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Well-posedness, regularity, asymptotic behavior and analyticity for some plate-membrane type transmission problems

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A dissertation submitted to the Doctorado en Ciencias Naturales in partial fulfillment of the requirements for the degree of doctor in sciences

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Barranquilla, October 2022

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To my wife and to my son

Abstract

In this thesis some plate-membrane type transmission problems are studied. See system (1.1)-(1.9). Three dampings are considered on the structure: thermal and structural for the plate, and global viscoelastic of Kelvin–Voigt type on the membrane. Sometimes some damping is removed from the structure. The plate may or may not have an inertial term. In the presence and/or absence of any of the elements mentioned above, we establish existence and uniqueness of solution of the system, which depends continuously on the initial data. We also obtain results of regularity, stability and analyticity. We use the semigroup approach to show the well-posedness our system. Following an idea of proof of regularity developed by Avalos and Lasiecka, we prove that if the inertial term is present or absent then the boundary and transmission conditions hold in the strong sense of the trace when the initial data are smooth enough. Then, using a general criteria of Arendt–Batty, we show the strong stability of our system when the membrane is damped and the plate is with or without rotational inertia. Employing a spectral approach, we indirectly prove exponential stability when the plate has rotational inertia and the structure is totally damped. This asymptotic behavior of the solutions is lost when we remove the viscoelastic component of the membrane. Under this situation, we impose a geometrical condition on the membrane boundary and obtain that the solutions decay polynomially with a rate of $t^{-1/25}$ when the plate has rotational inertia and structural damping. Finally, using a well-known Liu–Zheng criterion we prove by contradiction the analyticity of the system when the membrane has Kelvin–Voigt damping and the thermoelastic plate is considered without inertial term and without structural damping.

Keywords: Analyticity, exponential stability, lack of exponential stability, polynomial stability, regularity, strong stability, transmission problem, well-posedness.

Acknowledgments

I want to express huge thanks to my supervisors Bienvenido Barraza Martínez and Robert Denk. They always had a very good disposition in the moments that required them to discuss the topics of this document. This work allowed a rapprochement with them and I learned that they are people with valuable qualities.

I would also like to take this opportunity to thank Jairo Hernández Monzón for the discussions about my thesis and his accurate opinions. His contribution has been significant to my professional development.

Anggie and Nathan, you lived very closely the day to day of this work. Thank you for waiting, for patience, for tolerance and for the support provided. You are part of the impulse and the motivation that led me to start and finish my doctorate. This achievement belongs to the three of us.

Thanks to my parents, other family and friends who were attentive throughout the process of my doctorate with their voice of encouragement and motivation that strengthened me to continue in the most difficult moments. Your support is invaluable.

Finally, thanks to Minciencias (formerly Colciencias) for financing project *Problemas de transmisión asociados a placas termoelásticas acopladas con membranas* with award number 121571250194. This allowed me to start my doctoral studies and do a research stay at the University of Konstanz, where I had the pleasure of meeting Sophia Rau. Thanks to you also for the times we discussed math but above all for your kindness.

Jonathan González Ospino
Soledad - Atlántico, May 2022

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Chapter 1

Introduction

In recent years, many researchers have investigated transmission problems due to their applications in engineering, such as: aircrafts, satellite antennas and road traffic (see [63] and the references therein). The physical phenomenon consisting of the interaction between an acoustic medium and the vibrations of an elastic structure can be described with transmission equations (see [64]). There are also other interactive physical processes modeled through coupled partial differential equations, for example, composite laminates in smart materials and structures, and fluid-structure interactions (see [126]).

In this work, a plate-membrane type system within the L^2 theory is studied. In addition to proving the existence and uniqueness of the solution to the transmission problem, in certain appropriate spaces, its asymptotic behavior will be analyzed and subsequently the analyticity of a semigroup associated with the problem will be proved. The results depend on the dampings that are placed on the structure. The plate may or may not have thermal or structural damping, and the membrane may or may not have Kelvin–Voigt damping. The geometric description of the structure to be worked on is shown below.

1.1 Description of the problem

Let Ω and Ω_2 be bounded domains in $\mathbb{R}^2$ with boundaries $\Gamma := \partial\Omega$ and $I := \partial\Omega_2$, both of class C^4, such that $\overline{\Omega_2} \subset \Omega$. We set $\Omega_1 := \Omega \setminus \overline{\Omega_2}$. The unit normal vector pointing towards the exterior of Ω_1, both on Γ and on I, is denoted by ν (see Fig. 1.1). Note that unit outward normal vector to Ω_2 along the *interface* I is $-\nu$. We will assume that Ω_1 and Ω_2 are occupied by the

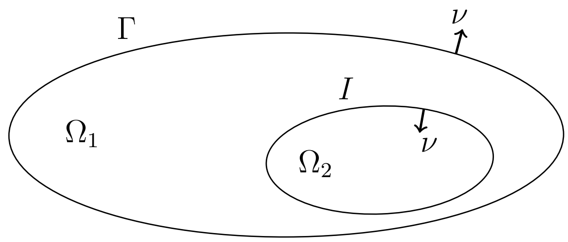

Fig. 1.1. The set $\Omega = \Omega_1 \cup I \cup \Omega_2$.

middle surface of the thermoelastic plate and by the membrane, respectively. In this case, $u = u(t, x)$ and $v = v(t, x)$ denote the vertical deflections of the plate and the membrane, respectively. The temperature difference of the plate is denoted by $\theta = \theta(t, x)$. We consider the following problem

$$\rho_1 u_{tt} - \gamma \Delta u_{tt} + \beta_1 \Delta^2 u - m_1 \Delta u_t + \alpha \Delta \theta = 0 \quad \text{in} \quad \mathbb{R}^+ \times \Omega_1, \tag{1.1}$$

$$\rho_0 \theta_t + \sigma \theta - \beta \Delta \theta - \alpha \Delta u_t = 0 \quad \text{in} \quad \mathbb{R}^+ \times \Omega_1, \tag{1.2}$$

$$\rho_2 v_{tt} - \beta_2 \Delta v - m_2 \Delta v_t = 0 \quad \text{in} \quad \mathbb{R}^+ \times \Omega_2. \tag{1.3}$$

The parameters γ, m_1, σ and m_2 are all non-negative, and the rest of the constants $\rho_1, \beta_1, \alpha, \rho_0, \beta, \rho_2$ and β_2 are positive. When $\gamma > 0$ it is because the rotational inertia of the plate filaments is considered and whose value is proportional to the square of the plate thickness, in this case (1.1) with $m_1 = \alpha = 0$ is known as the Kirchhoff plate equation (see (1) in [81, p. 2]). The case $\gamma = 0$ corresponds to a thin plate, and when $m_1 = \alpha = 0$ the equation (1.1) is called Euler–Bernoulli plate equation. The parameters $m_1 > 0$ and $m_2 > 0$ make the plate and the membrane have structural and Kelvin–Voigt damping, respectively. The constant σ depends on the thickness of the plate and the ratio of the external thermal conductivity to the thermal conductivity of the plate, in the case of being positive (see [79]). The coupling parameter α is called coefficient of thermal expansion (see [9]), β is the thermal conductivity (see [39]) and ρ_0 has the meaning of heat/thermal capacity (see [35, p. 244]). The constants β_1, β_2, ρ_1 and ρ_2 are the plate flexural rigidity, the in-plane stress, the plate mass/area and the membrane mass/area, respectively (see [129]). We will assume that the plate is embedded and attached to the membrane, this is interpreted by

$$u = \partial_\nu u = 0 \ \text{ on } \ \mathbb{R}^+ \times \Gamma \qquad \text{and} \qquad u = v \ \text{ on } \ \mathbb{R}^+ \times I. \tag{1.4}$$

We will assume that the temperature satisfies Newton's cooling law (see (2.5.7) in [95, p. 43]) with coefficient $\kappa > 0$ along $\partial\Omega_1$, this is,

$$\partial_\nu \theta + \kappa\theta = 0 \quad \text{on} \quad \mathbb{R}^+ \times \partial\Omega_1. \tag{1.5}$$

Besides the condition on the interface I in (1.4), we consider the following boundary and transmission conditions

$$\beta_1 \mathscr{B}_1 u + \alpha\theta = 0 \quad \text{on} \quad \mathbb{R}^+ \times I, \tag{1.6}$$

$$\gamma\partial_\nu u_{tt} - \beta_1 \mathscr{B}_2 u + m_1 \partial_\nu u_t - \alpha\partial_\nu\theta - \beta_2\partial_\nu v - m_2\partial_\nu v_t = 0 \quad \text{on} \quad \mathbb{R}^+ \times I, \tag{1.7}$$

where

$$\mathscr{B}_1 u := \Delta u + (1-\mu)B_1 u \quad \text{and} \quad \mathscr{B}_2 u := \partial_\nu \Delta u + (1-\mu)\partial_\tau B_2 u$$

are the boundary operators introduced in [79, p. 119], $\tau := (-\nu_2, \nu_1)^\top$ is the unit tangent vector along $\partial\Omega_1$ with ν_1 and ν_2 being the first and second components of ν, and μ is the Poisson ratio whose value belongs to the interval $(0, 1/2)$, which comes from certain physical considerations on the plate. The operators B_1 and B_2 are defined by the relations

$$\begin{aligned} B_1 u &:= 2\nu_1\nu_2 u_{x_1x_2} - \nu_1^2 u_{x_2x_2} - \nu_2^2 u_{x_1x_1}, \\ B_2 u &:= \nu_1\nu_2(u_{x_2x_2} - u_{x_1x_1}) + (\nu_1^2 - \nu_2^2)u_{x_1x_2}, \end{aligned}$$

which appear in [79, p. 2]. The initial conditions for the system are given by

$$u(0) = u_0, u_t(0) = u_1, \theta(0) = \theta_0 \quad \text{in} \quad \Omega_1, \tag{1.8}$$

$$v(0) = v_0, v_t(0) = v_1 \quad \text{in} \quad \Omega_2. \tag{1.9}$$

In this thesis it is guaranteed that the system (1.1)-(1.9) has a unique solution that can be weak or strong, this will depend on the choice of the initial data. Moreover, here there are results concerning regularity, strong stability, exponential stability, non-exponential stability, polynomial stability and analyticity, which depend on the presence or absence of the parameters γ, m_1 and m_2.

1.2 Physical aspects of thermoelastic plate-membrane model

Some paragraphs should be written regarding the derivation of the equations that govern the model.

Let us consider a thin plate which in equilibrium occupies the set $\overline{\Omega_1} \times [-h/2, h/2]$, where $h > 0$ is the plate's thickness. Following the ideas given in [79] and [81], let

$$\mathbf{U}(x_1, x_2, x_3, t) := \big(U_1(x_1, x_2, x_3, t), U_2(x_1, x_2, x_3, t), U_3(x_1, x_2, x_3, t)\big)$$

be the displacement vector at time t of the particle which when the plate is in equilibrium occupies position (x_1, x_2, x_3). Similarly, let $\mathbf{u}(x_1, x_2, t) := \big(u_1(x_1, x_2, t), u_2(x_1, x_2, t), u_3(x_1, x_2, t)\big)$ be the displacement vector at time t of the particle which when the plate is in equilibrium occupies position $(x_1, x_2, 0)$. Both $\mathbf{U}$ and $\mathbf{u}$ are supposed smooth enough for the present discussion. The *strain energy* in the plate is given by

$$\mathscr{P}^{\mathrm{P}} := \frac{1}{2} \int_{-h/2}^{h/2} \int_{\Omega_1} \sum_{i,j=1}^{3} \varepsilon_{ij} \sigma_{ij} \, dx' \, dx_3, \tag{1.10}$$

where ε_{ij} and σ_{ij} are, respectively, the components of the *strain* and *stress* tensors and $x' := (x_1, x_2)$. The components ε_{ij} depend on $\mathbf{U}$ and hence on (x_1, x_2, x_3, t). In the linear theory, the components of the stress tensor are linear combinations of the components of the strain tensor and the thermal strain (when the plate is subject to a temperature distribution). On the other hand, the *kinetic energy* of the plate is given by

$$\mathscr{K}^{\mathrm{P}} := \frac{1}{2} \int_{-h/2}^{h/2} \int_{\Omega_1} \rho_1 \sum_{i=1}^{3} (\partial_t U_i)^2 \, dx' \, dx_3, \tag{1.11}$$

where ρ_1 is the density (mass per unit volume) of the plate.

If the plate is subject to a temperature distribution $\tau(x_1, x_2, x_3, t)$, measured with respect to a reference temperature distribution τ_0, at which the thermal stresses and strains in the plate are zero, and assuming that the plate is homogeneous, elastically and thermally isotropic (see [79]), we have that the stress-strain relations are given by

$$\sigma_{ij} = \frac{E}{1+\mu}\Big(\varepsilon_{ij} + \frac{\mu}{1-2\mu}\delta_{ij} \sum_{k=1}^{3} \varepsilon_{kk}\Big) - \frac{E}{1-2\mu}\varepsilon_\tau \delta_{ij}, \quad i, j = 1, 2, 3. \tag{1.12}$$

Here, the constants E and μ are, respectively, the *Young's modulus* and the *Poisson's ratio* ($0 < \mu < 1/2$), and δ_{ij} is the well-known *Kronecker delta* notation. The symbol ε_τ denotes the *thermal strain* which satisfies $\varepsilon_\tau = 0$ when $\tau = 0$.

For the thin plates theory it is assumed that $\sigma_{33} = 0$. This implies in (1.12) that

$$0 = \frac{E}{1+\mu}\Big(\varepsilon_{33} + \frac{\mu}{1-2\mu}\sum_{k=1}^{3}\varepsilon_{kk}\Big) - \frac{E}{1-2\mu}\varepsilon_\tau$$

and therefore

$$\varepsilon_{33} = -\frac{\mu}{1-\mu}(\varepsilon_{11}+\varepsilon_{22}) + \frac{1+\mu}{1-\mu}\varepsilon_\tau.$$

Then, we have

$$\sigma_{11} = \frac{E}{1-\mu^2}\Big[\varepsilon_{11} + \mu\varepsilon_{22} - (1+\mu)\varepsilon_\tau\Big], \tag{1.13}$$

$$\sigma_{22} = \frac{E}{1-\mu^2}\Big[\mu\varepsilon_{11} + \varepsilon_{22} - (1+\mu)\varepsilon_\tau\Big], \tag{1.14}$$

and

$$\sigma_{ij} = \frac{E}{1+\mu}\varepsilon_{ij} \qquad (i \neq j,\ i,j = 1,2,3). \tag{1.15}$$

In the Kirchhoff model for a thin plate it is assumed that the components of the strain tensor are given by

$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\Big) \qquad (i,j = 1,2,3).$$

In addition transverse shear effects are neglected. Thus, $\sigma_{i3} = \sigma_{3i} = 0$ and $\varepsilon_{i3} = \varepsilon_{3i} = 0$ for $i = 1,2$. It follows that

$$0 = \varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\Big(\frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1}\Big)$$

and therefore

$$\frac{\partial U_1}{\partial x_3} = -\frac{\partial U_3}{\partial x_1}. \tag{1.16}$$

Now,

$$\begin{aligned} U_3(x_1,x_2,z,t) &= U_3(x_1,x_2,0,t) + \int_0^1 \frac{\partial}{\partial s}U_3(x_1,x_2,sz,t)\,ds \\ &= u_3(x_1,x_2,t) + z\int_0^1 (\partial_3 U_3)(x_1,x_2,sz,t)\,ds. \end{aligned} \tag{1.17}$$

From (1.16) and (1.17), we have

$$U_1(x_1,x_2,x_3,t) = U_1(x_1,x_2,0,t) - \int_0^{x_3} \partial_1 U_3(x_1,x_2,z,t)\,dz$$
$$= u_1(x_1,x_2,t) - x_3\frac{\partial u_3}{\partial x_1}(x_1,x_2,t) - \int_0^{x_3} z\int_0^1 (\partial_1\partial_3 U_3)(x_1,x_2,sz,t)\,ds\,dz.$$

After linearization (with respect to x_3), we obtain

$$U_1(x_1,x_2,x_3,t) = u_1(x_1,x_2,t) - x_3\frac{\partial u_3}{\partial x_1}(x_1,x_2,t). \tag{1.18}$$

In the same way it holds

$$U_2(x_1,x_2,x_3,t) = u_2(x_1,x_2,t) - x_3\frac{\partial u_3}{\partial x_2}(x_1,x_2,t) \tag{1.19}$$

and

$$U_3(x_1,x_2,x_3,t) = u_3(x_1,x_2,t). \tag{1.20}$$

Now we can write the components ε_{ij}, $i,j=1,2$, in terms of the u_i, $i=1,2,3$. In fact:

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} - 2x_3\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\Big), \tag{1.21}$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} - x_3\frac{\partial^2 u_3}{\partial x_1^2}, \tag{1.22}$$

and

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} - x_3\frac{\partial^2 u_3}{\partial x_2^2}. \tag{1.23}$$

From (1.13)-(1.15) and (1.21)-(1.23), it follows that

$$\sigma_{11} = \frac{E}{1-\mu^2}\Big[\frac{\partial u_1}{\partial x_1} - x_3\frac{\partial^2 u_3}{\partial x_1^2} + \mu\frac{\partial u_2}{\partial x_2} - \mu x_3\frac{\partial^2 u_3}{\partial x_2^2} - (1+\mu)\varepsilon_\tau\Big], \tag{1.24}$$

$$\sigma_{22} = \frac{E}{1-\mu^2}\Big[\mu\frac{\partial u_1}{\partial x_1} - \mu x_3\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial u_2}{\partial x_2} - x_3\frac{\partial^2 u_3}{\partial x_2^2} - (1+\mu)\varepsilon_\tau\Big], \tag{1.25}$$

and

$$\sigma_{12} = \sigma_{21} = \frac{E}{2(1+\mu)}\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} - 2x_3\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\Big). \tag{1.26}$$

We recall (1.10) for the strain energy in the plate. For the case of a thin plate in the Kirchhoff model, it reduces to

$$\mathscr{P}^{\mathrm{P}} = \frac{1}{2} \int_{-h/2}^{h/2} \int_{\Omega_1} \big(\varepsilon_{11}\sigma_{11} + 2\varepsilon_{12}\sigma_{12} + \varepsilon_{22}\sigma_{22} \big) \, dx' \, dx_3.$$

Using (1.21)-(1.26), after integration with respect to x_3, it can be found that $\mathscr{P}^{\mathrm{P}}$ splits into two terms $\mathscr{P}_b^{\mathrm{P}}$ and $\mathscr{P}_s^{\mathrm{P}}$, which uncouple the components u_1, u_2, representing in-plane stretching, and the component u_3 related to bending (see [79]). The term $\mathscr{P}_s^{\mathrm{P}}$ depends on u_1, u_2 and ε_τ, and it is the part of the strain energy due to the in-plane stretching. On the other side, $\mathscr{P}_b^{\mathrm{P}}$ depend only on u_3 and ε_τ, and it is the part of the strain energy due to bending. In this work we consider only the equations of motion for u_3. Exactly $\mathscr{P}_b^{\mathrm{P}}$ is given by

$$\begin{aligned}\mathscr{P}_b^{\mathrm{P}} := \frac{Eh^3}{24(1-\mu^2)} \int_{\Omega_1} \Bigg[\Big(\frac{\partial^2 u_3}{\partial x_1^2} \Big)^2 + \Big(\frac{\partial^2 u_3}{\partial x_2^2} \Big)^2 + 2(1-\mu) \Big(\frac{\partial^2 u_3}{\partial x_1 \partial x_2} \Big)^2 \\ + 2\mu \frac{\partial^2 u_3}{\partial x_1^2} \frac{\partial^2 u_3}{\partial x_2^2} + \frac{12(1+\mu)}{h^3} (\Delta u_3) \int_{-h/2}^{h/2} x_3 \varepsilon_\tau \, dx_3 \Bigg] \, dx',\end{aligned}$$

where Δ denotes the Laplacian with respect to the two variables x_1 and x_2. Using the expression for the *modulus of flexural rigidity* of the plate $D := Eh^3/12(1-\mu^2)$, setting $u := u_3$ and changing the notations for the derivatives to $\partial_i := \partial/\partial x_i$, we can write

$$\begin{aligned}\mathscr{P}_b^{P} = \frac{D}{2} \int_{\Omega_1} \Big[\big(\partial_1^2 u\big)^2 + \big(\partial_2^2 u\big)^2 + 2(1-\mu)\big(\partial_1 \partial_2 u\big)^2 \\ + 2\mu(\partial_1^2 u)(\partial_2^2 u) + (1+\mu)\theta \Delta u \Big] \, dx',\end{aligned}$$

where

$$\theta := \theta(x_1, x_2, t) := \frac{12}{h^3} \int_{-h/2}^{h/2} x_3 \varepsilon_\tau \, dx_3.$$

Now, substitution of (1.18), (1.19) and (1.20) in (1.11) and integration with respect to x_3 gives

$$\mathscr{K}^{\mathrm{P}} = \frac{h}{2} \int_{\Omega_1} \rho_1 \Big[\big(\partial_t u_1\big)^2 + \big(\partial_t u_2\big)^2 + \big(\partial_t u_3\big)^2 + \frac{h^2}{12} \big|\nabla(\partial_t u_3)\big|^2 \Big] \, dx',$$

where ∇ is the gradient with respect to the variables x_1 and x_2. We see that the components u_1, u_2 and u_3 are uncoupled in this expression for $\mathscr{K}^{\mathrm{P}}$. Again, we are only interested in the part of the kinetic energy due to bending, which is denoted by $\mathscr{K}_b^{\mathrm{P}}$ and it is given by

$$\mathscr{K}_b^{\mathrm{P}} := \frac{h}{2}\int_{\Omega_1} \rho_1 \Big[\big(\partial_t u\big)^2 + \frac{h^2}{12}\big|\nabla(\partial_t u)\big|^2 \Big]\, dx',$$

where $u := u_3$ again.

Suppose now that Ω_2 is occupied by an elastic membrane for which its vertical deflection is denoted by $v = v(x_1, x_2)$. The strain energy of the membrane is given by

$$\mathscr{P}^{\mathrm{M}} := \frac{C}{2}\int_{\Omega_2} |\nabla v|^2\, dx',$$

where C depends on the material and the initial tension in the membrane. On the other side, the kinetic energy for the membrane is given by

$$\mathscr{K}^{\mathrm{M}} := \frac{1}{2}\int_{\Omega_2} \rho_2 \big(\partial_t v\big)^2\, dx',$$

where ρ_2 is the surface density of the membrane.

We assume that the structure is clamped on Γ, that is $u = \partial_\nu u = 0$ on $\Gamma \times [0, T)$ for $0 < T \leq \infty$, where ν is the outward unit normal vector to the boundary of Ω_1. Furthermore, we assume that $u = v$ on $I \times [0, T)$. The meaning of this condition is that the structure does not break. We recall that small deflections are being considered.

In order to obtain an initial boundary value problem which models the coupled structure plate-membrane we have to set to zero the first variation in the time interval $[0, T)$ of the *Lagrangian*

$$L_b(u, v) := \int_0^T \big[\mathscr{K}_b^{\mathrm{P}}(t) + \mathscr{K}^{\mathrm{M}}(t) + \mathscr{W}_b(t) - \mathscr{P}_b^{\mathrm{P}}(t) - \mathscr{P}^{\mathrm{M}}(t) \big]\, dt \tag{1.27}$$

with respect to (u, v) in the space of kinematically admissible displacements, that is

$$\delta L_b(u, v) := \frac{\partial}{\partial \lambda} L_b(u + \lambda \widetilde{u}, v + \lambda \widetilde{v})\Big|_{\lambda = 0} = 0, \tag{1.28}$$

with $\widetilde{u}\big|_{\Gamma} = \partial_\nu \widetilde{u}\big|_{\Gamma} = 0$ and $\widetilde{u}\big|_{I} = \widetilde{v}\big|_{I}$. Here $\widetilde{u}$ and $\widetilde{v}$ are chosen, such that $\widetilde{u}(0) = \widetilde{u}'(0) = \widetilde{u}(T) = \widetilde{u}'(T) = 0$ in Ω_1 and $\widetilde{v}(0) = \widetilde{v}'(0) = \widetilde{v}(T) = \widetilde{v}'(T) = 0$ in Ω_2, where for a function $\varphi : \overline{\Omega_i} \times [0, \infty) \to \mathbb{R}$ we write $\varphi(t)(x_1, x_2) :=$ $\varphi(x_1, x_2, t)$ and $'$ denotes the derivative with respect to t. This approach is equivalent to the use of the Virtual Work Principle from the Classical Mechanic.

In (1.27) $\mathscr{W}_b$ is the part of the work done on the structure plate-membrane that contributes to bending due to external forces acting on it.

After renaming constants, equation (1.28) leads to the general equations (1.1)-(1.3) of the model, as well as the boundary and transmission conditions (1.4)-(1.7). Changing the space of kinematically admissible displacements, several models with different boundary and transmission conditions can be obtained. Finally, the damping which can be considered in each model can be incorporated in $\mathscr{W}_b$.

1.3 Literature

Transmission problems appear very frequently in various fields of physics and technics, e.g., there are applications in the electrodynamics of solid media when working with electromagnetic processes in ferromagnetic media with different dielectric constants. This type of problem also occurs in the solid mechanics of composite materials. The previous exposition can be found in [26]. Some authors also associate these problems with structures consisting of a finite number of interconnected flexible elements such as waves, beams, plates, casings and combinations thereof, which are representative of frames, robot arms, solar panels, antennas and deformable mirrors (see [80]). There are also transmission problems associated with acoustics (see [7, 75]), the automotive industry (see [13, 114]), polymers (see [24, 53, 54, 76, 82]), fluid behaviour in certain structures (see [34, 36, 115]), vibration suppression (see [42]), electrostatics and static magnets (see [88]), and material composition (see [101]).

The transmission problems were initiated by Meuro Picone in 1954 (see [61]), which are framed within the theory of partial differential equations and control theory, (see [35, 87]). Many researchers are interested in the asymptotic behavior of the solutions of the transmission problem in which they are investigating. In the literature we find this kind of problems with different types of decay, such as: exponential decay (see [14, 15, 22, 30, 31, 37, 42, 52, 63, 64, 65, 90, 102, 122]), polynomial decay (see [4, 14, 15, 19, 46, 52, 63, 64, 83, 90, 105, 122, 125, 127]) and logarithmic decay (see [46, 50, 68, 71, 90]). Of the works mentioned, only in [14, 15, 52, 64, 122], in addition to analyzing the exponential stability and the polynomial stability of their respective systems, the authors studied the absence of exponential stability. All this asymptotic analysis is done in this document for the system (1.1)-(1.9). In [68] and [71], the authors study a transmission problem where the membrane surrounds the plate that has localized Kelvin–Voigt damping. Instead, we consider the plate around the membrane and it is the latter that has the Kelvin–Voigt damping, but uniform.

The Kelvin–Voigt damping is a type of linear damping that suppresses the vibrations of elastic structures and is caused by the internal friction of the material of the vibrating structures and is thus called *internal damping*. In this case the material is viscoelastic because it has properties of viscosity and elasticity. The term “viscous” implies that it slowly deforms when exposed to an external force. The term “elastic” implies that once a deforming force has been removed, the material will return to its original configuration. For further explanations, see [69, p. 12]. Mathematically, the Kelvin–Voigt damping is an operator of the same order as the principal operator of the equation that describes the vibrating structure (see [71, p. 2242]). The model for a membrane occupying a bounded region $\widetilde{\Omega}$ of $\mathbb{R}^2$ is given by

$$w_{tt} - \Delta w + \widetilde{\beta} L_w = 0 \quad \text{in} \quad \widetilde{\Omega} \times \mathbb{R}^+,$$

where $\widetilde{\beta} \geq 0$ and L_w is a dissipative mechanism. In [110, Subsection 2.4.4], the authors determine the equation of motion of the vertically moving particles of the vibrating membrane when $\widetilde{\beta} = 0$. If the membrane has the frictional damping ($\widetilde{\beta} > 0$ and $L_w = w_t$), then the semigroup decays exponentially (see [111, p. 16]). In [73], the author proved that if the membrane has a global Kelvin–Voigt damping (i.e., $\widetilde{\beta} > 0$ and $L_w = -\Delta w_t$) then the corresponding semigroup is not only exponentially stable, but also is analytic. For this reason, Kelvin–Voigt damping is said to be stronger than frictional damping.

Another type of damping that is considered on some vibrating structures is thermoelastic damping, see [69, p. 13]. This damping originates from the coupling of the elasticity of the structure with a heat source. The thermal effect contributes to the deformation of the structure since its material expands when the temperature increases and contracts when the temperature decreases. Our attention is directed towards thermoelastic plates.

The first models for thin plates were developed by the German physicist Gustav Robert Kirchhoff (1824 - 1887). In the first chapter of [81], one finds linear and non-linear models for thin plates subject to different physical situations such as temperature, elasticity and viscosity, depending on different assumptions and conditions according to each model. A detailed study of deflections and plate deformations is in [119], here the authors work with thin plates with small deflections, thin plates with large deflections and thick plates. For plate deformations, see also [96]. On the theory of vibrating plates, we recommend to the reader [99].

The literature on thermoplastic plates is very extensive. The following are some of the works on this subject. The classic linear model for thermoelastic plates is given by

$$\begin{aligned} \rho u_{tt} - \widetilde{\mu}\Delta u_{tt} + \widetilde{\gamma}\Delta^2 u + \widetilde{\alpha}\Delta\theta = 0 \quad &\text{in} \quad \widetilde{\Omega} \times \mathbb{R}^+, \\ c\theta_t - \widetilde{\kappa}\Delta\theta - \widetilde{\alpha}\Delta u_t = 0 \quad &\text{in} \quad \widetilde{\Omega} \times \mathbb{R}^+, \end{aligned}$$

where $\rho, \widetilde{\mu}, \widetilde{\gamma}, \widetilde{\alpha}, c$ and $\widetilde{\kappa}$ are all positive constants and $\widetilde{\Omega} \subset \mathbb{R}^2$ is the region occupied by the middle surface of the plate. For the physical model we refer to [79]. In [74], Kim studies the Euler–Bernoulli plate equation ($\widetilde{\mu} = 0$) with thermal effect and homogeneous Dirichlet boundary conditions $u = \partial_\nu u = \theta = 0$ on $\partial\widetilde{\Omega}$. He proved exponential decay of the energy of his system. In [93], Liu and Renardy worked on the Kim's problem and obtain a much stronger result, showing that the semigroup associated is of analytic type. In [43], the authors do not account for rotational forces of the thermoelastic plate ($\widetilde{\mu} = 0$) and consider homogeneous Dirichlet boundary conditions. They generalize the result of Liu and Renardy, introducing the space $W^2_{p,D}(\widetilde{\Omega}) := \{u \in W^{2,p}(\widetilde{\Omega}) : u = \partial_\nu u = 0 \text{ on } \partial\widetilde{\Omega}\}$ with $\widetilde{\Omega}$ being a bounded domain of $\mathbb{R}^n$ ($n \geq 2$) and showing that if the initial data $(u, u_t, \theta)|_{t=0} \in W^2_{p,D}(\widetilde{\Omega}) \times L^p(\widetilde{\Omega}) \times L^p(\widetilde{\Omega})$ then the semigroup is analytic for any $p \in (1, \infty)$ and also exponentially stable. In [86], the authors set $\widetilde{\mu} = 0$ and, focusing on the case of free boundary conditions, proved that the associated semigroup is analytic. The addition of the term Δu_{tt} not only makes the problem physically more meaningful, it also makes it more mathematically interesting, because we need more regularity to establish the decay of the solutions. For the case $\widetilde{\mu} > 0$ and Dirichlet boundary conditions it was proved in [103] that the solutions decay exponentially.

Going back to analytical results, it is known that the literature pertaining to analytic semigroups is very extensive and has a wide variety of topics. For example, one finds works on: materials with porosity (see [20]), thermoelastic plate with variable coefficients (see [33]), viscoelastic plate equation of Moore–Gibson–Thompson type (see [39]), wave equation with Kelvin–Voigt damping and hyperbolic dynamic boundary conditions (see [60]), system of couple plate equations with different dampings (see [66]), laminated beam (see [67]), thermoelastic plate equations with different boundary conditions (see [84]), thermoelastic plate with hinged mechanical B.C. and Neumann thermal B.C. (see [85]), homogeneous and isotropic prestressed type III thermoelastic thin plate (see [91]), qualitative properties for the semigroup generated by certain matrix operator (see [94]) and thermoelastic plate with dynamical boundary conditions (see [128]). Other work on analyticity can be seen in [106]. Here the authors study a system consisting of a thermoelastic Euler–Bernoulli plate coupled with a membrane that has a global Kelvin–Voigt type damping. They consider appropriate initial conditions and certain boundary conditions, and show that the semigroup associated with the problem is analytic. Regarding transmission problems, it seems that the only work so far with analyticity has been [51]. There, the system studied by the authors is constituted by a pair of thermoelastic plates, none of which have the inertial term and their structure is like the one in Fig. 1.2. In the present manuscript is the analyticity proof of a transmission problem of a thermoelastic thin plate with a membrane that has global viscoelastic damping of Kelvin–Voigt type, this is one of the main results of this work. According to our search, it seems that there is no such result in the entire literature and for this reason it was submitted for a publication according to reference [16].

Many researchers studied transmission problems involving systems of the type: plate-plate, wave-wave and plate-wave. Below we recall some works concerning the three structures mentioned.

In [42], the authors investigate a coupled system of linear plate equations, one non-damped plate and the other structurally-damped plate in two sufficiently smooth and bounded subdomains, and consider a geometric configuration as Fig. 1.1. They showed, independently of the size of the damped part, that the damping is strong enough to produce exponential stability. In the literature, it is common to find the following configuration: Let Ω_1', Ω_2' and Ω' be bounded domains in $\mathbb{R}^2$ such that $\Gamma_0 \coloneqq \overline{\Omega_1'} \cap \overline{\Omega_2'}$, $\Gamma_1 \coloneqq \partial\Omega_1' \setminus \Gamma_0$ and $\Gamma_2 \coloneqq \partial\Omega_2' \setminus \Gamma_0$ are smooth curves. Set $\Omega' = \Omega_1' \cup \Omega_2' \cup \Gamma_0$ and $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$ (see Fig. 1.2). In [102], the authors consider a problem of transmission, with

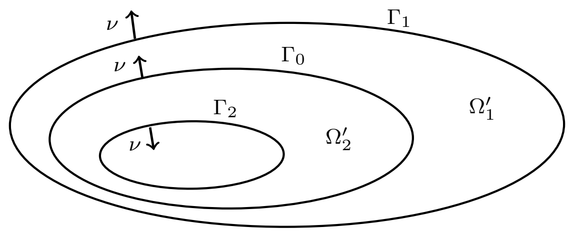

Fig. 1.2. The set $\Omega' = \Omega_1' \cup \Gamma_0 \cup \Omega_2'$.

the configuration of Fig. 1.2, of a plate Ω' constituted by two parts Ω_1' and Ω_2', being Ω_1' sensitive to temperature changes. They work on the following model

$$
\begin{aligned}
\widehat{\rho}_1 u_{tt} - \gamma_1 \Delta u_{tt} + \widehat{\beta}_1 \Delta^2 u + \widehat{\mu}\Delta\theta = 0 \quad &\text{in} \quad \Omega_1' \times \mathbb{R}^+,\\
\widehat{\rho}_0 \theta_t - \beta_0 \Delta\theta + \gamma_0\theta - \widehat{\mu}\Delta u_t = 0 \quad &\text{in} \quad \Omega_1' \times \mathbb{R}^+,\\
\widehat{\rho}_2 v_{tt} - \gamma_2 \Delta v_{tt} + \widehat{\beta}_2 \Delta^2 v = 0 \quad &\text{in} \quad \Omega_2' \times \mathbb{R}^+.
\end{aligned}
$$

Under certain initial conditions and with several boundary and transmission conditions different from those of problem (1.1)-(1.9), they show that there is a solution and that the local thermal effect is strong enough to produce exponential decay. Afterwards, the author of [112] studies a transmission problem (Fig. 1.2) for a plate consisting of a thermoelastic part Ω_1' and an isothermal part Ω_2'. He demonstrates the existence of a compact global attractor when the non-linearity is of the Berger or scalar type. The model worked was

$$
\begin{aligned}
\widehat{\rho}_1 u_{tt} + \widehat{\beta}_1 \Delta^2 u + \widehat{\mu}\Delta\theta + F_1(u,v) = 0 \quad &\text{in} \quad \Omega_1' \times \mathbb{R}^+,\\
\widehat{\rho}_0 \theta_t - \beta_0 \Delta\theta - \widehat{\mu}\Delta u_t = 0 \quad &\text{in} \quad \Omega_1' \times \mathbb{R}^+,\\
\widehat{\rho}_2 v_{tt} + \widehat{\beta}_2 \Delta^2 v + F_2(u,v) = 0 \quad &\text{in} \quad \Omega_2' \times \mathbb{R}^+,
\end{aligned}
$$

satisfying boundary conditions and initial conditions, where the functions $F_1 : H^2(\Omega_1') \times H^2(\Omega_2') \to L^2(\Omega_1')$ and $F_2 : H^2(\Omega_1') \times H^2(\Omega_2') \to L^2(\Omega_2')$

are both non-linear. The author studies three problems where F_1 and F_2 are taken differently. Another transmission problem with the configuration of Fig. 1.2 is treated in [121]. The authors consider a system constituted by a thermoelastic plate with localized thermal dissipation of memory type coupled with an isothermal plate, both plates have inertial term. They prove that the solutions have exponential decay.

Much study has been made of the wave equation with different dampings. When a vibrating source disturbs the medium, a wave is formed. Several dampings can be added to a system of wave equations to control the vibrations. In [80], Chapter VI, the authors consider the movement of interconnected elastic membranes and determine the equations of this physical phenomenon. In [31], the authors consider the configuration of Fig. 1.1 and study the propagation of waves on a domain consisting of two different materials: one component is elastic where a frictional damping is acting while the other one possesses a viscoelastic component with a memory with past history. They establish exponential stability for the solutions of the problem. In [65], the authors analyze a transmission problem of the wave equation (Fig. 1.2) with linear dynamical feedback control. They prove that the energy of system exponentially decays. In [122], the authors investigate a locally coupled wave equations with only one internal viscoelastic damping of Kelvin–Voigt type. The damping and the coupling coefficients are non smooth. They show that the energy of smooth solutions of the system decays polynomially. In [18], the authors consider a wave-wave system, in one space dimension, with frictional damping. They study the wave propagation in a medium with a component with attrition and another simply elastic, and show that for this type of material, the dissipation produced by the frictional part is strong enough to produce exponential decay of the solution, no matter how small is its size.

There are several works about transmission problems that involve these plate-wave equations, here we mention some. In [70], Hernández works on a semi-linear problem of initial and boundary values, this models a thin elastic plate coupled with an elastic membrane, considering homogeneous boundary conditions. Using semigroup theory, the author obtained existence and uniqueness of weak solutions. In [68], Hassine studies a transmission problem of a membrane coupled with an Euller–Bernoulli plate which has a localized Kelvin–Voigt damping, the membrane surrounds the plate (see Fig. 1.1). He proves that sufficiently smooth solutions decay logarithmically. Afterwards, in [71], the authors work in the Hassine system but adding the rotational term in the plate. They show that the energy of the transmission system is stable with logarithmic decay. In [63], the authors investigate a transmission wave-plate model with different localized frictional damping, in their structure the plate is next to the wave. They show respectively that the energy of the system decays polynomially under some geometric condition when the frictional damping only acts on the part of the plate, and the energy of the system is exponentially stable when the frictional damping acts only on the other part of the wave. These same authors, in [64], study another plate-wave system but now its structure is as in Fig. 1.2 and obtained the same results as in [63]. We further mention two works about plate-membrane transmission problems with transmission conditions different from each other but with exactly the same configuration of Fig. 1.1, see [14] and [15]. The plate in [15] is isothermal and without rotational inertia, while in [14] the plate may or may not have temperature and rotational term. Both works have stability results, but in neither of them is analyticity treated, perhaps because the frictional damping that they place on the membrane is not strong enough to achieve this type of result in the semigroup associated with their problem.

In this document there is an extensive analysis of a transmission problem constituted by a thermoelastic plate-membrane structure with the configuration of Fig. 1.1. In the presence and/or absence of inertial term, structural damping for the plate and Kelvin–Voigt damping on the membrane, we show results of existence and uniqueness of solutions, regularity and asymptotic behavior. These results complement what is already in the literature. Additionally, we establish that the semigroup associated with problem (1.1)-(1.9) is analytic when $\gamma = m_1 = 0$ and $m_2 > 0$. For more details see the next section.

1.4 Document structure

The document contains five chapters. In the first of them, the transmission problem is described, some literature on the subject is presented and are mentioned the results obtained in the thesis.

In Chapter 2, the reader will find the fundamental function spaces for the development of this work. We consider Sobolev spaces with some of their properties, among which we mention those regarding continuous immersion, integration formulas, and important inequalities that satisfy some of their elements. There is a brief mention of some semigroup results regarding generation, stability and analyticity. Subsequently, Stone's theorem for group generation is given. The theory of interpolation-extrapolation scales is presented concisely but providing references where the details of the mentioned results are carried out. The chapter continues with a section devoted to powers of positive self-adjoint operators and follows with some theory about regular elliptic problems to establish a characterization that satisfies the domains of powers of operators that are realizations of some elliptic operator. We end with a result taken from [15] containing the estimate (2.30), that comes from a boundary value problem, which is used in the last two chapters to obtain polynomial stability and analyticity.

