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HOOP-LIKE CONFIGURATIONS OF THE CLASSICAL STRING WITH RIGIDITY*

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Time evolution of hoop-like configurations of the classical string with rigidity is studied. Emphasis is put on effects which are due to the presence of higher derivatives in the theory. Trajectories close to Nambu–Goto trajectories and tachyonic trajectories of the string are found. It is pointed out that the dynamics of the hoop-like string with rigidity is probably non-integrable.

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1. Introduction

In the present paper we continue our studies of the classical string with rigidity. Recently, this kind of string has been considered in connection with a string-like limit of vortices [1-6], low energy QCD [7,8], and statistical physics of random surfaces [9-11]. In the previous paper [12] we have considered some rather formal problems of Lagrangian and Hamiltonian formulation of dynamics of the classical string with rigidity. In the present paper we study trajectories of this string in four-dimensional Minkowski space-time.

Equations of motion for the classical string with rigidity are very complicated. They form a set of four, nonlinear, partial differential equations with derivatives of the fourth order for four functions $X^{\mu}(\tau,\sigma)$, $\mu = 0,1,2,3$, of the two variables τ and σ. Here $\sigma \in [0,2\pi]$ is a parameter along the string, and τ is the evolution parameter. We assume that the string is closed, *i.e.*,

$$X^{\mu}(\tau,0) = X^{\mu}(\tau,2\pi)$$

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for all τ. Because the model possesses local reparametrization invariance, we have also to choose a gauge condition. Models with equations of motion of the order higher than second were considered many times in the past, see *e.g.* [13-15], and always they were rather problematic. We regard the string with rigidity as an approximation to a vortex. One might wonder whether some new terms could be included in the Lagrangian of the model to the effect that the model would be more satisfactory. Till now no such improvement of the string with rigidity model is known to us.

In order to simplify the problem we adopt a specific Ansatz for the trajectories of the string. Namely, we consider only closed strings of hoop shape with variable radius of the hoop. We also restrict possible motions of the center of the hoop to a straight-line, *e.g.* to the x^3-axis. Thus we abandon many other classes of trajectories. However the set of trajectories we are left with is sufficiently large to allow for nontrivial dynamics. As we shall see, there is a coupling between the radius and the position of the center of the hoop degrees of freedom. Such coupling is absent in the case of Nambu–Goto string. Because of this coupling the hoop-like string with rigidity can have tachyonic trajectories.

The main our findings in the present paper are the following. We find a family of solutions which are close to Nambu–Goto trajectories in a finite time interval. These solutions depend analytically on a rigidity parameter α. We give some characteristics of the tachyonic trajectories of the string. We point out that the equations of motion for the hoop-like string are of nonintegrable type.

In Section 2 of our paper we briefly recall basic formulae for the classical, closed string with rigidity, and we present the hoop-like Ansatz for the trajectories of the string. In Section 3 we show that the equations of motion for the hoop-like string can be written in the form of a nonintegrable set of Newton-type equations of motion with the second order derivatives only. Section 4 contains a discussion of the radial motions of the string. In particular, we present there the family of solutions which are close to Nambu–Goto trajectories. Section 5 is devoted to the tachyonic solutions. Section 6 contains several general remarks with which we would like to conclude our investigations of the string with rigidity.

2. Basic formulae for the string with rigidity and the Ansatz

Lagrangian for the string with rigidity has the following form: [7,9]

$$\mathcal{L} = \sqrt{-g}(-\gamma + \alpha \Delta X^{\mu} \Delta X_{\mu}), \tag{1}$$

where α and $\gamma > 0$ are constants. Parameter α is called the rigidity parameter. For $\alpha = 0$ we obtain the Nambu–Goto string. In formula (1)

$g = \det(g_{ab})$, where

$$g_{ab} = \partial_a X^\mu \partial_b X_\mu \tag{2}$$

is the metric tensor on a world sheet of the string, and

$$\triangle X^\mu = \frac{1}{\sqrt{-g}}\partial_a \left(\sqrt{-g} g^{ab}\partial_b X^\mu\right) \tag{3}$$

is the Laplacian of $X^\mu(\tau, \sigma)$. We use the compact notation

$$u^0 \equiv \tau, \;\; u^1 \equiv \sigma, \;\; \partial_a \equiv \frac{\partial}{\partial u^a}.$$

We assume that $g < 0$.

Euler–Lagrange equation of motion following from Lagrangian (1) can be written in the following form [12]:

$$(\gamma - \alpha \triangle X^\nu \triangle X_\nu)\triangle X_\mu + 2\alpha\left(\triangle(\triangle X_\mu) - g^{ab}X^\nu_{,a}X_{\mu,b}\triangle(\triangle X_\nu)\right)$$
$$-4\alpha g^{ab}g^{cd}(\triangle X_\nu)_{,b}X^\nu_{,c}\nabla_a X_{\mu,d} = 0, \tag{4}$$

where

$$\nabla_a X_{\mu,d} = X_{\nu,ab}(\delta^\nu_\mu - g^{cd}X^\nu_{,d}X_{\mu,c}) \tag{5}$$

is the covariant derivative of $X_{\mu,d}$ corresponding to the metric g_{ab} [12]. It is clear that any solution of the Nambu–Goto equation,

$$\triangle X_\mu = 0, \tag{6}$$

also obeys equation (4).

