Doctoral Thesis

On Kyle models with terminal trading constraints

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Abstract

We study Kyle models with terminal trading constraints that are variations of Kyle (1985) and Back (1992) where the insider has no trading constraint. We find that the constraint produces new features to our model. First, it turns out that we need a new state process in the structure of equilibria. Second, we show that our insider places a block order at terminal time, $\Delta\theta_T = \tilde{a} - \theta_{T-}$, to satisfy her constraint. We prove the existence of equilibria in both discrete time and continuous time settings. For the continuous time model, we establish the explicit equilibrium by deriving an autonomous system of first-order nonlinear ordinary differential equations (ODEs). Moreover, we obtain results associated with empirical findings, for example, autocorrelated aggregate holdings, decreasing price impact function, and U-shaped trading patterns.

Keywords: stochastic control, insider trading, market microstructure, trading constraint, Kyle equilibrium

Contents

Chapter 1

Introduction

The search for equilibrium price formulations in the presence of insiders with asymmetric information has been an important research topic in financial mathematics since the pioneering work of Albert S. Kyle (see [2]). In [2], Kyle considered three market participants: a risk-neutral insider, risk-neutral and competitive market makers, and random noise traders. He found equilibria in both discrete time and continuous time settings with fundamental asset value, $\tilde{v}$, which is assumed to be a normal random variable.

There are numerous variations and generalizations of the Kyle model. Kerry Back established the existence of unique equilibrium in a continuous time setting with fundamental asset value, $f(\tilde{v})$, where f is an increasing function (see [1]). The papers [3] and [4] consider equilibria for the case where multiple insiders exist. The paper [6] considers a market structure with a risk-averse insider and [7] and [5] consider equilibria with general price distribution and noise trading.

In fact, investors may have trading constraints, therefore, equilibrium models with trading constraints have been studied by many researchers. However, the constraints make it difficult to examine equilibrium structure and to prove the existence of equilibria. The papers [9], [10] and [8] consider Kyle models with terminal trading constraint and analyze equilibrium numerically.

We impose a trading constraint on our insider at terminal time, $t = T$. To be specific, we use a "hard" constraint formulation for the insider's stock holding process $(\theta_t)_{t\in[0,T]}$ such that the terminal value of the process satisfies $\theta_T = \tilde{a}$, almost surely. (i.e., $\mathbb{P}(\theta_T = \tilde{a}) = 1$).

Actually, trading restrictions have been used in many problems: for example, Radner equilibrium models and models based on "soft" constraints (see [8],[9] and [10]). However, there is no equilibrium existence proofs in the settings of [2] when the insider has either a soft or hard trading constraint. We prove global existence of an equilibrium when the insider has terminal trading constraint.

To sum up, the contributions of this thesis can be summarized as followings:

1. *Giving an existence proofs in discrete and continuous settings*: To the best of our knowledge, there is no equilibrium existence proof in the settings of Kyle and Back when the insider has a soft or hard trading target. We establish the existence of an equilibrium when the insider needs to meet hard trading constraint in both discrete time version and continuous one. Furthermore, in the continuous time setting, we express the equilibrium explicitly by solving an autonomous two-dimensional coupled system of ordinary differential equations.

2. *Introducing a new state variable*: Due to the restriction $\theta_T = \tilde{a}$, our equilibrium structure requires a new state variable $Q_t = \mathbb{E}[\tilde{a} - \theta_t | \mathcal{F}_t^M]$ in addition to the aggregate order process Y_t. Consequently, the pricing rules cannot be represented by a function of Y_t only, as in [2] and [1]. In addition, we see that the insider's optimal trading strategy is a linear function of Q_t and $\tilde{a} - \theta_t$. Based on these two state variables we create generalized pricing rules.

3. *Finding that the insider's optimal behavior includes placing a block order at the terminal time*: We define the admissible set which allows the insider's trading process to jump at any time on the trading interval. Whereas such jump processes are suboptimal in [1], our constrained insider trades continuously before the terminal time and then makes block order at the terminal time.

4. *Solving problems with fully informed/partially informed insider with constraint, simultaneously*: Initially, we assume that the insider in order model can observe the target $\tilde{a}$, but cannot observe the asset value $\tilde{v}$. Nevertheless, one has a partial information of $\tilde{v}$ since we assume that there is a positive correlation between $\tilde{a}$ and $\tilde{v}$. We then establish the existence of equilibrium with the insider in Theorem 5.4.1. Furthermore, when the insider has full information (i.e., one observes about both $\tilde{a}$ and $\tilde{v}$), we find that the equilibrium constructed in Theorem 5.4.1 continues to be an equilibrium.

5. *Verifying that properties of the equilibrium are consistent with several empirical findings*: In the equilibrium, we see that (i) the price impact decreases over time, (ii) the autocorrelation of all participants' holdings is positive and (iii) the optimal trading strategy of the insider follows U-shaped patterns.

In Chapter 2, we introduce mathematical concepts used to analyze our model. We review Kyle [2] and Back [1] in Chapter 3. In Chapter 4, we construct and find an equilibrium of our model in discrete time setting. Chapter 5 shows that an equilibrium exists in continuous time setting and discusses several properties of the equilibrium. The Appendix contains the justification of the appearance of the state variables $\tilde{a} - \theta_\cdot$ and $Q_\cdot$ and considers the market with a fully informed insider.

Chapter 2

Preliminary

This chapter summarizes necessary mathematical concepts that can be used to solve the problem. Please refer [13] and [12] for details.

This dissertation includes abuse of notation. Δ means difference in the discrete time setting and jump in the continuous time setting. For example, $\Delta X_n = X_n - X_{n-1}$ in the discrete time, and $\Delta X_t = X_t - X_{t-}$ in the continuous time.

Definition 2.0.1 (Brownian motion)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We define $W = \{W_t, \mathcal{F}_t;\ 0 \le t < \infty\}$ as a standard **Brownian Motion** if the followings hold:

(i) $W_0 = 0$ almost surely and $W_{t_2} - W_{t_1} \sim N(0, t_2 - t_1)$ for any $0 \le t_1 < t_2$.

(ii) For $0 = t_0 < t_1 < \cdots < t_n$, $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \cdots, W_{t_n} - W_{t_{n-1}}$ are independent.

(iii) Trajectories are continuous almost surely:
$\exists \Omega^* \in \mathcal{F}$ such that $\mathbb{P}(\Omega^*) = 1$ and $t \mapsto W_t(\omega)$ is continuous for $\omega \in \Omega^*$.

Definition 2.0.2 (Semimartingale)**.** We define a real valued process X defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ as a semimartingale if it can be decomposed as

$$X_t = M_t + V_t$$

where M is a local martingale and V is a càdlàg adapted process of locally bounded variation. We call an $\mathbb{R}^n$-valued process $X = (X^1, \cdots, X^n)$ as a semimartingale if each of its components X^i is a semimartingale.

Definition 2.0.3 (Quadratic covariation process)**.** Let X_t and Y_t be real-valued stochastic processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Its quadratic covariation is also a process, which is defined as

$$[X,Y]_t = \lim_{||P||\to 0} \sum_{k=1}^{n} \left(X_{t_k} - X_{t_{k-1}}\right)\left(Y_{t_k} - Y_{t_{k-1}}\right)$$

where P ranges over partitions of the interval $[0,t]$ and the norm of the partition P is the mesh. This limit is defined using convergence in probability.

Definition 2.0.4 (Itô's formula with jumps)**.** Let $X = (X^1, \cdots, X^n)$ be an n-tuple of semimartingales and let $f : \mathbb{R}^n \to \mathbb{R}$ have second order partial derivatives and they are continuous. Then $f(X)$ is a semimartingale and the following formula holds:

$$\begin{aligned}
&f(X_t) - f(X_0)\\
&= \sum_{i=1}^{n} \int_{0-}^{t} \frac{\partial f}{\partial x_i}(X_{s-}) dX_s^i + \frac{1}{2} \sum_{1\le i,j\le n} \int_{0-}^{t} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-}) d[X^i, X^j]_s^c\\
&+ \sum_{0<s\le t} \left\{ f(X_s) - f(X_{s-}) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{s-}) \Delta X_s^i \right\},
\end{aligned}$$

where $[\cdot,\cdot]^c$ denotes the continuous part of the quadratic covariation process.

Theorem 2.0.5 (Kalman-Bucy filter)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and let $(\mathcal{F}_t), 0 \le t \le T$ be a nondecreasing family of right continuous $\sigma-$algebras of $\mathcal{F}$-augmented by $\mathbb{P}$-null sets in $\mathcal{F}$. Suppose the observable process $X = (X_t, \mathcal{F}_t)$ follows an Itô process*

$$X_t = X_0 + \int_0^t A_s(\omega) ds + \int_0^t B_s(X) dW_s,$$

where $W = (W_t, \mathcal{F}_t)$ is a Brownian motion, the process $A = (A_t(\omega), \mathcal{F}_t)$ is $\mathbb{P}$-a.s. integrable and $B = (B_t(X), \mathcal{F}_t)$ is $\mathbb{P}$-a.s. square integrable. Assume that the process $Y = (Y_t, \mathcal{F}_t), t \le T$ is

$$Y_t = Y_0 + \int_0^t H_s ds + Z_t$$

where Z_t is $\mathcal{F}_t$-adapted martingale and $H = (H_t, \mathcal{F}_t)$ is a random process with $\int_0^T |H_s| ds < \infty$ $\mathbb{P}$-a.s. We also assume that

$$\begin{aligned}
&\sup_{0\le t\le T} \mathbb{E}[Y_t^2] < \infty,\\
&\int_0^T \mathbb{E}[H_t^2] dt < \infty,\\
&\int_0^T \mathbb{E}[A_t^2] dt < \infty,\\
&B_t^2(x) \ge C > 0.
\end{aligned}$$

Then for each t, $0 \leq t \leq T$, ($\mathbb{P}$-a.s.)

$$\mathbb{E}[Y|\mathcal{F}_t] = \mathbb{E}[Y|\mathcal{F}_0] + \int_0^T \mathbb{E}[H|\mathcal{F}_s]ds + \int_0^T \left\{\mathbb{E}[D|\mathcal{F}_s] + \big(\mathbb{E}[YA|\mathcal{F}_s] - \mathbb{E}[Y|\mathcal{F}_s]\mathbb{E}[A|\mathcal{F}_s]B_s^{-1}(X)\big)\right\} d\bar{W}_s,$$

where

$$\bar{W}_t = \int_0^t \frac{dX_s - \mathbb{E}[A|\mathcal{F}_s]ds}{B_s(X)}$$

is a Brownian motion (with respect to the system $(\mathcal{F}_t^X)$, $0 \leq t \leq T$), and $D = (D_t, \mathcal{F}_t)$ is a process with

$$D_t = \frac{d\langle Z, W\rangle_t}{dt}.$$

Chapter 3

Literature Review

The models proposed by [2] and [1] have been widely adopted in the literature on financial economics. In this chapter, we first review the sequential auction equilibrium in [2] and then review the continuous time version of Kyle equilibrium in [1].

3.1 Kyle (1985)

Trading starts at time $t = 0$ and ends at time $t = 1$. Assume that there are N auctions and $0 = t_0 < t_1 < \cdots < t_N = 1$ where the time of nth auction is t_n. The true asset value, $\tilde{v}$, is assumed to be normally distributed with a mean p_0 and variance Σ_0. Noise traders in the market trade $\Delta\tilde{u}_n$ at the nth auction. Assume that $\Delta\tilde{u}_n$ is normally distributed with zero mean and variance $\sigma_u^2\Delta t_n$ and is independent of $\tilde{v}$ for all $n = 0, 1, \cdots, N$. Denote $\Delta\tilde{x}_n$ the quantity traded by the insider at the nth iteration. Let $\tilde{p}_n$ be the market clearing price at the time n. By this formulation, the insider observes not only the $\tilde{v}$, but also all the past prices. We thus see that the insider's position at the time n is,

$$\tilde{x}_n = X_n(\tilde{v}, \tilde{p}_1, \cdots, \tilde{p}_{n-1}), \text{ for any } n = 1, \cdots, N, \tag{3.1}$$

where X_n is a measurable function. The market makers can observe the aggregate order of all traders, so the price $\tilde{p}_n$ is determined by

$$\tilde{p}_n = P_n(\tilde{x}_1 + \tilde{u}_1, \cdots, \tilde{x}_n + \tilde{u}_n), \text{ for any } n = 1, \cdots, N, \tag{3.2}$$

for some measurable function P_n.

Define the vectors of functions $X := \langle X_1, \cdots, X_N\rangle$ and $P := \langle P_1, \cdots, P_N\rangle$ as the informed trader's strategy and the market makers' pricing rule, respectively. For any n, define $\tilde{\pi}_n$ as the profits of the insider at time n. Therefore, $\tilde{\pi}_n$ is given by

$$\tilde{\pi}_n = \sum_{k=n}^{N} (\tilde{v} - \tilde{p}_k)\tilde{x}_k. \tag{3.3}$$

Now we can define the equilibrium concept:

Definition 3.1.1. A sequential auction equilibrium is defined as a pair (X, P) such that the following conditions hold:

(i) Profit maximization: For any $n = 1, \cdots, N$ and for any $X' = \langle X'_1, \cdots, X'_N \rangle$ which satisfies $X'_1 = X_1, \cdots, X'_{n-1} = X_{n-1}$, we have

$$\mathbb{E}[\tilde{\pi}_n(X, P)|\tilde{v}, \tilde{p}_1, \cdots, \tilde{p}_{n-1}] \geq \mathbb{E}[\tilde{\pi}_n(X', P)|\tilde{v}, \tilde{p}_1, \cdots, \tilde{p}_{n-1}] \tag{3.4}$$

(ii) Market efficiency: For all $n = 1, \cdots, N$,

$$\tilde{p}_n = \mathbb{E}[\tilde{v}|\tilde{x}_1 + \tilde{u}_1, \cdots, \tilde{x}_n + \tilde{u}_n] \tag{3.5}$$

Definition 3.1.2. If an equilibrium is called a **linear equilibrium** if the component functions of X and P are linear, and is called a **recursive linear equilibrium** if there exist a constants $\lambda_1, \cdots, \lambda_N$ such that for $n = 1, \cdots, N$,

$$\tilde{p}_n = \tilde{p}_{n-1} + \lambda_n(\Delta\tilde{x}_n + \Delta\tilde{u}_n).$$

Theorem 3.1.3 (Kyle (1985))**.** *There is a unique linear equilibrium that is recursive. In this equilibrium there are constants $\beta_n, \lambda_n, \alpha_n, \delta_n$ and Σ_n such that for $n = 1, \cdots, N$,*

$$\begin{aligned}
\Delta\tilde{x}_n &= \beta_n(\tilde{v} - \tilde{p}_{n-1})\Delta t_n, \\
\Delta\tilde{p}_n &= \lambda_n(\Delta\tilde{x}_n + \Delta\tilde{u}_n), \\
\Sigma_n &= \mathbb{V}(\tilde{v}|\Delta\tilde{x}_1 + \Delta\tilde{u}_1, \cdots, \Delta\tilde{x}_n + \Delta\tilde{u}_n), \\
\mathbb{E}[\tilde{\pi}_n|v, p_1, \cdots, p_{n-1}] &= \alpha_{n-1}(v - p_{n-1})^2 + \delta_{n-1}.
\end{aligned}$$

Given Σ_0, the constants $\beta_n, \lambda_n, \alpha_n, \delta_n, \Sigma_n$ are the unique solution to the difference equation system

$$\begin{aligned}
\alpha_{n-1} &= \frac{1}{4\lambda_n(1 - \alpha_n\lambda_n)}, \\
\delta_{n-1} &= \delta_n + \alpha_n\lambda_n^2\sigma_u^2\Delta t_n, \\
\beta_n\Delta t_n &= \frac{1 - 2\alpha_n\lambda_n}{2\lambda_n(1 - \alpha_n\lambda_n)}, \\
\lambda_n &= \frac{\beta_n\Sigma_n}{\sigma_u^2}, \\
\Sigma_n &= (1 - \beta_n\lambda_n\Delta t_n)\Sigma_{n-1}
\end{aligned}$$

subject to $\alpha_N = \delta_N = 0$ and the second order condition $\lambda_n(1 - \alpha_n\lambda_n) > 0$.

Kyle proved this theorem by applying both the backward induction argument and the projection theorem. In this equilibrium, the price impact function is a constant and the insider's optimal order rate process is constant, since $\frac{\mathbb{E}[\tilde{x}_n|\tilde{v}]}{\tilde{v}} = \frac{\sigma_u}{\sigma_v}$.

3.2 Back (1992)

As in [2], let the trading starts at time $t = 0$ and ends at time $t = 1$. Consider tradings to occur continuously throught the interval $[0, 1]$. The insider already possesses about the information $\tilde{v}$. Assume the distribution function of $\tilde{v}$ is denoted by F and the support of F is an interval, including the whole real line or half line, and that F is continuous on this interval. Therefore, the inverse of F is well defined on the interval $(0, 1)$. We also assume that $\int_{-\infty}^{\infty} v^2 dF < \infty$. Moreover, there are risk neutral market makers and noise traders. Let Z_t the cumulative orders of the noise traders at time t and this is independent of $\tilde{v}$. Assume that the process Z to be a Brownian motion with a mean zero and variance σ^2. Let X_t denote the cumulative orders of the insider and $Y = X + Z$. Back considered equilibria as having the property that the price at time t depends only on cumulative orders Y_t and not on the history of orders. Therefore, we assume $P_t = H(Y_t, t)$ for some function H. Let $\mathcal{H}$ denote the class of continuous functions $H : \mathbb{R} \times [0, 1] \to \mathbb{R}$ that are twice-continuously differentiable in y and continuously differentiable in t on $\mathbb{R} \times (0, 1)$ and for which $H(\cdot, t)$ is strictly monotone for each $t \in [0, 1]$ and

$$\mathbb{E}\left[H(Z_1, 1)^2\right] < \infty \text{ and } \mathbb{E}\left[\int_0^1 H(Z_t, t)^2 dt\right] < \infty.$$

A pricing rule is an element of $\mathcal{H}$. Let $\mathcal{X}$ denote the class of semimartingales X adapted to $\mathcal{F}$ such that

$$\mathbb{E}\left[\int_0^1 H(X_{t-} + Z_t, t)^2 dt\right] < \infty \text{ for all } H \in \mathcal{H}.$$

As in Kyle(1985), an equilibrium is a pair (H, X) for which

(i) Given a trading strategy $X \in \mathcal{X}$, a pricing rule H satisfies $H(Y_t, t) = \mathbb{E}[\tilde{v}|(Y_s)_{s \le t}]$ for all $t \in [0, 1]$.

(ii) Given a pricing rule $H \in \mathcal{H}$, a trading strategy $X \in \mathcal{X}$ maximizes

$$\mathbb{E}\left[\int_{[0,1]} (\tilde{v} - P_{t-}) dX_t - [P, X]_1\right]$$

Based on these structures, Back established the following equilibrium:

Theorem 3.2.1. *Define*

$$H(y,t) = \mathbb{E}[h(y + Z_1 - Z_t)], \tag{3.6}$$

where $h = F^{-1} \circ N$. *For each* $v \in V$, *define*

$$X_t = (1-t)\int_0^t \frac{h^{-1}(v) - Z_s}{(1-s)^2} ds. \tag{3.7}$$

Then (H, X) *is an equilibrium.*

The contribution of this paper is its finding that the distribution of the risky asset follows general form, not normal distribution.

Chapter 4

Discrete Time

4.1 Model

We develop a variation of Kyle's multi-period discrete time model where the insider has a trading constraint $\tilde{a}$. The trading day is normalized to the interval $[0, 1]$, and there are $N \in \mathbb{N}$ trading points with time step $\Delta = \frac{1}{N}$. As in [2], the true value of the asset $\tilde{v}$ is defined as a normal random variable with mean zero and variance $\sigma_v^2 > 0$. The insider's trading constraint $\tilde{a}$ is also a normal random variable with mean zero and variance $\sigma_a^2 > 0$. Assume that $\tilde{a}$ and $\tilde{v}$ are correlated with correlation $\rho \in (0, 1]$. Our model posits three types of market participants:

1. *Insider*: The (risk-neutral) insider's order for the stock at time $n = 0, 1, 2, \cdots, N$ is denoted by $\Delta\theta_n$ so that θ_n is her accumulated position at time n. We assume that the initial holding of the insider is zero, $\theta_0 = 0$. Moreover, this constrained insider needs to satisfy the trading target $\tilde{a}$ at the terminal time, i.e., $\theta_N = \tilde{a}$. At time $t = 0$ the insider knows $\tilde{a}$ and observes the stock-prices over time. The main difference between Kyle's insider and ours is that ours has no information about the exact asset value $\tilde{v}$ until all information is revealed. However, one still has partial information about the asset value since we assume that $\tilde{v}$ and $\tilde{a}$ have a positive correlation $\rho \in (0, 1]$. In other words, $\tilde{a}$ gives the insider initial private information about the asset value. Roughly speaking, the situation that an insider has partial information about the true asset value can be considered as a case where the insider knows secret information that affects the stock price.

2. *Noise Traders*: These traders submit net stock orders is exogenously given by $\Delta W_n (:= W_n - W_{n-1})$ at time $n = 1, \cdots, N$. These increments are normally distributed with zero mean and variance $\sigma_w^2\Delta$ for a positive constant σ_w. Assume that $\tilde{v}, \tilde{a}$ and $(W_n)_{n=1}^N$ are jointly Gaussian, $W_1, \cdots, W_N$ are independent, and they are independent of $\tilde{v}$ and $\tilde{a}$.