Chapter 3 is dedicated to an entire analysis that leads to the existence, uniqueness and regularity of the solutions of the transmission problem. This is achieved by introducing operators defined in appropriate spaces that satisfy properties that allow establishing a Cauchy problem to which semigroup theory can be applied to prove existence and uniqueness of solutions. These results are in Section 3.1 and correspond to the well-posedness of the problem (1.1)-(1.9) when $\gamma > 0$ and $m_1, m_2 \geq 0$, and in the case $\gamma = m_1 = 0$ with $m_2 \geq 0$ (see Theorem 3.15 and Remark 3.16). In the second section of the chapter regularity results appear that will be very useful to obtain asymptotic behavior of the solutions of some transmission problems. One of them can be seen in Remark 3.17, which contains the estimate (3.44) that we will use in the next two chapters in the proofs of strong stability, of polynomial stability and in the one of analyticity. The main result of Section 3.2 is Theorem 3.18, for its demonstration we follow the ideas of the proof of Theorem 1.2 in [10], there we get a shared regularity that v and v_t win when $m_2 > 0$ and we establish that for smooth data (1.6)-(1.7) are satisfied in the strong sense of the trace, which will depend on the consideration or not of the parameters γ, m_1 and m_2. Remark 3.19 indicates the regularity that the components of the solution have when the initial data are taken with less smoothness compared to those of Theorem 3.18.

In Chapter 4, the stability of several plate-membrane type transmission problems is analyzed, depending on the types of damping that the structure possesses. In Section 4.1, using a general criteria of Arendt and Batty (see

Theorem 2.39) we show that our problem possesses strong stability when the membrane has the Kelvin–Voigt damping ($m_2 > 0$) and in the structure it is considered the Kirchhoff plate with or without structural damping ($m_1 \geq 0$), or the Euler–Bernoulli plate without structural damping ($m_1 = 0$), see Proposition 4.2 and Proposition 4.5. The Remark 4.4 and Remark 4.7 highlight the role of the thermal effect on the plate in achieving this stability result. In Section 4.2, employing a well-known characterization (see Theorem 2.41) we establish by contradiction the exponential stability of the solutions of the plate-membrane system when the inertial term and the mechanical dampings are considered, that is, the constants γ, m_1 and m_2 are all positive, see Theorem 4.8. In Section 4.3, we prove that the system conformed by a undamped membrane coupled to a thermoelastic Kirchhoff plate with or without structural damping lacks exponential stability (see Theorem 4.15). To obtain the absence of this asymptotic behavior, we apply Theorem 4.11 and for this we make use of Stone's Theorem on unitary groups, the interpolation-extrapolation scales theory and compactness arguments. But in Section 4.4, for $\gamma, m_1 > 0$ and $m_2 = 0$ we prove polynomial stability when we consider the geometric condition (4.99). We base the proof on a multiplier method and then we apply a generalization due to Muñoz Rivera and Racke (see Theorem 2.44) of the well-known Borichev and Tomilov characterization on polynomial decay, see Theorem 2.4 in [25].

In Chapter 5, we proof by contradiction the existence of an analytic semigroup associated with a transmission problem of a plate without rotational term that has a thermal effect coupled to a membrane that has a global viscoelastic Kelvin–Voigt damping. In other words, the solutions of the system (1.1)-(1.9) are analytic functions when $\gamma = m_1 = 0$ and $m_2 > 0$. This can be proved thanks to the Liu and Zheng analyticity criterion, see Theorem 2.46. The main result of this chapter is Theorem 5.1 and as a consequence we have a regularizing effect of the solutions and a couple of corollaries (see Corollary 5.3 and Corollary 5.4). We end with Remark 5.5, which indicates the non-analyticity of a semigroup associated with our plate-membrane system.

1.5 Generalities

Throughout the thesis, C represents a positive constant which is not necessarily the same every time it appears, it can change from one line to another line. If X and Y are topological spaces, then the collection of all bounded linear operators from X to Y be denoted by $\mathcal{L}(X, Y)$. We shall write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. In this work the term *domain* means open and connected subset of $\mathbb{R}^n$, which is different from the concept of the domain of a function.

Let X and Y be two locally convex spaces such that $X \subset Y$ so that the identity operator $\mathrm{id} : X \to Y$ is continuous. Let $A : D(A) \subset Y \to Y$ be a linear operator. The *X-realization of A*, denoted by A_X, is the linear operator

$$A_X : D(A_X) \subset X \to X \quad \text{given by} \quad A_X x := Ax,$$

where $D(A_X) := \{x \in X \cap D(A) : Ax \in X\}$. It is easy to see that if A is closed then A_X is closed. This definition is taken from [5, p. 7].

Let X be a Banach space and $A : D(A) \subset X \to X$ be a linear operator. The smallest closed extension of A, if it exists, is called the *closure of A* and is denoted by $\overline{A}$. Operators having a closure are called *closable*. This definition is taken from [48, Definition A.7].

Chapter 2

Preliminaries

In this second chapter, the reader will find notations, definitions and results that will be used later.

2.1 Basic spaces

This section has the purpose of presenting the Sobolev spaces in a structured way. In addition, we will establish important properties of these spaces. It is known that these elements are used in the formulation of problems of partial differential equations. Some concepts and properties given here can be found in [1, 55, 72, 123, 131].

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. A *multi-index* α is an n-tuple of non-negative integers, this is, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$. We denote $|\alpha| := \sum_{j=1}^n \alpha_j$. As usual ∂_j represents the partial derivative with respect to the j-th variable x_j. We set

$$\partial^\alpha := \prod_{j=1}^n \partial_j^{\alpha_j} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

If $\alpha = 0_{\mathbb{N}_0^n}$, then ∂^α is the identity operator. Let $k \in \mathbb{N}_0$ and $\Omega \subset \mathbb{R}^n$ be an open set. $C^k(\Omega)$ denotes the vector space of all functions $f : \Omega \to \mathbb{C}$ such that $\partial^\alpha f : \Omega \to \mathbb{C}$ are continuous for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$. Note that $C(\Omega) := C^0(\Omega)$ is constituted by continuous complex-valued functions on Ω. We define $C^k(\overline{\Omega})$ as the space of all functions $f \in C^k(\Omega)$ such that $\partial^\alpha f$ extends continuously to the closure $\overline{\Omega}$ for $|\alpha| \leq k$. We fix $C^\infty(\Omega) := \bigcap_{k=0}^\infty C^k(\Omega)$. The *support* of $f : \Omega \to \mathbb{C}$ is given by

$$\operatorname{supp} f := \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

The space formed by the functions of $C^\infty(\Omega)$ that have compact support in Ω is denoted by $C_c^\infty(\Omega)$ and its elements are called *test functions*. We say that *a sequence* $(\varphi_j)_{j\in\mathbb{N}}$ *in* $C_c^\infty(\Omega)$ *converges to zero* if the following two conditions are satisfied:

i) There is a compact subset K of Ω such that $\operatorname{supp}\varphi_j \subset K$ for all $j \in \mathbb{N}$.

ii) $\partial^\alpha \varphi_j \to 0$ uniformly in K for every $\alpha \in \mathbb{N}_0^n$, this is,

$$\max_{x\in K} |\partial^\alpha \varphi_j(x)| \to 0 \text{ as } j \to \infty \text{ for all } \alpha \in \mathbb{N}_0^n.$$

The space $C_c^\infty(\Omega)$ equipped with this notion of convergence is symbolized by $\mathscr{D}(\Omega)$. The topological dual $\mathscr{D}'(\Omega) := \mathcal{L}(\mathscr{D}(\Omega), \mathbb{C})$ of $\mathscr{D}(\Omega)$ is called the *space of distributions* on Ω. The complex value of $T \in \mathscr{D}'(\Omega)$ on $\varphi \in C_c^\infty(\Omega)$ is denoted by $\langle T, \varphi\rangle$. Note that the continuity of a linear map T of $\mathscr{D}(\Omega)$ in $\mathbb{C}$ is equivalent to: For every sequence $(\varphi_j)_{j\in\mathbb{N}}$ convergent to zero in $\mathscr{D}(\Omega)$ we have that $\langle T, \varphi_j\rangle \to 0$ in $\mathbb{C}$. For $\alpha \in \mathbb{N}_0^n$ and $T \in \mathscr{D}'(\Omega)$, we define the *distributional derivative* $\partial^\alpha T$ by

$$\langle \partial^\alpha T, \varphi\rangle := (-1)^{|\alpha|}\langle T, \partial^\alpha \varphi\rangle \text{ for any } \varphi \in \mathscr{D}(\Omega).$$

Suppose that $1 \leq p < \infty$. The space of all equivalence classes of Lebesgue measurable functions $u : \Omega \to \mathbb{C}$ such that $|u|^p$ is a integrable function on Ω will be denoted by $L^p(\Omega)$. This space endowed with the norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx\right)^{1/p}$$

is a Banach space. The space $L^2(\Omega)$ considered with the inner product $(u, v)_{L^2(\Omega)} := \int_\Omega u(x)\overline{v(x)}dx$ is a Hilbert space.

The space $L^1_{loc}(\Omega)$ consists of all functions $f : \Omega \to \mathbb{C}$ such that the integral $\int_K |f|dx$ is finite for each compact subset K of Ω. If $f \in L^1_{loc}(\Omega)$, we say that f is *locally integrable* on Ω. Each $u \in L^1_{loc}(\Omega)$ defines a distribution $T_u \in \mathscr{D}'(\Omega)$ given by

$$\langle T_u, \varphi\rangle := \int_\Omega u(x)\varphi(x)dx, \quad \varphi \in \mathscr{D}(\Omega).$$

We shall write u instead of T_u. Due to $L^p(\Omega) \subset L^1_{loc}(\Omega)$, we can consider any function of $L^p(\Omega)$ as a distribution.

The *Sobolev space of non-negative integer order* $W^{k,p}(\Omega)$ is composed of all functions $u \in L^p(\Omega)$ whose distributional derivatives $\partial^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leq k$. We introduce the following norm on $W^{k,p}(\Omega)$:

$$\|u\|_{W^{k,p}(\Omega)} \coloneqq \left(\sum_{|\alpha| \leq k} \|\partial^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}.$$

In particular, $H^k(\Omega) \coloneqq W^{k,2}(\Omega)$ is a Hilbert space with the scalar product $(u, v)_{H^k(\Omega)} \coloneqq \sum_{|\alpha| \leq k} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}$. Note that $H^0(\Omega) = L^2(\Omega)$. Let $k_1, k_2 \in \mathbb{N}$ with $k_1 > k_2$, then we have the strict inclusions

$$H^{k_1}(\Omega) \subset H^{k_2}(\Omega) \subset L^2(\Omega),$$

see Remark 1.2 in [89, p. 3].

Next we define some *non-negative real order Sobolev spaces*. Let $s \geq 0$. We shall denote by $[s]$ the integer part of s (i.e., $[s] \coloneqq \max\{z \in \mathbb{Z} : z \leq s\}$) and by σ its fractional part. So, $s = [s] + \sigma$ with $0 \leq \sigma < 1$. The Sobolev space $H^s(\Omega)$ is defined as $W^{s,2}(\Omega)$ provided that $s \in \mathbb{N}_0$ and if $s \notin \mathbb{N}_0$ then $H^s(\Omega)$ contains all functions $u \in H^{[s]}(\Omega)$ that have finite norm $\|u\|_{H^s(\Omega)}$, which comes from the following inner product:

$$(u, v)_{H^s(\Omega)} \coloneqq (u, v)_{H^{[s]}(\Omega)} + (u, v)_{H^{s,\sigma}(\Omega)},$$

where

$$(u, v)_{H^{s,\sigma}(\Omega)} \coloneqq \sum_{|\alpha|=[s]} \int_\Omega \int_\Omega \frac{(\partial^\alpha u(x) - \partial^\alpha u(y))\overline{(\partial^\alpha v(x) - \partial^\alpha v(y))}}{|x-y|^{n+2\sigma}} dxdy.$$

The pair $(H^s(\Omega), (\cdot, \cdot)_{H^s(\Omega)})$ is a Hilbert space. Other definitions of these spaces are in [2, Subsection 5.1], [89, p. 40] and [118, p. 49]. In the literature we also find the $W^{s,p}(\Omega)$ spaces, see for example [45, Section 6]. We put $W^{s,p}_0(\Omega)$ to denote the closure of $C^\infty_c(\Omega)$ in $W^{s,p}(\Omega)$. We set $H^k_0(\Omega) \coloneqq W^{k,2}_0(\Omega)$. It can be proved that $(H^k_0(\Omega), (\cdot, \cdot)_{H^k(\Omega)})$ is a Hilbert space. The dual of $H^k_0(\Omega)$ is denoted by $H^{-k}(\Omega)$.

We continue this section with a space which is constituted by abstract functions or vector-valued functions (see, e.g., [35, Subsection 1.1.4], [44, Chapter 3] or [78, Chapter 1]). Let X be a Banach space, $1 \leq p < \infty$ and $(a, b) \subset \mathbb{R}$. The symbol $L^p((a, b), X)$ denotes the space of all equivalence classes of strongly Bochner-measurable functions $u : (a, b) \to X$ such that $t \mapsto \|u(t)\|_X$ belongs to $L^p(a, b)$, which is a Banach space with the norm

$$\|u\|_{L^p((a,b),X)} := \left(\int_a^b \|u(t)\|_X^p \, dt \right)^{1/p}.$$

When $p = 2$ and X is a Hilbert space, then $L^2((a, b), X)$ is a Hilbert space with the scalar product

$$(u, v)_{L^2((a,b),X)} := \int_a^b (u(t), v(t))_X \, dt.$$

$\mathscr{D}'((a, b), X) := \mathcal{L}(\mathscr{D}(a, b), X)$ denotes the space of all *vector-valued distributions* $T : \mathscr{D}(a, b) \to X$. The *vector distributional derivative of order* k of $T \in \mathscr{D}'((a, b), X)$, denoted by $T^{(k)}$, is defined as

$$\langle T^{(k)}, \varphi \rangle := (-1)^k \langle T, \varphi^{(k)} \rangle \quad \forall \varphi \in \mathscr{D}(a, b),$$

where $\varphi^{(k)} := d^k\varphi/dt^k$. The previous equality holds in X. Now, if $u \in L^p((a, b), X)$ then $\widetilde{u} : \mathscr{D}(a, b) \to X$ given by

$$\langle \widetilde{u}, \varphi \rangle := \int_a^b u(t)\varphi(t)dt$$

is a continuous linear mapping, this is, $\widetilde{u} \in \mathscr{D}'((a, b), X)$. The above integral should be understood as a Bochner integral (for details see [1, p. 206] or [47, Appendix C]). The mapping $L^p((a, b), X) \ni u \mapsto \widetilde{u} \in \mathscr{D}'((a, b), X)$ is injective and thus identifying $\widetilde{u}$ with u we obtain that

$$L^p((a, b), X) \subset \mathscr{D}'((a, b), X),$$

see Section 1.3 in [89, p. 6].

We conclude this section with the *Aubin–Lions–Simon lemma*. Interesting historical comments on this result can be found in [116].

Definition 2.1. Let X and Y be two normed spaces such that $X \subset Y$. If $\mathrm{id} : X \to Y$ is continuous, we say X is *continuously embedded* in Y and write $X \hookrightarrow Y$. We use the symbol $X \overset{c}{\hookrightarrow} Y$ when $\mathrm{id} : X \to Y$ is a compact operator.

Theorem 2.2 (Aubin–Lions–Simon lemma)**.** *Let B_0, B, B_1 be three Banach spaces such that $B_0 \overset{c}{\hookrightarrow} B$ and $B \hookrightarrow B_1$. Let $1 \leq p, q < \infty$ and $t > 0$. Let us consider the space*

$$\mathcal{W}_{p,q}((0,t); B_0, B_1) \coloneqq \{u \in L^p((0,t), B_0) : u_t \in L^q((0,t), B_1)\}$$

with the norm

$$\|u\|_{\mathcal{W}_{p,q}((0,t);B_0,B_1)} \coloneqq \|u\|_{L^p((0,t),B_0)} + \|u_t\|_{L^q((0,t),B_1)},$$

where $u_t \coloneqq du/dt$. Then,

$$\mathcal{W}_{p,q}((0,t); B_0, B_1) \overset{c}{\hookrightarrow} L^p((0,t), B).$$

Proof. See Theorem II.5.16 in [27]. □

2.2 Regular boundaries

The *boundary* of an open set $\Omega \subset \mathbb{R}^n$ is denoted by $\partial\Omega \coloneqq \overline{\Omega} \cap (\mathbb{R}^n \setminus \Omega)$. Here we will find properties, which depend on the regularity of $\partial\Omega$, that satisfy the Sobolev spaces defined over Ω. One of these results is known as *Rellich–Kondrachov theorem*. For a vector-valued version, see [6, Theorem 5.1]. Before presenting the theorem we will give Definition 1.2.1.1 of [62]. There a couple of types of functions are mentioned, among others, whose definitions we present below.

Let $\mathscr{O}$ be an open set of $\mathbb{R}^{n-1}$. We recall that a function $\varphi : \mathscr{O} \to \mathbb{R}$ is said to be *Lipschitz continuous function* or simply *Lipschitz function* if it satisfies

$$|\varphi(x') - \varphi(y')| \leq C|x' - y'| \quad \text{for all} \quad x', y' \in \mathscr{O},$$

where the constant $C > 0$ does not depend on x' and y'. We say that φ belongs to the class $C^{k,1}$ if it is k times continuously differentiable and its derivatives of order k are Lipschitz functions.

Definition 2.3. Let Ω be an open subset of $\mathbb{R}^n$. We say that $\partial\Omega$ is *Lipschitz* (resp. *of class* $C^{k,1}$) if for every $x \in \partial\Omega$ there exists a neighbourhood V of x in $\mathbb{R}^n$ and new orthogonal coordinates $\{y_1, \dots, y_n\}$ such that V is a hypercube in the new coordinates given by

$$V = \{(y_1, \dots, y_n) : -a_j < y_j < a_j, 1 \le j \le n\},$$

and moreover there exists a Lipschitz (resp. of class $C^{k,1}$) function φ, defined on the set

$$V' \coloneqq \{y' = (y_1, \dots, y_{n-1}) : -a_j < y_j < a_j, 1 \le j \le n-1\}$$

and such that

$$\begin{aligned} |\varphi(y')| &\le a_n/2 \text{ for every } y' \in V', \\ \Omega \cap V &= \{y = (y', y_n) \in V \;:\; y_n < \varphi(y')\}, \\ \partial\Omega \cap V &= \{y = (y', y_n) \in V \;:\; y_n = \varphi(y')\}. \end{aligned}$$

Remark 2.4. The above definition means that, in a neighbourhood of x, Ω is below the graph of φ and that the boundary $\partial\Omega$ coincides with the graph of φ.

Theorem 2.5 (cf. [108, Theorem 1.15])**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary. If $s > t \ge 0$ and $s - n/p > t - n/q$, then*

$$W^{s,p}(\Omega) \overset{c}{\hookrightarrow} W^{t,q}(\Omega).$$

For $1 < p < \infty$ and $s \in \mathbb{R}$, the *boundary spaces* $W^{s,p}(\partial\Omega)$ are defined in different ways which are endowed with a norm that makes them Banach spaces; see Remark 4 in [40, p. 146], [62, Definition 1.3.3.2] and [107, p. 90]. We fix $H^s(\partial\Omega) \coloneqq W^{s,2}(\partial\Omega)$, these spaces are rigorously introduced in [56, Section 9.2], [72, Section 4.2], [89, p. 34] and [98, Section 2.7]. We put $L^p(\partial\Omega) \coloneqq W^{0,p}(\partial\Omega)$. Due to (7.17) in [89, p. 35], we can affirm that $H^s(\partial\Omega)$ is dense in $L^2(\partial\Omega)$ for $s \ge 0$. When $s > 0$, the dual space of the antilinear continuous functionals on $H^s(\partial\Omega)$ is denoted by $H^{-s}(\partial\Omega)$. For the boundary spaces $H^s(\partial\Omega)$ there is a result of compact embedding and consequently also continuous.

Theorem 2.6 (cf. [72, Theorem 4.2.2])**.** *Let Ω be a bounded open set of $\mathbb{R}^n$ with a $C^{k,1}$ boundary and let $t, s \in \mathbb{R}$ with $|t|, |s| \le k + 1/2$. Then,*

$$H^s(\partial\Omega) \overset{c}{\hookrightarrow} H^t(\partial\Omega) \quad \text{for} \quad t < s.$$

There is a relation among the boundary spaces and the Sobolev spaces given by the *trace theorem*, which is stated below.

Theorem 2.7 (cf. [62, Theorem 1.5.1.2]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a $C^{k,1}$ boundary. Let $1 < p < \infty$ and $s \in \mathbb{R}$ such that $s - 1/p \notin \mathbb{Z}$, $s \leq k+1$ and $0 \leq [s - 1/p] =: l$. The mapping*

$$C^{k,1}(\overline{\Omega}) \ni u \mapsto \left(u\big|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}, \dots, \frac{\partial^l u}{\partial \nu^l}\Big|_{\partial\Omega} \right)$$

has a unique extension to a surjective continuous linear operator

$$\gamma : W^{s,p}(\Omega) \to \prod_{j=0}^{l} W^{s-j-1/2,p}(\partial\Omega),$$

where $\frac{\partial^j u}{\partial \nu^j}$ is the j-order normal derivative of u on $\partial\Omega$.

The operator γ is called *trace operator* and its components are denoted by $\gamma_0, \gamma_1, \dots, \gamma_l$ and known as *trace of order 0*, *trace of order 1*,..., *trace of order l*; respectively. So, for $u \in W^{s,p}(\Omega)$ we can write $\gamma u = (\gamma_0 u, \gamma_1 u, \dots, \gamma_l u)$. Due to Theorem 2.7, for certain Sobolev spaces $W_0^{s,p}(\Omega)$ with regular boundary $\partial\Omega$ we have a characterization stated in the following result.

Theorem 2.8 (cf. [62, Corollary 1.5.1.6]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a $C^{k,1}$ boundary. Let $1 < p < \infty$ and $s \in \mathbb{R}$ such that $s - 1/p \notin \mathbb{Z}$, $s \leq k+1$ and $0 \leq [s-1/p] =: l$. Then, $u \in W_0^{s,p}(\Omega)$ if and only if $u \in W^{s,p}(\Omega)$ and*

$$u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^l u}{\partial \nu^l} = 0 \text{ on } \partial\Omega.$$

Green's formulas for integration are given below. The following result is known as *integration by parts*. A more general version can be found in Section 3.1.2 from [107].

Theorem 2.9 (cf. [62, Theorem 1.5.3.1]). *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. For $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ with $1 < p < \infty$ and $1/p + 1/q = 1$, we have*

$$\int_\Omega (\partial_j u)\overline{v} dx = -\int_\Omega u(\partial_j \overline{v}) dx + \int_{\partial\Omega} u\overline{v}\nu_j dS$$

for $j = 1, \dots, n$. Here ν_j denotes the j-th component of ν which is the unit outer normal vector along $\partial\Omega$.

Remark 2.10. The product $u\overline{v}$ in the integral over $\partial\Omega$ should be understood as $(\gamma_0 u)(\gamma_0 \overline{v})$. The same holds in similar situations in the boundary integrals that appear later.

Next we present an integration formula that will be widely used in this work which is an easy consequence of Theorem 2.9.

Theorem 2.11 (cf. [62, Theorem 1.5.3.7])**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Then for every $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega (\Delta u)\overline{v}dx = -\int_\Omega \nabla u \cdot \nabla \overline{v} dx + \int_{\partial\Omega} (\partial_\nu u)\overline{v} dS, \tag{2.1}$$

where $\Delta u := \sum_{j=1}^n \partial_j^2 u$ is the Laplace operator *of u, $\nabla u := (\partial_1 u, \ldots, \partial_n u)$ is the* gradient *of u and $\partial_\nu u$ denotes the* derivative of u in the direction of ν.

Corollary 2.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary. Then,*

$$\int_\Omega (\Delta u)\overline{v}dx = \int_\Omega u(\Delta \overline{v})dx + \int_{\partial\Omega} [(\partial_\nu u)\overline{v} - u(\partial_\nu \overline{v})]dS \tag{2.2}$$

for all $u, v \in H^2(\Omega)$.

Proof. An application of (2.1) twice allows to get the desired formula. □

Identities (2.1) and (2.2) are known as *first Green's formula* and *second Green's formula*, respectively. The next theorem is an extension of Theorem 2.11.

Definition 2.13 (cf. [49, APPENDIX C])**.** Let $U \subset \mathbb{R}^n$ be open and bounded and $k \in \mathbb{N}$. We say that the boundary ∂U is C^k if for each point $x \in \partial U$ there exist $r > 0$ and a C^k function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that -upon relabeling and reorienting the coordinates axis if necessary- we have

$$U \cap B(x,r) = \{x \in B(x,r) : x_n > \phi(x_1, \ldots, x_{n-1})\},$$

where $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$. We also say that ∂U is a *boundary of class C^k* or simply *C^k-boundary ∂U*. The boundary ∂U is of *class C^∞* if it is C^k-boundary ∂U for all $k \in \mathbb{N}$.

Theorem 2.14 (cf. [21, Lemma 8.2.4])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary $\partial\Omega$. For $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ and $v \in H^1(\Omega)$, we have that*

$$\int_\Omega (\Delta u)\overline{v}dx = -\int_\Omega \nabla u \cdot \nabla \overline{v} dx + \langle \partial_\nu u, v \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}. \tag{2.3}$$

Remark 2.15. In Theorem 2.14, Δu is a regular distribution generated by a unique function $f \in L^2(\Omega)$ through the expression

$$\langle \Delta u, \varphi \rangle := \int_\Omega f\overline{\varphi} dx, \quad \varphi \in C_c^\infty(\Omega).$$

In this case, f is denoted by Δu.

In [62, p. 62], formula (2.3) appears but now: $u \in H^1(\Omega)$ with $\Delta u \in L^p(\Omega)$ and Ω being a bounded open subset of $\mathbb{R}^n$ with a $C^{1,1}$ boundary.

The following result is an integration formula known as *Rellich identity* and it is a immediate consequence of Corollary 2.1 in [100]. This formula will allow obtaining a key estimate in the proof of polynomial stability in Section 4.4.

Theorem 2.16. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^2. If $u \in H^2(\Omega)$ and $h \in C^1(\overline{\Omega}, \mathbb{R}^n)$, then*

$$\begin{aligned} 2\operatorname{Re}\int_\Omega \Delta u(h\cdot\nabla\overline{u})dx &= 2\operatorname{Re}\int_{\partial\Omega}\partial_\nu u(h\cdot\nabla\overline{u})dS - \int_{\partial\Omega}(h\cdot\nu)|\nabla u|^2 dS \\ &\quad + \int_\Omega \operatorname{div} h|\nabla u|^2 dx - 2\operatorname{Re}\sum_{i,j=1}^n\int_\Omega \partial_i h_j \partial_i u \partial_j \overline{u} dx, \end{aligned}$$

where $\operatorname{div} h := \partial_1 h_1 + \cdots + \partial_n h_n$.

We continue this section with an integration formula involving the bounded open set $\Omega_1 \subset \mathbb{R}^2$, which has a boundary of class C^4, from Section 1.1.

Proposition 2.17 (cf. [15, Lemma 2.1])**.** *For $u \in H^4(\Omega_1)$ and $v \in H^2(\Omega_1)$ such that $u = \partial_\nu u = v = \partial_\nu v = 0$ on Γ, it holds*

$$\begin{aligned} \int_{\Omega_1}(\Delta^2 u)\overline{v}dx &= \mu\int_{\Omega_1}\Delta u\Delta\overline{v}dx + (1-\mu)\int_{\Omega_1}\nabla^2 u : \nabla^2\overline{v}dx \\ &\quad - \int_I(\mathscr{B}_1 u)\partial_\nu\overline{v}dS + \int_I(\mathscr{B}_2 u)\overline{v}dS, \end{aligned} \tag{2.4}$$

where

$$\nabla^2 u : \nabla^2\overline{v} := u_{x_1x_1}\overline{v}_{x_1x_1} + u_{x_2x_2}\overline{v}_{x_2x_2} + 2u_{x_1x_2}\overline{v}_{x_1x_2}. \tag{2.5}$$

Generalizations of the previous result can be found in [3], formula (2.13), or in [128], identity (3.1). Next we give the version of the first reference, which we will use in Subsection 3.1.2.

Proposition 2.18. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with boundary of class C^4. If $u \in H^2(\Omega)$ with $\Delta^2 u \in L^2(\Omega)$ and $v \in H^2(\Omega)$, then*

$$\int_\Omega (\Delta^2 u)\overline{v}dx = \mu\left(\Delta u, \Delta v\right)_{L^2(\Omega)} + (1-\mu)\int_\Omega \nabla^2 u : \nabla^2 \overline{v} dx \\ - \left\langle \mathscr{B}_1 u, \partial_\nu v\right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} + \left\langle \mathscr{B}_2 u, v\right\rangle_{H^{-3/2}(\partial\Omega)\times H^{3/2}(\partial\Omega)}.$$

2.3 Some useful inequalities

In [49, p. 706] appears the well-known *Young's inequality*: Let $p, q > 1$ such that $1/p + 1/q = 1$. For $a, b \geq 0$ we have

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q. \tag{2.6}$$

Proposition 2.19. *Let $\alpha\beta > 0$. For $a, b \geq 0$ we have that*

$$a^\alpha b^\beta \leq \frac{\alpha}{\alpha+\beta}a^{\alpha+\beta} + \frac{\beta}{\alpha+\beta}b^{\alpha+\beta}. \tag{2.7}$$

Proof. If $\alpha\beta > 0$, then $p := \frac{\alpha+\beta}{\alpha} > 1$ and $q := \frac{\alpha+\beta}{\beta} > 1$ satisfy the equality $1/p + 1/q = 1$. Now, (2.7) follows from the application of (2.6) to the product $a^\alpha b^\beta$. □

Corollary 2.20. *Let $a, b \geq 0$ and $\alpha \in (0, 2)$. We have*

$$a^{2-\alpha} b^\alpha \leq \varepsilon a^2 + C_{\alpha,\varepsilon} b^2 \quad \forall \varepsilon > 0, \tag{2.8}$$

where $C_{\alpha,\varepsilon} := \frac{\alpha}{2}\left(\frac{2-\alpha}{2\varepsilon}\right)^{\frac{2-\alpha}{\alpha}}$.

Proof. As $0 < \alpha < 2$, then $(2-\alpha)\alpha > 0$. Let $\varepsilon > 0$. Applying (2.7) to the right-hand side of the expression

$$a^{2-\alpha} b^\alpha = \left(\sqrt{\frac{2\varepsilon}{2-\alpha}}a\right)^{2-\alpha} \left[\left(\frac{2\varepsilon}{2-\alpha}\right)^{\frac{\alpha-2}{2\alpha}} b\right]^\alpha,$$

we get (2.8). □

Now, we present a generalization of Theorem 1.4.4 in [95], see [17, Proposition 7.3] or [28, Theorem 1.6.6], followed by a simple proof.

Theorem 2.21. *Let $1 < p < \infty$ and let Ω be a bounded domain in $\mathbb{R}^n$ with C^1-boundary $\partial\Omega$. For $u \in W^{1,p}(\Omega)$, the following estimate holds:*

$$\|u\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}^{1/p} \|u\|_{L^p(\Omega)}^{1-1/p} \tag{2.9}$$

with C being a positive constant independent of u.

Proof. By Theorem 1.5.1.10 in [62], there exists a constant $C > 0$ such that

$$\|u\|_{L^p(\partial\Omega)}^p \leq C \Big[\varepsilon^{1-1/p} \|u\|_{W^{1,p}(\Omega)}^p + \varepsilon^{-1/p} \|u\|_{L^p(\Omega)}^p\Big] \tag{2.10}$$

for any $u \in W^{1,p}(\Omega)$ and for all $0 < \varepsilon < 1$. Note that (2.9) is trivially satisfied if $u = 0$. Let $u \in W^{1,p}(\Omega)$ with $u \neq 0$. Taking $\varepsilon = \frac{\|u\|_{L^p(\Omega)}^p}{\|u\|_{W^{1,p}(\Omega)}^p}$, we obtain (2.9) from (2.10). □

Corollary 2.22. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1-boundary $\partial\Omega$. For any $u \in H^2(\Omega)$, the following estimate holds:*

$$\|\partial_\nu u\|_{L^2(\partial\Omega)} \leq \widetilde{C} \|u\|_{H^2(\Omega)}^{1/2} \|\nabla u\|_{L^2(\Omega)^n}^{1/2}$$

with $\widetilde{C}$ being a positive constant independent of u.

Proof. Let $u \in C^2(\overline{\Omega})$. Because of Theorem 2.21, we can write

$$\begin{aligned}
\|\partial_\nu u\|_{L^2(\partial\Omega)} &\leq \sum_{j=1}^n \|\nu_j \partial_j u\|_{L^2(\partial\Omega)} \\
&\leq C \max_{\partial\Omega} |\nu| \sum_{j=1}^n \|\partial_j u\|_{H^1(\Omega)}^{1/2} \|\partial_j u\|_{L^2(\Omega)}^{1/2} \\
&\leq \widetilde{C} \|u\|_{H^2(\Omega)}^{1/2} \|\nabla u\|_{L^2(\Omega)^n}^{1/2},
\end{aligned}$$

where $\widetilde{C} := nC \max\{|\nu(x')| : x' \in \partial\Omega\}$. Since $\partial\Omega \in C^1$, then it satisfies the *segment property*[1] and thus $C^2(\overline{\Omega})$ is dense in $H^2(\Omega)$, see [1, p. 68]. This density allows us to finish the proof. □

[1] cf. this concept in [123, Definition 2.1].

In the next theorem, we present the well-known *interpolation inequality* for Sobolev spaces. This result is the product of the combination of Theorem 2 of Subsection 4.3.1 in [120, p. 317], Remark 5 of Subsection 4.2.3 in [120, p. 314] and part (f) of Theorem of Subsection 1.9.3 in [120, p. 59].

Theorem 2.23. *Let Ω be a bounded domain of $\mathbb{R}^n$ with C^1-boundary. For $0 \leq s_2 < s_1$ and $0 < \theta < 1$, there exists a constant $C := C_{\theta,s_1,s_2} > 0$ such that*

$$\|u\|_{H^{(1-\theta)s_1+\theta s_2}(\Omega)} \leq C \|u\|_{H^{s_1}(\Omega)}^{1-\theta} \|u\|_{H^{s_2}(\Omega)}^{\theta} \tag{2.11}$$

for all $u \in H^{s_1}(\Omega)$.

Next we present the *Friedrichs inequality* also known as the *generalized Poincaré inequality*.

Theorem 2.24 (cf. [107, Theorem 1.9])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $\Gamma \subset \partial\Omega$ with $\mu(\Gamma) \neq 0$, where μ is the $(n-1)$-dimensional Lebesgue surface measure. Then, for $u \in H^1(\Omega)$ we have that*

$$\|u\|_{H^1(\Omega)} \leq C \left(\|u\|_{L^2(\Gamma)}^2 + \|\nabla u\|_{L^2(\Omega)^n}^2 \right)^{1/2},$$

where the positive constant C depends only on Ω.

2.4 Semigroups and groups of bounded linear operators

We will give some concepts (taken from the books [47, 48, 58, 95, 109]) and results of our interest corresponding to the theory of semigroups and groups of bounded linear operators, which have to do with generation, stability and analyticity.

Throughout this section X will be a Banach space over $\mathbb{C}$. The space $\mathcal{L}(X)$ endowed with the usual *operator norm* is a Banach space.

Definition 2.25. A one-parameter family $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is a *semigroup of bounded linear operators* or simply *semigroup* on X if satisfies
i) $T(0) = I$, where I is the identity operator on X.
ii) $T(s+t) = T(s)T(t)$ for every $s, t \geq 0$.

Definition 2.26. An operator A is the *infinitesimal generator* or simply *generator* of a semigroup $(T(t))_{t\geq 0}$ when its domain is given by

$$D(A) := \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t} \quad \text{for} \quad x \in D(A).$$

A semigroup $(T(t))_{t\geq 0}$ on X is called *strongly continuous semigroup* or simply a C_0*-semigroup* if

$$\lim_{t\to 0^+} T(t)x = x \quad \text{for every} \quad x \in X.$$

Proposition 2.27. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup and A be its infinitesimal generator. If $x \in D(A)$, then $T(t)x \in D(A)$ and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

Proof. See part c) of Theorem 2.4 in [109, p. 4]. □

Proposition 2.28 (cf. [109, Corollary 2.5 on p. 5])**.** *If A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$, then $D(A)$ is dense in X and A is a closed linear operator.*

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup. Due to Theorem 2.2 in [109, p. 4], there exist constants $\omega \geq 0$ and $M \geq 1$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$. When $\omega = 0$ and $M = 1$, $(T(t))_{t\geq 0}$ is called C_0*-semigroup of contractions.*

Let X' be the dual of X. The value of $x' \in X'$ at $x \in X$ is denoted by $\langle x, x' \rangle_{X\times X'}$ or $\langle x', x \rangle_{X'\times X}$. For $x \in X$ the *duality set* $\mathcal{F}(x)$ is defined as

$$\mathcal{F}(x) := \left\{ x' \in X' : \langle x, x' \rangle_{X\times X'} = \|x\|_X^2 = \|x'\|_{X'}^2 \right\}.$$

The Hahn–Banach theorem (complex case) allows to obtain the Corollary 1.3 in [29] when the vector space is defined over $\mathbb{C}$. So, we have that for every $x \in X$ there exists $x' \in X'$ such that $\|x'\|_{X'} = \|x\|_X$ and $\langle x', x \rangle_{X'\times X} = \|x\|_X^2$. In consequence, $\mathcal{F}(x) \neq \emptyset$ for every $x \in X$.

Definition 2.29. A linear operator $A : D(A) \subset X \to X$ is *dissipative* if for every $x \in D(A)$ there exists $x' \in \mathcal{F}(x)$ such that $\mathrm{Re}\langle Ax, x'\rangle_{X\times X'} \leq 0$.

Theorem 2.30 (Lumer–Phillips)**.** *Let $A : D(A) \subset X \to X$ be a linear operator such that $\overline{D(A)} = X$.*

i) If A is dissipative and there is $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.

ii) If A is the infinitesimal generator of a C_0-semigroup of contractions on X, then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative.

Proof. See Theorem 4.3 in [109, p. 14]. □

Theorem 2.31 (cf. [109, Theorem 4.6 on p. 16])**.** *Let $A : D(A) \subset X \to X$ be dissipative with $R(I - A) = X$. If X is reflexive, then $\overline{D(A)} = X$.*

Remark 2.32. Let V and H be two Hilbert spaces over $\mathbb{C}$ such that $V \overset{d}{\hookrightarrow} H$, i.e., V is dense in H and is continuously embedded in H. Identifying H with its antidual, we have $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$ and moreover

$$\langle f, v\rangle_{V'\times V} = (f, v)_H \tag{2.12}$$

for all $f \in H$ and $v \in V$ (see Observação 4.11 in [32]). So, if in the definition of dissipativity we put H instead of X, we have that $A : D(A) \subset H \to H$ is *dissipative* if for any $x \in D(A)$,

$$\mathrm{Re}\,(Ax, x)_H \leq 0.$$

Compare this remark with item (iii) from [48, p. 83].

Now, let $A : D(A) \subset X \to X$ be a linear operator and $u_0 \in X$. The *abstract Cauchy problem* for A with initial data u_0 is given by

$$\begin{cases} \frac{du(t)}{dt} = Au(t) \quad \text{for} \quad t > 0, \\ u(0) = u_0. \end{cases} \tag{2.13}$$

Two types of solutions of the previous system can be considered, which will depend on the choice of the initial data, thanks to the following two theorems.

Definition 2.33. A *classic solution* or simply *solution* of (2.13) is a function $u : [0, \infty) \to X$ such that $u(t)$ is continuous for $t \geq 0$ and continuously differentiable for $t > 0$ with $u(t) \in D(A)$ for $t > 0$ and satisfying (2.13).

Theorem 2.34. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions on X and let be A its infinitesimal generator. If $u_0 \in D(A)$, then* (2.13) *has an unique solution $u(t) = T(t)u_0$, for $t \geq 0$, such that*

$$u \in C([0,\infty), D(A)) \cap C^1([0,\infty), X),$$

where $D(A)$ is equipped with the graph norm $|x|_{D(A)} \coloneqq \|x\|_X + \|Ax\|_X$ *for $x \in D(A)$.*

Proof. See Remark 2.2.1 and Theorem 2.2.2 in [130]. □

Definition 2.35. A continuous function $u : [0,\infty) \to X$ is called a *mild solution* of (2.13) if $\int_0^t u(s)ds \in D(A)$ for all $t \geq 0$ and moreover

$$u(t) = A\int_0^t u(s)ds + u_0.$$

Theorem 2.36. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X. If $u_0 \in X$, then $u(t) \coloneqq T(t)u_0$ is the unique mild solution of* (2.13).

Proof. See Proposition 6.4 in [47, p. 146]. □

Next we define the *iterations of unbounded operators*. Let $n \in \mathbb{N}$. The n-th power A^n of $A : D(A) \subset X \to X$ is defined successively as

$$\begin{aligned} A^n x &\coloneqq A(A^{n-1}x), \\ D(A^n) &\coloneqq \{x \in D(A) : A^{n-1}x \in D(A)\}, \end{aligned}$$

where A^0 is the identity operator.

Theorem 2.37 (Regularity of solutions)**.** *Let A be the infinitesimal generator of a C_0-semigroup of contractions on X. If $u_0 \in D(A^k)$, $k \in \mathbb{N}$, then the unique solution of problem* (2.13) *belongs to*

$$\bigcap_{j=0}^{k} C^{k-j}([0,\infty), D(A^j)).$$

Proof. See Theorem 7.5 in [29] or Theorem 2.3.1 in [130]. □

Now we present some notions of stability followed by certain classical characterizations. Subsequently, we give the concept of analytic semigroup and an analyticity result for C_0-semigroups of contractions.

Definition 2.38. Let $(T(t))_{t\geq 0}$ be a semigroup on X. Then, $(T(t))_{t\geq 0}$ is said to be *strongly stable* if

$$\lim_{t\to\infty} \|T(t)x\|_X = 0, \quad \forall x \in X.$$

The next theorem corresponds to a simpler version of the well-known general result for strong stability established by Arendt and Batty (see [8, Theorem 2.4]). Before presenting the theorem we define a pair of sets. Let $A : D(A) \subset X \to X$ be a linear operator. The *resolvent set* $\rho(A)$ of A is the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - A : D(A) \to X$ is bijective and $(\lambda I - A)^{-1} \in \mathcal{L}(X)$. The set $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is called *spectrum* of A.

