A Further Study of Vectorial Dual-Bent Functions†

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Abstract

Vectorial dual-bent functions have recently attracted some researchers' interest as they play a significant role in constructing partial difference sets, association schemes, bent partitions and linear codes. In this paper, we further study vectorial dual-bent functions $F: V_n^{(p)} \rightarrow V_m^{(p)}$, where $2 \leq m \leq \frac{n}{2}$, $V_n^{(p)}$ denotes an n-dimensional vector space over the prime field $\mathbb{F}_p$. We give new characterizations of certain vectorial dual-bent functions (called vectorial dual-bent functions with Condition A) in terms of amorphic association schemes, linear codes and generalized Hadamard matrices, respectively. When $p = 2$, we characterize vectorial dual-bent functions with Condition A in terms of bent partitions. Furthermore, we characterize certain bent partitions in terms of amorphic association schemes, linear codes and generalized Hadamard matrices, respectively. For general vectorial dual-bent functions $F: V_n^{(p)} \rightarrow V_m^{(p)}$ with $F(0) = 0, F(x) = F(-x)$ and $2 \leq m \leq \frac{n}{2}$, we give a necessary and sufficient condition on constructing association schemes. Based on such a result, more association schemes are constructed from vectorial dual-bent functions.

Index Terms

Vectorial dual-bent functions; Association schemes; Generalized Hadamard matrices; Linear codes; Bent partitions; Partial difference sets

I. Introduction

Boolean bent functions were introduced by Rothaus in [17], which have been extensively studied due to their important applications in cryptography, coding theory, combinatorics and

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sequences. Please refer to book [13] for further understanding Boolean bent functions and their generalizations, such as p-ary bent functions and vectorial bent functions, where p is an odd prime.

As a special class of vectorial bent functions, vectorial dual-bent functions introduced by Çeşmelioğlu *et al.* [8] have attracted some researchers' research interest due to their significant applications in constructing partial difference sets [6], [7], [21], association schemes [3], bent partitions [23] and linear codes [24]. Recently, for certain vectorial dual-bent functions $F : V_n^{(p)} \rightarrow V_m^{(p)}$ (called vectorial dual-bent functions with Condition A), where $V_n^{(p)}$ denotes an n-dimensional vector space over the prime field $\mathbb{F}_p$, Wang *et al.* in [23] provided a characterization in terms of partial difference sets. Furthermore, when p is an odd prime, they provided a characterization in terms of bent partitions. When $p = 2$, they showed that vectorial dual-bent functions with Condition A can be used to construct bent partitions, but they did not give a characterization of vectorial dual-bent functions with Condition A in terms of bent partitions. As far as we know, apart from the literature [23], there is a lack of research on the characterizations of vectorial dual-bent functions. As to the applications, Anbar *et al.* in [3] considered using vectorial dual-bent functions to construct association schemes. Also, they in [1] used bent partitions to construct association schemes. Anbar *et al.* showed that vectorial dual-bent functions $F : V_n^{(p)} \rightarrow V_m^{(p)}$ with $F(0) = 0$ and all component functions $F_c, c \in V_m^{(p)} \backslash \{0\}$ being regular or weakly regular but not regular (that is, the corresponding $\varepsilon_{F_c}, c \in V_m^{(p)} \backslash \{0\}$ are all the same) can induce association schemes. Note that for such vectorial dual-bent functions, n must be even. It is interesting to investigate whether there are other vectorial dual-bent functions which can be used to construct association schemes.

In this paper, we further study vectorial dual-bent functions $F : V_n^{(p)} \rightarrow V_m^{(p)}$, where $2 \leq m \leq \frac{n}{2}$. We summarize our contributions as below.

- For any prime p, we provide new characterizations of vectorial dual-bent functions $F : V_n^{(p)} \rightarrow V_m^{(p)}$ with Condition A in terms of amorphic association schemes, linear codes and generalized Hadamard matrices, respectively.
- We present the relations between bent partitions of $V_n^{(2)}$ of depth 2^m and the corresponding vectorial bent functions, based on which we characterize vectorial dual-bent functions with Condition A in terms of bent partitions when $p = 2$.
- Based on the relations between vectorial dual-bent functions with Condition A and bent

partitions, we give new characterizations of certain bent partitions in terms of amorphic association schemes, linear codes and generalized Hadamard matrices, respectively.

- For general vectorial dual-bent functions $F : V_n^{(p)} \to V_m^{(p)}$ with $F(0) = 0$, $F(x) = F(-x)$ and $2 \le m \le \frac{n}{2}$, a necessary and sufficient condition on constructing association schemes from F is presented. Based on such a result, more association schemes are constructed by using two classes of vectorial dual-bent functions $F : V_n^{(p)} \to V_m^{(p)}$ for which n can be odd, or n is even and the corresponding $\varepsilon_{F_c}, c \in V_m^{(p)} \backslash \{0\}$ are not all the same.

The rest of the paper is organized as follows. Section II provides necessary preliminaries. In Sections III-VI, we provide some new characterizations of certain vectorial dual-bent functions. In Section VII, some new characterizations of certain bent partitions are presented. In Section VIII, for certain vectorial dual-bent functions, a necessary and sufficient condition on constructing association schemes is given. In Section IX, we make a conclusion.

II. Preliminaries

In this section, we give the needed results on vectorial dual-bent functions, bent partitions, partial difference sets, association schemes, generalized Hadamard matrices and linear codes, respectively. First, we fix some notations used throughout this paper.

- p is a prime and $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$ is a complex primitive p-th root of unity.
- $\mathbb{F}_{p^n}$ is the finite field with p^n elements.
- $\mathbb{F}_p^n$ is the vector space of the n-tuples over $\mathbb{F}_p$.
- $V_n^{(p)}$ is an n-dimensional vector space over $\mathbb{F}_p$.
- $\langle \cdot \rangle_n$ denotes a (non-degenerate) inner product of $V_n^{(p)}$. In this paper, when $V_n^{(p)} = \mathbb{F}_{p^n}$, let $\langle a, b \rangle_n = Tr_1^n(ab)$, where $a, b \in \mathbb{F}_{p^n}$, $Tr_m^n(\cdot)$ denotes the trace function from $\mathbb{F}_{p^n}$ to $\mathbb{F}_{p^m}$, $m \mid n$; when $V_n^{(p)} = \mathbb{F}_p^n$, let $\langle a, b \rangle_n = a \cdot b = \sum_{i=1}^n a_i b_i$, where $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{F}_p^n$; when $V_n^{(p)} = V_{n_1}^{(p)} \times \cdots \times V_{n_s}^{(p)} (n = \sum_{i=1}^s n_i)$, let $\langle a, b \rangle_n = \sum_{i=1}^s \langle a_i, b_i \rangle_{n_i}$, where $a = (a_1, \ldots, a_s), b = (b_1, \ldots, b_s) \in V_n^{(p)}$.
- For any set $A \subseteq V_n^{(p)}$, let $A^* = A \backslash \{0\}$ and $\chi_u(A) = \sum_{x \in A} \chi_u(x), u \in V_n^{(p)}$, where χ_u denotes the character $\chi_u(x) = \zeta_p^{\langle u, x \rangle_n}$.
- For a function $F : V_n^{(p)} \to V_m^{(p)}$, let $D_{F,i} = \{x \in V_n^{(p)} : F(x) = i\}, i \in V_m^{(p)}$ and $F(V_n^{(p)*}) = \{F(x), x \in V_n^{(p)*}\}$.

- For any set A, let δ_A be the indicator function. In particular, if $A = \{a\}$, we simply denote $\delta_{\{a\}}$ by δ_a.

A. Vectorial dual-bent functions

A function from $V_n^{(p)}$ to $V_m^{(p)}$ is called a *vectorial p-ary function*, or simply *p-ary function* when $m = 1$. For a p-ary function $f : V_n^{(p)} \to \mathbb{F}_p$, the Walsh transform W_f is defined as

$$W_f(a) = \sum_{x \in V_n^{(p)}} \zeta_p^{f(x) - \langle a, x \rangle_n}, a \in V_n^{(p)}. \tag{1}$$

The p-ary function f can be recovered by the inverse transform

$$\zeta_p^{f(x)} = \frac{1}{p^n} \sum_{a \in V_n^{(p)}} W_f(a) \zeta_p^{\langle a, x \rangle_n}, x \in V_n^{(p)}. \tag{2}$$

A p-ary function $f : V_n^{(p)} \to \mathbb{F}_p$ is called *bent* if $|W_f(a)| = p^{\frac{n}{2}}$ for any $a \in V_n^{(p)}$. When $p = 2$, that is, f is a Boolean bent function, then n must be even. The Walsh transform of a p-ary bent function $f : V_n^{(p)} \to \mathbb{F}_p$ satisfies that when $p = 2$, then

$$W_f(a) = 2^{\frac{n}{2}} (-1)^{f^*(a)}, a \in V_n^{(2)}, \tag{3}$$

and when p is an odd prime, then

$$W_f(a) = \begin{cases} \pm p^{\frac{n}{2}} \zeta_p^{f^*(a)}, & \text{if } \ p \equiv 1 \pmod 4 \text{ or } n \text{ is even}, \\ \pm \sqrt{-1} p^{\frac{n}{2}} \zeta_p^{f^*(a)}, & \text{if } \ p \equiv 3 \pmod 4 \text{ and } n \text{ is odd}, \end{cases} \tag{4}$$

where f^* is a p-ary function from $V_n^{(p)}$ to $\mathbb{F}_p$, called the *dual* of f. A p-ary bent function $f : V_n^{(p)} \to \mathbb{F}_p$ is said to be *weakly regular* if $W_f(a) = \varepsilon_f p^{\frac{n}{2}} \zeta_p^{f^*(a)}$, where ε_f is a constant independent of a, otherwise f is called *non-weakly regular*. In particular, if $W_f(a) = p^{\frac{n}{2}} \zeta_p^{f^*(a)}$, that is, $\varepsilon_f = 1$, then f is called *regular*. All Boolean bent functions are regular. The dual f^* of a weakly regular bent function f is also a weakly regular bent function and

$$(f^*)^*(x) = f(-x), \varepsilon_{f^*} = \varepsilon_f^{-1}. \tag{5}$$

A vectorial p-ary function $F : V_n^{(p)} \to V_m^{(p)}$ is called *vectorial bent* if all *component functions* $F_c : V_n^{(p)} \to \mathbb{F}_p, c \in V_m^{(p)^*}$ defined as $F_c(x) = \langle c, F(x) \rangle_m$ are bent. It is known that if $F : V_n^{(p)} \to V_m^{(p)}$ is vectorial bent with all component functions $F_c, c \in V_m^{(p)^*}$ being regular or weakly regular but not regular (that is, ε_{F_c} is a constant independent of c), then n is even and $m \leq \frac{n}{2}$ (see [3],

[7]). A vectorial p-ary bent function $F: V_n^{(p)} \rightarrow V_m^{(p)}$ is called *vectorial dual-bent* if the set of the duals $(F_c)^*, c \in V_m^{(p)^*}$ of the component functions $F_c, c \in V_m^{(p)^*}$ of F (together with the zero function) forms a vector space $\mathcal{V}_F$ of bent functions of dimension m.

For a vectorial dual-bent function $F: V_n^{(p)} \rightarrow V_m^{(p)}$, let $\{(F_{c_1})^*, \ldots, (F_{c_m})^*\}$ be any basis of $\mathcal{V}_F$, where $c_i \in V_m^{(p)^*}, 1 \leq i \leq m$. Then for any $c \in V_m^{(p)^*}$, there is unique nonzero vector $(a_1^{(c)}, \ldots, a_m^{(c)}) \in \mathbb{F}_p^m$ such that $(F_c)^* = \sum_{i=1}^m a_i^{(c)} (F_{c_i})^*$. Define $G: V_n^{(p)} \rightarrow V_m^{(p)}$ as $G(x) = \sum_{i=1}^m (F_{c_i})^*(x)\alpha_i$, where $\{\alpha_1, \ldots, \alpha_m\}$ is any basis of $V_m^{(p)}$. For any $c \in V_m^{(p)^*}$, let $\sigma(c) \in V_m^{(p)^*}$ be given by the following equation system:

$$\begin{cases} \langle \sigma(c), \alpha_1 \rangle_m = a_1^{(c)}, \\ \langle \sigma(c), \alpha_2 \rangle_m = a_2^{(c)}, \\ \quad\quad \vdots \\ \langle \sigma(c), \alpha_m \rangle_m = a_m^{(c)}. \end{cases}$$

Then σ is a permutation over $V_m^{(p)^*}$ and $(F_c)^* = G_{\sigma(c)}, c \in V_m^{(p)^*}$. Since F is vectorial dual-bent, $(F_c)^*, c \in V_m^{(p)^*}$ are all bent functions and G is vectorial bent. By the argument, one can see that a vectorial bent function $F: V_n^{(p)} \rightarrow V_m^{(p)}$ is vectorial dual-bent if and only if there exists a vectorial bent function $G: V_n^{(p)} \rightarrow V_m^{(p)}$ such that $(F_c)^* = G_{\sigma(c)}, c \in V_m^{(p)^*}$, where σ is some permutation over $V_m^{(p)^*}$. The vectorial bent function G is called a *vectorial dual* of F and denoted by F^*. By the above analysis, one can see that the vectorial dual of a vectorial dual-bent function is not unique. In the following, we show that if F is a vectorial dual-bent function for some fixed permutation σ over $V_m^{(p)^*}$, then its vectorial dual F^* with $(F_c)^* = (F^*)_{\sigma(c)}, c \in V_m^{(p)^*}$ is unique.

Proposition 1. *Let $F: V_n^{(p)} \rightarrow V_m^{(p)}$ be a vectorial dual-bent function for some fixed permutation σ over $V_m^{(p)^*}$. Then its vectorial dual F^* with $(F_c)^* = (F^*)_{\sigma(c)}, c \in V_m^{(p)^*}$ is unique.*

Proof: Let $\{\alpha_1, \ldots, \alpha_m\}$ be any basis of $V_m^{(p)}$ and $c_i = \sigma^{-1}(\alpha_i), 1 \leq i \leq m$. Then $\{\sigma(c_1), \ldots, \sigma(c_m)\}$ is a basis of $V_m^{(p)}$ and for any $x \in V_n^{(p)}$, there is unique $G(x) \in V_m^{(p)}$ such that the following equation system hold:

$$\begin{cases} \langle G(x), \sigma(c_1)\rangle_m = (F_{c_1})^*(x), \\ \langle G(x), \sigma(c_2)\rangle_m = (F_{c_2})^*(x), \\ \quad \vdots \\ \langle G(x), \sigma(c_m)\rangle_m = (F_{c_m})^*(x). \end{cases}$$

Hence, the vectorial dual F^* with $(F_c)^* = (F^*)_{\sigma(c)}, c \in {V_m^{(p)}}^*$ is unique and $F^* = G$. ∎

In [23], Wang *et al.* studied vectorial dual-bent functions for which the corresponding permutation σ over ${V_m^{(p)}}^*$ is the identity map. We recall vectorial dual-bent functions with Condition A defined and studied in [23].

Condition A: Let $n \geq 4$ be even and $2 \leq m \leq \frac{n}{2}$. Let $F : V_n^{(p)} \to V_m^{(p)}$ be a vectorial dual-bent function for which

$$(F_c)^* = (F^*)_c, c \in {V_m^{(p)}}^*, \tag{6}$$

and all component functions $F_c, c \in {V_m^{(p)}}^*$ are regular or weakly regular but not regular. We denote by $\varepsilon = \varepsilon_{F_c}$ for all $c \in {V_m^{(p)}}^*$.