3. *Market makers*: The (competitive and risk-neutral) market makers observe the total net order Y_n at time $n = 1, \cdots, N$, where

$$Y_n = \Delta\theta_n + \Delta W_n, \quad W_0 = 0. \tag{4.1}$$

Therefore, the market makers' filtration is

$$\mathcal{F}_n^M := \sigma\left(Y_1, \cdots, Y_n\right), \ n = 1, \cdots, N. \tag{4.2}$$

Based on this filtration, the market makers set the stock price by

$$P_n = \mathbb{E}\left[\tilde{v} \middle| \mathcal{F}_n^M\right], \ n = 1, \cdots, N, \ P_0 = 0. \tag{4.3}$$

In addition, the market makers predict the insider's remaining trading demand $\tilde{a} - \theta_n$:

$$Q_n = \mathbb{E}\left[\tilde{a} - \theta_n \middle| \mathcal{F}_n^M\right], \ n = 1, \cdots, N, \ Q_0 = 0. \tag{4.4}$$

Note that the insider's filtration is

$$\mathcal{F}_n^I := \sigma\left(\tilde{a}, Y_1, \cdots, Y_{n-1}\right), \ n = 1, \cdots, N. \tag{4.5}$$

Based on this filtration, the insider wants to maximize her expected profit:

$$\sup_{\Delta\theta} \mathbb{E}\left[\sum_{n=1}^{N} (\tilde{v} - p_n)\Delta\theta_n \middle| \mathcal{F}_0^I\right] = \sup_{\Delta\theta} \mathbb{E}\left[\tilde{a}(\tilde{v} - P_N) + \theta_{N-1}\Delta P_N + \cdots + \theta_1 \Delta P_2 \middle| \mathcal{F}_0^I\right] \tag{4.6}$$

$$= \frac{\rho\sigma_v}{\sigma_a}\tilde{a}^2 - \inf_{\Delta\theta} \mathbb{E}\left[\sum_{n=1}^{N} (\tilde{a} - \theta_{n-1})\Delta P_n \middle| \mathcal{F}_0^I\right] \tag{4.7}$$

where the second equality is from joint normality of $\tilde{v}$ and $\tilde{a}$, $P_N = \sum_{n=1}^{N} P_n$ and $P_0 = 0$.

Definition 4.1.1. An equilibrium is defined as a pair (θ, P) such that the following two conditions hold:

(i) **Optimal Trading Strategy of the Insider**: Given the function P_n, the strategy θ_n maximizes the insider's profit (4.7).

(ii) **Pricing Rule**: Given the function θ_n, the pricing rule P_n satisfies (4.3).

4.2 Conjectured form of equilibrium

Consider a set of possible candidate values for an equilibrium:

$$\lambda_n, \mu_n, r_n, s_n, \beta_n, \alpha_n \text{ for } n = 1, \cdots, N \text{ with } \beta_N = \alpha_N = 1 \tag{4.8}$$

Our goal is to construct an **linear** equilibrium that satisfies the following systems:

$$\Delta P_n = \lambda_n Y_n + \mu_n Q_{n-1},\ P_0 = 0, \tag{4.9}$$

$$\Delta Q_n = r_n Y_n + s_n Q_{n-1},\ Q_0 = 0, \tag{4.10}$$

$$\Delta \theta_n = \beta_n \left(\tilde{a} - \theta_{n-1} - Q_{n-1}\right) + \alpha_n Q_{n-1},\ \theta_0 = 0. \tag{4.11}$$

$$Y_n = \Delta\theta_n + \Delta W_n, \quad W_0 = 0. \tag{4.12}$$

Moreover, we define some variance and covariance functions which will be used in later:

$$\Sigma_n^{(1)} = \mathbb{E}\left[\left(\tilde{a} - \theta_{n-1} - Q_{n-1}\right)^2\right] \tag{4.13}$$

$$\Sigma_n^{(2)} = \mathbb{E}\left[\left(\tilde{a} - \theta_{n-1} - Q_{n-1}\right)\left(\tilde{v} - P_{n-1}\right)\right] \tag{4.14}$$

The following result is an application of the classic Kalman filter from jointly normal distributions.

Lemma 4.2.1. *Consider the linear system that satisfies* (4.9)-(4.12) *with arbitrary coefficients* $\lambda_n, \mu_n, r_n, s_n$. *If this system satisfies*

$$\lambda_n = \frac{\beta_n \Sigma_{n-1}^{(2)}}{\beta_n^2 \Sigma_{n-1}^{(1)} + \sigma_w^2 \Delta}, \quad \mu_n = \frac{-\alpha_n \beta_n \Sigma_{n-1}^{(2)}}{\beta_n^2 \Sigma_{n-1}^{(1)} + \sigma_w^2 \Delta}, \tag{4.15}$$

$$r_n = \frac{(1-\beta_n)\beta_n \Sigma_{n-1}^{(1)}}{\beta_n^2 \Sigma_{n-1}^{(1)} + \sigma_w^2 \Delta}, \quad s_n = \frac{-\alpha_n (1-\beta_n)\beta_n \Sigma_{n-1}^{(1)}}{\beta_n^2 \Sigma_{n-1}^{(1)} + \sigma_w^2 \Delta} - \alpha_n, \tag{4.16}$$

then the process P *and* Q *satisfy the relation* (4.3) *and* (4.4), *respectively.*
Moreover, the variance and covariance functions for the market makers' prediction have the following relations:

$$\Sigma_n^{(1)} = (1 - (1 + r_n)\beta_n)^2 \Sigma_{n-1}^{(1)} + r_n^2 \sigma_w^2 \Delta \tag{4.17}$$

$$\Sigma_n^{(2)} = (1 - (1 + r_n)\beta_n)\Sigma_{n-1}^{(2)} - \lambda_n \beta_n (1 - (1 + r_n)\beta_n)\Sigma_{n-1}^{(1)} + \lambda_n r_n \sigma_w^2 \Delta \tag{4.18}$$

Proof. For any $n = 1, \cdots, N$, define a process $\hat{Y}_n$ as

$$\hat{Y}_n := Y_n - \alpha_n Q_{n-1} \tag{4.19}$$

$$= \Delta\theta_n + \Delta W_n - \alpha_n Q_{n-1} = \beta_n(\tilde{a} - \theta_{n-1} - Q_{n-1}) + \Delta W_n \tag{4.20}$$

where the second equality (4.19) follows from the definition of $\mathcal{F}_{n-1}^M = \sigma(Y_1, \cdots, Y_{n-1})$. These random variables $\hat{Y}_1, \cdots, \hat{Y}_N$ are mutually independent Gaussian random variables. Furthermore, we have $\sigma(\hat{Y}_1, \cdots, \hat{Y}_n) = \sigma\left(Y_1, \cdots, Y_n\right) = \mathcal{F}_n^M$ for all $n = 1, \cdots, N$. Therefore, by the projection theorem for joint normal random variables,

$$\begin{aligned}\Delta P_n &= \mathbb{E}\left[\tilde{v}|Y_1,\cdots,Y_n\right] - \mathbb{E}\left[\tilde{v}|Y_1,\cdots,Y_{n-1}\right]\\ &= \mathbb{E}\left[\tilde{v}\middle|\hat{Y}_1,\cdots,\hat{Y}_n\right] - \mathbb{E}\left[\tilde{v}\middle|\hat{Y}_1,\cdots,\hat{Y}_{n-1}\right]\\ &= \frac{Cov(\tilde{v},\hat{Y}_n)}{Var(\hat{Y}_n)}\hat{Y}_n = \frac{\mathbb{E}[\tilde{v}\hat{Y}_n]}{Var(\hat{Y}_n)}\hat{Y}_n\end{aligned} \tag{4.21}$$

$$\begin{aligned}\Delta Q_n &= \mathbb{E}\left[\tilde{a}-\theta_n|Y_1,\cdots,Y_n\right] - \mathbb{E}\left[\tilde{a}-\theta_n|Y_1,\cdots,Y_{n-1}\right]\\ &= \mathbb{E}\left[\tilde{a}-\theta_{n-1}\middle|\hat{Y}_1,\cdots,\hat{Y}_n\right] - \mathbb{E}\left[\tilde{a}-\theta_{n-1}\middle|\hat{Y}_1,\cdots,\hat{Y}_{n-1}\right] - \mathbb{E}\left[\Delta\theta_n\middle|\hat{Y}_1,\cdots,\hat{Y}_n\right]\\ &= \frac{Cov\left(\tilde{a}-\theta_{n-1},\hat{Y}_n-\mathbb{E}\left[\hat{Y}_n\middle|\hat{Y}_1,\cdots,\hat{Y}_{n-1}\right]\right)}{Var\left(\hat{Y}_n-\mathbb{E}\left[\hat{Y}_n|\hat{Y}_1,\cdots,\hat{Y}_{n-1}\right]\right)}\left(\hat{Y}_n-\mathbb{E}\left[\hat{Y}_n\middle|\hat{Y}_1,\cdots,\hat{Y}_{n-1}\right]\right) - \mathbb{E}\left[\Delta\theta_n\middle|\hat{Y}_1,\cdots,\hat{Y}_n\right]\\ &= \frac{\mathbb{E}\left[(\tilde{a}-\theta_{n-1})\hat{Y}_n\right]}{Var(\hat{Y}_n)}\hat{Y}_n - \mathbb{E}\left[\beta_n\left(\tilde{a}-\theta_{n-1}-Q_{n-1}\right)+\alpha_n Q_{n-1}\middle|\hat{Y}_1,\cdots,\hat{Y}_n\right]\\ &= \frac{\mathbb{E}\left[(\tilde{a}-\theta_{n-1})\,\hat{Y}_n\right]}{Var(\hat{Y}_n)}\hat{Y}_n - \beta_n\mathbb{E}\left[\tilde{a}-\theta_{n-1}-\hat{Y}_{n-1}\middle|\hat{Y}_n\right] - \alpha_n Q_{n-1}\\ &= \frac{\mathbb{E}\left[(\tilde{a}-\theta_{n-1}-Q_{n-1})\,\hat{Y}_n\right]}{Var(\hat{Y}_n)}\hat{Y}_n - \beta_n\frac{\mathbb{E}\left[(\tilde{a}-\theta_{n-1}-Q_{n-1})\,\hat{Y}_n\right]}{Var(\hat{Y}_n)}\hat{Y}_n - \alpha_n Q_{n-1}\\ &= (1-\beta_n)\frac{\mathbb{E}\left[(\tilde{a}-\theta_{n-1}-Q_{n-1})\,\hat{Y}_n\right]}{Var(\hat{Y}_n)}\hat{Y}_n - \alpha_n Q_{n-1}\end{aligned} \tag{4.22}$$

Moreover, from the linear construction, we can compute the expectations:

$$\begin{aligned}&Var(\hat{Y}_n) = \mathbb{E}\left[\left(\beta_n(\tilde{a}-\theta_{n-1}-Q_{n-1})+\Delta W_n\right)^2\right] = \beta_n^2\Sigma_{n-1}^{(1)} + \sigma_w^2\Delta\\ &\mathbb{E}\left[\tilde{v}\hat{Y}_n\right] = \mathbb{E}\left[(\tilde{v}-P_{n-1})\,\hat{Y}_n\right] = \mathbb{E}\left[(\tilde{v}-P_{n-1})\left(\beta_n(\tilde{a}-\theta_{n-1}-Q_{n-1})+\Delta W_n\right)\right] = \beta_n\Sigma_{n-1}^{(2)}\\ &\mathbb{E}\left[(\tilde{a}-\theta_{n-1}-Q_{n-1})\,\hat{Y}_n\right] = \mathbb{E}\left[(\tilde{a}-\theta_{n-1}-Q_{n-1})\left(\beta_n(\tilde{a}-\theta_{n-1}-Q_{n-1})+\Delta W_n\right)\right] = \beta_n\Sigma_{n-1}^{(1)}\end{aligned} \tag{4.23}$$

Therefore, combining (4.21)-(4.23) with (4.9) and (4.10), we can find the relations (4.15) and (4.16). Furthermore, we can compute the variance and covariance of market makers' prediction by using (4.15)-(4.14):

$$\begin{aligned}
\Sigma_n^{(1)} &= \mathbb{E}\left[(\tilde{a}-\theta_n-Q_n)^2\right] \\
&= \mathbb{E}\left[(\tilde{a}-\theta_{n-1}-Q_{n-1}-\Delta\theta_n-\Delta Q_n)^2\right] \\
&= \mathbb{E}\left[(\tilde{a}-\theta_{n-1}-Q_{n-1}-\Delta\theta_n-r_nY_n+(1+r_n)\alpha_nQ_{n-1})^2\right] \\
&= \mathbb{E}\left[(\tilde{a}-\theta_{n-1}-(1-(1+r_n)\alpha_n)Q_{n-1}-\Delta\theta_n-r_n\Delta\theta_n-r_n\Delta W_n)^2\right] \\
&= \mathbb{E}\left[(\tilde{a}-\theta_{n-1}-(1-(1+r_n)\alpha_n)Q_{n-1}-(1+r_n)\{\beta_n(\tilde{a}-\theta_{n-1}-Q_{n-1}+\alpha_nQ_{n-1})\}-r_n\Delta W_n)^2\right] \\
&= \mathbb{E}\left[((\tilde{a}-\theta_{n-1}-Q_{n-1})(1-(1+r_n)\beta_n)-r_n\Delta W_n)^2\right] \\
&= (1-(1+r_n)\beta_n)^2\,\Sigma_{n-1}^{(1)}+r_n^2\sigma_w^2\Delta
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
\Sigma_n^{(2)} &= \mathbb{E}\left[(\tilde{v}-P_n)(\tilde{a}-\theta_n-Q_n)\right] \\
&= \mathbb{E}\left[(\tilde{v}-\Delta P_n-P_{n-1})(\tilde{a}-\theta_{n-1}-Q_{n-1}-\Delta\theta_n-\Delta Q_n)\right] \\
&= \mathbb{E}\Big[\Big((\tilde{v}-P_{n-1})-\lambda_n\beta_n(\tilde{a}-\theta_{n-1}-Q_{n-1})-\lambda_n\Delta W_n\Big) \\
&\qquad\qquad \Big((1-(1+r_n)\beta_n)(\tilde{a}-\theta_{n-1}-Q_{n-1})-r_n\Delta W_n\Big)\Big] \\
&= (1-(1+r_n)\beta_n)\Sigma_{n-1}^{(2)}-\lambda_n\beta_n(1-(1+r_n)\beta_n)\Sigma_{n-1}^{(1)}+\lambda_nr_n\sigma_w^2\Delta
\end{aligned} \tag{4.25}$$

□

4.3 Existence of equilibrium

We conjecture that the state variables are $\tilde{a}-\theta_n$ and Q_n. Our treatment of Q as a state variable will be discussed in §4.4. Theorem 4.3.1 shows that the insider's value function in (4.7) for $n=0,1,\cdots,N$ follows the quadratic function

$$\inf_{\theta_k\in\mathcal{F}_{k-1}^I}\mathbb{E}\left[\sum_{k=n+1}^{N}(\tilde{a}-\theta_{k-1})\Delta P_k\middle|\mathcal{F}_n^I\right] = I_n(\tilde{a}-\theta_n-Q_n)^2+J_n(\tilde{a}-\theta_n-Q_n)Q_n+K_n \tag{4.26}$$

where I_n, J_n and K_n are constants.

Theorem 4.3.1. *Let $\Sigma_{N-1}^{(1)}>0$ and $\Sigma_{N-1}^{(2)}>0$. Suppose that $0<\frac{\beta_n}{1-\beta_n}<\sqrt{\frac{\sigma_w^2\Delta}{\Sigma_n^{(1)}}}$ holds for all $n=1,\cdots,N$. There exists a linear and recursive equilibrium in the system (4.9)-(4.14). In this equilibrium, there are constants $\beta_n,\alpha_n,I_n,J_n,K_n,\lambda_n,\mu_n,r_n,s_n,\Sigma_n^{(1)}$ and $\Sigma_n^{(2)}$ which satisfies (4.26),(4.17),(4.18), (4.15) and (4.16). In particular, the insider's value function* *has the quadratic form as in* (4.26) *and the coefficients for the value functions satisfy*

$$\begin{aligned} I_{n-1} &= \lambda_n\beta_n + I_n(1-(1+r_n)\beta_n)^2 + J_n((1-(1+r_n)\beta_n)r_n\beta_n) \\ J_{n-1} &= \lambda_n\beta_n + J_n((1-(1+r_n)\beta_n)(1-\alpha_n)) \\ K_{n-1} &= K_n + I_n r_n^2\sigma_w^2\Delta - J_n r_n^2\sigma_w^2\Delta \end{aligned} \tag{4.27}$$

subject to $\beta_N = \alpha_N = 1$ *with the second order condition*

$$(1+r_n)^2 I_n - r_n(1+r_n)J_n > 0. \tag{4.28}$$

Proof. Suppose that (4.26) holds for time $n+1$. Then the insider's value function in the n-th iteration becomes

$$\inf_{\theta_k\in\mathcal{F}^I_{k-1}} \mathbb{E}\left[\sum_{k=n}^{N}(\tilde{a}-\theta_{k-1})\Delta P_k \middle| \mathcal{F}^I_{n-1}\right] \tag{4.29}$$

$$= \inf_{\theta_k\in\mathcal{F}^I_{k-1}} \mathbb{E}\left[(\tilde{a}-\theta_{n-1})\Delta P_n + I_n(\tilde{a}-\theta_n-Q_n)^2 + J_n(\tilde{a}-\theta_n-Q_n)Q_n + K_n \middle| \mathcal{F}^I_{n-1}\right] \tag{4.30}$$

By the joint normality and definition of (4.9)-(4.12), we can compute the conditional expectations:

$$\begin{aligned} &\mathbb{E}[(\tilde{a}-\theta_{n-1})\Delta P_n|\mathcal{F}^I_{n-1}] = \mathbb{E}\left[(\tilde{a}-\theta_{n-1})(\lambda_n Y_n + \mu_n Q_{n-1})\middle|\mathcal{F}^I_{n-1}\right] \\ &= (\tilde{a}-\theta_{n-1})(\lambda_n\Delta\theta_n + \mu_n Q_{n-1}) \\ &\mathbb{E}\left[(\tilde{a}-\theta_n-Q_n)^2\middle|\mathcal{F}^I_{n-1}\right] \\ &= \mathbb{E}\left[(\tilde{a}-\theta_{n-1}-Q_{n-1}-\Delta\theta_n-\Delta Q_n)^2\middle|\mathcal{F}^I_{n-1}\right] \\ &= \mathbb{E}\left[(\tilde{a}-\theta_{n-1}-Q_{n-1}-\Delta\theta_n-r_n\Delta\theta_n-r_n\Delta W_n-s_nQ_{n-1})^2\middle|\mathcal{F}^I_{n-1}\right] \\ &= (\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n-s_nQ_{n-1})^2 + r_n^2\sigma_w^2\Delta \\ &\mathbb{E}\left[(\tilde{a}-\theta_n-Q_n)Q_n\middle|\mathcal{F}^I_{n-1}\right] \\ &= \mathbb{E}\left[(\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n-r_n\Delta W_n-s_nQ_{n-1})\left(Q_{n-1}+r_n\Delta\theta_n+r_n\Delta W_n+s_nQ_{n-1}\right)\middle|\mathcal{F}^I_{n-1}\right] \\ &= (\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n-s_nQ_{n-1})\left(Q_{n-1}+r_n\Delta\theta_n+s_nQ_{n-1}\right) - r_n^2\sigma_w^2\Delta \end{aligned} \tag{4.31}$$

Therefore, by using (4.31), (4.30) becomes

$$\begin{aligned} (4.30) = \inf_{\theta}\Big(&(\tilde{a}-\theta_{n-1})(\lambda_n\Delta\theta_n+\mu_nQ_{n-1}) + K_n \\ &+ I_n\left((\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n-s_nQ_{n-1})^2 + r_n^2\sigma_w^2\Delta\right) \\ &+ J_n\left((\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n-s_nQ_{n-1})\left(Q_{n+1}+r_n\Delta\theta_n+s_nQ_{n-1}\right) - r_n^2\sigma_w^2\Delta\right)\Big) \end{aligned} \tag{4.32}$$

By taking derivatives,

$$
\begin{aligned}
&\lambda_n(\tilde{a}-\theta_{n-1})+I_n\left(-2(1+r_n)\left(\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n+s_nQ_{n-1}\right)\right)\\
&+J_n\left(-(1+r_n)\left(Q_{n+1}+r_n\Delta\theta_n+s_nQ_{n-1}\right)+r_n\left(\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n-s_nQ_{n-1}\right)\right)=0
\end{aligned}
\tag{4.33}
$$

With the second order condition (4.28), we derive a candidate optimizer of the insider's problem: For $n=1,\cdots,N$,

$$
\Delta\theta_n=\beta_n\left(\tilde{a}-\theta_{n-1}-Q_{n-1}\right)+\alpha_nQ_{n-1}, \tag{4.34}
$$

where

$$
\beta_n=\frac{\lambda_n-2I_n(1+r_n)+r_nJ_n}{2\{r_n(1+r_n)J_n-I_n(1+r_n)^2\}}\text{ and }\alpha_n=\frac{\lambda_n+2(1+r_n)s_nI_n-J_n(1+s_n+r_n2r_ns_n)}{2\{r_n(1+r_n)J_n-I_n(1+r_n)^2\}} \tag{4.35}
$$

Furthermore, combining (4.30), (4.31) and (4.26) produce the difference equations for the coefficient functions (4.27). We confirm that the candidate equilibrium is a true equilibrium. Define $x_n:=\frac{\beta_n}{1-\beta_n}$ and by combining (4.35) with (4.15)-(4.16), we have for all $1\le n\le N-1$,

$$
\begin{aligned}
2\left(\frac{\Sigma_n^{(1)}}{\sigma_w^2\Delta}\right)^2(J_n-I_n)\,x_n^3+&\left(\frac{\Sigma_n^{(1)}}{\sigma_w^2\Delta}J_n-\frac{2\Sigma_n^{(1)}}{\sigma_w^2\Delta}I_n-\frac{\Sigma_n^{(2)}}{\sigma_w^2\Delta}\right)x_n^2\\
&+\left(\frac{2\Sigma_n^{(1)}}{\sigma_w^2\Delta}I_n-\frac{\Sigma_n^{(1)}}{\sigma_w^2\Delta}J_n-\frac{\Sigma_n^{(2)}}{\sigma_w\Delta}\right)x_n+2I_n=0.
\end{aligned}
\tag{4.36}
$$

