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From Ancient Greece to Beloch's Crease: The Delian Problem and Origami

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Peer Review

This work has undergone a double-blind review by a minimum of two faculty members from institutions of higher learning from around the world. The faculty reviewers have expertise in disciplines closely related to those represented by this work. If possible, the work was also reviewed by undergraduates in collaboration with the faculty reviewers.

Abstract

The problem of how to double the volume of a cube, also known as the Delian problem, has intrigued mathematicians for millennia. The great variety of solutions discovered over the centuries have used diverse tools, mathematical and otherwise. Recently, in 1936, Margherita Piazollo Beloch discovered a simple and elegant solution to this question. It uses a single piece of paper and a handful of folds. The solution has renewed interest in geometrical constructability problems, in particular those that incorporate origami. And ultimately, it's given rise to the field of origami mathematics.

Keywords

Cubic equations, Origami, The Delian Problem

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1. Introduction

Cranes, swans, boxes: Created with a few simple creases, these origami objects have long been thought of as nothing more than playthings. But with the introduction of a systematic universal language, origami has blossomed as an art capable of creating virtually any three-dimensional object. Bugs of any kind, fiddlers holding fiddles, and complex polyhedra are just a few of the things that can now be produced with a single piece of paper, a clutch of creases, and a lot of math.

In return, mathematicians have been able to use the elevated art form to provide new solutions to some old problems and to create entirely novel directions for mathematics. One old problem with a new origamic solution is the construction of the cube root of two. The problem is famous for being impossible to solve with straightedge and compass alone and has puzzled and intrigued mathematicians for millennia. Now the exploration of paper-folding has uncovered a little known solution from an otherwise little known mathematician. This solution has increased interest in origami constructability problems. Origami and math are now more intertwined than ever.

2. Origins of the Delian Problem

There are several theories about the origins of the question of how to construct the cube root of two. The most popular tale was born of plague and prophecy. Disease had wiped out thousands of Athenians in 430 B.C. Helpless to slow the mounting death toll, the Athenians sent a representative to the oracle at Delos. The message they received was this: The plague would end when the Athenians doubled the volume of their altar.

This altar was a cube. To increase its size, as ordered, the Athenians doubled the length of its sides. The error angered the gods, and, as the legend goes, the devastation wreaked by the plague increased. Confused, the Athenians turned to the wisest man they could find: Plato. He pointed out their error and explained that the oracle's instructions had nothing to do with any need for a larger altar. The task was meant to reproach the Greeks for their neglect of mathematics and their contempt of geometry.

At least one Greek took the reproach to heart. With no algebraic tools at his disposal, Hippocrates of Chios made significant progress by relating the problem to plane geometry. He showed that the problem could be reduced to the construction of two mean proportionals, x, and y, between a given line a and another line $b = 2a$.

In the notation of modern math, it can be expressed as follows: We aim to find x and y, such that

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}.$$

From the first equation we have $x^2 = ay$. From the second equation we have $y^2 = bx$. Square the first equation on both sides and we have $x^4 = a^2y^2$. Then replace y^2 with bx. We have $x^4 = a^2bx$, which implies $x^3 = a^2b$. So, we have $x = \sqrt[3]{a^2b}$. If we let $b = 2a = 2$, x is the cubic root we are seeking.

Plato, though, had stipulated that the solution should rely only on a straightedge and compass. With this restriction, there was no way for Hippocrates, or anyone else, to find the mean proportionals. Egyptian contemporaries, similarly algebraically deprived, declared the problem unsolvable.

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But without the restriction, many solutions were found. Most used the mean proportionals that Hippocrates of Chios developed. Plato's friend, Menaechmus, for instance, found two different theoretical methods to double the cube. As a by-product of his work, he discovered the conic sections.

His solutions, in the language of modern mathematics, are as follows:

$$\frac{a}{x} = \frac{y}{b} \Rightarrow xy = ab,$$
$$\frac{x}{y} = \frac{y}{b} \Rightarrow y^2 = bx,$$
$$\frac{a}{x} = \frac{x}{y} \Rightarrow x^2 = ay.$$

The first solution was to find the intersection of the hyperbola and the parabola that are the first two equations above. The second solution was to find the intersection of the parabolas from the second and third equations.

Another friend of Plato's, Archytas, solved the question by using the intersection of three dimensional surfaces. Philon and Heron found methods of solving the problem by using the intersection of a circle and a rectangular hyperbola. But these solutions, as well as similar ones from others, used more than the straightedge and compass that Plato preferred for geometry.

Other great minds continued to search for a solution that would satisfy his directive. Many curves were created, such as the conchoid curve found by Greek mathematician Nicomedes. He used the curve to both double the cube and to solve the trisecting of an angle. Diocles invented the cissoid curve to double the cube. [3,4,7].

