2023

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The Role of r-Modes in Pulsar Spindown, Pulsar Timing and Gravitational Waves

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ABSTRACT

Pulsars are fast spinning neutron stars that lose their rotational energy via various processes such as gravitational and magnetic radiation, particle acceleration and mass loss processes. This dissipation can be quantified by a spin-down equation that measures the rate of change of the frequency as a function of the rotational frequency itself. We explore the pulsar spin-down and consider the spin-down equation upto the seventh order in frequency. This seventh order term accounts for energy loss due to the gravitational radiation caused by a current type quadrupole in the pulsar due to r-modes. We derive the rotational frequency due to the r-modes and find a solution in terms of the Lambert function. We also present an analytic exact solution for the period from the spindown equation and numerically verify this for the Crab pulsar. This analysis will be relevant for the detection of continuous gravitational waves by 3G ground based and space based gravitational wave detectors.

1. INTRODUCTION

Pulsars are known to be highly magnetized, fast spinning neutron stars. They emit beams of EM radiation from their poles and are detected only when a beam shines on the earth. This sort of lighthouse effect has often led to them being referred to as Cosmic Lighthouses (Lorimer & Kramer 2004). Neutrons stars are the remnants of stars after their supernovae and tend to retain most of the angular momentum. Since the moment of inertia reduces greatly during the supernova, they end up with very high rotations speeds. They also have a magnetic axis (that may or may not be aligned with the axis of rotation) which determines the direction of the ejected electromagnetic beam. Over time, these pulsars tend to lose their energy and consequently, slow down until they eventually "turn off" (Taylor & Stinebring 1986).

Broadly speaking, neutron stars are divided into three main categories– rotation powered, accretion powered and magnetars. Rotation powered pulsars (or radio pulsars) generate their power from the physical rotation of the pulsar. The rotating magnetic field creates an electric field. The field then accelerates the charged particles on the surface that results in an electromagnetic beam; the energy loss comes due to the rotational deceleration. These types of pulsars typically have periods of the order of $1ms - 1s$ (Bhattacharya & van den Heuvel 1991; Beskin et al. 2015). Accretion powered pulsars (X-ray pulsars) use the gas particles accreting around the star to generate X-ray emissions that can then be detected by space telescopes. Since there is no direct conversion of rotational energy into EM radiation, X-ray pulsars, in contrast to radio pulsars, display a variety of spin behaviors (increasing spin or spin-up, decreasing spin or spin-down, nearly constant spin, erratic spin-ups and spin-downs etc.) (Bildsten et al. 1997). It is also common for accreting pulsars to spin up to transform into millisecond pulsars (MSPs), this is usually referred to as the "recycling" of neutron stars/pulsars (Strohmayer et al. 1996; Barkov & Komissarov 2011). Magnetars are extremely magnetic neutron stars (about 1000 times stronger than normal pulsars) that emit EM radiation in the X-ray and gamma ray regions.(Duncan & Thompson 1992). Their magnetic fields are theorized to be generated by magnetohydrodynamic dynamo processes in the highly dense and turbulent fluid in the neutron star (Thompson & Duncan 1993); there are about 31 magnetars that have been discovered so far (Kaspi & Beloborodov 2017).

Pulsars are used by astronomers and physicists for a variety of different applications. They emit radio waves in the form of pulses as they rotate. These pulses can be precisely timed, much like the ticking of a clock. Among pulsars, millisecond pulsars hold particular significance as they spin hundreds of times per second and exhibit remarkable sta-

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bility comparable to atomic clocks on Earth (Taylor 1991; Matsakis et al. 1997). Autonomous spacecraft navigation is another area where pulsars are utilized. The timing stabilities of pulsars allow them to be used as natural navigation beacons (similar to GPS on earth). The locations of spacecrafts can be determined autonomously by measuring the pulse arrival times from an array of pulsars. This approach has the potential to revolutionize deep space exploration by enabling spacecraft to operate independently and efficiently in remote regions of the universe (Becker et al. 2013). By far the most common and most heavily researched application of pulsars is their ability to detect gravitational waves using Pulsar Timing Arrays. PTAs consist of a group of pulsars whose correlated pulse arrival times are analyzed collectively. The inherent accuracy and consistency of pulsar arrival times make them sensitive probes for detecting deviations or discrepancies, which could indicate the presence of gravitational waves propagating through spacetime along with the pulsar beam. The detection of such variations would then be compelling evidence for gravitational waves. Notably, multiple consortia, including EPTA, InPTA, PPTA, CPTA and NANOGrav, are actively involved in utilizing pulsar timing arrays for gravitational wave detection in the nanohertz-microhertz frequencies (Manchester et al. 2013; Manchester 2013; Hobbs et al. 2010). The recent detections and data releases by the InPTA and EPTA (Antoniadis et al. 2023), the Parkes PTA (Zic et al. 2023) and the NANOGrav (Agazie et al. 2023) have been important keystones in the detection of continuous gravitational waves. The Square Kilometer Array is a planned large radio interferometer designed to operate over a wide range of frequencies, with sensitivities higher than any of the existing radio telescopes (Weltman et al. 2020). The SKA PTA detection of low frequency gravitational waves can provide robust constraints on the seeding and growth of the first supermassive black holes (Padmanabhan & Loeb 2023). In addition to the detection of gravitational waves using the PTAs, the introduction of the 3G ground based GW detectors like the Einstein Telescope (Branchesi et al. 2023), Cosmic Explorer (Evans et al. 2021), and the space-based LISA (Amaro-Seoane et al. 2017), Taiji (Ruan et al. 2020), and Tianqin (Luo et al. 2016) detectors, will provide the prospect of detecting continuous gravitational waves directly from radio pulsars with imbalanced mass distributions.