First, we consider time $n=N-1$ case. Since $\alpha_N=\beta_N=1$ and $I_N=J_N=0$, we have $I_{N-1}=J_{N-1}=-\lambda_N\beta_N=-\lambda_N=\frac{-\Sigma_{N-1}^{(2)}}{\Sigma_{N-1}^{(1)}+\sigma_w^2\Delta}$ from (4.27). Hence, the (4.36) for time $n=N-1$ becomes

$$
\left(\lambda_N\frac{\Sigma_{N-1}^{(1)}}{\sigma_w^2\Delta}+\frac{\Sigma_{N-1}^{(2)}}{\sigma_w^2\Delta}\right)x_{N-1}^2+\left(-\lambda_N\frac{\Sigma_{N-1}^{(1)}}{\sigma_w^2\Delta}+\frac{\Sigma_{N-1}^{(2)}}{\sigma_w^2\Delta}\right)x_{N-1}-2\lambda_N=0
$$

This equation can be more simply transcribed as,

$$
\left(2\Sigma_{N-1}^{(1)}+\sigma_w^2\Delta\right)x_{N-1}^2+\sigma_w^2\Delta x_{N-1}-2\sigma_w^2\Delta=0. \tag{4.37}
$$

Define a function $f(x_{N-1})=\left(2\Sigma_{N-1}^{(1)}+\sigma_w^2\Delta\right)x_{N-1}^2+\sigma_w^2\Delta x_{N-1}-2\sigma_w^2\Delta$. Then

$$
f(0)=-2\sigma_w^2\Delta<0, \tag{4.38}
$$

$$
f\left(\frac{\sigma_w^2\Delta}{\Sigma_{N-1}^{(1)}}\right)=\left(2\Sigma_{N-1}^{(1)}+\sigma_w^2\Delta\right)\frac{\sigma_w^2\Delta}{\Sigma_{N-1}^{(1)}}+\sigma_w^2\Delta\sqrt{\frac{\sigma_w^2\Delta}{\Sigma_{N-1}^{(1)}}}-2\sigma_w^2\Delta=\sigma_w^2\Delta\left(\frac{\sigma_w^2\Delta}{\Sigma_{N-1}^{(1)}}+\sqrt{\frac{\sigma_w^2\Delta}{\Sigma_{N-1}^{(1)}}}\right)>0. \tag{4.39}
$$

Therefore, between 0 and $\sqrt{\frac{\sigma_w\Delta}{\Sigma^{(1)}_{N-1}}}$ there is a solution x_{N-1}. And this x_{N-1} satisfies the second order condition:

$$\begin{aligned}(1+r_{N-1})^2 I_{N-1} - r_{N-1}(1+r_{N-1})J_{N-1} > 0 \Leftrightarrow I_{N-1}\left(\frac{\Sigma^{(1)}_{N-1}}{\sigma_w^2\Delta}x_{N-1}+1\right) > 0 \\ \Leftrightarrow x_{N-1} > -\frac{\sigma_w^2\Delta}{\Sigma^{(1)}_{N-1}}.\end{aligned} \tag{4.40}$$

Therefore, the solution occurs at $N-1$-th time step. We can also prove its existence by backward induction. Suppose that there exists a solution x_n such that $0 < x_n < \sqrt{\frac{\sigma_w^2\Delta}{\Sigma^{(1)}_n}}$ for $1 \leq n \leq N-1$. Then in $n-1$ iteration, as in (4.36), we get a cubic equation:

$$\begin{aligned}2\left(\frac{\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}\right)^2 \left(J_{n-1}-I_{n-1}\right)x^3_{n-1} + \left(\frac{\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}J_{n-1} - \frac{2\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}I_{n-1} - \frac{\Sigma^{(2)}_{n-1}}{\sigma_w^2\Delta}\right)x^2_{n-1} \\ + \left(\frac{2\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}I_{n-1} - \frac{\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}J_{n-1} - \frac{\Sigma^{(2)}_{n-1}}{\sigma_w\Delta}\right)x_{n-1} + 2I_{n-1} = 0.\end{aligned} \tag{4.41}$$

As in above, we define the cubic function in the left-hand side of (4.41) as $f_{n-1}(x_{n-1})$. Then,

$$f_{n-1}(0) = 2I_{n-1} > 0, \tag{4.42}$$

$$\begin{aligned}f_{n-1}\left(\sqrt{\frac{\sigma_w^2\Delta}{\Sigma^{(1)}_{n-1}}}\right) &= \sqrt{\frac{\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}}J_{n-1} + J_{n-1} - \frac{\Sigma^{(2)}_{n-1}}{\Sigma^{(1)}_{n-1}} - \frac{\Sigma^{(2)}_{n-1}}{\sigma_w^2\Delta}\sqrt{\frac{\sigma_w^2\Delta}{\Sigma^{(1)}_{n-1}}} \\ &= \left(1+\sqrt{\frac{\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}}\right)\left(J_{n-1} - \frac{\Sigma^{(2)}_{n-1}}{\Sigma^{(1)}_{n-1}}\right) < 0\end{aligned} \tag{4.43}$$

where the last inequality of (4.43) holds since

$$J_{n-1} - \frac{\Sigma^{(2)}_{n-1}}{\Sigma^{(1)}_{n-1}} = \frac{\lambda_n}{1+r_n} - \frac{\lambda_{n-1}}{r_{n-1}} = \frac{\beta_n\Sigma^{(2)}_{n-1}}{\beta_n\Sigma^{(1)}_{n-1}+\sigma_w^2\Delta} - \frac{\beta_n\Sigma^{(2)}_{n-1}}{\beta_n\Sigma^{(1)}_{n-1}} < 0. \tag{4.44}$$

(4.42) and (4.43) imply that there exists a solution x_{n-1} of the cubic equation (4.36) between the interval $\left(0, \sqrt{\frac{\sigma_w^2\Delta}{\Sigma^{(1)}_{n-1}}}\right)$. In addition, note that (4.36), (4.44) and the interval of x_{n-1} implies that

$$\frac{\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}\left(I_{n-1}-J_{n-1}\right)x_{n-1} + I_{n-1} = \frac{x_{n-1}(1+x_{n-1})\frac{\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}\left(J_{n-1}-\frac{\Sigma^{(2)}_{n-1}}{\Sigma^{(1)}_{n-1}}\right)}{2\left(\frac{\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}x^2_{n-1}-1\right)} > 0. \tag{4.45}$$

Therefore, the the solution x_{n-1} satisfies the second order condition (4.28):

$$\left(1+\frac{\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}x_{n-1}\right)\left(\frac{\Sigma^{(1)}_{n-1}}{\sigma_w^2\Delta}\left(I_{n-1}-J_{n-1}\right)x_{n-1}+I_{n-1}\right) > 0 \tag{4.46}$$

$$\Leftrightarrow (1+r_{n-1})\left((1+r_{n-1})I_{n-1} - r_{n-1}J_{n-1}\right) > 0 \Leftrightarrow (4.28). \tag{4.47}$$

By backward induction, there exists an equilibrium satisfying all the relations (4.9)-(4.18) and (4.26)-(4.27). □

In §4.4 , we address state variables. We explain that $Q_t = \mathbb{E}[\tilde{a}-\theta_t|\mathcal{F}_t^M]$ and $\tilde{a}-\theta_t$ are considered naturally as state variables in the linear setting.

4.4 Uniqueness of the linear structure

In the previous sections, $Q_t = \mathbb{E}[\tilde{a}-\theta_t|\mathcal{F}_t^M]$ and $\tilde{a}-\theta_t$ are considered naturally as state variables in a linear market. In this section, we will make sure that it is really natural to consider those state variables in our linear market.

First, the insider's expected profit is zero at time N since all information is revealed to all market participants at the terminal time N. In addition, suppose that for all $n = 1, \cdots, N$,

$$\begin{aligned}
&\mathcal{F}_n^I := \sigma(\tilde{a}, Y_1, \cdots, Y_{n-1}) \text{ and } \mathcal{F}_n^M := \sigma(Y_1, \cdots, Y_n),\\
&\Delta P_n = \lambda_n Y_n + h_{n-1},\\
&\Delta Y_n = \Delta\theta_n + \sigma_w \Delta W_n,\\
&\Delta\theta_N = \tilde{a} - \theta_{N-1} \text{ (Because of the terminal constraint)},
\end{aligned} \tag{4.48}$$

where λ_n is a constant and h_{n-1} is some linear function of $Y_1, \cdots, Y_{n-1}$ so that $\mathcal{F}_{n-1}^M$ measurable. The insider's expected profit at time $N-1$ is

$$\begin{aligned}
&\mathbb{E}[(\tilde{a}-\theta_{N-1})\Delta P_N|\mathcal{F}_{N-1}^I]\\
&= \mathbb{E}[(\tilde{a}-\theta_{N-1})(\lambda_N Y_N + h_{N-1})|\mathcal{F}_{N-1}^I]\\
&= \mathbb{E}[(\tilde{a}-\theta_{N-1})(\lambda_N(\tilde{a}-\theta_{N-1}+\sigma_w\Delta W_N) + h_{N-1})|\mathcal{F}_{N-1}^I]\\
&= \lambda_N(\tilde{a}-\theta_{N-1})^2 + (\tilde{a}-\theta_{N-1})h_{N-1}
\end{aligned} \tag{4.49}$$

By the market efficiency condition, we have

$$\begin{aligned}
0 &= \mathbb{E}\left[\Delta P_N \middle| \mathcal{F}_{N-1}^M\right]\\
&= \mathbb{E}\left[\lambda_N(\tilde{a}-\theta_{N-1}+\Delta W_N) + h_{N-1}\middle|\mathcal{F}_{N-1}^M\right] = \lambda_N \mathbb{E}\left[\tilde{a}-\theta_{N-1}\middle|\mathcal{F}_{N-1}^M\right] + h_{N-1}
\end{aligned}$$

Therefore we get $h_{N-1} = -\lambda_N\mathbb{E}[\tilde{a}-\theta_{N-1}|\mathcal{F}_{N-1}^M]$, implying that $\mathbb{E}[\tilde{a}-\theta_{N-1}|\mathcal{F}_{N-1}^M] = \mathbb{E}[\Delta\theta_N|\mathcal{F}_{N-1}^M]$ is linear in $Y_1, \cdots, Y_{N-1}$. Therefore, (4.49) becomes

$$\begin{aligned}
\mathbb{E}\left[(\tilde{a}-\theta_{N-1})\Delta P_N\middle|\mathcal{F}_{N-1}^M\right] &= \lambda_N(\tilde{a}-\theta_{N-1})^2 - \lambda_N(\tilde{a}-\theta_{N-1})\mathbb{E}[\tilde{a}-\theta_{N-1}|\mathcal{F}_{N-1}^M]\\
&= \lambda_N(\tilde{a}-\theta_{N-1}-Q_{N-1})^2 - 3\lambda_N(\tilde{a}-\theta_{N-1}-Q_{N-1})Q_{N-1}.
\end{aligned} \tag{4.50}$$

This coincides with the equation (4.26) in $n = N-1$. We suppose that $Q_n = \mathbb{E}[\tilde{a} - \theta_n | \mathcal{F}_n^M]$ is linear in $Y_1, \cdots, Y_n$. If we use this assumption to check whether Q_{n-1} is linear in $Y_1, \cdots, Y_{n-1}$, we can conclude that Q_n is linear in $Y_1, \cdots, Y_n$ for any $n = 1, \cdots, N$, thanks to backward induction. Let $Q_n = c_n Y_n + c_{n-1} Y_{n-1} + \cdots + c_1 Y_1$ with some constants $c_n, c_{n-1}, \cdots, c_1$. Again, by the market efficiency condition,

$$\begin{aligned} 0 &= \mathbb{E}\left[\Delta P_n \middle| \mathcal{F}_{n-1}^M\right] \\ &= \mathbb{E}\left[\lambda_n Y_n + h_{n-1} \middle| \mathcal{F}_{n-1}^M\right] = \lambda_n \mathbb{E}\left[\Delta\theta_n \middle| \mathcal{F}_{n-1}^M\right] + h_{n-1} \end{aligned}$$

Hence, we have $h_{n-1} = -\lambda_n \mathbb{E}[\Delta\theta_n | \mathcal{F}_{n-1}^M]$, that is, $\mathbb{E}\left[\Delta\theta_n \middle| \mathcal{F}_{n-1}^M\right]$ is linear in $Y_1, \cdots, Y_{n-1}$. Then we check that Q_{n-1} is also a linear function in $Y_1, \cdots, Y_{n-1}$:

$$\begin{aligned} Q_{n-1} &= \mathbb{E}\left[\tilde{a} - \theta_{n-1} \middle| \mathcal{F}_{n-1}^M\right] = \mathbb{E}\left[\tilde{a} - \theta_n \middle| \mathcal{F}_{n-1}^M\right] + \mathbb{E}\left[\Delta\theta_n \middle| \mathcal{F}_{n-1}^M\right] \\ &= \mathbb{E}\left[Q_n \middle| \mathcal{F}_{n-1}^M\right] + \mathbb{E}\left[\Delta\theta_n \middle| \mathcal{F}_{n-1}^M\right] \\ &= \mathbb{E}\left[c_n Y_n + c_{n-1} Y_{n-1} + \cdots + c_1 Y_1 \middle| \mathcal{F}_{n-1}^M\right] - \frac{1}{\lambda_n} h_{n-1} \\ &= c_n \mathbb{E}\left[\Delta\theta_n + \Delta W_n \middle| \mathcal{F}_{n-1}^M\right] + (c_{n-1} Y_{n-1} + \cdots + c_1 Y_1) - \frac{1}{\lambda_n} h_{n-1} \\ &= -\frac{c_n}{\lambda_n} h_{n-1} + (c_{n-1} Y_{n-1} + \cdots + c_1 Y_1) - \frac{1}{\lambda_n} h_{n-1} \end{aligned} \tag{4.51}$$

By backward induction, we confirm that Q_n is linear in $Y_1, \cdots, Y_n$ for all $n = 1, \cdots, N$. Thus:

$$\Delta Q_n = Q_n - Q_{n-1} = r_n Y_n + \tilde{h}_{n-1} \tag{4.52}$$

where $\tilde{h}_{n-1}$ is some linear function in $Y_1, \cdots, Y_{n-1}$. We then want to check that the form of insider's value function (4.26) is natural. In (4.50), we already checked that the value function for $n = N-1$ is the same as (4.26). Assume that the value function of time n satisfies (4.26) and consider the time $n-1$:

$$\begin{aligned} &\mathbb{E}\left[(\tilde{a} - \theta_{n-1})\Delta P_n + \sum_{k=n+1}^{N} (\tilde{a} - \theta_{k-1})\Delta P_k \middle| \mathcal{F}_{n-1}^I\right] \\ &= \mathbb{E}\left[(\tilde{a} - \theta_{n-1})\Delta P_n + I_n(\tilde{a} - \theta_n - Q_n)^2 + J_n(\tilde{a} - \theta_n - Q_n)Q_n + K_n \middle| \mathcal{F}_{n-1}^I\right] \end{aligned} \tag{4.53}$$

To compute these expectations, observe that

1. $\mathbb{E}[(\tilde{a}-\theta_{n-1})\Delta P_n|\mathcal{F}^I_{n-1}] = \mathbb{E}\left[(\tilde{a}-\theta_{n-1})(\lambda_n Y_n + h_{n-1})\middle|\mathcal{F}^I_{n-1}\right]$
$= (\tilde{a}-\theta_{n-1})(\lambda_n \Delta\theta_n + h_{n-1}) = (\tilde{a}-\theta_{n-1})(\lambda_n\Delta\theta_n - \lambda_n\mathbb{E}[\Delta\theta_n|\mathcal{F}^M_{n-1}])$

2. $\mathbb{E}\left[(\tilde{a}-\theta_n-Q_n)^2\middle|\mathcal{F}^I_{n-1}\right]$
$= \mathbb{E}\left[(\tilde{a}-\theta_{n-1}-Q_{n-1}-\Delta\theta_n-\Delta Q_n)^2\middle|\mathcal{F}^I_{n-1}\right]$
$= \mathbb{E}\left[\left(\tilde{a}-\theta_{n-1}-Q_{n-1}-\Delta\theta_n-r_n\Delta\theta_n-r_n\Delta W_n-\tilde{h}_{n-1}\right)^2\middle|\mathcal{F}^I_{n-1}\right]$
$= \left(\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n-\tilde{h}_{n-1}\right)^2 + r_n^2\sigma_w^2\Delta$

3. $\mathbb{E}\left[(\tilde{a}-\theta_n-Q_n)Q_n\middle|\mathcal{F}^I_{n-1}\right]$
$= \mathbb{E}\left[\left(\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n-r_n\Delta W_n-\tilde{h}_{n-1}\right)\left(Q_{n+1}+r_n\Delta\theta_n+r_n\Delta W_n+\tilde{h}_{n-1}\right)\middle|\mathcal{F}^I_{n-1}\right]$
$= \left(\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n-\tilde{h}_{n-1}\right)\left(Q_{n+1}+r_n\Delta\theta_n+\tilde{h}_{n-1}\right) - r_n^2\sigma_w^2\Delta$

where we used (4.48) and (4.52). Plugging these relations in (4.53) and taking derivatives with respect to $\Delta\theta_n$ yields

$$\lambda_n(\tilde{a}-\theta_{n-1}) + I_n\left(-2(1+r_n)\left(\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n-\tilde{h}_{n-1}\right)\right) \tag{4.54}$$
$$+ J_n\left(-(1+r_n)\left(Q_{n+1}+r_n\Delta\theta_n+\tilde{h}_{n-1}\right) + r_n\left(\tilde{a}-\theta_{n-1}-Q_{n-1}-(1+r_n)\Delta\theta_n-\tilde{h}_{n-1}\right)\right) = 0 \tag{4.55}$$

For $n = 1,\cdots,N$, we get

$$\Delta\theta_n = a_n(\tilde{a}-\theta_{n-1}) + b_nQ_{n-1} + c_n\tilde{h}_{n-1} \text{ for some constant } a_n, b_n, c_n. \tag{4.56}$$

Therefore, from the previous observation, h_{n-1} becomes a linear combination of Q_{n-1} and $\tilde{h}_{n-1}$.

$$\begin{aligned} h_{n-1} &= -\lambda_n\mathbb{E}\left[\Delta\theta_n\middle|\mathcal{F}^M_{n-1}\right] \\ &= -\lambda_n\left(a_n\mathbb{E}\left[\tilde{a}-\theta_{n-1}\middle|\mathcal{F}^M_{n-1}\right] + b_nQ_{n-1} + c_n\tilde{h}_{n-1}\right) \\ &= -\lambda_n\left((a_n+b_n)Q_{n-1} + c_n\tilde{h}_{n-1}\right) \end{aligned} \tag{4.57}$$

Observe that for any $n = 1,\cdots,N$,

$$\begin{aligned} \hat{Y}_n &:= Y_n - \mathbb{E}\left[Y_n\middle|\mathcal{F}^M_{n-1}\right] = \Delta\theta_n + \Delta W_n - \mathbb{E}\left[\Delta\theta_n+\Delta W_n\middle|\mathcal{F}^M_{n-1}\right] \\ &= a_n(\tilde{a}-\theta_{n-1}) + b_nQ_{n-1} + c_n\tilde{h}_{n-1} + \Delta W_n - \mathbb{E}\left[a_n(\tilde{a}-\theta_{n-1}) + b_nQ_{n-1} + c_n\tilde{h}_{n-1} + \Delta W_n\middle|\mathcal{F}^M_{n-1}\right] \\ &= a_n\left(\tilde{a}-\theta_{n-1}-\mathbb{E}\left[\tilde{a}-\theta_{n-1}\middle|\mathcal{F}^M_{n-1}\right]\right) + \Delta W_n \end{aligned}$$

So $\hat{Y}_1, \hat{Y}_2, \cdots, \hat{Y}_N$ are mutually independent, and $\tilde{a} - \theta_n$ and $\hat{Y}_n$ are also independent for all $n = 1, \cdots, N$. Finally, the projection theorem for joint normal random variables produces

$$\begin{aligned}
\Delta Q_n &= \mathbb{E}\left[\tilde{a} - \theta_n \middle| \mathcal{F}_n^M\right] - \mathbb{E}\left[\tilde{a} - \theta_{n-1} \middle| \mathcal{F}_{n-1}^M\right] \\
&= \mathbb{E}\left[\tilde{a} - \theta_{n-1} \middle| \mathcal{F}_n^M\right] - \mathbb{E}\left[\tilde{a} - \theta_{n-1} \middle| \mathcal{F}_{n-1}^M\right] - \mathbb{E}\left[\Delta\theta_n \middle| \mathcal{F}_n^M\right] \\
&= \frac{Cov(\tilde{a} - \theta_{n-1}, \hat{Y}_n)}{Var(\hat{Y}_n)}(Y_n - \mathbb{E}\left[\Delta\theta_n \middle| \mathcal{F}_{n-1}^M\right]) - a_n \mathbb{E}\left[\tilde{a} - \theta_{n-1} \middle| \mathcal{F}_n^M\right] - b_n Q_{n-1} - c_n \tilde{h}_{n-1} \\
&= \frac{Cov(\tilde{a} - \theta_{n-1}, \hat{Y}_n)}{Var(\hat{Y}_n)}(Y_n - \mathbb{E}\left[\Delta\theta_n \middle| \mathcal{F}_{n-1}^M\right]) - a_n \mathbb{E}\left[\tilde{a} - \theta_{n-1} \middle| \hat{Y}_n\right] - b_n Q_{n-1} - c_n \tilde{h}_{n-1} \\
&= (1 - a_n)\frac{Cov(\tilde{a} - \theta_{n-1}, \hat{Y}_n)}{Var(\hat{Y}_n)}\Big(Y_n - ((a_n + b_n)Q_{n-1} + c_n \tilde{h}_{n-1})\Big) - b_n Q_{n-1} - c_n \tilde{h}_{n-1}
\end{aligned} \tag{4.58}$$

Comparing (4.52) and (4.58), we have

$$\tilde{h}_{n-1} = \delta_n Q_{n-1} \text{ for some constant } \delta_n. \tag{4.59}$$

Moreover, this observation and (4.57) yields

$$h_{n-1} = \bar{\delta}_n Q_{n-1} \text{ for some constant } \bar{\delta}_n. \tag{4.60}$$

Ultimately, the insider's holding in (4.56) is now

$$\Delta\theta_n = a_n(\tilde{a} - \theta_{n-1}) + d_n Q_{n-1} \text{ for some constants } a_n, d_n \tag{4.61}$$

Combining (4.54), and (4.60)-(4.61), we conclude that insider's value function follows the form (4.8).