Theorem 2.39 (cf. [47, Corollary 2.22 on p. 327])**.** *Let us suppose that A is the infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geq 0}$ on a Hilbert space H. If A has no purely imaginary eigenvalues and $\sigma(A) \cap i\mathbb{R}$ is countable, then $(T(t))_{t\geq 0}$ is strongly stable.*

Definition 2.40. A semigroup $(T(t))_{t\geq 0}$ on X is called *exponentially stable* if there exist constants $\omega > 0$ and $M \geq 1$ such that

$$\|T(t)\|_{\mathcal{L}(X)} \leq M e^{-\omega t} \text{ for all } t \geq 0. \tag{2.14}$$

The following characterization for exponential stability is due to Gearhat [57] and Prüss [113], see [95, Theorem 1.3.2]. An elementary proof of this result, but now for bounded C_0-semigroups, can be seen in [41].

Theorem 2.41. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space H and A be its infinitesimal generator. Then, $(T(t))_{t\geq 0}$ is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and moreover*

$$\limsup_{\lambda\in\mathbb{R},\, |\lambda|\to\infty} \left\|(i\lambda I - A)^{-1}\right\|_{\mathcal{L}(H)} < \infty. \tag{2.15}$$

In case it is not possible to determine the exponential stability, one tries to look for another type of asymptotic decay. Next we present the notion of polynomial stability, which is slower than exponential stability.

Definition 2.42. Let $(T(t))_{t\geq 0}$ be a semigroup on X and A be its generator. Then, $(T(t))_{t\geq 0}$ is *polynomially stable* of order $\alpha > 0$ if there exists $C > 0$ such that

$$\|T(t)x\|_X \leq C t^{-\alpha} |x|_{D(A)} \tag{2.16}$$

for any $t > 0$ and for all $x \in D(A)$. In that case, one says that $(T(t))_{t\geq 0}$ *decays at a rate $t^{-\alpha}$.*

The following observation is based on comments made by Liu and Rao in Introduction from [92].

Remark 2.43. The norm on the right-hand side of (2.16) can not be the norm in X because if $(T(t))_{t\geq 0}$ is bounded, in particular it could be a C_0-semigroup of contractions, then (2.16) implies (2.14). Indeed, let us assume

$$\|T(t)x\|_X \leq Ct^{-\alpha}\|x\|_X\,, \quad t>0,$$

for all $x \in X$. As $\lim_{t\to\infty} t^{-\alpha} = 0$, then there exists $t_0 \in \mathbb{R}^+$ such that $t_0^{-\alpha} < 1/C$ and so $\|T(t_0)\|_{\mathcal{L}(X)} < 1$. If $(T(t))_{t\geq 0}$ is bounded, there is $\widetilde{C} \geq 1$ such that $\|T(s)\|_{\mathcal{L}(X)} \leq \widetilde{C}$ for all $s \geq 0$. Let $t > 0$. By the division algorithm, $t = n_t t_0 + r_t$ where $n_t \in \mathbb{N}_0$ and $0 \leq r_t < t_0$. In consequence,

$$\|T(t)\|_{\mathcal{L}(X)} = \|(T(t_0))^{n_t}\,T(r_t)\|_{\mathcal{L}(X)} \leq \widetilde{C}\,\|T(t_0)\|_{\mathcal{L}(X)}^{n_t}\,. \tag{2.17}$$

If $\|T(t_0)\|_{\mathcal{L}(X)} = 0$, then (2.14) holds trivially. Let us assume $\|T(t_0)\|_{\mathcal{L}(X)} \neq 0$. From (2.17), we get

$$\|T(t)\|_{\mathcal{L}(X)} \leq \widetilde{C}\,\|T(t_0)\|_{\mathcal{L}(X)}^{-r_t/t_0}\,\|T(t_0)\|_{\mathcal{L}(X)}^{t/t_0}\,.$$

Since $0 < \|T(t_0)\|_{\mathcal{L}(X)} < 1$, we have $\|T(t_0)\|_{\mathcal{L}(X)}^{-1} > \|T(t_0)\|_{\mathcal{L}(X)}^{-r_t/t_0} \geq 1$ (note that $-1 < -r_t/t_0 \leq 0$) and there is $\delta > 0$ such that $\frac{1}{t_0}\ln\|T(t_0)\|_{\mathcal{L}(X)} = -\delta$. Therefore,

$$\|T(t)\|_{\mathcal{L}(X)} < Me^{-\delta t},$$

where $M \coloneqq \widetilde{C}\,\|T(t_0)\|_{\mathcal{L}(X)}^{-1} > 1$.

In [25, Theorem 2.4], Borichev and Tomilov have a characterization for the polynomial stability of bounded C_0-semigroups. This result is extended by Muñoz Rivera and Racke, now considering C_0-semigroup of contractions.

Theorem 2.44 (cf. [105, Lemma 5.2])**.** *Let $(T(t))_{t\geq 0}$ be a contraction semigroup on a Hilbert space H with generator A such that $i\mathbb{R}\cap\sigma(A)$ is empty. Then, for $\alpha' \in \mathbb{N}_0$ and $\beta' > 0$ fixed the following assertions are equivalent:*

i) There exist $C > 0$ and $\lambda_0 > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| > \lambda_0$ and all $F \in D(A^{\alpha'})$ it holds

$$\left\|(i\lambda\mathcal{I} - A)^{-1}F\right\|_H \leq C|\lambda|^{\beta'}\|A^{\alpha'}F\|_H. \tag{2.18}$$

ii) There exists some $C > 0$ such that for all $t > 0$ it holds

$$\left\|T(t)A^{-1}\right\|_{\mathcal{L}(H)} \leq Ct^{-\frac{1}{\alpha'+\beta'}}. \tag{2.19}$$

Asymptotic behavior is also a characteristic of some analytic semigroups. More precisely, the decay would be of the exponential type (see Corollary 5.3). On the other side, analytic semigroups have the property of smoothing effect, that is, the solution of (2.13) is of class C^∞ regardless of the irregularity of the initial data (see [51, Introduction] and [104, Observação 7.1.1]).

Definition 2.45. For $0 < \vartheta \leq \pi$, we set $\Sigma_\vartheta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \vartheta\}$. A semigroup $(T(t))_{t\geq 0}$ on X is said to be *analytic* if it admits an extension $T(\lambda) \in \mathcal{L}(X)$ for $\lambda \in \Sigma_\theta \cup \{0\}$ for some $0 < \theta \leq \pi/2$ such that

i) $\lambda \mapsto T(\lambda)$ is analytic on the sector Σ_θ.

ii) $\lim\limits_{\Sigma_{\theta-\varepsilon} \ni \lambda \to 0} \|T(\lambda)x - x\|_X = 0$ for every $x \in X$ and each $0 < \varepsilon < \theta$.

iii) $T(\lambda + \mu) = T(\lambda)T(\mu)$ for all $\lambda, \mu \in \Sigma_\theta$.

Theorem 2.46 (cf. [95, Theorem 1.3.3])**.** *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space H and let A be its generator. Suppose that $i\mathbb{R} \subset \rho(A)$. Then, $T(t)$ is analytic if and only if*

$$\limsup_{\lambda\in\mathbb{R},\, |\lambda|\to\infty} \left\|\lambda(i\lambda I - A)^{-1}\right\|_{\mathcal{L}(H)} < \infty \tag{2.20}$$

holds.

We continue with some notions about groups of bounded linear operators and close with a classical result of group generation which was initially proved by the American mathematician Marshall Harvey Stone in 1932.

Definition 2.47. A one-parameter family $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is a C_0-*group* or simply *group* on X if it satisfies

i) $T(0) = I$.

ii) $T(t + s) = T(t)T(s)$ for any $t, s \in \mathbb{R}$.

iii) $\lim_{t\to 0} T(t)x = x$ for $x \in X$.

Definition 2.48. The *infinitesimal generator* $A : D(A) \subset X \to X$ of a group $(T(t))_{t\in\mathbb{R}}$ on X is the operator

$$Ax := \lim_{t\to 0} \frac{1}{t}\big[T(t)x - x\big]$$

defined for every x in its domain

$$D(A) := \Big\{x \in X : \lim_{t\to 0} \frac{1}{t}\big[T(t)x - x\big] \text{ exists}\Big\}.$$

Before giving Stone's theorem, we will recall some definitions. Let H be a complex Hilbert space and $A : D(A) \subset H \to H$ be a densely defined linear operator. The set

$$D(A^*) := \Big\{ y \in H : \exists y^* \in H \text{ such that } (Ax, y)_H = (x, y^*)_H \;\; \forall x \in D(A) \Big\}$$

is a vector subspace of H. Since $\overline{D(A)} = H$, then for each $y \in D(A^*)$ there exists an unique $y^* \in H$ such that $(Ax, y)_H = (x, y^*)_H$ for all $x \in D(A)$. The *adjoint* of A denoted A^* is given by $A^* : D(A^*) \to H$ with $y \mapsto A^*y := y^*$. It can be easily proved that A^* is a linear operator. Note that

$$(Ax, y)_H = (x, A^*y)_H$$

for all $x \in D(A)$ and for any $y \in D(A^*)$. For details of the above statements, see for example [32, p. 292–294] and [56, p. 313]. The operator A is said to be *self-adjoint* if $A = A^*$. The operator A is called *skew-adjoint* if and only if iA is self-adjoint, i.e., $A^* = -A$ since $(\lambda A)^* = \overline{\lambda} A^*$ for all $\lambda \in \mathbb{C}$ (see (i) of Proposição 5.97 in [32]).

Definition 2.49. Let H be a Hilbert space. A bijective linear operator $U : H \to H$ is *unitary* if $U^* = U^{-1}$.

It is easy to prove that U is unitary if and only if U is an isometry and $R(U) = H$. So, if U is unitary then U is a bounded linear operator.

Theorem 2.50 (Stone)**.** *Let $A : D(A) \subset H \to H$ be a densely defined linear operator on a complex Hilbert space H. Then, A generates a unitary group $(T(t))_{t\in\mathbb{R}}$ on H if and only if A is skew-adjoint.*

Proof. See Theorem 3.24 in [48]. □

2.5 Interpolation-extrapolation scales

Let X be a Banach space and $A : D(A) \subset X \to X$ be a closed linear operator with $0 \in \rho(A)$. Since $A^{-1} \in \mathcal{L}(X)$, then $\xi \mapsto \|A^{-1}\xi\|_X$ defines a norm on X such that $\|A^{-1}\xi\|_X \leq C\,\|\xi\|_X$ for each $\xi \in X$. Equivalence of norms cannot be guaranteed between $\|\cdot\|_X$ and $\|A^{-1}\cdot\|_X$ on X. By Theorem 2.3-2 in [77], the normed space $(X, \|A^{-1}\cdot\|_X)$ has a completion

$$\big(X_{-1}, \|\cdot\|_{-1}\big) := \big(X, \big\|A^{-1}\cdot\big\|_X\big)^{\sim},$$

called the *extrapolation space* of X generated by A, which is a Banach space such that $X \overset{d}{\hookrightarrow} X_{-1}$ with $\|\xi\|_{-1} = \|A^{-1}\xi\|_X$ for any $\xi \in X$, see [5, p. 262].

Let us suppose that the operator A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X. For each $t \geq 0$,

$$T(t) : X \subset X_{-1} \to X_{-1}$$

is a densely defined bounded linear operator. The extension by continuity (see, e.g., [32, Teorema 2.42]) implies that there exists a unique extension of $T(t)$ to X_{-1}, which is denoted by $T_{-1}(t)$. The family $(T_{-1}(t))_{t\geq 0}$ is a C_0-semigroup on X_{-1}, called *extrapolated semigroup* in X_{-1}, whose generator $A_{-1} : D(A_{-1}) \subset X_{-1} \to X_{-1}$ is an extension of A with domain $D(A_{-1}) = X$ such that $0 \in \rho(A_{-1})$, see [47, Theorem 5.5] and [117, Proposition 4.2].

Definition 2.51 ([48, Definition A.12])**.** Let X' be the dual space of X. As the domain of A is dense in X, the *adjoint operator* $A' : D(A') \subset X' \to X'$ exists and is given by

$$\begin{aligned} D(A') &:= \{x' \in X' : \exists y' \in X' \text{ such that } \langle Ax, x'\rangle = \langle x, y'\rangle \ \forall x \in D(A)\}, \\ A'x' &:= y' \text{ for all } x' \in D(A'). \end{aligned}$$

Here $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{X\times X'}$.

If X is reflexive, then $A' : D(A') \subset X' \to X'$ is a densely defined close linear operator such that $\rho(A') = \rho(A)$. We set $X^\sharp := X'$, $A^\sharp := A'$ and $X_1^\sharp := (D(A^\sharp), \|A^\sharp \cdot\|_{X^\sharp})$, see Section 1.2 and Section 1.4 in [5, Chapter V]. Thanks to Corollary 1.4.7 in [5, p. 271], the space X_{-1} is the dual space of $D(A')$, i.e., $X_{-1} = [D(A')]'$.

For $m \in \mathbb{N}_0$ arbitrary but fixed, we have that the *extrapolated discrete power scale* $[(X_k, A_k); k \in \mathbb{Z} \cap [-m, \infty)]$ *of order* m is well-defined and it is a densely injected *Banach scale.* For details, see Section 1.3 in [5, Chapter V]. Here,

$$X_k := \begin{cases} (D(A^k), \|\cdot\|_k) & \text{if } k \geq 0, \\ (X, \|\cdot\|_k)^\sim & \text{if } k < 0, \end{cases}$$

where $\|x\|_k := \left\|A^k x\right\|_X$ for $x \in D(A^k)$ and $k \in \mathbb{Z}$. A_k is the X_k-realization of A if $k \geq 0$ and the closure of A in X_k if $k < 0$. Putting $T_k(t) := T(t)\big|_{X_k}$ and denoting $T_{-k}(t)$ as the continuous extension of $T_{-k+1}(t)$ to X_{-k}, for $k \geq 0$, the following diagram we obtain

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{A_2} & X_2 & \xrightarrow{A_1} & X_1 & \xrightarrow{A_0} & X_0 & \xrightarrow{A_{-1}} & X_{-1} & \xrightarrow{A_{-2}} & \cdots \\
 & & \big\downarrow {\scriptstyle T_2(t)} & & \big\downarrow {\scriptstyle T_1(t)} & & \big\downarrow {\scriptstyle T_0(t)} & & \big\downarrow {\scriptstyle T_{-1}(t)} & & \\
\cdots & \xleftarrow[A_2^{-1}]{} & X_2 & \xleftarrow[A_1^{-1}]{} & X_1 & \xleftarrow[A_0^{-1}]{} & X_0 & \xleftarrow[A_{-1}^{-1}]{} & X_{-1} & \xleftarrow[A_{-2}^{-1}]{} & \cdots
\end{array}$$

For $0 < \theta < 1$ consider an *admissible interpolation functor* $(\cdot,\cdot)_\theta$ of exponent θ, e.g., the real interpolation functor, the complex interpolation functor or the continuous interpolation functor, see Section 2 in [5, Chapter I] and the references therein. Let $\alpha \coloneqq k + \theta$ with $k \in \mathbb{Z} \cap [-m, \infty)$. Defining

$$X_\alpha \coloneqq (X_k, X_{k+1})_\theta$$

and A_α as the X_α-realization of A_k, one has that X_α is an *intermediate space* between X_{k+1} and X_k, and $[(X_\alpha, A_\alpha); \alpha \in [-m, \infty)]$ is a densely injected Banach scale, which is called the *interpolation-extrapolation scale of order* m *generated by* (X, A) *and* $(\cdot,\cdot)_\theta$. See Theorem 1.5.1 in [5, p. 275]. Therefore,

$$X_\alpha \overset{d}{\hookrightarrow} X_\beta$$

provided that $\alpha > \beta$. Additionally, if $A^{-1} : X \to X$ is a compact operator then the previous interpolation-extrapolation scale is *compactly injected*, i.e., $X_\alpha \overset{c}{\hookrightarrow} X_\beta$ for $\alpha > \beta$.

2.6 Powers of positive self-adjoint operators

Here we indicate when a self-adjoint operator is positive and present some properties that satisfy its fractional powers. Along this section H denotes a Hilbert space.

A self-adjoint operator $A : D(A) \subset H \to H$ is said to be *positive* if there exists $\delta > 0$ such that

$$(Ax, x)_H \geq \delta \|x\|_H^2 \quad \text{for all} \quad x \in D(A). \tag{2.21}$$

Remark 2.52. If $A : D(A) \to H$ is a positive self-adjoint operator, then Lemma 4.31 in [97] implies that $(-\infty, \delta) \subset \rho(A)$ and moreover

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(H)} \leq \frac{1}{\delta - \lambda} \quad \text{for} \quad \lambda < \delta, \tag{2.22}$$

where δ is the positive constant of (2.21). Putting $\varphi(\lambda) \coloneqq \frac{1-\lambda}{\delta-\lambda}$, it follows that $\varphi(\lambda)$ converges to 1, as $\lambda \to -\infty$, and so there exists $\lambda_0 < 0$ such that $|\varphi(\lambda) - 1| < 1$ for any $\lambda < \lambda_0$. Note that the function $\lambda \mapsto \varphi(\lambda)$ is continuous on the compact interval $[\lambda_0, 0]$. Therefore, one obtains that there exists $M > 1$ such that

$$\frac{1-\lambda}{\delta - \lambda} \leq M \quad \text{for} \quad \lambda \leq 0. \tag{2.23}$$

Obviously $(-\infty, 0] \subset \rho(A)$, and the inequalities (2.22) and (2.23) imply that

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(H)} \leq \frac{M}{1 + |\lambda|} \quad \text{for} \quad \lambda \leq 0.$$

Suppose $A : D(A) \to H$ is a positive self-adjoint operator. Then, $0 \in \rho(A)$ and Remark 2.52 implies that A is a *positive operator* in the sense of Definition 4.1 in [97] or Definition in Subsection 1.14.1 of [120], and is also *positive of type* K (see [5, p. 147]). The spectral theory allows to define the fractional powers of A. The previous affirmation appears, for instance, in [118, p. 57]. It is known that $A^s : D(A^s) \subset H \to H$ is positive self-adjoint operator for $s > 0$. The space $D(A^s)$ is endowed with the inner product

$$(u, v)_{D(A^s)} \coloneqq (A^s u, A^s v)_H \tag{2.24}$$

for all $u, v \in D(A^s)$ and for any $s > 0$, which makes it a Hilbert space because A^s is closed and since the norm induced by (2.24), denoted $\|\cdot\|_{D(A^s)}$, is equivalent to the graph norm $|\cdot|_{D(A^s)}$. It can be easily proved that the isometric isomorphism

$$\widetilde{A} : H \to [D(A)]' \quad \text{given by} \quad \langle \widetilde{A}\xi, \zeta\rangle_{[D(A)]' \times D(A)} \coloneqq (\xi, A\zeta)_H \tag{2.25}$$

is a extension of A which is called *standard extension* of A.

The following theorem collects some of the properties that the positive self-adjoint operators satisfy, which can be found in [5, Section 4.6], [97, Chapter 4], [118, p. 57] and [120, Subsection 1.15.2].

Theorem 2.53. *Let $A : D(A) \subset H \to H$ be a positive self-adjoint operator. Their fractional powers satisfy the following properties:*

a) If $s_1, s_2 > 0$, then $A^{s_1} A^{s_2} x = A^{s_1+s_2} x$ for each $x \in D(A^{s_1+s_2})$.

b) If $s_2 > s_1 > 0$, then $D(A^{s_2}) \overset{d}{\hookrightarrow} D(A^{s_1}) \overset{d}{\hookrightarrow} H$.

c) If $D(A^{-s}) := [D(A^s)]'$ for $s > 0$, then $A^{s_1-s_2} : D(A^{s_1}) \to D(A^{s_2})$ is an isomorphism $\forall s_1, s_2 \in \mathbb{R}$ with $s_1 > s_2$.

2.7 Elliptic and boundary operators

This section recalls fundamental aspects of elliptic boundary value problems, and is largely based on Chapter 2 of [89]. Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary of class C^∞ and let

$$A(D)u := \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha u \tag{2.26}$$

be a linear differential operator in Ω of order $2m$. Here $u : \Omega \to \mathbb{C}$, $\alpha \in \mathbb{N}_0^n$, $m \in \mathbb{N}$, $a_\alpha \in \mathbb{C}$ and $D^\alpha := (-i)^{|\alpha|}\partial^\alpha$. Its *principal symbol* is the polynomial $A^0(\xi) := \sum_{|\alpha|=2m} a_\alpha \xi^\alpha$, where $\xi \in \mathbb{R}^n$ and $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

Definition 2.54. The operator A is said to be *elliptic* if the following holds

$$A^0(\xi) \neq 0, \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

If moreover for every linearly independent couple of vectors ξ_1 and ξ_2 of $\mathbb{R}^n$, the polynomial $A^0(\xi_1 + \zeta\xi_2)$ in the complex variable ζ has m roots with positive imaginary part, then A is called *properly elliptic*.

Example 2.55. It is well known that the Laplacian Δ and also the Bi-Laplacian Δ^2 are properly elliptic (see, e.g., Section 3.2 in [12]).

Let $B_1, B_2, \ldots, B_m$ be boundary operators of order m_j with $j = 1, \ldots, m$, respectively, defined as follows

$$B_j(D)v := \sum_{|\beta| \leq m_j} b_{j\beta} D^\beta v, \tag{2.27}$$

where $v : \overline{\Omega} \to \mathbb{C}$, $\beta \in \mathbb{N}_0^n$, $b_{j\beta} \in \mathbb{C}$ and $D^\beta v$ must be understood as $\gamma_0(D^\beta v)$, and whose *principal symbols* are given by $B_j^0(\xi) := \sum_{|\beta|=m_j} b_{j\beta}\xi^\beta$. In this part of the document Γ is considered as a subset of $\partial\Omega$.

Definition 2.56. The system of operators $\{B_j\}_{1\leq j\leq m}$ is a *normal system on* Γ if the system satisfies the following two conditions:

a) For $x \in \Gamma$, it holds $B_j^0(\xi) \neq 0$ for any $0 \neq \xi \in \mathbb{R}^n$ normal to Γ at x and for all $1 \leq j \leq m$.

b) $m_i \neq m_j$ for $i \neq j$ with $1 \leq i, j \leq m$.

Definition 2.57. The system $\{B_j\}_{1\leq j\leq m}$ *covers the operator* A *on* Γ if for all $x \in \Gamma$, all $0 \neq \tau \in \mathbb{R}^n$ tangent to Γ at x, and all $0 \neq \eta \in \mathbb{R}^n$ normal to Γ at x, the polynomials $\mathbb{C} \ni \zeta \mapsto \sum_{|\beta|=m_j} b_{j\beta}(\tau+\zeta\eta)^\beta$ with $j = 1, \ldots, m$ are linearly independent modulo the polynomial $\prod_{k=1}^m [\zeta - \zeta_k^+(\tau, \eta)]$, where $\zeta_k^+(\tau, \eta)$ are the roots of the polynomial $A^0(\tau + \zeta\eta)$ with positive imaginary part.

The operator A of (2.26) and the boundary operators of (2.27) constitute a system called *boundary value problem*, which is given by

$$\begin{cases} Au = f \quad \text{in} \quad \Omega, \\ B_j u = g_j \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.28}$$

and denoted by $(A, B_1, \ldots, B_m)$. Here f and g_j are given functions belonging to suitable spaces.

Remark 2.58. The general theory of boundary value problems is developed for boundaries of class C^∞. This assumption allows to prove general results. However, in some cases we do not need too much regularity on the boundary (see, e.g., Proposition 2.60 and Remark 3.17).

Definition 2.59. The problem (2.28) is a *regular elliptic problem* if and only if A is properly elliptic in $\overline{\Omega}$, the operators B_j have order $m_j \leq 2m - 1$, the system $\{B_j\}_{1\leq j\leq m}$ is normal on $\partial\Omega$ and covers the operator A on $\partial\Omega$.

Suppose the system $(A, B_1, \ldots, B_m)$ is a regular elliptic problem and now consider $\mathcal{A} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ defined as follows:

$$\begin{aligned} D(\mathcal{A}) &:= \{u \in H^{2m}(\Omega) : B_j(D)u = 0 \text{ on } \partial\Omega \text{ for } j = 1, \ldots, m\}, \\ \mathcal{A}u &:= A(D)u. \end{aligned}$$

The space $D(\mathcal{A})$ is endowed with the graph norm. The operator $\mathcal{A}$ is called the *realization* of A in $L^2(\Omega)$ under the boundary conditions $\{B_j\}_{1\leq j\leq m}$. The space $D(\mathcal{A})$ will be denoted by $H_B^{2m}(\Omega)$ when it is equipped with the norm of $H^{2m}(\Omega)$. As $|\cdot|_{D(\mathcal{A})} \sim \|\cdot\|_{H^{2m}}$ on $D(\mathcal{A})$, where $\sim$ means equivalence in norm, then the following characterizations hold:

$$\begin{aligned} D(\mathcal{A}^\theta) &= [D(\mathcal{A}), L^2(\Omega)]_{1-\theta} = [H^{2m}_B(\Omega), H^0(\Omega)]_{1-\theta} \\ &= \Big\{ u \in H^{2m\theta}(\Omega) : B_j(D)u = 0 \text{ on } \partial\Omega,\ m_j < 2m\theta - \tfrac{1}{2} \Big\} \end{aligned} \tag{2.29}$$

for $0 < \theta < 1$. A summary explanation of how to get the previous equalities can be seen in [23, p. 172] or [87, p. 284]. Here $[X, Y]_{\widetilde{\theta}}$, with $0 \le \widetilde{\theta} \le 1$, is the *complex interpolation space* defined for an *interpolation couple* (X, Y) consisting of complex Banach spaces X and Y. See [97, Chapter 2] and [120, Section 1.9] for details.

Next, we present a proposition that treats a boundary value problem defined in Ω_1 with mixed boundary conditions (Ω_1 as in Section 1.1). In addition to guaranteeing the existence and uniqueness of the problem, this result offers an estimate that will be very useful later on.

Proposition 2.60 (cf. [15, Corollary 4.3])**.** *Let $f \in L^2(\Omega_1)$, $g_1 \in H^{7/2}(\Gamma)$, $g_2 \in H^{5/2}(\Gamma)$, $h_1 \in H^{3/2}(I)$ and $h_2 \in H^{1/2}(I)$. Then for sufficiently large $\lambda_0 > 0$, we have that the boundary value problem*

$$\begin{aligned} (\lambda_0 + \Delta^2)u &= f \quad \text{in } \Omega_1, \\ u &= g_1 \quad \text{on } \Gamma, \\ \partial_\nu u &= g_2 \quad \text{on } \Gamma, \\ \mathscr{B}_1 u &= h_1 \quad \text{on } I, \\ \mathscr{B}_2 u &= h_2 \quad \text{on } I, \end{aligned}$$

has a unique solution $u \in H^4(\Omega_1)$. Moreover, the a priori-estimate

$$\begin{aligned} \|u\|_{H^4(\Omega_1)} \le C\big(\|f\|_{L^2(\Omega_1)} &+ \|g_1\|_{H^{7/2}(\Gamma)} + \|g_2\|_{H^{5/2}(\Gamma)} \\ &+ \|h_1\|_{H^{3/2}(I)} + \|h_2\|_{H^{1/2}(I)} \big) \end{aligned} \tag{2.30}$$

holds with a constant $C > 0$ which depends on λ_0 but not on u or on the data.

Chapter 3

Well-posedness and regularity of the solutions of a plate-membrane system

Using semigroup theory of linear operators it will be proved that the plate-membrane transmission problem with initial conditions (1.1)-(1.9) has a unique solution, either a classical or a weak solution depending on the choice of the initial datum, the Lumer–Phillips theorem being the key tool for this purpose. In addition, the regularity of the solutions will be established.

3.1 Existence and uniqueness of the solutions

In this section the system (1.1)-(1.9) will be written as an abstract Cauchy problem. For that, certain linear operators defined on appropriate function spaces are necessary. The energy of the system (1.1)-(1.7) is what allows these spaces to be determined, which is defined by

$$\begin{aligned} E_\gamma(t) := \frac{1}{2}\int_{\Omega_1} \beta_1\mu|\Delta u|^2 + \beta_1(1-\mu)|\nabla^2 u|^2 + \rho_1|u_t|^2 + \gamma|\nabla u_t|^2 \; dx \\ + \frac{1}{2}\int_{\Omega_2} \beta_2|\nabla v|^2 + \rho_2|v_t|^2 \; dx + \frac{1}{2}\int_{\Omega_1} \rho_0|\theta|^2 \; dx, \end{aligned} \tag{3.1}$$

where $|\nabla^2 u|^2 := \nabla^2 u : \nabla^2 \overline{u}$. The above notation was introduced in (2.5).

All the spaces given in this document are taken over the field of complex numbers. Let k be a natural number. We start by considering the following space $H^k_\Gamma(\Omega_1) := \left\{ w \in H^k(\Omega_1) : \frac{\partial^j w}{\partial \nu^j} = 0 \text{ on } \Gamma \text{ for } j = 0, \dots, k-1 \right\}$. For the particular case $k = 2$ one has that the space $H^2_\Gamma(\Omega_1)$ endowed with the inner product

$$(w, \widetilde{w})_{H^2_\Gamma(\Omega_1)} := \mu \left(\Delta w, \Delta \widetilde{w}\right)_{L^2(\Omega_1)} + (1-\mu)\left(\nabla^2 w, \nabla^2 \widetilde{w}\right)_{L^2(\Omega_1)^4},$$

where

$$\left(\nabla^2 u, \nabla^2 v\right)_{L^2(\Omega_1)^4} := \int_{\Omega_1} \nabla^2 u : \nabla^2 \overline{v} dx, \tag{3.2}$$

is a Hilbert space. Indeed, if $u, v \in H^2_\Gamma(\Omega_1)$ then

$$\begin{aligned}(u, v)_{H^2(\Omega_1)} = \sum_{|\alpha| \le 2} \int_{\Omega_1} \partial^\alpha u \partial^\alpha \overline{v}\, dx = (u, v)_{L^2(\Omega_1)} + (\nabla u, \nabla v)_{L^2(\Omega_1)^2} \\ + \int_{\Omega_1} u_{x_1x_1} \overline{v}_{x_1x_1} + u_{x_1x_2} \overline{v}_{x_1x_2} + u_{x_2x_2} \overline{v}_{x_2x_2}\, dx.\end{aligned} \tag{3.3}$$

Due to $u = \partial_\nu u = 0$ on Γ, it is easy to see that $\nabla u = 0$ on Γ and therefore $u_{x_1} = u_{x_2} = 0$ on Γ. By Friedrichs inequality or Corollary 5.5 in [17],

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega_1)^2} &= \|u_{x_1}\|^2_{L^2(\Omega_1)} + \|u_{x_2}\|^2_{L^2(\Omega_1)} \\ &\le C^2\left(\|\nabla u_{x_1}\|^2_{L^2(\Omega_1)^2} + \|\nabla u_{x_2}\|^2_{L^2(\Omega_1)^2} \right) \\ &\le 2C^2 \left\|\nabla^2 u\right\|^2_{L^2(\Omega_1)^4}.\end{aligned} \tag{3.4}$$

By (3.2)-(3.4) and again Friedrichs inequality,

$$\|u\|^2_{H^2(\Omega_1)} \le \|u\|^2_{L^2(\Omega_1)} + \|\nabla u\|^2_{L^2(\Omega_1)^2} + \left\|\nabla^2 u\right\|^2_{L^2(\Omega_1)^4} \le \widetilde{C}\, \|u\|^2_{H^2_\Gamma(\Omega_1)}. \tag{3.5}$$

Hölder's inequality allows us to write

$$\begin{aligned}\|u\|^2_{H^2_\Gamma(\Omega_1)} = \mu \int_{\Omega_1} |u_{x_1x_1}|^2 + 2\,\mathrm{Re}(u_{x_1x_1}\overline{u}_{x_2x_2}) + |u_{x_2x_2}|^2 dx \\ + (1-\mu) \int_{\Omega_1} |u_{x_1x_1}|^2 + |u_{x_2x_2}|^2 + 2|u_{x_1x_2}|^2 dx \le 4\, \|u\|^2_{H^2(\Omega_1)}.\end{aligned}$$

This shows that $\|\cdot\|_{H^2_\Gamma(\Omega_1)}$ and $\|\cdot\|_{H^2(\Omega_1)}$ are equivalent norms on $H^2_\Gamma(\Omega_1)$. Theorem 2.7 guarantees that $H^2_\Gamma(\Omega_1)$ is a closed subspace of $H^2(\Omega_1)$.

Let $\eta \coloneqq \frac{\gamma}{\rho_1}$. If the inertial term is present on the plate, that is $\gamma > 0$, then $H^1_{\Gamma,\eta}(\Omega_1)$ will be the space $H^1_\Gamma(\Omega_1)$ and when $\gamma = 0$ it will be $L^2(\Omega_1)$. For $\eta \geq 0$, the space $H^1_{\Gamma,\eta}(\Omega_1)$ is endowed with the scalar product

$$(w, \widetilde{w})_{H^1_{\Gamma,\eta}(\Omega_1)} \coloneqq (w, \widetilde{w})_{L^2(\Omega_1)} + \eta\, (\nabla w, \nabla \widetilde{w})_{L^2(\Omega_1)^2}\,.$$

Note that $\|\cdot\|_{H^1_\Gamma(\Omega_1)}$ and $\|\cdot\|_{H^1(\Omega_1)}$ are equivalent norms on the space $H^1_\Gamma(\Omega_1)$. The continuity of the zero order trace operator $H^1(\Omega_1) \ni u \mapsto \gamma_0 u \in H^{1/2}(\Gamma)$ allows us to prove that $H^1_\Gamma(\Omega_1)$ is a closed subspace in $H^1(\Omega_1)$. Therefore, $\left(H^1_{\Gamma,\eta}(\Omega_1), (\cdot,\cdot)_{H^1_{\Gamma,\eta}(\Omega_1)}\right)$ is a Hilbert space for any $\eta \geq 0$.

Based on the expression (3.1), the *energy space* or *phase space* is defined, which is presented below. For $\eta \geq 0$, we set

$$\mathscr{H}_\eta \coloneqq \Big\{\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)^\top \in \mathscr{X}_\eta : \varphi_1 = \varphi_3 \text{ on } I\Big\},$$

where $\mathscr{X}_\eta \coloneqq H^2_\Gamma(\Omega_1) \times H^1_{\Gamma,\eta}(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2) \times L^2(\Omega_1)$. The energy space is equipped with the inner product

$$\begin{aligned}(\varphi, \psi)_{\mathscr{H}_\eta} \coloneqq& \beta_1\,(\varphi_1, \psi_1)_{H^2_\Gamma(\Omega_1)} + \rho_1\,(\varphi_2, \psi_2)_{H^1_{\Gamma,\eta}(\Omega_1)} + \beta_2\,(\nabla\varphi_3, \nabla\psi_3)_{L^2(\Omega_2)^2} \\ &+ \rho_2\,(\varphi_4, \psi_4)_{L^2(\Omega_2)} + \rho_0\,(\varphi_5, \psi_5)_{L^2(\Omega_1)}\end{aligned}$$

for all $\varphi, \psi \in \mathscr{H}_\eta$. It is easy to see that the operator $T : \mathscr{X}_\eta \to H^{1/2}(I)$ given by $T(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \coloneqq \varphi_1 - \varphi_3\big|_I$ is linear, and also continuous thanks to the trace theorem. As $\mathscr{H}_\eta = T^{-1}(\{0\})$, then $\mathscr{H}_\eta$ is closed in $\mathscr{X}_\eta$. By Theorem 2.24 and the trace theorem,

$$\|\varphi_3\|^2_{H^1(\Omega_2)} \leq C\left(\|\varphi_1\|^2_{H^2_\Gamma(\Omega_1)} + \|\nabla\varphi_3\|^2_{L^2(\Omega_2)^2}\right)$$

for $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \in \mathscr{H}_\eta$. This allows us to prove that there is $C > 0$ such that $\|\varphi\|_{\mathscr{X}_\eta} \leq C\,\|\varphi\|_{\mathscr{H}_\eta}$ for any $\varphi \in \mathscr{H}_\eta$. From the definition of the space $\mathscr{X}_\eta$ follows immediately the inequality $\|\varphi\|_{\mathscr{H}_\eta} \leq C\,\|\varphi\|_{\mathscr{X}_\eta}$ for all $\varphi \in \mathscr{H}_\eta$. In consequence, $\left(\mathscr{H}_\eta, (\cdot,\cdot)_{\mathscr{H}_\eta}\right)$ is a Hilbert space.

3.1.1 Basic operators

At this point in the document, the contents of Section 2.6 and Section 2.7 are necessary to obtain properties of the operators that will be part of the abstract formulation of the problem (1.1)-(1.9). The following definition is taken from Subsection 7.1 of [29].

Definition 3.1. Let H be a Hilbert space. An unbounded linear operator $A : D(A) \subset H \to H$ is said to be *monotone* if and only if $(Au, u)_H \geq 0$ for all $u \in D(A)$. If also $R(\mathcal{I} + A) = H$, then A is called *maximal monotone.*

Throughout this document $\mathcal{I}$ denotes the identity operator. The following two results will allows us to show that certain operators that are of interest to us are self-adjoint.

Lemma 3.2. *Let H be a Hilbert space and $A : D(A) \subset H \to H$ a monotone operator. If there exists $\lambda_0 > 0$ such that $R(\lambda_0\mathcal{I} + A) = H$, then A is a maximal monotone operator.*

Proof. Let $B := \frac{1}{\lambda_0}A$. Note that $B : D(B) \subset H \to H$ is an unbounded linear operator with $D(B) := D(A)$. As $(Bu, u)_H \geq 0$ for all $u \in D(B)$ and $R(\mathcal{I} + B) = H$, we have that B is a maximal monotone operator. The part (c) of Proposition 7.1 from [29] implies $R(\mathcal{I} + \lambda B) = H$ for every $\lambda > 0$. In particular, taking $\lambda = \lambda_0$ we get $R(\mathcal{I} + A) = H$. □

Lemma 3.3 (cf. [29, Proposition 7.6])**.** *Let $A : D(A) \subset H \to H$ be a maximal monotone symmetric operator. Then A is self-adjoint.*

Proposition 3.4. *If $\mathcal{A}_B : L^2(\Omega_1) \supset D(\mathcal{A}_B) \to L^2(\Omega_1)$ is the Bi-Laplacian, this is $\mathcal{A}_B := \Delta^2$, with domain*

$$D(\mathcal{A}_B) := \left\{w \in H^4(\Omega_1) : w = \partial_\nu w = 0 \text{ on } \Gamma \text{ and } \mathscr{B}_1 w = \mathscr{B}_2 w = 0 \text{ on } I\right\},$$

then $\mathcal{A}_B$ is a positive self-adjoint operator.