Remark 1. *Let $n \geq 4$ be even and $2 \leq m \leq \frac{n}{2}$. When $p = 2$, since all Boolean bent functions are regular, $F : V_n^{(2)} \to V_m^{(2)}$ is a vectorial dual-bent function with Condition A if and only if F is a vectorial dual-bent function with $(F_c)^* = (F^*)_c, c \in {V_m^{(2)}}^*$.*

When $p > 3$, if $F : V_n^{(p)} \to V_m^{(p)}$ is a vectorial dual-bent function with Condition A, we show that all component functions $F_c, c \in {V_m^{(p)}}^*$ are regular.

Proposition 2. *Let $p > 3$ be an odd prime. If $F : V_n^{(p)} \to V_m^{(p)}$ is a vectorial dual-bent function with Condition A, then all component functions $F_c, c \in {V_m^{(p)}}^*$ are regular, that is, $\varepsilon = 1$.*

Proof: By the proof of Theorem 1 of [23], if F is a vectorial dual-bent function with Condition A, then $F(ax) = F(x), a \in \mathbb{F}_p^*$. Note that $F_c(x) - F_c(0), c \in {V_m^{(p)}}^*$ are all weakly regular bent functions with $F_c(ax) - F_c(0) = F_c(x) - F_c(0), a \in \mathbb{F}_p^*$ and $\varepsilon_{F_c(x) - F_c(0)} = \varepsilon$. By Corollary 3.5 of [10], for a weakly regular bent function $f : V_{2r}^{(p)} \to \mathbb{F}_p$ with $f(0) = 0, f(ax) = f(x), a \in \mathbb{F}_p^*$, f is regular if $p > 3$. Therefore, we have $\varepsilon = 1$ if $p > 3$. ∎

It was shown in [23] that the known bent partitions from (pre)semifields can be obtained from vectorial dual-bent functions with Condition A, and vectorial dual-bent functions with Condition

A can be used to construct partial difference sets (see also [21]). In [3], Anbar *et al.* showed that vectorial dual-bent functions with Condition A are able to construct amorphic association schemes. In Sections III-VI, we will further study vectorial dual-bent functions with Condition A.

B. Bent partitions

Let n be an even positive integer, K be a positive integer divisible by p. Let $\Gamma = \{A_1, \ldots, A_K\}$ be a partition of $V_n^{(p)}$. Assume that every p-ary function $f : V_n^{(p)} \rightarrow \mathbb{F}_p$ for which every $i \in \mathbb{F}_p$ has exactly $\frac{K}{p}$ of sets A_j in Γ in its preimage set, is a p-ary bent function. Then Γ is called a *bent partition* of $V_n^{(p)}$ of depth K and every such bent function f is called a *bent function constructed from bent partition* Γ.

For a bent partition $\Gamma = \{A_i, 1 \leq i \leq p^m\}$ of $V_n^{(p)}$, the following lemma gives the cardinality of A_i.

Lemma 1 ([4]). *Let n be an even positive integer. Let $\Gamma = \{A_i, 1 \leq i \leq p^m\}$ be a bent partition of $V_n^{(p)}$. Then except one set, denoted by A_{i_0}, all other sets A_i have the same cardinality, namely*

$$|A_{i_0}| = p^{\frac{n}{2}-m}(p^{\frac{n}{2}} \mp 1) \pm p^{\frac{n}{2}},\ |A_i| = p^{\frac{n}{2}-m}(p^{\frac{n}{2}} \mp 1), i \neq i_0.$$

In [23], Wang *et al.* studied the relations between vectorial dual-bent functions with Condition A and bent partitions with Condition $\mathcal{C}$. We recall bent partitions with Condition $\mathcal{C}$ defined and studied in [23].

Condition $\mathcal{C}$: Let $n \geq 4$ be even, $2 \leq m \leq \frac{n}{2}$. Let $\Gamma = \{A_i, i \in V_m^{(p)}\}$ be a bent partition of $V_n^{(p)}$, which satisfies that $aA_i = A_i$ for any $a \in \mathbb{F}_p^*$ and $i \in V_m^{(p)}$ and all bent functions constructed from Γ are regular or weakly regular but not regular. We denote by $\varepsilon = \varepsilon_f$ for all bent functions f constructed from Γ.

Remark 2. *Let $n \geq 4$ be even and $2 \leq m \leq \frac{n}{2}$. When $p = 2$, since all Boolean bent functions are regular, Condition $\mathcal{C}$ is trivial for every bent partition of $V_n^{(2)}$ of depth 2^m.*

When p is odd, it was proved in [23] that bent partitions with Condition $\mathcal{C}$ one-to-one correspond to vectorial dual-bent functions with Condition A.

Lemma 2 ([23]). *Let p be an odd prime. Let $\Gamma = \{A_i, i \in V_m^{(p)}\}$ be a partition of $V_n^{(p)}$, where $n \geq 4$ is even and $2 \leq m \leq \frac{n}{2}$. Define $F : V_n^{(p)} \to V_m^{(p)}$ as $F(x) = \sum_{i \in V_m^{(p)}} \delta_{A_i}(x) i$. Then Γ is a bent partition with Condition $\mathcal{C}$ if and only if F is a vectorial dual-bent function with Condition A.*

C. Partial difference sets and association schemes

Let $(G, +)$ be a finite abelian group of order v and D be a subset of G with k elements. Then D is called a (v, k, λ, μ) *partial difference set* of G, if $\overline{D}\ \overline{(-D)} = \mu\overline{G} + (\lambda - \mu)\overline{D} + \gamma 0$ with $\overline{D} = \sum_{g \in D} g$ denoting the element in the group ring $\mathbb{Z}[G]$ and $-D = \{-d, d \in D\}$, where $\gamma = k - \mu$ if $0 \notin D$ and $\gamma = k - \lambda$ if $0 \in D$. By Page 223 of [11], the empty set can be seen as a $(v, 0, \lambda, 0)$ partial difference set of any finite abelian group of order v, where λ is any integer. A partial difference set D is called *regular* if $-D = D$ and $0 \notin D$. A regular (v, k, λ, μ) partial difference set is called to be of *Latin square type* if $v = N^2, k = s(N-1), \lambda = N + s^2 - 3s, \mu = s^2 - s$, and a regular (v, k, λ, μ) partial difference set is called to be of *negative Latin square type* if $v = N^2, k = s(N+1), \lambda = -N + s^2 + 3s, \mu = s^2 + s$. We allow $s = 0$, which corresponds to the empty set.

There is an important tool to characterize partial difference sets in terms of characters.

Lemma 3 ([11], [18]). *Let G be an abelian group of order v. Suppose that D is a subset of G with k elements which satisfies $-D = D$ and $0 \notin D$. Then D is a (v, k, λ, μ) partial difference set if and only if for each non-principal character χ of G,*

$$\chi(D) = \frac{\beta \pm \sqrt{\Delta}}{2},$$

where $\chi(D) = \sum_{x \in D} \chi(x)$, $\beta = \lambda - \mu, \gamma = k - \mu, \Delta = \beta^2 + 4\gamma$.

Let X be a nonempty finite set. A *d-class association scheme* on X is a sequence $R_0, R_1, \ldots, R_d$ of nonempty subsets of $X \times X$, satisfying

1. $R_0 = \{(x, x) : x \in X\}$;
2. $X \times X = R_0 \bigcup R_1 \bigcup \cdots \bigcup R_d$ and $R_i \bigcap R_j = \emptyset$ for $i \neq j$;
3. for any $i \in \{0, \ldots, d\}$, there is j such that $R_i^\top = R_j$, where $R_i^\top = \{(y, x) : (x, y) \in R_i\}$;
4. for all integers $k, i, j \in \{0, 1, \ldots, d\}$, and for all $x, y \in X$ such that $(x, y) \in R_k$, the number $p_{i,j}^k = |\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|$ depends only on k, i, j and not on (x, y).

The numbers $p_{i,j}^k$ are called *intersection numbers* of an association scheme. If for any $i \in \{0, \dots, d\}$, $R_i^\top = R_i$, then the association scheme is called *symmetric*.

A *fusion* of an association scheme $\{R_0, R_1, \dots, R_d\}$ on X is a partition $\{A_0, A_1, \dots, A_t\}$ of $X \times X$ such that $A_0 = R_0$ and each A_i ($1 \leq i \leq t$) is the union of some of $R_j, 1 \leq j \leq d$. An association scheme is called *amorphic* if its any fusion is again an association scheme. The following lemma gives a characterization of amorphic association schemes induced from partitions.

Lemma 4 ([19], [20]). *Let nonempty sets $D_0 = \{0\}, D_1, \dots, D_d$ form a partition of a finite abelian group G, where $d \geq 3$. Define $R_i, 0 \leq i \leq d$ as*

$$R_i = \{(x, y) \in G \times G : x - y \in D_i\}.$$

The following two statements are equivalent.

(1) $R_0, R_1, \dots, R_d$ form an amorphic association scheme.

(2) $D_1, \dots, D_d$ are regular partial difference sets, all of which are of Latin square type, or all of which are of negative Latin square type.

D. Generalized Hadamard matrices

Let $\zeta_m = e^{\frac{2\pi\sqrt{-1}}{m}}$ be a complex primitive m-th root of unity. A complex matrix H of size $n \times n$ consisting of integer powers of ζ_m is called a *generalized Hadamard matrix* if $H\overline{H}^\top = nI_n$, where $\overline{H}$ is the conjugate matrix of H, $\overline{H}^\top$ is the transpose matrix of $\overline{H}$, and I_n is the identity matrix of size $n \times n$. When $m = 2$, H is simply called a *Hadamard matrix*.

There is a characterization of p-ary bent functions in terms of generalized Hadamard matrices.

Lemma 5 ([9], [14]). *Let $f : V_n^{(p)} \to \mathbb{F}_p$. Define $H = \left[\zeta_p^{f(x-y)}\right]_{x,y \in V_n^{(p)}}$. Then f is a p-ary bent function if and only if H is a generalized Hadamard matrix.*

E. Linear codes

For a vector $a = (a_1, \dots, a_n) \in \mathbb{F}_p^n$, the *Hamming weight* of a is defined as $wt(a) = |\{1 \leq i \leq n : a_i \neq 0\}|$. For two vectors $a, b \in \mathbb{F}_p^n$, the *Hamming distance* between a and b is defined as $d(a, b) = wt(a - b)$.

Let C be a p-ary $[n,k]$ linear code, that is, C is a subspace of $\mathbb{F}_p^n$ with dimension k. The *minimum Hamming distance* d of C is defined as $d=\min\{d(a,b) : a,b\in C, a\neq b\} = \min\{wt(c) : c\in C, c\neq 0\}$. The dual code of C is defined by $C^{\perp}=\{u\in\mathbb{F}_p^n : u\cdot c = 0 \text{ for all } c\in C\}$. If the minimum Hamming weight $d^{\perp}$ of the dual code $C^{\perp}$ satisfies $d^{\perp}\geq 3$, then C is called *projective*. For any $1\leq i\leq n$, let A_i denote the number of codewords in C whose Hamming weight is i. The sequence $(1,A_1,\ldots,A_n)$ is called the *weight distribution* of C. The code C is called *t-weight* if $|\{1\leq i\leq n : A_i\neq 0\}|=t$.

The following lemma gives a characterization of a two-weight projective p-ary linear code in terms of a partial difference set.

Lemma 6 ([11]). *Let $\widetilde{D}=\{d_1,\ldots,d_m\}$, where $d_i, 1\leq i\leq m$ are pairwise linearly independent vectors in $V_n^{(p)}$. Define*

$$C_{\widetilde{D}}=\{(\langle x,d_1\rangle_n,\ldots,\langle x,d_m\rangle_n) : x\in V_n^{(p)}\}.$$

Then $C_{\widetilde{D}}$ is a two-weight $[m,n]$ projective linear code if and only if $D=\mathbb{F}_p^*\widetilde{D}=\{yd_i : y\in\mathbb{F}_p^*, 1\leq i\leq m\}$ is a regular partial difference set in $V_n^{(p)}$. Furthermore, if the two nonzero weights of $C_{\widetilde{D}}$ are w_1 and w_2, then the parameters of the (v,k,λ,μ) partial difference set D are $v=p^n, k=m(p-1), \lambda=k^2+3k-p(k+1)(w_1+w_2)+p^2w_1w_2, \mu=k^2+k-pk(w_1+w_2)+p^2w_1w_2$.

III. A Characterization of vectorial dual-bent functions with Condition A in terms of amorphic association schemes

In this section, we give a characterization of vectorial dual-bent functions with Condition A in terms of amorphic association schemes.

In Theorem 6 of [23], Wang *et al.* characterized vectorial dual-bent functions F with Condition A in terms of partial difference sets $D_{F,I}^*$, where I is an arbitrary nonempty subset of $V_m^{(p)}$ and $D_{F,I}=\bigcup_{i\in I}D_{F,i}$. In the following, based on Theorem 6 of [23], we give a characterization of vectorial dual-bent functions with Condition A in terms of partial difference sets $D_{F,i}^*, i\in V_m^{(p)}$.

Proposition 3. *Let $F: V_n^{(p)}\to V_m^{(p)}$, where $n\geq 4$ is even and $2\leq m\leq \frac{n}{2}$. The following two statements are equivalent.*

(1) *F is a vectorial dual-bent function with Condition A.*

(2) *For any $i\in V_m^{(p)}$, $D_{F,i}^*$ is a regular $(p^n, s_i(p^{\frac{n}{2}}-\varepsilon), \varepsilon p^{\frac{n}{2}}+s_i^2-3\varepsilon s_i, s_i^2-\varepsilon s_i)$ partial difference set, where $s_i=p^{\frac{n}{2}-m}+\varepsilon\delta_{F(0)}(i)$, $\varepsilon\in\{\pm 1\}$ is a constant with $\varepsilon=1$ if $p\neq 3$.*

Proof: If (1) holds, then (2) holds by Theorem 6 of [23] and Proposition 2. In the following, we prove that if (2) holds, then (1) holds.