We discuss the spin-down and various mechanisms by which a pulsar loses its energy in Sec. 2. We also derive the braking indices as functions of the spin-down coefficients in this section. We present an analytic solution for the period and the frequency assuming r-modes along with some numerical estimates in Sec. 3. We discuss the results in Sec. 4 and present the conclusions in Sec. 5

2. SPINDOWN

2.1. *Spindown Mechanisms*

As radio pulsars get older, they tend to lose their rotational energy via various physical phenomena. The most dominant process through which pulsars spin-down is the magnetic dipole radiation (Gold 1968; Pacini 1968). Another mechanism which becomes relevant when younger pulsars are considered is the emission of gravitational waves. This type of emission is usually a quadrupole type radiation. (Ostriker & Gunn 1969; Ferrari & Ruffini 1969). Pulsars also lose some rotational energy due to mass loss and particle acceleration processes (Michel 1969). An interesting yet relatively less explored way pulsars lose energy is through r-modes (Papaloizou & Pringle 1978; Andersson 1998). We will discuss the impact of these r-modes on the spin-down in the subsequent sections. These are usually inferred from the energy change rates (Alvarez, C. & Carramiñana, A. 2004),

$$\dot{E}_{\rm mono} = -\beta \left(\frac{\mu}{cR^2}\right) \omega^2 \tag{1}$$

$$\dot{E}_{\rm dip} = -\left(\frac{2\mu^2}{3c^3}\right) \omega^4 \tag{2}$$

$$\dot{E}_{\rm quad} = -\left(\frac{32GI^2e^2}{5c^5}\right) \omega^6 \tag{3}$$

where $\omega = 2\pi f$, β is a dimensionless factor, μ is the magnetic moment perpendicular to the angular frequency vector ω, R is the pulsar radius, I is the pulsar's moment of inertia and e is its ellipticity. The frequency rate can be related to the rate of change of energy by taking the rotational kinetic energy into account,

$$
\begin{aligned}
E_{rot} &= \frac{1}{2} I\omega^2 \\
\dot{E}_{rot} &= I\omega\dot{\omega} \\
\dot{\omega} &= \frac{\dot{E}_{rot}}{I\omega} \\
\implies \dot{f} &= \left(\frac{1}{4\pi^2 I}\right) \frac{\dot{E}_{rot}}{f}
\end{aligned} \tag{4}
$$

Eq. (4) is then used to calculate the spin-down contributions due to different energy loss mechanisms. It is useful to express the spindown equation (Eq.34 in a more general form using a Taylor series expansion and setting some constraints on the spindown rate. From here on, we build on the models discussed in (Alvarez, C. & Carramiñana, A. 2004) which considers the monopolar and dipole terms, and (Chishtie et al. 2018) which takes into account the effects of gravitational wave emissions. The spindown can then be formulated using the coefficients,

$$\dot{f} = -s(t)f - r(t)f^3 - g(t)f^5 \tag{5}$$

where $\{s, r, g\}$ are referred to as "spin-down coefficients". These coefficients hold physical significance and can be derived from the rate of energy change equations. These spindown coefficients are assumed to be time dependent to follow an inverse decay (this comes from the inverse decay of the magnetic field of the pulsar itself).

2.2. *r-Modes*

r-Modes were first proposed in 1978 by Papaloizou & Pringle (1978). They proposed a new class of low frequency modes in rotating neutron stars and white dwarfs which could account for short period oscillations of cataclysmic variables. These modes were termed r-modes and they are analogs of Rossby waves in the Earth's oceans and atmosphere. Andersson (1998) then showed that these would be a class of unstable modes for all types of rotating stars due to the emission of gravitational waves. It was also derived that the r-mode frequency is directly proportional to the rotational frequency. These results were based on mechanisms proposed by Chandrasekhar (1970); Friedman & Schutz (1978). Gravitational waves usually amplify the oscillations of these modes in neutron stars and this instability is called the CFS-instability. There were also some numerical studies that explored the increasing instability of these r-modes (Lindblom et al. 1998, 2001). Andersson et al. (1999) discussed the impact of r-modes on accreting neutron stars and white dwarfs to find that r-mode instability drastically slows down the rotation of young neutron stars and is active in short period white dwarfs. More recently too, there have been several researchers interested in r-modes. Mahmoodifar & Strohmayer (2017) calculated the upper limits for the r-mode oscillations in 2 pulsars (J1640, J1709) using *Chandra* Observations. Mytidis et al. (2015) derive physical quantities like the moment of interia as a function of observables from (hypothetical) r-mode detections. They also discuss a detection strategy to efficiently search for r-modes in gravitational-wave data. Gittins & Andersson (2023) discuss r-modes in stratified neutron stars and how the barotropicity of stars affects the modes.

The frequency at which a rotating pulsar emits gravitational waves is of the order of the rotational frequency (Bonazzola & Gourgoulhon 1996). More precisely, the gravitational wave (ν) and rotational frequency (f) are related as $\nu = (2/3\pi)f$. During the non-linear saturation phase of evolution of a pulsar, the time derivative of the gravitational wave frequency $\dot{\nu}$ is related to the frequency ν by the following power law (Owen et al. 1998):

$$\frac{d\nu}{dt} \approx -1.8\kappa \left(\frac{\nu}{1\ \text{kHz}}\right)^7 \text{Hz/s} \tag{6}$$

κ here is square of the r-mode amplitude. It is straightforward to write this equation in terms of the rotational frequency of the pulsar,

$$\frac{df}{dt} = -lf^7 \tag{7}$$

PSR	Crab	J1023-5746	J1418-6058	B2234+61
f (Hz)	29.947	8.971	9.044	2.019
s (Hz)	1.28×10^{-10}	2.97×10^{-6}	5.37×10^{-6}	2.58×10^{-6}
r ($\mathbf{Hz^{-1}}$)	-3.43×10^{-13}	-1.09×10^{-7}	-1.91×10^{-7}	-1.87×10^{-6}
g ($\mathbf{Hz^{-3}}$)	3.22×10^{-16}	1.34×10^{-9}	2.27×10^{-9}	4.49×10^{-7}
l ($\mathbf{Hz^{-5}}$)	-9.36×10^{-20}	-5.46×10^{-12}	-8.96×10^{-12}	-3.60×10^{-8}

Table 1. Spin-down parameters for four selected pulsars from the ATNF Pulsar Database (Manchester et al. 2005).

where l is some positive constant. Considering this addition to Eq. (5), it is useful to analyze how the period and frequency evolution of the pulsar changes.

