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Editorial

Preface to the Special Issue "Algebraic Structures and Graph Theory"

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1. Introduction

Connections between algebraic structure theory and graph theory have been established in order to solve open problems in one theory with the help of the tools existing in the other, emphasizing the remarkable properties of one theory with techniques involving the second. This has provided new methods for solving several open problems, and has proposed new ones. One remarkable example in this direction is the contribution of Artur Cayley, who defined the concept of a group in 1854 (the composition table of the operation on the group takes his name, i.e., the Cayley table) and described in 1878 the structure of a group with a special graph, called a Cayley graph. There are many ways to define an algebraic structure (as a group, ring, hypergroup, hyperfield, lattice, etc.) starting from a graph and also vice versa, with the algebraic structures leading to the various types of graph. Many such constructions are discussed and illustrated with several non-trivial examples in [1–4].

This Special Issue aims to collect recent theoretical and applied studies on the interrelations between algebraic structures and graphs. This topic has attracted the interest of many researchers from different branches of algebra, and among the 63 submissions, 18 articles have been selected and published in this book. In the next section, we will briefly summarize their findings; for more detail, we recommend the readers to consult the original articles and the related bibliographies.

2. Contributions

The articles published in this Special Issue present new and up-to-date theoretical and applied research topics related to the following: (1) algebraic structures, (2) graphs and hypergraphs, and (3) connections between graphs and algebraic structures. We will start with the first group, containing seven articles dealing with semigroups, differential graded algebras, *BL*-algebras, hypergroups and hyperfields.

Regular elements in a semigroup play a fundamental role, being the key focus of the study of regular semigroups. As an example of such a structure, we recall the total transformation semigroup $T(X)$ on a nonempty set X. In 1975, Symons [5] introduced a subsemigroup of $T(X)$, defined as $T(X,Y) = \{\alpha \in T(X) \mid X\alpha \subseteq Y\}$, for a nonempty subset Y of X, determining all its automorphisms. The regularity of this subsemigroup and its implications for the computation of the number of the left/right regular elements of $T(X,Y)$ are discussed in [6]. A second paper [7] related to the theory of semigroups presents a novel method for generating the M-tri basis of an ordered Γ-semigroup. Here, the authors showed how the elements and subsets of an ordered Γ-semigroup yield to M-tri-ideals and the M-tri basis. Another two articles are related to the theory of algebras. The first [8] regards the differential graded algebras. Based on the notion of the Hopf Galois extension, the author emphasizes the relationships between the derived categories $\mathcal{D}(R\#H)$ of the smash product $R\#H$ and $\mathcal{D}(R^H)$, where H is a finite dimensional semisimple Hopf algebra and R a left H-module algebra. The second paper [9] studies the structure of a

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block code generated by a BL-algebra. In particular, H. Bordbar defines a new order for the generated code associated with a BL-algebra, and shows that the structure of the BL-algebra with its initial order and the one of the corresponding generated code with the new defined order coincide.

The last three manuscripts within this first group are in the framework of hypercompositional algebra. This is a relatively new field of abstract algebra, studying algebraic structures endowed with at least one hyperoperation, i.e., a multivalued operation associating with any pair of elements a subset of the support set. Thus, the hypercompositional structures are natural generalizations of the classical algebraic structures. For example, the hypergroups are a generalization of groups, while hyperfields are a generalization of the concept of fields. In [10] the authors work with the most well-known class of hyperfields, Krasner hyperfields, studying the notions of positive cone, characteristic and C-characteristic. Using these notions, they provide a criterion for deciding whether certain hyperfields cannot be obtained via Krasner's quotient construction. Furthermore, they prove that for any positive integer n greater than 1, there exists an infinite quotient hyperfield of characteristic n. A similar result holds for the C-characteristic. The manuscripts [11,12] cover some applications of hypercompositional algebra to automata theory. Massouros et al. [11] study the binary state machines with magma of two elements as their environment. Another aspect of automata theory is discussed in [12], where the authors propose several conditions for simplifying the verification of the GMAC condition for systems of quasi-multiautomata. Furthermore, using the concatenation, they construct quasi-multiautomata corresponding to the deterministic automata of the theory of formal languages.

We continue our presentation of the second group of contributions published in this Special Issue. They discuss innovative aspects in graph and hypergraph theory. Addition signed Cayley graphs are investigated in [13] with respect to the balancing, clusterability and sign compatibility properties. Here, the author presents the necessary conditions such that an addition signed Cayley graph is balanced, sign-compatible, clusterable, a line signed graph or $\mathcal{C}$-consistent. The Cayley graphs are also the focus in [14]. In particular, it is shown that the Cayley graphs generated by transposition trees on the set $\{1, 2, \ldots, n\}$ are $n-2$-extendable and their extendability number is $n-2$ for any integer $n \geq 3$. Based on the notion of the trace norm of a $\{0, 1\}$-Brauer configuration, the authors of [15] compute the graph energy of some families of graphs defined by Brauer configuration algebras. In [16], the author counts the crossing number of the joint product $G^* \, D_n$ of the disconnected graph G^* consisting of two components isomorphic to K_2 and K_3 and the discrete graph D_n with n isolated vertices. Moreover, a lower bound of the distance of the Laplacian spectral radius of the n-vertex bipartite graphs, with a diameter equal to 4, is determined in [17]. The paper concludes with the conjecture that the graph $G(1, \ldots, 1, n-d, 1, \ldots, 1)$ is unique, minimizing the distance of the Laplacian spectral radius among the n-vertex bipartite graphs, with all having a diameter greater than or equal to 4. A similar argument is posed by the authors of [18], who find upper and lower bounds on the spectral radius of the generalized reciprocal distance matrix of a connected graph with n vertices. The main goal of Solovyev's paper [19] is to establish a counting formula for a 2-dimensional lattice path model with filter restrictions in the presence of long steps. The last manuscript [20] within this group is in the area of hypergraphs, originally introduced by Berge as extensions of graphs, where the edges are substituted by hyperedges, being nonempty subsets of the set of vertices. Using the innovative concepts of knot and knot-hyperpath, the authors study the behaviour of the hyperpaths under hyper-continuous mappings and pseudo-open mappings, and find the sufficient conditions under which a hypergraph becomes a hypertree. The paper ends with an algorithm extracting a host graph from a hypertree.

We conclude this section with a description of the third group of manuscripts, presenting different relationships between algebraic structures and graph theory. The first paper [21] introduces a construction of a new graph associated with a semihypergroup, using the fundamental relation γ^*. Several properties such as completeness, regularity, being Eulerian or Hamiltonian, and Cartesian products are studied. Much the same direction is followed in the second paper [22], where a t-graph associated with a finitely generated group, using the Minkowski metric, is defined. The groups involved here are the two-generator finite groups, and the authors characterize the chromatic number of a t-graph depending exclusively on the parity of t. Finally, the study presented in the eighteenth paper [23] of this collection is focused on the construction and properties of an ideal-based dot total graph associated with a commutative ring with nonzero unity.

3. Conclusions

Based on the number of views and citations received, we are confident that the selected manuscripts of this edited book have aroused considerable interest among researchers in this field, and will open new lines of investigation not only in the domain of algebraic structure theory or graph theory, but also in other research topics. Therefore, this Special Issue will continue with a second edition, edited by Irina Cristea and Alessandro Linzi.

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