By Lemma 3, if (2) holds, then for any $u \in V_n^{(p)}$ and $i \in V_m^{(p)}$, we have $\chi_u(D_{F,i}) = p^{n-m}\delta_0(u) + \varepsilon p^{\frac{n}{2}} - \varepsilon p^{\frac{n}{2}-m}$ or $\chi_u(D_{F,i}) = p^{n-m}\delta_0(u) - \varepsilon p^{\frac{n}{2}-m}$. For any $u \in V_n^{(p)}$, let

$$n_u = |\{i \in V_m^{(p)} : \chi_u(D_{F,i}) = p^{n-m}\delta_0(u) + \varepsilon p^{\frac{n}{2}} - \varepsilon p^{\frac{n}{2}-m}\}|.$$

Since $p^n\delta_0(u) = \sum_{x \in V_n^{(p)}} \zeta_p^{\langle u,x\rangle_n} = \chi_u(V_n^{(p)})$ and

$$\begin{aligned}\chi_u(V_n^{(p)}) &= \sum_{i \in V_m^{(p)}} \chi_u(D_{F,i}) \\ &= n_u(p^{n-m}\delta_0(u) + \varepsilon p^{\frac{n}{2}} - \varepsilon p^{\frac{n}{2}-m}) + (p^m - n_u)(p^{n-m}\delta_0(u) - \varepsilon p^{\frac{n}{2}-m}) \\ &= p^n\delta_0(u) + \varepsilon p^{\frac{n}{2}}(n_u - 1),\end{aligned}$$

we have $n_u = 1$. Therefore, for any nonempty set $I \subseteq V_m^{(p)}$ and $u \in V_n^{(p)}$,

$$\chi_u(D_{F,I}) = p^{n-m}\delta_0(u)|I| + \varepsilon p^{\frac{n}{2}} - \varepsilon p^{\frac{n}{2}-m}|I| \text{ or } \chi_u(D_{F,I}) = p^{n-m}\delta_0(u)|I| - \varepsilon p^{\frac{n}{2}-m}|I|, \quad (7)$$

where $D_{F,I} = \sum_{i \in I} D_{F,i}$. For any $i \in V_m^{(p)}$, define $E_i = \{u \in V_n^{(p)} : \chi_u(D_{F,i}) = p^{n-m}\delta_0(u) + \varepsilon p^{\frac{n}{2}} - \varepsilon p^{\frac{n}{2}-m}\}$. We claim that $E_i \bigcap E_j = \emptyset$ for any $i \neq j$ and $\bigcup_{i \in V_m^{(p)}} E_i = V_n^{(p)}$. If there exists $i \neq j$ such that $E_i \bigcap E_j \neq \emptyset$, then there is $u \in V_n^{(p)}$ such that $\chi_u(D_{F,i}) = \chi_u(D_{F,j}) = p^{n-m}\delta_0(u) + \varepsilon p^{\frac{n}{2}} - \varepsilon p^{\frac{n}{2}-m}$ and $\chi_u(D_{F,i} \bigcup D_{F,j}) = 2p^{n-m}\delta_0(u) + 2\varepsilon p^{\frac{n}{2}} - 2\varepsilon p^{\frac{n}{2}-m}$, which contradicts Eq. (7). Thus, $E_i \bigcap E_j = \emptyset$ for any $i \neq j$. If there is $u \in V_n^{(p)}$ such that $u \notin E_i$ for any $i \in V_m^{(p)}$, then $\chi_u(D_{F,i}) = p^{n-m}\delta_0(u) - \varepsilon p^{\frac{n}{2}-m}$ for any $i \in V_m^{(p)}$ and $\chi_u(V_n^{(p)}) = \sum_{i \in V_m^{(p)}} \chi_u(D_{F,i}) = p^n\delta_0(u) - \varepsilon p^{\frac{n}{2}}$, which contradicts $\chi_u(V_n^{(p)}) = \sum_{x \in V_n^{(p)}} \zeta_p^{\langle u,x\rangle_n} = p^n\delta_0(u)$. Thus, $\bigcup_{i \in V_m^{(p)}} E_i = V_n^{(p)}$. By the above arguments, we can obtain

$$\chi_u(D_{F,I}) = p^{n-m}\delta_0(u)|I| + \varepsilon p^{\frac{n}{2}-m}(p^m\delta_{E_I}(u) - |I|),$$

where $E_I = \sum_{i \in I} E_i$. Then by Lemma 1 of [23], F is a vectorial dual-bent function with Condition A. ■

Remark 3. *For a vectorial dual-bent function $F : V_n^{(p)} \to V_m^{(p)}$ with Condition A, by Proposition 3, $D_{F,i}^*, i \in V_m^{(p)}$ are all regular partial difference sets of Latin square type if $\varepsilon = 1$, and $D_{F,i}^*, i \in V_m^{(p)}$ are all regular partial difference sets of negative Latin square type if $\varepsilon = -1$.*

The following corollary is directly from Proposition 3 and Remark 1.

Corollary 1. *Let $F : V_n^{(2)} \to V_m^{(2)}$, where $n \geq 4$ is even and $2 \leq m \leq \frac{n}{2}$. The following two statements are equivalent.*

(1) *F is a vectorial dual-bent function with $(F_c)^* = (F^*)_c, c \in {V_m^{(2)}}^*$.*

(2) *For any $i \in V_m^{(2)}$, $D^*_{F,i}$ is a regular $(2^n, s_i(2^{\frac{n}{2}} - 1), 2^{\frac{n}{2}} + s_i^2 - 3s_i, s_i^2 - s_i)$ partial difference set, where $s_i = 2^{\frac{n}{2}-m} + \delta_{F(0)}(i)$.*

Based on Proposition 3 and Lemma 4, we give the following theorem, which characterizes vectorial dual-bent functions with Condition A in terms of amorphic association schemes.

Theorem 1. *Let $F : V_n^{(p)} \to V_m^{(p)}$, where $n \geq 4$ is even and $2 \leq m \leq \frac{n}{2}$. Denote $I = F({V_n^{(p)}}^*)$. Define*

$$R_{id} = \{(x, x) : x \in V_n^{(p)}\},$$

$$R_i = \{(x, y) : x, y \in V_n^{(p)}, x - y \in D^*_{F,i}\}, i \in I.$$

The following two statements are equivalent.

(1) *F is a vectorial dual-bent function with Condition A.*

(2) *$\{R_{id}, R_i, i \in I\}$ is an $|I|$-class amorphic association scheme for which $|I| \geq 3$ and for any $i \in I$, the intersection number $p_{i,i}^{id} = p^{n-m} - \varepsilon p^{\frac{n}{2}-m} + \delta_{F(0)}(i)(\varepsilon p^{\frac{n}{2}} - 1)$, where $\varepsilon \in \{\pm 1\}$ is a constant with $\varepsilon = 1$ if $p \neq 3$.*

Furthermore, if (1) *or* (2) *holds, then the following statement holds:*

(3) *$I = V_m^{(p)}$ and $|I| = p^m$ except one case that $p = 3$, $n = 2m$ and $\varepsilon = -1$ (in such a case, $I = V_m^{(3)} \backslash \{F(0)\}$ and $|I| = 3^m - 1$).*

Proof: If (1) holds, then (2) and (3) follow from Theorem 3 of [3], Proposition 4 of [23] and Proposition 2. In the following, we prove that if (2) holds, then (1) and (3) hold.

Note that $i \in I$ if and only if $D^*_{F,i}$ is nonempty. By Lemma 4, $\{R_{id}, R_i, i \in I\}$ is an amorphic association scheme if and only if for any $i \in I$, $D^*_{F,i}$ is a regular $(p^n, s_i(p^{\frac{n}{2}} - \varepsilon'), \varepsilon' p^{\frac{n}{2}} + s_i^2 - 3\varepsilon' s_i, s_i^2 - \varepsilon' s_i)$ partial difference set, where s_i is a positive integer, $\varepsilon' \in \{\pm 1\}$ is a constant. From $|D^*_{F,i}| = p_{i,i}^{id} = p^{n-m} - \varepsilon p^{\frac{n}{2}-m} + \delta_{F(0)}(i)(\varepsilon p^{\frac{n}{2}} - 1), i \in I$, we have

$$s_i(p^{\frac{n}{2}} - \varepsilon') = p^{n-m} - \varepsilon p^{\frac{n}{2}-m} + \delta_{F(0)}(i)(\varepsilon p^{\frac{n}{2}} - 1), i \in I.$$

Assume that $\varepsilon' \neq \varepsilon$, that is, $\varepsilon' = -\varepsilon$. Let $i \in I \backslash \{F(0)\}$, then $s_i(p^{\frac{n}{2}} + \varepsilon) = p^{\frac{n}{2}-m}(p^{\frac{n}{2}} - \varepsilon)$. Since $gcd(p^{\frac{n}{2}} + \varepsilon, p^{\frac{n}{2}-m}) = 1$, then $(p^{\frac{n}{2}} + \varepsilon) \mid (p^{\frac{n}{2}} - \varepsilon)$, which implies that $\varepsilon = -1$ and

$p=3, n=2$, which contradicts $n\geq 4$. Therefore, $\varepsilon'=\varepsilon$ and for any $i\in I$, $D^*_{F,i}$ is a regular $(p^n, s_i(p^{\frac{n}{2}}-\varepsilon), \varepsilon p^{\frac{n}{2}}+s_i^2-3\varepsilon s_i, s_i^2-\varepsilon s_i)$ partial difference set, where $s_i=p^{\frac{n}{2}-m}+\varepsilon\delta_{F(0)}(i)$. Note that $0\in D_{F,F(0)}$ and $|D^*_{F,i}|=0$ if $i\in V_m^{(p)}\backslash I$.

If $D_{F,F(0)}=\{0\}$, then $F(0)\notin I$ and

$$\begin{aligned} p^n=|V_n^{(p)}| &= \sum_{i\in V_m^{(p)}}|D_{F,i}| \\ &= \sum_{i\in V_m^{(p)}\backslash\{F(0)\}}|D^*_{F,i}|+|D_{F,F(0)}| \\ &= \sum_{i\in I}|D^*_{F,i}|+1 \\ &= p^{\frac{n}{2}-m}(p^{\frac{n}{2}}-\varepsilon)|I|+1. \end{aligned} \tag{8}$$

From Eq. (8), we have $p^{\frac{n}{2}-m}\mid(p^n-1)$, which implies that $n=2m$. Further, by Eq. (8) and $|I|\leq p^m-1=p^{\frac{n}{2}}-1$, we obtain $\varepsilon=-1, |I|=p^m-1, I=V_m^{(p)}\backslash\{F(0)\}$. Note that $p=3$ since $\varepsilon=1$ when $p\neq 3$. In this case, for any $i\in V_m^{(3)}$, $D^*_{F,i}$ is a regular $(3^n, s_i(3^{\frac{n}{2}}+1), -3^{\frac{n}{2}}+s_i^2+3s_i, s_i^2+s_i)$ partial difference set, where $s_i=1-\delta_{F(0)}(i)$. By Proposition 3, F is a vectorial dual-bent function with Condition A.

If $D^*_{F,F(0)}$ is nonempty, then $F(0)\in I$ and

$$\begin{aligned} p^n=|V_n^{(p)}| &= \sum_{i\in V_m^{(p)}}|D_{F,i}| \\ &= \sum_{i\in V_m^{(p)}\backslash\{F(0)\}}|D^*_{F,i}|+|D_{F,F(0)}| \\ &= \sum_{i\in I\backslash\{F(0)\}}|D^*_{F,i}|+|D_{F,F(0)}| \\ &= p^{\frac{n}{2}-m}(p^{\frac{n}{2}}-\varepsilon)(|I|-1)+(p^{\frac{n}{2}-m}+\varepsilon)(p^{\frac{n}{2}}-\varepsilon)+1, \end{aligned}$$

which implies that $|I|=p^m$, that is, $I=V_m^{(p)}$. Thus for any $i\in V_m^{(p)}$, $D^*_{F,i}$ is a regular $(p^n, s_i(p^{\frac{n}{2}}-\varepsilon), \varepsilon p^{\frac{n}{2}}+s_i^2-3\varepsilon s_i, s_i^2-\varepsilon s_i)$ partial difference set, where $s_i=p^{\frac{n}{2}-m}+\varepsilon\delta_{F(0)}(i)$. By Proposition 3, F is a vectorial dual-bent function with Condition A.

Furthermore, the analysis mentioned above shows that statement (3) holds. ■

Remark 4. *Keep the same notation as in Theorem 1. If F is a vectorial dual-bent function with Condition A, then by Corollary 1 of* [19]*, the intersection numbers of the amorphic association* *scheme induced from F are given by the following equations:*

$$p_{id,id}^{i}=p_{id,i}^{j}=p_{i,id}^{j}=p_{i,j}^{id}=0,p_{id,i}^{i}=p_{i,id}^{i}=p_{id,id}^{id}=1,p_{i,i}^{id}=s_i(p^{\frac{n}{2}}-\varepsilon),$$

$$p_{i,i}^{i}=\varepsilon p^{\frac{n}{2}}-2+(s_i-\varepsilon)(s_i-2\varepsilon),p_{i,i}^{j}=s_i(s_i-\varepsilon),p_{i,j}^{j}=s_i(s_j-\varepsilon),p_{i,j}^{k}=s_is_j,$$

where $i,j,k\in F(V_n^{(p)^*})$ are distinct, $s_i=p^{\frac{n}{2}-m}+\varepsilon\delta_{F(0)}(i)$, $\varepsilon\in\{\pm 1\}$ is a constant with $\varepsilon=1$ if $p\neq 3$.

The following corollary is directly from Theorem 1 and Remark 1.

Corollary 2. *Let $F:V_n^{(2)}\to V_m^{(2)}$, where $n\geq 4$ is even and $2\leq m\leq\frac{n}{2}$. Define*

$$R_{id}=\{(x,x):x\in V_n^{(2)}\},$$

$$R_i=\{(x,y):x,y\in V_n^{(2)},x+y\in D_{F,i}^{*}\},i\in V_m^{(2)}.$$

The following two statements are equivalent.

(1) *F is a vectorial dual-bent function with $(F_c)^*=(F^*)_c,c\in V_m^{(2)^*}$.*

(2) *$\{R_{id},R_i,i\in V_m^{(2)}\}$ is a 2^m-class amorphic association scheme for which for any $i\in V_m^{(2)}$, the intersection number $p_{i,i}^{id}=2^{n-m}-2^{\frac{n}{2}-m}+\delta_{F(0)}(i)(2^{\frac{n}{2}}-1)$.*

IV. A CHARACTERIZATION OF VECTORIAL DUAL-BENT FUNCTIONS WITH CONDITION A IN TERMS OF LINEAR CODES

In this section, we give a characterization of vectorial dual-bent functions with Condition A in terms of linear codes.

First, we introduce a notation. For a set $D\subseteq V_n^{(p)^*}$, let $\widetilde{D}$ be a subset of D, denoted by $\widetilde{D}=\{x_1,\ldots,x_t\}$, for which $x_j,1\leq j\leq t$ are pairwise linearly independent, and for any $x\in D$, there exist $a\in\mathbb{F}_p^*$ and x_j such that $x=ax_j$. Note that when $p=2$, $\widetilde{D}=D$.

Theorem 2. *Let $F:V_n^{(p)}\to V_m^{(p)}$, where $n\geq 4$ is even and $2\leq m\leq\frac{n}{2}$. Denote $I=F(V_n^{(p)^*})$. Define*

$$C_{\widetilde{D_{F,i}^{*}}}=\{c_\alpha=(\langle\alpha,x\rangle_n)_{x\in\widetilde{D_{F,i}^{*}}}:\alpha\in V_n^{(p)}\},i\in I.$$

The following two statements are equivalent.

(1) *F is a vectorial dual-bent function with Condition A.*

(2) *For any* $i \in I$, $C_{\widetilde{D_{F,i}^*}}$ *is a two-weight* $[\frac{p^{n-m}-\varepsilon p^{\frac{n}{2}-m}+\delta_{F(0)}(i)(\varepsilon p^{\frac{n}{2}}-1)}{p-1}, n]$ *projective linear code and the two nonzero weights are*

$$w_1 = p^{n-m-1} + \frac{1-\varepsilon+2\varepsilon\delta_{F(0)}(i)}{2} p^{\frac{n}{2}-1},$$
$$w_2 = p^{n-m-1} + \frac{-1-\varepsilon+2\varepsilon\delta_{F(0)}(i)}{2} p^{\frac{n}{2}-1},$$

where $\varepsilon \in \{\pm 1\}$ *is a constant with* $\varepsilon = 1$ *if* $p \neq 3$.

