TRAPPED SURFACES IN NONSPHERICAL OPEN UNIVERSES*

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(Received February 11, 1992)

We continue our investigation of formation of trapped surfaces in strongly curved geometries which do not contain gravitational waves. The expansion of open, flat universes does not change substantially the results obtained hitherto in the case of asymptotically and conformally flat space-time. The necessary and sufficient conditions for the formation of trapped surfaces are given, which explicitly demonstrate that the quicker universes are expanding, the more matter is required to develop a trapped surface.

PACS numbers: 04.20.Cv

1. Introduction

The attempt to prove the Cosmic Censorship Hypothesis [1] may be divided into two steps. The first step would be the proof that singularities are surrounded by trapped surfaces. That of course requires a suitable definition of what is meant by the term "singularities"; our results suggest that it is possible to prove the so called "trapped surface conjecture", according to which massive singularities have to be trapped. The word "massive" means, in our previous work, geometries that comprise a large amount of matter (or, more generally, a lot of scalar 3-curvature) inside a fixed volume.

The next step should be the proof that trapped surfaces are enclosed, on initial Cauchy slices, by the portion of an event horizon lying in the Cauchy slice. Thus, one would wish to prove, that trapped surfaces, once they appear, always imply the existence of event horizons, that is, the existence of a black hole.

The second part of this programme has not been done, at least in sufficiently general situations, but the first problem is relatively well shaped (see [2] for a review).

* This work is supported by the Government Grant No PB 2526/2/91.

In this paper we will examine the formation of trapped surfaces in nonspherical cosmologies. We will study an open, expanding universe, which locally is inhomogeneous and anisotropic, but very far from bounded regions coincides with a flat, homogeneous and isotropic Friedman–Robertson–Walker universe. We combine techniques of two previous papers [3,4], to prove in Sections 3 and 4 that in conformally flat universes (initially) time-independent perturbations give rise to the formation of trapped surfaces if and only if the amount of energy is large comparing to the size of region in which it is compacted. A part of Section 4 pursues further the previously obtained results on the formation of trapped surfaces in asymptotically and conformally flat geometries.

The order of the rest of this work is following. Section 2 is introductory one. Section 5 shows that there is an upper limit for the amount of matter inside a volume of a fixed size. That result has been proven earlier (see [2] for the discussion of known results), and it suggests that the old idea of Einstein [5] that singularities do not exist, may be true, at least for some types of singularities, e.g., massive ones. More conclusive statements would require the solving of the problem of the existence of maximal slices, or its equivalent used below, slices of constant mean curvature.

Section 6 discusses the formation of surfaces that are pointwise trapped. The last Section comprises some comments on the obtained results and on possible generalizations.

2. Preliminaries

Initial data for the evolution of Einstein equations consist of energy density ρ and matter current J_b, which are prescribed on a space-like hypersurface Σ with a three-geometry defined by induced metric tensor g_{ab} and with an immersion of Σ into 4-dimensional space-time given by an external curvature $K_{ab} = \frac{1}{2}\pounds_t g_{ab}$. This whole set of functions is not arbitrary, since they have to satisfy the initial value equations:

$$ {}^{(3)}R[g] - K_{ab}K^{ab} + (K^a{}_a)^2 = 16\pi\rho, \tag{2.1}$$

$$ D_aK^a{}_b - D_bK^a{}_a = -8\pi J_b. \tag{2.2}$$

In (2.1) ${}^{(3)}R[g]$ is the scalar curvature of Σ. D_a in (2.2) denotes covariant derivative of the 3-dimensional metric. We put $c = 1, G = 1$.

As it was mentioned, we consider a cosmological Friedman–Lemaître model (with k=0) with *conformally flat* perturbations. We describe solutions in terms of their finite deviation from homogeneous-isotropic background. It is necessary to point out that we do not apply linearization procedure and our analysis is general one.