Chapter 5

Continuous Time

This chapter includes the published contents:

J. Choi, H. Kwon and K. Larsen (2023): *Trading Constraints in Continuous-Time Kyle Models*, SIAM Journal on Control and Optimization **61**, 1494–1512

5.1 Model

Basically, we assume the same structure as that used in a discrete-time system. The time index denoted by n in the discrete-time system is replaced by $t \in [0, T]$ in continuous-time one. Here, the terminal time T is arbitrary and finite; that is, $T \in (0, \infty)$. Again, our model includes three types of market participants : noise traders, an insider with trading constraint, and (competitive) market makers:

1. *Noise Traders*: As in the discrete-time system, these traders' cumulative order process is exogenously given by $\sigma_w W_t$ at time $t \in [0, T]$.

2. *Insider*: This trader's cumulative order process is denoted by $(\theta_t)_{t\in[0,T]}$. The insider's initial holding is zero, $\theta_{0-} = 0$, and the insider should meet the trading target at the terminal time, that is, $\theta_T = \tilde{a} \sim \mathcal{N}(0, \sigma_a^2)$. The insider knows the terminal target and can observe the price process P. Therefore, the insider's filtration is

$$\mathcal{F}_t^I = \sigma\left(\tilde{a}, (P_s)_{s\in[0,t]}\right) \text{ for all } t \in [0, T].$$

Similarly, we can assume that the insider directly observes the aggregate orders $Y_t = \sigma_w W_t + \theta_t$; hence, the insider's filtration is replaced with

$$\mathcal{F}_t^I = \sigma\left(\tilde{a}, (Y_s)_{s\in[0,t]}\right) = \sigma\left(\tilde{a}, (W_s)_{s\in[0,t]}\right) \text{ for all } t \in [0, T].$$

The insider wants to maximize her expected profit subject to the constraint $\theta_T = \tilde{a}$:

$$\sup_{\theta \in \mathcal{A}} \mathbb{E}\left[\int_0^T (\tilde{v} - P_{t-})\, d\theta_t \middle| \mathcal{F}_0^I\right] = \sup_{\theta \in \mathcal{A}} \mathbb{E}\left[(\tilde{v} - P_T)\,\theta_T - \int_0^T P_{t-} d\theta_t \middle| \mathcal{F}_0^I\right] \tag{5.1}$$

$$= \rho \frac{\sigma_v}{\sigma_a}\tilde{a}^2 - \inf_{\theta \in \mathcal{A}} \mathbb{E}\left[\int_0^T (\tilde{a} - \theta_{t-})\, dP_t \middle| \mathcal{F}_0^I\right] \tag{5.2}$$

where $\mathcal{A}$ is set of all admissible strategies, which is defined in Definition 5.1.1. Here, the equality in (5.1) in comes from the integration of parts as in Back(1992) and last one comes from the joint normality of $\tilde{v}$ and $\tilde{a}$.

3. *Market Makers*: These traders observe the aggregate orders dY_t over time where $Y_t = \sigma_w W_t + \theta_t$. Therefore, the market makers' filtration is

$$\mathcal{F}_t^M = \sigma\left((Y_s)_{s \in [0,t]}\right) \text{ for all } t \in [0, T].$$

As in the discrete setting, all the market participants can observe this aggregate process Y over time but only the insider can distinguish the amount of θ and W from Y. Moreover, the market makers set the stock price by using the information of Y in order to clear the market; that is, they will set the market price as

$$P_t = \mathbb{E}\left[\tilde{v} \middle| \mathcal{F}_t^M\right] \text{ for all } t \in [0, T]. \tag{5.3}$$

Before defining a concept of an equilibrium, we have to consider the insider's state variables. In this setting, we conjecture that the insider's state variables are $\tilde{a} - \theta_t$ and $Q_t =: \mathbb{E}\left[\tilde{a} - \theta_t \middle| \mathcal{F}_t^M\right]$, which represent the remaining trading amount of the insider to the constraint, and the estimated value of the amount is considered by the market maker. We will see that the insider's optimal trading strategy is a linear combination of those state variables:

$$d\theta_t = (\beta(t)\,(\tilde{a} - \theta_{t-} - Q_{t-}) + \alpha(t) Q_{t-})\, dt \tag{5.4}$$

As in the discrete time setting in Chapter 4, we only consider linear equilibria:

$$dP_t = \lambda(t) dY_t + \mu(t) dQ_{t-} dt, \text{ for } t \in (0, T),\ P_0 = 0 \tag{5.5}$$

$$\Delta P_T = \lambda(T)\,(\tilde{a} - \theta_{T-} - Q_{T-}) \tag{5.6}$$

$$dQ_t = r(t) dY_t + s(t) dQ_{t-} dt, \text{ for } t \in (0, T),\ Q_0 = 0. \tag{5.7}$$

The functions $\lambda(t), \mu(t), r(t)$ and $s(t)$ will be determined in equilibrium. To solve the insider's optimization problem with this linear structure, we need to exclude doubling strategies from our definition of the set of admissible strategies:

Definition 5.1.1 (Admissible Strategies)**.** A process $(\theta_t)_{t \in [0,T]}$ with initial condition $\theta_{0-} = 0$ and the terminal constraint $\theta_T = \tilde{a}$ is an admissible strategy if it satisfies the following:

(i) The order process θ_t is cádlág semimartingale which is adapted to insider's filtration $\mathcal{F}_t^I$ and square integrable $\mathbb{E}\left[\int_0^T \theta_t^2 dt\right] < \infty$.

(ii) For given continuous functions $r, s : [0,T) \to \mathbb{R}$, a cádlág solution $(Q_t)_{t\in[0,T)}$ of the SDE (5.7) exists and the solution $(Q_t)_{t\in[0,T)}$ is square integrable $\mathbb{E}\left[\int_0^T Q_t^2 dt\right] < \infty$ and have an almost surely finite limit $\lim_{t\uparrow T} Q_t =: Q_{T-}$.

(iii) For given continuous functions $\mu : [0,T) \to \mathbb{R}$ and $\lambda : [0,T] \to \mathbb{R}$, a cádlág solution $(P_t)_{t\in[0,T]}$ of the SDE (5.5)-(5.6) exists such that the stochastic integral $\int_0^t (\tilde{a}-\theta_{s-})dP_s$, $t \in [0,T]$ is a well-defined semimartingale with integrable $\int_0^T (\tilde{a}-\theta_{s-})dP_s$ and an almost surely finite limit $\lim_{t\uparrow T} P_t =: P_{T-}$.

Moreover, define $\mathcal{A}$ as the set that contains all admissible strategies.

Definition 5.1.2 (Kyle Equilibrium)**.** Continuous functions $\mu, r, s, \beta, \alpha : [0,T) \to \mathbb{R}$ and $\lambda : [0,T] \to \mathbb{R}$ constitute an equilibrium if

(i) For the pricing rule (5.5) and (5.6) with Q_t in (5.7), the stock-holding process

$$d\theta_t = \beta(t)\left(\tilde{a} - \theta_{t-} - Q_{t-}\right) + \alpha(t)Q_{t-},\ \theta_{0-} = 0 \tag{5.8}$$

$$\Delta\theta_T = \tilde{a} - \theta_{T-}. \tag{5.9}$$

is in $\mathcal{A}$ and maximizes the insider's expected profit (5.2).

(ii) For the insider's strategy (5.8)-(5.9) with Q_t in (5.7), P_t that follows the dynamics (5.5)-(5.6) satisfies the market clearing condition (5.3).

5.2 Candidate Equilibrium

This section includes a derivation of value function, candidate equilibrium and corresponding ODEs for the insider's optimization problem in (5.2). The next result shows relations between the functions $\lambda, \mu, r, s, \Sigma_1, \Sigma_2, \beta, \alpha$ by using the classic Kalman-Bucy filter.

Lemma 5.2.1. *We assume that θ, P, Q satisfy the system* (5.5)-(5.9).

(i) Suppose that the conditional expectations $P_t = \mathbb{E}\left[\tilde{v}\middle|\mathcal{F}_t^M\right]$ and $Q_t = \mathbb{E}\left[\tilde{a} - \theta_t\middle|\mathcal{F}_t^M\right]$ hold. Define $\Sigma_1, \Sigma_2 : [0,T] \to \mathbb{R}$ be

$$\Sigma_1(t) = \mathbb{E}\left[\left(\tilde{a} - \theta_t - Q_t\right)^2\right] \tag{5.10}$$

$$\Sigma_2(t) = \mathbb{E}\left[\left(\tilde{v} - P_t\right)\left(\tilde{a} - \theta_t - Q_t\right)\right] \tag{5.11}$$

Then the terminal time pricing rule coefficient satisfies

$$\lambda(T)=\begin{cases}\text{any constant} & \text{if}\quad \Sigma_1(T-)=0\\ \dfrac{\Sigma_2(T-)}{\Sigma_1(T-)}(\tilde{a}-\theta_{T-}-Q_{T-}) & \text{if}\quad \Sigma_1(T-)\neq 0\end{cases} \tag{5.12}$$

For $t\in[0,T)$, the pricing rule coefficients satisfy

$$\lambda(t)=\frac{\beta(t)\Sigma_2(t)}{\sigma_w^2},\quad \mu(t)=-\alpha(t)\lambda(t) \tag{5.13}$$

$$r(t)=\frac{\beta(t)\Sigma_1(t)}{\sigma_w^2},\quad s(t)=-\alpha(t)(1+r(t)) \tag{5.14}$$

and the differential equations for the dynamics are given by

$$\Sigma_1'(t)=-\sigma_w^2\left(r(t)^2+2r(t)\right),\quad \Sigma_1(0)=\sigma_a^2 \tag{5.15}$$

$$\Sigma_2'(t)=-\sigma_w^2\left(1+r(t)\right)\lambda(t),\quad \Sigma_2(0)=\rho\sigma_a\sigma_v. \tag{5.16}$$

(ii) Suppose that $\lambda,\mu,r,s,\Sigma_1,\Sigma_2$ satisfy (5.10)-(5.14)*, then $P_t=\mathbb{E}\left[\tilde{v}\middle|\mathcal{F}_t^M\right],Q_t=\mathbb{E}\left[\tilde{a}-\theta_t\middle|\mathcal{F}_t^M\right]$ and* (5.10)-(5.11) *hold.*

Proof. (i) We can replace $t-$ by t in (5.5)-(5.8) due to the continuity of the processes θ,P,Q. First, the dynamics of θ,P,Q in (5.5)-(5.9) imply the joint normality of $\tilde{v}-P_t$, $\tilde{a}-\theta_t-Q_t$ and Y_s for $s\in[0,t]$. From this, we observe that $\mathbb{E}\left[(\tilde{v}-P_t)Y_s\right]=\mathbb{E}\left[(\tilde{a}-\theta_t-Q_t)Y_s\right]=0$ for $s\in[0,t]$. Therefore, we conclude that $\tilde{v}-P_t,\tilde{a}-\theta_t-Q_t$ and $(\tilde{v}-P_t)(\tilde{a}-\theta_t-Q_t)$ are independent of $\mathcal{F}_t^M$. Therefore, using this independence, we obtain

$$\mathbb{E}\left[\tilde{v}\left(\tilde{a}-\theta_t-Q_t\right)\middle|\mathcal{F}_t^M\right]=\mathbb{E}\left[\left(\tilde{v}-P_t\right)\left(\tilde{a}-\theta_t-Q_t\right)\middle|\mathcal{F}_t^M\right]=\Sigma_2(t), \tag{5.17}$$

$$\mathbb{E}\left[\left(\tilde{a}-\theta_t\right)\left(\tilde{a}-\theta_t-Q_t\right)\middle|\mathcal{F}_t^M\right]=\mathbb{E}\left[\left(\tilde{a}-\theta_t-Q_t\right)^2\middle|\mathcal{F}_t^M\right]=\Sigma_1(t). \tag{5.18}$$

and Theorem 8.1 in Liptser and Shiryaev(2001), we derive the SDEs for P_t and Q_t :

$$\begin{aligned} dP_t &= \frac{\mathbb{E}\left[\tilde{v}\left(\beta(t)(\tilde{a}-\theta_t-Q_t)+\alpha(t)Q_t\right)\middle|\mathcal{F}_t^M\right]-\alpha(t)P_tQ_t}{\sigma_w^2}\left(dY_t-\alpha(t)Q_tdt\right)\\ &= \frac{\beta(t)\Sigma_2(t)}{\sigma_w^2}\left(dY_t-\alpha(t)Q_tdt\right)\quad \text{for}\quad t\in[0,T). \end{aligned} \tag{5.19}$$

$$\begin{aligned} dQ_t &= \frac{\mathbb{E}\left[\left(\tilde{a}-\theta_t\right)\left(\beta(t)(\tilde{a}-\theta_t-Q_t)+\alpha(t)Q_t\right)\middle|\mathcal{F}_t^M\right]-\alpha(t)Q_t^2}{\sigma_w^2}\left(dY_t-\alpha(t)Q_tdt\right)-\alpha(t)Q_tdt\\ &= \frac{\beta(t)\Sigma_1(t)}{\sigma_w^2}\left(dY_t-\alpha(t)Q_tdt\right)-\alpha(t)Q_tdt\quad \text{for}\quad t\in[0,T). \end{aligned} \tag{5.20}$$

where (5.19) and (5.20) comes from the joint normality relations (5.17) and (5.18), respectively. Therefore, P_t and Q_t satisfies the pricing rule (5.13) and (5.14).

We observe that the expressions (5.13) and (5.14) produce

$$
\begin{aligned}
d(\tilde{a}-\theta_t-Q_t) &= -d\theta_t - dQ_t = -d\theta_t - r(t)(d\theta_t+\sigma_w dW_t) - s(t)Q_t dt \\
&= -(1+r(t))(\beta(t)(\tilde{a}-\theta_t-Q_t)dt+\alpha(t)Q_t dt) + (1+r(t))\alpha(t)Q_t dt - \sigma_w r(t)dW_t \\
&= -(1+r(t))\beta(t)(\tilde{a}-\theta_t-Q_t)dt - \sigma_w r(t)dW_t \\
d(\tilde{v}-P_t) &= -\lambda(t)(d\theta_t+\sigma_w dW_t) - \mu(t)Q_t dt \\
&= -\lambda(t)\beta(t)(\tilde{a}-\theta_t-Q_t)dt - \sigma_w\lambda(t)dW_t
\end{aligned}
\tag{5.21}
$$

Applying Ito's formula and (5.21) to (5.10) and (5.11), we get

$$
\begin{aligned}
\Sigma_1(t) &= \mathbb{E}\Big[\int_0^t 2(\tilde{a}-\theta_s-Q_s)\Big(-(1+r(s))\beta(s)(\tilde{a}-\theta_s-Q_s)ds - \sigma_w r(s)dW_s\Big) + \sigma_w^2 r(s)^2 ds\Big] \\
&= \int_0^t \Big(-2(1+r(s))\beta(s)\Sigma_1(s) + \sigma_w^2 r(s)^2\Big)ds \\
&= -\sigma_w^2\int_0^t (2r(s)+r(s)^2)ds,
\end{aligned}
$$

where the last equality is due to (5.14). The above equation produces (5.15).

Similarly, we obtain

$$
\begin{aligned}
\Sigma_2(t) = \mathbb{E}\Big[\int_0^t &(\tilde{v}-P_s)\Big(-(1+r(s))\beta(s)(\tilde{a}-\theta_s-Q_s)ds - \sigma_w r(s)dW_s\Big) \\
&+ (\tilde{a}-\theta_s-Q_s)\Big(-\lambda(s)\beta(s)(\tilde{a}-\theta_s-Q_s)ds - \sigma_w\lambda(s)dW_s\Big) + \sigma_w^2 r(s)\lambda(s)ds\Big] \\
-\sigma_w^2\int_0^t &(1+r(s))\lambda(s)ds
\end{aligned}
$$

and the above equation produces (5.16).

Moreover, we derive the condition for the terminal pricing rule:

$$
\begin{aligned}
\Delta P_T &= \mathbb{E}\left[\tilde{v}\middle|\mathcal{F}_T^M\right] - \mathbb{E}\left[\tilde{v}\middle|\mathcal{F}_{T-}^M\right] \\
&= \mathbb{E}\left[\tilde{v}-P_{T-}|\mathcal{F}_{T-}^M \cup \sigma(\tilde{a}-\theta_{T-})\right] \\
&= \begin{cases} \mathbb{E}\left[\tilde{v}-P_{T-}|\mathcal{F}_{T-}^M\right] & \text{if} \quad \tilde{a}-\theta_{T-} = Q_{T-} \\ \mathbb{E}\left[\tilde{v}-P_{T-}|\sigma(\tilde{a}-\theta_{T-}-Q_{T-})\right] & \text{if} \quad \tilde{a}-\theta_{T-} \neq Q_{T-} \end{cases} \\
&= \begin{cases} 0 & \text{if} \quad \Sigma_1(T-) = 0 \\ \dfrac{\Sigma_2(T-)}{\Sigma_1(T-)}(\tilde{a}-\theta_{T-}-Q_{T-}) & \text{if} \quad \Sigma_1(T-) \neq 0 \end{cases},
\end{aligned}
\tag{5.22}
$$

where the first inequality is due to $P_{T-}, Q_{T-} \in \mathcal{F}_{T-}^M$ and the second equality is due to the joint normality and independence structure we mentioned above. The third equality holds because $\Sigma_1(T-) = 0$ if and only if $\tilde{a}-\theta_{T-}-Q_{T-} = 0$ almost surely. Therefore, comparing with (5.6) we get (5.12).