However inventive, these Greeks were unable to solve the Delian problem according to Plato's constraints. Though they may have sensed that a solution was impossible, they did not have the mathematical tools to prove it. These wouldn't come about until 1837. That is when French mathematician Pierre Wantzel proved that the Delian problem could not be solved with a straightedge and compass [5].

Wantzel showed the following general theorem: If r is a number that can be constructed by straightedge and compass only, it must be the root of an irreducible polynomial of degree 2^n for some non-negative integer n. With this theorem, we can easily deduce that $\sqrt[3]{2}$ cannot be constructed by a compass and straightedge since $\sqrt[3]{2}$ is a root of $x^3 - 2 = 0$, which is irreducible, but not to degree 2^n [16, 17]. The proof was integral to the development of constructible numbers and played an important role in the birth of abstract algebra [18].

3. The history of Origami and Math

Unlike the Delian problem and other classical geometric construction problems, paper-folding received little academical interest until recently. For most of its history, origami has been a tool for amusing children, not for solving math problems.

Origami hasn't faired much better in the world of art, and its origins remain unclear. Some historians have put its birthplace in China, sometime in the second century when paper first appeared. Paper made its way to Japan in the sixth century, but the paper of that time was likely too brittle for any real folding. The earliest known folding comes from Japan's Shinto priests. They folded strips of paper into a kind of lightening bolt, called a *shide*, to designate holy areas. In the Heian era (c.1000), men used folded paper purses. As an art form, though, origami

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didn't appear in earnest until after 1600, when paper became cheaper and more widespread [16].

Exactly when paper-folding was first used as a teaching tool will forever be a mystery, but due to the geometric nature of the pastime, it was inevitable that it would be introduced to the classroom. A German educator, Friedrich Frobel (1782-1852), who established the kindergarten system, was a paper-folding enthusiast. He incorporated paper-folding into the kindergarten curriculum and thought it was one of the most important skills that a child could learn [16].

Inspired by the educational value of paper-folding, educator Row T. Sundara wrote a now well-known book about paper-folding and geometry, *Geometric Exercises in Paper Folding*, published in 1893 [17]. It kindled an interest in mathematicians and educators for paper-folding. In particular, the famous geometer Felix Klein read Sundara's book. He mentioned it in his own book, *Famous Problems of Elementary Geometry* [10]. Having discovered origami through Klein's citation of Sundara, several authors went on to examine techniques for solving quadratic equations using paper-folding. However, Sundara's book mistakenly said that paper-folding techniques could not find the construction of cubic roots of integers [17].

4. Beloch's solution

In 1930, an Italian algebraic geometer, Margherita Piazollo Beloch, proved Sundara wrong. Born in 1879, Beloch was the daughter of a famous historian. She earned her Ph.D. at the University of Rome and went on to teach in Pavia and Palermo. From 1927 to 1955, she served as Chair of Geometry at the University of Ferrara, where she finished her career. Though she explored other areas of geometry, she is chiefly know for her paper of 1936, "Sul metodo del ripiegamento della carta per la risoluzione dei problemi geometrici", or "On the method of paper folding for the resolution of geometric problems." This paper was drawn from lectures for a course she taught in the early thirties. In it, she describes her solution to the Delian problem using nothing but a square piece of paper and a few folds. The paper also shows how her origami solution is a tactile version of a solution offered by Eduard Lill in 1867 [2]. Beloch died in Rome in 1976, and her paper-folding solution to the Delian problem remained undiscovered for many years [14].

Her method is as follows: Given two points, A and B, we can construct two orthogonal lines, r and s, such that A is on the line s, and B is on the line r. Let point O be the intersection point of the lines r and s. To construct a Beloch Square $CDEF$ as shown in figure 1, first we'll construct a line r' that is parallel to line r, whose distance from line r is the same as the distance between point A and line r. To construct line r', we can make a fold along line r. Then we make a small pinch mark where point A touches the paper. Now we make a fold that passes through the pinch mark, perpendicular to line s. The fold is line r'. Similarly, we can create line s', which is parallel to line s. It can be shown that its distance from line s is the same as the distance between point B and line s. Now we will perform a Beloch fold, which is defined by the following: Place point A on line r' and point B on line s' simultaneously. We call images of A and B A' and B', respectively [2, 8].