Conserved energy-momentum and four-dimensional angular momentum $M_{\mu\nu}$ of the string are given by the following formulae [12]:

$$P_\mu = \int_0^{2\pi} d\sigma p_\mu, \tag{7}$$

$$M_{\mu\nu} = \int_0^{2\pi} d\sigma(X_\mu p_\nu + 2\alpha\sqrt{-g}(X_{\nu,0}g^{00} + 2X_{\nu,1}g^{01})\triangle X_\mu) - (\mu \leftrightarrow \nu), \tag{8}$$

where

$$p_\mu = \sqrt{-g}g^{0b}(\gamma - \alpha\triangle X_\sigma \triangle X^\sigma)X_{\mu,b} + 2\alpha\partial_0(\sqrt{-g}g^{00}\triangle X_\mu)$$
$$+ 2\alpha\sqrt{-g}g^{0a}g^{bc}(2\triangle X_\sigma X^\sigma_{,ab}X_{\mu,c} + X^\lambda_{,bc}X_{\lambda,a}\triangle X_\mu). \tag{9}$$

We see from formulae (7), (8), (9) that the energy-momentum and the angular momentum of the string with rigidity in the case of Nambu–Goto trajectories, *i.e.*, the ones obeying equation (6), have the same values as for the Nambu–Goto string described by Lagrangian with $\alpha = 0$. In this sense, the Nambu–Goto string is fully embedded in the model of the string with rigidity.

It is clear that equation (4) is too complicated for a general analysis of its solutions. In particular it is not known, whether there exists a gauge choice which would linearize equation (4).

In the following we will consider the string with rigidity of the circular shape, with the center which can move along the x^3-axis. The radius of the circle may vary in time. The appropriate Ansatz for the $X^\mu(\tau,\sigma)$ functions has the following form:

$$(X^\mu(\tau,\sigma)) = (\tau, r(\tau)\cos\sigma, r(\tau)\sin\sigma, z(\tau)). \tag{10}$$

We have chosen the evolution parameter τ to be the physical time x^0,

$$\tau = x^0 \equiv t.$$

In the case of Ansatz (10) the induced metrics (2) has the simple diagonal form,

$$(g_{ab}) = \begin{pmatrix} h^2(t) & 0 \\ 0 & -r^2(t) \end{pmatrix}, \tag{11}$$

where

$$h^2 \equiv 1 - \dot{r}^2 - \dot{z}^2. \tag{12}$$

Notice that g_{ab} does not depend on σ.

For $r = 0$ the metric tensor is singular. Therefore we restrict the range of the r variable to $r > 0$. In the case of metric tensor (11) the Laplacian $\triangle f$ of a function $f(t,\sigma)$ is given by the following formula:

$$\triangle f = \frac{1}{hr}\partial_0\left(\frac{r}{h}\partial_0 f\right) - \frac{1}{r^2}\partial_1^2 f. \tag{13}$$

Direct derivation of equations of motion for the hoop-like string with rigidity by inserting Ansatz (10) into Eq. (4) is quite tedious. We prefer to proceed in the following manner. First, adapting a reasoning presented by S.Coleman [16] we prove that we may first insert the Ansatz (10) in Lagrangian (1) and to consider only the two Euler–Lagrange equations for $r(t)$,$z(t)$ following from such reduced Lagrangian. Next we shall use canonical formalism for theories with higher derivatives, [17,18]. In this way, instead of two equations for $r(t)$,$z(t)$ with the fourth order derivatives with respect to time we shall have a set of simpler equations of motion with derivatives of the first and second order. Canonical formalism for the string with rigidity in the general case is discussed in papers [19,12].

Let us consider action S for the string with rigidity,

$$S = \int_{t_1}^{t_2} dt \int_0^{2\pi} d\sigma \mathcal{L},$$

where Lagrangian $\mathcal{L}$ is given by formula (1). General variation of the action S can be written in the following form [12]

$$\delta S = \int_{t_1}^{t_2} dt \int_0^{2\pi} d\sigma R_\mu \delta x^\mu + \text{boundary terms}, \tag{14}$$

where R_μ denotes the l.h.s. of Euler–Lagrange equation (4). In our case the boundary terms vanish because we consider the closed string and because we require that

$$\delta x^\mu = 0 = \delta \dot{x}^\mu$$

at the ends t_1, t_2 of the considered time interval. It is easy to check that for our Ansatz (10) R_μ depends on σ in the same manner as x^μ does, *i.e.*, R_0 and R_3 are constant with respect to σ,

$$R_1 = f(t)\cos\sigma, \quad R_2 = f(t)\sin\sigma, \tag{15}$$

where $f(t)$ depends on $r(t), z(t)$ and their derivatives. Notice that the dependence on σ has the form of particular terms from Fourier expansion. Also a generic variation $\delta x^\mu(t,\sigma)$ which obeys the imposed boundary conditions can be written in the form of Fourier series

$$\delta x^\mu(t,\sigma) = \delta c^\mu(t) + \sum_{n=1}^{\infty} \left(\delta a_n^\mu(t)\cos n\sigma + \delta b_n^\mu(t)\sin n\sigma\right). \tag{16}$$