Proof. By Proposition 2.17, we have $(\mathcal{A}_B w, w)_{L^2(\Omega_1)} = \|w\|^2_{H^2_\Gamma(\Omega_1)} \geq 0$ for all $w \in D(\mathcal{A}_B)$. The operator $\mathcal{A}_B$ is maximal monotone. In fact: According to Lemma 3.2 it is enough to prove that $R(\lambda\mathcal{I} + \mathcal{A}_B) = L^2(\Omega_1)$ for some $\lambda > 0$, this is, there is $\lambda > 0$ such that for any $f \in L^2(\Omega_1)$ there exists $w \in D(\mathcal{A}_B)$ so that $(\lambda\mathcal{I} + \mathcal{A}_B)w = f$ and this in turn is equivalent to the following formulation

$$\begin{cases} \text{Given } f \in L^2(\Omega_1) \text{ there exists } w \in H^4(\Omega_1) \text{ such that} \\ \lambda w + \Delta^2 w = f \quad \text{in} \quad \Omega_1, \\ w = \partial_\nu w = 0 \quad \text{on} \quad \Gamma, \\ \mathscr{B}_1 w = \mathscr{B}_2 w = 0 \quad \text{on} \quad I. \end{cases} \tag{3.6}$$

Proposition 2.60 guarantees the existence of $\lambda > 0$ so that (3.6) holds. It is immediate from formula (2.4) that $(\mathcal{A}_B w, \widetilde{w})_{L^2(\Omega_1)} = (w, \mathcal{A}_B \widetilde{w})_{L^2(\Omega_1)}$ for all $w, \widetilde{w} \in D(\mathcal{A}_B)$. Thus, $\mathcal{A}_B$ is a symmetric operator. Due to Lemma 3.3, $\mathcal{A}_B$ is a self-adjoint operator. Estimation (3.5) implies the existence of $C > 0$ such that $(\mathcal{A}_B w, w)_{L^2(\Omega_1)} \geq C \|w\|^2_{L^2(\Omega_1)}$ for all $w \in D(\mathcal{A}_B)$. □

The operator $\mathcal{A}_B^s : D(\mathcal{A}_B^s) \to L^2(\Omega_1)$ is a positive self-adjoint operator for any $s > 0$, see Section 2.6 and Proposition 3.4. Recall that $D(\mathcal{A}_B^s)$ is endowed with the scalar product $(w, \widetilde{w})_{D(\mathcal{A}_B^s)} := (\mathcal{A}_B^s w, \mathcal{A}_B^s \widetilde{w})_{L^2(\Omega_1)}$ for $w, \widetilde{w} \in D(\mathcal{A}_B^s)$, see (2.24), and becomes a Hilbert space. For $s_1 > s_2 > 0$, we have

$$D(\mathcal{A}_B^{s_1}) \overset{d}{\hookrightarrow} D(\mathcal{A}_B^{s_2}) \overset{d}{\hookrightarrow} L^2(\Omega_1), \tag{3.7}$$

see part b) of Theorem 2.53. Characterization (2.29) allows us to explicitly write the domain of some fractional powers of the operator $\mathcal{A}_B$. Hereinafter, the symbol $\cong$ means equality between sets and equivalence in norm. We have,

$$\begin{aligned} &D(\mathcal{A}_B^{1/4}) \cong H^1_\Gamma(\Omega_1), \\ &D(\mathcal{A}_B^{1/2}) \cong H^2_\Gamma(\Omega_1), \\ &D(\mathcal{A}_B^{3/4}) \cong \left\{ w \in H^3(\Omega_1) \cap H^2_\Gamma(\Omega_1) : \mathscr{B}_1 w = 0 \text{ on } I \right\}. \end{aligned} \tag{3.8}$$

Due to (3.7), we have that $D(\mathcal{A}_B)$ is dense in $D(\mathcal{A}_B^{1/2})$. If $\xi, \zeta \in D(\mathcal{A}_B^{1/2})$, there exist two sequences $(\xi_n)_{n\in\mathbb{N}}$ and $(\zeta_n)_{n\in\mathbb{N}}$ contained in $D(\mathcal{A}_B)$ such that $\xi_n \to \xi$ and $\zeta_n \to \zeta$ in $D(\mathcal{A}_B^{1/2})$. Taking into account (2.12), part a) of Theorem 2.53 and the fact that $\mathcal{A}_B^{1/2}$ is a self-adjoint operator, we obtain

$$\langle \mathcal{A}_B \xi_n, \zeta_n \rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})} = (\mathcal{A}_B \xi_n, \zeta_n)_{L^2(\Omega_1)} = (\mathcal{A}_B^{1/2} \xi_n, \mathcal{A}_B^{1/2} \zeta_n)_{L^2(\Omega_1)} \tag{3.9}$$

for each $n \in \mathbb{N}$. Note that $\mathcal{A}_B \xi \in [D(\mathcal{A}_B^{1/2})]'$, see part c) of Theorem 2.53. We compute that

$$\begin{aligned} &\left| \langle \mathcal{A}_B \xi_n, \zeta_n \rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})} - \langle \mathcal{A}_B \xi, \zeta \rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})} \right| \\ &= \left| \langle \mathcal{A}_B \xi_n, \zeta_n - \zeta \rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})} + \langle \mathcal{A}_B (\xi_n - \xi), \zeta \rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})} \right| \\ &\leq \|\mathcal{A}_B \xi_n\|_{[D(\mathcal{A}_B^{1/2})]'} \|\zeta_n - \zeta\|_{D(\mathcal{A}_B^{1/2})} + \|\mathcal{A}_B(\xi_n - \xi)\|_{[D(\mathcal{A}_B^{1/2})]'} \|\zeta\|_{D(\mathcal{A}_B^{1/2})}. \end{aligned}$$

Now the continuity of the operator $\mathcal{A}_B : D(\mathcal{A}_B^{1/2}) \to [D(\mathcal{A}_B^{1/2})]'$ implies that

$$\langle \mathcal{A}_B \xi_n, \zeta_n \rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})} \to \langle \mathcal{A}_B \xi, \zeta \rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})}. \tag{3.10}$$

By Proposition 2.17 and the equivalence of norms of the second characterization in (3.8), we get the convergence

$$(\mathcal{A}_B \xi_n, \zeta_n)_{L^2(\Omega_1)} = (\xi_n, \zeta_n)_{H^2_\Gamma(\Omega_1)} \to (\xi, \zeta)_{H^2_\Gamma(\Omega_1)} . \tag{3.11}$$

We find that

$$(\mathcal{A}_B^{1/2}\xi_n, \mathcal{A}_B^{1/2}\zeta_n)_{L^2(\Omega_1)} = (\xi_n, \zeta_n)_{D(\mathcal{A}_B^{1/2})} \to (\xi, \zeta)_{D(\mathcal{A}_B^{1/2})} = (\mathcal{A}_B^{1/2}\xi, \mathcal{A}_B^{1/2}\zeta)_{L^2(\Omega_1)}.$$

By (3.9)-(3.11) and the last limit, we can write the following equalities

$$\langle \mathcal{A}_B \xi, \zeta \rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})} = (\mathcal{A}_B^{1/2}\xi, \mathcal{A}_B^{1/2}\zeta)_{L^2(\Omega_1)} = (\xi, \zeta)_{H^2_\Gamma(\Omega_1)} \tag{3.12}$$

for all $\xi, \zeta \in D(\mathcal{A}_B^{1/2})$.

Proposition 3.5. *The operator $\mathcal{A}_L : L^2(\Omega_1) \supset D(\mathcal{A}_L) \to L^2(\Omega_1)$ given by $\mathcal{A}_L := -\Delta$ is a positive self-adjoint operator, where*

$$D(\mathcal{A}_L) := \left\{ w \in H^2(\Omega_1) : w = 0 \text{ on } \Gamma \quad \text{and} \quad \partial_\nu w = 0 \text{ on } I \right\}.$$

Proof. Using the integration by parts formula (2.1) and Friedrichs inequality, one has that $(\mathcal{A}_L u, u)_{L^2(\Omega_1)} \geq C \|u\|^2_{L^2(\Omega_1)}$ for any $u \in D(\mathcal{A}_L)$. Thus, $\mathcal{A}_L$ is monotone. By elliptic regularity theory (see, e.g., Theorem 3.2 and Remark 3.3 in [14]) one has that for sufficiently large $\lambda > 0$, the solution u of the boundary value problem

$$\begin{cases} \lambda u - \Delta u = f \quad \text{in} \quad \Omega_1, \\ u = 0 \quad \text{on} \quad \Gamma, \\ \partial_\nu u = 0 \quad \text{on} \quad I, \end{cases}$$

belongs to $H^2(\Omega_1)$ for any $f \in L^2(\Omega_1)$. Thus, $\mathcal{A}_L$ is maximal monotone and it is easy to see that it is symmetric. □

The square root operator $\mathcal{A}_L^{1/2}$ of $\mathcal{A}_L$ is self-adjoint and positive, and its domain has the following characterization:

$$D(\mathcal{A}_L^{1/2}) \cong H^1_\Gamma(\Omega_1). \tag{3.13}$$

It is clear that $(\mathcal{A}_L w, \widetilde{w})_{L^2(\Omega_1)} = (\nabla w, \nabla \widetilde{w})_{L^2(\Omega_1)^2}$ $\forall w, \widetilde{w} \in D(\mathcal{A}_L)$. Using the density of $D(\mathcal{A}_L)$ in $D(\mathcal{A}_L^{1/2})$, the equivalence of norms $\|\cdot\|_{H^1_\Gamma(\Omega_1)} \sim \|\cdot\|_{H^1(\Omega_1)}$ on $H^1_\Gamma(\Omega_1)$, (3.13) and an argument as in the previous page, we derive

$$\langle \mathcal{A}_L \xi, \zeta \rangle_{[D(\mathcal{A}_L^{1/2})]' \times D(\mathcal{A}_L^{1/2})} = (\mathcal{A}_L^{1/2}\xi, \mathcal{A}_L^{1/2}\zeta)_{L^2(\Omega_1)} = (\nabla \xi, \nabla \zeta)_{L^2(\Omega_1)^2} \tag{3.14}$$

for all $\xi, \zeta \in D(\mathcal{A}_L^{1/2})$.

Now, we define the *inertia operator* $\mathcal{M}_\eta \coloneqq \mathcal{I} + \eta \mathcal{A}_L$ with $D(\mathcal{M}_\eta) \coloneqq D(\mathcal{A}_L)$ if $\eta > 0$ and $D(\mathcal{M}_0) \coloneqq L^2(\Omega_1)$. For $\xi \in D(\mathcal{A}_L^{1/2})$, we have that the part *c*) of Theorem 2.53 implies $\mathcal{A}_L \xi \in [D(\mathcal{A}_L^{1/2})]'$ and in consequence $\mathcal{M}_\eta \xi = \xi + \eta \mathcal{A}_L \xi \in H^{-1}_{\Gamma,\eta}(\Omega_1) \coloneqq [H^1_{\Gamma,\eta}(\Omega_1)]'$. From (2.12), (3.13) and (3.14) it follows that

$$\langle \mathcal{M}_\eta \xi, \zeta \rangle_{H^{-1}_{\Gamma,\eta}(\Omega_1) \times H^1_{\Gamma,\eta}(\Omega_1)} = (\xi, \zeta)_{H^1_{\Gamma,\eta}(\Omega_1)} \text{ for all } \xi, \zeta \in H^1_{\Gamma,\eta}(\Omega_1). \tag{3.15}$$

By the Lax–Milgram theorem, see [40, Theorem 7 on p. 368], we obtain that $\mathcal{M}_\eta : H^1_{\Gamma,\eta}(\Omega_1) \to H^{-1}_{\Gamma,\eta}(\Omega_1)$ is an isomorphism. In particular, $\mathcal{M}_\eta^{-1}$ exists for $\eta \geq 0$, where $\mathcal{M}_0^{-1}$ is the identity in $L^2(\Omega_1)$.

Proposition 3.6. *The operator* $\mathcal{A}_T : L^2(\Omega_1) \supset D(\mathcal{A}_T) \to L^2(\Omega_1)$ *defined by* $\mathcal{A}_T \coloneqq -\Delta + \frac{\sigma}{\beta}\mathcal{I}$, *with domain*

$$D(\mathcal{A}_T) \coloneqq \left\{ \vartheta \in H^2(\Omega_1) : \partial_\nu \vartheta + \kappa \vartheta = 0 \text{ on } \partial\Omega_1 \right\},$$

is self-adjoint and positive.

Proof. Using integration by parts, see Theorem 2.11, it is easy to get that

$$(\mathcal{A}_T \vartheta, \vartheta)_{L^2(\Omega_1)} = \|\nabla \vartheta\|^2_{L^2(\Omega_1)^2} + \kappa \|\vartheta\|^2_{L^2(\partial\Omega_1)} + \frac{\sigma}{\beta} \|\vartheta\|^2_{L^2(\Omega_1)} \tag{3.16}$$

for all $\vartheta \in D(\mathcal{A}_T)$. Hence, $(\mathcal{A}_T \widetilde{\vartheta}, \widetilde{\vartheta})_{L^2(\Omega_1)} \geq C \|\widetilde{\vartheta}\|^2_{L^2(\Omega_1)}$ for any $\widetilde{\vartheta} \in D(\mathcal{A}_T)$, which is trivially true if $\sigma > 0$ and when $\sigma = 0$ is also true thanks to Theorem 2.24 since $\kappa > 0$. By Theorem 3.2 and part (b) of Remark 3.3 in [14]: For sufficiently large $\lambda_0 > 0$, the boundary value problem

$$\begin{aligned} \lambda_0 u - \Delta u &= f \quad \text{in} \quad \Omega_1, \\ \partial_\nu u + \kappa u &= 0 \quad \text{on} \quad \partial\Omega_1, \end{aligned}$$

has a unique solution $u \in H^2(\Omega_1)$ whenever $f \in L^2(\Omega_1)$. This allows to assert that $R(\lambda \mathcal{I} + \mathcal{A}_T) = L^2(\Omega_1)$ for some $\lambda > 0$. It is very simple to check that $\mathcal{A}_T$ is symmetric. An application of Lemma 3.2 and Lemma 3.3 completes the proof. □

The same arguments used for the operators $\mathcal{A}_B$ and $\mathcal{A}_L$ are considered for the operator $\mathcal{A}_T$ to obtain the following characterization

$$D(\mathcal{A}_T^{1/2}) \cong H^1(\Omega_1). \tag{3.17}$$

Therefore, one can establish via density and integration by parts the identity

$$(\xi,\zeta)_{D(\mathcal{A}_T^{1/2})} = (\nabla\xi,\nabla\zeta)_{L^2(\Omega_1)^2} + \kappa\,(\xi,\zeta)_{L^2(\partial\Omega_1)} + \frac{\sigma}{\beta}\,(\xi,\zeta)_{L^2(\Omega_1)} \tag{3.18}$$

for all $\xi,\zeta \in D(\mathcal{A}_T^{1/2})$.

3.1.2 Abstract formulation of the problem

The operators $\mathcal{A}_B$, $\mathcal{A}_L$, $\mathcal{A}_T$ and $\mathcal{M}_\eta$ together with others, will be part of a matrix operator which will allow to write system (1.1)-(1.9) as a Cauchy problem on the Hilbert space $\mathscr{H}_\eta$. We will make use of two *Green maps* introduced in [10, p. 158]. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be defined by the formulas $\mathcal{G}_1 x := \widetilde{u}$ and $\mathcal{G}_2 y := \widetilde{v}$, where $\widetilde{u}$ and $\widetilde{v}$ solve the following problems

$$\begin{cases} \Delta^2\widetilde{u} = 0 \ \text{ in } \ \Omega_1, \\ \widetilde{u} = \partial_\nu \widetilde{u} = 0 \ \text{ on } \ \Gamma, \\ \mathscr{B}_1\widetilde{u} = x \ \text{ on } \ I, \\ \mathscr{B}_2\widetilde{u} = 0 \ \text{ on } \ I, \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta^2\widetilde{v} = 0 \ \text{ in } \ \Omega_1, \\ \widetilde{v} = \partial_\nu \widetilde{v} = 0 \ \text{ on } \ \Gamma, \\ \mathscr{B}_1\widetilde{v} = 0 \ \text{ on } \ I, \\ \mathscr{B}_2\widetilde{v} = y \ \text{ on } \ I. \end{cases} \tag{3.19}$$

We define the *Neumann map* $\mathcal{N}$ (introduced in [11, p. 405]) given by $\mathcal{N}z := \widetilde{w}$ where $\widetilde{w}$ is solution of the problem

$$\begin{cases} \Delta\widetilde{w} = 0 \ \text{ in } \ \Omega_1, \\ \widetilde{w} = 0 \ \text{ on } \ \Gamma, \\ \partial_\nu\widetilde{w} = z \ \text{ on } \ I. \end{cases} \tag{3.20}$$

From elliptic theory, see Section 7.3 of Chapter 2 in [89], we obtain that $\mathcal{N} : H^s(I) \to H^{s+\frac{3}{2}}(\Omega_1)$ and $\mathcal{G}_i : H^s(I) \to H^{s+\frac{3}{2}+i}(\Omega_1)$, with $i = 1,2$, are bounded linear maps for any $s \geq 0$.

Lemma 3.7. *For all $f \in D(\mathcal{A}_B)$, we have that* a) $\mathcal{G}_1^*\mathcal{A}_B f = \gamma_1 f$ *on I and* b) $\mathcal{G}_2^*\mathcal{A}_B f = -\gamma_0 f$ *on I, where $\gamma_0 w := w\big|_I$, $\gamma_1 w := \partial_\nu w\big|_I$ and $\mathcal{G}_i^*$ denotes the adjoint of $\mathcal{G}_i$ in the sense*

$$(\mathcal{G}_i\varphi,\psi)_{L^2(\Omega_1)} = (\varphi,\mathcal{G}_i^*\psi)_{L^2(I)} \quad \textit{for} \ \ \varphi \in L^2(I) \ \ \textit{and} \ \ \psi \in L^2(\Omega_1).$$

Proof. Let $f \in D(\mathcal{A}_B)$ and $g \in L^2(I)$. Note that $\mathcal{G}_1 g \in H^{5/2}(\Omega_1)$. Taking into account (2.12) and the integration formula of Proposition 2.18, we obtain

$$\begin{aligned}\left(\Delta^2 f, \mathcal{G}_1 g\right)_{L^2(\Omega_1)} - \left(f, \Delta^2 \mathcal{G}_1 g\right)_{L^2(\Omega_1)} &= -\left(\mathscr{B}_1 f, \partial_\nu \mathcal{G}_1 g\right)_{L^2(I)} \\ + \left(\mathscr{B}_2 f, \mathcal{G}_1 g\right)_{L^2(I)} + \left(\partial_\nu f, \mathscr{B}_1 \mathcal{G}_1 g\right)_{L^2(I)} &- \left(f, \mathscr{B}_2 \mathcal{G}_1 g\right)_{L^2(I)}.\end{aligned}$$

On account of $\Delta^2 \mathcal{G}_1 g = 0$ in Ω_1 and $\mathscr{B}_1 f = \mathscr{B}_2 f = \mathscr{B}_2 \mathcal{G}_1 g = 0$ on I, we deduce

$$\left(\mathcal{G}_1^* \mathcal{A}_B f, g\right)_{L^2(I)} = \left(\mathcal{A}_B f, \mathcal{G}_1 g\right)_{L^2(\Omega_1)} = \left(\Delta^2 f, \mathcal{G}_1 g\right)_{L^2(\Omega_1)} = \left(\gamma_1 f, g\right)_{L^2(I)}.$$

This last one shows a). The proof of b) is similar. □

Proposition 3.8. *For any $w \in D(\mathcal{A}_B^{1/2})$ and $g \in L^2(I)$, we have*

$$\left\langle \mathcal{A}_B \mathcal{G}_i g, w\right\rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})} = \left(g, (-1)^{i-1} \gamma_{2-i} w\right)_{L^2(I)} \tag{3.21}$$

for $i = 1, 2$.

Proof. Let $w \in D(\mathcal{A}_B^{1/2})$ and $g \in L^2(I)$. As $D(\mathcal{A}_B)$ is dense in $D(\mathcal{A}_B^{1/2})$, then there is $(w_n)_{n\in\mathbb{N}} \subset D(\mathcal{A}_B)$ such that $\lim_{n\to\infty} w_n = w$ in $D(\mathcal{A}_B^{1/2})$. For any $n \in \mathbb{N}$, the part a) of Lemma 3.7 implies

$$\left\langle \mathcal{A}_B \mathcal{G}_1 g, w_n\right\rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})} = \left(\mathcal{G}_1 g, \mathcal{A}_B w_n\right)_{L^2(\Omega_1)} = \left(g, \gamma_1 w_n\right)_{L^2(I)}. \tag{3.22}$$

As $\mathcal{G}_1 g, w_n \in D(\mathcal{A}_B^{1/2})$, then identity (3.12) allows us to write the convergence

$$\begin{aligned}&\lim_{n\to\infty} \left\langle \mathcal{A}_B \mathcal{G}_1 g, w_n\right\rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})} = \lim_{n\to\infty} \left(\mathcal{G}_1 g, w_n\right)_{D(\mathcal{A}_B^{1/2})} \\ &= \left(\mathcal{G}_1 g, w\right)_{D(\mathcal{A}_B^{1/2})} = \left\langle \mathcal{A}_B \mathcal{G}_1 g, w\right\rangle_{[D(\mathcal{A}_B^{1/2})]' \times D(\mathcal{A}_B^{1/2})}.\end{aligned} \tag{3.23}$$

The continuous embedding $H^{1/2}(I) \hookrightarrow L^2(I)$, see Theorem 2.6, and the continuity of the trace operator $\gamma_1 : H^2(\Omega_1) \to H^{1/2}(I)$ imply

$$\|\gamma_1 w_n - \gamma_1 w\|_{L^2(I)} \le C \|w_n - w\|_{H^2(\Omega_1)} \le C \|w_n - w\|_{D(\mathcal{A}_B^{1/2})}$$

because $\|\cdot\|_{H^2_\Gamma(\Omega_1)}$ and $\|\cdot\|_{H^2(\Omega_1)}$ are equivalent norms on $H^2_\Gamma(\Omega_1)$. Therefore, $\lim_{n\to\infty} \gamma_1 w_n = \gamma_1 w$ in $L^2(I)$. Thus,

$$\lim_{n\to\infty} \left(g, \gamma_1 w_n\right)_{L^2(I)} = \left(g, \gamma_1 w\right)_{L^2(I)}. \tag{3.24}$$

From (3.22)-(3.24) it follows (3.21) when $i = 1$. Using the same arguments as before, (3.21) is proved when $i = 2$. □

Lemma 3.9. *The* $L^2(\Omega_1)$-adjoint $\mathcal{N}^*$ *of the Neumann map* $\mathcal{N}$ *is given by* $(\varphi, \mathcal{N}^*\psi)_{L^2(I)} = (\mathcal{N}\varphi, \psi)_{L^2(\Omega_1)}$ *for* $\varphi \in L^2(I)$ *and* $\psi \in L^2(\Omega_1)$*. Then, for any* $f \in D(\mathcal{A}_L)$*, we have*

$$\mathcal{N}^*\mathcal{A}_L f = \gamma_0 f. \tag{3.25}$$

Proof. Let $g \in L^2(I)$ and $f \in D(\mathcal{A}_L)$. Note that $\mathcal{N}g \in H^{3/2}(\Omega_1)$ satisfies $\Delta\mathcal{N}g \in L^2(\Omega_1)$. Applying the generalization of Green's first formula (see Theorem 2.14) twice, we may write

$$\begin{aligned}
(\mathcal{N}^*\mathcal{A}_L f, g)_{L^2(I)} &= (\mathcal{A}_L f, \mathcal{N}g)_{L^2(\Omega_1)} \\
&= (\nabla f, \nabla\mathcal{N}g)_{L^2(\Omega_1)^2} - (\partial_\nu f, \mathcal{N}g)_{L^2(\Gamma)} - (\partial_\nu f, \mathcal{N}g)_{L^2(I)} \\
&= -(f, \Delta\mathcal{N}g)_{L^2(\Omega_1)} + (f, \partial_\nu\mathcal{N}g)_{L^2(\Gamma)} + (f, \partial_\nu\mathcal{N}g)_{L^2(I)} \\
&= (f, g)_{L^2(I)}.
\end{aligned}$$

Above we have considered the definitions of $D(\mathcal{A}_L)$ and $\mathcal{N}$, see Proposition 3.5 and (3.20). Therefore, (3.25) is true. □

Proposition 3.10. *For all* $w \in D(\mathcal{A}_L^{1/2})$ *and* $g \in L^2(I)$*, it holds*

$$\langle \mathcal{A}_L\mathcal{N}g, w\rangle_{[D(\mathcal{A}_L^{1/2})]' \times D(\mathcal{A}_L^{1/2})} = (g, \gamma_0 w)_{L^2(I)}. \tag{3.26}$$

Proof. Similar to the proof of Proposition 3.8. Here we use the fact that $D(\mathcal{A}_L)$ is dense in $D(\mathcal{A}_L^{1/2})$ and apply Lemma 3.9, also consider (3.14). □

Proposition 3.11. *For* $g \in L^2(I)$*, we have*

$$\mathcal{A}_B\mathcal{G}_2 g + \mathcal{A}_L\mathcal{N}g = 0 \quad \textit{in} \quad H_\Gamma^{-1}(\Omega_1).$$

Proof. Let $g \in L^2(I)$. Note that $\mathcal{G}_2 g \in D(\mathcal{A}_B^{3/4})$ and $\mathcal{N}g \in D(\mathcal{A}_L^{1/2})$. So, $\mathcal{A}_B^{3/4}\mathcal{G}_2 g, \mathcal{A}_L^{1/2}\mathcal{N}g \in L^2(\Omega_1)$. Parts *a*) and *c*) of Theorem 2.53, and the first characterization of (3.8) imply

$$\mathcal{A}_B\mathcal{G}_2 g = \mathcal{A}_B^{1/4}\mathcal{A}_B^{3/4}\mathcal{G}_2 g \in H_\Gamma^{-1}(\Omega_1).$$

Similar arguments allows to affirm

$$\mathcal{A}_L\mathcal{N}g \in H_\Gamma^{-1}(\Omega_1).$$

Let $f \in D(\mathcal{A}_B^{1/4})$. Now, we will argue analogously as in the proof of Proposition 3.8. The density of $D(\mathcal{A}_B)$ in $D(\mathcal{A}_B^{1/4})$ implies the existence of a sequence

$(f_n)_{n\in\mathbb{N}} \subset D(\mathcal{A}_B)$ such that $\lim_{n\to\infty} f_n = f$ in $D(\mathcal{A}_B^{1/4})$. Using the fact that $\mathcal{A}_B^{3/4}$ is self-adjoint and part b) of Lemma 3.7, we get that

$$\langle \mathcal{A}_B\mathcal{G}_2 g, f_n\rangle_{[D(\mathcal{A}_B^{1/4})]'\times D(\mathcal{A}_B^{1/4})} = (\mathcal{A}_B^{3/4}\mathcal{G}_2 g, \mathcal{A}_B^{1/4} f_n)_{L^2(\Omega_1)} = (g, -\gamma_0 f_n)_{L^2(I)}$$

for all $n\in\mathbb{N}$. Now, putting $\mathcal{A}_B^{1/4}\mathcal{A}_B^{1/2}$ instead of $\mathcal{A}_B^{3/4}$ below, we obtain that

$$\langle \mathcal{A}_B\mathcal{G}_2 g, f_n\rangle_{[D(\mathcal{A}_B^{1/4})]'\times D(\mathcal{A}_B^{1/4})} = (\mathcal{A}_B^{3/4}\mathcal{G}_2 g, \mathcal{A}_B^{1/4} f_n)_{L^2(\Omega_1)} = (\mathcal{A}_B^{1/2}\mathcal{G}_2 g, f_n)_{D(\mathcal{A}_B^{1/4})}.$$

for any $n\in\mathbb{N}$. Taking limit as $n\to\infty$ in the two equalities above, we have

$$\langle \mathcal{A}_B\mathcal{G}_2 g, f\rangle_{[D(\mathcal{A}_B^{1/4})]'\times D(\mathcal{A}_B^{1/4})} = (g, -\gamma_0 f)_{L^2(I)}\,. \tag{3.27}$$

To end, let $w\in H^1_\Gamma(\Omega_1)$. An application of identities (3.26) and (3.27) together with the characterizations (3.8), the first of them, and (3.13) allows us to write the following equalities

$$\begin{aligned}
&\langle \mathcal{A}_B\mathcal{G}_2 g + \mathcal{A}_L\mathcal{N}g, w\rangle_{H^{-1}_\Gamma(\Omega_1)\times H^1_\Gamma(\Omega_1)}\\
&= \langle \mathcal{A}_B\mathcal{G}_2 g, w\rangle_{[D(\mathcal{A}_B^{1/4})]'\times D(\mathcal{A}_B^{1/4})} + \langle \mathcal{A}_L\mathcal{N}g, w\rangle_{[D(\mathcal{A}_L^{1/2})]'\times D(\mathcal{A}_L^{1/2})}\\
&= (g, -\gamma_0 w)_{L^2(I)} + (g, \gamma_0 w)_{L^2(I)} = 0.
\end{aligned}$$

This shows what we wanted. □

Now, let $u,\theta : \mathbb{R}^+\times\overline{\Omega_1}\to\mathbb{C}$ and $v:\mathbb{R}^+\times\overline{\Omega_2}\to\mathbb{C}$ be three arbitrary functions with sufficient regularity. Taking into account the definition of the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ in (3.19), it follows in Ω_1 that

$$\begin{aligned}
\beta_1\Delta^2 u = \beta_1\Delta^2 u + \alpha\Delta^2\mathcal{G}_1\theta - \gamma\Delta^2\mathcal{G}_2\partial_\nu u_{tt} - m_1\Delta^2\mathcal{G}_2\partial_\nu u_t\\
+ \alpha\Delta^2\mathcal{G}_2\partial_\nu\theta + \beta_2\Delta^2\mathcal{G}_2\partial_\nu v + m_2\Delta^2\mathcal{G}_2\partial_\nu v_t = \mathcal{A}_B\Psi,
\end{aligned} \tag{3.28}$$

where $\Psi \coloneqq \beta_1 u + \alpha\mathcal{G}_1\theta - \gamma\mathcal{G}_2\partial_\nu u_{tt} - m_1\mathcal{G}_2\partial_\nu u_t + \alpha\mathcal{G}_2\partial_\nu\theta + \beta_2\mathcal{G}_2\partial_\nu v + m_2\mathcal{G}_2\partial_\nu v_t$. The first conditions of (1.4) are equivalent to $\Psi = \partial_\nu\Psi = 0$ on $\mathbb{R}^+\times\Gamma$ and the transmission conditions on the interface (1.6) and (1.7) can be replaced by $\mathscr{B}_1\Psi = \mathscr{B}_2\Psi = 0$ on $\mathbb{R}^+\times I$. As $\mathcal{A}_B$ is a positive self-adjoint operator, it follows from (2.25) that it has a standard extension $\widetilde{\mathcal{A}}_B : L^2(\Omega_1)\to[D(\mathcal{A}_B)]'$. For simplicity of notation, we shall write $\mathcal{A}_B$ instead of $\widetilde{\mathcal{A}}_B$, with no fear of confusion. From (3.28) it follows that

$$\begin{aligned}
\beta_1\Delta^2 u = \beta_1\mathcal{A}_B u + \alpha\mathcal{A}_B\mathcal{G}_1\theta - \gamma\mathcal{A}_B\mathcal{G}_2\partial_\nu u_{tt} - m_1\mathcal{A}_B\mathcal{G}_2\partial_\nu u_t\\
+ \alpha\mathcal{A}_B\mathcal{G}_2\partial_\nu\theta + \beta_2\mathcal{A}_B\mathcal{G}_2\partial_\nu v + m_2\mathcal{A}_B\mathcal{G}_2\partial_\nu v_t
\end{aligned} \tag{3.29}$$

in $[D(\mathcal{A}_B)]'$. Let $u_{tt} \in H^2(\Omega_1)$ with $u_{tt} = 0$ on Γ. Note that $\Delta\mathcal{N}\partial_\nu u_{tt} = 0$ in Ω_1, $\mathcal{N}\partial_\nu u_{tt} = 0$ on Γ and $\partial_\nu\mathcal{N}\partial_\nu u_{tt} = \partial_\nu u_{tt}$ on I. Then, $u_{tt} - \mathcal{N}\partial_\nu u_{tt} = 0$ on Γ and $\partial_\nu(u_{tt} - \mathcal{N}\partial_\nu u_{tt}) = 0$ on I. So, $u_{tt} - \mathcal{N}\partial_\nu u_{tt} \in D(\mathcal{A}_L)$. Using the standard extension of $\mathcal{A}_L$, writing in the last equality below $\mathcal{A}_L$ instead of $\widetilde{\mathcal{A}}_L$, we get

$$\Delta u_{tt} = \Delta u_{tt} - \Delta\mathcal{N}\partial_\nu u_{tt} = -\mathcal{A}_L(u_{tt} - \mathcal{N}\partial_\nu u_{tt}) = -\mathcal{A}_L u_{tt} + \mathcal{A}_L\mathcal{N}\partial_\nu u_{tt} \tag{3.30}$$

in $[D(\mathcal{A}_L)]'$. Let $u_t \in H^2(\Omega_1)$ with $u_t = 0$ on Γ. Reasoning similarly as above,

$$\Delta u_t = -\mathcal{A}_L u_t + \mathcal{A}_L\mathcal{N}\partial_\nu u_t \ \text{ in } \ [D(\mathcal{A}_L)]'. \tag{3.31}$$

The equalities (3.29)-(3.31) together with Proposition 3.11 allow us to rewrite the problem (1.1)-(1.7) as

$$\begin{cases} \rho_1\mathcal{M}_\eta u_{tt} + \beta_1\mathcal{A}_B u + \alpha\mathcal{A}_B\mathcal{G}_1\gamma_0\theta - \alpha\kappa\mathcal{A}_B\mathcal{G}_2\gamma_0\theta + \beta_2\mathcal{A}_B\mathcal{G}_2\gamma_1 v \\ +m_2\mathcal{A}_B\mathcal{G}_2\gamma_1 v_t + m_1\mathcal{A}_L u_t - \alpha\big(\mathcal{A}_T - \frac{\sigma}{\beta}\mathcal{I}\big)\theta = 0, \\ \rho_0\theta_t + \beta\mathcal{A}_T\theta + \alpha\mathcal{A}_L u_t - \alpha\mathcal{A}_L\mathcal{N}\partial_\nu u_t = 0, \\ \rho_2 v_{tt} - \beta_2\Delta v - m_2\Delta v_t = 0. \end{cases} \tag{3.32}$$

If $w \coloneqq (u, u_t, v, v_t, \theta)^\top$ is in an appropriate space, we can write (3.32) together with the initial conditions (1.8) and (1.9) as the following Cauchy problem

$$\partial_t w(t) = \mathscr{A}_\eta w(t),\ t > 0, \ \text{ and } \ w(0) = w_0, \tag{3.33}$$

where $w_0 \coloneqq (u_0, u_1, v_0, v_1, \theta_0)^\top$. For $\eta > 0$, the operator $\mathscr{A}_\eta$ is given by

$$\mathscr{A}_\eta \coloneqq D\begin{pmatrix} 0 & \mathcal{I} & 0 & 0 & 0 \\ -\beta_1\mathcal{A}_B & -m_1\mathcal{A}_L & -\beta_2\mathcal{A}_B\mathcal{G}_2\gamma_1 & -m_2\mathcal{A}_B\mathcal{G}_2\gamma_1 & \alpha\mathscr{P} \\ 0 & 0 & 0 & \mathcal{I} & 0 \\ 0 & 0 & \beta_2\Delta & m_2\Delta & 0 \\ 0 & -\alpha\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1) & 0 & 0 & -\beta\mathcal{A}_T \end{pmatrix},$$

where D is the diagonal matrix given by $\operatorname{diag}\big(\mathcal{I}, \frac{1}{\rho_1}\mathcal{M}_\eta^{-1}, \mathcal{I}, \frac{1}{\rho_2}\mathcal{I}, \frac{1}{\rho_0}\mathcal{I}\big)$ and $\mathscr{P} \coloneqq \mathcal{A}_T - \frac{\sigma}{\beta}\mathcal{I} - \mathcal{A}_B\mathcal{G}_1\gamma_0 + \kappa\mathcal{A}_B\mathcal{G}_2\gamma_0$. Now, introducing the term $\mathcal{W} \coloneqq \beta_1\mathcal{A}_B w_1 + \beta_2\mathcal{A}_B\mathcal{G}_2\gamma_1 w_3 + m_2\mathcal{A}_B\mathcal{G}_2\gamma_1 w_4 + \alpha\mathcal{A}_B\mathcal{G}_1\gamma_0 w_5 - \alpha\kappa\mathcal{A}_B\mathcal{G}_2\gamma_0 w_5$ allows us to write

$$\mathscr{A}_\eta w = \begin{pmatrix} w_2 \\ -\frac{1}{\rho_1}\mathcal{M}_\eta^{-1}\mathcal{W} - \frac{m_1}{\rho_1}\mathcal{M}_\eta^{-1}\mathcal{A}_L w_2 + \frac{\alpha}{\rho_1}\mathcal{M}_\eta^{-1}\mathcal{A}_T w_5 - \frac{\alpha\sigma}{\beta\rho_1}\mathcal{M}_\eta^{-1} w_5 \\ w_4 \\ \frac{\beta_2}{\rho_2}\Delta w_3 + \frac{m_2}{\rho_2}\Delta w_4 \\ -\frac{\alpha}{\rho_0}\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1)w_2 - \frac{\beta}{\rho_0}\mathcal{A}_T w_5 \end{pmatrix},$$

where $w \coloneqq (w_1, w_2, w_3, w_4, w_5)^\top$. The domain of the operator $\mathscr{A}_\eta$ is defined in the following way

$$D(\mathscr{A}_\eta) \coloneqq \Big\{ w = (w_j)^\top_{1,\dots,5} \in \mathscr{H}_\eta : w_2 \in H^2_\Gamma(\Omega_1), w_4 \in H^1(\Omega_2), w_5 \in D(\mathcal{A}_T), \\ \mathcal{W} \in H^{-1}_{\Gamma,\eta}(\Omega_1), \beta_2 \Delta w_3 + m_2 \Delta w_4 \in L^2(\Omega_2) \text{ and } w_2 = w_4 \text{ on } I \Big\}.$$

Remark 3.12. If $w = (w_1, w_2, w_3, w_4, w_5)^\top \in D(\mathscr{A}_\eta)$, then $\mathscr{A}_\eta w \in \mathscr{H}_\eta$. In effect, since $w_2 \in H^1_\Gamma(\Omega_1)$ we get from (3.13) and part c) of Theorem 2.53 that $\mathcal{A}_L w_2 \in H^{-1}_{\Gamma,\eta}(\Omega_1)$. As $w_2 \in H^2_\Gamma(\Omega_1)$ we have that $\partial_\nu w_2\big|_{\partial\Omega_1} \in H^{1/2}(\partial\Omega_1)$ and so $\mathcal{N}\gamma_1 w_2 \in H^2(\Omega_1)$. Therefore, $w_2 - \mathcal{N}\gamma_1 w_2 \in H^2(\Omega_1)$. From the definition of the Neumann map $\mathcal{N}$ it follows that $w_2 - \mathcal{N}\gamma_1 w_2 = 0$ on Γ and $\partial_\nu(w_2 - \mathcal{N}\gamma_1 w_2) = 0$ on I. Thus, $\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1)w_2 \in L^2(\Omega_1)$.

When $\eta = 0$, we not consider structural damping (i.e., $m_1 = 0$). Under these conditions, we define the operator $\mathscr{A}_0 : \mathscr{H}_0 \supset D(\mathscr{A}_0) \to \mathscr{H}_0$ as follows

$$\mathscr{A}_0 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} \coloneqq \begin{pmatrix} w_2 \\ -\frac{1}{\rho_1}\mathcal{A}_B \mathcal{W}_0 + \frac{\alpha}{\rho_1}\mathcal{A}_T w_5 - \frac{\alpha\sigma}{\beta\rho_1} w_5 \\ w_4 \\ \frac{\beta_2}{\rho_2}\Delta w_3 + \frac{m_2}{\rho_2}\Delta w_4 \\ -\frac{\alpha}{\rho_0}\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1)w_2 - \frac{\beta}{\rho_0}\mathcal{A}_T w_5 \end{pmatrix}, \tag{3.34}$$

where $\mathcal{W}_0 \coloneqq \beta_1 w_1 + \beta_2 \mathcal{G}_2 \gamma_1 w_3 + m_2 \mathcal{G}_2 \gamma_1 w_4 + \alpha \mathcal{G}_1 \gamma_0 w_5 - \alpha\kappa \mathcal{G}_2 \gamma_0 w_5$. The domain of $\mathscr{A}_0$ is given by

$$D(\mathscr{A}_0) \coloneqq \Big\{ w = (w_j)^\top_{1,\dots,5} \in [D(\mathcal{A}_B^{1/2})]^2 \times [H^1(\Omega_2)]^2 \times D(\mathcal{A}_T) : \mathcal{W}_0 \in D(\mathcal{A}_B), \\ \beta_2 \Delta w_3 + m_2 \Delta w_4 \in L^2(\Omega_2) \text{ and } w_j = w_{j+2} \text{ on } I \text{ for } j = 1, 2 \Big\}.$$

3.1.3 The semigroup approach

We will prove that the linear operator $\mathscr{A}_\eta : D(\mathscr{A}_\eta) \subset \mathscr{H}_\eta \to \mathscr{H}_\eta$, for $\eta \geq 0$, is dissipative. This result is associated with the derivative of the energy of our system (3.1), which is given by

$$\partial_t E_\gamma(t) = - m_1 \|\nabla u_t\|^2_{L^2(\Omega_1)^2} - m_2 \|\nabla v_t\|^2_{L^2(\Omega_2)^2} \\ - \sigma \|\theta\|^2_{L^2(\Omega_1)} - \beta \|\nabla\theta\|^2_{L^2(\Omega_1)^2} - \beta\kappa \|\theta\|^2_{L^2(\partial\Omega_1)}$$

and for this reason the energy of the system decreases as time passes. Physically, the system is said to be *dissipative*. The Lumer–Phillips theorem, see Theorem 2.30, will establish that $\mathscr{A}_\eta$ is the generator of a C_0-semigroup of contractions on $\mathscr{H}_\eta$ and as a consequence we will have that the Cauchy problem (3.33) has a unique solution.

Proposition 3.13. *Let $\eta \geq 0$. For $w = (w_1, w_2, w_3, w_4, w_5)^\top \in D(\mathscr{A}_\eta)$, we have*

$$\begin{aligned}\operatorname{Re}\left(\mathscr{A}_\eta w, w\right)_{\mathscr{H}_\eta} = &- m_1 \left\|\nabla w_2\right\|^2_{L^2(\Omega_1)^2} - m_2 \left\|\nabla w_4\right\|^2_{L^2(\Omega_2)^2} \\ &- \sigma \left\|w_5\right\|^2_{L^2(\Omega_1)} - \beta \left\|\nabla w_5\right\|^2_{L^2(\Omega_1)^2} - \beta\kappa \left\|w_5\right\|^2_{L^2(\partial\Omega_1)} .\end{aligned} \tag{3.35}$$

Therefore, $\mathscr{A}_\eta$ is dissipative.