Furthermore, if (1) *or* (2) *holds, then the following statement holds:*

(3) $I = V_m^{(p)}$ *except one case that* $p = 3$, $n = 2m$ *and* $\varepsilon = -1$ *(in such a case,* $I = V_m^{(3)} \backslash \{F(0)\}$*).*

Proof: (1) $\Rightarrow$ (2): If (1) holds, then (3) holds by Proposition 4 of [23] and Proposition 2. Since F is a vectorial dual-bent function with Condition A, by the proof of Theorem 1 of [23], we have $F(ax) = F(x), a \in \mathbb{F}_p^*$, which implies that $D_{F,i}^* = \mathbb{F}_p^* \widetilde{D_{F,i}^*}, i \in I$. For any $i \in I$, by Proposition 3, we have that $D_{F,i}^*$ is a regular $(p^n, s_i(p^{\frac{n}{2}}-\varepsilon), \varepsilon p^{\frac{n}{2}}+s_i^2-3\varepsilon s_i, s_i^2-\varepsilon s_i)$ partial difference set, where $s_i = p^{\frac{n}{2}-m}+\varepsilon\delta_{F(0)}(i)$, $\varepsilon \in \{\pm 1\}$ is a constant with $\varepsilon = 1$ if $p \neq 3$. By Lemma 6, $C_{\widetilde{D_{F,i}^*}}$ is a two-weight projective linear code with parameters $[\frac{p^{n-m}-\varepsilon p^{\frac{n}{2}-m}+\delta_{F(0)}(i)(\varepsilon p^{\frac{n}{2}}-1)}{p-1}, n]$ and the two nonzero weights are $w_1 = p^{n-m-1} + \frac{1-\varepsilon+2\varepsilon\delta_{F(0)}(i)}{2} p^{\frac{n}{2}-1}$, $w_2 = p^{n-m-1} + \frac{-1-\varepsilon+2\varepsilon\delta_{F(0)}(i)}{2} p^{\frac{n}{2}-1}$.

(2) $\Rightarrow$ (1): If (2) holds, then by Lemma 6, for any $i \in I$, $\mathbb{F}_p^* \widetilde{D_{F,i}^*}$ is a regular $(p^n, s_i(p^{\frac{n}{2}} - \varepsilon), \varepsilon p^{\frac{n}{2}} + s_i^2 - 3\varepsilon s_i, s_i^2 - \varepsilon s_i)$ partial difference set, where $s_i = p^{\frac{n}{2}-m} + \varepsilon\delta_{F(0)}(i)$. Note that for any $i \in I$, $D_{F,i}^* \subseteq \mathbb{F}_p^* \widetilde{D_{F,i}^*}$ and $|D_{F,i}^*| \leq |\mathbb{F}_p^* \widetilde{D_{F,i}^*}| = s_i(p^{\frac{n}{2}} - \varepsilon)$. Then

$$p^n - 1 = \sum_{i \in I} |D_{F,i}^*| \leq \sum_{i \in I} |\mathbb{F}_p^* \widetilde{D_{F,i}^*}| = \sum_{i \in I} s_i(p^{\frac{n}{2}} - \varepsilon) = (p^{\frac{n}{2}} - \varepsilon) \sum_{i \in I} p^{\frac{n}{2}-m} + \varepsilon\delta_{F(0)}(i),$$

which yields that

$$|I| p^{\frac{n}{2}-m} + \varepsilon\delta_I(F(0)) \geq p^{\frac{n}{2}} + \varepsilon. \tag{9}$$

If $F(0) \notin I$, then $D_{F,F(0)} = \{0\}$ and by Inequality (9), $|I| p^{\frac{n}{2}-m} \geq p^{\frac{n}{2}} + \varepsilon$. By $|I| \leq p^m - 1$, we obtain $\varepsilon = -1, n = 2m, |I| = p^m - 1, I = V_m^{(p)} \backslash \{F(0)\}$. Note that $p = 3$ since $\varepsilon = 1$ when $p \neq 3$. In this case, $\sum_{i \in V_m^{(3)} \backslash \{F(0)\}} |D_{F,i}^*| = \sum_{i \in V_m^{(3)} \backslash \{F(0)\}} |\mathbb{F}_3^* \widetilde{D_{F,i}^*}|$, which implies that for any $i \in V_m^{(3)} \backslash \{F(0)\}$, we have $D_{F,i}^* = \mathbb{F}_3^* \widetilde{D_{F,i}^*}$. Therefore, for any $i \in V_m^{(3)}$, $D_{F,i}^*$ is a regular $(3^n, s_i(3^{\frac{n}{2}} + 1), -3^{\frac{n}{2}} + s_i^2 + 3s_i, s_i^2 + s_i)$ partial difference set, where $s_i = 1 - \delta_{F(0)}(i)$. By Proposition 3, F is a vectorial dual-bent function with Condition A.

If $F(0) \in I$, then $D^*_{F,F(0)}$ is nonempty and by Inequality (9), $|I|p^{\frac{n}{2}-m} + \varepsilon \geq p^{\frac{n}{2}} + \varepsilon$, that is, $|I| \geq p^m$. By $|I| \leq p^m$, we have $|I| = p^m$, $I = V_m^{(p)}$. In this case, $\sum_{i \in V_m^{(p)}} |D^*_{F,i}| = \sum_{i \in V_m^{(p)}} |\mathbb{F}_p^* \widetilde{D^*_{F,i}}|$, which implies that for any $i \in V_m^{(p)}$, we have $D^*_{F,i} = \mathbb{F}_p^* \widetilde{D^*_{F,i}}$. Therefore, for any $i \in V_m^{(p)}$, $D^*_{F,i}$ is a regular $(p^n, s_i(p^{\frac{n}{2}} - \varepsilon), \varepsilon p^{\frac{n}{2}} + s_i^2 - 3\varepsilon s_i, s_i^2 - \varepsilon s_i)$ partial difference set, where $s_i = p^{\frac{n}{2}-m} + \varepsilon \delta_{F(0)}(i)$. By Proposition 3, F is a vectorial dual-bent function with Condition A.

Furthermore, the analysis mentioned above shows that statement (3) holds. ■

The following corollary is directly from Theorem 2 and Remark 1.

Corollary 3. *Let $F : V_n^{(2)} \to V_m^{(2)}$, where $n \geq 4$ is even and $2 \leq m \leq \frac{n}{2}$. Define*

$$C_{D^*_{F,i}} = \{c_\alpha = (\langle \alpha, x \rangle_n)_{x \in D^*_{F,i}} : \alpha \in V_n^{(2)}\}, i \in V_m^{(2)}.$$

The following two statements are equivalent.

(1) *F is a vectorial dual-bent function with $(F_c)^* = (F^*)_c, c \in {V_m^{(2)}}^*$.*

(2) *For any $i \in V_m^{(2)}$, $C_{D^*_{F,i}}$ is a two-weight $[2^{n-m} - 2^{\frac{n}{2}-m} + \delta_{F(0)}(i)(2^{\frac{n}{2}} - 1), n]$ projective linear code and the two nonzero weights are $w_1 = 2^{n-m-1}, w_2 = 2^{n-m-1} - 2^{\frac{n}{2}-1} + \delta_{F(0)}(i)2^{\frac{n}{2}}$.*

V. A CHARACTERIZATION OF VECTORIAL DUAL-BENT FUNCTIONS WITH CONDITION A IN TERMS OF GENERALIZED HADAMRAD MATRICES

In this section, we give a characterization of vectorial dual-bent functions with Condition A in terms of generalized Hadamard matrices.

Since the case of p being odd is more complicated, we first consider the case of $p = 2$.

Theorem 3. *Let $F : V_n^{(2)} \to V_m^{(2)}$, where $n \geq 4$ is even and $2 \leq m \leq \frac{n}{2}$. For any $c \in {V_m^{(2)}}^*$, define*

$$H_c = \left[(-1)^{F_c(x+y)}\right]_{x,y \in V_n^{(2)}},$$

where $F_c(x) = \langle c, F(x) \rangle_m$. The following two statements are equivalent.

(1) *F is a vectorial dual-bent function with $(F_c)^* = (F^*)_c, c \in {V_m^{(2)}}^*$.*

(2) *$H_c, c \in {V_m^{(2)}}^*$ are all Hadamard matrices and for any $c \neq d \in {V_m^{(2)}}^*$, $H_c H_d = 2^{\frac{n}{2}} H_{c+d}$.*

Proof: (1) ⇒ (2): Since F is a vectorial bent function, that is, $F_c, c \in {V_m^{(2)}}^*$ are all Boolean bent functions, by Lemma 5 we have that $H_c, c \in {V_m^{(2)}}^*$ are all Hadamard matrices. For a matrix

$M=[a_{i,j}]$, denote $a_{i,j}$ by $(M)_{i,j}$. For any $c\neq d\in {V_m^{(2)}}^*$ and $i,j\in V_n^{(2)}$, since F is vectorial bent, we have

$$\begin{aligned}(H_cH_d)_{i,j}&=\sum_{u\in V_n^{(2)}}(-1)^{F_c(u+i)+F_d(u+j)}\\&=2^{-2n}\sum_{u\in V_n^{(2)}}\sum_{x\in V_n^{(2)}}W_{F_c}(x)(-1)^{\langle u+i,x\rangle_n}\sum_{y\in V_n^{(2)}}W_{F_d}(y)(-1)^{\langle u+j,y\rangle_n}\\&=2^{-n}\sum_{u\in V_n^{(2)}}\sum_{x,y\in V_n^{(2)}}(-1)^{(F_c)^*(x)+(F_d)^*(y)+\langle u+i,x\rangle_n+\langle u+j,y\rangle_n}\\&=2^{-n}\sum_{x,y\in V_n^{(2)}}(-1)^{(F_c)^*(x)+(F_d)^*(y)+\langle i,x\rangle_n+\langle j,y\rangle_n}\sum_{u\in V_n^{(2)}}(-1)^{\langle u,x+y\rangle_n}\\&=\sum_{x\in V_n^{(2)}}(-1)^{(F_c)^*(x)+(F_d)^*(x)+\langle i+j,x\rangle_n}\\&=W_{(F_c)^*+(F_d)^*}(i+j).\end{aligned}\tag{10}$$

Since F is vectorial dual-bent with $(F_c)^*=(F^*)_c, c\in {V_m^{(2)}}^*$, we have $(F_c)^*+(F_d)^*=(F^*)_c+(F^*)_d=(F^*)_{c+d}=(F_{c+d})^*$. Thus by Eq. (5) and Eq. (10) we obtain

$$(H_cH_d)_{i,j}=W_{(F_{c+d})^*}(i+j)=2^{\frac{n}{2}}(-1)^{F_{c+d}(i+j)}=2^{\frac{n}{2}}(H_{c+d})_{i,j},$$

which implies that $H_cH_d=2^{\frac{n}{2}}H_{c+d}$.

(2) $\Rightarrow$ (1): Since $H_c, c\in {V_m^{(2)}}^*$ are all Hadamard matrices, by Lemma 5 we have that $F_c, c\in {V_m^{(2)}}^*$ are all Boolean bent functions, that is, F is vectorial bent. For any $c\neq d\in {V_m^{(2)}}^*$ and $i,j\in V_n^{(2)}$, from Eq. (10) and $H_cH_d=2^{\frac{n}{2}}H_{c+d}$, we have

$$W_{(F_c)^*+(F_d)^*}(i+j)=2^{\frac{n}{2}}(-1)^{F_{c+d}(i+j)}.\tag{11}$$

By Eq. (11), for any $c\neq d\in {V_m^{(2)}}^*$, we have that $(F_c)^*+(F_d)^*$ is a Boolean bent function and $((F_c)^*+(F_d)^*)^*=F_{c+d}$, which implies that

$$(F_c)^*+(F_d)^*=(F_{c+d})^*.\tag{12}$$

Let $\{\alpha_1,\ldots,\alpha_m\}$ be an arbitrary fixed basis of $V_m^{(2)}$. For any $x\in V_n^{(2)}$, let $G(x)\in V_m^{(2)}$ be given by the following equation system

$$\begin{cases} \langle \alpha_1, G(x) \rangle_m = (F_{\alpha_1})^*(x), \\ \langle \alpha_2, G(x) \rangle_m = (F_{\alpha_2})^*(x), \\ \quad \vdots \\ \langle \alpha_m, G(x) \rangle_m = (F_{\alpha_m})^*(x). \end{cases}$$

Then G is a function from $V_n^{(2)}$ to $V_m^{(2)}$ satisfying $G_{\alpha_i} = (F_{\alpha_i})^*, 1 \leq i \leq m$. For any $c \in V_m^{(2)^*}$, denote c by $c = \alpha_{i_1} + \cdots + \alpha_{i_t}$. Then

$$\begin{aligned} G_c(x) &= \langle c, G(x) \rangle_m \\ &= \langle \alpha_{i_1}, G(x) \rangle_m + \cdots + \langle \alpha_{i_t}, G(x) \rangle_m \\ &= G_{\alpha_{i_1}}(x) + \cdots + G_{\alpha_{i_t}}(x) \\ &= (F_{\alpha_{i_1}})^*(x) + \cdots + (F_{\alpha_{i_t}})^*(x). \end{aligned} \tag{13}$$

Combine Eq. (12) and Eq. (13), we obtain

$$G_c(x) = (F_{\alpha_{i_1} + \cdots + \alpha_{i_t}})^*(x) = (F_c)^*(x).$$

For any $c \in V_m^{(2)^*}$, since $G_c = (F_c)^*$ is a Boolean bent function, we have that G is vectorial bent. Therefore, F is vectorial dual-bent with $(F_c)^* = (F^*)_c, c \in V_m^{(2)^*}$, where $F^* = G$. ■

Below we give an example to illustrate Theorem 3.

Example 1. *Let* $F : \mathbb{F}_{2^6} \times \mathbb{F}_{2^6} \to \mathbb{F}_{2^2}$ *be defined by*

$$F(x_1, x_2) = Tr_2^6(x_1 x_2^{58}).$$

Then by Proposition 3 of [23]*,* F *is a vectorial dual-bent function with Condition A. For any* $c \in \mathbb{F}_{2^2}^*$*, define*

$$H_c = \left[(-1)^{Tr_1^6(c(x_1+y_1)(x_2+y_2)^{58})} \right]_{(x_1,x_2),(y_1,y_2) \in \mathbb{F}_{2^6} \times \mathbb{F}_{2^6}}.$$

Then by Theorem 3, $H_c, c \in \mathbb{F}_{2^2}^*$ *are all Hadamard matrices and* $H_c H_d = 64 H_{c+d}$ *for any* $c \neq d \in \mathbb{F}_{2^2}^*$*.*

In the following, we consider the case of p being odd. First, we need a lemma.

For an odd prime p, define $U_p^{(1)} = \{\zeta_p^i : 0 \leq i \leq p-1\}$ and $U_p^{(-1)} = \{-\zeta_p^i : 0 \leq i \leq p-1\}$.

Lemma 7. *Let p be an odd prime. Let $F : V_n^{(p)} \rightarrow V_m^{(p)}$, where $n \geq 4$ is even and $2 \leq m \leq \frac{n}{2}$. For any $c \in V_m^{(p)^*}$ and $z \in V_n^{(p)}$, define*

$$H_c^{(z)} = \left[\zeta_p^{F_c(x-y)-\langle z,x-y\rangle_n}\right]_{x,y\in V_n^{(p)}}, \tag{14}$$

where $F_c(x) = \langle c, F(x)\rangle_m$. The following two statements are equivalent.