(ii) Define $\hat{P}_t := \mathbb{E}\left[\tilde{v} \,|\, \mathcal{F}_t^M\right]$ and $\hat{Q}_t := \mathbb{E}\left[\tilde{a} - \theta_t \,|\, \mathcal{F}_t^M\right]$, and

$$\hat{\Sigma}_1(t) := \mathbb{E}[(\tilde{a} - \theta_t - \hat{Q}_t)^2], \quad \hat{\Sigma}_2(t) := \mathbb{E}[(\tilde{v} - \hat{P}_t)(\tilde{a} - \theta_t - \hat{Q}_t)].$$

In this case, we the market makers' innovation process is

$$\begin{aligned} dY_t - \mathbb{E}\left[Y_t \middle| \mathcal{F}_t^M\right] dt &= dY_t - \mathbb{E}\left[\beta(t)(\tilde{a} - \theta_t - Q_t) + \alpha(t)Q_t \middle| \mathcal{F}_t^M\right] dt \\ &= dY_t - \Big(\beta(t)(\hat{Q}_t - Q_t) + \alpha(t)Q_t\Big)\, dt \end{aligned}$$

Hence, similarly as in part (i) proof, by Theorem 8.1 in Liptser and Shiryaev 2001, we obtain the SDEs for $\hat{P}_t$ and $\hat{Q}_t$:

$$\begin{aligned} d\hat{P}_t &= \frac{\beta(t)\hat{\Sigma}_2(t)}{\sigma_w^2}\left(dY_t - \Big(\beta(t)(\hat{Q}_t - Q_t) + \alpha(t)Q_t\Big)\, dt\right), \\ d\hat{Q}_t &= -\Big(\beta(t)(\hat{Q}_t - Q_t) + \alpha(t)Q_t\Big)\, dt + \frac{\beta(t)\hat{\Sigma}_1(t)}{\sigma_w^2}\left(dY_t - \Big(\beta(t)(\hat{Q}_t - Q_t) + \alpha(t)Q_t\Big)\, dt\right), \end{aligned}$$

and the differential equations for $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$:

$$\hat{\Sigma}_1'(t) = -2\beta(t)\hat{\Sigma}_1(t) - \frac{\beta(t)^2\hat{\Sigma}_1(t)^2}{\sigma_w^2}, \quad \hat{\Sigma}_1(0) = \sigma_{\tilde{a}}^2, \tag{5.23}$$

$$\hat{\Sigma}_2'(t) = -\beta(t)\hat{\Sigma}_2(t) + \frac{\beta(t)^2\hat{\Sigma}_1(t)\hat{\Sigma}_2(t)}{\sigma_w^2}, \quad \hat{\Sigma}_2(0) = \rho\sigma_{\tilde{a}}\sigma_{\tilde{v}}. \tag{5.24}$$

Due to (5.13)-(5.14), we observe that (5.15)-(5.16) and (5.23)-(5.24) are the same system of ODEs. Therefore, we conclude that $\hat{\Sigma}_1 = \Sigma_1$ and $\hat{\Sigma}_2 = \Sigma_2$. Hence, the SDEs (5.5)-(5.7), together with the relations (5.13)-(5.14), produce the SDEs for $\hat{P}_t - P_t$ and $\hat{Q}_t - Q_t$:

$$\begin{aligned} d(\hat{P}_t - P_t) &= -\frac{\beta(t)^2\Sigma_2(t)}{\sigma_w^2}(\hat{Q}_t - Q_t)dt, \\ d(\hat{Q}_t - Q_t) &= -\left(\beta(t) + \frac{\beta(t)^2\Sigma_1(t)}{\sigma_w^2}\right)(\hat{Q}_t - Q_t)dt. \end{aligned}$$

Therefore, we get $P_t = \hat{P}_t$ and $Q_t = \hat{Q}_t$ since $P_0 = Q_0 = \hat{P}_0 = \hat{Q}_0 = 0$. $\square$

5.3 Derivation of Differential Equations

Recall that the insider want to minimize (See (5.2)):

$$\mathbb{E}\left[\int_0^T (\tilde{a} - \theta_{t-})\, dP_t \middle| \mathcal{F}_0^I\right]. \tag{5.25}$$

To derive the HJB equation corresponding to (5.25), it suffices to consider holding processes θ_t with $\theta_0 := 0$ and dynamics

$$d\theta_t = \begin{cases} \theta_t' dt, & t \in (0,T), \\ \tilde{a} - \theta_{T-}, & t = T, \end{cases} \tag{5.26}$$

where θ_t' is an arbitrary order-rate process. From previous calculation, we see that the state process

$$X_t := \tilde{a} - \theta_t - Q_t, \quad t \in [0,T] \tag{5.27}$$

has dynamics

$$dX_t = -\theta_t' dt - r(t)\left(\theta_t' dt + \sigma_w dW_t\right) - s(t) Q_t dt, \quad t \in (0,T). \tag{5.28}$$

We conjecture that the value function corresponding to the infimum of (5.25) has the following quadratic structure:

$$V(t,x,q) := Ix^2 + J(t)xq + K(t), \quad t \in [0,T], \quad x, q \in \mathbb{R}, \tag{5.29}$$

where $I > 0$ is a constant and (J, K) are deterministic functions of time. Assume that all processes are continuous. From the HJB equation, by equating the drift, we get:

$$\begin{aligned}
&(\tilde{a} - \theta_t)\, dP_t + dV(t, X_t, Q_t) \\
&= (X_t + Q_t)\Big(\lambda(t) d\theta_t - \lambda(t)\alpha(t) Q_t dt + \sigma_w \lambda(t) dW_t\Big) + \Big(J'(t) X_t Q_t + K'(t)\Big) dt \\
&\quad + (2IX_t + J(t)Q_t)\Big(- (1 + r(t)) d\theta_t + (1 + r(t))\alpha(t) Q_t dt - \sigma_w r(t) dW_t\Big) \\
&\quad + J(t) X_t \Big(r(t) d\theta_t - (1 + r(t))\alpha(t) Q_t dt + \sigma_w r(t) dW_t\Big) + \sigma_w^2 (I - J(t)) r(t)^2 dt \\
&= \Big((\lambda(t) - 2(1 + r(t))I + J(t) r(t))\, X_t + (\lambda(t) - (1 + r(t))J(t))\, Q_t \Big) d\theta_t \\
&\quad + \Big(I' X^2 + (J(t)(1 + r(t) - \lambda(t))\,\alpha(t) Q_t^2 + \big(J'(t) - \lambda(t)\alpha(t) + (1 + r(t))\alpha(t)(2I - J(t))\big)\, X_t Q_t \Big) dt \\
&\quad + \left(K'(t) + \sigma_w^2 (I - J(t)) r(t)^2\right) dt + \Big(\lambda(t)(X_t + Q_t) + r(t)\left(J(t) X_t - 2IX_t - J(t) Q_t\right) \Big) \sigma_w dW_t
\end{aligned} \tag{5.30}$$

where dP_t and dQ_t are from (5.5) and (5.7) and dX_t is from (5.28). From Ito's lemma, we see that the drift of (5.30) is linear of θ_t', so the slope and intercept must be zero separately:

$$\begin{aligned}
&\lambda(t) - 2(1 + r(t))I + J(t) r(t) = 0, \\
&\lambda(t) - (1 + r(t))J(t) = 0, \\
&J'(t) - \lambda(t)\alpha(t) + (1 + r(t))\alpha(t)(2I - J(t)) = 0, \\
&K'(t) + \sigma_w^2 (I - J(t)) r(t)^2 = 0.
\end{aligned} \tag{5.31}$$

Equivalently,

$$\begin{aligned} J(t) &= \frac{\lambda(t)}{1+r(t)}, \\ I &= \frac{\lambda(t)(1+2r(t))}{2(1+r(t))^2} \text{ (In fact, this is a constant.)}, \\ \alpha &= \frac{J'(t)}{J(t)}, \\ K'(t) &= -\sigma_w^2 (I - J(t)) r(t)^2. \end{aligned} \tag{5.32}$$

Our equilibrium existence proof is based on two dimensional coupled system of ODEs. To derive this system, consider the second equation of (5.32):

$$2I(1+r(t))^2 = \lambda(t)(1+2r(t)) = \frac{\Sigma_2(t)}{\Sigma_1(t)} r(t)(1+2r(t)) \tag{5.33}$$

where the last equality is from (5.13) and (5.14). By using the ODEs in (5.15) and (5.16),

$$\begin{aligned} &\Longrightarrow 4I(1+r(t))r'(t) = \frac{\Sigma_2(t)}{\Sigma_1(t)^2}\sigma_w^2 r(t)^2(1+2r(t)) + \frac{\Sigma_2(t)}{\Sigma_1(t)}(1+4r(t))r'(t) \\ &\Longrightarrow 2\frac{\Sigma_2(t)}{\Sigma_1(t)} \frac{r(t)(1+2r(t))}{1+r(t)} r'(t) = \frac{\Sigma_2(t)}{\Sigma_1(t)^2}\sigma_w^2 r(t)^2(1+2r(t)) + \frac{\Sigma_2(t)}{\Sigma_1(t)}(1+4r(t))r'(t) \\ &\Longrightarrow \Big(\frac{2r(t)(1+2r(t))}{1+r(t)} - (1+4r(t))\Big) r'(t) = \frac{\sigma_w^2 r(t)^2(1+2r(t))}{\Sigma_1(t)} \\ &\Longrightarrow r'(t) = -\frac{\sigma_w^2 r(t)^2(1+r(t))(1+2r(t))}{(1+3r(t))\Sigma_1(t)} \end{aligned} \tag{5.34}$$

Therefore, we have a ODE system: For $t \in (0,T)$,

$$\begin{cases} \Sigma_1'(t) = -\sigma_w^2(r(t)^2 + 2r(t)), & \Sigma_1(0) = \sigma_a^2, \\ r'(t) = -\dfrac{\sigma_w^2 r(t)^2(1+r(t))(1+2r(t))}{(1+3r(t))\Sigma_1(t)}, & r(0) = r_0 \end{cases} \tag{5.35}$$

Here, we can choose the initial condition $r(0) = r_0$ such that $\Sigma_1'(t)$ in (5.35) satisfies the terminal limit $\lim\limits_{t\uparrow T} \Sigma_1(t) = 0$. We will see in Theorem 5.4.1 that the market makers can predict the insider's terminal block order $\Delta\theta_T = \tilde{a} - \theta_{T-} \neq 0$. The next lemma shows that the existence of the above coupled ODE system.

Lemma 5.3.1. *There exists a constant $r_0 \in (0,\infty)$ such that the coupled ODE system* (5.35) *with initial conditions*

$$r(0) = r_0 \quad \textit{and} \quad \Sigma_1(0) = \sigma_a^2, \tag{5.36}$$

have global solutions in $\mathcal{C}^1([0,T])$ that satisfy

$$\Sigma_1(T) := \Sigma_1(T-) = 0, \quad r(T) := r(T-) = 0, \quad r(t), \Sigma_1(t) > 0 \quad \textit{for } t \in [0,T). \tag{5.37}$$

Proof. Define $f(r) := -\frac{\sigma_w^2 r(t)^2(1+r(t))(1+2r(t))}{1+3r(t)}$. Then $r'(t) = \frac{f(r(t))}{\Sigma_1(t)}$. Suppose that $\Sigma_1(t) = g(r(t))$. Then

$$\begin{aligned}
-\sigma_w^2\left(r(t)^2+2r(t)\right) = \Sigma_1'(t) &= \frac{g'(r(t))f(r(t))}{g(r(t))} \\
\Rightarrow \frac{g'(r(t))}{g(r(t))} &= \frac{(1+3r(t))(r(t)+2)}{r(t)(1+r(t))(1+2r(t))} \\
\Rightarrow g(r) &= \frac{\sigma_a^2}{k(r_0)}\cdot k(r), \qquad (5.38)
\end{aligned}$$

where $k(r) = \dfrac{r(t)^2(1+2r(t))^{3/2}}{(1+r(t))^2}$. Therefore, the ODE for $r(t)$ can be written as

$$\begin{aligned}
r'(t) = \frac{f(r(t))}{\Sigma_1(t)} = \frac{f(r)}{g(r)} &= -\frac{k(r_0)}{\sigma_a^2}\frac{f(r)}{k(r)} \\
&= -\frac{k(r_0)}{\sigma_a^2}\sigma_w^2\frac{(1+r(t))^2}{r(t)^2(1+2r(t))^{3/2}}\frac{r(t)^2(1+r(t))(1+2r(t))}{1+3r(t)} \\
&= -\frac{\sigma_w^2}{\sigma_a^2}k(r_0)\frac{(1+r(t))^3}{\sqrt{1+2r(t)}(1+3r(t))}
\end{aligned}$$

Therefore, we get

$$\frac{d}{dt}\Big(F(r(t))\Big) = -\frac{\sigma_w^2}{\sigma_a^2}k(r_0) \text{ where } F(r) := 4\tan^{-1}\left(\sqrt{1+2r(t)}\right) - \frac{\sqrt{1+2r(t)}(3+4r(t))}{(1+r(t))^2}. \qquad (5.39)$$

Observe that

$$F'(r) = \frac{(1+3r(t))\sqrt{1+2r(t)}}{(1+r(t))^3} > 0 \text{ for } r \geq 0.$$

Hence, the function $F : [0,\infty) \longrightarrow [\pi-3, 2\pi)$ is strictly increasing and bijective. Therefore, the inverse function $F^{-1} : [\pi-3, 2\pi) \longrightarrow [0,\infty)$ is well-defined. Therefore, (5.39) produces

$$\begin{aligned}
F(r(t)) &= F(r_0) - \frac{\sigma_w^2}{\sigma_a^2}k(r_0)t \\
\Rightarrow r(t) &= F^{-1}\left(F(r_0) - \frac{\sigma_w^2}{\sigma_a^2}k(r_0)t\right) = 0 \text{ for } t \in [0, \tau(r_0)],
\end{aligned}$$

where $\tau(r_0)$ is defined by $\tau(r_0) := \dfrac{\sigma_a^2}{\sigma_w^2 k(r_0)}(F(r_0) - F(0))$. (Here, we choose the function τ such that $r(\tau(r_0)) = 0$ and $r(t) > 0$ for $0 \leq t < \tau(r_0)$.) This observation and (5.39) imply that $\Sigma_1(\tau(r_0)) = 0$ and $\Sigma_1(t) > 0$ for $0 \leq t < \tau(r_0)$. Finally, for given $r_0 > 0$, we conclude that

$$\begin{cases} r(t) = F^{-1}\left(F(r_0) - \dfrac{\sigma_w^2}{\sigma_a^2}k(r_0)t\right) \\ \Sigma_1(t) = g(r(t)) \end{cases} \quad \text{for } t \in [0, \tau(r_0)] \qquad (5.40)$$

is a unique solution of (5.35) on $t \in [0, \tau(r_0))$, and this solution has the property that

$$r(\tau(r_0)) = \Sigma_1(\tau(r_0)) = 0, r(t) > 0, \Sigma_1(t) > 0 \text{ for } t \in [0, \tau(r_0)). \tag{5.41}$$

Now it remains to check that for any given $T > 0$, there is unique $r_0 \in (0, \infty)$ such that $T = \tau(r_0)$. By the L'Hopital's rule,

$$\begin{aligned} \lim_{r\downarrow 0} \tau(r) &= \frac{\sigma_a^2}{\sigma_w^2} \frac{\lim_{r\downarrow 0} F'(r)}{\lim_{r\downarrow 0} k'(r)} = \lim_{r\downarrow 0} \frac{\sigma_a^2}{\sigma_w^2} \frac{1}{r(t)(r(t)+2)} = \infty \\ \lim_{r\to\infty} \tau(r) &= \lim_{r\to\infty} \frac{\sigma_a^2}{\sigma_w K(r)} (F(r) - F(0)). \end{aligned} \tag{5.42}$$

Note that

$$\tau'(r) = -\frac{\sigma_a^2}{\sigma_w^2} \frac{(1+r(t))(1+3r(t))}{r(t)^3(1+2r(t))^{5/2}} h(r)$$

where $h(r) := \left(4\tan^{-1}\left(\sqrt{1+2r(t)}\right) - \pi + 3\right)(r(t)+2) - 6\sqrt{1+2r(t)}$. Since $h''(r) = \frac{2\sqrt{1+2r(t)}}{(1+r(t))^2} > 0$ for $r \geq 0$, the initial values $h(0) = 0$ and $h'(0) = 1 > 0$ imply that $h(r) > 0$ for $r > 0$. Therefore, we get $\tau'(r) < 0$ for $r > 0$ and we conclude that $\tau : (0, \infty) \longrightarrow \mathbb{R}$ is strictly decreasing. This and (5.42) imply that τ ensures the unique existence of r_0 which satisfies $\tau(r_0) = T$. □

The combination Lemma 7.1.1 and Lemma 7.1.2 in appendix is alternative proof of the existence of the ODE system (5.35).

5.4 Verification

The next result is our main contribution and the theorem ensures the existence of an equilibrium in the sense of Definition 5.1.2.

Theorem 5.4.1. *Let $\rho \in (0, 1]$, let $r(t)$ and $\Sigma_1(t)$ be as in Lemma 5.3.1, and define the constant $I := \frac{\rho\sigma_v}{\sigma_a} \frac{r_0(1+2r_0)}{2(1+r_0)^2} > 0$ and the functions*

$$\begin{aligned} \lambda(t) &= 2I \frac{\left(1+r(t)\right)^2}{1+2r(t)}, \quad t \in [0, T], \\ \beta(t) &= \frac{\sigma_w^2 r_0^2 (1+2r_0)^{\frac{3}{2}}}{\sigma_{\tilde{a}}^2 (1+r_0)^2} \cdot \frac{(1+r(t))^2}{r(t)(1+2r(t))^{\frac{3}{2}}}, \quad t \in [0, T), \\ \alpha(t) &= \frac{\sigma_w^2 r_0^2 (1+2r_0)^{\frac{3}{2}}}{\sigma_{\tilde{a}}^2 (1+r_0)^2} \cdot \frac{(1+r(t))^2}{(1+3r(t))(1+2r(t))^{\frac{3}{2}}}, \quad t \in [0, T], \\ \mu(t) &= -\frac{\rho\sigma_w^2 \sigma_{\tilde{v}} r_0^3 (1+2r_0)^{\frac{5}{2}}}{\sigma_{\tilde{a}}^3 (1+r_0)^4} \cdot \frac{(1+r(t))^4}{(1+3r(t))(1+2r(t))^{\frac{5}{2}}}, \quad t \in [0, T], \\ s(t) &= -\frac{\sigma_w^2 r_0^2 (1+2r_0)^{\frac{3}{2}}}{\sigma_{\tilde{a}}^2 (1+r_0)^2} \cdot \frac{(1+r(t))^3}{(1+3r(t))(1+2r(t))^{\frac{3}{2}}}, \quad t \in [0, T]. \end{aligned} \tag{5.43}$$

Then the functions $\lambda, \mu, r, s, \beta, \alpha$ constitute an equilibrium in Definition 5.1.2. Additionally, as we shall see, the process Q_t satisfies $Q_t = \mathbb{E}\left[\tilde{a} - \theta_t \middle| \mathcal{F}_t^M\right]$.

Remark 5.4.2. We allow jumps at any time for admissible strategies. We will see that jumping before the terminal time is sub-optimal (See (5.8)-(5.9)). The optimal strategy has a block order at the terminal time, but the jump size is Q_{T-}, which is predictable to the market maker since $Q_{T-} \in \mathcal{F}_{T-}^M$ (See Proposition 5.5.1-(4).)

Remark 5.4.3. However, this doesn't mean that the market maker knows the exact value of $\tilde{a}$. The market maker knows only the terminal block order at the terminal time (i.e., $\tilde{a} - \theta_{T-}$). The insider still has some private information for the true value $\tilde{a}$ (or $\tilde{v}$ in some sense) until the end. (See Proposition 5.5.1-(5).)

Proof. We start by defining the function

$$\begin{aligned} \Sigma_2(t) :=& \frac{\lambda(t)\Sigma_1(t)}{r(t)} \\ =& \frac{\rho\sigma_{\tilde{a}}\sigma_{\tilde{v}}}{r_0\sqrt{1+2r_0}} r(t)\sqrt{1+2r(t)}, \end{aligned} \tag{5.44}$$

for $t \in [0, T]$. Since $r(t)$ is continuous on $t \in [0, T]$ and $r(T) = 0$, $\Sigma_2(t)$ is continuous on $t \in [0, T]$ with $\Sigma_2(T) = 0$.

We divide the proof into three steps: The first step shows that θ in (5.8)-(5.9) is optimal. Step 2 verifies whether the θ is admissible strategy. Finally, we verify the market clearing condition (5.3).

Step 1/3: In this step, we show that the value function corresponding to (5.2) is greater than or equal to $V(t, x, q)$ in (5.29) for the coefficient functions $J, K : [0, T] \to \mathbb{R}$ defined by

$$\begin{aligned} J(t) &:= \frac{\lambda(t)}{1 + r(t)}, \\ K(t) &:= \sigma_w^2 \int_t^T \big(I - J(u)\big) r(u)^2 du. \end{aligned} \tag{5.45}$$

We let X_t be as in (5.27). Then, for any $\theta \in \mathcal{A}$, we have

$$\begin{aligned} d[X, X]_t^c &= \big(1 + r(t)\big)^2 d[\theta, \theta]_t^c + \sigma_w^2 r(t)^2 dt + 2\sigma_w \big(1 + r(t)\big) r(t) d[\theta, W]_t^c, \\ d[X, Q]_t^c &= -r(t)\big(1 + r(t)\big) d[\theta, \theta]_t^c - \sigma_w^2 r(t)^2 dt - \sigma_w r(t)\big(2r(t) + 1\big) d[\theta, W]_t^c. \end{aligned} \tag{5.46}$$

For $t \in [0,T)$, Itô's formula gives

$$
\begin{aligned}
&\int_{[0,t]} (\tilde{a} - \theta_{u-}) dP_u + V(t, X_t, Q_t) \\
&= \int_{[0,t]} (\tilde{a} - \theta_{u-}) \lambda(u) \big(d\theta_u + \sigma_w dW_u - \alpha(u) Q_u du \big) \\
&\quad + I\tilde{a}^2 + K(0) + \int_{[0,t]} \big(J'(u) X_u Q_u + K'(u) \big) du \\
&\quad + \int_{[0,t]} \Big(\big(2IX_{u-} + J(u) Q_{u-}\big) dX_u + J(u) X_{u-} dQ_u + I d[X,X]^c_u + J(u) d[X,Q]^c_u \Big) \\
&\quad + \sum_{0 \le u \le t} \Big(\Delta V(u, X_u, Q_u) - \big(2IX_{u-} + J(u) Q_{u-}\big) \Delta X_u - J(u) X_{u-} \Delta Q_u \Big) \\
&= I\tilde{a}^2 + K(0) + \int_0^t \Big(\big(\lambda(u) - 2r(u) I + r(u) J(u)\big) X_u + \big(\lambda(u) - r(u) J(u)\big) Q_u \Big) \sigma_w dW_u \\
&\quad + \tfrac{1}{2} \int_0^t \lambda(u) d[\theta, \theta]^c_u + \tfrac{1}{2} \sum_{0 \le u \le t} \lambda(u) (\Delta \theta_u)^2,
\end{aligned} \tag{5.47}
$$

where we used $\tilde{a} - \theta_{u-} = X_{u-} + Q_{u-}$, (5.43), (5.45), and (5.46) to obtain the second equality. In (5.47), the stochastic integral with respect to dW_u is a martingale on $t \in [0,T]$ because of the square integrability requirement in Definition 5.1.2. The Lemma 5.3.1 and $\lambda(t)$ in (5.43) show that $\lambda(t)$ is positive on the interval $[0,T]$. Therefore, for any $\theta \in \mathcal{A}$, taking a limit $t \uparrow T$ and expectation with respect to the insider's filtration produce

$$
\mathbb{E}\left[\int_{[0,T)} (\tilde{a} - \theta_{u-}) dP_u + \lim_{t \uparrow T} V(t, X_t, Q_t) \Big| \mathcal{F}^I_0 \right] \ge I\tilde{a}^2 + K(0). \tag{5.48}
$$

By using $r(T-) = 0$, $J(t)$ in (5.45), and $\lambda(t)$ in (5.43) we obtain[1]

$$
J(T-) = \lambda(T) = 2I > 0. \tag{5.49}
$$

Combining (5.49) with (5.6) produces the inequality

$$
\begin{aligned}
(\tilde{a} - \theta_{T-}) \Delta P_T &= \lambda(T) X^2_{T-} + \lambda(T) X_{T-} Q_{T-} \\
&= 2I X^2_{T-} + J(T-) X_{T-} Q_{T-} \\
&\ge \lim_{t \uparrow T} V(t, X_t, Q_t).
\end{aligned} \tag{5.50}
$$

We combine (5.48) and (5.50) to conclude that

$$
\begin{aligned}
\inf_{\theta \in \mathcal{A}} \mathbb{E}\left[\int_{[0,T]} (\tilde{a} - \theta_{u-}) dP_u \Big| \mathcal{F}^I_0 \right] &= \inf_{\theta \in \mathcal{A}} \mathbb{E}\left[\int_{[0,T)} (\tilde{a} - \theta_{u-}) dP_u + (\tilde{a} - \theta_{T-}) \Delta P_T \Big| \mathcal{F}^I_0 \right] \\
&\ge \inf_{\theta \in \mathcal{A}} \mathbb{E}\left[\int_{[0,T)} (\tilde{a} - \theta_{u-}) dP_u + \lim_{t \uparrow T} V(t, X_t, Q_t) \Big| \mathcal{F}^I_0 \right] \\
&\ge I\tilde{a}^2 + K(0).
\end{aligned} \tag{5.51}
$$

[1] The functions $\lambda(t)$ and $J(t)$ are continuous functions on $t \in [0,T]$, so we have $\lambda(T-) = \lambda(T)$ and $J(T-) = J(T)$. Except for β, the other functions $(r, \Sigma_1, \Sigma_2, \mu, s, \alpha)$ are also continuous on $t \in [0,T]$.