We thus move on to analyse how the spindown analysis in Alvarez, C. & Carramiñana, A. (2004); Chishtie et al. (2018) would change when a new term, representing these modes, is introduced in the spindown equation.

3. ANALYTICAL EXPRESSIONS CONSIDERING R-MODES

In this section, we analytically solve for the rotational frequency and period taking into account the r-modes. We introduce a fourth spindown coefficient $l(t)$ that would account for this addition (Owen et al. 1998). This addition would transform the previous spin-down equation, Eq.(5) as:

$$\dot{f} = -sf - rf^3 - gf^5 - lf^7 \tag{8}$$

Estimates for these spin-down coefficients can be found in Table 1. The values were estimated by differentiating Eq.(8) and eliminating the spin-down coefficients subsequently. In Sec.(3.1, we look at an expression for the angular velocity of a pulsar in the presence of r-modes (Owen et al. 1998) and solve to get an expression for the same in terms of the Lambert W function. In Sec. (3.2, we write the equation (8) in terms of the period and then solve to get the rotation period of the pulsar as a function of time.

3.1. *Lambert solution of the Rotational Frequency*

The neutron star or pulsar can be treated as a system with two degrees of freedom– the angular velocity Ω and the r-mode amplitude α. We use the equations detailing the evolution of these parameters from (Owen et al. 1998)

$$\frac{d\alpha}{dt} = -\frac{\alpha}{\tau_{GR}} - \frac{\alpha}{\tau_V}\frac{1-\alpha^2 Q}{1+\alpha^2 Q} \tag{9}$$

$$\frac{d\Omega}{dt} = -\frac{2\Omega}{\tau_V}\frac{\alpha^2 Q}{1+\alpha^2 Q} \tag{10}$$

Q is an equation of state dependent parameter and is defined by $Q = 3\tilde{J}/2\tilde{I}$. τ_{GR}, τ_V are the gravitational wave and viscous timescales We first solve for α and then use that expression to solve for the angular velocity.

$$\frac{d\alpha}{\alpha\left(\frac{1}{\tau_{GR}} + \frac{1}{\tau_V}\frac{1-\alpha^2 Q}{1+\alpha^2 Q}\right)} = -dt \tag{11}$$

$$\frac{\tau_{GR}\tau_V(1+\alpha^2 Q)d\alpha}{\alpha((\tau_V + \tau_{GR} + \alpha^2(\tau_V - \tau_{GR}Q)} = -dt \tag{12}$$

Making some simple substitutions for brevity, $\xi_1 = \tau_{GR}\tau_V$, $A = \tau_{GR} + \tau_V$ and $B = (\tau_V - \tau_{GR}Q$

$$\frac{\xi_1(1+\alpha^2 Q)d\alpha}{\alpha(A+\alpha^2 B)} = -dt$$
$$\frac{1}{\alpha}\frac{\xi_1 d\alpha}{A+B\alpha^2} + \frac{\alpha Q d\alpha}{A+B\alpha^2} = -dt$$
$$\frac{\xi_1}{A\alpha} - \frac{(\xi_1 B/A)\alpha d\alpha}{A+B\alpha^2} + \frac{\alpha Q d\alpha}{A+B\alpha^2} = -dt$$
$$\frac{\xi_1}{A}\int\frac{d\alpha}{\alpha} + \left(Q - \frac{\xi_1 B}{A}\right)\int\frac{\alpha d\alpha}{A+B\alpha^2} = -\int dt$$
$$\frac{\xi_1}{A}\ln\alpha + \left(\frac{Q}{2} - \frac{\xi_1 B}{A}\right)\frac{\ln(A+B\alpha^2)}{B} = -t + c_1 \tag{13}$$

c_1 here is the constant of integration. We also further substitute $\xi_A = \xi_1/A$ and $\xi_B = \xi_1/B$

$$\xi_A \ln\alpha + \xi_B \ln(A+B\alpha^2) = -t + c_1$$
$$\xi_A \ln\alpha + \xi_B(\ln A + \ln(1 + B\alpha^2/A)) = -t + c_1$$

The term $B\alpha^2/A << 1$ and a Taylor expansion can be made for the logarithmic term as follows,

$$\xi_A \ln\alpha + \xi_B(\ln A + B\alpha^2/A) \approx -t + c_1$$
$$\ln\alpha + \frac{\xi_B}{\xi_A}\left(\frac{B\alpha^2}{A}\right) = \frac{-t - \xi_B \ln A + c}{\xi_A} \tag{14}$$

Putting in $\xi_c = B\xi_B/A\xi_A$,

$$\ln\alpha + \xi_c\alpha^2 = \frac{-t - \xi_B \ln A + c_1}{\xi_A}$$
$$\alpha e^{\xi_c\alpha^2} = \exp\left(\frac{-t - \xi_B \ln A + c_1}{\xi_A}\right)$$
$$(2\xi_c\alpha^2)e^{(2\xi_c\alpha^2)} = 2\xi_c \exp\left(\frac{-2(t + \xi_B \ln A - c_1)}{\xi_A}\right) \tag{15}$$