Let us relate the background quantities (denoted by a hat) and perturbed ones as follows:

$$g_{ab} = \Phi^4 \hat{g}_{ab}, \qquad g^{ab} g_{bc} = \delta^a_c, \qquad g^{ab} = \Phi^{-4} \hat{g}^{ab}, \tag{2.3}$$

$$\hat{g}_{ab} = a(t) \tilde{g}_{ab}, \tag{2.4}$$

$$\widehat{K}_{ab} = \frac{1}{2} \frac{da}{dt} \tilde{g}_{ab} = \beta(t) \hat{g}_{ab}, \tag{2.5}$$

$$K_{ab} = \Phi^4 \widehat{K}_{ab} + \delta K_{ab} = \beta(t) g_{ab} + \delta K_{ab}, \tag{2.6}$$

$$\rho = \hat{\rho} + \delta\rho, \qquad J_b = \delta J_b. \tag{2.7}$$

All indices of non-hatted quantities are raised and lowered with the metric (2.3). Φ is a conformal factor. In (2.4) $\tilde{g}_{ab}$ denotes a flat metric and $a(t)$ is a scalar function determined by Friedman equation. In (2.5) $\beta(t) = \frac{d}{dt} \ln a/2$.

We employ, following [7], special coordinates adapted to equipotential surfaces S (*i.e.*, surfaces of constant Φ). In these coordinates the three-dimensional line element reads

$$ds^2 = \Phi^4(\sigma) a(t) (\tilde{g}_{\sigma\sigma} d\sigma^2 + \tilde{g}_{\tau\tau} d\tau^2 + \tilde{g}_{\phi\phi} d\phi^2 + 2\tilde{g}_{\tau\phi} d\tau d\phi). \tag{2.8}$$

Here $\sigma \geq 0$, σ foliates the level surfaces of the Φ, that from now one are *assumed to be convex*, and τ, ϕ are quasi-angle variables.

Remark. One can show, that outside matter the surfaces of constant Φ coincide with surfaces of constant g_{00}. Therefore, the equipotential surfaces have a transparent physical meaning — these are surfaces of constant redshift — and might be detected experimentally.

The cosmological background satisfies the following relation

$$(\hat{g}^{ab} \widehat{K}_{ab})^2 - (\hat{g}^{ad} \hat{g}^{bc} \widehat{K}_{dc} \widehat{K}_{ab}) = 6\beta^2 = 16\pi\hat{\rho}. \tag{2.9}$$

We restrict our attention to the case, when

$$\delta K_{ab} = 0, \tag{2.10}$$

which guarantees us the absence of gravitational waves. This allows us to use the total proper rest mass as the quasilocal measure of the gravitational energy. The constraints (2.1, 2.2) reduce to a single equation

$${}^{(3)}R = 16\pi\delta\rho. \tag{2.11}$$

The metric $\hat{g}_{ab}$ is flat, hence the scalar curvature of the conformally related metric g_{ab} reads

$$^{(3)}R = -8\frac{\nabla^2\phi}{\phi^5}.$$

Inserting this into (2.11) we obtain the Lichnerowicz equation:

$$\nabla^2\Phi = -2\pi\delta\rho\Phi^5. \tag{2.12}$$

3. Necessary condition

In this chapter we will find a necessary condition for a surface S to be *an averaged trapped surface.* We start with expressions for the orthonormal unit vectors to S, in all the three metrics that were defined hitherto:

$$n^i = \left(\frac{1}{\phi^2\sqrt{a\tilde{g}_{\sigma\sigma}}}, 0, 0\right), \qquad \hat{n}^i = \left(\frac{1}{\sqrt{a\tilde{g}_{\sigma\sigma}}}, 0, 0\right), \qquad \tilde{n}^i = \left(\frac{1}{\sqrt{\tilde{g}_{\sigma\sigma}}}, 0, 0\right). \tag{3.1}$$

We define the mass of the inhomogeneity as follows

$$\delta M = \int\limits_V \delta\rho dV = \int\limits_{\widehat{V}} \delta\rho\Phi^6 d\widehat{V}. \tag{3.2}$$

Let us multiply equation (2.12) by Φ and integrate over the volume $\widehat{V}$. Using the above definition of δM we get

$$\delta M = -\frac{1}{2\pi}\int\limits_{\widehat{V}} \Phi\nabla^2\Phi d\widehat{V} = \frac{1}{2\pi}\int\limits_{\widehat{V}} \partial_i\Phi\partial^i\Phi d\widehat{V} - \frac{1}{2\pi}\int\limits_{\widehat{S}} \Phi\partial_i\Phi\hat{n}^i d\widehat{S} =$$

$$\frac{1}{2\pi}\int\limits_{\widehat{V}} \partial_i\Phi\partial^i\Phi d\widehat{V} - \frac{\sqrt{a}}{2\pi}\int\limits_{\widetilde{S}} \Phi\partial_\sigma\Phi\frac{1}{\sqrt{\tilde{g}_{\sigma\sigma}}} d\widetilde{S} \tag{3.3}$$