If θ satisfies (5.8), then $\Delta\theta_t = 0$ and $[\theta,\theta]_t = 0$ for all $t \in [0,T)$. Consequently, the second inequality in (5.51) becomes equality.

Moreover, from the Kalman-Bucy result in Lemma 5.2.1, the expression (5.10) and the boundary condition $\Sigma_1(T-) = 0$ give

$$0 = \lim_{t\uparrow T} \mathbb{E}\left[(\tilde{a} - \theta_t - Q_t)^2\right] \geq \mathbb{E}\left[(\tilde{a} - \theta_{T-} - Q_{T-})^2\right]$$

where the inequality is from the Fatou's lemma. Finally, we get

$$X_{T-} = \tilde{a} - \theta_{T-} - Q_{T-} = 0, \text{ almost surely,} \tag{5.52}$$

implying that the first inequality in (5.51) becomes an equality.

In conclusion, θ_t in (5.8)-(5.9) is an optimal solution and satisfies

$$\mathbb{E}\left[\int_{[0,T]} (\tilde{a} - \theta_{u-})dP_u \middle| \mathcal{F}_0^I\right] = I\tilde{a} + K(0).$$

Step 2/3: We need to confirm that the optimal θ in (5.8)-(5.9) is an admissible strategy. Let (P_t, Q_t, θ_t) be the solution of SDE (5.5), (5.7) and (5.8), respectively. The existence of the solution is guaranteed by the continuity of $\lambda, \mu, r, s, \beta, \alpha$ on the interval $[0,T)$. Before we check the admissibility, observe that for $t \in [0,T)$,

$$\begin{aligned}
\Sigma_3(t) &:= \mathbb{E}\left[Q_t^2\right] \\
&= \mathbb{E}\left[\int_0^t (2Q_u dQ_u + \sigma_w^2 r(u)^2 du)\right] \\
&= \mathbb{E}\left[\int_0^t \left(2r(u)\beta(u)Q_u(\tilde{a} - \theta_u - Q_u)du - 2\alpha(u)Q_u^2 du + 2r(u)Q_u dW_u + \sigma_w^2 r(u)^2 du\right)\right] \\
&= \int_0^t \left(-2\alpha(u)\Sigma_3(u)du + \sigma_w^2 r(u)du\right)
\end{aligned} \tag{5.53}$$

where the last equality holds since

$$\mathbb{E}\left[Q_t(\tilde{a} - \theta_t - Q_t)\right] = \mathbb{E}[Q_t \mathbb{E}[\tilde{a} - \theta_t - Q_t | \mathcal{F}_t^M]] = 0, \quad t \in [0,T) \tag{5.54}$$

and (5.13)-(5.14). By taking derivative with respect to time t, we have

$$\Sigma_3'(t) = -2\alpha(t)\Sigma_3(t) + \sigma_w^2 r(t)^2,\ \Sigma_3(0) = 0. \tag{5.55}$$

Likewise, for the function $\Sigma_4(t) := \mathbb{E}[(\tilde{v} - P_t)^2]$, Itô's lemma produce

$$\begin{aligned}
\Sigma_4(t) &= \mathbb{E}\left[\int_0^t \big(2(\tilde{v} - P_u)(-dP_u) + \sigma_w^2 \lambda(u)^2 du\big)\right] \\
&= \mathbb{E}\left[\int_0^t \big(-2\lambda(u)\beta(u)(\tilde{a} - \theta_u - Q_u)(\tilde{v} - P_u)du - 2\lambda(u)(\tilde{v} - P_u)dW_u + \sigma_w^2 \lambda(u)^2 du\big)\right] \\
&= \int_0^t \big(-2\lambda(u)\beta(u)\Sigma_1(u)du + \sigma_w^2 \lambda(u)^2 du\big) \\
&= -\int_0^t \sigma_w^2 \lambda(u)^2 du
\end{aligned}$$

Hence,

$$\Sigma_4'(t) = -\sigma_w^2 \lambda(t)^2 dt, \ \Sigma_4(0) = \sigma_v^2. \tag{5.56}$$

Given that we can see (5.43), the function r, α, λ are bounded on the interval $[0, T]$. Hence,

$$\sup_{t\in[0,T)} \Sigma_i(t) < \infty \tag{5.57}$$

holds for all $i \in \{1, 2, 3, 4\}$. (See (5.15),(5.16),(5.55) and (5.56).) Moreover, the explicit expressions of β and Σ_1 in (5.43) and (5.38) imply that

$$\begin{aligned}
\sup_{t\in[0,T)} \beta(t)^2\Sigma_1(t) &= \sup_{t\in[0,T)} \frac{\sigma_w^4 r_0^2 (1 + 2r_0)^{3/2}}{\sigma_a^2 (1 + r_0)^2} \frac{(1 + r(t))^2}{(1 + 2r(t))^{3/2}} < \infty \\
\sup_{t\in[0,T)} \beta(t)r(t) &= \sup_{t\in[0,T)} \frac{\sigma_w^2 r_0^2 (1 + 2r_0)^{3/2}}{\sigma_a^2 (1 + r_0)^2} \frac{(1 + r(t))^2}{(1 + 2r(t))^{3/2}} < \infty.
\end{aligned} \tag{5.58}$$

Now, we can verify the conditions in Definition 5.1.1 to check the admissibility of θ in (5.8)-(5.9).

(i) For $t \in [0, T)$,

$$\begin{aligned}
\mathbb{E}\left[\theta_t^2\right] &= \mathbb{E}\left[\left(\int_0^t (\beta(u)(\tilde{a} - \theta_u - Q_u) + \alpha(u)Q_u)du\right)^2\right] \\
&= C \cdot \int_0^t \big(\beta(u)^2\Sigma_1(u) + \alpha(u)^2\Sigma_3(u)\big)\, du
\end{aligned}$$

where a constant C is independent of t.

(ii) (5.57) ensures that $\mathbb{E}\left[\int_0^T Q_t^2 dt\right]$ is finite. Define a process $Z_t := e^{\int_0^t \alpha(u)du} Q_t$ for $t \in [0, T)$. By Itô's formula,

$$\begin{aligned}
dZ_t &= e^{\int_0^t \alpha(u)du} dQ_t + \alpha(t)e^{\int_0^t \alpha(u)du} Q_t dt \\
&= e^{\int_0^t \alpha(u)du} r(t) \left(\beta(t)(\tilde{a} - \theta_t - Q_t)dt + \sigma_w dW_t\right)
\end{aligned}$$

where $\beta(t)(\tilde{a} - \theta_t - Q_t)dt + \sigma_w dW_t = Y_t - \mathbb{E}[Y_t|\mathcal{F}_t^M] \in \mathcal{F}_t^M$. Therefore, Z_t is a martingale with respect to $\mathcal{F}_t^M$. Moreover, (5.58) and the boundedness of $r(t)$ and $\alpha(t)$ on $t \in [0, T)$

imply that Z_t is a square integrable martingale:

$$\begin{aligned}
&\sup_{t\in[0,T)} \mathbb{E}[Z_t^2] \\
&= \sup_{t\in[0,T)} \mathbb{E}\left[\left(\int_0^t e^{\int_0^u \alpha(v)dv} r(u)(\beta(u)(\tilde{a}-\theta_u - Q_u)du + r(u)dW_u)\right)^2\right] \\
&= \sup_{t\in[0,T)} \left(\int_0^t e^{\int_0^u \alpha(v)dv} r(u)^2\beta(u)^2\Sigma_1(u)du + \left(e^{\int_0^u \alpha(v)dv} r(u)\right)^2 du\right) < \infty
\end{aligned}$$

Hence, the limit $\lim_{t\uparrow T} Z_t$ exists almost surely and finite and so is Q_{T-}.

(iii) Similar to (ii), the process $(P_t)_{t\in[0,T)}$ is a square integrable martingale uniformly bounded in $\mathcal{L}_2(\mathbb{P})$, implying the existence of P_{T-}.

Step 3/3: The market clearing condition in Definition 5.1.2 has yet to be proven. For $t \in [0,T)$, which follows from the Kalman-Bucy result in Lemma 5.2.1. We need only to check that (5.3) holds for $t = T$:

$$\begin{aligned}
\mathbb{E}[\tilde{v}|\mathcal{F}_T^M] &= \mathbb{E}[\tilde{v}|\mathcal{F}_{T-}^M \cup \sigma(\tilde{a}-\theta_{T-})] \\
&= \mathbb{E}[\tilde{v}|\mathcal{F}_{T-}^M] \\
&= P_{T-} \qquad (5.59) \\
&= P_{T-} + \lambda(T)(\tilde{a}-\theta_{T-} - Q_{T-}) \\
&= P_T
\end{aligned}$$

where the first equality is due to

$$\mathcal{F}_T^M = \sigma\left(\mathcal{F}_{T-}^M \cup \sigma(\Delta Y_T)\right) = \sigma\left(\mathcal{F}_{T-}^M \cup \sigma(\Delta \theta_T)\right) = \sigma\left(\mathcal{F}_{T-}^M \cup \sigma(\tilde{a}-\theta_{T-})\right),$$

the second equality is due to $X_{T-} = 0$ and $Q_{T-} = \mathcal{F}_{T-}^M$, the fourth equality is due to $X_{T-} = \tilde{a} - \theta_{T-} - Q_{T-} = 0$, and the last equality comes from (5.6).

□

5.5 Properties and Numerics of Equilibrium

From now on, we discuss new and desirable features produced by the insider's terminal trading constraint $\theta_T = \tilde{a}$.

Proposition 5.5.1. In equilibrium with the setting of Theorem 5.4.1, we have the following:

1. The scaled order autocorrelation of all trader's aggregate holdings is positive

$$\lim_{h\downarrow 0} \frac{1}{h} \frac{\mathbb{E}[(Y_t - Y_{t-h})(Y_{t+h}-Y_t)]}{\sqrt{\mathbb{V}[Y_t - Y_{t-h}]\mathbb{V}[Y_{t+h}-Y_t]}} = \alpha(t)\left(\frac{\alpha(t)\Sigma_3(t)}{\sigma_w^2} + r(t)\right) > 0, \quad t \in (0,T), \qquad (5.60)$$

where the positive function $\Sigma_3(t)$ is defined in (5.53). Moreover, the process Q_t is mean reverting.

2. The price-impact function $\lambda(t)$ is decreasing for $t \in [0,T]$.

3. For $\tilde{a} \neq 0$, the mapping $[0,T) \ni t \mapsto \frac{1}{\tilde{a}}\mathbb{E}[\theta'_t|\mathcal{F}_0^I]$ is U shaped where θ'_t is the insider's equilibrium order-rate process[2]

$$\theta'_t := \beta(t)(\tilde{a} - \theta_t) + \big(\alpha(t) - \beta(t)\big)Q_t, \quad t \in [0,T), \tag{5.61}$$

   and the terminal block order satisfies $0 < \frac{1}{\tilde{a}}\mathbb{E}[\Delta\theta_T|\mathcal{F}_0^I] < 1$.

4. $\Delta\theta_T = Q_{T-} \neq 0$ almost surely, and $Q_{T-} \in \mathcal{F}_{T-}^M$ and $\Delta P_T = 0$.

5. $P_T \neq \mathbb{E}[\tilde{v}|\mathcal{F}_0^I]$ almost surely.

Proof. (1): To simplify, we give the proof for $\sigma_w := 1$. Based on dQ_t in (5.7), the dynamics of θ'_t in (5.61) have the form

$$d\theta'_t = A_t dt + \big(\alpha(t) - \beta(t)\big)r(t)dW_t, \quad t \in [0,T), \tag{5.62}$$

for some integrable process A_t. For $h > 0$, we have

$$\begin{aligned}
\mathbb{E}[(Y_t - Y_{t-h})(Y_{t+h} - Y_t)] &= \mathbb{E}\left[\left(\int_{t-h}^t \theta'_s ds + W_t - W_{t-h}\right)\left(\int_t^{t+h} \theta'_s ds + W_{t+h} - W_t\right)\right] \\
&= \mathbb{E}\left[\left(\int_{t-h}^t \theta'_s ds + W_t - W_{t-h}\right)\int_t^{t+h} \theta'_s ds\right] \\
&= \mathbb{E}\left[\int_{t-h}^t \theta'_s ds \int_t^{t+h} \theta'_s ds\right] + \int_t^{t+h} \mathbb{E}[(W_t - W_{t-h})\theta'_s]ds.
\end{aligned}$$

The first term above can be approximated as $h^2\mathbb{E}[(\theta'_t)^2]$ for $h > 0$ close to 0. For the second term, we let $s \in [t, t+h]$ and compute

$$\begin{aligned}
\mathbb{E}[(W_t - W_{t-h})\theta'_s] &= \mathbb{E}\left[(W_t - W_{t-h})\left(\theta'_0 + \int_0^s \Big(A_u du + \big(\alpha(u) - \beta(u)\big)r(u)dW_u\Big)\right)\right] \\
&= \mathbb{E}\left[(W_t - W_{t-h})\int_{t-h}^s \Big(A_u du + \big(\alpha(u) - \beta(u)\big)r(u)dW_u\Big)\right] \\
&= \int_{t-h}^t \big(\alpha(u) - \beta(u)\big)r(u)du + O(h^{3/2}),
\end{aligned}$$

[2]It turns out that $\mathbb{E}[\theta'_t|\mathcal{F}_0^I]$ is linear in $\tilde{a}$; hence, the ratio $\mathbb{E}[\theta'_t|\mathcal{F}_0^I]/\tilde{a}$ does not depend on $\tilde{a}$.

where we used the following observations:

$$
\begin{aligned}
&\mathbb{E}\left[(W_t - W_{t-h})\left(\int_{t-h}^{t}\Big(A_u du + \big(\alpha(u)-\beta(u)\big)r(u)dW_u\Big)\right)\right]\\
&= \mathbb{E}\left[(W_t - W_{t-h})\int_{t-h}^{t} A_u du\right] + \int_{t-h}^{t}\big(\alpha(u)-\beta(u)\big)r(u)du\\
&= \int_{t-h}^{t}\big(\alpha(u)-\beta(u)\big)r(u)du + O(h^{3/2}),\\
&\mathbb{E}\left[(W_t - W_{t-h})\left(\int_{t}^{s} A_u du + \big(\alpha(u)-\beta(u)\big)r(u)dW_u\right)\right]\\
&= \mathbb{E}\left[(W_t - W_{t-h})\int_{t}^{s} A_u du\right]\\
&= O(h^{3/2}).
\end{aligned}
$$

Therefore, we obtain the scaled autocorrelation:

$$
\begin{aligned}
\lim_{h\downarrow 0}\frac{1}{h}\frac{\mathbb{E}[(Y_t - Y_{t-h})(Y_{t+h}-Y_t)]}{\sqrt{\mathbb{V}[Y_t - Y_{t-h}]\mathbb{V}[Y_{t+h}-Y_t]}} &= \lim_{h\downarrow 0}\frac{\mathbb{E}[(Y_t - Y_{t-h})(Y_{t+h}-Y_t)]}{h^2}\\
&= \mathbb{E}[(\theta_t')^2] + \big(\alpha(t)-\beta(t)\big)r(t)\\
&= \beta(t)^2\Sigma_1(t) + \alpha(t)^2\Sigma_3(t) + \big(\alpha(t)-\beta(t)\big)r(t)\\
&= \alpha(t)\left(\alpha(t)\Sigma_3(t) + r(t)\right) > 0,
\end{aligned}
$$

where the third equality is due to (5.10),(5.11) and (5.54), and the last equality is due to $r(t) = \beta(t)\Sigma_1(t)$ from (5.14).

(2): Using the ODE (5.35) and the expressions of $\Sigma_1(t)$ and $\lambda(t)$ in (5.40) and (5.43), we obtain

$$
\lambda'(t) = -\frac{2\rho\sigma_w^2\sigma_{\tilde v}r_0^3(1+2r_0)^{\frac{5}{2}}}{\sigma_{\tilde a}^3(1+r_0)^4}\frac{r(t)(1+r(t))^4}{(1+3r(t))(1+2r(t))^{\frac{5}{2}}} < 0 \quad \text{for} \quad t \in (0,T).
$$

(3): Let $f, g : [0,T) \to \mathbb{R}$ be defined as

$$
f(t) := \frac{\mathbb{E}[\theta_t|\mathcal{F}_0^I]}{\tilde a}, \quad g(t) := \frac{\mathbb{E}[Q_t|\mathcal{F}_0^I]}{\tilde a}.
$$

The SDEs (5.7) and (5.8) and the relation $s(t) = -(1+r(t))\alpha(t)$ from (5.14) produce the following ODEs for f and g:

$$
\begin{aligned}
f'(t) &= \beta(t)\big(1 - f(t) - g(t)\big) + \alpha(t)g(t), \quad f(0) = 0,\\
g'(t) &= r(t)\beta(t)\big(1 - f(t) - g(t)\big) - \alpha(t)g(t), \quad g(0) = 0.
\end{aligned}
$$

We can find explicit expressions of the unique solution of the above ODE system by using (5.43):

$$
\begin{aligned}
f(t) &= 1 - \frac{(1+2r(t))^{\frac{3}{2}}}{r_0\sqrt{1+2r_0}(1+r(t))} + \frac{(1+r_0-r_0^2)(1+2r(t))}{r_0(1+2r_0)(1+r(t))},\\
g(t) &= \frac{1+2r(t)}{1+r(t)}\left(\frac{1+r(t)-r(t)^2}{r_0\sqrt{1+2r_0}\sqrt{1+2r(t)}} - \frac{1+r_0-r_0^2}{r_0(1+2r_0)}\right).
\end{aligned}
$$

These expressions give

$$
\begin{aligned}
f''(0) &= -\frac{\sigma_w^4 r_0^4}{\sigma_{\tilde{a}}^4(1+3r_0)} < 0, \\
f''(T-) &= \frac{3\sigma_w^4 r_0^3(1+2r_0)^2}{\sigma_{\tilde{a}}^4(1+r_0)^4}\Big(\sqrt{1+2r_0}-1+r_0(r_0-1)\Big) > 0,
\end{aligned}
\tag{5.63}
$$

where the second equality is due to $r(T-)=0$ and the second inequality is due to

$$
\begin{cases}
\big(\sqrt{1+2r_0}-1+r_0(r_0-1)\big)\Big|_{r_0=0} = 0, \\
\dfrac{d}{dr_0}\big(\sqrt{1+2r_0}-1+r_0(r_0-1)\big)\Big|_{r_0=0} = 0, \\
\dfrac{d^2}{dr_0^2}\big(\sqrt{1+2r_0}-1+r_0(r_0-1)\big) = 2 - \dfrac{1}{(1+2r_0)^{\frac{3}{2}}} > 0, \quad \text{for} \quad r_0 \in (0,\infty).
\end{cases}.
$$

Let $H : [0,\infty)^2 \to \mathbb{R}$ be defined as

$$
H(x,y) := \left(\frac{\sigma_{\tilde{a}}^6(1+r_0)^6(1+2r(t))^{\frac{7}{2}}(1+3r(t))^5}{\sigma_w^6 r_0^5(1+2r_0)^3(1+r(t))^5} f'''(t)\right)\Bigg|_{r(t)=x,\ r_0=y}.
$$

Direct computations produce for $0 < x \le y$:

$$
\begin{cases}
H(x,x) = (1+x)(1+3x)^3(1+2x+4x^2)\sqrt{1+2x} > 0, \\
H_y(x,x) = \dfrac{2(1+3x)(1+x(20+x(4+3x)(22+3x(7+4x))))}{\sqrt{1+2x}} > 0, \\
H_{yy}(x,y) = \dfrac{2(1+3y(3+5y))(11+x(55+x(83+33x)))}{(1+2y)^{\frac{3}{2}}} > 0,
\end{cases},
$$

where H_y and H_{yy} denote partial derivatives. These inequalities imply that

$$
H(x,y) > 0 \quad \text{for} \quad 0 < x \le y. \tag{5.64}
$$

Since $0 < r(t) \le r_0$ for $t \in [0,T)$, the definition of H and (5.64) produce

$$
0 < H(r(t),r_0) = \frac{\sigma_{\tilde{a}}^6(1+r_0)^6(1+2r(t))^{\frac{7}{2}}(1+3r(t))^5}{\sigma_w^6 r_0^5(1+2r_0)^3(1+r(t))^5} f'''(t) \quad \text{for} \quad t \in [0,T),
$$

and we obtain

$$
f'''(t) > 0 \quad t \in [0,T). \tag{5.65}
$$

Combining (5.63) and (5.65), we conclude that the map $t \mapsto \dfrac{\mathbb{E}[\theta_t' | \mathcal{F}_0^I]}{\tilde{a}} = f'(t)$ is U shaped for $t \in [0,T)$.