Let D denote the intersection point of line r and $\overline{AA'}$ in figure 1. Let E denote the intersection point of line s and $\overline{BB'}$ in figure 1. Then we fold $\overline{DE}$ onto $\overline{DA}$,

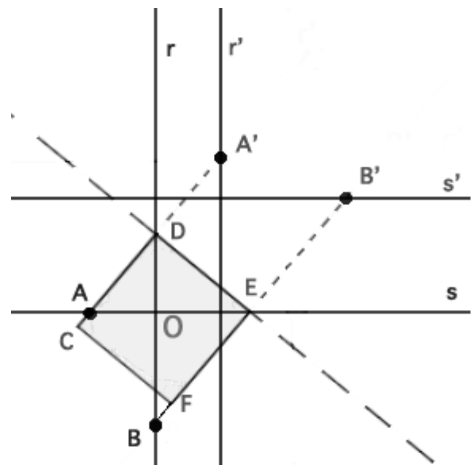

Figure 1. Beloch's Square

make a pinch mark at the image of the point E on the line that passes through $\overline{DA}$. We call the pinch mark C. Similarly, we can find the point F. The square $CDEF$ is the square we are looking for. We can choose A and B such that $OA = 1$ and $OB = 2 \times OA = 2$. Then we can show that OD is $\sqrt[3]{2}$.

Using the fact that the triangles ODA, OED, and OBE are similar triangles we have

$$\frac{OD}{OA} = \frac{OE}{OD} = \frac{OB}{OE},$$

$$\frac{OD}{OA} = \frac{OE}{OD} \Rightarrow OD^2 = OA \times OE,$$

$$\frac{OE}{OD} = \frac{OB}{OE} \Rightarrow OE^2 = OD \times OB,$$

$$OD^4 = OA^2 \times OE^2 = OA^2 \times OD \times OB \Rightarrow OD^3 = 2 \Rightarrow OD = \sqrt[3]{2}.$$

However clever, Beloch's work went largely unnoticed. Fifty years after Beloch's solution, several independent discoveries cropped up nearly simultaneously. In 1986, for instance, Peter Messer discovered a similar method of cube doubling with origami [15].

We produce Peter Messer's method of finding the cube root of 2 here, as it is interesting in its own right. First, we fold a square piece of paper into thirds (see figure 2). One way to do this is to make a middle-point mark, M, on one edge of the paper, $\overline{AD}$, by folding the line $\overline{AB}$ on the line $\overline{DC}$, and making a pinch mark on the side of $\overline{AD}$. Then we make a fold that connects point B and point M. We make another fold that passes through point A and point C. Denote the intersection of $\overline{AC}$ and $\overline{BM}$ as O. Then make a fold, denoted as $\overline{PQ}$, passing through O and

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perpendicular to $\overline{AB}$. Folding $\overline{BC}$ on the line $\overline{PQ}$, the resulting fold is denoted as $\overline{RS}$. Then $\overline{PQ}$ and $\overline{RS}$ divide the square into three equal strips.

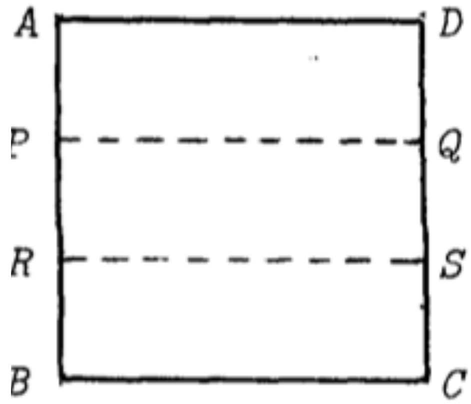

FIGURE 2. Divide a square paper into three equal parts

Using Beloch's fold, we make a fold to place point C onto line $\overline{AB}$ (the left side of the square) and point S onto the line $\overline{PQ}$, simultaneously (see figure 3). Let CB be one unit. Let AC be x and BT be y. The side of the square is $s = 1 + x$. From the construction, BP equals two thirds of a side of a square. That is,

$$BP = \frac{2}{3}s = \frac{2(x+1)}{3} \Rightarrow CP = BP - CB = \frac{2(x+1)}{3} - 1 = \frac{2x-1}{3}.$$

From the way the paper is folded, we can see that CT is $x + 1 - y$. Apply the Pythagorean theorem to the right triangle, CBT, and we have

$$CB^2 + BT^2 = CT^2 \Rightarrow 1 + y^2 = (x + 1 - y)^2.$$

Solving for y, we obtain

$$y = \frac{x^2 + 2x}{2x + 2}.$$

Again, from the construction we can see that AP is a third of the side of the square. Thus

$$AP = \frac{1}{3}s = \frac{x+1}{3} \Rightarrow CP = x - \frac{1}{3}s = x - \frac{1+x}{3} = \frac{2x-1}{3}.$$

Using the fact that triangle CBT is similar to triangle PSC, we have $\frac{BT}{CT} = \frac{CP}{CS}$. That is,

$$\frac{y}{x+1-y} = \frac{\frac{2x-1}{3}}{\frac{x+1}{3}} \Rightarrow \frac{y}{x+1-y} = \frac{2x-1}{x+1}.$$