This expression is a representation of the Lambert W function and we can thus express α as,

$$2\xi_c\alpha^2 = W\left(2\xi_c \exp\left(\frac{-2(t + \xi_B \ln A - c_1)}{\xi_A}\right)\right)$$
$$\text{let } w = 2\xi_c \exp\left(\frac{-2(t + \xi_B \ln A - c_1)}{\xi_A}\right) \implies dt = -\frac{\xi_A dw}{4w\xi_c}$$
$$\implies \alpha^2 = \frac{1}{2\xi_c}W(w) \tag{16}$$

Using this expression for alpha in the frequency equation:

$$\frac{d\Omega}{dt} = -\frac{2\Omega}{\tau_V}\frac{\alpha^2 Q}{1+\alpha^2 Q}$$
$$\frac{d\Omega}{\Omega} = \frac{-2}{\tau_V}\frac{W(w)Q}{2\xi_c + W(w)Q}\left(-\frac{\xi_A dw}{4w\xi_c}\right)$$
$$\frac{d\Omega}{\Omega} = \left(\frac{\xi_A}{2\tau_V\xi_c}\right)\frac{W(w)Q dw}{(2\xi_c + W(w)Q)w}$$
$$\frac{d\Omega}{\Omega} = \left(\frac{\xi_A}{2\tau_V\xi_c}\right)\frac{(Q/2\xi_c)W(w)dw}{(1 + (Q/2\xi_c)W(w)Q)w} \tag{17}$$

The derivative of a Lambert function is given as:

$$\frac{dW(w)}{dw} = \frac{W(w)}{w(1+W(w))}$$
$$\implies kW'(w) = \frac{kW(w)}{w(1+kW(w))}\text{ , } k \in \mathbb{R}$$

Using this in the freq eq:

$$\int \frac{d\Omega}{\Omega} = \left(\frac{\xi_A}{2\tau_V \xi_c}\right) \frac{Q}{2\xi_c} \int W'(w)dw$$
$$\ln \Omega = \left(\frac{\xi_A Q}{4\tau_V \xi_c^2}\right) W(w) + c_2$$
$$\Omega = \exp\left(c_2 \left(\frac{\xi_A Q}{4\tau_V \xi_c^2}\right) W(w)\right)$$
$$\implies \Omega = \left(\frac{4\tau_V \xi_c^2}{c_2 \xi_A Q}\right) \frac{w}{W(w)} \text{ , where } w = 2\xi_c \exp\left(\frac{-2(t+\xi_B \ln A - c_1)}{\xi_A}\right) \tag{18}$$

The rotational frequency f relates to the angular velocity as $\Omega = 2\pi f$ and we thus have an expression for the frequency as a function of time.

3.2. *Period Analysis with Time Dependent Spin-down Coefficients*

Here we do a similar analysis with the period instead of the frequency and assume that the coefficients $\{s, r, g, l\}$ are time dependent (we assume an inverse linear decay for the magnetic field as done in (Chishtie et al. 2018). Transforming Eq. (8) by using $P = 1/f$, we get,

$$\frac{dP}{dt} = \left(s_0 P + \frac{r_0}{P} + \frac{g_0}{P^3} + \frac{l_0}{P^5}\right)\left(1 + \frac{t}{t_c}\right)^{-2}$$
$$P^5 \frac{dP}{dt} = (s_0 P^6 + r_0 P^4 + g_0 P^2 + l_0)\left(1 + \frac{t}{t_c}\right)^{-2} \tag{19}$$

Here, t_c is the characteristic timescale for field decay. Making an appropriate substitution, $Q = P^2, \quad dQ = 2PdP$, we then rewrite this as

$$\frac{Q^2 dQ}{2(s_0 Q^3 + r_0 Q^2 + g_0 Q + l_0)} = \left(1 + \frac{t}{t_c}\right)^{-2} dt$$
$$\implies \frac{Q^2 dQ}{2s_0(Q^3 + \frac{r_0}{s_0}Q^2 + \frac{g_0}{s_0}Q + \frac{l_0}{s_0}} = \left(1 + \frac{t}{t_c}\right)^{-2} dt$$
$$\implies \frac{Q^2 dQ}{2s_0((Q-a)(Q-b)(Q-c))} = \left(1 + \frac{t}{t_c}\right)^{-2} dt \tag{20}$$

(a, b, c) are the roots of the cubic in the denominator. The expression can then be simplified using partial fractions. Integrating on both sides after simplifying leaves us with:

$$-\frac{(b-a)c^2 \ln(Q-c) + (a-c)b^2 \ln(Q-b) + (c-b)a^2 \ln(Q-a)}{(b-a)(a-c)(c-b)} = \frac{-2t_c s_0}{1 + t/t_c} + \mathcal{C} \tag{21}$$

We assume (P_0, t_0) as the initial conditions for the period and time respectively, and then put these into the expression to calculate $\mathcal{C}$:

$$-\frac{(b-a)c^2 \ln\left(\frac{Q-c}{Q_0-c}\right) + (a-c)b^2 \ln\left(\frac{Q-b}{Q_0-b}\right) + (c-b)a^2 \ln\left(\frac{Q-a}{Q_0-a}\right)}{(b-a)(a-c)(c-b)} = 2t_c s_0 \left(\frac{1}{1 + t_0/t_c} - \frac{1}{1 + t/t_c}\right)$$
$$\implies (b-a)c^2 \ln\left(\frac{Q-c}{Q_0-c}\right) + (a-c)b^2 \ln\left(\frac{Q-b}{Q_0-b}\right) + (c-b)a^2 \ln\left(\frac{Q-a}{Q_0-a}\right) = -2s_0 t(b-a)(a-c)(c-b) \tag{22}$$

Where, $Q_0 = P_0^2$ and the limits $t_c >> t, t_0$ are applied.