We need to calculate the following quantities, *(i)* the divergence of the normal unit vector n^i, *i.e.*, the mean curvature of a surface S in the physical metric g_{ik}

$$D_i n^i = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}n^i) = \frac{\partial_\sigma \ln\sqrt{\xi}}{\Phi^2\sqrt{a\tilde{g}_{\sigma\sigma}}} + \frac{4\Phi\partial_\sigma\Phi}{\Phi^4\sqrt{a\tilde{g}_{\sigma\sigma}}}, \tag{3.4}$$

(ii) the mean curvature of S with respect to the background cosmological metric

$$\tilde{p} = \frac{1}{\sqrt{\tilde{g}}}\partial_i(\sqrt{\tilde{g}}\tilde{n}^i) = \frac{\partial_\sigma \ln\sqrt{\xi}}{\sqrt{\tilde{g}_{\sigma\sigma}}}. \tag{3.5}$$

Above $\xi = g_{\tau\tau}g_{\varphi\varphi} - g^2_{\tau\varphi}$. Inserting (3.4) to (3.5) we obtain the identity

$$\Phi\partial_\sigma\Phi = \left(D_i n^i - \frac{1}{\Phi^2\sqrt{a}}\tilde{p}\right)\frac{\Phi^4\sqrt{a\bar{g}_{\sigma\sigma}}}{4}. \tag{3.6}$$

We replace the integrand of the last term of (3.3) by the right hand side of (3.6), in order to get

$$\delta M = \frac{1}{2\pi}\int_{\widehat{V}} \partial_i\Phi\partial^i\Phi\, d\widehat{V} - \frac{\Phi^4 a}{8\pi}\int_{\widetilde{S}} D_i n^i\, d\widetilde{S} + \frac{\Phi^2\sqrt{a}}{8\pi}\int_{\widetilde{S}} \tilde{p}\, d\widetilde{S}. \tag{3.7}$$

A surface S is called an averaged trapped surface, if the mean expansion ϑ of outgoing future directed null geodesics which are orthogonal to S is nonpositive,

$$\int_S \vartheta dS \le 0,$$

where ϑ relates to the initial data of Einstein equations g_{ab}, K_{ab} by

$$\begin{aligned}\vartheta = D_a n^a - K_{ab}n^a n^b + g_{ab}K^{ab} &= D_a n^a - \widehat{K}^{ab}\hat{n}_a\hat{n}_b + \hat{g}^{ab}\widehat{K}_{ab}\\ &= D_a n^a + 2\beta.\end{aligned}$$

In the last two equations we take into account (2.6) and $\delta K_{ab} = 0$. The condition that S is an averaged trapped surface reads now

$$\int_{\widetilde{S}} (D_a n^a + 2\beta) d\widetilde{S} \le 0 \tag{3.8}$$

We rewrite (3.7) in the following form, by adding and subtracting a term containing β:

$$\begin{aligned}\delta M =& \frac{1}{2\pi}\int_{\widehat{V}} \partial_i\Phi\partial^i\Phi d\widehat{V} - \frac{\Phi^4 a}{8\pi}\int_{\widetilde{S}} (D_i n^i + 2\beta)\, d\widetilde{S}\\ &+ \frac{\Phi^4 a}{8\pi}\int_{\widetilde{S}} 2\beta d\widetilde{S} + \frac{\Phi^2\sqrt{a}}{8\pi}\int_{\widetilde{S}} \tilde{p}\, d\widetilde{S}.\end{aligned} \tag{3.9}$$

Now let us assume that S is an averaged trapped surface. In such a case the first two terms of the above equation are positive. A classical isoperimetric inequality (see, *e.g.*, [7]) gives for a convex surface $\widetilde{S}$ in the three-dimensional flat space the estimation

$$\int_{\widetilde{S}} \tilde{p} d\widetilde{S} \geq 4\sqrt{\pi \widetilde{S}}$$

($\widetilde{S}$ denotes an area of $\widetilde{S}$ and $\tilde{p}$ is just the mean curvature of $\widehat{S}$ with respect to the Euclidean metric $\tilde{g}_{ik}$). Thus we may conclude that

$$\Phi^2 \frac{\sqrt{a}}{(8\pi)} \int_{\widetilde{S}} \tilde{p} d\widetilde{S} \geq \sqrt{\frac{S}{(4\pi)}}.$$