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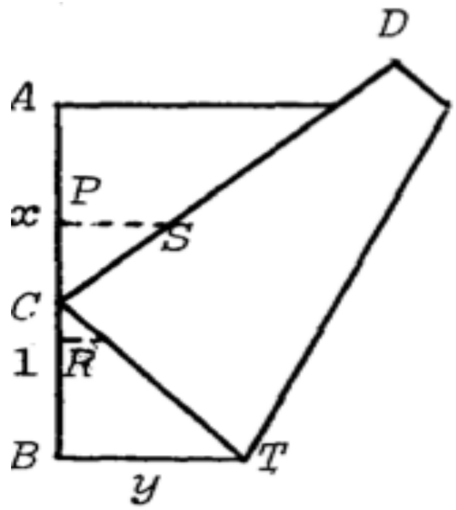

FIGURE 3. Peter Messer's method for finding the cube root of 2

We then acquire a second expression for y, namely,

$$y = \frac{2x^2 + x - 1}{3x},$$

Equating the two expressions for y, $\frac{x^2+2x}{2x+2} = \frac{2x^2+x-1}{3x}$., and solving for x, we obtain $x = \sqrt[3]{2}$.

5. Modern development

Origami blossomed in the 20th century because of the efforts of a single man, Akira Yoshizawa. Born in 1911, Yoshizawa turned paper-folding into a fine art. When a teenager, Yoshizawa quit his factory job to devote his life to origami. Subsequently, he designed thousands of models and pioneered a system of notations for origami folds. Using mainly lines and arrows, it requires no words to explain each step. "A version of this system has since become the worldwide standard and international visual language for paper-folding instruction," writes Meher McArthur in *Folding Paper: The infinite possibilities of origami.* He also created a sculptural folding technique called wet–folding. By the 1950s, his creations were featured in numerous publications [16].

Interest in origami grew both in and outside of Japan thanks to Yoshizawa's work. Eventually, physicists, mathematicians, and engineers brought more complex folds to origami, uncovering its power.

One of the most important events to bring origami and mathematicians together was a 1989 conference, "The First International Meeting of Origami Science and Technology," organized by Humiaki Huzita. Huzita served as a bridge to bring origami from Japan and the West together. He said that it was Beloch's work that inspired him to study the geometry of paper folding. To honor her, he held

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the conference in the building where Beloch had announced the discovery of the solution of the Delian problem with origami. The proceedings reproduced several papers by Beloch and Huzita dedicated the conference to her [13].

The proceedings of the conference and the talks given there essentially created the field of the mathematics of origami. (Since that first conference, there have been five more.) Crucial to this new field were the basic axioms of origami introduced there. The first six axioms were discovered by Jacques Justin and Huzita, who reported them in the proceedings. Later, Koshiro Hatori and Robert Lang added one more axiom. Together, the seven axioms completely describe all the possible ways to make a single fold.

The seven axioms are [1, 9]:

(a) Given two points, p_1 and p_2, we can make a fold passing through both points.
(b) Given two points, p_1 and p_2, we can make a fold that places p_1 onto p_2.
(c) Given two lines, l_1 and l_2, we can make a fold that places l_1 onto l_2.
(d) Given a point, p_1, and a line, l_1, we can make a unique fold that is perpendicular to l_1 that passes through point p_1.
(e) Given two points, p_1 and p_2, and a line, l_1, there is a fold that places p_1 onto l_1 and passes through p_2.
(f) Given two points, p_1 and p_2, and two lines, l_1 and l_2, we can make a fold to place p_1 onto l_1 and p_2 onto l_2 at the same time.
(g) Given one point, p, and two lines, l_1 and l_2, we can perform a fold to place p onto l_1. It is perpendicular to l_2.

These axioms inspired many mathematicians to look at origami as a geometric construction tool. As the ancient but revitalized area of research expanded, new recruits brought more theory to the possibilities contained within paper folding. Soon math was using origami to create complex polyhedra, shapes in rigid materials, multiple folds in a single action, and curved folding. In recent years, the field has exploded, creating new math problems, connecting graph theory, group theory, abstract algebra, and, of course, geometry to paper folding. But today's high level math and stunning paper sculptures all spring from the simplicity found in a straight crease. This complexity from simplicity is at the very heart of mathematics, and it mirrors Plato's preference for solutions that use nothing more than a straightedge and compass. If Plato could see the origami solution of Beloch, and the subtle math and art that have sprung from it, would he disapprove of a tool that strayed from straightedge and compass or would he embrace its elegant simplicity?

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