In most cases, there are two complex and one real roots (as is the case for most cubic polynomials) with the complex roots being conjugates of eachother. We then write (b, c) as: $b = x + iy, \quad c = x - iy$. Eq. (22) can then be rewritten like so

$$\begin{aligned}(x-a+iy)\left(x^2-y^2-2ixy\right)\ln\left(\frac{Q-c}{Q_0-c}\right)+\left(-(x-a)+iy\right)\left(x^2-y^2+2ixy\right)\ln\left(\frac{Q-b}{Q_0-b}\right)\\ -2iya^2\ln\left(\frac{Q-a}{Q_0-a}\right)=-2s_0t(x+iy-a)(a-x+iy)(-2iy)=-2s_0t(x^2-2ax+a^2+y^2)(2iy)\end{aligned} \tag{23}$$

$$\begin{aligned}\implies [(x-a)(x^2-y^2)+2xy^2]\left(\ln\left(\frac{Q-c}{Q_0-c}\right)-\ln\left(\frac{Q-b}{Q_0-b}\right)\right)-2iya^2\ln\left(\frac{Q-a}{Q_0-a}\right)\\ +i[y(x^2-y^2)-(x-a)2xy]\left(\ln\left(\frac{Q-c}{Q_0-c}\right)+\ln\left(\frac{Q-b}{Q_0-b}\right)\right)=-4iys_0t(x^2-2ax+a^2+y^2)\end{aligned} \tag{24}$$

We now simplify the log terms using basic properties of complex logarithms.

$$\begin{aligned}\ln(z_R+iz_I)&=\ln\left(\sqrt{z_R^2+z_I^2}\right)+i\tan^{-1}\left(\frac{z_I}{z_R}\right)\\ \implies \ln\left(\frac{Q-c}{Q_0-c}\right)&=\ln\left(\frac{Q-x+iy}{Q_0-x+iy}\right)=\ln(Q-x+iy)-\ln(Q_0-x+iy)\\ &=\frac{1}{2}\ln\left(\frac{(Q-x)^2+y^2}{(Q_0-x)^2+y^2}\right)+i\left(\tan^{-1}\left(\frac{y}{Q-x}\right)-\tan^{-1}\left(\frac{y}{Q_0-x}\right)\right)\\ &=\frac{1}{2}\ln\left(\frac{(Q-x)^2+y^2}{(Q_0-x)^2+y^2}\right)+i\left(\tan^{-1}\left(\frac{y(Q_0-Q)}{(Q-x)(Q_0-x)+y^2}\right)\right)\end{aligned} \tag{25}$$

Similarly,

$$\ln\left(\frac{Q-b}{Q_0-b}\right)=\frac{1}{2}\ln\left(\frac{(Q-x)^2+y^2}{(Q_0-x)^2+y^2}\right)-+i\left(\tan^{-1}\left(\frac{y(Q_0-Q)}{(Q-x)(Q_0-x)+y^2}\right)\right) \tag{26}$$

Plugging Eqs. (25), (26) into Eq (24), we get:

$$\begin{aligned}2i[(x-a)(x^2-y^2)+2xy^2]\tan^{-1}\left(\frac{y(Q_0-Q)}{(Q-x)(Q_0-x)+y^2}\right)+i[y(x^2-y^2)-2xy(x-a)]\ln\left(\frac{(Q-x)^2+y^2}{(Q_0-x)^2+y^2}\right)\\ -2iya^2\ln\left(\frac{Q-a}{Q_0-a}\right)=-4iys_0t(x^2-2ax+a^2+y^2)\\ \implies 2[(x-a)(x^2-y^2)+2xy^2]\tan^{-1}\left(\frac{y(Q_0-Q)}{(Q-x)(Q_0-x)+y^2}\right)+[y(x^2-y^2)-2xy(x-a)]\ln\left(\frac{(Q-x)^2+y^2}{(Q_0-x)^2+y^2}\right)\\ -2ya^2\ln\left(\frac{Q-a}{Q_0-a}\right)=-4ys_0t(x^2-2ax+a^2+y^2)\end{aligned} \tag{27}$$

The above equation is an implicit expression for the period as a function of time and the cubic roots (which in turn depend on the spin-down coefficients). In order to proceed, we make some estimates of the terms including the period by taking the Crab pulsar into account (Lyne et al. 1993). The terms including the period in Eq. (27) are:

$$\ln\left(\frac{(Q-x)^2+y^2}{(Q_0-x)^2+y^2}\right),\ \tan^{-1}\left(\frac{y(Q_0-Q)}{(Q-x)(Q_0-x)+y^2}\right),\ \ln\left(\frac{Q-a}{Q_0-a}\right)$$

We numerically evaluate each of the terms for the Crab pulsar and proceed further. The numerical values for each term are expressed in Table 2.

Function	Value for the Crab Pulsar
$\ln\left(\frac{(Q-x)^2+y^2}{(Q_0-x)^2+y^2}\right)$	0.0158
$\tan^{-1}\left(\frac{y(Q_0-Q)}{(Q-x)(Q_0-x)+y^2}\right)$	-0.066
$\ln\left(\frac{Q-a}{Q_0-a}\right)$	0.0540

Table 2. Estimates for the arctan and logarithm terms in Eq.(27). Since the values are small enough, we rewrite the functions using their Taylor expansions.