Let us remark also that the third term on the right hand side of (3.9) is equal to $(\beta/4\pi)S$. Taking into account this fact, the last inequality and Eq. (2.9), we may infer the following

Theorem 1. (a necessary condition). Under the assumption of:

$$\delta K^a{}_a = 0,$$

if an equipotential convex surface S is trapped then

$$\delta M \geq \frac{1}{2}\sqrt{\frac{S}{\pi}} + S\sqrt{\frac{\widehat{\rho}}{6\pi}},$$

where S denotes the area of a surface S.

4. Sufficient condition

Let us start from the assumption that an equipotential surface S *is not trapped.* Then its averaged expansion is positive,

$$\int_{\widetilde{S}} (D_i n^i + 2\beta) d\widetilde{S} > 0.$$

The equation (3.9) and the above inequality imply

$$\delta M \leq S\sqrt{\frac{\hat{\rho}}{6\pi}} + \frac{1}{2\pi}\int_{\widehat{V}} \partial_i \Phi \partial^i \Phi d\widehat{V} + \frac{\Phi^2\sqrt{a}}{8\pi}\int_{\widetilde{S}} \tilde{p} d\widetilde{S}. \tag{4.1}$$

We define quantities

$$D(S(\sigma)) = \frac{1}{2\pi}\int_{\widehat{V}} \partial_i\Phi\partial^i\Phi d\widehat{V} + \frac{\Phi^2\sqrt{a}}{8\pi}\int_{\widetilde{S}} \tilde{p} d\widetilde{S}, \tag{4.2}$$

$$\mathrm{Rad}(S(\sigma)) = D(S(0)) + \frac{\sqrt{a}}{8\pi}\int_0^\sigma \Phi^2(s)\partial_s\left(\int_{\widetilde{S}} \tilde{p} d\widetilde{S}\right) ds. \tag{4.3}$$

$\mathrm{Rad}(S(\sigma))$ equals to the proper radius in the case of spherical symmetry [4]. It is reasonable to conjecture that $\mathrm{Rad}(S(\sigma))$ is bounded from above by the largest proper radius of the volume enclosed by the surface S in all geometries when the equipotential surfaces are convex [4,7]. We prove now an important technical estimation:

Lemma. Assume that $\delta\rho \geq 0$. Then

$$\mathrm{Rad}(S(\sigma)) \geq D(S(\sigma)). \tag{4.4}$$

Proof.

For $\sigma = 0$ the equation (4.4) becomes an identity. Thence it is sufficient to show that $\partial_\sigma \mathrm{Rad}(S(\sigma)) \geq \partial_\sigma D(S(\sigma))$, *i.e.*,

$$\frac{\sqrt{a}}{8\pi}\Phi^2(\sigma)\partial_\sigma \int_{\widetilde{S}} \tilde{p} d\widetilde{S} \geq \frac{\sqrt{a}}{2\pi}\int_{\widetilde{S}} \tilde{n}^\sigma (\partial_\sigma\Phi)^2 d\widetilde{S}$$
$$+ \frac{\sqrt{a}\Phi\partial_\sigma\Phi}{4\pi}\int_{\widetilde{S}} \tilde{p} d\widetilde{S} + \frac{\Phi^2\sqrt{a}}{8\pi}\partial_\sigma \int_{\widetilde{S}} \tilde{p} d\widetilde{S}.$$

From the maximum principle and $\delta\rho \geq 0 : \quad \partial_\sigma\Phi \leq 0$, hence the above equation holds true assuming that

$$\int_{\widetilde{S}} (2\partial_\sigma\Phi\tilde{n}^\sigma + \Phi\tilde{p}) d\widetilde{S} \geq 0. \tag{4.5}$$

It is possible to prove even a stronger inequality

$$\int_{\widetilde{S}} \left[2\partial_\sigma\Phi\tilde{n}^\sigma - (1-\Phi)\tilde{p}\right] d\widetilde{S} \geq 0. \tag{4.6}$$

Eq. (4.6) was proven in [4], under certain (conjectured) property of convex foliations in flat geometry. Here we shall present a different proof, communicated to one of us (EM) by Eanna Flanagan.