Proof. We first assume that $\eta > 0$. If $w = (w_1, w_2, w_3, w_4, w_5)^\top \in D(\mathscr{A}_\eta)$, then it follows from the inner product of $\mathscr{H}_\eta$ that

$$\begin{aligned}&\left(\mathscr{A}_\eta w, w\right)_{\mathscr{H}_\eta} = \beta_1 \left(w_2, w_1\right)_{H^2_\Gamma(\Omega_1)} + \left(\beta_2 \Delta w_3 + m_2 \Delta w_4, w_4\right)_{L^2(\Omega_2)} \\ &+ \big(- \mathcal{M}_\eta^{-1}\mathcal{W} + \mathcal{M}_\eta^{-1}\big(- m_1 \mathcal{A}_L w_2 + \alpha \mathcal{A}_T w_5 - \tfrac{\alpha\sigma}{\beta} w_5\big), w_2\big)_{H^1_{\Gamma,\eta}(\Omega_1)} \\ &+ \beta_2 \left(\nabla w_4, \nabla w_3\right)_{L^2(\Omega_2)^2} + \left(-\alpha \mathcal{A}_L(w_2 - \mathcal{N}\partial_\nu w_2) - \beta \mathcal{A}_T w_5, w_5\right)_{L^2(\Omega_1)} .\end{aligned} \tag{3.36}$$

Using the first Green's formula (2.3) and the fact that $w_2 = w_4$ on I, we get

$$\begin{aligned}&\left(\beta_2 \Delta w_3 + m_2 \Delta w_4, w_4\right)_{L^2(\Omega_2)} \\ &= - \left(\nabla(\beta_2 w_3 + m_2 w_4), \nabla w_4\right)_{L^2(\Omega_2)^2} - \left(\partial_\nu(\beta_2 w_3 + m_2 w_4), w_4\right)_{L^2(I)} \\ &= -\beta_2 \left(\nabla w_3, \nabla w_4\right)_{L^2(\Omega_2)^2} - m_2 \left\|\nabla w_4\right\|^2_{L^2(\Omega_2)^2} - \beta_2 \left(\partial_\nu w_3, w_2\right)_{L^2(I)} \\ &\quad - m_2 \left(\partial_\nu w_4, w_2\right)_{L^2(I)} .\end{aligned} \tag{3.37}$$

By (2.12), (3.14), (3.15) and integration by parts, we obtain

$$\begin{aligned}&\big(- \mathcal{M}_\eta^{-1}\mathcal{W} + \mathcal{M}_\eta^{-1}\big(- m_1 \mathcal{A}_L w_2 + \alpha \mathcal{A}_T w_5 - \tfrac{\alpha\sigma}{\beta} w_5\big), w_2\big)_{H^1_{\Gamma,\eta}(\Omega_1)} \\ &= - \left\langle \mathcal{W}, w_2\right\rangle_{H^{-1}_{\Gamma,\eta}(\Omega_1)\times H^1_{\Gamma,\eta}(\Omega_1)} - m_1 \left\langle \mathcal{A}_L w_2, w_2\right\rangle_{H^{-1}_{\Gamma,\eta}(\Omega_1)\times H^1_{\Gamma,\eta}(\Omega_1)} \\ &\quad + \alpha \left(\mathcal{A}_T w_5, w_2\right)_{L^2(\Omega_1)} - \tfrac{\alpha\sigma}{\beta} \left(w_5, w_2\right)_{L^2(\Omega_1)} \\ &= - \left\langle \mathcal{W}, w_2\right\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})} - m_1 \left\|\nabla w_2\right\|^2_{L^2(\Omega_1)^2} \\ &\quad + \alpha \left(\nabla w_5, \nabla w_2\right)_{L^2(\Omega_1)^2} - \alpha \left(\partial_\nu w_5, w_2\right)_{L^2(I)} .\end{aligned} \tag{3.38}$$

We have used that $\mathcal{W} \in H^{-1}_{\Gamma,\eta}(\Omega_1) \subset [D(\mathcal{A}_B^{1/2})]'$ and $-\Delta w_5 = (\mathcal{A}_T - \frac{\sigma}{\beta}\mathcal{I})w_5$ (see definition of $D(\mathscr{A}_\eta)$, (3.8) and Proposition 3.6). On the other hand,

$$\begin{aligned}&\langle \mathcal{W}, w_2\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})} = \beta_1 \langle \mathcal{A}_B w_1, w_2\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})}\\ &+ \beta_2 \langle \mathcal{A}_B \mathcal{G}_2\gamma_1 w_3, w_2\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})} + m_2 \langle \mathcal{A}_B \mathcal{G}_2\gamma_1 w_4, w_2\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})}\\ &+ \alpha \langle \mathcal{A}_B \mathcal{G}_1\gamma_0 w_5, w_2\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})} - \alpha\kappa \langle \mathcal{A}_B \mathcal{G}_2\gamma_0 w_5, w_2\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})}.\end{aligned}$$

Due to (3.12) and (3.21), we get

$$\begin{aligned}&\langle \mathcal{W}, w_2\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})} = \beta_1 (w_1, w_2)_{H^2_\Gamma(\Omega_1)} - \beta_2 (\partial_\nu w_3, w_2)_{L^2(I)}\\ &- m_2 (\partial_\nu w_4, w_2)_{L^2(I)} + \alpha (w_5, \partial_\nu w_2)_{L^2(I)} + \alpha\kappa (w_5, w_2)_{L^2(I)}.\end{aligned} \tag{3.39}$$

From (3.16), (3.30) and integration by parts it follows that

$$\begin{aligned}&\big(- \alpha\mathcal{A}_L(w_2 - \mathcal{N}\partial_\nu w_2) - \beta\mathcal{A}_T w_5, w_5\big)_{L^2(\Omega_1)}\\ &= -\alpha (\nabla w_2, \nabla w_5)_{L^2(\Omega_1)^2} + \alpha (\partial_\nu w_2, w_5)_{L^2(I)} - \beta \|\nabla w_5\|^2_{L^2(\Omega_1)^2}\\ &\quad - \beta\kappa \|w_5\|^2_{L^2(\partial\Omega_1)} - \sigma \|w_5\|^2_{L^2(\Omega_1)}.\end{aligned} \tag{3.40}$$

A combination of equalities (3.36)-(3.40) implies

$$\begin{aligned}&(\mathscr{A}_\eta w, w)_{\mathscr{H}_\eta} = i2\beta_1 \operatorname{Im} (w_2, w_1)_{H^2_\Gamma(\Omega_1)} + i2\alpha \operatorname{Im} (\partial_\nu w_2, w_5)_{L^2(I)}\\ &+ i2\alpha \operatorname{Im} (\nabla w_5, \nabla w_2)_{L^2(\Omega_1)^2} + i2\beta_2 \operatorname{Im} (\nabla w_4, \nabla w_3)_{L^2(\Omega_2)^2} - m_1 \|\nabla w_2\|^2_{L^2(\Omega_1)^2}\\ &- m_2 \|\nabla w_4\|^2_{L^2(\Omega_2)^2} - \sigma \|w_5\|^2_{L^2(\Omega_1)} - \beta \|\nabla w_5\|^2_{L^2(\Omega_1)^2} - \beta\kappa \|w_5\|^2_{L^2(\partial\Omega_1)}.\end{aligned}$$

Taking real part above we see that (3.35) holds. For $\eta = 0$ and $m_1 = 0$ the proof is similar. □

Proposition 3.14. *Let $\mathscr{P}$ be as in the definition of $\mathscr{A}_\eta$, see page 56. For $w_1 \in D(\mathcal{A}_B^{1/2})$ and $w_5 \in H^1(\Omega_1)$, we have*

$$\langle \mathscr{P} w_5, w_1\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})} = (\nabla w_5, \nabla w_1)_{L^2(\Omega_1)^2} - (w_5, \partial_\nu w_1)_{L^2(I)}. \tag{3.41}$$

Proof. Let $w_1 \in D(\mathcal{A}_B^{1/2})$ and $w_5 \in H^1(\Omega_1)$. Characterization (3.17) implies that $w_5 \in D(\mathcal{A}_T^{1/2})$ and thus $\mathcal{A}_T w_5 \in [D(\mathcal{A}_T^{1/2})]'$. As $D(\mathcal{A}_B^{1/2}) \subset D(\mathcal{A}_T^{1/2})$, we have that $\mathcal{A}_T w_5 \in [D(\mathcal{A}_B^{1/2})]'$. Note that $\mathcal{G}_1\gamma_0 w_5, \mathcal{G}_2\gamma_0 w_5 \in D(\mathcal{A}_B^{1/2})$ and thus

$\mathcal{A}_B\mathcal{G}_1\gamma_0 w_5, \mathcal{A}_B\mathcal{G}_2\gamma_0 w_5 \in [D(\mathcal{A}_B^{1/2})]'$. Hence, $\mathscr{P}w_5 \in [D(\mathcal{A}_B^{1/2})]'$. Using (2.12), we obtain

$$\begin{aligned}&\langle \mathscr{P}w_5, w_1\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})} = \langle \mathcal{A}_T w_5, w_1\rangle_{[D(\mathcal{A}_T^{1/2})]'\times D(\mathcal{A}_T^{1/2})} - \tfrac{\sigma}{\beta}(w_5, w_1)_{L^2(\Omega_1)}\\ &- \langle \mathcal{A}_B\mathcal{G}_1\gamma_0 w_5, w_1\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})} + \kappa\,\langle \mathcal{A}_B\mathcal{G}_2\gamma_0 w_5, w_1\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})}.\end{aligned}$$

Now, (3.18) and (3.21) leads to the identity (3.41). □

Theorem 3.15. *For all $\eta > 0$ and $m_1, m_2 \geq 0$, the operator $\mathscr{A}_\eta$ generates a C_0-semigroup $(\mathscr{T}_\eta(t))_{t\geq 0}$ of contractions on $\mathscr{H}_\eta$. Therefore, for any $w_0 \in \mathscr{H}_\eta$ there exists a unique mild solution $w \in C([0,\infty), \mathscr{H}_\eta)$ of (3.33). Furthermore, for any $w_0 \in D(\mathscr{A}_\eta^k)$ with $k \in \mathbb{N}$, there exists a unique classical solution w to the problem (3.33) that belongs to $\cap_{j=0}^{k} C^{k-j}([0,\infty), D(\mathscr{A}_\eta^j))$.*

Proof. Because of Proposition 3.13, Theorem 2.31 and Theorem 2.30 it is sufficient to prove that $\mathcal{I} - \mathscr{A}_\eta : D(\mathscr{A}_\eta) \to \mathscr{H}_\eta$ is a surjective operator. Let $f = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathscr{H}_\eta$. We will find $w = (w_1, w_2, w_3, w_4, w_5)^\top \in D(\mathscr{A}_\eta)$ such that $(\mathcal{I} - \mathscr{A}_\eta)w = f$, this is,

$$\begin{aligned} w_1 - w_2 &= f_1 \quad \text{in} \quad D(\mathcal{A}_B^{1/2}),\\ w_2 + \tfrac{1}{\rho_1}\mathcal{M}_\eta^{-1}\big(\mathcal{W} + m_1\mathcal{A}_L w_2 - \alpha\mathcal{A}_T w_5 + \tfrac{\alpha\sigma}{\beta}w_5\big) &= f_2 \quad \text{in} \quad H^1_{\Gamma,\eta}(\Omega_1),\\ w_3 - w_4 &= f_3 \quad \text{in} \quad H^1(\Omega_2),\\ w_4 - \tfrac{\beta_2}{\rho_2}\Delta w_3 - \tfrac{m_2}{\rho_2}\Delta w_4 &= f_4 \quad \text{in} \quad L^2(\Omega_2),\\ w_5 + \tfrac{\alpha}{\rho_0}\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1)w_2 + \tfrac{\beta}{\rho_0}\mathcal{A}_T w_5 &= f_5 \quad \text{in} \quad L^2(\Omega_1).\end{aligned}$$

Plugging in $w_{j+1} = w_j - f_j$ for $j = 1, 3$, we have the following matrix equation

$$\begin{aligned}&\begin{pmatrix} \rho_1\mathcal{M}_\eta + \beta_1\mathcal{A}_B + m_1\mathcal{A}_L & (\beta_2 + m_2)\mathcal{A}_B\mathcal{G}_2\gamma_1 & -\alpha\mathscr{P}\\ 0 & \rho_2\mathcal{I} - (\beta_2 + m_2)\Delta & 0\\ \alpha\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1) & 0 & \rho_0\mathcal{I} + \beta\mathcal{A}_T \end{pmatrix}\begin{pmatrix} w_1\\ w_3\\ w_5\end{pmatrix}\\ &+ \begin{pmatrix} 0\\ m_2\Delta f_3\\ 0\end{pmatrix} = \begin{pmatrix} \rho_1\mathcal{M}_\eta f_1 + m_1\mathcal{A}_L f_1 + \rho_1\mathcal{M}_\eta f_2 + m_2\mathcal{A}_B\mathcal{G}_2\gamma_1 f_3\\ \rho_2 f_3 + \rho_2 f_4\\ \alpha\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1)f_1 + \rho_0 f_5\end{pmatrix}\end{aligned}$$

in $H^{-2}_\Gamma(\Omega_1) \times L^2(\Omega_2) \times L^2(\Omega_1)$. We set $\mathcal{Y} := \mathcal{X} \times H^1(\Omega_1)$, where the Hilbert space

$$\mathcal{X} := \Big\{(w_1, w_3)^\top \in D(\mathcal{A}_B^{1/2}) \times H^1(\Omega_2) : w_1 = w_3 \text{ on } I\Big\} \tag{3.42}$$

is equipped with the scalar product

$$((w_1, w_3), (\widetilde{w}_1, \widetilde{w}_3))_{\mathcal{X}} \coloneqq \beta_1 \left(w_1, \widetilde{w}_1\right)_{D(\mathcal{A}_B^{1/2})} + \beta_2 \left(\nabla w_3, \nabla \widetilde{w}_3\right)_{L^2(\Omega_2)^2}.$$

The sesquilinear form $b : \mathcal{Y} \times \mathcal{Y} \to \mathbb{C}$, defined by

$$\begin{aligned}
&b((w_1, w_3, w_5), (\phi_1, \phi_3, \phi_5)) \coloneqq \rho_1 \left(w_1, \phi_1\right)_{H^1_{\Gamma,\eta}(\Omega_1)} + \beta_1 \left(w_1, \phi_1\right)_{H^2_{\Gamma}(\Omega_1)} \\
&+ m_1 \left(\nabla w_1, \nabla \phi_1\right)_{L^2(\Omega_1)^2} - \alpha \left(\nabla w_5, \nabla \phi_1\right)_{L^2(\Omega_1)^2} + \alpha \left(w_5, \partial_\nu \phi_1\right)_{L^2(I)} \\
&+ \rho_2 \left(w_3, \phi_3\right)_{L^2(\Omega_2)} + (\beta_2 + m_2) \left(\nabla w_3, \nabla \phi_3\right)_{L^2(\Omega_2)^2} + \alpha \left(\nabla w_1, \nabla \phi_5\right)_{L^2(\Omega_1)^2} \\
&- \alpha \left(\partial_\nu w_1, \phi_5\right)_{L^2(I)} + \rho_0 \left(w_5, \phi_5\right)_{L^2(\Omega_1)} + \beta \left(\nabla w_5, \nabla \phi_5\right)_{L^2(\Omega_1)^2} \\
&+ \beta\kappa \left(w_5, \phi_5\right)_{L^2(\partial\Omega_1)} + \sigma \left(w_5, \phi_5\right)_{L^2(\Omega_1)},
\end{aligned}$$

is continuous. The coercivity of b is immediate, i.e.,

$$\operatorname{Re} b((w_1, w_3, w_5), (w_1, w_3, w_5)) \geq C \left\|(w_1, w_3, w_5)\right\|_{\mathcal{Y}}^2$$

for all $(w_1, w_3, w_5) \in \mathcal{Y}$. The mapping $\mathcal{K} : \mathcal{Y} \to \mathbb{C}$ given by

$$\begin{aligned}
\mathcal{K}(\phi_1, \phi_3, \phi_5) \coloneqq\ & \rho_1 \left(f_1 + f_2, \phi_1\right)_{H^1_{\Gamma,\eta}(\Omega_1)} + m_1 \left(\nabla f_1, \nabla \phi_1\right)_{L^2(\Omega_1)^2} \\
&+ \rho_2 \left(f_3 + f_4, \phi_3\right)_{L^2(\Omega_2)} + m_2 \left(\nabla f_3, \nabla \phi_3\right)_{L^2(\Omega_2)^2} \\
&- \alpha \left(\Delta f_1, \phi_5\right)_{L^2(\Omega_1)} + \rho_0 \left(f_5, \phi_5\right)_{L^2(\Omega_1)}
\end{aligned}$$

is antilinear and continuous. By the Lax–Milgram theorem, there exists a unique $(w_1, w_3, w_5) \in \mathcal{Y}$ such that

$$b((w_1, w_3, w_5), (\phi_1, \phi_3, \phi_5)) = \mathcal{K}(\phi_1, \phi_3, \phi_5) \tag{3.43}$$

for all $(\phi_1, \phi_3, \phi_5) \in \mathcal{Y}$. By (3.12), (3.14), (3.15), (3.18), (3.21), Proposition 3.14, the first Green's formula and (3.43), we obtain the matrix equation and its third row implies

$$\mathcal{A}_T w_5 = \tfrac{\alpha}{\beta} \mathcal{A}_L (\mathcal{I} - \mathcal{N}\gamma_1)(f_1 - w_1) + \tfrac{\rho_0}{\beta}(f_5 - w_5) \in L^2(\Omega_1)$$

and as $\mathcal{A}_T : D(\mathcal{A}_T) \to L^2(\Omega_1)$ is a bijection because it is a positive self-adjoint operator, see Remark 2.52, then $w_5 \in D(\mathcal{A}_T)$. We set $w_2 \coloneqq w_1 - f_1$ and $w_4 \coloneqq w_3 - f_3$. From the first and second rows of the matrix equation, it follows that $\mathcal{W} \in H^{-1}_{\Gamma,\eta}(\Omega_1)$ and $\beta_2 \Delta w_3 + m_2 \Delta w_4 \in L^2(\Omega_2)$, respectively. Note that $w_2 = w_4$ on I. Hence, $w \coloneqq (w_1, w_2, w_3, w_4, w_5)^\top \in D(\mathscr{A}_\eta)$. Finally, we have proven that the range $R(\mathcal{I} - \mathscr{A}_\eta)$ is equal to $\mathscr{H}_\eta$.

The unique solution of the Cauchy problem (3.33), classical or weak, is of the form $w(t) = \mathscr{T}_\eta(t) w_0$ $(t \geq 0)$ and its regularity depends on the choice of the initial datum (see Theorem 2.34, Theorem 2.36 and Theorem 2.37). □

Remark 3.16. The statement of the previous theorem is also true when $\eta = m_1 = 0$ and $m_2 \geq 0$. The proof is analogous to that presented above.

3.2 Regularity

The section starts with a regularity result which contains a useful estimate that we will use in the next two chapters. Said result will allow the proof of Theorem 3.18. This theorem indicates that when the initial data have good regularity, then some components of the solution of the problem (1.1)-(1.9) gain regularity.

Remark 3.17. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$ with a $C^{2,1}$ boundary. We recall that the classic map

$$H^2(\mathcal{O}) \ni u \mapsto (-\Delta u, u\big|_{\partial\mathcal{O}}) \in L^2(\mathcal{O}) \times H^{3/2}(\partial\mathcal{O})$$

is an isomorphism, see [123, p. 253], which implies that for each $f \in L^2(\mathcal{O})$ and $g \in H^{3/2}(\partial\mathcal{O})$ there exists a unique $u \in H^2(\mathcal{O})$ such that $\Delta u = f$ in $\mathcal{O}$ with $u\big|_{\partial\mathcal{O}} = g\big|_{\partial\mathcal{O}}$. Moreover,

$$\|u\|_{H^2(\mathcal{O})} \leq C(\|f\|_{L^2(\mathcal{O})} + \|g\|_{H^{3/2}(\partial\mathcal{O})}). \tag{3.44}$$

Theorem 3.18. *Let $\eta > 0$ and $m_1, m_2 \geq 0$. If $w \in D(\mathscr{A}_\eta^2)$, then $w_1 \in H^4(\Omega_1)$, $w_2 \in H^3(\Omega_1)$, $\beta_2 w_3 + m_2 w_4 \in H^2(\Omega_2)$ and $w_5 \in H^3(\Omega_1)$. Therefore, if $w_0 \in D(\mathscr{A}_\eta^2)$ then $w(t) := \mathscr{T}_\eta(t) w_0$ $(t \geq 0)$ is the unique solution of the problem* (1.1)-(1.9)*, which belongs to $D(\mathscr{A}_\eta^2)$ and satisfies the boundary and transmission conditions,* (1.6) *and* (1.7)*, in the strong sense of traces.*

Proof. If $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)^\top \in D(\mathscr{A}_\eta)$, we have that $\beta_1 \mathcal{A}_B \xi_1 + \beta_2 \mathcal{A}_B \mathcal{G}_2 \gamma_1 \xi_3 + m_2 \mathcal{A}_B \mathcal{G}_2 \gamma_1 \xi_4 + \alpha \mathcal{A}_B \mathcal{G}_1 \gamma_0 \xi_5 - \alpha\kappa \mathcal{A}_B \mathcal{G}_2 \gamma_0 \xi_5 =: \mathcal{W}_\xi \in [D(\mathcal{A}_B^{1/4})]'$. It is well known that $\mathcal{A}_B : D(\mathcal{A}_B^{3/4}) \to [D(\mathcal{A}_B^{1/4})]'$ is an isomorphism, see part $c)$ of Theorem 2.53. Thus, $\mathcal{A}_B^{-1} \mathcal{W}_\xi \in H^3(\Omega_1)$ due to the third characterization in (3.8). Due to the trace operators γ_{i-1} and the regularity of the Green maps $\mathcal{G}_i$, $i = 1, 2$, we obtain

$$\xi_1 = \tfrac{1}{\beta_1} \mathcal{A}_B^{-1} \mathcal{W}_\xi - \tfrac{1}{\beta_1} \mathcal{G}_2 \gamma_1 (\beta_2 \xi_3 + m_2 \xi_4) - \tfrac{\alpha}{\beta_1} (\mathcal{G}_1 \gamma_0 \xi_5 - \kappa \mathcal{G}_2 \gamma_0 \xi_5) \in H^3(\Omega_1).$$

Let $w = (w_1, w_2, w_3, w_4, w_5)^\top \in D(\mathscr{A}_\eta^2)$. Since $\mathscr{A}_\eta w \in D(\mathscr{A}_\eta)$, then the above allows us to affirm that $w_2 \in H^3(\Omega_1)$. We also have

$$\tfrac{1}{\rho_1} \mathcal{M}_\eta^{-1} \big(\mathcal{W} + m_1 \mathcal{A}_L w_2 - \alpha \mathcal{A}_T w_5 + \tfrac{\alpha\sigma}{\beta} w_5 \big) =: \phi \in D(\mathcal{A}_B^{1/2}). \tag{3.45}$$

Note that $\mathcal{G}_2\gamma_1\phi \in D(\mathcal{A}_B^{3/4})$ and so $\mathcal{A}_B\mathcal{G}_2\gamma_1\phi \in H^{-1}_{\Gamma,\eta}(\Omega_1)$. Taking $\widetilde{\phi} \in D(\mathcal{A}_B^{1/2})$ and applying integration by parts together with (2.12), (3.14) and (3.21), we get the following equalities

$$\begin{aligned}
&\langle \Delta\phi + \mathcal{A}_B\mathcal{G}_2\gamma_1\phi, \widetilde{\phi}\rangle_{H^{-1}_{\Gamma,\eta}(\Omega_1)\times H^1_{\Gamma,\eta}(\Omega_1)} \\
&= (\Delta\phi, \widetilde{\phi})_{L^2(\Omega_1)} + \langle \mathcal{A}_B\mathcal{G}_2\gamma_1\phi, \widetilde{\phi}\rangle_{[D(\mathcal{A}_B^{1/2})]'\times D(\mathcal{A}_B^{1/2})} \\
&= -\langle \mathcal{A}_L\phi, \widetilde{\phi}\rangle_{H^{-1}_{\Gamma,\eta}(\Omega_1)\times H^1_{\Gamma,\eta}(\Omega_1)}
\end{aligned} \tag{3.46}$$

and thus

$$\mathcal{M}_\eta\phi = \phi - \eta\Delta\phi - \eta\mathcal{A}_B\mathcal{G}_2\gamma_1\phi \ \text{ in } \ H^{-1}_{\Gamma,\eta}(\Omega_1). \tag{3.47}$$

Doing $w_{jk} := \beta_2 w_j + m_2 w_k$, we obtain that $\Delta w_{34} =: \widetilde{f} \in L^2(\Omega_2)$. Note that $w_{34} = w_{12}$ on I with $\gamma_0 w_{12} \in H^{3/2}(I)$. By Remark 3.17, there exists a unique $\widetilde{w}_{34} \in H^2(\Omega_2)$ such that $\Delta\widetilde{w}_{34} = \widetilde{f}$ in Ω_2 and $\widetilde{w}_{34}\big|_I = w_{12}\big|_I$. Since $w_{34} - \widetilde{w}_{34} \in H^1_0(\Omega_2)$ is a weak solution of $\Delta(w_{34} - \widetilde{w}_{34}) = 0$, we immediately obtain $w_{34} = \widetilde{w}_{34}$, which leads to $\beta_2 w_3 + m_2 w_4 \in H^2(\Omega_2)$. From (3.45)-(3.47) it follows that

$$\mathcal{W} - m_1\mathcal{A}_B\mathcal{G}_2\gamma_1 w_2 + \gamma\mathcal{A}_B\mathcal{G}_2\gamma_1\phi = \rho_1\phi - \gamma\Delta\phi + m_1\Delta w_2 - \alpha\Delta w_5 \in L^2(\Omega_1).$$

In consequence,

$$\beta_1 w_1 - m_1\mathcal{G}_2\gamma_1 w_2 + \mathcal{G}_2\gamma_1(\beta_2 w_3 + m_2 w_4) + \alpha(\mathcal{G}_1 - \kappa\mathcal{G}_2)\gamma_0 w_5 + \gamma\mathcal{G}_2\gamma_1\phi \in D(\mathcal{A}_B)$$

because $\mathcal{A}_B : D(\mathcal{A}_B) \to L^2(\Omega_1)$ is bijective, see Remark 2.52. Once again, appealing to the regularity of the operators γ_{i-1} and maps $\mathcal{G}_i$ we obtain that $w_1 \in H^4(\Omega_1)$. Now, the last expression above leads us to that the boundary and transmission conditions (1.6) and (1.7) hold in the strong sense of the traces with w_2 and w_4 instead of u_t and v_t, respectively. On the other hand, we have that

$$\tfrac{\alpha}{\rho_0}\widehat{\varphi} - \tfrac{\beta}{\rho_0}\mathcal{A}_T w_5 =: \widehat{\phi} \in D(\mathcal{A}_T), \tag{3.48}$$

where $\widehat{\varphi} := -\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1)w_2$. This is also due to the fact that $\mathscr{A}_\eta w \in D(\mathscr{A}_\eta)$. As $w_2 \in H^3(\Omega_1)$, then $\widehat{\varphi} = \Delta w_2 \in H^1(\Omega_1)$. Because of $\widehat{\phi} \in H^2(\Omega_1)$, we get from (3.17) and (3.48) that

$$\mathcal{A}_T w_5 = \tfrac{\alpha}{\beta}\widehat{\varphi} - \tfrac{\rho_0}{\beta}\widehat{\phi} \in D(\mathcal{A}_T^{1/2}).$$

Finally, since the operator $\mathcal{A}_T : D(\mathcal{A}_T^{3/2}) \to D(\mathcal{A}_T^{1/2})$ is bijective (see again part c) of Theorem 2.53) we conclude that $w_5 \in D(\mathcal{A}_T^{3/2}) \subset H^3(\Omega_1)$.

If $w_0 \in D(\mathscr{A}_\eta^2)$, then $w_0 \in D(\mathscr{A}_\eta)$ and $\mathscr{A}_\eta w_0 \in D(\mathscr{A}_\eta)$. By Proposition 2.27, $\mathscr{T}_\eta(t)w_0 \in D(\mathscr{A}_\eta)$ and $\mathscr{A}_\eta \mathscr{T}_\eta(t)w_0 = \mathscr{T}_\eta(t)\mathscr{A}_\eta w_0 \in D(\mathscr{A}_\eta)$. Hence, the unique solution of the problem (1.1)-(1.9) satisfies $w(t) = \mathscr{T}_\eta(t)w_0 \in D(\mathscr{A}_\eta^2)$ for any $t \geq 0$. □

Remark 3.19. Let $\eta \geq 0$. Due to the proof of Theorem 3.18, we have: If $w = (w_1, w_2, w_3, w_4, w_5)^\top \in D(\mathscr{A}_\eta)$ then $\beta_2 w_3 + m_2 w_4 \in H^2(\Omega_2)$ and moreover $w \in H^{4-\chi}(\Omega_1) \times H^2(\Omega_1) \times [H^1(\Omega_2)]^2 \times H^2(\Omega_1)$, where $\chi \coloneqq \chi_{(0,\infty)}(\eta)$ stands for the characteristic function of $(0, \infty)$.

Chapter 4

Asymptotic behavior of the solutions of some plate-membrane problems

In this chapter the reader will find an extensive study of the asymptotic behavior of the solutions of the plate-membrane transmission problem (1.1)-(1.9). We will see that for $\gamma \geq 0$ and $m_2 > 0$ the solutions will have strong stability, see Corollary 4.3 and Corollary 4.6. When the structural damping on the plate is not considered ($m_1 = 0$) it will be the thermal effect that will help to obtain the strong stability, see part b) of Remark 4.4 and Remark 4.7. We prove by contradiction the exponential stability of the energy of the system when the plate has rotational inertial ($\gamma > 0$) and the structure is damped ($m_1 > 0$ and $m_2 > 0$), see Theorem 4.8. When we remove the Kelvin–Voigt damping ($m_2 = 0$) and the inertial term is present, an absence of exponential stability is caused (see Theorem 4.15). In this case, the solutions have polynomial stability when we add a geometric condition (see Theorem 4.16).

4.1 Strong stability

First we will prove that the operator $\mathscr{A}_\eta$ is continuously invertible, this is, 0 is in the resolvent set $\rho(\mathscr{A}_\eta)$. This result will be key to prove the strong stability of the solutions of our problem for when $\gamma \geq 0$ and $m_2 > 0$.

Proposition 4.1. $0 \in \rho(\mathscr{A}_\eta)$ *if* a) $\eta > 0$ *and* $m_1, m_2 \geq 0$ *or* b) $\eta = m_1 = 0$ *and* $m_2 \geq 0$.

Proof. Due to the similarity between the proofs of the two cases a) and b), we will only prove part a).

We show that the operator $\mathscr{A}_\eta : D(\mathscr{A}_\eta) \to \mathscr{H}_\eta$ is bijective. Let $f = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathscr{H}_\eta$. We will find a unique $w = (w_1, w_2, w_3, w_4, w_5)^\top \in D(\mathscr{A}_\eta)$ such that $\mathscr{A}_\eta w = f$. Conveniently we set $w_2 := f_1$, $w_4 := f_3$ and $w_5 := -\mathcal{A}_T^{-1}\big[\frac{\alpha}{\beta}\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1)f_1 + \frac{\rho_0}{\beta}f_5\big]$. We recall again that Remark 2.52 implies that $\mathcal{A}_T$ is bijective. We now consider the following system

$$\begin{cases} \beta_1 \mathcal{A}_B w_1 + \beta_2 \mathcal{A}_B \mathcal{G}_2 \gamma_1 w_3 = h & \text{in} \quad H_\Gamma^{-2}(\Omega_1), \\ -\beta_2 \Delta w_3 - m_2 \Delta f_3 = -\rho_2 f_4 & \text{in} \quad L^2(\Omega_2), \end{cases} \tag{4.1}$$

where $h := -m_1\mathcal{A}_L f_1 - \rho_1 \mathcal{M}_\eta f_2 - m_2 \mathcal{A}_B \mathcal{G}_2 \gamma_1 f_3 + \alpha \mathscr{P} w_5$. With $\mathcal{X}$ as in (3.42), we define $\widetilde{b} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ by $\widetilde{b}((w_1, w_3), (\phi_1, \phi_3)) := ((w_1, w_3), (\phi_1, \phi_3))_{\mathcal{X}}$. The function $\widetilde{\mathcal{K}} : \mathcal{X} \to \mathbb{C}$ given by

$$\begin{aligned} \widetilde{\mathcal{K}}(\phi_1, \phi_3) := &- m_1 (\nabla f_1, \nabla \phi_1)_{L^2(\Omega_1)^2} - \rho_1 (f_2, \phi_1)_{H^1_{\Gamma,\eta}(\Omega_1)} + \alpha (\nabla w_5, \nabla \phi_1)_{L^2(\Omega_1)^2} \\ &- \alpha (w_5, \partial_\nu \phi_1)_{L^2(I)} - m_2 (\nabla f_3, \nabla \phi_3)_{L^2(\Omega_2)^2} - \rho_2 (f_4, \phi_3)_{L^2(\Omega_2)} \end{aligned}$$

is an element of the antidual space $\mathcal{X}'$. By the Riesz's representation theorem there exists a unique $(w_1, w_3) \in \mathcal{X}$ such that

$$\widetilde{b}((w_1, w_3), (\phi_1, \phi_3)) = \widetilde{\mathcal{K}}(\phi_1, \phi_3) \tag{4.2}$$

for all $(\phi_1, \phi_3) \in \mathcal{X}$. From (3.12), (3.14), (3.15), (3.21), Proposition 3.14, the first Green's formula and (4.2), we obtain that (4.1) is true. Reasoning as in the last part of the proof of Theorem 3.15, we get $w := (w_1, w_2, w_3, w_4, w_5)^\top \in D(\mathscr{A}_\eta)$. Thus, $\mathscr{A}_\eta$ is surjective. As the constructed w above is unique, then $\mathscr{A}_\eta$ is injective.

Since $\mathscr{A}_\eta : D(\mathscr{A}_\eta) \to \mathscr{H}_\eta$ is a closed linear operator (due to Theorem 3.15 and Proposition 2.28) and bijective, we conclude that $\mathscr{A}_\eta^{-1} \in \mathcal{L}(\mathscr{H}_\eta)$. □

The following proposition together with Theorem 2.39 allow us to affirm that $(\mathscr{T}_\eta(t))_{t\geq 0}$ is strongly stable in $\mathscr{H}_\eta$, when there is Kelvin–Voigt damping on the membrane.

Proposition 4.2. *Let* $\eta > 0$, $m_1 \geq 0$ *and* $m_2 > 0$. *The imaginary axis is contained in the resolvent set of* $\mathscr{A}_\eta$, *this is,* $i\mathbb{R} \subset \rho(\mathscr{A}_\eta)$.

Proof. Let η and m_2 be positive, and m_1 be non-negative. As $0 \in \rho(\mathscr{A}_\eta)$ and $\rho(\mathscr{A}_\eta)$ is open in $\mathbb{C}$, see Proposition 4.1 and Theorem 1 in [124, p. 211], then there is $R > 0$ such that $B(0, R) \subset \rho(\mathscr{A}_\eta)$. For real δ such that $0 < \delta < R$, we have $i[-\delta, \delta] \subset \rho(\mathscr{A}_\eta)$. Thus, $\mathcal{R} \coloneqq \{\lambda > 0 : i[-\lambda, \lambda] \subset \rho(\mathscr{A}_\eta)\}$ is not empty. Let $\lambda^* \coloneqq \sup \mathcal{R}$. If $\lambda^* = \infty$, we have nothing to prove. Let us suppose $\lambda^* < \infty$. If $\lambda^* \in \mathcal{R}$, then $i[-\lambda^*, \lambda^*]$ is contained in $\rho(\mathscr{A}_\eta)$ and so it is possible to find $r > 0$ such that $i[-\lambda^* - r, \lambda^* + r] \subset \rho(\mathscr{A}_\eta)$. The above indicates that $\lambda^* + r \in \mathcal{R}$ which contradicts the assumption that λ^* is the supremum of $\mathcal{R}$. Hence, $\lambda^* \notin \mathcal{R}$. Then, there exists $(\lambda_n)_{n\in\mathbb{N}} \subset \mathcal{R}$ such that $\lim_{n\to\infty} \lambda_n = \lambda^*$ and $\lim_{n\to\infty} \|(i\lambda_n \mathcal{I} - \mathscr{A}_\eta)^{-1}\|_{\mathcal{L}(\mathscr{H}_\eta)} = \infty$, see proof of Theorem 4.1 in [105]. Hence, there exists $(\widetilde{f}_n)_{n\in\mathbb{N}} \subset \mathscr{H}_\eta$ with

$$\|\widetilde{f}_n\|_{\mathscr{H}_\eta} = 1 \quad \text{and} \quad \lim_{n\to\infty} \|(i\lambda_n \mathcal{I} - \mathscr{A}_\eta)^{-1} \widetilde{f}_n\|_{\mathscr{H}_\eta} = \infty.$$

Here putting $\widetilde{w}_n \coloneqq (i\lambda_n \mathcal{I} - \mathscr{A}_\eta)^{-1} \widetilde{f}_n$, $w_n \coloneqq \widetilde{w}_n / \|\widetilde{w}_n\|_{\mathscr{H}_\eta}$ and $f_n \coloneqq \widetilde{f}_n / \|\widetilde{w}_n\|_{\mathscr{H}_\eta}$, we get that

$$(i\lambda_n \mathcal{I} - \mathscr{A}_\eta)\, w_n = f_n \tag{4.3}$$

with

$$\|w_n\|_{\mathscr{H}_\eta} = 1 \tag{4.4}$$

and

$$\|(i\lambda_n \mathcal{I} - \mathscr{A}_\eta) w_n\|_{\mathscr{H}_\eta} \xrightarrow[n\to\infty]{} 0. \tag{4.5}$$

It is clear that $(w_n)_{n\in\mathbb{N}} \subset D(\mathscr{A}_\eta)$. From (4.3) it follows that

$$i\lambda_n w_1^n - w_2^n = f_1^n, \tag{4.6}$$

$$i\lambda_n \rho_1 w_2^n + \mathcal{M}_\eta^{-1}\big(\mathcal{W}^n + m_1 \mathcal{A}_L w_2^n - \alpha \mathcal{A}_T w_5^n + \tfrac{\alpha\sigma}{\beta} w_5^n\big) = \rho_1 f_2^n, \tag{4.7}$$

$$i\lambda_n w_3^n - w_4^n = f_3^n, \tag{4.8}$$

$$i\lambda_n \rho_2 w_4^n - \beta_2 \Delta w_3^n - m_2 \Delta w_4^n = \rho_2 f_4^n, \tag{4.9}$$

$$i\lambda_n \rho_0 w_5^n + \alpha \mathcal{A}_L (\mathcal{I} - \mathcal{N}\gamma_1) w_2^n + \beta \mathcal{A}_T w_5^n = \rho_0 f_5^n, \tag{4.10}$$

where $\mathcal{W}^n \coloneqq \beta_1 \mathcal{A}_B w_1^n + \mathcal{A}_B \mathcal{G}_2 \gamma_1 (\beta_2 w_3^n + m_2 w_4^n) + \alpha \mathcal{A}_B \mathcal{G}_1 \gamma_0 w_5^n - \alpha\kappa \mathcal{A}_B \mathcal{G}_2 \gamma_0 w_5^n$. Moreover,

$$\begin{aligned} \|(i\lambda_n \mathcal{I} - \mathscr{A}_\eta) w_n\|^2_{\mathscr{H}_\eta} = \beta_1 \|f_1^n\|^2_{H^2_\Gamma(\Omega_1)} + \rho_1 \|f_2^n\|^2_{H^1_{\Gamma,\eta}(\Omega_1)} + \beta_2 \|\nabla f_3^n\|^2_{L^2(\Omega_2)^2} \\ + \rho_2 \|f_4^n\|^2_{L^2(\Omega_2)} + \rho_0 \|f_5^n\|^2_{L^2(\Omega_1)}. \end{aligned}$$

The limit (4.5) and (4.6)-(4.10) imply that

$$i\lambda_n w_1^n - w_2^n \to 0 \;\text{ in }\; H^2_\Gamma(\Omega_1), \tag{4.11}$$
$$i\lambda_n \rho_1 \mathcal{M}_\eta w_2^n + \mathcal{W}^n + m_1 \mathcal{A}_L w_2^n - \alpha\big(\mathcal{A}_T - \tfrac{\sigma}{\beta}\big) w_5^n \to 0 \;\text{ in }\; H^{-1}_{\Gamma,\eta}(\Omega_1), \tag{4.12}$$
$$i\lambda_n \nabla w_3^n - \nabla w_4^n \to 0 \;\text{ in }\; L^2(\Omega_2), \tag{4.13}$$
$$i\lambda_n \rho_2 w_4^n - \beta_2 \Delta w_3^n - m_2 \Delta w_4^n \to 0 \;\text{ in }\; L^2(\Omega_2), \tag{4.14}$$
$$i\lambda_n \rho_0 w_5^n + \alpha \mathcal{A}_L (\mathcal{I} - \mathcal{N}\gamma_1) w_2^n + \beta \mathcal{A}_T w_5^n \to 0 \;\text{ in }\; L^2(\Omega_1). \tag{4.15}$$

Due to the dissipativity (3.35) of the operator $\mathscr{A}_\eta$, we have

$$\begin{aligned}&\operatorname{Re}\left((i\lambda_n \mathcal{I} - \mathscr{A}_\eta) w_n, w_n\right)_{\mathscr{H}_\eta} = m_1 \left\|\nabla w_2^n\right\|^2_{L^2(\Omega_1)^2} + m_2 \left\|\nabla w_4^n\right\|^2_{L^2(\Omega_2)^2}\\ &+ \sigma \left\|w_5^n\right\|^2_{L^2(\Omega_1)} + \beta \left\|\nabla w_5^n\right\|^2_{L^2(\Omega_1)^2} + \beta\kappa \left\|w_5^n\right\|^2_{L^2(\partial\Omega_1)}.\end{aligned} \tag{4.16}$$

By Theorem 2.24, Cauchy–Schwarz inequality and (4.4), we get the estimate

$$\left\|w_5^n\right\|^2_{H^1(\Omega_1)} \le C \operatorname{Re}\left((i\lambda_n \mathcal{I} - \mathscr{A}_\eta) w_n, w_n\right)_{\mathscr{H}_\eta} \le C \left\|(i\lambda_n \mathcal{I} - \mathscr{A}_\eta) w_n\right\|_{\mathscr{H}_\eta}.$$

In consequence,

$$w_5^n \to 0 \;\text{ in }\; H^1(\Omega_1). \tag{4.17}$$

Assuming $m_1 > 0$, from (4.5) and (4.16) we obtain

$$\nabla w_2^n \to 0 \;\text{ in }\; L^2(\Omega_1) \quad\text{and}\quad \nabla w_4^n \to 0 \;\text{ in }\; L^2(\Omega_2). \tag{4.18}$$

The limits $\lambda_n \to \lambda^*$, (4.13) and the right-hand side of (4.18) imply

$$\nabla w_3^n \to 0 \;\text{ in }\; L^2(\Omega_2). \tag{4.19}$$

On the other hand,

$$\begin{aligned}&(i\lambda_n \rho_2 w_4^n - \beta_2 \Delta w_3^n - m_2 \Delta w_4^n, w_4^n)_{L^2(\Omega_2)}\\ &= i\lambda_n \rho_2 \left\|w_4^n\right\|^2_{L^2(\Omega_2)} - (\beta_2 \Delta w_3^n + m_2 \Delta w_4^n, w_4^n)_{L^2(\Omega_2)}.\end{aligned} \tag{4.20}$$