The values are appropriate for series expansions for the terms and we thus rewrite Eq. (27) as

$$2[(x-a)(x^2-y^2)+2xy^2]\left(\frac{-(Q-Q_0)}{(Q-x)(Q_0-x)-y^2}\right)+[y(x^2-y^2)-2xy(x-a)]\left(\frac{(Q+Q_0-2x)(Q-Q_0)}{(Q_0-x)^2+y^2}\right) - 2a^2\left(\frac{Q-Q_0}{Q_0-a}\right) = -4s_0t(x^2-2ax+a^2+y^2) \tag{28}$$

$$(Q-Q_0)\left\{\left(\frac{-2((x-a)(x^2-y^2)+2xy^2)}{(Q-x)(Q_0-x)-y^2}\right)+\left(\frac{((x^2-y^2)-2x(x-a))(Q+Q_0-2x)}{(Q_0-x)^2+y^2}\right)-\left(\frac{2a^2}{Q_0-a}\right)\right\} = -4s_0t(x^2-2ax+a^2+y^2) \tag{29}$$

$$\implies (Q-Q_0)\left\{\left(\frac{\lambda_3}{(Q-x)(Q_0-x)-y^2}\right)+\lambda_1(Q+Q_0-2x)+\lambda_2\right\} = -\lambda_s t \tag{30}$$

$$\text{where, } \lambda_1 = \frac{2ax-x^2-y^2}{(q_0-x)^2+y^2}\ \lambda_2 = \frac{-2a^2}{Q_0-a}\ \lambda_3 = 2(x^3-ax^2+ay^2+xy^2)\ \lambda_s = 4s_0\left((x-a)^2+y^2\right)$$

At this point we perform simple calculations for pulsars from the ATNF database to get numerical estimates on the values of the λ's. We find that the term containing the λ_1, tends to be much smaller than the other two on the LHS. We thus neglect that term and continue with the analysis with the λ_2 and λ_3 terms. The exact values for various pulsars can be found in Table 3

PSR	$\lambda_1(Q+Q_0-2x)$	λ_2	$\lambda_3/((Q-x)(Q_0-x)-y^2)$	λ_s
J0007+7303	0.012	-0.60	6160.00	2.167×10^{-7}
B0531+21 (Crab)	-4.815×10^{-7}	-9.586×10^{-4}	-8.638×10^{-3}	5.589×10^{-9}
J1023-5746	0.00028	-0.52	-43.37	5.71×10^{-9}
J1418-6058	0.00025	-0.23	-650.16	1.10×10^{-8}

Table 3. Estimates for the λ terms for different pulsars

Eq. (30) can then be simplified as,

$$(Q - Q_0)\left\{\left(\frac{\lambda_3}{(Q-x)(Q_0-x)-y^2}\right) + \lambda_2\right\} = -\lambda_s t$$
$$\implies (Q-Q_0)(\lambda_3 + \lambda_2(Q-x)(Q_0-x) - \lambda_2 y^2) = -\lambda_s t\left((Q-x)(Q_0-x)-y^2\right)$$
$$\implies Q^2[\lambda_2(Q_0-x)] + Q[\lambda_3 + \lambda_2(x^2-y^2-Q_0^2) - \lambda_s t(Q_0-x)] + Q_0(\lambda_2(Q_0x - x^2+y^2)-\lambda_3) - \lambda_s t(Q_0 x - x^2 + y^2) = 0$$
$$\implies \alpha_1 Q^2 + (\alpha_2 + \lambda_a t)Q + \alpha_3 + \lambda_b t = 0 \tag{31}$$
$$\text{where, } \alpha_1 = \lambda_2(Q_0 - x)\ \alpha_2 = \lambda_3 + \lambda_2(x^2-y^2-Q_0^2)\ \alpha_3 = Q_0(\lambda_2(Q_0x-x^2+y^2)-\lambda_3)$$
$$\text{and, } \lambda_a = -\lambda_s(Q_0-x)\ \lambda_b = -\lambda_s(Q_0x - x^2+y^2)$$

We can find the roots of Eq.(31) using the quadratic formula,

$$Q = \frac{1}{2\alpha_1}\left\{-\alpha_2 - \lambda_a t \pm \sqrt{(\alpha_2+\lambda_a t)^2 - 4\alpha_1(\alpha_3+\lambda_b t)}\right\} \tag{32}$$

Only one of the two roots gives values that agree with the data, the period can then be written as:

$$P(t) = \sqrt{\frac{1}{2\alpha_1}\left\{-\alpha_2 - \lambda_a t + \sqrt{(\alpha_2+\lambda_a t)^2 - 4\alpha_1(\alpha_3+\lambda_b t)}\right\}} \tag{33}$$

The estimates for the different parameters in Eq. (33) can be found in Table 4. We compute the parameter values and estimate the period at different times for one of the most extensively studied pulsar – Crab (B0531+21). This pulsar has also been studied in the context of r-modes in the past (Rajbhandari et al. 2021; Rezania & Jahan-Miri 2000).

PSR	Parameter/Variable	Value
	α_1	-3.967×10^{-8}
	α_2	2.284×10^{-9}
	α_3	-2.489×10^{-12}
Crab (PSR B0531+21)	λ_a	1.104×10^{-20}
	λ_b	-7.376×10^{-23}
	P_0	33.333 ms
	P (calculated)	33.808 ms
	P (observed)	33.814 ms

Table 4. Values for the different parameters in Eq. (33) The initial and final values for the rotational periods were taken from the Jodrell Bank data. The initial and final periods are taken at MJDs 46812 and 60050 respectively. The relative error between the calculated and observed values of the period is 0.02% (Lyne et al. 1993)

3.3. *Braking Indices*

The simplest formulation of pulsar spin-down relates the rate of change of frequency to a power of the frequency itself. The power is referred to as the braking index and for a purely magnetic dipole radiation in a vacuum, this index would take a value of 3 (Manchester et al. 1985).