(1) F is a vectorial bent function with all component functions $F_c, c \in V_m^{(p)^*}$ being regular or weakly regular but not regular, that is, $\varepsilon_{F_c} = \varepsilon$ for all $c \in V_m^{(p)^*}$, where $\varepsilon \in \{\pm 1\}$ is a constant.

(2) $H_c^{(z)}, c \in V_m^{(p)^*}, z \in V_n^{(p)}$ are all generalized Hadamard matrices for which there exists a constant $\varepsilon \in \{\pm 1\}$ such that

$$p^{-\frac{n}{2}} \sum_{i\in V_n^{(p)}} (H_c^{(z)})_{i,0} \in U_p^{(\varepsilon)} \ \text{ for all } \ c \in V_m^{(p)^*}, z \in V_n^{(p)}, \tag{15}$$

where for a matrix $M = [a_{i,j}]$, denote $a_{i,j}$ by $(M)_{i,j}$.

Proof: (1) $\Rightarrow$ (2): Since F is a vectorial bent function, $F_c, c \in V_m^{(p)^*}$ are all bent functions. Further, $F_c(x) - \langle z, x\rangle_n$ is bent for any $c \in V_m^{(p)^*}$ and $z \in V_n^{(p)}$ since $W_{F_c(x)-\langle z,x\rangle_n}(a) = W_{F_c}(z + a), a \in V_n^{(p)}$. By Lemma 5, we have that $H_c^{(z)}, c \in V_m^{(p)^*}, z \in V_n^{(p)}$ are all generalized Hadamard matrices. For any $c \in V_m^{(p)^*}$ and $z \in V_n^{(p)}$, we have

$$p^{-\frac{n}{2}} \sum_{i\in V_n^{(p)}} (H_c^{(z)})_{i,0} = p^{-\frac{n}{2}} \sum_{i\in V_n^{(p)}} \zeta_p^{F_c(i)-\langle z,i\rangle_n} = p^{-\frac{n}{2}} W_{F_c}(z). \tag{16}$$

Since F is vectorial bent with $\varepsilon_{F_c} = \varepsilon, c \in V_m^{(p)^*}$, by Eq. (16) we have that Eq. (15) holds.

(2) $\Rightarrow$ (1): Since $H_c^{(0)}, c \in V_m^{(p)^*}$ are all generalized Hadamard matrices, by Lemma 5 we have that $F_c, c \in V_m^{(p)^*}$ are all bent functions, then F is vectorial bent. By Eq. (15) and Eq. (16), we have that for any $c \in V_m^{(p)^*}$, the component function F_c is weakly regular with $\varepsilon_{F_c} = \varepsilon$. ∎

Based on Lemma 7, with a similar proof as Theorem 3, we give the following theorem, which provides a characterization of vectorial dual-bent functions with Condition A in terms of generalized Hadamard matrices when p is an odd prime.

Theorem 4. *Let p be an odd prime. Let $F : V_n^{(p)} \rightarrow V_m^{(p)}$, where $n \geq 4$ is even and $2 \leq m \leq \frac{n}{2}$. Let matrices $H_c^{(z)}, c \in V_m^{(p)^*}, z \in V_n^{(p)}$ be defined by* Eq. (14). *For simplicity, denote $H_c^{(0)}$ by H_c. The following two statements are equivalent.*

(1) F is a vectorial dual-bent function with Condition A.

(2) $H_c^{(z)}, c \in {V_m^{(p)}}^*, z \in V_n^{(p)}$ *are all generalized Hadamard matrices for which there exists a constant* $\varepsilon \in \{\pm 1\}$ *with* $\varepsilon = 1$ *if* $p \neq 3$ *such that* Eq. (15) *holds and* $H_c \overline{H_d}^\top = \varepsilon p^{\frac{n}{2}} H_{c-d}$ *for any* $c \neq d \in {V_m^{(p)}}^*$.

Proof: (1) $\Rightarrow$ (2): Since F is a vectorial dual-bent function with Condition A, by Proposition 2, we have that for any $c \in {V_m^{(p)}}^*$, F_c is weakly regular bent with $\varepsilon_{F_c} = \varepsilon$, where $\varepsilon \in \{\pm 1\}$ is a constant with $\varepsilon = 1$ if $p \neq 3$. By Lemma 7, we have that $H_c^{(z)}, c \in {V_m^{(p)}}^*, z \in V_n^{(p)}$ are all generalized Hadamard matrices and Eq. (15) holds. For any $c \neq d \in {V_m^{(p)}}^*$ and $i, j \in V_n^{(p)}$, since F is vectorial bent with $\varepsilon_{F_c} = \varepsilon, c \in {V_m^{(p)}}^*$, we have

$$\begin{aligned}
(H_c \overline{H_d}^\top)_{i,j} &= \sum_{u \in V_n^{(p)}} \zeta_p^{F_c(i-u) - F_d(j-u)} \\
&= p^{-2n} \sum_{u \in V_n^{(p)}} \sum_{x \in V_n^{(p)}} W_{F_c}(x) \zeta_p^{\langle i-u, x \rangle_n} \sum_{y \in V_n^{(p)}} \overline{W_{F_d}(y)} \zeta_p^{-\langle j-u, y \rangle_n} \\
&= p^{-n} \sum_{u \in V_n^{(p)}} \sum_{x \in V_n^{(p)}} \varepsilon \zeta_p^{(F_c)^*(x) + \langle i-u, x \rangle_n} \sum_{y \in V_n^{(p)}} \varepsilon \zeta_p^{-(F_d)^*(y) - \langle j-u, y \rangle_n} \\
&= p^{-n} \sum_{x, y \in V_n^{(p)}} \zeta_p^{(F_c)^*(x) - (F_d)^*(y) + \langle i, x \rangle_n - \langle j, y \rangle_n} \sum_{u \in V_n^{(p)}} \zeta_p^{\langle u, y-x \rangle_n} \\
&= \sum_{x \in V_n^{(p)}} \zeta_p^{(F_c)^*(x) - (F_d)^*(x) + \langle i-j, x \rangle_n} \\
&= W_{(F_c)^* - (F_d)^*}(j-i).
\end{aligned} \tag{17}$$

Since F is vectorial dual-bent with $(F_c)^* = (F^*)_c, c \in {V_m^{(p)}}^*$, for any $c \neq d \in {V_m^{(p)}}^*$ we have $(F_c)^* - (F_d)^* = (F^*)_c - (F^*)_d = (F^*)_{c-d} = (F_{c-d})^*$. Hence by Eq. (5) and Eq. (17) we have

$$(H_c \overline{H_d}^\top)_{i,j} = W_{(F_{c-d})^*}(j-i) = \varepsilon p^{\frac{n}{2}} \zeta_p^{F_{c-d}(i-j)} = \varepsilon p^{\frac{n}{2}} (H_{c-d})_{i,j},$$

which implies that $H_c \overline{H_d}^\top = \varepsilon p^{\frac{n}{2}} H_{c-d}$.

(2) $\Rightarrow$ (1): Since $H_c^{(z)}, c \in {V_m^{(p)}}^*, z \in V_n^{(p)}$ are all generalized Hadamard matrices and there is a constant $\varepsilon \in \{\pm 1\}$ such that Eq. (15) holds, by Lemma 7 we have that F is a vectorial bent function for which for any $c \in {V_m^{(p)}}^*$, F_c is weakly regular with $\varepsilon_{F_c} = \varepsilon$. For any $c \neq d \in {V_m^{(p)}}^*$ and $i, j \in V_n^{(p)}$, by Eq. (17) and $H_c \overline{H_d}^\top = \varepsilon p^{\frac{n}{2}} H_{c-d}$, we have

$$W_{(F_c)^* - (F_d)^*}(j-i) = \varepsilon p^{\frac{n}{2}} \zeta_p^{F_{c-d}(i-j)}. \tag{18}$$

By Eq. (18), for any $c \neq d \in V_m^{(p)^*}$, we have that $(F_c)^* - (F_d)^*$ is a weakly regular bent function with $\varepsilon_{(F_c)^*-(F_d)^*} = \varepsilon$ and $((F_c)^* - (F_d)^*)^*(x) = F_{c-d}(-x)$, which implies that

$$(F_c)^* - (F_d)^* = (F_{c-d})^*. \tag{19}$$

Since Eq. (19) holds for all $c \neq d \in V_m^{(p)^*}$, for any $c, d \in V_m^{(p)^*}$ with $c + d \neq 0$, we have

$$(F_c)^* + (F_d)^* = (F_{c+d})^*. \tag{20}$$

Let $\{\alpha_1, \ldots, \alpha_m\}$ be an arbitrary fixed basis of $V_m^{(p)}$. For any $x \in V_n^{(p)}$, let $G(x) \in V_m^{(p)}$ be given by the following equation system

$$\begin{cases} \langle \alpha_1, G(x) \rangle_m = (F_{\alpha_1})^*(x), \\ \langle \alpha_2, G(x) \rangle_m = (F_{\alpha_2})^*(x), \\ \qquad \vdots \\ \langle \alpha_m, G(x) \rangle_m = (F_{\alpha_m})^*(x). \end{cases}$$

Then G is a function from $V_n^{(p)}$ to $V_m^{(p)}$ satisfying $G_{\alpha_i} = (F_{\alpha_i})^*, 1 \leq i \leq m$. For any $c \in V_m^{(p)^*}$, denote c by $c = a_{i_1}\alpha_{i_1} + \cdots + a_{i_t}\alpha_{i_t}$, where $a_{i_j} \in \mathbb{F}_p^*, 1 \leq j \leq t$. Then by Eq. (20) we have

$$\begin{aligned} G_c(x) &= \langle c, G(x) \rangle_m \\ &= a_{i_1}\langle \alpha_{i_1}, G(x) \rangle_m + \cdots + a_{i_t}\langle \alpha_{i_t}, G(x) \rangle_m \\ &= a_{i_1}G_{\alpha_{i_1}}(x) + \cdots + a_{i_t}G_{\alpha_{i_t}}(x) \\ &= a_{i_1}(F_{\alpha_{i_1}})^*(x) + \cdots + a_{i_t}(F_{\alpha_{i_t}})^*(x) \\ &= (F_{a_{i_1}\alpha_{i_1}})^*(x) + \cdots + (F_{a_{i_t}\alpha_{i_t}})^*(x) \\ &= (F_{a_{i_1}\alpha_{i_1}+\cdots+a_{i_t}\alpha_{i_t}})^*(x) \\ &= (F_c)^*(x). \end{aligned}$$

For any $c \in V_m^{(p)^*}$, since $G_c = (F_c)^*$ is bent, we have that G is vectorial bent. Therefore, F is vectorial dual-bent with $(F_c)^* = (F^*)_c, c \in V_m^{(p)^*}$, where $F^* = G$. ■

Below we give an example to illustrate Theorem 4.

Example 2. *Let* $F : \mathbb{F}_{3^6} \times \mathbb{F}_{3^6} \to \mathbb{F}_{3^2}$ *be defined by*

$$F(x_1, x_2) = Tr_2^6(x_1 x_2^{717}).$$

Then by Proposition 3 of [23]*, F is a vectorial dual-bent function with Condition A and the corresponding $\varepsilon = 1$. For any $c \in \mathbb{F}_{3^2}^*, z = (z_1, z_2) \in \mathbb{F}_{3^6} \times \mathbb{F}_{3^6}$, define*

$$H_c^{(z)} = \left[\zeta_3^{Tr_1^6(c(x_1-y_1)(x_2-y_2)^{717}) - Tr_1^6(z_1(x_1-y_1)+z_2(x_2-y_2))}\right]_{(x_1,x_2),(y_1,y_2)\in\mathbb{F}_{3^6}\times\mathbb{F}_{3^6}}.$$

Denote $H_c^{(0)}$ by H_c. Then by Theorem 4, $H_c^{(z)}, c \in \mathbb{F}_{3^2}^*, z = (z_1, z_2) \in \mathbb{F}_{3^6} \times \mathbb{F}_{3^6}$ are all generalized Hadamard matrices for which

$$729^{-1} \sum_{i\in\mathbb{F}_{3^6}\times\mathbb{F}_{3^6}} (H_c^{(z)})_{i,(0,0)} \in \{1, \zeta_3, \zeta_3^2\} \text{ for all } c \in \mathbb{F}_{3^2}^*, z = (z_1, z_2) \in \mathbb{F}_{3^6} \times \mathbb{F}_{3^6},$$

and $H_c\overline{H_d}^\top = 729H_{c-d}$ for any $c \neq d \in \mathbb{F}_{3^2}^*$.

VI. A CHARACTERIZATION OF VECTORIAL DUAL-BENT FUNCTIONS WITH CONDITION A IN TERMS OF BENT PARTITIONS WHEN $p = 2$

When p is an odd prime, a characterization of vectorial dual-bent functions with Condition A in terms of bent partitions has been given in [23], see Lemma 2. In this section, when $p = 2$, we give a characterization of vectorial dual-bent functions with Condition A in terms of bent partitions. First, we give a lemma, which characterizes bent partitions of $V_n^{(2)}$ of depth 2^m in terms of vectorial bent functions.

Lemma 8. *Let $\Gamma = \{A_i, i \in V_m^{(2)}\}$ be a partition of $V_n^{(2)}$, where $n \geq 4$ is even, $2 \leq m \leq \frac{n}{2}$. Define $F : V_n^{(2)} \to V_m^{(2)}$ as*

$$F(x) = \sum_{i\in V_m^{(2)}} \delta_{A_i}(x)i.$$

The following two statements are equivalent.