Using integration by parts and the fact that $w_2^n = w_4^n$ on I, we compute

$$\begin{aligned}&(\beta_2 \Delta w_3^n + m_2 \Delta w_4^n, w_4^n)_{L^2(\Omega_2)} = -\beta_2 \left(\nabla w_3^n, \nabla w_4^n\right)_{L^2(\Omega_2)^2}\\ &- m_2 \left\|\nabla w_4^n\right\|^2_{L^2(\Omega_2)^2} - (\partial_\nu(\beta_2 w_3^n + m_2 w_4^n), w_2^n)_{L^2(I)},\end{aligned} \tag{4.21}$$

by Remark 3.19 we have $\beta_2 w_3^n + m_2 w_4^n \in H^2(\Omega_2)$. Thanks to Cauchy–Schwarz inequality and the trace theorem,

$$\begin{aligned} &\big| \left(\partial_\nu(\beta_2 w_3^n + m_2 w_4^n), w_2^n\right)_{L^2(I)} \big| \\ &\le C \left\|\beta_2 w_3^n + m_2 w_4^n\right\|_{H^2(\Omega_2)} \left\|w_2^n\right\|_{H^1(\Omega_1)} . \end{aligned} \tag{4.22}$$

Note that $w_2^n \in H^1_\Gamma(\Omega_1)$ and in consequence $\|w_2^n\|_{H^1(\Omega_1)} \le C \|\nabla w_2^n\|_{L^2(\Omega_1)^2}$, see Theorem 2.24. By the left-hand side of (4.18),

$$w_2^n \to 0 \ \text{ in } \ H^1(\Omega_1). \tag{4.23}$$

As $\Delta(\beta_2 w_3^n + m_2 w_4^n) = i\lambda_n \rho_2 w_4^n - \rho_2 f_4^n$ in Ω_2 and $\beta_2 w_3^n + m_2 w_4^n = \beta_2 w_1^n + m_2 w_2^n$ on I, see (4.9), Remark 3.17 implies

$$\begin{aligned} &\|\beta_2 w_3^n + m_2 w_4^n\|_{H^2(\Omega_2)} \\ &\le C\big(\|i\lambda_n \rho_2 w_4^n - \rho_2 f_4^n\|_{L^2(\Omega_2)} + \|\beta_2 w_1^n + m_2 w_2^n\|_{H^{3/2}(I)} \big) \\ &\le C\big(|\lambda_n| \, \|w_4^n\|_{L^2(\Omega_2)} + \|f_4^n\|_{L^2(\Omega_2)} + \|w_1^n\|_{H^2(\Omega_1)} + \|w_2^n\|_{H^2(\Omega_1)} \big). \end{aligned} \tag{4.24}$$

We know that $f_4^n \to 0$ in $L^2(\Omega_2)$. By (4.4), the sequences $(w_1^n)_{n\in\mathbb{N}}$ and $(w_4^n)_{n\in\mathbb{N}}$ are bounded in $H^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively. The limit (4.11) implies that $(w_2^n)_{n\in\mathbb{N}}$ is bounded in $H^2(\Omega_1)$. Hence,

$$\|\beta_2 w_3^n + m_2 w_4^n\|_{H^2(\Omega_2)} \le C. \tag{4.25}$$

From (4.14), (4.18)-(4.23) and (4.25), it follows that

$$w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{4.26}$$

Now, let us consider the following equality

$$\begin{aligned} &\big\langle i\lambda_n \rho_1 \mathcal{M}_\eta w_2^n + \mathcal{W}^n + m_1 \mathcal{A}_L w_2^n - \alpha \mathcal{A}_T w_5^n + \tfrac{\alpha\sigma}{\beta} w_5^n, w_1^n \big\rangle_{H^{-1}_{\Gamma,\eta} \times H^1_{\Gamma,\eta}} \\ &= i\lambda_n \rho_1 \left\langle \mathcal{M}_\eta w_2^n, w_1^n \right\rangle_{H^{-1}_{\Gamma,\eta} \times H^1_{\Gamma,\eta}} + \left\langle \mathcal{W}^n, w_1^n \right\rangle_{H^{-1}_{\Gamma,\eta} \times H^1_{\Gamma,\eta}} \\ &\quad + m_1 \left\langle \mathcal{A}_L w_2^n, w_1^n \right\rangle_{H^{-1}_{\Gamma,\eta} \times H^1_{\Gamma,\eta}} + \big\langle -\alpha \mathcal{A}_T w_5^n + \tfrac{\alpha\sigma}{\beta} w_5^n, w_1^n \big\rangle_{H^{-1}_{\Gamma,\eta} \times H^1_{\Gamma,\eta}}, \end{aligned} \tag{4.27}$$

where $H^{-1}_{\Gamma,\eta} \times H^1_{\Gamma,\eta} \coloneqq H^{-1}_{\Gamma,\eta}(\Omega_1) \times H^1_{\Gamma,\eta}(\Omega_1)$. By (3.15), we have the following

$$\big| \left\langle \mathcal{M}_\eta w_2^n, w_1^n \right\rangle_{H^{-1}_{\Gamma,\eta}(\Omega_1) \times H^1_{\Gamma,\eta}(\Omega_1)} \big| = \big| \left(w_2^n, w_1^n\right)_{H^1_{\Gamma,\eta}(\Omega_1)} \big| \le C \left\|w_2^n\right\|_{H^1(\Omega_1)} . \tag{4.28}$$

Proceeding as in the proof of Proposition 3.13, we obtain

$$\begin{aligned}\langle \mathcal{W}^n, w_1^n \rangle_{H^{-1}_{\Gamma,\eta} \times H^1_{\Gamma,\eta}} &= \beta_1 \|w_1^n\|^2_{H^2_\Gamma(\Omega_1)} - (\partial_\nu(\beta_2 w_3^n + m_2 w_4^n), w_1^n)_{L^2(I)} \\ &\quad + \alpha\,(w_5^n, \partial_\nu w_1^n)_{L^2(I)} + \alpha\kappa\,(w_5^n, w_1^n)_{L^2(I)}\,.\end{aligned} \tag{4.29}$$

Arguing as in (4.22) and considering (4.25),

$$\big|\,(\partial_\nu(\beta_2 w_3^n + m_2 w_4^n), w_1^n)_{L^2(I)}\,\big| \le C\,\|w_1^n\|_{H^1(\Omega_1)}\,. \tag{4.30}$$

The equivalence of norms $\|\cdot\|_{H^2_\Gamma(\Omega_1)} \sim \|\cdot\|_{H^2(\Omega_1)}$ and $\|\cdot\|_{H^1_\Gamma(\Omega_1)} \sim \|\cdot\|_{H^1(\Omega_1)}$ on $H^2_\Gamma(\Omega_1)$ and $H^1_\Gamma(\Omega_1)$, respectively, together with Theorem 2.5 allow to obtain immediately the continuous embedding $H^2_\Gamma(\Omega_1) \hookrightarrow H^1_\Gamma(\Omega_1)$. Now, the limits (4.11) and (4.23) imply

$$w_1^n \to 0 \;\text{ in }\; H^1(\Omega_1). \tag{4.31}$$

By the trace theorem, we get

$$\big|\,(w_5^n, \partial_\nu w_1^n)_{L^2(I)}\,\big| \le C\,\|w_5^n\|_{H^1(\Omega_1)}\,\|w_1^n\|_{H^2(\Omega_1)} \le C\,\|w_5^n\|_{H^1(\Omega_1)} \tag{4.32}$$

and

$$\big|\,(w_5^n, w_1^n)_{L^2(I)}\,\big| \le C\,\|w_5^n\|_{H^1(\Omega_1)}\,. \tag{4.33}$$

From (3.14), it follows that

$$\big|\,\langle \mathcal{A}_L w_2^n, w_1^n\rangle_{H^{-1}_{\Gamma,\eta}\times H^1_{\Gamma,\eta}}\,\big| = \big|\,(\nabla w_2^n, \nabla w_1^n)_{L^2(\Omega_1)^2}\,\big| \le C\,\|\nabla w_2^n\|_{L^2(\Omega_1)^2}\,. \tag{4.34}$$

Using integration by parts,

$$\begin{aligned}\big\langle -\mathcal{A}_T w_5^n + \tfrac{\sigma}{\beta} w_5^n, w_1^n\big\rangle_{H^{-1}_{\Gamma,\eta}(\Omega_1)\times H^1_{\Gamma,\eta}(\Omega_1)} &= (\Delta w_5^n, w_1^n)_{L^2(\Omega_1)} \\ &= -\,(\nabla w_5^n, \nabla w_1^n)_{L^2(\Omega_1)^2} + (\partial_\nu w_5^n, w_1^n)_{L^2(\partial\Omega_1)}\,.\end{aligned} \tag{4.35}$$

Applying Cauchy–Schwarz inequality,

$$\big|\,(\nabla w_5^n, \nabla w_1^n)_{L^2(\Omega_1)^2}\,\big| \le \|\nabla w_5^n\|_{L^2(\Omega_1)^2}\,\|\nabla w_1^n\|_{L^2(\Omega_1)^2}\,. \tag{4.36}$$

As $\partial_\nu w_5^n + \kappa w_5^n = 0$ on $\partial\Omega_1$, then

$$\big|\,(\partial_\nu w_5^n, w_1^n)_{L^2(\partial\Omega_1)}\,\big| \le C\,\|w_5^n\|_{H^1(\Omega_1)}\,\|w_1^n\|_{H^1(\Omega_1)}\,. \tag{4.37}$$

From (4.12), (4.17), (4.23) and (4.27)-(4.37), we can affirm

$$w_1^n \to 0 \quad \text{in} \quad H^2(\Omega_1). \tag{4.38}$$

The limits (4.17), (4.19), (4.23), (4.26) and (4.38) imply $w_n \to 0$ in $\mathscr{H}_\eta$, this contradicts (4.4). Hence, $\lambda^* = \infty$.

Let us now consider $m_1 = 0$. In this case, we initially only have the right-hand side of (4.18) and consequently (4.19) holds. Note that (4.20), (4.21) and (4.25) are true here too. The boundedness of the sequence $(w_2^n)_{n\in\mathbb{N}}$ in $H^2(\Omega_1)$, the trace theorem and Corollary 2.22 allows us to write

$$\begin{aligned}
&\left| (\partial_\nu(\beta_2 w_3^n + m_2 w_4^n), w_2^n)_{L^2(I)} \right| \\
&\leq C \, \|\beta_2 w_3^n + m_2 w_4^n\|_{H^2(\Omega_2)}^{1/2} \, \|\nabla(\beta_2 w_3^n + m_2 w_4^n)\|_{L^2(\Omega_2)^2}^{1/2} \, \|w_2^n\|_{H^1(\Omega_1)} \\
&\leq C\big(\|\nabla w_3^n\|_{L^2(\Omega_2)^2}^{1/2} + \|\nabla w_4^n\|_{L^2(\Omega_2)^2}^{1/2} \big).
\end{aligned} \tag{4.39}$$

By (4.4), (4.14), right-hand side of (4.18), (4.19)-(4.21) and (4.39) we get

$$w_4^n \to 0 \quad \text{in} \quad L^2(\Omega_2). \tag{4.40}$$

From (4.15) and (4.17), it follows that

$$\alpha \Delta w_2^n + \beta \Delta w_5^n \to 0 \quad \text{in} \quad L^2(\Omega_1).$$

The integration by parts formula (2.1) allows the calculation

$$\begin{aligned}
(\alpha \Delta w_2^n + \beta \Delta w_5^n, w_2^n)_{L^2(\Omega_1)} = &-\alpha \, \|\nabla w_2^n\|_{L^2(\Omega_1)^2}^2 + \alpha \, (\partial_\nu w_2^n, w_2^n)_{L^2(I)} \\
&- \beta \, (\nabla w_5^n, \nabla w_2^n)_{L^2(\Omega_1)^2} + \beta \, (\partial_\nu w_5^n, w_2^n)_{L^2(I)} \, .
\end{aligned}$$

It is easy to see that

$$(\nabla w_5^n, \nabla w_2^n)_{L^2(\Omega_1)^2} \to 0 \quad \text{and} \quad (\partial_\nu w_5^n, w_2^n)_{L^2(I)} \to 0.$$

Since $w_2^n = w_4^n$ on I,

$$\begin{aligned}
\left| (\partial_\nu w_2^n, w_2^n)_{L^2(I)} \right| &= \left| (\partial_\nu w_2^n, w_4^n)_{L^2(I)} \right| \leq \|\partial_\nu w_2^n\|_{L^2(I)} \, \|w_4^n\|_{L^2(I)} \\
&\leq C \, \|w_2^n\|_{H^2(\Omega_1)} \, \|w_4^n\|_{H^{1/2}(I)} \leq C \, \|w_4^n\|_{H^1(\Omega_2)} \, .
\end{aligned} \tag{4.41}$$

The limit of the right part of (4.18), (4.40) and (4.41) imply

$$(\partial_\nu w_2^n, w_2^n)_{L^2(I)} \to 0. \tag{4.42}$$

Therefore, $\nabla w_2^n \to 0$ in $L^2(\Omega_1)$. Reasoning as in the previous case, we obtain again the same contradiction. □

Corollary 4.3. *Let $\eta > 0$, $m_1 \geq 0$ and $m_2 > 0$. Then $\left(\mathscr{T}_\eta(t)\right)_{t\geq 0}$ is strongly stable in $\mathscr{H}_\eta$.*

Remark 4.4. a) In absence of temperature on the plate we also have strong stability when the rotational term is present and the structure is damped.

b) The thermal effect on the plate guarantees the strong stability when we remove the structural damping on the plate and leave the Kelvin–Voigt damping on the membrane and there is presence of the rotational term.

c) With an appropriate geometric condition on Ω_2, see (4.99), for the situation $\eta, m_1 > 0$ and $m_2 = 0$ we also have strong stability (see the proof of Theorem 4.16).

Now we will see the second strong stability result of the system (1.1)-(1.9) when both the inertial term and the structural damping are not present, but there is Kelvin–Voigt damping on the membrane.

Proposition 4.5. *If $\eta = m_1 = 0$ and $m_2 > 0$, then the imaginary axis is contained in the resolvent set of $\mathscr{A}_0$, i.e., $i\mathbb{R} \subset \rho(\mathscr{A}_0)$.*

Proof. Arguing as in Proposition 4.2, there exists $\left(\lambda_n\right)_{n\in\mathbb{N}} \subset \mathcal{R}$ and $\lambda^* \in \mathbb{R}^+$ such that $\lim_{n\to\infty} \lambda_n = \lambda^*$. Moreover, there are sequences $\left(w_n\right)_{n\in\mathbb{N}} \subset D(\mathscr{A}_0)$ and $\left(f_n\right)_{n\in\mathbb{N}} \subset \mathscr{H}_0$ such that

$$\left(i\lambda_n\mathcal{I} - \mathscr{A}_0\right) w_n = f_n \tag{4.43}$$

with

$$\|w_n\|_{\mathscr{H}_0} = 1 \quad \text{and} \quad \|(i\lambda_n\mathcal{I} - \mathscr{A}_0)w_n\|_{\mathscr{H}_0} \xrightarrow[n\to\infty]{} 0. \tag{4.44}$$

From (3.34) and (4.43) it follows that

$$i\lambda_n w_1^n - w_2^n = f_1^n, \tag{4.45}$$

$$i\lambda_n\rho_1 w_2^n + \mathcal{A}_B\mathcal{W}_0^n - \alpha\mathcal{A}_T w_5^n + \tfrac{\alpha\sigma}{\beta} w_5^n = \rho_1 f_2^n, \tag{4.46}$$

$$i\lambda_n w_3^n - w_4^n = f_3^n, \tag{4.47}$$

$$i\lambda_n\rho_0 w_5^n + \alpha\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1)w_2^n + \beta\mathcal{A}_T w_5^n = \rho_0 f_5^n, \tag{4.48}$$

where $\mathcal{W}_0^n \coloneqq \beta_1 w_1^n + \beta_2\mathcal{G}_2\gamma_1 w_3^n + m_2\mathcal{G}_2\gamma_1 w_4^n + \alpha\mathcal{G}_1\gamma_0 w_5^n - \alpha\kappa\mathcal{G}_2\gamma_0 w_5^n$. Notice,

$$\begin{aligned}\|(i\lambda_n\mathcal{I} - \mathscr{A}_0)w_n\|_{\mathscr{H}_0}^2 = \beta_1 \|f_1^n\|_{H^2_\Gamma(\Omega_1)}^2 + \rho_1 \|f_2^n\|_{L^2(\Omega_1)}^2 + \beta_2 \|\nabla f_3^n\|_{L^2(\Omega_2)^2}^2 \\ + \rho_2 \|f_4^n\|_{L^2(\Omega_2)}^2 + \rho_0 \|f_5^n\|_{L^2(\Omega_1)}^2 .\end{aligned}$$

The limit of (4.44) and (4.45)-(4.48) imply that

$$
\begin{aligned}
i\lambda_n w_1^n - w_2^n \to 0 \ \text{ in } \ H^2_\Gamma(\Omega_1), \qquad (4.49)\\
i\lambda_n \rho_1 w_2^n + \mathcal{A}_B \mathcal{W}_0^n - \alpha \mathcal{A}_T w_5^n + \tfrac{\alpha\sigma}{\beta} w_5^n \to 0 \ \text{ in } \ L^2(\Omega_1), \qquad (4.50)\\
i\lambda_n \nabla w_3^n - \nabla w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2), \qquad (4.51)\\
i\lambda_n \rho_0 w_5^n + \alpha \mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1) w_2^n + \beta \mathcal{A}_T w_5^n \to 0 \ \text{ in } \ L^2(\Omega_1). \qquad (4.52)
\end{aligned}
$$

Due to the dissipativity (3.35) of the operator $\mathscr{A}_0$, we have

$$
\begin{aligned}
\operatorname{Re}\left((i\lambda_n \mathcal{I} - \mathscr{A}_0) w_n, w_n\right)_{\mathscr{H}_0} = m_2 \left\|\nabla w_4^n\right\|^2_{L^2(\Omega_2)^2} + \sigma \left\|w_5^n\right\|^2_{L^2(\Omega_1)} \\
+ \beta \left\|\nabla w_5^n\right\|^2_{L^2(\Omega_1)^2} + \beta\kappa \left\|w_5^n\right\|^2_{L^2(\partial\Omega_1)}.
\end{aligned} \qquad (4.53)
$$

In consequence,

$$
w_5^n \to 0 \ \text{ in } \ H^1(\Omega_1). \qquad (4.54)
$$

As $m_2 > 0$, from (4.44) and (4.53) we obtain

$$
\nabla w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2). \qquad (4.55)
$$

The limits $\lambda_n \to \lambda^*$, (4.51) and (4.55) imply

$$
\nabla w_3^n \to 0 \ \text{ in } \ L^2(\Omega_2). \qquad (4.56)
$$

Here, we can make use of (4.39), (4.40) and (4.42). Now, we will work on the following equation

$$
\begin{aligned}
&\left(i\lambda_n \rho_1 w_2^n + \mathcal{A}_B \mathcal{W}_0^n - \alpha \mathcal{A}_T w_5^n + \alpha\sigma\beta^{-1} w_5^n, w_2^n\right)_{L^2(\Omega_1)} \\
&= i\lambda_n \rho_1 \left\|w_2^n\right\|^2_{L^2(\Omega_1)} + \left(\mathcal{A}_B \mathcal{W}_0^n, w_2^n\right)_{L^2(\Omega_1)} + \alpha \left(\Delta w_5^n, w_2^n\right)_{L^2(\Omega_1)}.
\end{aligned} \qquad (4.57)
$$

Using integration by parts, Cauchy–Schwarz inequality, $\partial_\nu w_5^n = -\kappa w_5^n$ on I, the trace theorem and (4.54) we get

$$
\left(\Delta w_5^n, w_2^n\right)_{L^2(\Omega_1)} \to 0. \qquad (4.58)
$$

Due to Proposition 3.8,

$$
\begin{aligned}
\left(\mathcal{A}_B \mathcal{W}_0^n, w_2^n\right)_{L^2(\Omega_1)} = \beta_1 \left(w_1^n, w_2^n\right)_{H^2_\Gamma(\Omega_1)} - \left(\partial_\nu(\beta_2 w_3^n + m_2 w_4^n), w_2^n\right)_{L^2(I)} \\
+ \alpha \left(w_5^n, \partial_\nu w_2^n\right)_{L^2(I)} + \alpha\kappa \left(w_5^n, w_2^n\right)_{L^2(I)}. \qquad (4.59)
\end{aligned}
$$

The trace theorem and (4.54) imply

$$(w_5^n, \partial_\nu w_2^n)_{L^2(I)} \to 0 \quad \text{and} \quad (w_5^n, w_2^n)_{L^2(I)} \to 0. \tag{4.60}$$

Joining (4.39), the left-hand side of (4.44), (4.50) and (4.55)-(4.60) we obtain the following convergence

$$i\lambda_n \rho_1 \|w_2^n\|^2_{L^2(\Omega_1)} + \beta_1 (w_1^n, w_2^n)_{H^2_\Gamma(\Omega_1)} \to 0. \tag{4.61}$$

From (4.49), it follows that

$$i\lambda_n \|w_1^n\|^2_{H^2_\Gamma(\Omega_1)} - (w_2^n, w_1^n)_{H^2_\Gamma(\Omega_1)} \to 0. \tag{4.62}$$

Taking into account (4.54) and adding (4.50) with (4.52), we have

$$i\lambda_n \beta \rho_1 w_2^n + \beta \mathcal{A}_B \mathcal{W}_0^n - \alpha^2 \Delta w_2^n \to 0 \ \text{ in } \ L^2(\Omega_1)$$

and thus

$$\beta \left[i\lambda_n \rho_1 \|w_2^n\|^2_{L^2(\Omega_1)} + \beta_1 (w_1^n, w_2^n)_{H^2_\Gamma(\Omega_1)} \right] - \alpha^2 (\Delta w_2^n, w_2^n)_{L^2(\Omega_1)} \to 0. \tag{4.63}$$

By (4.61) and (4.63),

$$(\Delta w_2^n, w_2^n)_{L^2(\Omega_1)} \to 0$$

and in consequence

$$\|\nabla w_2^n\|^2_{L^2(\Omega_1)^2} - (\partial_\nu w_2^n, w_2^n)_{L^2(I)} \to 0. \tag{4.64}$$

From (4.42) and (4.64), it follows that $\nabla w_2^n \to 0$ in $L^2(\Omega_1)$. By Friedrichs inequality, we obtain

$$w_2^n \to 0 \ \text{ in } \ L^2(\Omega_1). \tag{4.65}$$

From (4.61), (4.62) and (4.65), it follows that

$$w_1^n \to 0 \ \text{ in } \ H^2(\Omega_1). \tag{4.66}$$

Because of (4.40), (4.54), (4.56), (4.65) and (4.66) we have $w_n \to 0$ in $\mathscr{H}_0$. This contradicts the first assertion in (4.44). □

Corollary 4.6. *Let $\eta = m_1 = 0$ and $m_2 > 0$. Then $(\mathscr{T}_0(t))_{t\geq 0}$ is strongly stable in $\mathscr{H}_0$.*

Remark 4.7. The temperature on the plate played a very important role in obtaining the result of Proposition 4.5.

4.2 Exponential stability

The main result of the section is the exponential stability of our problem (1.1)-(1.9) when $\gamma > 0$ and we maintain all dampings on the structure. In consequence, for the solutions $w(t) = \mathscr{T}_\eta(t)w_0$ of this system, we will have that $\|w(t)\|_{\mathscr{H}_\eta} \leq Ce^{-\delta t}\|w_0\|_{\mathscr{H}_\eta}$ for all $t \geq 0$, being C and δ positive constants. We will use the characterization of Theorem 2.41. The proof will be done by contradiction.

Theorem 4.8. *If $\eta > 0$, $m_1 > 0$ and $m_2 > 0$, then the semigroup $(\mathscr{T}_\eta(t))_{t\geq 0}$ generated by $\mathscr{A}_\eta$ is exponentially stable.*

Proof. From Proposition 4.2, we have $i\mathbb{R} \subset \rho(\mathscr{A}_\eta)$. Let us suppose (2.15) is not true. Then, there exists a sequence $(\lambda_n, w_n)_{n\in\mathbb{N}} \subset \mathbb{R} \times D(\mathscr{A}_\eta)$ with $\|w_n\|_{\mathscr{H}_\eta} = 1$ such that

$$\|(i\lambda_n\mathcal{I} - \mathscr{A}_\eta)w_n\|_{\mathscr{H}_\eta} \xrightarrow[n\to\infty]{} 0.$$

As the resolvent of $\mathscr{A}_\eta$ is holomorphic (see Theorem 1 in [124, p. 211]) and therefore bounded on compact subsets of the imaginary axis, we see that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is unbounded, thereby we may assume $\lim_{n\to\infty}\lambda_n = \infty$. If $f_n := (i\lambda_n\mathcal{I} - \mathscr{A}_\eta)w_n$, then (4.6)-(4.18) hold. Note that in this proof we can make use of (4.20)-(4.24). From (4.13),

$$i\nabla w_3^n - \lambda_n^{-1}\nabla w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2).$$

By the right-hand side of (4.18), we obtain

$$\nabla w_3^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{4.67}$$

From (4.11),

$$iw_1^n - \lambda_n^{-1}w_2^n \to 0 \ \text{ in } \ H^2(\Omega_1)$$

and in consequence $(\lambda_n^{-1}w_2)_{n\in\mathbb{N}}$ is bounded in $H^2(\Omega_1)$. On the other hand, using (4.24) we get

$$\lambda_n^{-1}\|\beta_2 w_3^n + m_2 w_4^n\|_{H^2(\Omega_2)} \leq C. \tag{4.68}$$

A combining of (4.14), (4.20)-(4.23) and (4.68) causes that

$$w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{4.69}$$

From (4.11),

$$i\left\|w_1^n\right\|_{H_\Gamma^2(\Omega_1)}^2-\lambda_n^{-1}\left(w_2^n, w_1^n\right)_{H_\Gamma^2(\Omega_1)} \rightarrow 0. \tag{4.70}$$

Now, let us consider the following equality

$$\begin{aligned} & \left\langle i\lambda_n\rho_1\mathcal{M}_\eta w_2^n+\mathcal{W}^n+m_1\mathcal{A}_L w_2^n-\alpha\mathcal{A}_T w_5^n+\alpha\sigma\beta^{-1}w_5^n, w_2^n\right\rangle_{H_{\Gamma,\eta}^{-1}\times H_{\Gamma,\eta}^1} \\ & =i\lambda_n\rho_1\left\langle\mathcal{M}_\eta w_2^n, w_2^n\right\rangle_{H_{\Gamma,\eta}^{-1}\times H_{\Gamma,\eta}^1}+\left\langle\mathcal{W}^n, w_2^n\right\rangle_{H_{\Gamma,\eta}^{-1}\times H_{\Gamma,\eta}^1} \\ & \quad+m_1\left\langle\mathcal{A}_L w_2^n, w_2^n\right\rangle_{H_{\Gamma,\eta}^{-1}\times H_{\Gamma,\eta}^1}+\left\langle-\alpha\mathcal{A}_T w_5^n+\alpha\sigma\beta^{-1}w_5^n, w_2^n\right\rangle_{H_{\Gamma,\eta}^{-1}\times H_{\Gamma,\eta}^1}. \end{aligned} \tag{4.71}$$

By (3.15),

$$\left|\left\langle\mathcal{M}_\eta w_2^n, w_2^n\right\rangle_{H_{\Gamma,\eta}^{-1}(\Omega_1)\times H_{\Gamma,\eta}^1(\Omega_1)}\right|=\left\|w_2^n\right\|_{H_{\Gamma,\eta}^1(\Omega_1)}^2. \tag{4.72}$$

Proceeding as in the proof of Proposition 3.13,

$$\begin{aligned} \left\langle\mathcal{W}^n, w_2^n\right\rangle_{H_{\Gamma,\eta}^{-1}\times H_{\Gamma,\eta}^1}= & \beta_1\left(w_1^n, w_2^n\right)_{H_\Gamma^2(\Omega_1)}-\left(\partial_\nu(\beta_2 w_3^n+m_2 w_4^n), w_2^n\right)_{L^2(I)} \\ & +\alpha\left(w_5^n, \partial_\nu w_2^n\right)_{L^2(I)}+\alpha\kappa\left(w_5^n, w_2^n\right)_{L^2(I)}. \end{aligned} \tag{4.73}$$

Taking into account (4.22) and (4.68),

$$\left|\lambda_n^{-1}\left(\partial_\nu(\beta_2 w_3^n+m_2 w_4^n), w_2^n\right)_{L^2(I)}\right| \leq C\left\|w_2^n\right\|_{H^1(\Omega_1)}. \tag{4.74}$$

By the trace theorem,

$$\left|\lambda_n^{-1}\left(w_5^n, \partial_\nu w_2^n\right)_{L^2(I)}\right| \leq C\lambda_n^{-1}\left\|w_2^n\right\|_{H^2(\Omega_1)}\left\|w_5^n\right\|_{H^1(\Omega_1)} \leq C\left\|w_5^n\right\|_{H^1(\Omega_1)} \tag{4.75}$$

and

$$\left|\lambda_n^{-1}\left(w_5^n, w_2^n\right)_{L^2(I)}\right| \leq C\lambda_n^{-1}\left\|w_2^n\right\|_{H^1(\Omega_1)}\left\|w_5^n\right\|_{H^1(\Omega_1)} \leq C\left\|w_5^n\right\|_{H^1(\Omega_1)}. \tag{4.76}$$

From (3.14),

$$\left|\lambda_n^{-1}\left\langle\mathcal{A}_L w_2^n, w_2^n\right\rangle_{H_{\Gamma,\eta}^{-1}(\Omega_1)\times H_{\Gamma,\eta}^1(\Omega_1)}\right|=\lambda_n^{-1}\left\|\nabla w_2^n\right\|_{L^2(\Omega_1)^2}^2. \tag{4.77}$$

Using integration by parts,

$$\begin{aligned} \left\langle-\mathcal{A}_T w_5^n+\frac{\sigma}{\beta}w_5^n, w_2^n\right\rangle_{H_{\Gamma,\eta}^{-1}(\Omega_1)\times H_{\Gamma,\eta}^1(\Omega_1)} & =\left(\Delta w_5^n, w_2^n\right)_{L^2(\Omega_1)} \\ & =-\left(\nabla w_5^n, \nabla w_2^n\right)_{L^2(\Omega_1)^2}+\left(\partial_\nu w_5^n, w_2^n\right)_{L^2(\partial\Omega_1)}. \end{aligned} \tag{4.78}$$

Applying Cauchy–Schwarz inequality,

$$\left|\lambda_n^{-1}\left(\nabla w_5^n, \nabla w_2^n\right)_{L^2(\Omega_1)^2}\right| \leq \lambda_n^{-1}\left\|\nabla w_2^n\right\|_{L^2(\Omega_1)^2}\left\|\nabla w_5^n\right\|_{L^2(\Omega_1)^2}. \tag{4.79}$$

As $\partial_\nu w_5^n + \kappa w_5^n = 0$ on $\partial\Omega_1$, then

$$\left|\lambda_n^{-1} \left(\partial_\nu w_5^n, w_2^n\right)_{L^2(\partial\Omega_1)}\right| \leq C\lambda_n^{-1} \left\|w_2^n\right\|_{H^1(\Omega_1)} \left\|w_5^n\right\|_{H^1(\Omega_1)}. \tag{4.80}$$

From (4.12), (4.17), (4.23) and (4.71)-(4.80), we get

$$\lambda_n^{-1} \left(w_1^n, w_2^n\right)_{H^2_\Gamma(\Omega_1)} \to 0$$

and therefore

$$w_1^n \to 0 \ \text{ in } \ H^2(\Omega_1), \tag{4.81}$$

see (4.70). Combining (4.17), (4.23), (4.67), (4.69) and (4.81), we can write $\|w_n\|_{\mathscr{H}_\eta} \to 0$. This is a contradiction to $\|w_n\|_{\mathscr{H}_\eta} = 1$. □

Remark 4.9. In the previous proof we took into account the temperature. However, Theorem 4.8 continues to be true when we remove the temperature on the plate. But when we remove the structural damping on the plate and we leave the other conditions of the theorem, we know nothing about exponential stability, even with temperature in the plate.

Remark 4.10. In the next chapter we will see that if $\eta = m_1 = 0$ and $m_2 > 0$, then the semigroup $\left(\mathscr{T}_0(t)\right)_{t\geq 0}$ is exponentially stable.

4.3 Lack of exponential stability

In this section, we will prove that our system (1.1)-(1.9) is not exponentially stable, if we remove the Kelvin–Voigt damping on the membrane and we leave the rotational term on the plate. Our proof is supported by the following theorem.

Theorem 4.11 (cf. [59, Theorem 3.1])**.** *Let H_0 be a closed subspace of a Hilbert space H. Let $\left(T_0(t)\right)_{t\in\mathbb{R}}$ be a unitary group on H_0 and $\left(T(t)\right)_{t\geq 0}$ be a C_0-semigroup over H. If the difference $T(t) - T_0(t) : H_0 \to H$ is a compact operator for all $t > 0$, then $\left(T(t)\right)_{t\geq 0}$ is not exponentially stable.*

We define $\mathscr{H} := \{0\} \times \{0\} \times H_0^1(\Omega_2) \times L^2(\Omega_2) \times \{0\}$ endowed with the norm $\|\widehat{w}\|^2_{\mathscr{H}} := \beta_2 \left\|\nabla \widehat{w}_3\right\|^2_{L^2(\Omega_2)^2} + \rho_2 \left\|\widehat{w}_4\right\|^2_{L^2(\Omega_2)}$. Let us consider the system determined by the wave equation with zero Dirichlet boundary condition

$$\begin{cases} \rho_2 \widehat{v}_{tt} - \beta_2 \Delta \widehat{v} = 0 \ \text{ in } \ \mathbb{R}^+ \times \Omega_2, \\ \widehat{v} = 0 \ \text{ on } \ \mathbb{R}^+ \times I, \\ \widehat{v}(0,\cdot) = \widehat{v}^0, \ \widehat{v}_t(0,\cdot) = \widehat{v}^1 \ \text{ in } \ \Omega_2, \end{cases} \tag{4.82}$$

where the initial data $\widehat{v}^0$ and $\widehat{v}^1$ lie in appropiate Hilbert spaces. We introduce the operator

$$\mathscr{A}_{\mathscr{G}} : D(\mathscr{A}_{\mathscr{G}}) \subset \mathscr{H} \to \mathscr{H} \text{ given by } \mathscr{A}_{\mathscr{G}} \begin{pmatrix} 0 \\ 0 \\ \widehat{w}_3 \\ \widehat{w}_4 \\ 0 \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \\ \widehat{w}_4 \\ \frac{\beta_2}{\rho_2}\Delta \widehat{w}_3 \\ 0 \end{pmatrix}$$

with domain $D(\mathscr{A}_{\mathscr{G}}) := \{0\} \times \{0\} \times H^2(\Omega_2) \cap H_0^1(\Omega_2) \times H_0^1(\Omega_2) \times \{0\}$. Putting $\widehat{w} = (\widehat{w}_j)^\top_{j=1,\dots,5} := (0, 0, \widehat{v}, \widehat{v}_t, 0)^\top$ we have that the system (4.82) can be written as the Cauchy problem

$$\partial_t \widehat{w}(t) = \mathscr{A}_{\mathscr{G}} \widehat{w}(t) \ (t > 0) \ \text{ with } \ \widehat{w}(0) = (0, 0, \widehat{v}^0, \widehat{v}^1, 0)^\top.$$

The operator $i\mathscr{A}_{\mathscr{G}} : D(\mathscr{A}_{\mathscr{G}}) \to \mathscr{H}$ is densely defined due to $H^2(\Omega_2) \cap H_0^1(\Omega_2)$ and $H_0^1(\Omega_2)$ are dense in $H_0^1(\Omega_2)$ and $L^2(\Omega_2)$, respectively, symmetric because $(i\mathscr{A}_{\mathscr{G}}\widehat{w}, \widehat{w})_{\mathscr{H}} \in \mathbb{R}$ for any $\widehat{w} \in D(\mathscr{A}_{\mathscr{G}})$, see [77, p. 534], and surjective (Remark 3.17 allows to conclude that). Proposição 5.122 in [32] implies that $i\mathscr{A}_{\mathscr{G}}$ is self-adjoint and therefore $\mathscr{A}_{\mathscr{G}}$ is skew-adjoint. By Theorem 2.50, we obtain that $\mathscr{A}_{\mathscr{G}}$ generates a unitary group $(\mathscr{U}_{\mathscr{G}}(t))_{t\in\mathbb{R}}$ on $\mathscr{H}$.

Lemma 4.12. *Let $\widehat{v}$ be a sufficiently regular solution of* (4.82) *with $\widehat{v}^0 \in H^1(\Omega_2)$ and $\widehat{v}^1 \in L^2(\Omega_2)$. For $t > 0$ the following estimate holds*

$$\|\partial_\nu \widehat{v}\|^2_{L^2((0,t),L^2(I))} \leq C\big(\|\widehat{v}^1\|^2_{L^2(\Omega_2)} + \|\nabla \widehat{v}^0\|^2_{L^2(\Omega_2)^2}\big) \tag{4.83}$$

with C being a positive constant that depends on $\rho_2, \beta_2, \Omega_2$ and t but independent of the solution and the initial data.

Proof. Let $t > 0$. Multiplying the differential equation in (4.82) by the conjugate of $\widehat{v}_t$, integrating over Ω_2, employing integration by parts and after taking real part yields

$$\frac{1}{2}\frac{d}{ds}\big(\rho_2 \|\widehat{v}_t(s)\|^2_{L^2(\Omega_2)} + \beta_2 \|\nabla \widehat{v}(s)\|^2_{L^2(\Omega_2)^2}\big) = 0 \tag{4.84}$$

for $0 \leq s \leq t$. The initial data in (4.82) and (4.84) imply

$$\int_{\Omega_2} \rho_2 |\widehat{v}_t(s)|^2 + \beta_2 |\nabla \widehat{v}(s)|^2 dx = \int_{\Omega_2} \rho_2 |\widehat{v}^1|^2 + \beta_2 |\nabla \widehat{v}^0|^2 dx. \tag{4.85}$$

Let $\hbar : \overline{\Omega_2} \to \mathbb{R}^2$ be a vector field of class C^1 in $\overline{\Omega_2}$ with $\hbar\big|_I = -\nu$. Using the conditions that satisfy $\hbar$ and $\widehat{v}$ on I, theorem of divergence and the identity $\hbar \cdot \nabla|\widehat{v}_t(s)|^2 = \operatorname{div}(\hbar|\widehat{v}_t(s)|^2) - (\operatorname{div}\hbar)|\widehat{v}_t(s)|^2$, we obtain

$$\begin{aligned}\int_{\Omega_2} \operatorname{Re}(\widehat{v}_{tt}(s)\hbar \cdot \overline{\nabla\widehat{v}(s)})dx = \frac{1}{2}\int_{\Omega_2} (\operatorname{div}\hbar)|\widehat{v}_t(s)|^2 dx \\ + \frac{d}{ds}\int_{\Omega_2} \operatorname{Re}(\widehat{v}_t(s)\hbar \cdot \overline{\nabla\widehat{v}(s)})dx.\end{aligned} \tag{4.86}$$

On the other side, using the convention of summation over repeated indices and integration by parts, we have

$$\begin{aligned}\int_{\Omega_2} \operatorname{Re}(\Delta\widehat{v}(s)\hbar \cdot \overline{\nabla\widehat{v}(s)})dx = -\operatorname{Re}\int_{\Omega_2} (\partial_j\widehat{v}(s))(\partial_j\hbar_k)(\overline{\partial_k\widehat{v}(s)})dx \\ + \frac{1}{2}\int_{\Omega_2} (\operatorname{div}\hbar)|\nabla\widehat{v}(s)|^2 dx + \frac{1}{2}\int_I |\partial_\nu\widehat{v}(s)|^2 dS.\end{aligned} \tag{4.87}$$

Multiplying the differential equation in (4.82) by $\hbar \cdot \overline{\nabla\widehat{v}}$ and using (4.86) together with (4.87), we get

$$\begin{aligned}\int_I |\partial_\nu\widehat{v}(s)|^2 dS = 2\frac{\rho_2}{\beta_2}\frac{d}{ds}\int_{\Omega_2} \operatorname{Re}(\widehat{v}_t(s)\hbar \cdot \overline{\nabla\widehat{v}(s)})dx + \frac{\rho_2}{\beta_2}\int_{\Omega_2} (\operatorname{div}\hbar)|\widehat{v}_t(s)|^2 dx \\ + 2\operatorname{Re}\int_{\Omega_2} (\partial_j\widehat{v}(s))(\partial_j\hbar_k)(\overline{\partial_k\widehat{v}(s)})dx - \int_{\Omega_2} (\operatorname{div}\hbar)|\nabla\widehat{v}(s)|^2 dx.\end{aligned}$$

Note that $|\hbar|$, $|\operatorname{div}\hbar|$ and $|\partial_j\hbar_k|$ are bounded scalar fields on $\overline{\Omega_2}$. Integrating above with respect to the variable s from 0 to t, and then using the inequalities of Cauchy–Schwarz and Young together with equality (4.85), we deduce (4.83). □

We next state and prove a result that will be useful in the proof of the next theorem. Moreover, we take the opportunity to introduce Remark 4.14 which will be important for the next section.