$$\dot{f} = kf^n \tag{34}$$

Here, f is the rotational frequency of the pulsar, k is a constant and n is referred to as the braking index. The braking index of a pulsar is determined by the physical processes that cause the pulsar to spin down. $n = 1$ implies mass loss/pulsar wind/ particle acceleration processes, $n = 3$ arises due to a pure magnetic dipole moment, and $n = 5$

corresponds the lowest order gravitational wave emission (or an electromagnetic quadrupole moment). The braking indices in terms of the frequency derivatives are given as:

$$n = \frac{\ddot{f} f}{\dot{f}^2} \tag{35}$$

$$m = \frac{\dddot{f} f^2}{\dot{f}^3} \tag{36}$$

Using Eq.(8), we can rewrite the braking indices in terms of the spin-down coefficients. Differentiating the equation and subsitituting the expressions for the frequency derivatives gives the following expressions for the braking indices:

$$n = \frac{s + 3rf^2 + 5gf^4 + 7lf^6}{s + rf^2 + gf^4 + lf^6} \tag{37}$$

$$m = \left(\frac{s + 3rf^2 + 5gf^4 + 7lf^6}{s + rf^2 + gf^4 + lf^6}\right)^2 + 2f^2\left(\frac{3r + 10gf^2 + 21lf^4}{s + rf^2 + gf^4 + lf^6}\right) \tag{38}$$

The numerical estimates for n and m using the spin-down equations can be found in Table 5

PSR	Type	n	m
B0531+21 (Crab)	Estimation	2.33	45.33
	Observation	2.32	45.33
B1509-58	Estimation	2.83	13.53
	Observation	2.84	14.5
J1023-5746	Estimation	66.71	297314.50
	Observation	66.8	2.98×10^5
J1418-6058	Estimation	29.96	2436392.81
	Observation	30.02	2.46×10^6

Table 5. The numerical estimates for the braking indices using the spin-down coefficients for 4 pulsars from the ATNF database. Values labelled 'Observation' were calculated directly from the frequency derivatives whereas the ones labelled 'Estimation' were calculated using the spindown coefficients. The high values for the braking indices for J1023-5746 and J1418-6058 could imply the presence of glitches at the time of measurements (Espinoza 2017).

These expressions provide a way to represent the braking indices of a pulsar in terms of the spindown coefficients for an analysis that takes the r-modes in a pulsar into account.

4. DISCUSSION

Continuous gravitational waves are widely accepted to be emitted by systems with well defined frequencies. Rotating neutron stars with irregularities or"mountains" on their surfaces are a notable example of systems that emit these waves (others include binary systems of black holes or stars). These waves can be used to study the star's physical properties and further offers potential to study their inner physics (Lu et al. 2023). Although CWs have not been detected yet, both the techniques to detect them (Tenorio et al. 2021), Jaranowski et al. (1998), as well as the gravitational wave detector sensitivities (Abbott et al. 2017), Pitkin et al. (2015), Abbott et al. (2022a), Abbott et al. (2022b), Abbott et al. (2022c), Abbott et al. (2021a) continue to be improved. Searches for gravitational wave emissions caused due to r-modes have also been carried out for specific pulsars, albeit unsuccessful in detecting them they have reported constraints on the gravitational wave amplitudes (Abbott et al. 2021b). Pulsars are also used extensively in detecting low frequency gravitational waves (generally in the nanohertz range) by analyzing correlated deviations in their timing signals (Agazie et al. 2023),Antoniadis et al. (2023).

Analyzing the spindown in puslars and studying their braking indices offers insight into both the detection of continuous waves as well as in the analysis of their timing residuals (Shaw et al. 2018). We have modelled the rotational frequency and period of pulsars in terms of their spin-down coefficients and time. The inclusion of r-modes presents a more accurate and fuller picture of the mechanisms that cause pulsars to lose their rotational energies. Our work applies to pulsars in their non-linear saturation phases, as during this period, the r-modes contribute to the frequency rate with a seventh order frequency term (Lindblom et al. 1998). Among the four terms we consider in the spindown equation, we find that the highest contribution for these pulsars generally comes from the magnetic dipole radiation term whereas the lowest comes from the r-modes. However, modelling the spindown due to the inclusion of r-modes could lead to more precise PTA frequency estimations and more accurate data analysis of continuous gravitational waves. The spindown coefficients get smaller for higher powers of the frequency and further investigation is required to understand what this means for the physical parameters responsible for the mechanisms. Unlike in Alvarez, C. & Carramiñana, A. (2004), we open the possibility for negative spindown coefficients to account for glitches and spin-ups. For the pulsars we computed the coefficients, we find that s_0, g_0 take positive values whereas r_0, l_0 take negative values (Table 1. We also present the braking indices as functions of the spindown coefficients and the estimates for these agree with observational values (Table 5

We believe that this analysis can also be extended to pulsar timing. The expected amplitude of timing residuals at some epoch t due to an abrupt change at t_1 is given as (Shaw et al. 2018),

$$\phi = -\Delta f(t - t_1) - \Delta \dot{f}\frac{(t - t_1)^2}{2} + \cdots \tag{39}$$

Our expression for the frequency and its derivatives as functions of time can help better model the residuals, thereby accounting for any spin changes or glitches in the timing analysis.

5. CONCLUSIONS

In this work, we have discussed, analytically the mechanisms through which pulsars spin down. We considered the impact of r-modes (analogous to Rossby waves in the Earth's oceans and atmosphere) in pulsars and their spindown. We have solved the non-linear differential equation incorporating the contribution of the r-modes to obtain the the rotational frequency and periods as functions of time. We find that our numerically analysis is in accord with observations for the Crab pulsar. We also presented the expresssion indices in terms of the spindown coefficients.

Gravitational radiation emitted by young pulsars might be detected as strong sources from single spindown events or as a stochastic background made up of many weaker sources (Owen et al. 1998). With the advent of 3G GW detectors and rapid improvements in detector technology, the spindown analysis taking r-modes into account offers a strong possibility of disentangling individual events from the stochastic GW background.