(1) Γ *is a bent partition.*

(2) *F is a vectorial bent function for which there exists a function $G : V_n^{(2)} \to V_m^{(2)}$ and a set $S \subseteq V_n^{(2)}$ such that*

$$(F_c)^*(x) = G_c(x) + \delta_S(x), c \in {V_m^{(2)}}^*, x \in V_n^{(2)}.$$

Proof: (1) $\Rightarrow$ (2): By the result in [9], for any Boolean bent function $f : V_n^{(2)} \to \mathbb{F}_2$ and $u \in V_n^{(2)}, j \in \mathbb{F}_2$, we have

$$\chi_u(D_{f,j}) = \begin{cases} 2^{n-1}\delta_0(u) + 2^{\frac{n}{2}-1}, & \text{if } f^*(u) = j, \\ 2^{n-1}\delta_0(u) - 2^{\frac{n}{2}-1}, & \text{if } f^*(u) = j+1. \end{cases} \tag{21}$$

By Eq. (21) and the definition of bent partitions, we have that for any fixed $u \in V_n^{(2)}$,

$$\chi_u(D_{F,i}) = \chi_u(A_i) = \begin{cases} 2^{n-m}\delta_0(u) - 2^{\frac{n}{2}-m}, & \text{if } i \in V_m^{(2)} \backslash \{G(u)\}, \\ 2^{n-m}\delta_0(u) - 2^{\frac{n}{2}-m} + 2^{\frac{n}{2}}, & \text{if } i = G(u), \end{cases} \tag{22}$$

or

$$\chi_u(D_{F,i}) = \chi_u(A_i) = \begin{cases} 2^{n-m}\delta_0(u) + 2^{\frac{n}{2}-m}, & \text{if } i \in V_m^{(2)} \backslash \{G(u)\}, \\ 2^{n-m}\delta_0(u) + 2^{\frac{n}{2}-m} - 2^{\frac{n}{2}}, & \text{if } i = G(u), \end{cases} \tag{23}$$

where G is some function from $V_n^{(2)}$ to $V_m^{(2)}$. Let

$$S = \{u \in V_n^{(2)} : \chi_u(D_{F,i}), i \in V_m^{(2)} \text{ satisfy Eq. (23)}\}.$$

Then for any $c \in V_m^{(2)*}$ and $u \in V_n^{(2)}$ we obtain

$$\begin{aligned} W_{F_c}(u) &= \sum_{x \in V_n^{(2)}} (-1)^{\langle c, F(x)\rangle_m + \langle u, x\rangle_n} \\ &= \sum_{i \in V_m^{(2)}} \sum_{x \in V_n^{(2)} : F(x) = i} (-1)^{\langle c, F(x)\rangle_m + \langle u, x\rangle_n} \\ &= \sum_{i \in V_m^{(2)}} (-1)^{\langle c, i\rangle_m} \sum_{x \in V_n^{(2)} : F(x) = i} (-1)^{\langle u, x\rangle_n} \\ &= \sum_{i \in V_m^{(2)}} (-1)^{\langle c, i\rangle_m} \chi_u(D_{F,i}) \\ &= \begin{cases} \sum_{i \in V_m^{(2)}} (-1)^{\langle c, i\rangle_m} (2^{n-m}\delta_0(u) + 2^{\frac{n}{2}-m} - 2^{\frac{n}{2}}\delta_{G(u)}(i)), & \text{if } u \in S, \\ \sum_{i \in V_m^{(2)}} (-1)^{\langle c, i\rangle_m} (2^{n-m}\delta_0(u) - 2^{\frac{n}{2}-m} + 2^{\frac{n}{2}}\delta_{G(u)}(i)), & \text{if } u \notin S, \end{cases} \\ &= \begin{cases} (2^{n-m}\delta_0(u) + 2^{\frac{n}{2}-m}) \sum_{i \in V_m^{(2)}} (-1)^{\langle c, i\rangle_m} - 2^{\frac{n}{2}}(-1)^{\langle c, G(u)\rangle_m}, & \text{if } u \in S, \\ (2^{n-m}\delta_0(u) - 2^{\frac{n}{2}-m}) \sum_{i \in V_m^{(2)}} (-1)^{\langle c, i\rangle_m} + 2^{\frac{n}{2}}(-1)^{\langle c, G(u)\rangle_m}, & \text{if } u \notin S, \end{cases} \\ &= \begin{cases} 2^{\frac{n}{2}}(-1)^{1 + \langle c, G(u)\rangle_m}, & \text{if } u \in S, \\ 2^{\frac{n}{2}}(-1)^{\langle c, G(u)\rangle_m}, & \text{if } u \notin S, \end{cases} \end{aligned}$$

which implies that F is a vectorial bent function with $(F_c)^*(x) = G_c(x) + \delta_S(x), c \in V_m^{(2)*}, x \in V_n^{(2)}$.

(2) $\Rightarrow$ (1): With the same argument as in the proof of Proposition 3 of [21], for any $u \in V_n^{(2)}, i \in V_m^{(2)}$ we have

$$\chi_u(D_{F,i}) = 2^{n-m}\delta_0(u) + 2^{-m} \sum_{c \in {V_m^{(2)}}^*} W_{F_c}(u)(-1)^{\langle c,i\rangle_m}. \tag{24}$$

Since F is a vectorial bent function with $(F_c)^*(x) = G_c(x) + \delta_S(x), c \in {V_m^{(2)}}^*$, by Eq. (24) we have

$$\begin{aligned}
\chi_u(A_i) &= \chi_u(D_{F,i}) \\
&= 2^{n-m}\delta_0(u) + 2^{\frac{n}{2}-m} \sum_{c \in {V_m^{(2)}}^*} (-1)^{(F_c)^*(u)+\langle c,i\rangle_m} \\
&= 2^{n-m}\delta_0(u) + 2^{\frac{n}{2}-m} \sum_{c \in {V_m^{(2)}}^*} (-1)^{G_c(u)+\delta_S(u)+\langle c,i\rangle_m} \\
&= 2^{n-m}\delta_0(u) + (-1)^{\delta_S(u)} 2^{\frac{n}{2}-m} \sum_{c \in {V_m^{(2)}}^*} (-1)^{\langle c,G(u)+i\rangle_m} \\
&= 2^{n-m}\delta_0(u) + (-1)^{\delta_S(u)} 2^{\frac{n}{2}-m}(2^m\delta_{G(u)}(i) - 1).
\end{aligned} \tag{25}$$

For any union D of 2^{m-1} sets of $\{A_i, i \in V_m^{(2)}\}$, we have

$$\chi_u(D) = \begin{cases} 2^{n-1}\delta_0(u) + (-1)^{\delta_S(u)} 2^{\frac{n}{2}-1}, & \text{if } A_{G(u)} \subseteq D, \\ 2^{n-1}\delta_0(u) - (-1)^{\delta_S(u)} 2^{\frac{n}{2}-1}, & \text{if } A_{G(u)} \nsubseteq D. \end{cases} \tag{26}$$

Let $f : V_n^{(2)} \to \mathbb{F}_2$ be a function for which for each $j \in \mathbb{F}_2$, there are exactly 2^{m-1} sets A_i in Γ in its preimage set. By Eq. (26), for any $u \in V_n^{(2)}$ we have

$$\chi_u(D_{f,j}) = \begin{cases} 2^{n-1}\delta_0(u) + (-1)^{\delta_S(u)} 2^{\frac{n}{2}-1}, & \text{if } j = g(u), \\ 2^{n-1}\delta_0(u) - (-1)^{\delta_S(u)} 2^{\frac{n}{2}-1}, & \text{if } j = g(u) + 1, \end{cases}$$

where $g(u) = f(A_{G(u)})$. Then we obtain

$$\begin{aligned}
W_f(u) &= \sum_{x \in V_n^{(2)}} (-1)^{f(x)+\langle u,x\rangle_n} \\
&= \sum_{j \in \mathbb{F}_2} \sum_{x \in V_n^{(2)} : f(x)=j} (-1)^{f(x)+\langle u,x\rangle_n} \\
&= \sum_{j \in \mathbb{F}_2} (-1)^j \sum_{x \in V_n^{(2)} : f(x)=j} (-1)^{\langle u,x\rangle_n} \\
&= \sum_{j \in \mathbb{F}_2} (-1)^j \chi_u(D_{f,j})
\end{aligned}$$

$$= (2^{n-1}\delta_0(u) + (-1)^{\delta_S(u)}2^{\frac{n}{2}-1})(-1)^{g(u)} + (2^{n-1}\delta_0(u) - (-1)^{\delta_S(u)}2^{\frac{n}{2}-1})(-1)^{g(u)+1}$$

$$= 2^{\frac{n}{2}}(-1)^{g(u)+\delta_S(u)},$$

which implies that f is a Boolean bent function, and thus Γ is a bent partition. ∎

The following theorem gives a characterization of vectorial dual-bent functions $F : V_n^{(2)} \rightarrow V_m^{(2)}$ with Condition A in terms of bent partitions.

Theorem 5. *Let $F : V_n^{(2)} \rightarrow V_m^{(2)}$, where $n \geq 4$ is even and $2 \leq m \leq \frac{n}{2}$. The following two statements are equivalent.*

(1) F is a vectorial dual-bent function with $(F_c)^* = (F^*)_c, c \in V_m^{(2)*}$.

(2) $\Gamma = \{D_{F,i}, i \in V_m^{(2)}\}$ is a bent partition of $V_n^{(2)}$ with $\chi_u(D_{F,i}) \in \{-2^{\frac{n}{2}-m}, -2^{\frac{n}{2}-m} + 2^{\frac{n}{2}}\}, u \in V_n^{(2)*}, i \in V_m^{(2)}$.

Proof: By the proof of Lemma 8, F is a vectorial dual-bent function with $(F_c)^* = (F^*)_c, c \in V_m^{(2)*}$ if and only if $\Gamma = \{D_{F,i}, i \in V_m^{(2)}\}$ is a bent partition with $\chi_u(D_{F,i}) \in \{2^{n-m}\delta_0(u) - 2^{\frac{n}{2}-m}, 2^{n-m}\delta_0(u) - 2^{\frac{n}{2}-m} + 2^{\frac{n}{2}}\}, u \in V_n^{(2)}, i \in V_m^{(2)}$. In the following, we only need to show that when $\Gamma = \{D_{F,i}, i \in V_m^{(2)}\}$ is a bent partition and $\chi_u(D_{F,i}) \in \{-2^{\frac{n}{2}-m}, -2^{\frac{n}{2}-m} + 2^{\frac{n}{2}}\}, u \in V_n^{(2)*}, i \in V_m^{(2)}$, then $\chi_0(D_{F,i}) = |D_{F,i}| \in \{2^{n-m} - 2^{\frac{n}{2}-m}, 2^{n-m} - 2^{\frac{n}{2}-m} + 2^{\frac{n}{2}}\}, i \in V_m^{(2)}$.

For any $i \in V_m^{(2)}$, let $b_i = |\{u \in V_n^{(2)*} : \chi_u(D_{F,i}) = -2^{\frac{n}{2}-m} + 2^{\frac{n}{2}}\}|$. Assume that there is i such that $|D_{F,i}| \notin \{2^{n-m} - 2^{\frac{n}{2}-m}, 2^{n-m} - 2^{\frac{n}{2}-m} + 2^{\frac{n}{2}}\}$. Then by Lemma 1, there exists i_0 such that $|D_{F,i_0}| = 2^{n-m} + 2^{\frac{n}{2}-m} - 2^{\frac{n}{2}}, |D_{F,i}| = 2^{n-m} + 2^{\frac{n}{2}-m}, i \neq i_0$. Let $j \in V_m^{(2)}$ with $j \neq F(0), i_0$. Then $0 \notin D_{F,j}$ and $|D_{F,j}| = 2^{n-m} + 2^{\frac{n}{2}-m}$. Since

$$\sum_{u \in V_n^{(2)}} \chi_u(D_{F,j}) = \sum_{u \in V_n^{(2)}} \sum_{x \in D_{F,j}} (-1)^{\langle u,x \rangle_n} = \sum_{x \in D_{F,j}} \sum_{u \in V_n^{(2)}} (-1)^{\langle u,x \rangle_n} = 2^n \delta_{D_{F,j}}(0) = 0,$$

and

$$\sum_{u \in V_n^{(2)}} \chi_u(D_{F,j}) = |D_{F,j}| + (-2^{\frac{n}{2}-m} + 2^{\frac{n}{2}})b_j - 2^{\frac{n}{2}-m}(2^n - 1 - b_j)$$

$$= 2^{\frac{n}{2}-m}(2^{\frac{n}{2}} - 2^n + 2 + 2^m b_j),$$

we have

$$2^n = 2^{\frac{n}{2}} + 2^m b_j + 2. \tag{27}$$

Note that $b_j \neq 0$ by $n \geq 4$. Since $m \leq \frac{n}{2}$, we have $2^m \mid 2^n, 2^m \mid 2^{\frac{n}{2}}, 2^m \mid 2^m b_j$. Thus by Eq. (27), $2^m \mid 2$, which contradicts $m \geq 2$. Therefore, $|D_{F,i}| \in \{2^{n-m} - 2^{\frac{n}{2}-m}, 2^{n-m} - 2^{\frac{n}{2}-m} + 2^{\frac{n}{2}}\}, i \in V_m^{(2)}$.

∎

Below we give an example to illustrate Theorem 5.

Example 3. *Let* $F : \mathbb{F}_{2^6} \times \mathbb{F}_{2^6} \times \mathbb{F}_{2^4} \times \mathbb{F}_{2^4} \to \mathbb{F}_{2^2}$ *be defined by*

$$F(x_1, x_2, x_3, x_4) = (Tr_2^4(x_3 x_4^{14}))^3 Tr_2^6(x_1^{52} x_2 - x_1 x_2^{58}) + Tr_2^6(x_1 x_2^{58}) + Tr_2^4(\alpha x_3 x_4^{14}),$$

where α *is a primitive element of* $\mathbb{F}_{2^4}$*. By Theorem 5 of* [23]*,* F *is a vectorial dual-bent function with Condition A. By Theorem 5,* $\{D_{F,i}, i \in \mathbb{F}_{2^2}\}$ *is a bent partition with* $\chi_u(D_{F,i}) \in \{-256, 768\}, u \in (\mathbb{F}_{2^6} \times \mathbb{F}_{2^6} \times \mathbb{F}_{2^4} \times \mathbb{F}_{2^4})^*, i \in \mathbb{F}_{2^2}$.

VII. New characterizations of certain bent partitions

In this section, we give some new characterizations of bent partitions with Condition $\mathcal{C}$ when p is an odd prime, and bent partitions with condition given in Theorem 5 when $p = 2$, respectively.

Theorem 6. *Let* p *be an odd prime. Let* $\Gamma = \{A_i, i \in V_m^{(p)}\}$ *be a partition of* $V_n^{(p)}$*, where* $n \geq 4$ *is even and* $2 \leq m \leq \frac{n}{2}$*. Denote* $0 \in A_{i_0}$ *and* $I = \{\sum_{i \in V_m^{(p)}} \delta_{A_i}(x) i : x \in {V_n^{(p)}}^*\}$*. The following statements are pairwise equivalent.*

(1) Γ *is a bent partition with Condition* $\mathcal{C}$*.*

(2) *For any* $i \in V_m^{(p)}$*,* A_i^* *is a regular* $(p^n, s_i(p^{\frac{n}{2}} - \varepsilon), \varepsilon p^{\frac{n}{2}} + s_i^2 - 3\varepsilon s_i, s_i^2 - \varepsilon s_i)$ *partial difference set, where* $s_i = p^{\frac{n}{2} - m} + \varepsilon \delta_{i_0}(i)$*,* $\varepsilon \in \{\pm 1\}$ *is a constant with* $\varepsilon = 1$ *if* $p \neq 3$.

(3) *Let*

$$R_{id} = \{(x, x) : x \in V_n^{(p)}\},$$

$$R_i = \{(x, y) : x, y \in V_n^{(p)}, x - y \in A_i^*\}, i \in I.$$

Then $\{R_{id}, R_i, i \in I\}$ *is an* $|I|$*-class amorphic association scheme for which* $|I| \geq 3$ *and for any* $i \in I$*, the intersection number* $p_{i,i}^{id} = p^{n-m} - \varepsilon p^{\frac{n}{2} - m} + \delta_{i_0}(i)(\varepsilon p^{\frac{n}{2}} - 1)$*, where* $\varepsilon \in \{\pm 1\}$ *is a constant with* $\varepsilon = 1$ *if* $p \neq 3$.