Proposition 4.13. *If $\eta > 0$, $m_1 \geq 0$ and $m_2 = 0$, then the operator $\mathscr{A}_\eta$ has compact resolvent.*

Proof. Let $\eta > 0$, $m_1 \geq 0$ and $m_2 = 0$. By Remark 3.19 and Lemma 3.5 in [14], we have the continuous embedding

$$D(\mathscr{A}_\eta) \hookrightarrow H^3(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2) \times H^2(\Omega_1).$$

Reasoning as in Proposition 3.9 from [14], we get that id : $D(\mathscr{A}_\eta) \to \mathscr{H}_\eta$ is compact. Now, the statement is a consequence of Proposition 5.8 in [48, p. 107]. □

Remark 4.14. Thanks to the previous proposition and Corollary 1.15 in [48, p. 162], we have that the spectrum $\sigma(\mathscr{A}_\eta)$ is composed only by eigenvalues when $\eta > 0$, $m_1 \geq 0$ and $m_2 = 0$.

Theorem 4.15. *For $\eta > 0$, $m_1 \geq 0$ and $m_2 = 0$, we have that the system* (1.1)-(1.9) *does not have exponential decay.*

Proof. It is very simple to see that $\mathscr{H}$ is a closed subspace of $\mathscr{H}_\eta$. We will show that $\mathscr{T}_\eta(t) - \mathscr{U}_\mathscr{G}(t) \colon \mathscr{H} \to \mathscr{H}_\eta$ $(t > 0)$ is compact. It is enough to prove that $\mathscr{T}_\eta(t) - \mathscr{U}_\mathscr{G}(t) : \mathscr{D} \to \mathscr{H}_\eta$ is compact because $\mathscr{D} := \{0\}^2 \times [\mathscr{D}(\Omega_2)]^2 \times \{0\}$ is a dense subspace of $\mathscr{H}$. For $w_0 \in \mathscr{D}$ and $t \geq 0$, we set

$$\mathscr{E}(t) := \frac{1}{2}\|\mathscr{T}_\eta(t)w_0 - \mathscr{U}_\mathscr{G}(t)w_0\|^2_{\mathscr{H}_\eta}.$$

Note that $D(\mathscr{A}_\eta) \ni w(t) := \mathscr{T}_\eta(t)w_0$ and $D(\mathscr{A}_\mathscr{G}) \ni \widehat{w}(t) := \mathscr{U}_\mathscr{G}(t)w_0$ because $w_0 \in D(\mathscr{A}_\eta) \cap D(\mathscr{A}_\mathscr{G})$, see Theorem 2.27. By (3.33), we can write

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(t) = \operatorname{Re}\left(\mathscr{A}_\eta w(t), w(t)\right)_{\mathscr{H}_\eta} + \operatorname{Re}(\mathscr{A}_\mathscr{G}\widehat{w}(t), \widehat{w}(t))_{\mathscr{H}_\eta} \\ - \operatorname{Re}\left(\mathscr{A}_\eta w(t), \widehat{w}(t)\right)_{\mathscr{H}_\eta} - \operatorname{Re}(\mathscr{A}_\mathscr{G}\widehat{w}(t), w(t))_{\mathscr{H}_\eta}.\end{aligned} \tag{4.88}$$

From (3.35) we know $\operatorname{Re}\left(\mathscr{A}_\eta w(t), w(t)\right)_{\mathscr{H}_\eta} \leq 0$. Using integration by parts and taking into account that $\widehat{w}_4(t) = 0$ on I, we get the following expression

$$(\mathscr{A}_\mathscr{G}\widehat{w}(t), \widehat{w}(t))_{\mathscr{H}_\eta} = i2\beta_2 \operatorname{Im}\left(\nabla\widehat{w}_4(t), \nabla\widehat{w}_3(t)\right)_{L^2(\Omega_2)^2}$$

and thus $\operatorname{Re}(\mathscr{A}_\mathscr{G}\widehat{w}(t), \widehat{w}(t))_{\mathscr{H}_\eta} = 0$. From the definition of the operator $\mathscr{A}_\eta$, see Subsection 3.1.2, it is immediate that

$$\left(\mathscr{A}_\eta w(t), \widehat{w}(t)\right)_{\mathscr{H}_\eta} = \beta_2(\nabla w_4(t), \nabla\widehat{w}_3(t))_{L^2(\Omega_2)^2} - \beta_2(\nabla w_3(t), \nabla\widehat{w}_4(t))_{L^2(\Omega_2)^2}.$$

Employing integration by parts, we obtain

$$\begin{aligned}(\mathscr{A}_\mathscr{G}\widehat{w}(t), w(t))_{\mathscr{H}_\eta} = \beta_2(\nabla\widehat{w}_4(t), \nabla w_3(t))_{L^2(\Omega_2)^2} \\ - \beta_2(\nabla\widehat{w}_3(t), \nabla w_4(t))_{L^2(\Omega_2)^2} - \beta_2(\partial_\nu\widehat{w}_3(t), w_4(t))_{L^2(I)}.\end{aligned}$$

The last two equalities produce the following expression

$$-\left(\mathscr{A}_\eta w(t), \widehat{w}(t)\right)_{\mathscr{H}_\eta} - (\mathscr{A}_{\mathscr{G}}\widehat{w}(t), w(t))_{\mathscr{H}_\eta} = i2\beta_2 \operatorname{Im}(\nabla \widehat{w}_3(t), \nabla w_4(t))_{L^2(\Omega_2)^2}$$
$$+ i2\beta_2 \operatorname{Im}(\nabla w_3(t), \nabla \widehat{w}_4(t))_{L^2(\Omega_2)^2} + \beta_2(\partial_\nu \widehat{w}_3(t), w_4(t))_{L^2(I)}.$$

Taking real part in the last equality, inserting this into (4.88) and integrating over $(0,t)$, we obtain the estimate

$$\mathscr{E}(t) \leq \beta_2 \operatorname{Re}\left(\partial_\nu \widehat{w}_3, w_2\right)_{L^2((0,t),L^2(I))}. \tag{4.89}$$

The equality $\mathscr{E}(0) = 0$ was used and the fact that $w_2(t) = w_4(t)$ on I.

Let $(w_0^k)_{k\in\mathbb{N}} \subset \mathscr{D}$ be a bounded sequence in $\mathscr{H}$. We must prove that $(\mathscr{T}_\eta(t)w_0^k - \mathscr{U}_{\mathscr{G}}(t)w_0^k)_{k\in\mathbb{N}}$ possesses a convergent subsequence in $\mathscr{H}_\eta$. This would show the compactness of $\mathscr{T}_\eta(t) - \mathscr{U}_{\mathscr{G}}(t)\colon \mathscr{D} \to \mathscr{H}_\eta$. We set $w^k(t) := \mathscr{T}_\eta(t)w_0^k$ and $\widehat{w}^k(t) := \mathscr{U}_{\mathscr{G}}(t)w_0^k$. Due to Lemma 4.12,

$$\|\partial_\nu \widehat{w}_3^k\|_{L^2((0,t),L^2(I))} \leq C\|\widehat{w}^k(0)\|_{\mathscr{H}} = C\|w_0^k\|_{\mathscr{H}} \leq C \quad \text{for all} \quad k \in \mathbb{N}. \tag{4.90}$$

Let $\phi \in \mathscr{Y} := H_0^4(\Omega_1) \times H_0^2(\Omega_1) \times H_0^2(\Omega_2) \times H_0^1(\Omega_2) \times H_0^2(\Omega_1)$. Taking $w \in D(\mathscr{A}_\eta)$ and reasoning as in the proof of Proposition 3.13, we get

$$\begin{aligned}(\mathscr{A}_\eta w, \phi)_{\mathscr{H}_\eta} &= \beta_1 (w_2, \phi_1)_{H^2_\Gamma(\Omega_1)} - \beta_1 (w_1, \phi_2)_{H^2_\Gamma(\Omega_1)} - m_1 (\nabla w_2, \nabla \phi_2)_{L^2(\Omega_1)^2} \\ &\quad - \alpha (w_5, \Delta\phi_2)_{L^2(\Omega_1)} - \beta_2 (w_4, \Delta\phi_3)_{L^2(\Omega_2)} - \beta_2 (\nabla w_3, \nabla\phi_4)_{L^2(\Omega_2)^2} \\ &\quad - \alpha (\nabla w_2, \nabla\phi_5)_{L^2(\Omega_1)^2} + \beta_1 (w_5, \Delta\phi_5)_{L^2(\Omega_1)} - \sigma (w_5, \phi_5)_{L^2(\Omega_1)}.\end{aligned}$$

Let $\varpi \in H^2(\Omega_1)$ with $\varpi = 0$ on $\partial\Omega_1$. We have that $\partial_\tau^k \varpi = 0$ on $\partial\Omega_1$ for any $k \in \mathbb{N}$. Because of Propositions 3C.7 and 3C.11 in [87], we obtain

$$\mathscr{B}_1\varpi = \partial_\nu^2\varpi + \mu(\operatorname{div}\nu)\partial_\nu\varpi, \tag{4.91}$$
$$\mathscr{B}_2\varpi = \partial_\nu^3\varpi + (2-\mu)\partial_\tau^2\partial_\nu\varpi + [\partial_\nu(\operatorname{div}\nu)]\partial_\nu\varpi + (\operatorname{div}\nu)\partial_\nu^2\varpi. \tag{4.92}$$

Indeed: (4.91) is immediate. Thanks to the part (ii) of Corollary 3C.10 in [87], we obtain $\partial_\nu\partial_\tau\varpi = \partial_\tau\partial_\nu\varpi$ and $\partial_\nu\partial_\tau^2\varpi = \partial_\tau\partial_\nu\partial_\tau\varpi$. Introducing these last equalities in (3C.69) from [87], we get (4.92).

As $\phi_1 \in H_0^4(\Omega_1)$, then $\partial_\nu^j\phi_1 = 0$ on $\partial\Omega_1$ for $j = 0,1,2,3$ (see Theorem 2.8). From (4.91) and (4.92), it follows that $\mathscr{B}_1\phi_1 = \mathscr{B}_2\phi_1 = 0$ on $\partial\Omega_1$. So, Proposition 2.17 implies

$$(\phi_1, w_2)_{H^2_\Gamma(\Omega_1)} = \left(\Delta^2\phi_1, w_2\right)_{L^2(\Omega_1)}.$$

Therefore,

$$\left| (\mathscr{A}_\eta w, \phi)_{\mathscr{H}_\eta} \right| \le C_\phi \left\| w \right\|_{\mathscr{H}_\eta},$$

where $C_\phi := C \left\| \phi \right\|_{\mathscr{Y}}$. In consequence, the linear application $\mathbb{A} : D(\mathscr{A}_\eta) \to \mathbb{C}$ defined by $\mathbb{A}w := (\mathscr{A}_\eta w, \phi)_{\mathscr{H}_\eta}$ is continuous. As $D(\mathscr{A}_\eta)$ is dense in $\mathscr{H}_\eta$ there exists a unique extension linear and continuous of $\mathbb{A}$ to $\mathscr{H}_\eta$, which is denoted by $\widetilde{\mathbb{A}} : \mathscr{H}_\eta \to \mathbb{C}$. Note that $\phi \in \mathscr{H}_\eta'$, and for $\widetilde{w} \in D(\mathscr{A}_\eta)$ we have

$$\langle \widetilde{w}, \widetilde{\mathbb{A}} \rangle_{\mathscr{H}_\eta \times \mathscr{H}_\eta'} = \widetilde{\mathbb{A}} \widetilde{w} = \mathbb{A} \widetilde{w} = (\mathscr{A}_\eta \widetilde{w}, \phi)_{\mathscr{H}_\eta} = \langle \mathscr{A}_\eta \widetilde{w}, \phi \rangle_{\mathscr{H}_\eta \times \mathscr{H}_\eta'}.$$

Thus, $\phi \in D(\mathscr{A}_\eta')$. With this it is shown that $\mathscr{Y} \subset D(\mathscr{A}_\eta')$. We will now use the interpolation-extrapolation scales theory presented in Section 2.5. Let $X_0 := \mathscr{H}_\eta$, $X_{-1} := (X_0, \|\mathscr{A}_\eta^{-1} \cdot \|_{\mathscr{H}_\eta})^\sim$ and $X_{-\widetilde{\alpha}} := [X_0, X_{-1}]_{\widetilde{\alpha}}$ for $\widetilde{\alpha} \in (0,1)$, where $[\cdot,\cdot]_{\widetilde{\alpha}}$ stands for the complex interpolation functor. We have that

$$X_0 \overset{c}{\hookrightarrow} X_{-\widetilde{\alpha}} \hookrightarrow X_{-1}$$

for $\widetilde{\alpha} \in (0,1)$. The embedding above is compact because $\mathscr{A}_\eta^{-1} : \mathscr{H}_\eta \to \mathscr{H}_\eta$ is a compact operator, see Proposition 4.13. As $X_{-1} = [D(\mathscr{A}_\eta')]'$, we can insure that the embedding

$$X_{-1} \hookrightarrow H^{-4}(\Omega_1) \times H^{-2}(\Omega_1) \times H^{-2}(\Omega_2) \times H^{-1}(\Omega_2) \times H^{-2}(\Omega_1) \tag{4.93}$$

holds and since from the definition of $\mathscr{H}_\eta$ we have

$$X_0 \hookrightarrow H^2(\Omega_1) \times H^1(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2) \times L^2(\Omega_1), \tag{4.94}$$

then (4.93), (4.94) and Theorem 2.4 in [89] imply that

$$X_{-\widetilde{\alpha}} \hookrightarrow H^{2-5\widetilde{\alpha}}(\Omega_1) \times H^{1-3\widetilde{\alpha}}(\Omega_1) \times H^{1-3\widetilde{\alpha}}(\Omega_2) \times H^{-\widetilde{\alpha}}(\Omega_2) \times H^{-2\widetilde{\alpha}}(\Omega_1). \tag{4.95}$$

Let $t > 0$ be fixed. For $s \in [0,t]$ we have that $\|\mathscr{T}_\eta(s)\|_{\mathcal{L}(\mathscr{H}_\eta)} \le 1$. Therefore,

$$\|w^k\|^2_{L^2((0,t),\mathscr{H}_\eta)} = \int_0^t \|w^k(s)\|^2_{\mathscr{H}_\eta} ds \le \int_0^t \|\mathscr{T}_\eta(s)\|^2_{\mathcal{L}(\mathscr{H}_\eta)} \|w_0^k\|^2_{\mathscr{H}_\eta} ds \le C.$$

Thus, the sequence $(w^k)_{k \in \mathbb{N}}$ is bounded in the Hilbert space $L^2((0,t),\mathscr{H}_\eta)$. Now, we use $\partial_t w^k = \mathscr{A}_\eta w^k$ to get the following

$$\sup_{s \in [0,t]} \|\partial_s w^k(s)\|_{X_{-1}} = \sup_{s \in [0,t]} \|\mathscr{A}_\eta w^k(s)\|_{X_{-1}} = \sup_{s \in [0,t]} \|w^k(s)\|_{\mathscr{H}_\eta} \le C.$$

So, $(\partial_t w^k)_{k\in\mathbb{N}}$ is bounded in $L^2((0,t),X_{-1})$. We have proven that the sequence $(w^k)_{k\in\mathbb{N}}$ is bounded in the space $\mathcal{W}_{2,2}((0,t);X_0,X_{-1})$. By Aubin–Lions–Simon lemma, see Theorem 2.2, yields that there exists a subsequence $(w^{k_j})_{j\in\mathbb{N}}$ of $(w^k)_{k\in\mathbb{N}}$, which is convergent in $L^2((0,t),X_{-\widetilde{\alpha}})$. From (4.95) we see that the second component $(w_2^{k_j})_{j\in\mathbb{N}}$ converges in $L^2((0,t),H^{1-3\widetilde{\alpha}}(\Omega_1))$. Choosing $\widetilde{\alpha}<\frac{1}{6}$ and taking the trace on I we obtain convergence in $L^2((0,t),H^{\frac{1}{2}-3\widetilde{\alpha}}(I))$ and therefore in $L^2((0,t),L^2(I))$ for the subsequence $(w_2^{k_j})_{j\in\mathbb{N}}$ of $(w_2^k\big|_I)_{k\in\mathbb{N}}$.

Note $((\mathscr{T}_\eta(t)-\mathscr{U}_{\mathscr{G}}(t))w_0^{k_j})_{j\in\mathbb{N}}$ is a subsequence of $(\mathscr{T}_\eta(t)w_0^k-\mathscr{U}_{\mathscr{G}}(t)w_0^k)_{k\in\mathbb{N}}$. We write $a_{nm}\coloneqq a_n-a_m$ for any sequence $(a_n)_{n\in\mathbb{N}}$. With this notation, we have

$$w^{k_ik_j}(t)-\widehat{w}^{k_ik_j}(t)=\mathscr{T}_\eta(t)(w_0^{k_i}-w_0^{k_j})-\mathscr{U}_{\mathscr{G}}(t)(w_0^{k_i}-w_0^{k_j}). \tag{4.96}$$

For $i,j\in\mathbb{N}$ and $t\geq 0$, we now consider

$$\mathscr{E}^{ij}(t)\coloneqq\frac{1}{2}\|w^{k_ik_j}(t)-\widehat{w}^{k_ik_j}(t)\|^2_{\mathscr{H}_\eta}. \tag{4.97}$$

By (4.89) and (4.90), we compute that

$$\begin{aligned}\mathscr{E}^{ij}(t)&\leq\beta_2|(\partial_\nu\widehat{w}_3^{k_ik_j},w_2^{k_ik_j})_{L^2((0,t),L^2(I))}|\\&\leq\beta_2\|\partial_\nu\widehat{w}_3^{k_ik_j}\|_{L^2((0,t),L^2(I))}\|w_2^{k_ik_j}\|_{L^2((0,t),L^2(I))}\\&\leq C\|w_2^{k_i}-w_2^{k_j}\|_{L^2((0,t),L^2(I))}\to 0\quad(\text{as } i,j\to\infty).\end{aligned} \tag{4.98}$$

Thanks to (4.96)-(4.98), we get that $((\mathscr{T}_\eta(t)-\mathscr{U}_{\mathscr{G}}(t))w_0^{k_j})_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{H}_\eta$ and therefore converges in this Hilbert space. Accordingly, $\mathscr{T}_\eta(t)-\mathscr{U}_{\mathscr{G}}(t)$ is a compact operator from $\mathscr{H}$ to $\mathscr{H}_\eta$ for any $t>0$. In consequence, Theorem 4.11 leads to the conclusion of the present theorem. □

4.4 Polynomial stability

In this section we will prove that our system (1.1)-(1.9) is polynomially stable under certain geometric condition, imposed on the domain Ω_2, when we do not have the Kelvin–Voigt damping on the membrane, but the inertial term and structural damping are present. For the proof, we will use the characterization of Theorem 2.44.

In order to establish the polynomial stability, we need the following usual geometrical condition: There exists a point $x_0 \in \mathbb{R}^2$ such that

$$q(x) \cdot \nu(x) \leq 0 \quad \text{for} \quad x \in I, \tag{4.99}$$

where q is the vector field defined by $q(x) \coloneqq x - x_0$ for $x \in \overline{\Omega_2}$.

Theorem 4.16. *Let $\eta > 0$, $m_1 > 0$ and $m_2 = 0$ and assume that* (4.99) *is satiesfied. Then, the semigroup $(\mathscr{T}_\eta(t))_{t\geq 0}$ decays polynomially of order at least* $1/25$. *Furthermore, if $w_0 \in D(\mathscr{A}_\eta^k)$, $k \in \mathbb{N}$, then there exists a constant $C_k > 0$ such that*

$$\|\mathscr{T}_\eta(t)w_0\|_{\mathscr{H}_\eta} \leq \frac{C_k}{t^{k/25}} |w_0|_{D(\mathscr{A}_\eta^k)} \tag{4.100}$$

for all $t > 0$.

Proof. First we will prove that the intersection $i\mathbb{R} \cap \sigma(\mathscr{A}_\eta)$ is empty, then we will look for an estimate of the type (2.18) and this will allow us to obtain an inequality like (2.19). Finally, a standard argument leads to (4.100).

Since $0 \in \rho(\mathscr{A}_\eta)$ there is $\lambda_0 > 0$ such that $\{i\zeta : -\lambda_0 < \zeta < \lambda_0\} \subset \rho(\mathscr{A}_\eta)$. Let $\lambda \in \mathbb{R}$ with $|\lambda| \geq \lambda_0$. To prove that $i\mathbb{R} \cap \sigma(\mathscr{A}_\eta) = \emptyset$ it is sufficient to show that $i\lambda \notin \sigma(\mathscr{A}_\eta)$. Taking $w = (w_1, w_2, w_3, w_4, w_5)^\top \in D(\mathscr{A}_\eta^2)$ and setting

$$f \coloneqq (i\lambda\mathcal{I} - \mathscr{A}_\eta)w, \tag{4.101}$$

we derive that $f \in D(\mathscr{A}_\eta)$ and further

$$i\lambda w_1 - w_2 = f_1 \quad \text{in } H^2_\Gamma(\Omega_1), \tag{4.102}$$

$$i\lambda\rho_1\mathcal{M}_\eta w_2 + \mathcal{W} + m_1\mathcal{A}_L w_2 + \alpha\Delta w_5 = \rho_1\mathcal{M}_\eta f_2 \quad \text{in } H^{-1}_{\Gamma,\eta}(\Omega_1), \tag{4.103}$$

$$i\lambda w_3 - w_4 = f_3 \quad \text{in } H^1(\Omega_2), \tag{4.104}$$

$$i\lambda\rho_2 w_4 - \beta_2\Delta w_3 = \rho_2 f_4 \quad \text{in } L^2(\Omega_2), \tag{4.105}$$

$$i\lambda\rho_0 w_5 + \alpha\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1)w_2 + \beta\mathcal{A}_T w_5 = \rho_0 f_5 \quad \text{in } L^2(\Omega_1). \tag{4.106}$$

Here we set $f \coloneqq (f_1, f_2, f_3, f_4, f_5)^\top$. From (4.101), we get the following

$$-\operatorname{Re}(\mathscr{A}_\eta w, w)_{\mathscr{H}_\eta} \leq \|f\|_{\mathscr{H}_\eta} \|w\|_{\mathscr{H}_\eta}. \tag{4.107}$$

By (3.35) and (4.107),

$$\|w_2\|_{H^1_{\Gamma,\eta}(\Omega_1)} \leq C\|f\|^{1/2}_{\mathscr{H}_\eta} \|w\|^{1/2}_{\mathscr{H}_\eta} \tag{4.108}$$

and

$$\|w_5\|_{H^1(\Omega_1)} \leq C \|f\|_{\mathscr{H}_\eta}^{1/2} \|w\|_{\mathscr{H}_\eta}^{1/2}. \tag{4.109}$$

Replacing (4.102) in (4.103) and (4.106), we obtain

$$\begin{aligned} &-\lambda^2 \rho_1 \mathcal{M}_\eta w_1 + \mathcal{W} + i\lambda m_1 \mathcal{A}_L w_1 - \alpha \mathcal{A}_T w_5 + \alpha\beta^{-1}\sigma w_5 \\ &= i\lambda\rho_1 \mathcal{M}_\eta f_1 + m_1 \mathcal{A}_L f_1 + \rho_1 \mathcal{M}_\eta f_2 \end{aligned} \tag{4.110}$$

and

$$i\lambda\rho_0 w_5 - i\lambda\alpha\Delta w_1 + \beta\mathcal{A}_T w_5 = -\alpha\Delta f_1 + \rho_0 f_5. \tag{4.111}$$

Inserting (4.104) into (4.105) leads to

$$-\lambda^2\rho_2 w_3 - \beta_2\Delta w_3 = i\lambda\rho_2 f_3 + \rho_2 f_4. \tag{4.112}$$

Multiplying (4.110) by w_1, under the duality between $H^{-1}_{\Gamma,\eta}(\Omega_1)$ and $H^1_{\Gamma,\eta}(\Omega_1)$, considering the identities (3.12)-(3.15) and (3.21), using integration by parts and knowing that $\partial_\nu w_5 + \kappa w_5 = 0$ on I, we get

$$\begin{aligned} &-\lambda^2\rho_1 \|w_1\|^2_{H^1_{\Gamma,\eta}(\Omega_1)} + \beta_1 \|w_1\|^2_{H^2_\Gamma(\Omega_1)} - \beta_2 (\partial_\nu w_3, w_1)_{L^2(I)} \\ &+ \alpha (w_5, \partial_\nu w_1)_{L^2(I)} + i\lambda m_1 \|\nabla w_1\|^2_{L^2(\Omega_1)^2} - \alpha (\nabla w_5, \nabla w_1)_{L^2(\Omega_1)^2} \\ &= i\lambda\rho_1 (f_1, w_1)_{H^1_{\Gamma,\eta}(\Omega_1)} + m_1 (\nabla f_1, \nabla w_1)_{L^2(\Omega_1)^2} + \rho_1 (f_2, w_1)_{H^1_{\Gamma,\eta}(\Omega_1)}. \end{aligned} \tag{4.113}$$

Multiplying (4.111) by w_5, using integration by parts, (3.16) and (3.18) we get the following equality

$$\begin{aligned} &i\lambda\rho_0 \|w_5\|^2_{L^2(\Omega_1)} + i\lambda\alpha (\nabla w_1, \nabla w_5)_{L^2(\Omega_1)^2} \\ &- i\lambda\alpha (\partial_\nu w_1, w_5)_{L^2(I)} + \beta \|w_5\|^2_{D(\mathcal{A}_T^{1/2})} = \alpha (\nabla f_1, \nabla w_5)_{L^2(\Omega_1)^2} \\ &- \alpha (\partial_\nu f_1, w_5)_{L^2(I)} + \rho_0 (f_5, w_5)_{L^2(\Omega_1)}. \end{aligned} \tag{4.114}$$

Multiplying (4.112) by w_3 and using integration by parts, we obtain

$$\begin{aligned} &-\lambda^2\rho_2 \|w_3\|^2_{L^2(\Omega_2)} + \beta_2 \|\nabla w_3\|^2_{L^2(\Omega_2)^2} + \beta_2 (\partial_\nu w_3, w_3)_{L^2(I)} \\ &= i\lambda\rho_2 (f_3, w_3)_{L^2(\Omega_2)} + \rho_2 (f_4, w_3)_{L^2(\Omega_2)}. \end{aligned} \tag{4.115}$$

Multiplying (4.114) by $-i\lambda^{-1}$, adding together with (4.113) and (4.115), and taking into account that $w_1 = w_3$ on I, we see

$$-\lambda^2\rho_1 \|w_1\|^2_{H^1_{\Gamma,\eta}(\Omega_1)} + \beta_1 \|w_1\|^2_{H^2_\Gamma(\Omega_1)} + i2\alpha \operatorname{Im} (w_5, \partial_\nu w_1)_{L^2(I)}$$

$$
\begin{aligned}
&+ i\lambda m_1 \|\nabla w_1\|^2_{L^2(\Omega_1)^2} + i2\alpha \operatorname{Im}(\nabla w_1, \nabla w_5)_{L^2(\Omega_1)^2} + \rho_0 \|w_5\|^2_{L^2(\Omega_1)} \\
&- i\lambda^{-1}\beta \|w_5\|^2_{D(\mathcal{A}_T^{1/2})} - \lambda^2\rho_2 \|w_3\|^2_{L^2(\Omega_2)} + \beta_2 \|\nabla w_3\|^2_{L^2(\Omega_2)^2} \\
&= i\lambda\rho_1 (f_1, w_1)_{H^1_{\Gamma,\eta}(\Omega_1)} + m_1 (\nabla f_1, \nabla w_1)_{L^2(\Omega_1)^2} + \rho_1 (f_2, w_1)_{H^1_{\Gamma,\eta}(\Omega_1)} \\
&- i\lambda^{-1}\alpha (\nabla f_1, \nabla w_5)_{L^2(\Omega_1)^2} + i\lambda^{-1}\alpha (\partial_\nu f_1, w_5)_{L^2(I)} - i\lambda^{-1}\rho_0 (f_5, w_5)_{L^2(\Omega_1)} \\
&+ i\lambda\rho_2 (f_3, w_3)_{L^2(\Omega_2)} + \rho_2 (f_4, w_3)_{L^2(\Omega_2)}.
\end{aligned}
$$

Because of Friedrichs's inequality there exists a constant $C > 0$ such that $\|\Phi_3\|_{L^2(\Omega_2)} \leq C \|\Phi\|_{\mathscr{H}_\eta}$ for any $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5) \in \mathscr{H}_\eta$. Taking real part above, using (4.109) and remembering that $1 \leq \lambda_0^{-1}|\lambda|$ we achieve

$$
\begin{aligned}
&\beta_1 \|w_1\|^2_{H^2_\Gamma(\Omega_1)} + \beta_2 \|\nabla w_3\|^2_{L^2(\Omega_2)^2} + \rho_0 \|w_5\|^2_{L^2(\Omega_1)} \leq \lambda^2\rho_1 \|w_1\|^2_{H^1_{\Gamma,\eta}(\Omega_1)} \\
&+ \lambda^2\rho_2 \|w_3\|^2_{L^2(\Omega_2)} + C\big(|\lambda| \|f\|_{\mathscr{H}_\eta} \|w\|_{\mathscr{H}_\eta} + \|f\|^{3/2}_{\mathscr{H}_\eta} \|w\|^{1/2}_{\mathscr{H}_\eta}\big).
\end{aligned}
\tag{4.116}
$$

By (4.104), we find

$$
\rho_2 \|w_4\|^2_{L^2(\Omega_2)} = \rho_2 \|i\lambda w_3 - f_3\|^2_{L^2(\Omega_2)} \leq C\big(\lambda^2 \|w_3\|^2_{L^2(\Omega_2)} + \|f\|^2_{\mathscr{H}_\eta}\big). \tag{4.117}
$$

By (4.102) and (4.108), we derive

$$
\lambda^2 \|w_1\|^2_{H^1_{\Gamma,\eta}(\Omega_1)} = \|f_1 + w_2\|^2_{H^1_{\Gamma,\eta}(\Omega_1)} \leq C\big(\|f\|^2_{\mathscr{H}_\eta} + \|f\|_{\mathscr{H}_\eta} \|w\|_{\mathscr{H}_\eta}\big). \tag{4.118}
$$

Combining (4.108) and (4.116)-(4.118), we obtain

$$
\|w\|^2_{\mathscr{H}_\eta} \leq C\big(\lambda^2 \|w_3\|^2_{L^2(\Omega_2)} + |\lambda| \|f\|_{\mathscr{H}_\eta} \|w\|_{\mathscr{H}_\eta} + \|f\|^{3/2}_{\mathscr{H}_\eta} \|w\|^{1/2}_{\mathscr{H}_\eta} + \|f\|^2_{\mathscr{H}_\eta}\big). \tag{4.119}
$$

Now, we will establish an estimate for $\lambda^2 \|w_3\|^2_{L^2(\Omega_2)}$. Thanks to the Rellich identity, see Theorem 2.16, choosing there $h = q$, we can write

$$
\operatorname{Re}\int_{\Omega_2} \Delta w_3(q\cdot\nabla\overline{w}_3)dx = -\operatorname{Re}\int_I \partial_\nu w_3(q\cdot\nabla\overline{w}_3) - \frac{1}{2}(q\cdot\nu)|\nabla w_3|^2\, dS. \tag{4.120}
$$

Multiplying (4.112) by the scalar field $q \cdot \nabla\overline{w}_3$, integrating, taking real part and using (4.120) we get

$$
\begin{aligned}
&- \lambda^2\rho_2 \operatorname{Re}\int_{\Omega_2} w_3(q\cdot\nabla\overline{w}_3)dx + \beta_2 \operatorname{Re}\int_I \partial_\nu w_3(q\cdot\nabla\overline{w}_3)dS \\
&- \frac{\beta_2}{2}\int_I (q\cdot\nu)|\nabla w_3|^2 dS = \operatorname{Re}\int_{\Omega_2} (i\lambda\rho_2 f_3 + \rho_2 f_4)(q\cdot\nabla\overline{w}_3)dx.
\end{aligned}
\tag{4.121}
$$

Making use of identity $q \cdot \nabla w_3 = \operatorname{div}(q w_3) - 2w_3$ and employing integrating by parts, it holds

$$\operatorname{Re} \int_{\Omega_2} w_3 (q \cdot \nabla \overline{w}_3) dx = - \|w_3\|^2_{L^2(\Omega_2)} - \frac{1}{2} \int_I (q \cdot \nu) |w_3|^2 dS. \tag{4.122}$$

By (4.121) and (4.122),

$$\begin{aligned}\lambda^2 \rho_2 \|w_3\|^2_{L^2(\Omega_2)} = {} & \operatorname{Re} \int_{\Omega_2} (i\lambda \rho_2 f_3 + \rho_2 f_4)(q \cdot \nabla \overline{w}_3) dx - \frac{1}{2} \lambda^2 \rho_2 \int_I (q \cdot \nu) |w_3|^2 dS \\ & - \beta_2 \operatorname{Re} \int_I \partial_\nu w_3 (q \cdot \nabla \overline{w}_3) dS + \frac{1}{2} \beta_2 \int_I (q \cdot \nu) |\nabla w_3|^2 dS.\end{aligned}$$

Keeping in mind that $q \cdot \nu \leq 0$ on I, $w_1 = w_3$ on I and (4.118), we obtain

$$\lambda^2 \|w_3\|^2_{L^2(\Omega_2)} \leq C \Big(|\lambda| \|f\|_{\mathscr{H}_\eta} \|w\|_{\mathscr{H}_\eta} + \|f\|^2_{\mathscr{H}_\eta} + \int_I |\beta_2 \partial_\nu w_3 (q \cdot \nabla \overline{w}_3)| dS \Big). \tag{4.123}$$

Next we are going to estimate the last integral denoted this by $\mathscr{I}$. Let $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3, \widetilde{f}_4, \widetilde{f}_5)$ and $\widetilde{f} := \mathscr{A}_\eta w$. Since $w \in D(\mathscr{A}_\eta^2)$, we have that $\widetilde{f}_2 \in H^2_\Gamma(\Omega_1)$ and also

$$\beta_1 w_1 - m_1 \mathcal{G}_2 \gamma_1 w_2 + \beta_2 \mathcal{G}_2 \gamma_1 w_3 + \alpha(\mathcal{G}_1 - \kappa \mathcal{G}_2)\gamma_0 w_5 - \gamma \mathcal{G}_2 \gamma_1 \widetilde{f}_2 \in D(\mathcal{A}_B), \tag{4.124}$$

see the proof of Theorem 3.18. In consequence,

$$\beta_2 \partial_\nu w_3 = \gamma \partial_\nu \widetilde{f}_2 - \beta_1 \mathscr{B}_2 w_1 + m_1 \partial_\nu w_2 + \alpha \kappa w_5 \quad \text{on} \quad I.$$

By (4.101), $i\lambda w - \widetilde{f} = f$. Thus, $\widetilde{f}_2 = i\lambda w_2 - f_2$. Therefore,

$$\begin{aligned}\mathscr{I} &= \int_I |\beta_2 \partial_\nu w_3| |q \cdot \nabla \overline{w}_3| dS \leq C \|\beta_2 \partial_\nu w_3\|_{L^2(I)} \|\nabla w_3\|_{L^2(I)^2} \\ &\leq C \big(|\lambda| \|\partial_\nu w_2\|_{L^2(I)} + \|\partial_\nu f_2\|_{L^2(I)} + \|\mathscr{B}_2 w_1\|_{L^2(I)} + \|w_5\|_{L^2(I)} \big) \|\nabla w_3\|_{L^2(I)^2}.\end{aligned}$$

From (4.105), $\Delta w_3 = i\lambda \beta_2^{-1} \rho_2 w_4 - \beta_2^{-1} \rho_2 f_4$ in Ω_2 and as $w_3 = w_1$ on I, then by Remark 3.17 and the trace theorem we get

$$\begin{aligned}\|w_3\|_{H^2(\Omega_2)} &\leq C \big(\big\| i\lambda \beta_2^{-1} \rho_2 w_4 - \beta_2^{-1} \rho_2 f_4 \big\|_{L^2(\Omega_2)} + \|w_1\|_{H^{3/2}(I)} \big) \\ &\leq C \big(|\lambda| \|w\|_{\mathscr{H}_\eta} + \|f\|_{\mathscr{H}_\eta} \big).\end{aligned} \tag{4.125}$$

Applying Theorem 2.21 to the partial derivatives of first order of w_3, we obtain

$$\begin{aligned}\|\nabla w_3\|_{L^2(I)^2} &\le C\,\|w_3\|_{H^2(\Omega_2)}^{1/2}\,\|w_3\|_{H^1(\Omega_2)}^{1/2}\\ &\le C\big(|\lambda|^{1/2}\,\|w\|_{\mathscr{H}_\eta} + \|f\|_{\mathscr{H}_\eta}^{1/2}\,\|w\|_{\mathscr{H}_\eta}^{1/2}\big).\end{aligned} \tag{4.126}$$

By (4.102),

$$\|w_2\|_{H^2(\Omega_1)} \le C\,\|i\lambda w_1 - f_1\|_{H^2_\Gamma(\Omega_1)} \le C\big(|\lambda|\,\|w\|_{\mathscr{H}_\eta} + \|f\|_{\mathscr{H}_\eta}\big). \tag{4.127}$$

Applying Corollary 2.22 to w_2 and considering (4.108) together with (4.127), we get

$$\begin{aligned}|\lambda|\,\|\partial_\nu w_2\|_{L^2(I)} &\le C|\lambda|\,\|w_2\|_{H^2(\Omega_1)}^{1/2}\,\|w_2\|_{H^1(\Omega_1)}^{1/2}\\ &\le C\big(|\lambda|^{3/2}\,\|f\|_{\mathscr{H}_\eta}^{1/4}\,\|w\|_{\mathscr{H}_\eta}^{3/4} + |\lambda|\,\|f\|_{\mathscr{H}_\eta}^{3/4}\,\|w\|_{\mathscr{H}_\eta}^{1/4}\big).\end{aligned} \tag{4.128}$$

Doing $\widehat{f} := \mathscr{A}_\eta f$ with $\widehat{f} = (\widehat{f}_1, \widehat{f}_2, \widehat{f}_3, \widehat{f}_4, \widehat{f}_5)$, we have that $\widehat{f}_1 = f_2$ and by the trace theorem

$$\|\partial_\nu f_2\|_{L^2(I)} \le C\,\|f_2\|_{H^2_\Gamma(\Omega_1)} = C\|\widehat{f}_1\|_{H^2_\Gamma(\Omega_1)} \le C\,\|\mathscr{A}_\eta f\|_{\mathscr{H}_\eta}. \tag{4.129}$$

By (3C.53) in [87], we can write $B_2 w_1 = \partial_\nu \partial_\tau w_1$. Applying Corollary 2.22 to Δw_1 and Theorem 2.21 to the partial derivatives of third order of w_1, and then using Sobolev's interpolation inequality, we deduce that

$$\begin{aligned}\|\mathscr{B}_2 w_1\|_{L^2(I)} &\le C\big(\,\|\partial_\nu \Delta w_1\|_{L^2(I)} + \|\partial_\tau\partial_\nu\partial_\tau w_1\|_{L^2(I)}\,\big)\\ &\le C\big(\,\|\Delta w_1\|_{H^2(\Omega_1)}^{1/2}\,\|\nabla\Delta w_1\|_{L^2(\Omega_1)^2}^{1/2} + \|w_1\|_{H^4(\Omega_1)}^{1/2}\,\|w_1\|_{H^3(\Omega_1)}^{1/2}\,\big)\\ &\le C\,\|w_1\|_{H^4(\Omega_1)}^{5/6}\,\|w_1\|_{H^1(\Omega_1)}^{1/6}.\end{aligned} \tag{4.130}$$

Using the definitions of $\mathcal{M}_\eta$, $\mathcal{W}$, $\mathcal{A}_T$, $\mathcal{A}_B$, (3.30), (3.31), Proposition 3.11 and (4.124) we can establish that equation (4.103) is equivalent to the following system

$$\begin{cases}\Delta^2 w_1 = \beta_1^{-1}(\rho_1 f_2 - i\lambda\rho_1 w_2 + \gamma\Delta\widetilde{f}_2 + m_1\Delta w_2 - \alpha\Delta w_5) =: \widetilde{f}_* \ \text{ in } \ \Omega_1,\\ w_1 = 0 \ \text{ on } \ \Gamma,\\ \partial_\nu w_1 = 0 \ \text{ on } \ \Gamma,\\ \mathscr{B}_1 w_1 = -\alpha\beta_1^{-1} w_5 \ \text{ on } \ I,\\ \mathscr{B}_2 w_1 = \beta_1^{-1}(\gamma\partial_\nu\widetilde{f}_2 + m_1\partial_\nu w_2 - \beta_2\partial_\nu w_3 + \alpha\kappa w_5) \ \text{ on } \ I.\end{cases}$$

By Proposition 2.60 there are positive constants $\lambda_0^{(2)}$ and C such that

$$\begin{aligned}\|w_1\|_{H^4(\Omega_1)} \le C\big(\|\lambda_0^{(2)} w_1 + \widetilde{f}_*\|_{L^2(\Omega_1)} + \big\|-\alpha\beta_1^{-1} w_5\big\|_{H^{3/2}(I)} \\ + \|\beta_1^{-1}(\gamma\partial_\nu \widetilde{f}_2 + m_1\partial_\nu w_2 - \beta_2\partial_\nu w_3 + \alpha\kappa w_5)\|_{H^{1/2}(I)}\big).\end{aligned} \tag{4.131}$$

From (4.127), (4.129) and the fact that $0 \in \rho(\mathscr{A}_\eta)$ it follows that

$$\begin{aligned}\|\Delta\widetilde{f}_2\|_{L^2(\Omega_1)} &\le C\|\widetilde{f}_2\|_{H^2_\Gamma(\Omega_1)} = C\,\|i\lambda w_2 - f_2\|_{H^2_\Gamma(\Omega_1)} \\ &\le C\big(\lambda^2\,\|w\|_{\mathscr{H}_\eta} + |\lambda|\,\|\mathscr{A}_\eta f\|_{\mathscr{H}_\eta}\big).\end{aligned} \tag{4.132}$$

As $\Delta w_5 = \beta^{-1}(i\lambda\rho_0 w_5 - \alpha\Delta w_2 + \sigma w_5 - \rho_0 f_5) =: \widehat{f}_*$ in Ω_1 and $\partial_\nu w_5 + \kappa w_5 = 0$ on $\partial\Omega_1$, this is due to w_5 satisfies (4.106) and is an element of $D(\mathcal{A}_T)$, by regularity theory there exist positive constants $\lambda_0^{(3)}$ and C such that

$$\|w_5\|_{H^2(\Omega_1)} \le C\|\lambda_0^{(3)} w_5 - \widehat{f}_*\|_{L^2(\Omega_1)} \le C\big(|\lambda|\,\|w\|_{\mathscr{H}_\eta} + \|f\|_{\mathscr{H}_\eta}\big). \tag{4.133}$$

In the last inequality it was used (4.127). By (4.125), (4.127), (4.129), (4.131)-(4.133) and the trace theorem we see that

$$\|w_1\|_{H^4(\Omega_1)} \le C\big(\lambda^2\,\|w\|_{\mathscr{H}_\eta} + |\lambda|\,\|\mathscr{A}_\eta f\|_{\mathscr{H}_\eta}\big).$$

By (4.118),

$$\|w_1\|_{H^1(\Omega_1)} \le C|\lambda|^{-1}\big(\,\|\mathscr{A}_\eta f\|_{\mathscr{H}_\eta} + \|\mathscr{A}_\eta f\|^{1/2}_{\mathscr{H}_\eta}\,\|w\|^{1/2}_{\mathscr{H}_\eta}\big).$$

From (4.130),

$$\begin{aligned}\|\mathscr{B}_2 w_1\|_{L^2(I)} \le C\big(|\lambda|^{3/2}\,\|\mathscr{A}_\eta f\|^{1/6}_{\mathscr{H}_\eta}\,\|w\|^{5/6}_{\mathscr{H}_\eta} + |\lambda|^{3/2}\,\|\mathscr{A}_\eta f\|^{1/12}_{\mathscr{H}_\eta}\,\|w\|^{11/12}_{\mathscr{H}_\eta} \\ + |\lambda|^{2/3}\,\|\mathscr{A}_\eta f\|_{\mathscr{H}_\eta} + |\lambda|^{2/3}\,\|\mathscr{A}_\eta f\|^{11/12}_{\mathscr{H}_\eta}\,\|w\|^{1/12}_{\mathscr{H}_\eta}\big).\end{aligned} \tag{4.134}$$

By the trace theorem and (4.109), we get

$$\|w_5\|_{L^2(I)} \le C\,\|w_5\|_{H^1(\Omega_1)} \le C\,\|f\|^{1/2}_{\mathscr{H}_\eta}\,\|w\|^{1/2}_{\mathscr{H}_\eta}. \tag{4.135}$$

Due to (4.126), (4.128), (4.129), (4.134), (4.135) and Corollary 2.20, we write

$$\mathscr{I} \le \varepsilon C\,\|w\|^2_{\mathscr{H}_\eta} + C_\varepsilon|\lambda|^{48}\,\|\mathscr{A}_\eta f\|^2_{\mathscr{H}_\eta} \quad \text{for any } \ \varepsilon > 0. \tag{4.136}$$

Due to (4.119), (4.123) and (4.136) we obtain that

$$\|w\|_{\mathscr{H}_\eta} \le C|\lambda|^{24} \|\mathscr{A}_\eta f\|_{\mathscr{H}_\eta}. \tag{4.137}$$

Let $\widetilde{w} \in D(\mathscr{A}_\eta)$ and $\widetilde{f} := (i\lambda\mathcal{I} - \mathscr{A}_\eta)\widetilde{w} = 0$. In this situation, we have that $\mathscr{A}_\eta\widetilde{w} = i\lambda\widetilde{w} \in D(\mathscr{A}_\eta)$ and so $\widetilde{w} \in D(\mathscr{A}_\eta^2)$. Now, an application of (4.137) allows us to obtain $\widetilde{w} = 0$. In consequence, $\mathrm{Ker}(i\lambda\mathcal{I} - \mathscr{A}_\eta) = \{0\}$ and thus $i\lambda$ is not an eigenvalue of $\mathscr{A}_\eta$. By Remark 4.14, we get that $i\lambda \notin \sigma(\mathscr{A}_\eta)$.