ACKNOWLEDGEMENTS

We are thankful to Professors Shantanu Desai, Soumya Mohanty, and Dr. Hamsa Padmanabhan for helpful suggestions in the improvement of this manuscript. We are grateful to the MITACS Globalink program for providing the platform and funding for this project.

APPENDIX

A. FREQUENCY ANALYSIS WITH TIME INDEPENDENT COEFFICIENTS

For constant (time independent) spin-down coefficients, Eq. (5 can be rewrittten. We use a fourth spin-down coefficient l for the f^7 term here.

$$\begin{aligned}\dot{f} &= -sf - rf^3 - gf^5 - lf^7 \\ \frac{df}{dt} &= -f(s + rf^2 + gf^4 + lf^6) \\ \implies -dt &= \frac{1}{f}\frac{df}{s + rf^2 + gf^4 + lf^6}\end{aligned} \tag{A1}$$

We then make the substitution $f^2 = x,\ df = dx/2\sqrt{x}$. This allows to write the polynomial in the denominator as a cubic that can then be factorized.

$$\begin{aligned}\frac{1}{2x}\frac{dx}{s + rx + gx^2 + lx^3} &= -dt \\ \frac{1}{2lx}\frac{dx}{\frac{s}{l} + \frac{r}{l}x + \frac{g}{l}x^2 + x^3} &= -dt \\ \implies \frac{1}{2lx}\frac{dx}{(x-a)(x-b)(x-c)} &= -dt\end{aligned}$$

Where, (a, b, c) are the roots of the cubic polynomial. These roots can be expressed analytically (as shown later) or computed numerically. Due to the usually complicated analytic expression for cubic roots, it is often more apt to do a numerical computation instead. The fraction can then be simplified using the concept of partial fractions.

$$\frac{1}{x(x-a)(x-b)(x-c)} = \frac{A_0}{x} + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

$$\text{where}\quad A_0 = \frac{-1}{abc}, \quad A = \frac{-1}{a(c-a)(a-b)}, \quad B = \frac{-1}{b(a-b)(b-c)}, \quad C = \frac{-1}{c(b-c)(c-a)}$$

We then use this expression in the integral and integrate on both sides indefinitely.

$$\begin{aligned}\frac{1}{2l}\left(\frac{A_0}{x} + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}\right) &= -dt \\ \implies \frac{1}{2l}\left(A_0 \ln x + A\ln(x-a) + B\ln(x-b) + C\ln(x-c)\right) &= -t + \mathcal{C}\end{aligned} \tag{A2}$$

where $\mathcal{C}$ is the constant of integration. In order to figure out an expression for the constant of integration, we need to set some boundary conditions. We assume that at some initial time t_0, the frequency of the pulsar has some value f_0. Plugging these initial conditions in Eq.(A2), we then express the constant in terms of the other variables and initial parameters.

$$\begin{aligned}\mathcal{C} &= \frac{1}{2l}\left(A_0 \ln f_0^2 + A\ln(f_0^2 - a) + B\ln(f_0^2 - b) + C\ln(f_0^2 - c)\right) + t_0 \\ \implies A_0 \ln\left(\frac{f^2}{f_0^2}\right) + A\ln\left(\frac{f^2-a}{f_0^2-a}\right) + B\ln\left(\frac{f^2-b}{f_0^2-b}\right) + C\ln\left(\frac{f^2-c}{f_0^2-c}\right) &= -2l(t - t_0)\end{aligned} \tag{A3}$$

Eq. (A3) is an implicit equation that relates the frequency to time with the help of the spin-down coefficients and the initial conditions. It is relatively straightforward to make estimates for the frequency using the above expression by coding up a simple program. These estimates and calculations have been done after the analytic analysis.

The cubic roots (a, b, c) can be analytically expressed using the general cubic formula:

$$\{x_i\} = -\frac{1}{3}\left(\frac{r}{s} + \epsilon^i C + \frac{\Delta_0}{\epsilon^i C}\right) \tag{A4}$$

$$\text{where} \quad \epsilon = \frac{-1 + i\sqrt{3}}{2}, \quad C = \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}$$

$$\Delta_0 = \left(\frac{g}{l}\right)^2 - 3\left(\frac{r}{l}\right), \quad \Delta_1 = 2\left(\frac{g}{l}\right)^3 - 9\left(\frac{gr}{l^2}\right) + 27\left(\frac{s}{l}\right)$$

More explicitly, we can substitute the values $\{0, 1, 2\}$ for i to get exact expressions for a, b and c

$$a = \frac{-1}{3}\left(\frac{r}{s} + C + \frac{\Delta_0}{C}\right)$$

$$b = \frac{-1}{3}\left(\frac{r}{s} + \frac{(-1 + i\sqrt{3}C}{2} + \frac{2\Delta_0}{(-1 + i\sqrt{3}C}\right)$$

$$c = \frac{-1}{3}\left(\frac{r}{s} + \frac{(-1 - i\sqrt{3}C}{2} + \frac{2\Delta_0}{(-1 - i\sqrt{3}C}\right) \tag{A5}$$

The same can be written in a compact form using the complex cube root:

$$\{a, b, c\} = \sqrt[3]{\left(\frac{-g^3}{27l^3} + \frac{gr}{6l^2} - \frac{s}{2l}\right) + \sqrt{\left(\frac{-g^3}{27l^3} + \frac{gr}{6l^2} - \frac{s}{2l}\right)^2 + \left(\frac{r}{3l} - \frac{g^2}{9l^2}\right)^2}} + \sqrt[3]{\left(\frac{-g^3}{27l^3} + \frac{gr}{6l^2} - \frac{s}{2l}\right) - \sqrt{\left(\frac{-g^3}{27l^3} + \frac{gr}{6l^2} - \frac{s}{2l}\right)^2 + \left(\frac{r}{3l} - \frac{g^2}{9l^2}\right)^2}} - \frac{g}{3l} \tag{A6}$$

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