(4) *Let*

$$C_{\widetilde{A_i^*}} = \{c_\alpha = (\langle \alpha, x \rangle_n)_{x \in \widetilde{A_i^*}} : \alpha \in V_n^{(p)}\}, i \in I,$$

where $\widetilde{A_i^*}$ *is a subset of* A_i^* *for which any two elements in* $\widetilde{A_i^*}$ *are linearly independent and for any* $x \in A_i^*$*, there exist* $a \in \mathbb{F}_p^*, x' \in \widetilde{A_i^*}$ *such that* $x = a x'$*. Then for any* $i \in I$*,* $C_{\widetilde{A_i^*}}$ *is a* *two-weight* $[\frac{p^{n-m}-\varepsilon p^{\frac{n}{2}-m}+\delta_{i_0}(i)(\varepsilon p^{\frac{n}{2}}-1)}{p-1}, n]$ *projective linear code and the two nonzero weights are*

$$w_1 = p^{n-m-1} + \frac{1-\varepsilon+2\varepsilon\delta_{i_0}(i)}{2}p^{\frac{n}{2}-1},$$
$$w_2 = p^{n-m-1} + \frac{-1-\varepsilon+2\varepsilon\delta_{i_0}(i)}{2}p^{\frac{n}{2}-1},$$

where $\varepsilon \in \{\pm 1\}$ *is a constant with* $\varepsilon = 1$ *if* $p \neq 3$*.*

(5) *Let*

$$H_c^{(z)} = \left[\zeta_p^{\langle c, \sum_{i \in V_m^{(p)}} \delta_{A_i}(x-y)i\rangle_m - \langle z, x-y\rangle_n}\right]_{x,y \in V_n^{(p)}}, c \in V_m^{(p)*}, z \in V_n^{(p)},$$

and $H_c = H_c^{(0)}$*. Then* $H_c^{(z)}, c \in V_m^{(p)*}, z \in V_n^{(p)}$ *are all generalized Hadamard matrices for which there exists a constant* $\varepsilon \in \{\pm 1\}$ *with* $\varepsilon = 1$ *if* $p \neq 3$ *such that* Eq. (15) *holds and* $H_c\overline{H_d}^{\top} = \varepsilon p^{\frac{n}{2}} H_{c-d}$ *for any* $c \neq d \in V_m^{(p)*}$*.*

Furthermore, if any one of the above statements holds, then $I = V_m^{(p)}$ *and* $|I| = p^m$ *except one case that* $p = 3$*,* $n = 2m$ *and* $\varepsilon = -1$ *(in such a case,* $I = V_m^{(3)} \backslash \{i_0\}$ *and* $|I| = 3^m - 1$*).*

Proof: By Lemma 2, statement (1) holds if and only if $F : V_n^{(p)} \to V_m^{(p)}$ is a vectorial dual-bent function with Condition A, where

$$F(x) = \sum_{i \in V_m^{(p)}} \delta_{A_i}(x)i.$$

Then the result follows from Proposition 3 and Theorems 1, 2, 4. ■

Theorem 7. *Let* $\Gamma = \{A_i, i \in V_m^{(2)}\}$ *be a partition of* $V_n^{(2)}$*, where* $n \geq 4$ *is even and* $2 \leq m \leq \frac{n}{2}$*. Denote* $0 \in A_{i_0}$*. The following statements are pairwise equivalent.*

(1) Γ *is a bent partition with* $\chi_u(A_i) \in \{-2^{\frac{n}{2}-m}, -2^{\frac{n}{2}-m} + 2^{\frac{n}{2}}\}, u \in V_n^{(2)*}, i \in V_m^{(2)}$*.*

(2) *For any* $i \in V_m^{(2)}$*,* A_i^* *is a regular* $(2^n, s_i(2^{\frac{n}{2}}-1), 2^{\frac{n}{2}} + s_i^2 - 3s_i, s_i^2 - s_i)$ *partial difference set, where* $s_i = 2^{\frac{n}{2}-m} + \delta_{i_0}(i)$*.*

(3) *Let*

$$R_{id} = \{(x,x) : x \in V_n^{(2)}\},$$
$$R_i = \{(x,y) : x, y \in V_n^{(2)}, x + y \in A_i^*\}, i \in V_m^{(2)}.$$

Then $\{R_{id}, R_i, i \in V_m^{(2)}\}$ *is a* 2^m*-class amorphic association scheme for which for any* $i \in V_m^{(2)}$*, the intersection number* $p_{i,i}^{id} = 2^{n-m} - 2^{\frac{n}{2}-m} + \delta_{i_0}(i)(2^{\frac{n}{2}} - 1)$*.*

(4) *Let*

$$C_{A_i^*} = \{c_\alpha = (\langle \alpha, x\rangle_n)_{x \in A_i^*} : \alpha \in V_n^{(2)}\}, i \in V_m^{(2)}.$$

Then for any $i \in V_m^{(2)}$, $C_{A_i^*}$ is a two-weight $[2^{n-m} - 2^{\frac{n}{2}-m} + \delta_{i_0}(i)(2^{\frac{n}{2}} - 1), n]$ projective linear code and the two nonzero weights are

$$w_1 = 2^{n-m-1},$$

$$w_2 = 2^{n-m-1} - 2^{\frac{n}{2}-1} + \delta_{i_0}(i)2^{\frac{n}{2}}.$$

(5) *Let*

$$H_c = \left[(-1)^{\langle c, \sum_{i \in V_m^{(2)}} \delta_{A_i}(x+y)i \rangle_m}\right]_{x,y \in V_n^{(2)}}, c \in V_m^{(2)*}.$$

Then $H_c, c \in V_m^{(2)*}$ are all Hadamard matrices and $H_c H_d = 2^{\frac{n}{2}} H_{c+d}$ for any $c \neq d \in V_m^{(2)*}$.

Proof: By Theorem 5, statement (1) holds if and only if $F : V_n^{(2)} \to V_m^{(2)}$ is a vectorial dual-bent function with $(F_c)^* = (F^*)_c, c \in V_m^{(2)*}$, where

$$F(x) = \sum_{i \in V_m^{(2)}} \delta_{A_i}(x)i.$$

Then the result follows from Corollaries 1, 2, 3 and Theorem 3. ■

Remark 5. *As far as we know, the known bent partitions Γ of $V_n^{(2)}$ of depth 2^m with $m \geq 2$ given in* [2], [12], [23] *all satisfy the statement* (1) *of Theorem 7.*

VIII. Association schemes from general vectorial dual-bent functions with $F(0) = 0, F(x) = F(-x)$ and $2 \leq m \leq \frac{n}{2}$

In [3], Anbar *et al.* showed that vectorial dual-bent functions $F : V_n^{(p)} \to V_m^{(p)}$ with $F(0) = 0$ and all component functions $F_c, c \in V_m^{(p)*}$ being regular or weakly regular but not regular (that is, $\varepsilon_{F_c}, c \in V_m^{(p)*}$ are all the same) can induce association schemes. Note that for such vectorial dual-bent functions, n must be even and $m \leq \frac{n}{2}$. In this section, we give a necessary and sufficient condition on constructing association schemes from general vectorial dual-bent functions $F : V_n^{(p)} \to V_m^{(p)}$ with $F(0) = 0, F(x) = F(-x)$ and $2 \leq m \leq \frac{n}{2}$. Note that the known vectorial dual-bent functions F all satisfy $F(x) = F(-x)$. Based on our result, more association schemes can be yielded from some vectorial dual-bent functions $F : V_n^{(p)} \to V_m^{(p)}$ for which n can be odd, or n is even and $\varepsilon_{F_c}, c \in V_m^{(p)*}$ are not all the same. First, we need two lemmas.

Lemma 9. *Let $F : V_n^{(p)} \to V_m^{(p)}$ be a vectorial dual-bent function with $F(0) = 0, F(x) = F(-x)$, and F^* be a vectorial dual of F. Then F^* is also a vectorial dual-bent function with $F^*(0) = 0, F^*(x) = F^*(-x)$.*

Proof: Since F^* is a vectorial dual of F, $(F_c)^* = (F^*)_{\sigma(c)}, c \in {V_m^{(p)}}^*$ for some permutation σ over ${V_m^{(p)}}^*$. Since F is vectorial dual-bent, we have that $(F_c)^*, c \in {V_m^{(p)}}^*$ are all bent functions. By Theorem 3.1 of [15], for any p-ary bent function f whose dual f^* is also bent, $(f^*)^*(x) = f(-x)$ holds. Thus, for any $c \in {V_m^{(p)}}^*$,

$$((F^*)_c)^*(x) = ((F_{\sigma^{-1}(c)})^*)^*(x) = F_{\sigma^{-1}(c)}(-x) = F_{\sigma^{-1}(c)}(x),$$

which implies that F^* is a vectorial dual-bent function and a vectorial dual of F^* is F. When $p = 2$, obviously $F^*(x) = F^*(-x)$, and by the proof of Corollary 2 and Proposition 5 of [7], we have $F^*(0) = 0$. When p is an odd prime, for any p-ary bent function f with $f(x) = 0, f(x) = f(-x)$, by Proposition II. 1 of [16], $f^*(0) = 0, f^*(x) = f^*(-x)$. Thus for any $c \in {V_m^{(p)}}^*$, from $F_c(0) = 0, F_c(x) = F_c(-x)$, we have $(F^*)_{\sigma(c)}(0) = (F_c)^*(0) = 0, (F^*)_{\sigma(c)}(-x) = (F_c)^*(-x) = (F_c)^*(x) = (F^*)_{\sigma(c)}(x)$, which implies that $F^*(0) = 0, F^*(x) = F^*(-x)$. ■

Lemma 10. *Let $F : V_n^{(p)} \to V_m^{(p)}$ be a vectorial dual-bent function with $F(0) = 0, F(x) = F(-x)$ and $2 \le m \le \frac{n}{2}$, and F^* be a vectorial dual of F. Denote $\varepsilon_{F_c}(0) = p^{-\frac{n}{2}} \zeta_p^{-(F_c)^*(0)} W_{F_c}(0)$, $c \in {V_m^{(p)}}^*$. Then*

- *When $m < \frac{n}{2}$, then $|F({V_n^{(p)}}^*)| = |F^*({V_n^{(p)}}^*)| = p^m$;*
- *When n is even and $m = \frac{n}{2}$, if $\varepsilon_{F_c}(0) = -1$ for all $c \in {V_m^{(p)}}^*$, then $|F({V_n^{(p)}}^*)| = |F^*({V_n^{(p)}}^*)| = p^m - 1$, and if $\varepsilon_{F_c}(0), c \in {V_m^{(p)}}^*$ are not all -1, then $|F({V_n^{(p)}}^*)| = |F^*({V_n^{(p)}}^*)| = p^m$.*

Proof: By Proposition 3 of [21] and Eq. (24), for any $u \in V_n^{(p)}, i \in V_m^{(p)}$ we have

$$|D_{F,i}^*| = p^{n-m} + p^{-m} \sum_{c \in {V_m^{(p)}}^*} W_{F_c}(0) \zeta_p^{-\langle c,i \rangle_m} - \delta_0(i). \tag{28}$$

Since F^* is a vectorial dual of F, $(F_c)^* = (F^*)_{\sigma(c)}, c \in {V_m^{(p)}}^*$ for some permutation σ over ${V_m^{(p)}}^*$. For any $c \in {V_m^{(p)}}^*$, $W_{F_c}(0) = \varepsilon_{F_c}(0) p^{\frac{n}{2}} \zeta_p^{(F_c)^*(0)} = \varepsilon_{F_c}(0) p^{\frac{n}{2}} \zeta_p^{(F^*)_{\sigma(c)}(0)}$. Since F is vectorial dual-bent with $F(0) = 0, F(x) = F(-x)$, by Lemma 9 we have $F^*(x) = 0$ and $W_{F_c}(0) = \varepsilon_{F_c}(0) p^{\frac{n}{2}}, c \in {V_m^{(p)}}^*$. By Eq. (28), for any $i \in V_m^{(p)}$ we have

$$|D_{F,i}^*| = p^{n-m} + p^{\frac{n}{2}-m} \sum_{c \in {V_m^{(p)}}^*} \varepsilon_{F_c}(0) \zeta_p^{-\langle c,i \rangle_m} - \delta_0(i). \tag{29}$$

By Eq. (29), for any $i \in V_m^{(p)^*}$, if $|D_{F,i}^*| = 0$, then

$$|\sum_{c \in V_m^{(p)^*}} \varepsilon_{F_c}(0) \zeta_p^{-\langle c,i \rangle_m}| = p^{\frac{n}{2}}.$$

Since $m \leq \frac{n}{2}$, we have $|\sum_{c \in V_m^{(p)^*}} \varepsilon_{F_c}(0) \zeta_p^{-\langle c,i \rangle_m}| \leq p^m - 1 < p^{\frac{n}{2}}$. Hence, for any $i \in V_m^{(p)^*}$, $|D_{F,i}^*| \neq 0$. When $i = 0$, by Eq. (29) we have that $|D_{F,0}^*| = 0$ if and only if

$$p^{\frac{n}{2}-m} \sum_{c \in V_m^{(p)^*}} \varepsilon_{F_c}(0) = 1 - p^{n-m}.$$

When n is odd, by Theorem 1 of [5],

$$\varepsilon_{F_{ac}}(0) = \varepsilon_{F_c}(0) \eta_1(a) \quad \text{for any } a \in \mathbb{F}_p^*, c \in V_m^{(p)^*}, \tag{30}$$

where η_1 denotes the quadratic character of $\mathbb{F}_p$. From Eq. (30) and $\sum_{a \in \mathbb{F}_p^*} \eta_1(a) = 0$, we can obtain $\sum_{c \in V_m^{(p)^*}} \varepsilon_{F_c}(0) = 0$. Thus, when n is odd, $|D_{F,0}^*| \neq 0$ and $|F(V_n^{(p)^*})| = p^m$. When n is even and $m < \frac{n}{2}$, $p \mid p^{\frac{n}{2}-m} \sum_{c \in V_m^{(p)^*}} \varepsilon_{F_c}(0)$ (Note that $\varepsilon_{F_c}(0) \in \{\pm 1\}$ when n is even) and $p \nmid (1 - p^{n-m})$, thus $|D_{F,0}^*| \neq 0$ and $|F(V_n^{(p)^*})| = p^m$. When n is even, $m = \frac{n}{2}$ and $\varepsilon_{F_c}(0) = -1, c \in V_m^{(p)^*}$, $p^{\frac{n}{2}-m} \sum_{c \in V_m^{(p)^*}} \varepsilon_{F_c}(0) = 1 - p^{n-m}$, thus $|D_{F,0}^*| = 0$ and $|F(V_n^{(p)^*})| = p^m - 1$. When n is even, $m = \frac{n}{2}$ and $\varepsilon_{F_c}(0), c \in V_m^{(p)^*}$ are not all -1, $p^{\frac{n}{2}-m} \sum_{c \in V_m^{(p)^*}} \varepsilon_{F_c}(0)$ $\neq 1 - p^{n-m}$, thus $|D_{F,0}^*| \neq 0$ and $|F(V_n^{(p)^*})| = p^m$. From the above arguments, we have that the result of Lemma 10 holds for F. By Lemma 9, we have that F^* is also vectorial dual-bent with $F^*(0) = 0, F^*(-x) = F^*(x)$. By Proposition 2 of [22], for any p-ary bent function $f : V_n^{(p)} \to \mathbb{F}_p$ which satisfies that n is even, $f(x) = f(-x)$ and the dual f^* is also bent, $\varepsilon_{f^*}(0) = \varepsilon_f(0)$ holds. When n is even, since $F_c, c \in V_m^{(p)^*}$ are all bent with $F_c(x) = F_c(-x)$ and the duals $(F_c)^*, c \in V_m^{(p)^*}$ are also bent, we have $\varepsilon_{(F^*)_c}(0) = \varepsilon_{(F_{\sigma^{-1}(c)})^*}(0) = \varepsilon_{F_{\sigma^{-1}(c)}}(0), c \in V_m^{(p)^*}$ and $\{\varepsilon_{(F^*)_c}(0), c \in V_m^{(p)^*}\} = \{\varepsilon_{F_c}(0), c \in V_m^{(p)^*}\}$. Therefore, the result of Lemma 10 also holds for F^*. ■

The following theorem gives a necessary and sufficient condition on constructing association schemes from general vectorial dual-bent functions $F : V_n^{(p)} \to V_m^{(p)}$ with $F(0) = 0, F(x) = F(-x)$ and $2 \leq m \leq \frac{n}{2}$.