At the beginning, we prove (4.6) under a more stringent condition than stated in Lemma, namely we assume that $\delta\rho = 0$ outside S.

Let Ψ be a function satisfying the Laplace equation outside S and having the values -1 on S and 0 at infinity. It can be seen that $\Phi = 1 + (1 - A)\Psi$, where A is a value of Φ on S. The inequality (4.6) is now equivalent to

$$\int_{\tilde{S}} \left(\tfrac{1}{2}\tilde{p} - \partial_\sigma \Psi \tilde{n}^\sigma\right) d\tilde{S} \geq 0.$$

Let us define quantities

$$C = \frac{1}{4\pi} \int_{\tilde{S}} \partial_\sigma \Psi \tilde{n}^\sigma \, d\tilde{S},$$

$$M = \frac{1}{2} \int_{\tilde{S}} \tilde{p} \, d\tilde{S}.$$

The first quantity is related to the electrostatic capacity and the relation between the two was discussed by Szegö [9], who proved that for any convex surface embedded in Euclidean 3-space $M \geq 4\pi C$. This ends the proof of Lemma in the case of $\delta\rho = 0$.

(4.6) holds also in the case of non-vanishing (outside S) matter density. Let us assume that Φ and χ have the same boundary values, A at S and 1 at infinity and $\nabla^2\chi = 0$, $\nabla^2\Phi = -2\pi\delta\rho\chi^5$. Notice that $\nabla^2(\Phi - \chi) \leq 0$; thus, from the maximum principle, $u = \Phi - \chi$ achieves its maximal value somewhere between S and infinity. Since $u = 0$ at S, this implies that at S we have $\partial_\sigma(\Phi - \chi) \geq 0$, *i.e.*,

$$\int_{\tilde{S}} (\partial_\sigma \chi) \tilde{n}^\sigma d\tilde{S} \leq \int_{\tilde{S}} (\partial_\sigma \Phi) \tilde{n}^\sigma d\tilde{S}.$$

χ satisfies the inequality (4.6) at S, according to the preceding proof. Because of the last inequality and the boundary condition $u = 0$ at S, we conclude that also Φ fulfills the property (4.6). This ends the proof of Lemma.

Now we may replace the last two terms in (4.1) by $\mathrm{Rad}(S(\sigma))$, as to get

$$\delta M \leq S\sqrt{\frac{\hat{\rho}}{6\pi}} + \mathrm{Rad}(S(\sigma)). \tag{4.7}$$

If S is not trapped, then the amount of perturbative mass inside it can not exceed expression on the right hand side of (4.7). Thus we have proven the following:

Theorem 2. (a sufficient condition). Assume that:

$$\delta\rho \geq 0,$$

$$\delta K^a{}_b = 0.$$

If the amount of perturbative mass satisfies the inequality

$$\delta M \geq S\sqrt{\frac{\widehat{\rho}}{6\pi}} + \mathrm{Rad}(S(\sigma)),$$

then a convex surface S is trapped.

5. Mass of perturbations is bounded

From Eq. (3.6) we have

$$D_i n^i = \frac{\tilde{p}}{\Phi^2\sqrt{a}} + \frac{4}{\Phi^4\sqrt{a\tilde{g}_{\sigma\sigma}}}\Phi\partial_\sigma\Phi.$$

Inserting the above equation in (3.7) and replacing the first and the last terms of (3.7) by $D(S(\sigma))$ (see Eq. (4.2)) we obtain

$$\delta M = D(S(\sigma)) - \frac{\Phi^2\sqrt{a}}{4\pi}\int\limits_{\widetilde{S}}\left(\frac{1}{2}\Phi\tilde{p} + 2\tilde{n}^\sigma\partial_\sigma\Phi\right)d\widetilde{S}.$$

This equation can be rewritten in the form

$$\delta M = D(S(\sigma)) + \frac{\Phi^2\sqrt{a}}{8\pi}\int\limits_{\widetilde{S}}\tilde{p}d\widetilde{S} - \frac{\Phi\sqrt{a}}{4\pi}\int\limits_{\widetilde{S}}(\Phi\tilde{p} + 2\tilde{n}^\sigma\partial_\sigma\Phi)d\widetilde{S}.$$

From (4.2) and the inequality (4.5) we get

$$\delta M \leq 2D(S(\sigma)).$$

By the preceding Lemma, the following holds true.

