0,0)}}{\theta_{g(\cdot,0)}}[\sup(h_u)]^2 + 2\sup(g_v)\sup(h_u) < 4,$$

where $\alpha = \frac{1+\sup(g_v)}{1-\sup(g_v)\sup(h_u)}$ and $\beta = \frac{1+\sup(h_u)}{1-\sup(g_v)\sup(h_u)}$, then there is a neighborhood V of (g,h) in B such that if $(\bar{g},\bar{h}) \in V$, then (4.1) *with $(\bar{g},\bar{h})$ has a unique positive solution.*

We can establish other perturbation results using theorems 6.1 and 6.2.

Corollary 7.2. *If $(g,h) \in B$ is such that $g(\cdot,0) \geq h(0,\cdot), g(0,0) \geq h(0,0) > \lambda_1$, $\alpha\beta\sup(g_v)\sup(h_u) < \inf(1,\delta^2)$ and*

$$\sup \frac{\theta_{g(0,0)}}{\theta_{h(0,\cdot)}}[\sup(g_v)]^2 + \sup \frac{\theta_{g(0,0)}}{\theta_{g(\cdot,0)}}[\sup(h_u)]^2 + 2\sup(g_v)\sup(h_u) < 4,$$

then there is a neighborhood V of (g,h) in B such that if $(\bar{g},\bar{h}) \in V$, then (4.1) *with $(\bar{g},\bar{h})$ has a unique positive solution.*

Corollary 7.3. *If* $(g,h) \in B$ *is such that* $g(0,0) = h(0,0) > \lambda_1, \sup(g_u) = \sup(h_v) = -1$, $\frac{\sup(g_v)\sup(h_u)[\sup(g_v)+1][\sup(h_u)+1]}{[\inf(g_v)+1][\inf(h_u)+1]} < 1$ *and*

$$\sup \frac{\theta_{g(0,0)}}{\theta_{h(0,\cdot)}}[\sup(g_v)]^2 + \sup \frac{\theta_{g(0,0)}}{\theta_{g(\cdot,0)}}[\sup(h_u)]^2 + 2\sup(g_v)\sup(h_u) < 4,$$

then there is a neighborhood V *of* (g,h) *in* B *such that if* $(\bar{g},\bar{h}) \in V$*, then* (4.1) *with* $(\bar{g},\bar{h})$ *has a unique positive solution.*

8. Uniqueness with Perturbation of Region

The following theorem is the more generalized perturbation result.

Theorem 8.1. *Suppose* Γ *be a closed, bounded, convex region in* B *such that*
(A) for each $(g,h) \in \Gamma$, $g(0,0) \geq h(0,0) > \lambda_1$,
(B) for every $(g,h) \in \partial_L\Gamma$, (4.1) *has a unique positive solution, where* $\partial_L\Gamma = \{(\lambda_h,h) \in \Gamma | \text{for any fixed } h, \| \lambda_h \| = \inf\{\| g \| \,|(g,h) \in \Gamma\}\}$,
(C) for each $(g,h) \in \Gamma$*, the Fréchet derivative of* (4.1) *at every positive solution* (u,v) *is invertible.*
Then for all $(g,h) \in \Gamma$, (4.1) *has a unique positive solution. Furthermore, there is an open set* W *in* B *such that* $\Gamma \subseteq W$ *and for every* $(g,h) \in W$, (4.1) *has a unique positive solution.*

Proof. For each fixed h, consider $(g,h) \in \partial_L\Gamma$ and $(\bar{g},h) \in \Gamma$. We need to show that for all $0 \leq t \leq 1$, (4.1) with $(1-t)(g,h) + t(\bar{g},h)$ has a unique positive solution. Since (4.1) with (g,h) has a unique positive solution (u,v) and the Frèchet derivative of (4.1) at (u,v) is invertible, by the Theorem 7.1, there is an open neighborhood V of (g,h) in B such that if $(g_0,h_0) \in V$, then (4.1)) with (g_0,h_0) has a unique positive solution. Let $\lambda_s = \sup\{0 \leq \lambda \leq 1 | \text{ (4.1) with } (1-t)(g,h) + t(\bar{g},h) \text{ has a unique coexistence state for } 0 \leq t \leq \lambda\}$. We need to show that $\lambda_s = 1$. Suppose $\lambda_s < 1$. By the definition of λ_s, there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda_s^-$ and a sequence (u_n,v_n) of the unique positive solutions of (4.1) with $(1-\lambda_n)(g,h) + \lambda_n(\bar{g},h)$. Then by the elliptic theory, there is (u_0,v_0) such that (u_n,v_n) converges to (u_0,v_0) uniformly and (u_0,v_0) is the solution to (4.1) with $(1-\lambda_s)(g,h) + \lambda_s(\bar{g},h)$. But, by the same proof as in the section 6, $u_0 > 0, v_0 > 0$. We claim that (4.1) has a unique coexistence state with $(1-\lambda_s)(g,h) + \lambda_s(\bar{g},h)$. In fact, if not, assume that $(\bar{u}_0,\bar{v}_0) \neq (u_0,v_0)$ is another coexistence state. By the Implicit Function Theorem, there exists $0 < \tilde{a} < \lambda_s$ and very close to λ_s such that (4.1) with $(1-\tilde{a})(g,h) + \tilde{a}(\bar{g},h)$ has a coexistence state very close to $(\bar{u}_0,\bar{v}_0)$, which means that (4.1) with $(1-\tilde{a})(g,h)+\tilde{a}(\bar{g},h)$ has more than one coexistence state. This is a contradiction to the definition of λ_s. But, since (4.1) with $(1-\lambda_s)(g,h)+\lambda_s(\bar{g},h)$ has a unique coexistence state and the Fréchet derivative is invertible, Theorem 7.1 implies that λ_s can not be as defined. Therefore, for each $(g,h) \in \Gamma$, (4.1) with (g,h) has a unique coexistence state (u,v). Furthermore, by the assumption, for each $(g,h) \in \Gamma$, the Fréchet derivative of (4.1) with (g,h) at the unique solution (u,v) is invertible. Hence, Theorem 7.1 concluded that for each $(g,h) \in \Gamma$, there is an open neighborhood $V_{(g,h)}$ of (g,h) in B such that if $(\tilde{g},\tilde{h}) \in V_{(g,h)}$, then (4.1) with $(\tilde{g},\tilde{h})$ has a unique coexistence state. Let $W = \bigcup_{(g,h)\in\Gamma} V_{(g,h)}$. Then W is an open set in B such that $\Gamma \subseteq W$ and for each $(\tilde{g},\tilde{h}) \in W$, (4.1) with $(\tilde{g},\tilde{h})$ has a unique coexistence state. $\square$

Apparently, Theorem 8.1 generalizes Theorem 7.1.

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