Finally, to prove $0 < \frac{1}{\tilde{a}}\mathbb{E}[\Delta\theta_T | \mathcal{F}_0^I] < 1$, we observe

$$
\begin{aligned}
\frac{1}{\tilde{a}}\mathbb{E}[\Delta\theta_T | \mathcal{F}_0^I] &= 1 - f(T-) \\
&= \frac{\sqrt{1+2r_0}-1+r_0(r_0-1)}{r_0(1+2r_0)},
\end{aligned}
\tag{5.66}
$$

where the first equality uses $\Delta\theta_T = \tilde{a} - \theta_{T-}$ and the definition of f, and the second equality uses the explicit expression of f and $r(T-) = 0$. The conclusion follows because $\frac{\sqrt{1+2r_0}-1+r_0(r_0-1)}{r_0(1+2r_0)} \in (0,1)$ for $r_0 > 0$.

(4): $\Delta\theta_T = Q_{T-}$ is from (5.52) and $\Delta P_T = 0$ is from (5.59). We obtain $Q_{T-} \neq 0$ a.s. because $\dfrac{1}{\tilde{a}}\mathbb{E}[Q_{T-}|\mathcal{F}_0^I] = \dfrac{1}{\tilde{a}}\mathbb{E}[\Delta\theta_T|\mathcal{F}_0^I] \neq 0$ by part (3).

(5): The explicit solution of (5.56) is given by

$$\Sigma_4(t) = \frac{\rho^2\sigma_{\tilde{v}}^2\sqrt{1+2r_0}}{(1+r_0)^2}\frac{(1+r(t))^2}{\sqrt{1+2r(t)}} + (1-\rho^2)\sigma_{\tilde{v}}^2. \tag{5.67}$$

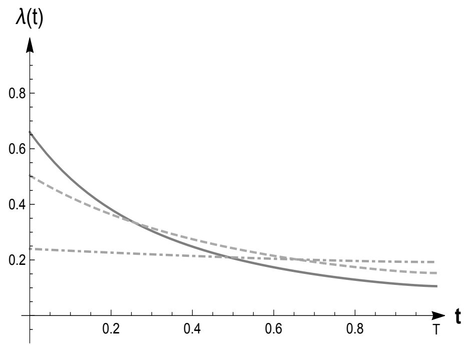
Because $\tilde{v} - \rho\dfrac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}$ is independent of $\mathcal{F}_t^I$, we obtain for $t \in [0,T)$ that

$$\begin{aligned}\mathbb{E}\left[\left(\rho\tfrac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a} - P_t\right)^2\right] &= \mathbb{E}\left[(\tilde{v} - P_t)^2\right] - \mathbb{E}\left[\left(\tilde{v} - \rho\tfrac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}\right)^2\right] \\ &= \Sigma_4(t) - (1-\rho^2)\sigma_{\tilde{v}}^2 \\ &= \frac{\rho^2\sigma_{\tilde{v}}^2\sqrt{1+2r_0}}{(1+r_0)^2}\frac{(1+r(t))^2}{\sqrt{1+2r(t)}}. \end{aligned} \tag{5.68}$$

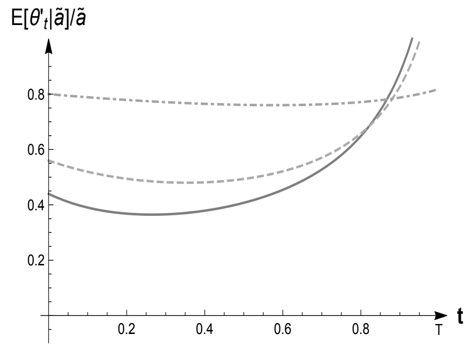
The expression in (5.68) and Fatou's lemma produce

$$\mathbb{E}\left[\left(\rho\frac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a} - P_T\right)^2\right] \geq \frac{\rho^2\sigma_{\tilde{v}}^2\sqrt{1+2r_0}}{(1+r_0)^2} > 0,$$

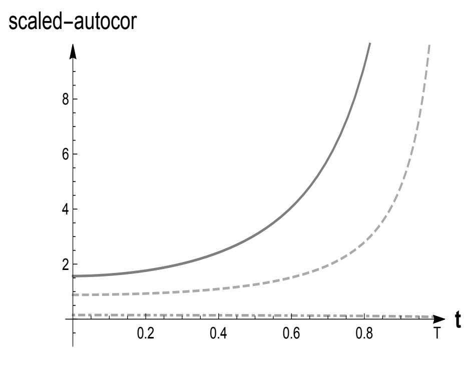
where we have used $r(T-) = 0$ and $P_T = P_{T-}$. □



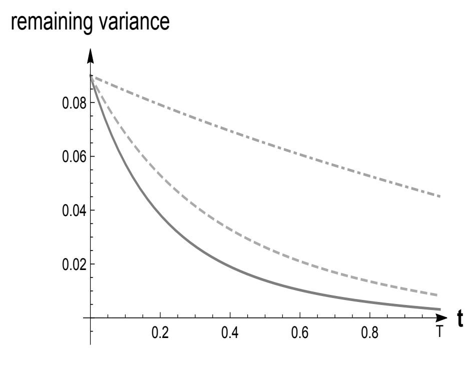
1A: price impact $\lambda(t)$ in (5.43)



1B: expected order rate $\dfrac{1}{\tilde{a}}\mathbb{E}[\theta'_t|\sigma(\tilde{a})]$



1C: scaled autocorrelation in (5.60)



1D: remaining variance $\mathbb{E}[(\rho\frac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a} - P_t)^2]$ in (5.68)

The above figures illustrate the price impact function $\lambda(t)$, the insider's expected order rates, the scaled autocorrelation of aggregate holdings, and the remaining unconditional variance of

$$P_T - \mathbb{E}[\tilde{v}|\mathcal{F}_0^I] = P_T - \rho\frac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}.$$

The parameters are $\sigma_w := 1$, $\sigma_{\tilde{v}} := 1$, $\rho := 0.3$, $T := 1$, and $\sigma_{\tilde{a}} := 5$ (—), $\sigma_{\tilde{a}} := 3$ (- - -), and $\sigma_{\tilde{a}} := 1$ (- · -).

5.6 Convergence of discrete time equilibrium to continuous time equilibrium

As in [17], we can consider that the discrete time system in Chapter 4 converges to the continuous time system in Chapter 5. We haven't been able to prove this yet, so we are leaving it for future work. Let $\sigma_w^2 = 1$ for simplicity. If we can show that the limits of $\dfrac{\beta_n}{\Delta}$ and $\dfrac{\alpha_n}{\Delta}$ are exists and indeed converge to $\beta(t)$ and $\alpha(t)$ respectively, then all other functions in discrete time converge to corresponding one in the continuous time setting. For example,

$$\begin{aligned} r_n &= \frac{(1-\beta_n)\beta_n\Sigma_{n-1}^{(1)}}{\beta_n^2\Sigma_{n-1}^{(1)} + \sigma_w^2\Delta} = \frac{(1-\beta_n)\frac{\beta_n}{\Delta}\Sigma_{n-1}^{(1)}}{\beta_n\frac{\beta_n}{\Delta}\Sigma_{n-1}^{(1)} + 1} \xrightarrow{\Delta\to 0} r(t) = \beta(t)\Sigma_1(t) \\ \lambda_n &= \frac{\beta_n\Sigma_{n-1}^{(3)}}{\beta_n^2\Sigma_{n-1}^{(1)} + \sigma_w^2\Delta} = \frac{\frac{\beta_n}{\Delta}\Sigma_{n-1}^{(3)}}{\beta_n\frac{\beta_n}{\Delta}\Sigma_{n-1}^{(1)} + 1} \xrightarrow{\Delta\to 0} \lambda(t) = \beta(t)\Sigma_3(t) \end{aligned}$$

Figure 5.1 shows graphs of r and λ in the discrete equilibrium and the continuous equilibrium. The green line is the continuous time solution when the terminal time $T = 1$, and the blue line is the discrete time solution when the time step is $\Delta t = \dfrac{1}{20}, \dfrac{1}{100}$ and $\dfrac{1}{2000}$, respectively.

5.7 Generalization to nonzero means of $\tilde{v}$ and $\tilde{a}$

Up to now, we considered the case that both the asset value and the terminal trading target have zero mean. In this section, we are going to see the case which is $\tilde{v}$ and $\tilde{a}$ have nonzero mean. Let $\tilde{a}_0 \sim N(0, \sigma_a^2)$, $\tilde{v}_0 \sim N(0, \sigma_v^2)$, $\tilde{a} := m + \tilde{a}_0$ and $\tilde{v} := n + \tilde{v}_0$ where $m, n \neq 0$. (The reason why we can assume $\tilde{a}$ and $\tilde{v}$ like this is that $m + \tilde{a}_0 \overset{d}{=} \tilde{a}$ and $n + \tilde{v}_0 \overset{d}{=} \tilde{v}$ hold). Let $\cdot^{(0)}$ be a process of the mean zero case, for example, $\theta^{(0)}$ is the proess of the insider with $\tilde{a}_0$ and $\tilde{v}_0$. Processes which don't have script (0) are the case with $\tilde{a}$ and $\tilde{v}$.

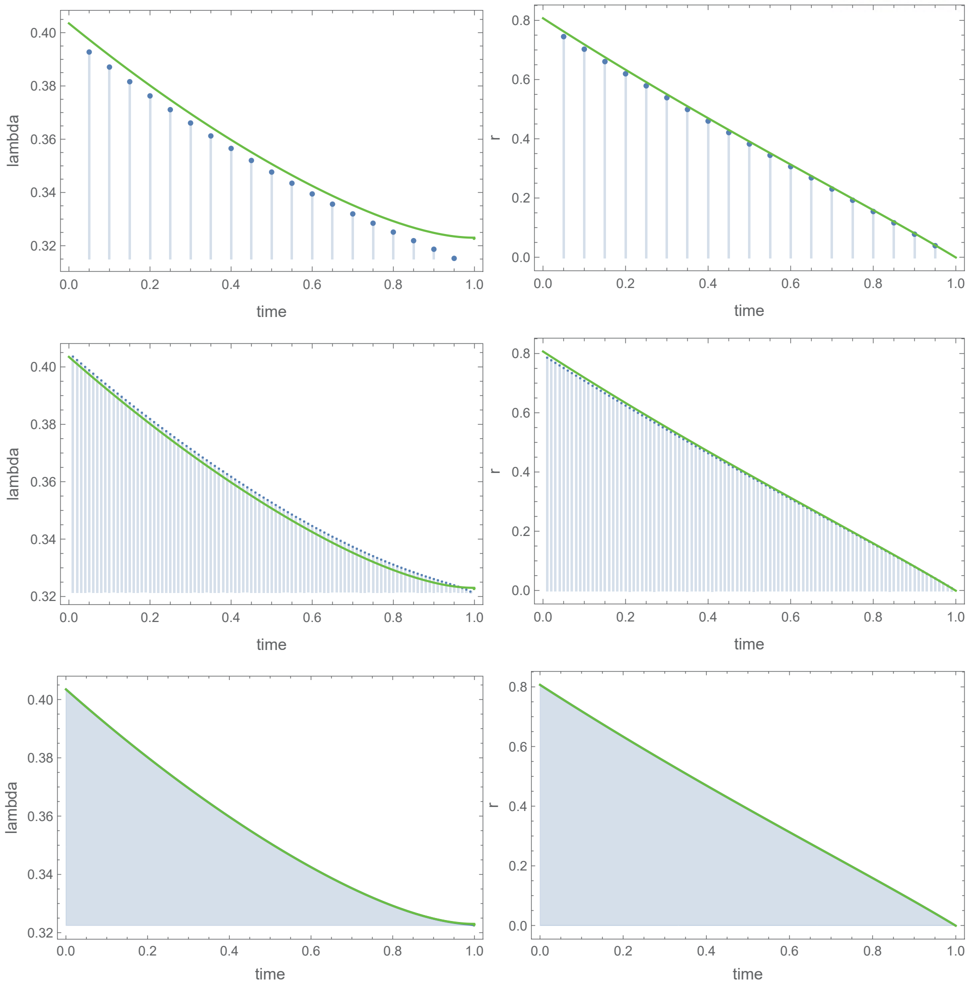


Figure 5.1: Graphs of r and λ in the discrete equilibrium and the continuous equilibrium. The terminal time is $T = 1$ and the initial values are $r(0) = 1$ and $\Sigma_1(0) = 0.8070001$.

Then, the following linear system constitutes an equilibrium:

$$
\begin{aligned}
&\theta_t = f(t) + \theta_t^{(0)} \text{ where } f(t) = m\left(1 - e^{-\int_0^T \alpha(s)ds}\right) \\
&\Delta\theta_T = \tilde{a} - \theta_{T-} = \Delta\theta_T^{(0)} + me^{-\int_0^T \alpha(s)ds} \\
&dY_t = d\theta_t + dW_t \quad (\text{For simplicity, let } \sigma_w^2 = 1.) \\
&dP_t = \lambda(t)\left(dY_t - f'(t)dt\right) + \mu(t)\left(Q_t - m + f(t)\right)dt \\
&dQ_t = r(t)dY_t + s(t)\left(Q_t - m + f(t)\right)dt - (1 + r(t))f'(t)dt
\end{aligned} \tag{5.69}
$$

where $\lambda, \mu, r, s, \beta, \alpha$ are same as in *Theorem* 5.4.1. Note that

$$\mathbb{E}\left[\tilde{v}\middle|\mathcal{F}_t^M\right] = \mathbb{E}\left[n + \tilde{v}_0\middle|\mathcal{F}_t^M\right] = n + P_t^{(0)} \tag{5.70}$$

$$\mathbb{E}\left[\tilde{a} - \theta_t\middle|\mathcal{F}_t^M\right] = \mathbb{E}\left[m + \tilde{a}_0 - f(t) - \theta_t^{(0)}\middle|\mathcal{F}_t^M\right] = m - f(t) + Q_t^{(0)} \tag{5.71}$$

$$\begin{aligned}\mathbb{E}\left[\tilde{v}\left(\tilde{a}_0-\theta_t^{(0)}-Q_t^{(0)}\right)\middle|\mathcal{F}_t^M\right]&=\mathbb{E}\left[(n+\tilde{v}_0)\left(\tilde{a}_0-\theta_t^{(0)}-Q_t^{(0)}\right)\middle|\mathcal{F}_t^M\right]\\&=\mathbb{E}\left[\left(\tilde{v}_0-P_t^{(0)}\right)\left(\tilde{a}_0-\theta_t^{(0)}-Q_t^{(0)}\right)\middle|\mathcal{F}_t^M\right]=\Sigma_2(t)\end{aligned}\tag{5.72}$$

$$\begin{aligned}\mathbb{E}\left[(\tilde{a}-\theta_t)\left(\tilde{a}_0-\theta_t^{(0)}-Q_t^{(0)}\right)\middle|\mathcal{F}_t^M\right]&=\mathbb{E}\left[\left(m+\tilde{a}_0-f(t)-\theta_t^{(0)}\right)\left(\tilde{a}_0-\theta_t^{(0)}-Q_t^{(0)}\right)\middle|\mathcal{F}_t^M\right]\\&=\mathbb{E}\left[\left(\tilde{a}_0-\theta_t^{(0)}\right)\left(\tilde{a}_0-\theta_t^{(0)}-Q_t^{(0)}\right)\middle|\mathcal{F}_t^M\right]=\Sigma_1(t)\end{aligned}\tag{5.73}$$

where $\Sigma_1(t)$ and $\Sigma_2(t)$ are defined in 5.10 and 5.11. We derive SDEs for $\mathbb{E}\left[\tilde{v}\middle|\mathcal{F}_t^M\right]$ and $\mathbb{E}\left[\tilde{a}-\theta_t\middle|\mathcal{F}_t^M\right]$:

$$\begin{aligned}d\mathbb{E}\left[\tilde{v}\middle|\mathcal{F}_t^M\right]&=\beta(t)\mathbb{E}\left[\tilde{v}\left(\tilde{a}_0-\theta_t^{(0)}-Q_t^{(0)}\right)\middle|\mathcal{F}_t^M\right]\left(dY_t-f'(t)dt-\alpha(t)Q_t^{(0)}dt\right)\\&=\beta(t)\Sigma_2(t)\left(dY_t-f'(t)dt\right)-\beta(t)\Sigma_1(t)\alpha(t)\left(m-f(t)-\mathbb{E}[\tilde{a}-\theta_t|\mathcal{F}_t^M]\right)dt\end{aligned}\tag{5.74}$$

$$\begin{aligned}&d\mathbb{E}\left[\tilde{a}-\theta_t\middle|\mathcal{F}_t^M\right]\\&\quad=\beta(t)\mathbb{E}\left[(\tilde{a}-\theta_t)\left(\tilde{a}_0-\theta_t^{(0)}-Q_t^{(0)}\right)\middle|\mathcal{F}_t^M\right]\left(dY_t-f'(t)dt-\alpha(t)Q_t^{(0)}dt\right)-f'(t)dt-\alpha(t)Q_t^{(0)}dt\\&\quad=\beta(t)\Sigma_2(t)dY_t-(1+\beta(t))\,\Sigma_2(t))f'(t)dt-\beta(t)\Sigma_2(t)\alpha(t)\left(Q_t-m+f(t)\right)dt\end{aligned}\tag{5.75}$$

As in Lemma 5.2.1, we can get $P_t = \mathbb{E}[\tilde{v}|\mathcal{F}_t^M]$ and $Q_t = \mathbb{E}[\tilde{a} - \theta_t|\mathcal{F}_t^M]$ hold.

For $t \in [0, T)$,

$$
\begin{aligned}
&\int_{[0,t]} (\tilde{a} - \theta_{u-}) dP_u + V(t, X_t, Q_t) \\
&= I\tilde{a}^2 + K(0) + \int_{[0,t]} (J'(u) X_{u-} Q_{u-} + K'(u)) du \\
&\quad + \int_{[0,t]} (\tilde{a} - \theta_{u-}) \big(\lambda(u)(d\theta_u - f'(u)du + dW_u) + \mu(u)(Q_u - m + f(u))du\big) \\
&\quad + \int_{[0,t]} (2IX_{u-} + J(u)Q_{u-}) \big(-(1 + r(u))d\theta_u - r(u)dW_u - s(u)(Q_u - m + f(u))du + (1 + r(u))f'(u)du\big) \\
&\quad + \int_{[0,t]} J(u)X_{u-} \big(r(u)d\theta_u + r(u)dW_u + s(u)(Q_u - m + f(u))du - (1 + r(u))f'(u)du\big) \\
&\quad + \int_{[0,t]} (Id[X, X]^c_u + J(u)d[X, X]^c_u) \\
&\quad + \sum_{0 \le u \le t} (\Delta V(u, X_u, Q_u) - (2IX_{u-} + J(u)Q_{u-})\Delta X_u - J(u)X_{u-}\Delta Q_u) \\
&= \int_{[0,t]} ((\lambda(u) - 2(1 + r(u))I + J(u)r(u))X_{u-} + (\lambda(u) - J(u)(1 + r(u)))Q_{u-}) \, d\theta_u \\
&\quad + \int_{[0,t]} ((-\lambda(u) + 2I(1 + r(u)) - J(u)(1 + r(u)))X_{u-} + (-\lambda(u) + J(u)(1 + r(u)))Q_{u-}) \, f'(t)dt \\
&\quad + \int_{[0,t]} \Big((-\mu(u)m + \mu(u)f + J(u)s(u)m - J(u)s(u)f)Q_{u-} + (K'(u) + (I - J(u))r(u)^2) \\
&\qquad\qquad + (-m\mu(u) + \mu(u)f + 2Ims(u) - 2Is(u)f - mJ(u)s(u) + s(u)J(u)f)X_{u-} \\
&\qquad\qquad + (\mu(u) + J'(u) - 2Is(u) + s(u)J(u))X_{u-}Q_{u-} + (\mu(u) - s(u)J(u))(Q_{u-})^2\Big) du \\
&\quad + \int_{[0,t]} ((\lambda(u) - 2Ir(u) + J(u)r(u))X_{u-} + (\lambda(u) - r(u)J(u))Q_{u-}) \, dW_u \\
&\quad + \frac{1}{2} \int_{[0,t]} \lambda(u) d[\theta, \theta]^c_u + \frac{1}{2} \sum_{0 \le u \le t} \lambda(u)(\Delta\theta_u)^2 \\
&= I\tilde{a}^2 + K(0) + \int_0^t ((\lambda(u) - 2Ir(u) + J(u)r(u))X_{u-} + (\lambda(u) - J(u)r(u))Q_{u-}) \, dW_u \\
&\quad + \frac{1}{2} \int_0^t \lambda(u) d[\theta, \theta]^c_u + \frac{1}{2} \sum_{0 \le u \le t} \lambda(u)(\Delta\theta_u)^2
\end{aligned}
$$

Taking $t \uparrow T$ produces the limit

$$
\begin{aligned}
&\int_{[0,T)} (\tilde{a} - \theta_{u-}) dP_u + V(T, X_{T-}, Q_{T-}) \\
&= \lim_{t \uparrow T} \left(\int_{[0,t]} (\tilde{a} - \theta_{u-}) dP_u + V(t, X_t, Q_t) \right) \\
&= I\tilde{a}^2 + K(0) + \int_0^t ((\lambda(u) - 2Ir(u) + J(u)r(u))X_{u-} + (\lambda(u) - J(u)r(u))Q_{u-}) \, dW_u \\
&\quad + \frac{1}{2} \int_0^t \lambda(u) d[\theta, \theta]^c_u + \frac{1}{2} \sum_{0 \le u \le t} \lambda(u)(\Delta\theta_u)^2
\end{aligned}
$$