Theorem 8. *Let $F : V_n^{(p)} \to V_m^{(p)}$ be a vectorial dual-bent function with $F(0) = 0, F(x) = F(-x)$ and $2 \leq m \leq \frac{n}{2}$, and F^* be a vectorial dual of F. Denote $I = F(V_n^{(p)^*})$ and $\varepsilon_{F_c}(x) =$*

$p^{-\frac{n}{2}}\zeta_p^{-(F_c)^*(x)}W_{F_c}(x), c \in V_m^{(p)^*}, x \in V_n^{(p)}$. *Define*

$$R_{id} = \{(x, x) : x \in V_n^{(p)}\},$$

$$R_i = \{(x, y) : x, y \in V_n^{(p)}, x - y \in D_{F,i}^*\}, i \in I.$$

Then

- $I = V_m^{(p)}$ *and* $|I| = p^m$ *except one case that* n *is even,* $m = \frac{n}{2}$ *and* $\varepsilon_{F_c}(0) = -1, c \in V_m^{(p)^*}$ *(in such a case,* $I = V_m^{(p)^*}$ *and* $|I| = p^m - 1$*).*
- *The following two statements are equivalent.*

  (1) $\{R_{id}, R_i, i \in I\}$ *is an* $|I|$*-class association scheme.*

  (2) *For any* $\beta, \beta' \in V_n^{(p)^*}$ *with* $F^*(\beta) = F^*(\beta')$*,* $\varepsilon_{F_c}(\beta) = \varepsilon_{F_c}(\beta'), c \in V_m^{(p)^*}$.

Proof: By Lemma 10 and its proof, we have that $I = V_m^{(p)}$ and $|I| = p^m$ except one case that n is even, $m = \frac{n}{2}$ and $\varepsilon_{F_c}(0) = -1, c \in V_m^{(p)^*}$ (in such a case, $I = V_m^{(p)^*}$ and $|I| = p^m - 1$).

Since F^* is a vectorial dual of F, $(F_c)^* = (F^*)_{\sigma(c)}, c \in V_m^{(p)^*}$ for some permutation σ over $V_m^{(p)^*}$. By

$$W_{F_c}(x) = \varepsilon_{F_c}(x)p^{\frac{n}{2}}\zeta_p^{(F_c)^*(x)} = \varepsilon_{F_c}(x)p^{\frac{n}{2}}\zeta_p^{(F^*)_{\sigma(c)}(x)}, c \in V_m^{(p)^*}, x \in V_n^{(p)},$$

where $\varepsilon_{F_c}(x) \in \{\pm 1, \pm\sqrt{-1}\}$ with $\varepsilon_{F_c}(x) = 1$ if $p = 2$, we have that for any $\beta, \beta' \in V_n^{(p)^*}$,

$$\begin{aligned} & W_{F_c}(\beta) = W_{F_c}(\beta'), c \in V_m^{(p)^*} \\ & \Leftrightarrow \varepsilon_{F_c}(\beta) = \varepsilon_{F_c}(\beta'), (F^*)_{\sigma(c)}(\beta) = (F^*)_{\sigma(c)}(\beta'), c \in V_m^{(p)^*} \\ & \Leftrightarrow \varepsilon_{F_c}(\beta) = \varepsilon_{F_c}(\beta'), c \in V_m^{(p)^*}, F^*(\beta) = F^*(\beta'). \end{aligned} \tag{31}$$

By Lemma 10, $|F(V_n^{(p)^*})| = |F^*(V_n^{(p)^*})|$. Therefore, by relation (31),

$$\begin{aligned} & |\{(W_{F_c}(\beta))_{c \in V_m^{(p)^*}} : \beta \in V_n^{(p)^*}\}| = |I| \\ & \Leftrightarrow \text{ for any } \beta, \beta' \in V_n^{(p)^*} \text{ with } F^*(\beta) = F^*(\beta'), \varepsilon_{F_c}(\beta) = \varepsilon_{F_c}(\beta'), c \in V_m^{(p)^*}. \end{aligned}$$

By Theorem 2 of [3] (Note that $I = F(V_n^{(p)})$ in Theorem 2 of [3] should be corrected as $I = F(V_n^{(p)^*})$), $\{R_{id}, R_i, i \in I\}$ is an $|I|$-class association scheme if and only if

$$|\{(W_{F_c}(\beta))_{c \in V_m^{(p)^*}} : \beta \in V_n^{(p)^*}\}| = |I|.$$

Hence, $\{R_{id}, R_i, i \in I\}$ is an $|I|$-class association scheme if and only if for any $\beta, \beta' \in V_n^{(p)^*}$ with $F^*(\beta) = F^*(\beta')$, we have $\varepsilon_{F_c}(\beta) = \varepsilon_{F_c}(\beta'), c \in V_m^{(p)^*}$. ■

The following corollary is directly from Theorem 8, which states that for a vectorial dual-bent function $F: V_n^{(p)} \rightarrow V_m^{(p)}$ with $F(0)=0, F(x)=F(-x)$ and $2 \leq m \leq \frac{n}{2}$, association schemes can be induced from F if F_c is weakly regular for any $c \in {V_m^{(p)}}^*$.

Corollary 4. *Let $F: V_n^{(p)} \rightarrow V_m^{(p)}$ be a vectorial dual-bent function with $F(0)=0, F(x)=F(-x)$ and $2 \leq m \leq \frac{n}{2}$. Denote $I=F({V_n^{(p)}}^*)$. Define*

$$R_{id}=\{(x,x): x \in V_n^{(p)}\},$$

$$R_i=\{(x,y): x, y \in V_n^{(p)}, x-y \in D_{F,i}^*\}, i \in I.$$

If F_c is weakly regular for any $c \in {V_m^{(p)}}^*$, then $\{R_{id}, R_i, i \in I\}$ is an $|I|$-class association scheme, where $I=V_m^{(p)}$ and $|I|=p^m$ except one case that n is even, $m=\frac{n}{2}$ and $\varepsilon_{F_c}=-1, c \in {V_m^{(p)}}^*$ (in such a case, $I={V_m^{(p)}}^*$ and $|I|=p^m-1$).

By using two classes of vectorial dual-bent functions $F: V_n^{(p)} \rightarrow V_m^{(p)}$ given in [8], [21] for which n can be odd, or n is even and $\varepsilon_{F_c}, c \in {V_m^{(p)}}^*$ are not all the same, we can obtain more association schemes.

Corollary 5. *Let p be an odd prime. Let $F: \mathbb{F}_{p^n} \rightarrow \mathbb{F}_{p^m}$ be defined as $F(x)=Tr_m^n(\alpha x^2)$, where $m \geq 2, m \mid n, m \neq n$. Denote $I=F(\mathbb{F}_{p^n}^*)$. Define*

$$R_{id}=\{(x,x): x \in \mathbb{F}_{p^n}\},$$

$$R_i=\{(x,y): x, y \in \mathbb{F}_{p^n}, x-y \in D_{F,i}^*\}, i \in I.$$

Then $\{R_{id}, R_i, i \in I\}$ is an $|I|$-class association scheme, where $I=\mathbb{F}_{p^m}$ and $|I|=p^m$ except one case that n is even, $m=\frac{n}{2}$ and $\eta_n(\alpha)=\xi^n$ (in such a case, $I=\mathbb{F}_{p^m}^*$ and $|I|=p^m-1$), where η_n denotes the quadratic character of $\mathbb{F}_{p^n}$, $\xi=1$ if $p \equiv 1 \pmod 4$ and $\xi=\sqrt{-1}$ if $p \equiv 3 \pmod 4$.

Proof: Obviously, $F(0)=0, F(x)=F(-x)$. By Example 1 of [8], F is a vectorial dual-bent function for which for any $c \in \mathbb{F}_{p^m}^*$, the component function $F_c(x)=Tr_1^n(\alpha c x^2)$ is weakly regular with $\varepsilon_{F_c}=(-1)^{n-1}\xi^n\eta_n(\alpha c)$. Then the result follows from Corollary 4. ■

Below we give an example to illustrate Corollary 5.

Example 4. *Let $p=3, n=6, m=2$. Define $F(x)=Tr_2^6(x^2), x \in \mathbb{F}_{3^6}$. Then F is a vectorial dual-bent function for which for any $c \in \mathbb{F}_{3^2}^*$, the component function $F_c(x)=Tr_1^6(cx^2)$ is* *weakly regular with* $\varepsilon_{F_c} = (-1)^{6-1}(\sqrt{-1})^6\eta_6(c) = \eta_6(c)$*. Note that* $\{\eta_6(c) : c \in \mathbb{F}_{3^2}^*\} = \{\pm 1\}$*. Let*

$$R_{id} = \{(x, x) : x \in \mathbb{F}_{3^6}\},$$

$$R_i = \{(x, y) : x, y \in \mathbb{F}_{3^6}, x - y \in D_{F,i}^*\}, i \in \mathbb{F}_{3^2}.$$

By Corollary 5, $\{R_{id}, R_i, i \in \mathbb{F}_{3^2}\}$ *is a* 9*-class association scheme.*

Corollary 6. *Let* p *be an odd prime. Let* r_1, r_2, m *be positive integers with* $m \geq 2, m \mid r_1, m \mid r_2$*. For* $i \in \mathbb{F}_{p^m}$*, define* $H(i; x) : \mathbb{F}_{p^{r_1}} \rightarrow \mathbb{F}_{p^m}$ *as* $H(0; x) = Tr_m^{r_1}(\alpha_1 x^2)$*,* $H(i; x) = Tr_m^{r_1}(\alpha_2 x^2)$ *if* i *is a square in* $\mathbb{F}_{p^m}^*$*,* $H(i; x) = Tr_m^{r_1}(\alpha_3 x^2)$ *if* i *is a non-square in* $\mathbb{F}_{p^m}^*$*, where* $\alpha_j, 1 \leq j \leq 3$ *are all squares or all non-squares in* $\mathbb{F}_{p^{r_1}}^*$*. Define* $G : \mathbb{F}_{p^{r_2}} \times \mathbb{F}_{p^{r_2}} \rightarrow \mathbb{F}_{p^m}$ *as* $G(y_1, y_2) = Tr_m^{r_2}(\beta y_1 L(y_2))$*, where* $\beta \in \mathbb{F}_{p^{r_2}}^*$ *and* $L(x) = \sum a_i x^{q^i} (q = p^m)$ *is a* q*-polynomial over* $\mathbb{F}_{p^{r_2}}$ *inducing a permutation of* $\mathbb{F}_{p^{r_2}}$*. Let* $F : \mathbb{F}_{p^{r_1}} \times \mathbb{F}_{p^{r_2}} \times \mathbb{F}_{p^{r_2}} \rightarrow \mathbb{F}_{p^m}$ *be defined as*

$$F(x, y_1, y_2) = H(Tr_m^{r_2}(\gamma y_2^2); x) + G(y_1, y_2),$$

where $\gamma \in \mathbb{F}_{p^{r_2}}^*$*. Define*

$$R_{id} = \{(x, x) : x \in \mathbb{F}_{p^{r_1}} \times \mathbb{F}_{p^{r_2}} \times \mathbb{F}_{p^{r_2}}\},$$

$$R_i = \{(x, y) : x, y \in \mathbb{F}_{p^{r_1}} \times \mathbb{F}_{p^{r_2}} \times \mathbb{F}_{p^{r_2}}, x - y \in D_{F,i}^*\}, i \in \mathbb{F}_{p^m}.$$

Then $\{R_{id}, R_i, i \in \mathbb{F}_{p^m}\}$ *is a* p^m*-class association scheme.*

Proof: It is easy to see that $F(0, 0, 0) = 0, F(x, y_1, y_2) = F(-x, -y_1, -y_2)$. By Theorem 1 of [21] and its proof, F is a vectorial dual-bent function for which for any $c \in \mathbb{F}_{p^m}^*$, the component function F_c is weakly regular with $\varepsilon_{F_c} = (-1)^{r_1 - 1}\xi^{r_1}\eta_{r_1}(\alpha_1 c)$, where η_{r_1} denotes the quadratic character of $\mathbb{F}_{p^{r_1}}$, $\xi = 1$ if $p \equiv 1 \pmod 4$ and $\xi = \sqrt{-1}$ if $p \equiv 3 \pmod 4$. Then the result follows from Corollary 4. ■

Below we give an example to illustrate Corollary 6.

Example 5. *Let* $p = 5, r_1 = r_2 = 9, m = 3$ *and* α *be a primitive element in* $\mathbb{F}_{5^9}$*. Then* $n = r_1 + 2r_2 = 27$ *is odd. For* $i \in \mathbb{F}_{5^3}$*, let* $H(i; x) = Tr_3^9(x^2), x \in \mathbb{F}_{5^9}$ *if* $i = 0$ *and* $H(i; x) = Tr_3^9(\alpha^2 x^2), x \in \mathbb{F}_{5^9}$ *if* $i \in \mathbb{F}_{5^3}^*$*. Define* $F : V_{27}^{(5)} \rightarrow \mathbb{F}_{5^3}$ *as* $F(x, y_1, y_2) = H(Tr_3^9(y_2^2); x) + Tr_3^9(y_1 y_2) = (Tr_3^9(y_2^2))^{124} Tr_3^9((\alpha^2 - 1)x^2) + Tr_3^9(x^2 + y_1 y_2)$*, where* $V_{27}^{(5)} = \mathbb{F}_{5^9} \times \mathbb{F}_{5^9} \times \mathbb{F}_{5^9}$*. Let*

$$R_{id} = \{(x, x) : x \in V_{27}^{(5)}\},$$

$$R_i = \{(x, y) : x, y \in V_{27}^{(5)}, x - y \in D_{F,i}^*\}, i \in \mathbb{F}_{5^3}.$$

By Corollary 6, $\{R_{id}, R_i, i \in \mathbb{F}_{5^3}\}$ *is a* 125*-class association scheme.*

IX. Conclusion

In this paper, we further studied vectorial dual-bent functions $F : V_n^{(p)} \rightarrow V_m^{(p)}$, where $2 \leq m \leq \frac{n}{2}$. First, we gave new characterizations of vectorial dual-bent functions with Condition A in terms of amorphic association schemes (Theorem 1), linear codes (Theorem 2), generalized Hadamard matrices (Theorems 3 and 4), and bent partitions when $p = 2$ (Theorem 5). Second, based on the relations between vectorial dual-bent functions with Condition A and bent partitions, new characterizations of certain bent partitions in terms of amorphic association schemes, linear codes and generalized Hadamard matrices were presented (Theorems 6 and 7). Finally, for general vectorial dual-bent functions $F : V_n^{(p)} \rightarrow V_m^{(p)}$ with $F(0) = 0, F(x) = F(-x), 2 \leq m \leq \frac{n}{2}$, we gave a necessary and sufficient condition on constructing association schemes (Theorem 8) and more association schemes were constructed (Corollaries 5 and 6).

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