Let us consider $\mathscr{F} \in D(\mathscr{A}_\eta)$ and $\lambda \in \mathbb{R}$ with $|\lambda| > \lambda_0$. As $i\lambda \in \rho(\mathscr{A}_\eta)$, then $i\lambda\mathcal{I} - \mathscr{A}_\eta : D(\mathscr{A}_\eta) \to \mathscr{H}_\eta$ is invertible and so there exists $\mathscr{U} \in D(\mathscr{A}_\eta)$ such that $(i\lambda\mathcal{I} - \mathscr{A}_\eta)\mathscr{U} = \mathscr{F}$. Since $\mathscr{A}_\eta\mathscr{U} = i\lambda\mathscr{U} - \mathscr{F} \in D(\mathscr{A}_\eta)$, then $\mathscr{U} \in D(\mathscr{A}_\eta^2)$ and thus (4.137) holds replacing w and f by $\mathscr{U}$ and $\mathscr{F}$, respectively. Hence,

$$\left\|(i\lambda\mathcal{I} - \mathscr{A}_\eta)^{-1}\mathscr{F}\right\|_{\mathscr{H}_\eta} \le C|\lambda|^{24} \|\mathscr{A}_\eta\mathscr{F}\|_{\mathscr{H}_\eta}.$$

From Theorem 2.44 with $\alpha' = 1$ and $\beta' = 24$ it follows the first statement of our theorem,

$$\left\|\mathscr{T}_\eta(t)\mathscr{A}_\eta^{-1}\right\|_{\mathcal{L}(\mathscr{H}_\eta)} \le Ct^{-1/25} \quad \text{for all } t > 0. \tag{4.138}$$

Let $k \in \mathbb{N}$, $w_0 \in D(\mathscr{A}_\eta^k)$ and $f_0 := \mathscr{A}_\eta^k w_0$. Note that $f_0 \in \mathscr{H}_\eta$. Since $\mathscr{A}_\eta^k$ is invertible, we have that $w_0 = \mathscr{A}_\eta^{-k} f_0$. Using part c) of Theorem 2.4 in [109, p. 5], one can prove that $\mathscr{T}_\eta^k(t)\mathscr{A}_\eta^{-k}\phi = [\mathscr{T}_\eta(t)\mathscr{A}_\eta^{-1}]^k\phi$ for any $\phi \in \mathscr{H}_\eta$ and for all $t \ge 0$. Thus, from (4.138) it follows that

$$\|\mathscr{T}_\eta(t)w_0\|_{\mathscr{H}_\eta} = \left\|\left(\mathscr{T}_\eta(t/k)\mathscr{A}_\eta^{-1}\right)^k f_0\right\|_{\mathscr{H}_\eta} \le (Ck^{1/25})^k t^{-k/25} |w_0|_{D(\mathscr{A}_\eta^k)}$$

for any $t > 0$. □

Remark 4.17. a) The estimate (4.100) indicates that the more regular are the initial data, the decay of the energy (3.1) is faster. Indeed: If $w_0 \in D(\mathscr{A}_\eta^k)$, then (4.100) implies that

$$E_\gamma(t) = \frac{1}{2}\|\mathscr{T}_\eta(t)w_0\|_{\mathscr{H}_\eta}^2 \le \frac{\widetilde{C}_k}{t^{2k/25}}|w_0|_{D(\mathscr{A}_\eta^k)}^2 \quad \text{for all} \quad t > 0,$$

where $\widetilde{C}_k := C_k^2/2$.

b) The result of the previous theorem continues to be true if the plate is isothermal, i.e., the temperature is not necessary to obtain the decay rate of Theorem 4.16.

c) If we remove the structural damping ($m_1 = 0$) of our thermoelastic plate-membrane system, we do not know anything about asymptotic polynomial behavior for parameters $\eta > 0$ and $m_2 \ge 0$, and when also $\eta = m_2 = 0$.

Remark 4.18. In the proof of Theorem 4.16, we did not aim at optimal polynomial rate.

Chapter 5

Analyticity of the semigroup associated with a transmission problem

The goal of this chapter (Theorem 5.1) is an analytical result when both the inertial term and the structural damping are not present, but there is Kelvin–Voigt damping on the membrane and temperature on the plate. As a consequence we will have that $w(t) = \mathscr{T}_0(t)w_0 \in D(\mathscr{A}_0^\infty) := \cap_{k=0}^\infty D(\mathscr{A}_0^k)$ for $w_0 \in \mathscr{H}_0$, i.e., no matter how irregular the initial data always the corresponding solution is of class C^∞ (see [51, p. 2] and [104, p. 179]). Other implications of the analyticity of a semigroup can be found in [94]. We will achieve our analytical purpose by making use of the well known result of Theorem 2.46. The proof will be done by contradiction.

Theorem 5.1. *If $\eta = m_1 = 0$ and $m_2 > 0$, then the semigroup $(\mathscr{T}_0(t))_{t\geq 0}$ generated by $\mathscr{A}_0$ is analytic.*

Proof. From Proposition 4.5, we have that $i\mathbb{R} \subset \rho(\mathscr{A}_0)$. Let us suppose (2.20) is not true. According to Remark 2.7 in [39], there are sequences $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ with $|\lambda_n| \to \infty$ and $(\widehat{f}_n)_{n\in\mathbb{N}} \subset \mathscr{H}_0$ with $\|\widehat{f}_n\|_{\mathscr{H}_0} = 1$ such that

$$\lim_{n\to\infty} \|\lambda_n(i\lambda_n\mathcal{I} - \mathscr{A}_0)^{-1}\widehat{f}_n\|_{\mathscr{H}_0} = \infty.$$

Without loss of generality, we will assume that λ_n is positive for each $n \in \mathbb{N}$. The opposite situation, $\lambda_n < 0$ for any $n \in \mathbb{N}$, is discussed in Remark 5.2.

Setting $\widehat{w}_n := \lambda_n(i\lambda_n\mathcal{I} - \mathscr{A}_0)^{-1}\widehat{f}_n$, $w_n := \widehat{w}_n/\|\widehat{w}_n\|_{\mathscr{H}_0}$ and $f_n := \widehat{f}_n/\|\widehat{w}_n\|_{\mathscr{H}_0}$, we obtain the following

$$\|w_n\|_{\mathscr{H}_0} = 1 \quad \text{and} \quad \left(i\mathcal{I} - \lambda_n^{-1}\mathscr{A}_0\right) w_n = f_n \tag{5.1}$$

with

$$\left\|(i\mathcal{I} - \lambda_n^{-1}\mathscr{A}_0)w_n\right\|_{\mathscr{H}_0} \xrightarrow[n\to\infty]{} 0. \tag{5.2}$$

Note that $(w_n)_{n\in\mathbb{N}} \subset D(\mathscr{A}_0)$. From the second assertion in (5.1),

$$iw_1^n - \lambda_n^{-1}w_2^n = f_1^n, \tag{5.3}$$
$$i\rho_1 w_2^n + \lambda_n^{-1}\big(\mathcal{A}_B\mathcal{W}_0^n - \alpha\mathcal{A}_T w_5^n + \tfrac{\alpha\sigma}{\beta}w_5^n\big) = \rho_1 f_2^n, \tag{5.4}$$
$$iw_3^n - \lambda_n^{-1}w_4^n = f_3^n, \tag{5.5}$$
$$i\rho_2 w_4^n - \beta_2\lambda_n^{-1}\Delta w_3^n - m_2\lambda_n^{-1}\Delta w_4^n = \rho_2 f_4^n, \tag{5.6}$$
$$i\rho_0 w_5^n + \alpha\lambda_n^{-1}\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1)w_2^n + \beta\lambda_n^{-1}\mathcal{A}_T w_5^n = \rho_0 f_5^n, \tag{5.7}$$

where $\mathcal{W}_0^n := \beta_1 w_1^n + \beta_2\mathcal{G}_2\gamma_1 w_3^n + m_2\mathcal{G}_2\gamma_1 w_4^n + \alpha\mathcal{G}_1\gamma_0 w_5^n - \alpha\kappa\mathcal{G}_2\gamma_0 w_5^n$. Note that

$$\begin{aligned}\left\|(i\mathcal{I} - \lambda_n^{-1}\mathscr{A}_0)w_n\right\|_{\mathscr{H}_0}^2 = {} & \beta_1\|f_1^n\|^2_{H^2_\Gamma(\Omega_1)} + \rho_1\|f_2^n\|^2_{L^2(\Omega_1)} + \beta_2\|\nabla f_3^n\|^2_{L^2(\Omega_2)^2} \\ & + \rho_2\|f_4^n\|^2_{L^2(\Omega_2)} + \rho_0\|f_5^n\|^2_{L^2(\Omega_1)}.\end{aligned}$$

Then, the limit (5.2) and (5.3)-(5.7) imply that

$$iw_1^n - \lambda_n^{-1}w_2^n \to 0 \ \text{ in } \ H^2_\Gamma(\Omega_1), \tag{5.8}$$
$$i\rho_1 w_2^n + \lambda_n^{-1}\big(\mathcal{A}_B\mathcal{W}_0^n - \alpha\mathcal{A}_T w_5^n + \tfrac{\alpha\sigma}{\beta}w_5^n\big) \to 0 \ \text{ in } \ L^2(\Omega_1), \tag{5.9}$$
$$i\nabla w_3^n - \lambda_n^{-1}\nabla w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2), \tag{5.10}$$
$$i\rho_2 w_4^n - \beta_2\lambda_n^{-1}\Delta w_3^n - m_2\lambda_n^{-1}\Delta w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2), \tag{5.11}$$
$$i\rho_0 w_5^n + \alpha\lambda_n^{-1}\mathcal{A}_L(\mathcal{I} - \mathcal{N}\gamma_1)w_2^n + \beta\lambda_n^{-1}\mathcal{A}_T w_5^n \to 0 \ \text{ in } \ L^2(\Omega_1). \tag{5.12}$$

Due to (3.35),

$$\begin{aligned}\operatorname{Re}\left((i\mathcal{I} - \lambda_n^{-1}\mathscr{A}_0)w_n, w_n\right)_{\mathscr{H}_0} = {} & \frac{m_2}{\lambda_n}\|\nabla w_4^n\|^2_{L^2(\Omega_2)^2} + \frac{\sigma}{\lambda_n}\|w_5^n\|^2_{L^2(\Omega_1)} \\ & + \frac{\beta}{\lambda_n}\|\nabla w_5^n\|^2_{L^2(\Omega_1)^2} + \frac{\beta\kappa}{\lambda_n}\|w_5^n\|^2_{L^2(\partial\Omega_1)}.\end{aligned} \tag{5.13}$$

By the first assertion in (5.1) and (5.13), we get

$$\lambda_n^{-1}\|w_5^n\|^2_{H^1(\Omega_1)} \le C\operatorname{Re}\left((i\mathcal{I} - \lambda_n^{-1}\mathscr{A}_0)w_n, w_n\right)_{\mathscr{H}_0} \le C\left\|(i\mathcal{I} - \lambda_n^{-1}\mathscr{A}_0)w_n\right\|_{\mathscr{H}_0}.$$

Now, (5.2) implies

$$\lambda_n^{-1/2} w_5^n \to 0 \ \text{ in } \ H^1(\Omega_1). \tag{5.14}$$

For being m_2 positive, it follows from (5.2) and (5.13) that

$$\lambda_n^{-1/2} \nabla w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{5.15}$$

The limit (5.10) and (5.15) imply

$$\nabla w_3^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{5.16}$$

We will obtain a key convergence in this proof through the following equality.

$$\begin{aligned} &\left(i\rho_0 w_5^n + \alpha\lambda_n^{-1}\mathcal{A}_L(\mathcal{I}-\mathcal{N}\gamma_1)w_2^n + \beta\lambda_n^{-1}\mathcal{A}_T w_5^n, w_5^n\right)_{L^2(\Omega_1)} \\ &= i\rho_0 \left\|w_5^n\right\|_{L^2(\Omega_1)}^2 - \alpha\lambda_n^{-1}\left(\Delta w_2^n, w_5^n\right)_{L^2(\Omega_1)} + \beta\lambda_n^{-1}\left(\mathcal{A}_T w_5^n, w_5^n\right)_{L^2(\Omega_1)}. \end{aligned} \tag{5.17}$$

Integration by parts implies

$$\left(\Delta w_2^n, w_5^n\right)_{L^2(\Omega_1)} = -\left(\nabla w_2^n, \nabla w_5^n\right)_{L^2(\Omega_1)^2} + \left(\partial_\nu w_2^n, w_5^n\right)_{L^2(I)}. \tag{5.18}$$

As $(w_1^n)_{n\in\mathbb{N}}$ is bounded in $H^2(\Omega_1)$, the limit (5.8) implies that $\left(\lambda_n^{-1} w_2^n\right)_{n\in\mathbb{N}}$ is also bounded in $H^2(\Omega_1)$. By interpolation inequality (2.11), we compute that

$$\frac{1}{\lambda_n^{1/2}}\left\|w_2^n\right\|_{H^1(\Omega_1)} \le C\frac{\left\|w_2^n\right\|_{H^2(\Omega_1)}^{1/2}}{\lambda_n^{1/2}}\left\|w_2^n\right\|_{L^2(\Omega_1)}^{1/2} \le C. \tag{5.19}$$

Using Cauchy–Schwarz inequality and (5.19),

$$\left|\lambda_n^{-1}\left(\nabla w_2^n, \nabla w_5^n\right)_{L^2(\Omega_1)^2}\right| \le C\lambda_n^{-1/2}\left\|w_5^n\right\|_{H^1(\Omega_1)}. \tag{5.20}$$

Next we will apply Theorem 2.21 to w_5^n and Corollary 2.22 to w_2^n. Considering again (5.19), we have

$$\begin{aligned} \left|\lambda_n^{-1}\left(\partial_\nu w_2^n, w_5^n\right)_{L^2(I)}\right| &\le \lambda_n^{-1}\left\|\partial_\nu w_2^n\right\|_{L^2(I)}\left\|w_5^n\right\|_{L^2(I)} \\ &\le C\frac{\left\|w_2^n\right\|_{H^2(\Omega_1)}^{1/2}}{\lambda_n^{1/2}}\frac{\left\|w_2^n\right\|_{H^1(\Omega_1)}^{1/2}}{\lambda_n^{1/4}}\frac{\left\|w_5^n\right\|_{H^1(\Omega_1)}^{1/2}}{\lambda_n^{1/4}}\left\|w_5^n\right\|_{L^2(\Omega_1)}^{1/2} \\ &\le C\lambda_n^{-1/4}\left\|w_5^n\right\|_{H^1(\Omega_1)}^{1/2}. \end{aligned} \tag{5.21}$$

Thanks to the fact that $\mathcal{A}_T^{1/2}$ is self-adjoint and to (3.17), we can write

$$\left|\lambda_n^{-1}\left(\mathcal{A}_T w_5^n, w_5^n\right)_{L^2(\Omega_1)}\right| = \lambda_n^{-1}\left\|w_5^n\right\|_{D(\mathcal{A}_T^{1/2})}^2 \le C\lambda_n^{-1}\left\|w_5^n\right\|_{H^1(\Omega_1)}^2. \tag{5.22}$$

Due to boundedness of $(w_5^n)_{n\in\mathbb{N}}$ in $L^2(\Omega_1)$, (5.12), (5.14), (5.17), (5.18) and (5.20)-(5.22), we conclude

$$w_5^n \to 0 \;\text{ in }\; L^2(\Omega_1). \tag{5.23}$$

On the other hand,

$$\begin{aligned}&\left(i\rho_2 w_4^n - \beta_2\lambda_n^{-1}\Delta w_3^n - m_2\lambda_n^{-1}\Delta w_4^n, w_4^n\right)_{L^2(\Omega_2)}\\ &= i\rho_2 \left\|w_4^n\right\|^2_{L^2(\Omega_2)} - \lambda_n^{-1}\left(\beta_2\Delta w_3^n + m_2\Delta w_4^n, w_4^n\right)_{L^2(\Omega_2)}.\end{aligned} \tag{5.24}$$

By Remark 3.19, integration by parts and the fact that $w_2^n = w_4^n$ on I we derive

$$\begin{aligned}\lambda_n^{-1}\left(\beta_2\Delta w_3^n + m_2\Delta w_4^n, w_4^n\right)_{L^2(\Omega_2)} &= -\beta_2\lambda_n^{-1}\left(\nabla w_3^n, \nabla w_4^n\right)_{L^2(\Omega_2)^2}\\ - m_2\lambda_n^{-1}\left\|\nabla w_4^n\right\|^2_{L^2(\Omega_2)^2} &- \lambda_n^{-1}\left(\partial_\nu(\beta_2 w_3^n + m_2 w_4^n), w_2^n\right)_{L^2(I)}.\end{aligned} \tag{5.25}$$

Obviously,

$$\left|\lambda_n^{-1}\left(\nabla w_3^n, \nabla w_4^n\right)_{L^2(\Omega_2)^2}\right| \le \lambda_n^{-1/2}\left\|\nabla w_3^n\right\|_{L^2(\Omega_2)^2}\lambda_n^{-1/2}\left\|\nabla w_4^n\right\|_{L^2(\Omega_2)^2}. \tag{5.26}$$

As $\Delta(\beta_2 w_3^n + m_2 w_4^n) = i\rho_2\lambda_n w_4^n - \rho_2\lambda_n f_4^n$ in Ω_2 and $\beta_2 w_3^n + m_2 w_4^n = \beta_2 w_1^n + m_2 w_2^n$ on I, see (5.6), Remark 3.17 implies

$$\begin{aligned}&\left\|\beta_2 w_3^n + m_2 w_4^n\right\|_{H^2(\Omega_2)}\\ &\le C\left(\left\|i\rho_2\lambda_n w_4^n - \rho_2\lambda_n f_4^n\right\|_{L^2(\Omega_2)} + \left\|\beta_2 w_1^n + m_2 w_2^n\right\|_{H^{3/2}(I)}\right)\\ &\le C\left(\lambda_n\left\|w_4^n\right\|_{L^2(\Omega_2)} + \lambda_n\left\|f_4^n\right\|_{L^2(\Omega_2)} + \left\|w_1^n\right\|_{H^2(\Omega_1)} + \left\|w_2^n\right\|_{H^2(\Omega_1)}\right).\end{aligned}$$

Because of $(w_4^n)_{n\in\mathbb{N}}$ and $(f_4^n)_{n\in\mathbb{N}}$ are bounded sequences in $L^2(\Omega_2)$, then

$$\lambda_n^{-1}\left\|\beta_2 w_3^n + m_2 w_4^n\right\|_{H^2(\Omega_2)} \le C. \tag{5.27}$$

Thanks to Cauchy–Schwarz inequality, Theorem 2.21 and (5.19) we obtain

$$\begin{aligned}&\left|\lambda_n^{-1}\left(\partial_\nu(\beta_2 w_3^n + m_2 w_4^n), w_2^n\right)_{L^2(I)}\right|\\ &\le C\lambda_n^{-3/4}\left\|\partial_\nu(\beta_2 w_3^n + m_2 w_4^n)\right\|_{L^2(I)}\lambda_n^{-1/4}\left\|w_2^n\right\|^{1/2}_{H^1(\Omega_1)}\left\|w_2^n\right\|^{1/2}_{L^2(\Omega_1)}\\ &\le C\lambda_n^{-3/4}\left\|\partial_\nu(\beta_2 w_3^n + m_2 w_4^n)\right\|_{L^2(I)}.\end{aligned} \tag{5.28}$$

Taking into account (5.27) and applying Corollary 2.22 to $\beta_2 w_3^n + m_2 w_4^n$,

$$\lambda_n^{-3/4}\left\|\partial_\nu(\beta_2 w_3^n + m_2 w_4^n)\right\|_{L^2(I)}$$

$$\leq C\frac{1}{\lambda_n^{1/2}}\left\|\beta_2 w_3^n + m_2 w_4^n\right\|_{H^2(\Omega_2)}^{1/2}\frac{1}{\lambda_n^{1/4}}\left\|\nabla(\beta_2 w_3^n + m_2 w_4^n)\right\|_{L^2(\Omega_2)^2}^{1/2}$$
$$\leq C\big(\lambda_n^{-1/4}\left\|\nabla w_3^n\right\|_{L^2(\Omega_2)^2}^{1/2} + \lambda_n^{-1/4}\left\|\nabla w_4^n\right\|_{L^2(\Omega_2)^2}^{1/2}\big). \tag{5.29}$$

From (5.11), (5.15), (5.16), (5.24)-(5.26), (5.28) and (5.29), it follows that

$$w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{5.30}$$

Now, let us consider the following equality

$$\begin{aligned}&\big(i\rho_1 w_2^n + \lambda_n^{-1}\left(\mathcal{A}_B\mathcal{W}_0^n - \alpha\mathcal{A}_T w_5^n + \alpha\beta^{-1}\sigma w_5^n\right), w_2^n\big)_{L^2(\Omega_1)}\\ &= i\rho_1\left\|w_2^n\right\|_{L^2(\Omega_1)}^2 + \lambda_n^{-1}\left(\mathcal{A}_B\mathcal{W}_0^n, w_2^n\right)_{L^2(\Omega_1)} + \alpha\lambda_n^{-1}\left(\Delta w_5^n, w_2^n\right)_{L^2(\Omega_1)}.\end{aligned} \tag{5.31}$$

Applying integration by parts,

$$\left(\Delta w_5^n, w_2^n\right)_{L^2(\Omega_1)} = -\left(\nabla w_5^n, \nabla w_2^n\right)_{L^2(\Omega_1)^2} + \left(\partial_\nu w_5^n, w_2^n\right)_{L^2(I)}. \tag{5.32}$$

Because of $\partial_\nu w_5^n + \kappa w_5^n = 0$ on I, the trace theorem and (5.19) we obtain that

$$\begin{aligned}\left|\lambda_n^{-1}\left(\partial_\nu w_5^n, w_2^n\right)_{L^2(I)}\right| &\leq C\lambda_n^{-1}\left\|w_5^n\right\|_{L^2(I)}\left\|w_2^n\right\|_{L^2(I)}\\ &\leq C\frac{\left\|w_5^n\right\|_{H^1(\Omega_1)}}{\lambda_n^{1/2}}\frac{\left\|w_2^n\right\|_{H^1(\Omega_1)}}{\lambda_n^{1/2}}\\ &\leq C\lambda_n^{-1/2}\left\|w_5^n\right\|_{H^1(\Omega_1)}.\end{aligned} \tag{5.33}$$

Gathering (4.59), (5.9), (5.14)-(5.16), (5.20), (5.21), (5.28), (5.29) and (5.31)-(5.33) we have that

$$i\rho_1\left\|w_2^n\right\|_{L^2(\Omega_1)}^2 + \beta_1\lambda_n^{-1}\left(w_1^n, w_2^n\right)_{H_\Gamma^2(\Omega_1)} \to 0.$$

Multiplying (5.8) by w_1^n, we obtain

$$i\left\|w_1^n\right\|_{H_\Gamma^2(\Omega_1)}^2 - \lambda_n^{-1}\left(w_2^n, w_1^n\right)_{H_\Gamma^2(\Omega_1)} \to 0.$$

The last two limits allow to write

$$\rho_1\left\|w_2^n\right\|_{L^2(\Omega_1)}^2 - \beta_1\left\|w_1^n\right\|_{H_\Gamma^2(\Omega_1)}^2 \to 0. \tag{5.34}$$

Our goal now is to show that the sequence $(w_2^n)_{n\in\mathbb{N}}$ converges to zero in $L^2(\Omega_1)$. Developing an analogous argument as in the limit (4.64), taking into account that $\lambda_n \to \infty$, we find that

$$\lambda_n^{-1}\left\|\nabla w_2^n\right\|_{L^2(\Omega_1)^2}^2 - \lambda_n^{-1}\left(\partial_\nu w_2^n, w_2^n\right)_{L^2(I)} \to 0. \tag{5.35}$$

Using the fact that $w_2^n = w_4^n$ on I, Theorem 2.21, Corollary 2.22 and (5.19) we obtain that

$$\begin{aligned}\left|\lambda_n^{-1}\left(\partial_\nu w_2^n, w_2^n\right)_{L^2(I)}\right| &\le \lambda_n^{-1}\left\|\partial_\nu w_2^n\right\|_{L^2(I)}\left\|w_4^n\right\|_{L^2(I)} \\ &\le C\frac{\|w_2^n\|_{H^2(\Omega_1)}^{1/2}}{\lambda_n^{1/2}}\frac{\|w_2^n\|_{H^1(\Omega_1)}^{1/2}}{\lambda_n^{1/4}}\frac{\|w_4^n\|_{H^1(\Omega_2)}^{1/2}}{\lambda_n^{1/4}}\left\|w_4^n\right\|_{L^2(\Omega_2)}^{1/2} \\ &\le C\lambda_n^{-1/4}\left\|w_4^n\right\|_{H^1(\Omega_2)}^{1/2}. \end{aligned} \tag{5.36}$$

Due to $(\lambda_n^{-1/2} w_4^n)_{n\in\mathbb{N}}$ and $(\lambda_n^{-1/2}\nabla w_4^n)_{n\in\mathbb{N}}$ are sequences that converge to zero in $L^2(\Omega_2)$, see the first assertion of (5.1) and (5.15), then

$$\lambda_n^{-1/2} w_4^n \to 0 \quad \text{in} \quad H^1(\Omega_2). \tag{5.37}$$

From (5.35)-(5.37), it follows that

$$\lambda_n^{-1/2}\nabla w_2^n \to 0 \quad \text{in} \quad L^2(\Omega_1). \tag{5.38}$$

By (5.4),

$$\mathcal{A}_B \mathcal{W}_0^n = \rho_1\lambda_n f_2^n - i\rho_1\lambda_n w_2^n - \alpha\Delta w_5^n.$$

Since $(w_n)_{n\in\mathbb{N}} \subset D(\mathscr{A}_0)$, it is immediate from the definition of the domain of the operator $\mathscr{A}_0$ that $\mathcal{W}_0^n \in D(\mathcal{A}_B)$. Because of the latter and the definitions of the Green maps $\mathcal{G}_1$ and $\mathcal{G}_2$ given in (3.19), we have that the last equation is equivalent to the following elliptic boundary value problem

$$\begin{cases} \Delta^2 w_1^n = \beta_1^{-1}\rho_1\lambda_n f_2^n - i\beta_1^{-1}\rho_1\lambda_n w_2^n - \alpha\beta_1^{-1}\Delta w_5^n =: f^* \quad \text{in} \quad \Omega_1, \\ w_1^n = 0 \quad \text{on} \quad \Gamma, \\ \partial_\nu w_1^n = 0 \quad \text{on} \quad \Gamma, \\ \mathscr{B}_1 w_1^n = -\alpha\beta_1^{-1} w_5^n \quad \text{on} \quad I, \\ \mathscr{B}_2 w_1^n = -\beta_1^{-1}\partial_\nu(\beta_2 w_3^n + m_2 w_4^n) + \alpha\beta_1^{-1}\kappa w_5^n \quad \text{on} \quad I. \end{cases} \tag{5.39}$$

By Proposition 2.60, there are two positive constants $\lambda_0^{(1)}$ and C that satisfy

$$\begin{aligned}\|w_1^n\|_{H^4(\Omega_1)} &\le C\big(\|\lambda_0^{(1)} w_1^n + f^*\|_{L^2(\Omega_1)} + \|-\alpha\beta_1^{-1} w_5^n\|_{H^{3/2}(I)} \\ &\quad + \|-\beta_1^{-1}\partial_\nu(\beta_2 w_3^n + m_2 w_4^n) + \alpha\beta_1^{-1}\kappa w_5^n\|_{H^{1/2}(I)}\big) \\ &\le C\big(\|w_1^n\|_{L^2(\Omega_1)} + \lambda_n\|f_2^n\|_{L^2(\Omega_1)} + \lambda_n\|w_2^n\|_{L^2(\Omega_1)} \\ &\quad + \|w_5^n\|_{H^2(\Omega_1)} + \|\beta_2 w_3^n + m_2 w_4^n\|_{H^2(\Omega_2)}\big). \end{aligned} \tag{5.40}$$

From (5.7) and the fact $w_5^n \in D(\mathcal{A}_T)$, it follows that: $\Delta w_5^n = i\beta^{-1}\rho_0\lambda_n w_5^n - \alpha\beta^{-1}\Delta w_2^n + \sigma\beta^{-1}w_5^n - \rho_0\beta^{-1}\lambda_n f_5^n =: g^*$ in Ω_1 and $\partial_\nu w_5^n + \kappa w_5^n = 0$ on $\partial\Omega_1$. By regularity theory, see for instance part b) of Remark 3.3 in [14], there are constants $\lambda_0^{(2)} > 0$ and $C > 0$ such that

$$\begin{aligned}\|w_5^n\|_{H^2(\Omega_1)} &\leq C\|\lambda_0^{(2)} w_5^n - g^*\|_{L^2(\Omega_1)} \\ &\leq C\big(\|w_5^n\|_{L^2(\Omega_1)} + \lambda_n \|w_5^n\|_{L^2(\Omega_1)} + \|w_2^n\|_{H^2(\Omega_1)} + \lambda_n \|f_5^n\|_{L^2(\Omega_1)} \big).\end{aligned}$$

Therefore,

$$\lambda_n^{-1} \|w_5^n\|_{H^2(\Omega_1)} \leq C. \tag{5.41}$$

Thanks to (5.27), (5.40) and (5.41) we can affirm that $\big(\lambda_n^{-1} w_1^n\big)_{n\in\mathbb{N}}$ is bounded in $H^4(\Omega_1)$. By interpolation inequality,

$$\frac{1}{\lambda_n^{1/2}} \|w_1^n\|_{H^3(\Omega_1)} \leq C \frac{\|w_1^n\|_{H^4(\Omega_1)}^{1/2}}{\lambda_n^{1/2}} \|w_1^n\|_{H^2(\Omega_1)}^{1/2} \leq C. \tag{5.42}$$

Due to the first equation in (5.39),

$$i w_2^n = -\tfrac{\beta_1}{\rho_1}\lambda_n^{-1}\Delta^2 w_1^n + f_2^n - \tfrac{\alpha}{\rho_1}\lambda_n^{-1}\Delta w_5^n \quad \text{in} \quad \Omega_1. \tag{5.43}$$

Taking the inner product in $L^2(\Omega_1)$ of (5.43) with w_2^n and applying integration by parts, we obtain

$$\begin{aligned} i\|w_2^n\|_{L^2(\Omega_1)}^2 = -\tfrac{\beta_1}{\rho_1}\lambda_n^{-1}\big[-(\nabla\Delta w_1^n, \nabla w_2^n)_{L^2(\Omega_1)^2} + (\partial_\nu \Delta w_1^n, w_2^n)_{L^2(I)} \big] \\ + (f_2^n, w_2^n)_{L^2(\Omega_1)} - \tfrac{\alpha}{\rho_1}\lambda_n^{-1}(\Delta w_5^n, w_2^n)_{L^2(\Omega_1)}. \end{aligned} \tag{5.44}$$

Due to (5.42), we can write

$$\begin{aligned} \big|\lambda_n^{-1}(\nabla\Delta w_1^n, \nabla w_2^n)_{L^2(\Omega_1)^2}\big| &\leq \lambda_n^{-1}\|\nabla\Delta w_1^n\|_{L^2(\Omega_1)^2}\|\nabla w_2^n\|_{L^2(\Omega_1)^2} \\ &\leq C\lambda_n^{-1/2}\|w_1^n\|_{H^3(\Omega_1)}\lambda_n^{-1/2}\|\nabla w_2^n\|_{L^2(\Omega_1)^2} \\ &\leq C\lambda_n^{-1/2}\|\nabla w_2^n\|_{L^2(\Omega_1)^2}. \end{aligned} \tag{5.45}$$

The boundedness of the sequences $(\lambda_n^{-1}w_1^n)_{n\in\mathbb{N}}$ in $H^4(\Omega_1)$ and $(w_2^n)_{n\in\mathbb{N}}$ in $L^2(\Omega_1)$ together with Theorem 2.21, Corollary 2.22, Friedrichs inequality and (5.42) allows us to write

$$\big|\lambda_n^{-1}(\partial_\nu\Delta w_1^n, w_2^n)_{L^2(I)}\big| \leq \lambda_n^{-1}\|\partial_\nu\Delta w_1^n\|_{L^2(I)}\|w_2^n\|_{L^2(I)}$$

$$\begin{aligned}&\leq C\frac{\|w_1^n\|_{H^4(\Omega_1)}^{1/2}}{\lambda_n^{1/2}}\frac{\|w_1^n\|_{H^3(\Omega_1)}^{1/2}}{\lambda_n^{1/4}}\frac{\|w_2^n\|_{H^1(\Omega_1)}^{1/2}}{\lambda_n^{1/4}}\|w_2^n\|_{L^2(\Omega_1)}^{1/2}\\&\leq C\lambda_n^{-1/4}\|\nabla w_2^n\|_{L^2(\Omega_1)^2}^{1/2}.\end{aligned}\tag{5.46}$$

The limit (5.38) together with (5.45) and (5.46) imply

$$\lambda_n^{-1}\left(\nabla\Delta w_1^n,\nabla w_2^n\right)_{L^2(\Omega_1)^2}\to 0\quad\text{and}\quad\lambda_n^{-1}\left(\partial_\nu\Delta w_1^n,w_2^n\right)_{L^2(I)}\to 0.\tag{5.47}$$

Because $f_2^n\to 0$ in $L^2(\Omega_1)$ and $\|w_2^n\|_{L^2(\Omega_1)}\leq C$ for all $n\in\mathbb{N}$,

$$\left(f_2^n,w_2^n\right)_{L^2(\Omega_1)}\to 0.\tag{5.48}$$

By (5.14), (5.20), (5.32) and (5.33),

$$\lambda_n^{-1}\left(\Delta w_5^n,w_2^n\right)_{L^2(\Omega_1)}\to 0.\tag{5.49}$$

Because of (5.34), (5.44) and (5.47)-(5.49) we obtain the convergences

$$w_1^n\to 0\ \text{ in }\ H^2(\Omega_1)\quad\text{and}\quad w_2^n\to 0\ \text{ in }\ L^2(\Omega_1).\tag{5.50}$$

Joining the limits (5.16), (5.23), (5.30) and (5.50) we get $w_n\to 0$ in $\mathscr{H}_0$, this contradicts the first assertion in (5.1). □

Remark 5.2. If $\lambda_n<0$ for all $n\in\mathbb{N}$, we multiply (5.13) by -1. From this point on, the proof is similar to the one presented above with the difference that some λ_n would now be $-\lambda_n$.

Corollary 5.3. *If $\eta=m_1=0$ and $m_2>0$, then the analytic semigroup $(\mathscr{T}_0(t))_{t\geq 0}$ generated by $\mathscr{A}_0$ is exponentially stable. Thus, there exist constants $M>0$ and $\omega<0$ such that $\mathcal{E}(t)\leq Me^{\omega t}\mathcal{E}(0)$ for all $t\geq 0$, where $\mathcal{E}\coloneqq E_0$ and E_0 is as in* (3.1).

Proof. By Remark 3.16, $\mathscr{A}_0$ is the infinitesimal generator of the C_0-semigroup of contractions $(\mathscr{T}_0(t))_{t\geq 0}$. Due to Corollary 3.6 in [109, p. 11], we have that $\rho(\mathscr{A}_0)\supset\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}$ and as $i\mathbb{R}\subset\rho(\mathscr{A}_0)$, see Proposition 4.5, then

$$\sigma(\mathscr{A}_0)\subset\mathbb{C}\setminus\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geq 0\}.$$

Now, Theorem 5.1 and Theorem 5.2 in [38] imply the exponential stability of the semigroup $(\mathscr{T}_0(t))_{t\geq 0}$. □

Another consequence of the analyticity of the solutions is the impossibility of localization. That means that the only solution that can be identically zero after a finite period of time is the null solution. This can be found in [20, p. 162].

Corollary 5.4. *Let* $\eta = m_1 = 0$ *and* $m_2 > 0$. *Let us assume that* $w = (u, u_t, v, v_t, \theta)$ *is a solution of the system* (1.1)-(1.3) *that satisfies* (1.4)-(1.7) *with initial conditions* (1.8) *and* (1.9) *such that* $u = v = \theta = 0$ *after a finite time* $t_0 > 0$. *Then,* $u = v = \theta = 0$ *for every* $t \geq 0$.

Remark 5.5. In the proof of Theorem 4.16 it was shown that if we assume condition (4.99) with $\eta > 0$, $m_1 > 0$ and $m_2 = 0$, then $i\mathbb{R} \subset \rho(\mathscr{A}_\eta)$. On the other hand, Theorem 4.15 implies that the C_0-semigroup of contractions $(\mathscr{T}_\eta(t))_{t\geq 0}$ generated by $\mathscr{A}_\eta$ does not have exponential stability. Consequently, under the given conditions, we have that $(\mathscr{T}_\eta(t))_{t\geq 0}$ is not analytic.

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$\mathcal{L}(X)$ 18
id 18
$\mathbb{N}_0$ 19
$|\alpha|$ 19
∂^α 19
$C^k(\Omega)$ 19
$C^k(\overline{\Omega})$ 19
$C^\infty(\Omega)$ 19
$\operatorname{supp} f$ 20
$C_c^\infty(\Omega)$ 20
$\mathscr{D}(\Omega)$ 20
$\mathscr{D}'(\Omega)$ 20
$L^p(\Omega)$ 20
$L^1_{loc}(\Omega)$ 20

$W^{k,p}(\Omega)$ 21
$H^k(\Omega)$ 21
$[s]$ 21
$H^s(\Omega)$ 21
$W^{s,p}(\Omega)$ 21
$W_0^{s,p}(\Omega)$ 21
$H_0^k(\Omega)$ 21
$L^p((a,b),X)$ 22
$\mathscr{D}'((a,b),X)$ 22
$\hookrightarrow, \overset{c}{\hookrightarrow}$ 23
$\mathcal{W}_{p,q}((0,t);B_0,B_1)$ 23
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$C^{k,1}$ 24
$W^{s,p}(\partial\Omega)$ 24
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