By taking expectations, we get

$$\mathbb{E}\left[\int_{[0,T)}(\tilde{a}-\theta_{u-})dP_u+V(T,X_{T-},Q_{T-})\middle|\mathcal{F}_0^I\right]\geq I\tilde{a}^2+K(0)$$

Note that

$$\begin{aligned}
(\tilde{a}-\theta_{T-})\Delta P_T &= (\tilde{a}-\theta_{T-})\Delta P_T^{(0)}\\
&= (\tilde{a}-\theta_{T-})\lambda(T)(\tilde{a}_0-\theta_{T-}^{(0)}-Q_{T-}^{(0)})\\
&= \lambda(T)(\tilde{a}-\theta_{T-})\left(\tilde{a}-m-\theta_{T-}+f(T-)-Q_{T-}+m-f(T-)\right)\\
&= \lambda(T)(\tilde{a}-\theta_{T-})(\tilde{a}-\theta_{T-}-Q_{T-})\\
&= \lambda(T)(\tilde{a}-\theta_{T-}-Q_{T-})^2+\lambda(T)(\tilde{a}-\theta_{T-}-Q_{T-})Q_{T-}\\
&\geq V(T,X_{T-},Q_{T-})
\end{aligned}$$

Therefore,

$$\begin{aligned}
I\tilde{a}^2+K(0) &\leq \inf_{\theta\in\mathcal{A}}\mathbb{E}\left[\int_{[0,T)}(\tilde{a}-\theta_{u-})dP_u+V(T,X_{T-},Q_{T-})\middle|\mathcal{F}_0^I\right]\\
&\leq \inf_{\theta\in\mathcal{A}}\mathbb{E}\left[\int_{[0,T)}(\tilde{a}-\theta_{u-})dP_u+(\tilde{a}-\theta_{T-})\Delta P_T\middle|\mathcal{F}_0^I\right]\\
&= \inf_{\theta\in\mathcal{A}}\mathbb{E}\left[\int_{[0,T]}(\tilde{a}-\theta_{u-})dP_u\middle|\mathcal{F}_0^I\right]
\end{aligned}$$

Since $\theta_t=0$ and $[\theta,\theta]_t=0$ for all $t\in[0,T)$, the first inequality becomes an equality. Moreover, the second inequality becomes an equality if we can check that $X_{T-}=\tilde{a}-\theta_{T-}-Q_{T-}=0$ almost surely. As in the previous chapter, we get $X_{T-}=\tilde{a}-\theta_{T-}-Q_{T-}=0$ a.s. because

$$0=\lim_{t\uparrow T}\mathbb{E}\left[(\tilde{a}-\theta_t-Q_t)^2\right]\geq\mathbb{E}\left[(\tilde{a}-\theta_{T-}-Q_{T-})^2\right].$$

Chapter 6

Conclusion

This dissertation studies the market microstructure theory based on the Kyle (1985) model in both discrete time and continuous time sense. The main point of our dissertation is that we establish the existence of an equilibrium if the insider have to satisfy hard trading constraint at the end of the market. As far as we can tell that there is no equilibrium existence proof in the settings of [2] and [1] when the insider has a soft or hard trading target. Moreover, we construct an equilibrium with an insider with a terminal target at the end of trading. Based on theoritical derivation, we show some characteristics in our equilibrium:
(i) The market impact function is time-decreasing,
(ii) The scaled autocorrelation of all trader's order is positive,
(iii) The insider has U-shaped trading patterns over daytime,
(iv) The insider's terminal block order is predictable by market makers,
(v) The equilibrium market price at the terminal time is different from the insider's initial expectation of the true value of the asset.
Since we have fully derived the equilibrium in a continuous time model, one might ask why the discrete time model is important. The reason lies in §4.4. In the continuous time model, we have not checked whether it is natural to use the state variable Q in the linear structure (i.e., about the uniqueness of the linear structure). Therefore, we could say the §4.4 could explain the reason why we use the state variable Q in the continuous time model. Therefore, one of my future work is to prove the uniqueness of the linear structure in the continuous time model. Moreover, to prove the convergence from the discrete time model to the continuous time model will also one of the important future works.

Chapter 7

Appendix

7.1 Alternative proof for the existence of ODE (5.35)

This is the alternative proof for the existence of ODE (5.35)(Let $\sigma_w = 1$ for simplicity.):

Lemma 7.1.1. *Let the initial value $r(0) > 0$ fixed. Then there exists $\tau \in \mathbb{R}^+$ such that (5.35) has a unique solution on $[0, \tau)$, and the solution satisfies*

$$\begin{aligned} &\Sigma_1(t) > 0 \text{ and } r(t) > 0, \text{ for } t \in [0, \tau) \\ &\lim_{t\uparrow\tau} \Sigma_1(t) = \lim_{t\uparrow\tau} r(t) = 0 \end{aligned} \tag{7.1}$$

Proof. We can check that if $r > 0$ and $\Sigma_1 > 0$,

$$\left(\frac{r}{\Sigma_1}\right)' = \frac{r^2(1 + 4r + r^2)}{(1 + 3r)\Sigma_1^2} > 0, \quad \left(\frac{\Sigma_1}{r^3}\right)' = \frac{1 + 2r + 3r^2}{r^2 + 3r^3} > 0 \tag{7.2}$$

Starting from $r(0) > 0$ and $\Sigma_1(0) > 0$, the solution of (5.35) decreases since $r' < 0$ and $\Sigma_1' < 0$. Let

$$\tau := \inf\{t > 0 : \Sigma_1(t) = 0 \text{ or } r(t) = 0\} \tag{7.3}$$

From (7.2),

$$\frac{\Sigma_1(0)}{r(0)^3} r(t)^3 < \Sigma_1(t) < \frac{\Sigma_1(0)}{r(0)} r(t) \text{ for } t \in (0, \tau) \tag{7.4}$$

Therefore, we have three possibilities:

(a) $\tau < \infty$ and $\lim_{t\uparrow\tau} \Sigma_1(t) = \lim_{t\uparrow\tau} r(t) = 0$.
(b) $\tau = \infty$ and $\lim_{t\uparrow\tau} \Sigma_1(t) > 0$.
(c) $\tau = \infty$ and $\lim_{t\uparrow\tau} \Sigma_1(t) = \lim_{t\uparrow\tau} r(t) = 0$. The case (b) is impossible because $\Sigma_1''(t) > 0$ for

$t \in [0,\tau)$, and (b) implies $\lim_{t\uparrow\tau} \Sigma_1'(t) = 0$, which implies $\lim_{t\uparrow\tau} r(t) = 0$, which is contradicts to (7.4). Moreover, we have

$$\left(\frac{r(t)^2}{\Sigma_1(t)}\right)' = \frac{(1-r)r^2}{(1+3r)\Sigma_1^2} > 0 \text{ for } 0 < r < 1 \tag{7.5}$$

If (c) is true, then there exists $t_0 > 0$ such that $\frac{r(t)^2}{\Sigma_1(t)}$ increases on $[t_0, \infty)$. Then $r'(t) < -\frac{r(t_0)^2}{\Sigma_1(t_0)}$ for $t \in [t_0, \infty)$, which contradicts $\tau = \infty$ (since r will touch 0 in finite time). Therefore, (a) is the only possibility, and the proof is complete. □

Lemma 7.1.2. *In Lemma 7.1.1, τ is a function of $r(0)$. The following inequality holds:*

$$\frac{\Sigma_1(0)}{r(0)^2 + 2r(0)} < \tau(r(0)) < \frac{\Sigma_1(0)}{r(0)} + \frac{\Sigma_1(0)}{\sqrt{r(0)}} \tag{7.6}$$

In particular,

$$\lim_{r(0)\downarrow 0} \tau(r(0)) = \infty, \qquad \lim_{r(0)\uparrow\infty} \tau(r(0)) = 0. \tag{7.7}$$

Proof. Since r is decreasing function,

$$\Sigma_1'(t) = -r(t)^2 - 2r(t) > -r(0)^2 - 2r(0)$$

By the definition of τ and the above inequality, we obtain

$$\frac{\Sigma_1(0)}{r(0)^2 + 2r(0)} < \tau(r(0)).$$

Define $\tau_1 := \inf\{t > 0 : r(t) = 1\}$ and observe that

$$\left(\frac{r(t)^{3/2}}{\Sigma_1(t)}\right)' = \frac{(1+5r(t))r(t)^{5/2}}{2(1+r(t))\Sigma_1(t)^2} > 0.$$

Therefore,

$$\begin{aligned} r'(t) &= -\frac{r(t)^{3/2}}{\Sigma_1(t)} \frac{\sqrt{r(t)}(1+r(t))(1+2r(t))}{1+3r(t)} < -\frac{r(0)^{3/2}}{\Sigma_1(0)} \text{ for } t \in [0, \tau_1] \\ \Longrightarrow \tau_1 &< \frac{r(0)}{\frac{r(0)^{3/2}}{\Sigma_1(0)}} = \frac{\Sigma_1(0)}{\sqrt{r(0)}} \end{aligned} \tag{7.8}$$

Finally, using (7.4) and (7.5) produces

$$\begin{aligned} &\frac{r(t)^2}{\Sigma_1(t)} > \frac{r(\tau_1)^2}{\Sigma_1(\tau_1)} > \frac{r(0)}{\Sigma_1(0)} r(\tau_1) = \frac{r(0)}{\Sigma_1(0)} \\ &\Longrightarrow r'(t) = -\frac{r(t)^2(1+r(t))(1+2r(t))}{(1+3r(t))\Sigma_1(t)} < -\frac{r(0)}{\Sigma_1(0)} \\ &\Longrightarrow \tau - \tau_1 < \frac{\Sigma_1(0)}{r(0)} \end{aligned} \tag{7.9}$$

Combining (7.8) and (7.9) yields another side of the inequality. □

7.2 Insider with full information case

It is very natural to question the case which the insider has the information about both the exact asset value $\tilde{v}$ and the terminal trading constraint $\theta_T = \tilde{a}$. In this section we consider the insider with filtration $\mathcal{F}_t^I := \sigma\left(\tilde{v}, \tilde{a}, (W_s)_{s\in[0,t]}\right)$. First, we conjecture that state variables of the insider's opimitzation problem are $\tilde{a} - \theta_t - Q_t$, Q_t and $\tilde{v} - P_t$ and conjecture that the pricing rules follow dynamics (simplified as, $\sigma_w^2 = 1$) :

$$dY_t = d\theta_t + dW_t, \tag{7.10}$$

$$dP_t = \lambda(t)dY_t + \mu(t)Q_{t-}dt, \tag{7.11}$$

$$dQ_t = r(t)dY_t + s(t)Q_{t-}dt, \tag{7.12}$$

$$d\theta_t = \Big(\beta(t)\left(\tilde{a} - \theta_{t-} - Q_{t-}\right) + \alpha(t)Q_{t-} + \gamma(t)\left(\tilde{v} - P_{t-}\right)\Big)dt, \tag{7.13}$$

$$\Delta\theta_T = \tilde{a} - \theta_{T-}. \tag{7.14}$$

Furthermore, let's assume that all processes are continuous in $t \in (0, T)$. As in the Lemma 5.2.1, $dP_t = \mathbb{E}\left[\tilde{v}\middle|\mathcal{F}_t^M\right]$ and $dQ_t = \mathbb{E}\left[\tilde{a} - \theta_t\middle|\mathcal{F}_t^M\right]$ produce

$$\begin{aligned} dP_t &= (\beta(t)\Sigma_2(t) + \gamma(t)\Sigma_3(t))\left(dY_t - \alpha(t)Q_t dt\right), \\ dQ_t &= (\beta(t)\Sigma_1(t) + \gamma(t)\Sigma_2(t))\left(dY_t - \alpha(t)Q_t dt\right) - \alpha(t)Q_t dt. \end{aligned} \tag{7.15}$$

where

$$\begin{aligned} \Sigma_1(t) &:= \mathbb{E}[(\tilde{a} - \theta_t - Q_t)^2], \\ \Sigma_2(t) &:= \mathbb{E}[(\tilde{v} - P_t)\left(\tilde{a} - \theta_t - Q_t\right)], \\ \Sigma_3(t) &:= \mathbb{E}[(\tilde{v} - P_t)^2]. \end{aligned} \tag{7.16}$$

Therefore, we have following relations:

$$\begin{aligned} \lambda(t) &= \beta(t)\Sigma_3(t) + \gamma(t)\Sigma_2(t), \quad \mu(t) = -\alpha(t)\lambda(t), \\ r(t) &= \beta(t)\Sigma_1(t) + \gamma(t)\Sigma_3(t), \quad s(t) = -\alpha(t)(1 + r(t)). \end{aligned} \tag{7.17}$$

Observe that the state processes have following dynamics with the relations (7.17):

$$\begin{aligned} d(\tilde{a} - \theta_t - Q_t) &= -d\theta_t - dQ_t = -d\theta_t - r(t)d\theta_t - r(t)dW_t - s(t)Q_t dt \\ &= -(1 + r(t))d\theta_t - r(t)dW_t - s(t)Q_t dt \\ &= -(1 + r(t))\beta(t)(\tilde{a} - \theta - Q_t)dt - (1 + r(t))\gamma(\tilde{v} - P_t)dt - r(t)dW_t \\ d(\tilde{v} - P_t) &= -dP_t = -\lambda(t)(d\theta + dW_t) - \mu(t)Q_t dt \\ &= \lambda(t)(\beta(t)(\tilde{a} - \theta_t - Q_t) + \gamma(t)(\tilde{v} - P_t))dt - \lambda(t)dW_t \end{aligned} \tag{7.18}$$

By using (7.15)-(7.18), the variance and covariance of market maker's estimation follow

$$\begin{aligned} d\Sigma_1(t) &= \Big(- 2r(t) - r(t)^2\Big) dt \\ d\Sigma_2(t) &= -\Big(1 + r(t)\Big)\lambda(t)dt \\ d\Sigma_3(t) &= -\lambda(t)^2 dt \end{aligned} \tag{7.19}$$

The insider want to maximize her expected profit subject to the constraint $\theta_T = \tilde{a}$. Since $\mathcal{F}_0^I = \sigma(\tilde{v}, \tilde{a})$, the left-hand-side of (5.1) becomes

$$\sup_{\theta\in\mathcal{A}} \mathbb{E}\left[(\tilde{v} - P_T)\theta_T + \int_{[0,T]} \theta_{t-} dP_t \middle| \mathcal{F}_0^I\right] = \tilde{v}\tilde{a} - \inf_{\theta\in\mathcal{A}} \mathbb{E}\left[\int_{[0,T]} (\tilde{a} - \theta_{t-}) dP_t \middle| \mathcal{F}_0^I\right] \tag{7.20}$$

with the admissible set $\mathcal{A}$ is as in Definition 5.1.1. Define $X_t := \tilde{a} - \theta_t - Q_t$ and $Z_t = \tilde{v} - P_t$. Suppose that the processes are continuous and let

$$\begin{aligned} &\inf_{\theta\in\mathcal{A}} \mathbb{E}\left[\int_{[0,T]} (\tilde{a} - \theta_{t-}) dP_t \middle| \mathcal{F}_0^I\right] \\ &= A(t)X_t^2 + B(t)Q_t^2 + C(t)Z_t^2 + D(t)X_tQ_t + E(t)X_tZ_t + F(t)Q_tZ_t + G(t) \qquad (7.21) \\ &=: V(t, X_t, Q_t, Z_t) \qquad (7.22) \end{aligned}$$

The HJB equation implies that

$$\begin{aligned} &(\tilde{a} - \theta_t)\, dP_t + dV \\ &= d\theta\Big(\big(\lambda(t) - 2(1 + r(t))A(t) + r(t)D(t) - \lambda(t)E(t)\big)X_t \\ &\qquad + \big(\lambda(t) - (1 + r(t))D(t) + 2r(t)B(t) - \lambda(t)F(t)\big)Q_t \\ &\qquad + \big(- (1 + r(t))E(t) + r(t)F(t) - 2\lambda(t)C(t)\big) Z_t\Big) \\ &\quad + dW_t\Big(\big(\lambda(t) - 2r(t)A(t) + r(t)D(t) - \lambda(t)E(t)\big)X_t + \\ &\qquad \big(\lambda(t) - r(t)D(t) + 2r(t)B(t) - \lambda(t)F(t)\big)Q_t + \big(- r(t)E(t) + r(t)F(t) - 2\lambda(t)C(t)\big) Z_t\Big) \\ &\quad + dt\Big(A'(t)X_t^2 + \big(B'(t)(1 + r(t))\alpha(t)D(t) - 2(1 + r(t))\alpha(t)B(t) + \alpha(t)\lambda(t)F(t)\big)Q_t^2 \\ &\qquad + \big(D'(t) + 2(1 + r(t))\alpha(t)A(t) - (1 + r(t))\alpha(t)D(t) + \alpha(t)\lambda(t)E(t)\big)X_tQ_t \\ &\qquad + \big(F'(t) + (1 + r(t))\alpha(t)E(t) - (1 + r(t))\alpha(t)F(t) + 2\alpha(t)\lambda(t)C(t)\big)Q_tZ_t \\ &\qquad + \big(G'(t) + r(t)^2A(t) + r(t)^2B(t) + \lambda(t)^2C(t) - r(t)^2D(t) + r(t)\lambda(t)E(t) - r(t)\lambda(t)F(t)\big) \\ &\qquad + C'(t)Z_t^2 + E'(t)X_tZ_t\Big) \end{aligned}$$

Equating coefficients for X_t, Q_t and Z_t in the $d\theta_t$ term and coefficients for $X_t^2, Q_t^2, Z_t^2, X_tZ_t, X_tQ_t$, Q_tZ_t and deterministic terms in the dt term to zero gives

$$\begin{aligned}
&A'(t) = 0,\ C'(t) = 0,\ E'(t) = 0,\\
&B'(t) - 2\alpha(t)B(t) = 0,\\
&D'(t) - \alpha(t)D(t) = 0,\\
&F'(t) - \alpha(t)F(t) = 0,\\
&B(t) = \frac{(1+r(t))^2}{r(t)^2}A(t) + \frac{\lambda(t)^2}{r(t)^2}C + \frac{\lambda(1+r(t))}{r(t)^2}E(t) - \frac{\lambda(t)(1+2r(t))}{2r(t)^2},\\
&D(t) = \frac{2(1+r(t))}{r(t)}A(t) + \frac{\lambda(t)}{r(t)}E(t) - \frac{\lambda(t)}{r(t)},\\
&F(t) = \frac{1+r(t)}{r(t)}E(t) + \frac{2\lambda(t)}{r(t)}C(t).
\end{aligned} \tag{7.23}$$

Note that $A(t) = \dfrac{\lambda(t)(1+2r(t))}{(1+r(t))^2}, D(t) = \dfrac{\lambda(t)}{1+r(t)}$ and $B = C = E = F \equiv 0$ solves the equations (7.23). These functions are exactly the same as (5.32). Therefore, as in Theorem 5.4.1, Itô's formula produces for $t \in (0, T)$,

$$\begin{aligned}
&\int_{[0,t]} (\tilde{a} - \theta_{s-})dP_s + V(t, X_t, Q_t, Z_t)\\
&= V(0, X_0, Q_0, Z_0) + \int_0^t \Big(2A(s)X_{s-} + D(s)Q_{s-} + E(s)Z_{s-}\Big)dW_s\\
&\quad + \int_0^t \Big((1+r(s))^2A(s) + r(s)^2B(s) + \lambda(s)^2C(s)\\
&\qquad\qquad - r(s)(1+r(s))D(s) + \lambda(1+r(s))E(s) - \lambda(s)r(s)F(s)\Big)d[\theta,\theta]_s^c\\
&\quad + \int_0^t \Big(r(s)(1+2r(s))A(s) + 2r(s)^2B(s) + 2\lambda(s)^2C(s)\\
&\qquad\qquad - r(s)(1+2r(s))D(s) + \lambda(s)(1+2r(s))E(s) - 2\lambda(s)r(s)F(s)\Big)d[\theta,W]_s^c\\
&\quad + \sum_{s\le t}\Big(\Delta V(s, X_s, Q_s, Z_s) - (2A(s)X_{s-} + D(s)Q_{s-} + E(s)Z_{s-})\Delta X_s\\
&\qquad - (2B(s)Q_{s-} + D(s)X_{s-} + F(s)Z_{s-})\Delta Y_s - (2C(s)Z_{s-} + E(s)X_{s-} + F(s)Q_{s-})\Delta Z_s\Big)\\
&= V(0, X_0, Y_0, Z_0) + \int_0^t \Big(2A(s)X_{s-} + D(s)Q_{s-} + E(s)Z_{s-}\Big)dW_s\\
&\qquad + \int_0^t \frac{\lambda(s)}{2}d[\theta,\theta]_s^c + \sum_{s\le t}\frac{\lambda(s)}{2}(\Delta\theta_s)^2.
\end{aligned} \tag{7.24}$$

The stochastic integral with respect to dW_s is a martingale on $t \in [0, T]$ and, the last two terms are positive for any $\theta \in \mathcal{A}$. Therefore, the same equilibrium in Theorem 5.4.1 continues to be an equilibrium with full informed insider case. While prediction of price terms like $\gamma(t)\,(\tilde{v} - P_t)$

with deterministic function $\gamma(t)$ have a important role in Kyle(1985) and Back(1992), they are not important in our case because the infimum in (7.20) is not related to $\tilde{v}$. Therefore, the equilibrium constructed in Theorem 5.4.1 is still valid even when the insider intially knows both $\tilde{v}$ and $\tilde{a}$.

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