Theorem 3. Under the assumptions:

$$\delta\rho \geq 0, \delta K^a{}_b = 0.$$

the amount of perturbing mass is bounded from above,

$$\delta M \leq 2\mathrm{Rad}(S(\sigma)).$$

6. The existence of pointwise trapped surfaces

Using (3.4), one gets the following from the definition of the mean expansion ϑ

$$4\int_S \hat{n}^\sigma d\widehat{S}(-\Phi\partial_\sigma\Phi) = -\int_S \vartheta dS + \int_S (\hat{p}+\beta)\Phi^2 d\widehat{S}. \tag{6.1}$$

If S is not pointwise trapped, then certainly the largest value of the expansion ϑ is positive, *i.e.*,

$$4\Phi(-\partial_\sigma\Phi) \leq \Phi^2 \max(\hat{n}\hat{p}). \tag{6.2}$$

Here and below $\max(f)$ means the maximal value of $f(x)$, for $x \in S$. Combining (6.1) and (6.2) one obtains then

$$\int_S \hat{n}^\sigma[\max(\hat{p}\hat{n})\Phi^2 + \beta\Phi^2 + 4\Phi\partial_\sigma\Phi - (\hat{p}\hat{n}_\sigma\Phi^2 + \beta\Phi^2 + 4\Phi\partial_\sigma\Phi)]d\widehat{S}$$
$$= \int_S n^\sigma[\max(n_\sigma\vartheta) - n_\sigma\vartheta]dS \geq -\int_S \vartheta dS = 8\pi\Big(\delta M - D(S(\sigma)) - \frac{1}{4\pi}\beta S\Big). \tag{6.3}$$

The last equality follows from Eq. (3.9), as expressed in terms of the mean expansion and $D(S(\sigma))$ (see Eq. (4.2)).

The inequality (6.3) has always to be true, if S is not pointwise trapped, as indicated above. Thus, if (6.3) is broken, then S has to be trapped. In particular, from the inequality (4.4) of our Lemma, we may conclude the following

Theorem 4. Assume that inside an equipotential surface S the following conditions are fulfilled:

(i) $\delta K_a^b = 0, \delta\rho \geq 0$,
(ii) $\delta M \geq \mathrm{Rad}(S(\sigma)) + \frac{1}{4\pi}\beta S + \frac{1}{8\pi}\int_S n^\sigma[\max(n_\sigma p) - n_\sigma p]$.

Then S is pointwise trapped.

7. Concluding remarks

We have discussed the formation of trapped surfaces in the class of conformally flat deformations (which initially do not change the expansion of the geometry) of Friedman flat expanding universes. Trapped surfaces have a well defined physical sense, namely a bundle of light rays which emanates from a trapped compact surface S (orthogonally to it) increases its intensity, at least initially, when moving outwards. That is unlike our everyday's experience; we know surfaces (say, concave mirrors) that have the property of focusing light rays, but they are not closed. A compact trapped surface S plays a role of a compact reflector; from this intuitive definition it is obvious that trapped surfaces can exist only in strongly curved geometries.

We have proven that, assuming the absence of gravitational waves, trapped surfaces can appear only in the presence of large masses (Section 3) and that they must appear if concentration of matter in a fixed volume is large. Surprisingly, the amount of perturbed mass in a finite volume has to be bounded from above (Section 5). The formation of trapped surfaces requires the more matter the larger is their nonsphericality (Section 6). Moreover, the expansion of the Universe (as represented in the above statements by the Hubble constant β or the averaged matter density $\hat{\rho}$) makes it more difficult to form a trapped surface, in the perfect agreement with intuition.

The all results are strict, since all inequalities are saturated in some spherically symmetric situations [3].

These results might be proved also in another foliations of the spacetime. A generalization of the radial gauge (in which $\mathrm{Tr}\, K = K^\sigma_\sigma$ — no summation over σ) allows one to consider spacetimes with initial data off time symmetry, since then the notions of minimal surfaces and apparent horizons are identical[1].

The existence problem of these foliations is not solved as yet, similarly as for the constant mean curvature foliations that are used in this paper.

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[1] I thank Niall O' Murchadha and Jemal Guven for discussing that point.