Because different terms in the Fourier series are orthogonal to each other with respect to the integration $\int_0^{2\pi} d\sigma$, nonvanishing contribution to the r.h.s. of formula (14) give only the following terms from the expansion (16):

$$\delta c^\mu \ \text{ with } \mu = 0,3, \ \ \delta a_1^1, \ \ \delta b_1^2 = \delta a_1^1.$$

The relation between δa_1^1 and δb_1^2 is due to the special form (15) of R_1 and R_2. Therefore it is sufficient to consider variations δx^μ of the following form:

$$(\delta x^\mu) = \left(\delta c^0(t), \delta a_1^1(t)\cos\sigma, \delta a_1^1(t)\sin\sigma, \delta c^3(t)\right). \tag{17}$$

Variations of this form are more general than the variations which follow from the Ansatz (10) by varying $r(t)$ and $z(t)$. In the latter case

$$(\delta x^{\mu}) = (0, \delta r(t)\cos\sigma, \delta r(t)\sin\sigma, \delta z(t)). \tag{18}$$

Variations (18) leave R_0 arbitrary, while variations (17) lead to the equation $R_0 = 0$. However, this equation follows from a Noether identity coming from the reparametrization invariance of the action S, and from equations $R_i = 0$, $i = 1, 2, 3$. The required identity has the following form [12]

$$R_{\mu}\dot{x}^{\mu} = 0. \tag{19}$$

It is obeyed by all smooth functions $x^{\mu}(t, \sigma)$, irrespectively whether do they satisfy the equations of motion or not. For our Ansatz (10) this identity implies that

$$R_o = \dot{r}(t)f(t) + \dot{z}(t)R^3.$$

Therefore it is sufficient to consider equations

$$f(t) = 0, \quad R^3 = 0.$$

These equations follow from variations (18).

Thus we have shown that we may insert the Ansatz (10) into Lagrangian (1) and to compute Euler-Lagrange equations for the functions $r(t)$ and $z(t)$ from such reduced Lagrangian. The reduced Lagrangian has the following relatively simple form:

$$\begin{aligned} L = &- (\gamma r + \frac{\alpha}{r})h - \frac{\alpha \dot{r}^2}{rh} - \frac{\alpha r}{h^5}\left((1-\dot{z}^2)\ddot{r}^2 + (1-\dot{r}^2)\ddot{z}^2 + 2\dot{r}\dot{z}\ddot{r}\ddot{z}\right) \\ &- \frac{2\alpha}{h^3}\left((1-\dot{z}^2)\ddot{r} + \dot{r}\dot{z}\ddot{z}\right). \end{aligned} \tag{20}$$

It is easy to check that the last term on the r.h.s. of formula (20) is the full time derivative of $\frac{-2\alpha\dot{r}}{h}$. In a theory with Lagrangian involving the second order derivatives such term does not contribute to equations of motion and to Hamiltonian. Therefore we drop it out. Thus the final form of the Lagrangian is

$$L = -(\gamma r + \frac{\alpha}{r})h - \frac{\alpha \dot{r}^2}{rh} - \frac{\alpha r}{h^5}\left((1-\dot{z}^2)\ddot{r}^2 + (1-\dot{r}^2)\ddot{z}^2 + 2\dot{r}\dot{z}\ddot{r}\ddot{z}\right), \tag{21}$$

where

$$h = (1 - \dot{r}^2 - \dot{z}^2)^{\frac{1}{2}}.$$

Euler–Lagrange equations following from this Lagrangian have the following general form

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} = 0, \tag{22}$$

where $q_1 \equiv r$, $q_2 \equiv z$. These equations are nonlinear and they involve derivatives of the fourth order with respect to time.

For $\alpha = 0$ we obtain reduced Lagrangian for the hoop-like configurations of the Nambu–Goto string:

$$L_0 = -\gamma r(1 - \dot{r}^2 - \dot{z}^2)^{\frac{1}{2}}. \tag{23}$$

3. Equations of motion for the hoop-like string with rigidity

Let us first take a look at the Nambu–Goto case. Euler–Lagrange equations following from Lagrangian (23) have the following form:

$$\frac{dp_z}{dt} = 0, \quad \frac{dp_r}{dt} = -\gamma(1 - \dot{z}^2 - \dot{r}^2)^{\frac{1}{2}}, \tag{24}$$

where

$$p_z = E\dot{z}, \quad p_r = E\dot{r}, \tag{25}$$

and

$$E = \gamma r(1 - \dot{r}^2 - \dot{z}^2)^{-\frac{1}{2}} \tag{26}$$

is the conserved energy following from Lagrangian (23). It is related to the conserved energy $E_{\mathrm{N-G}}$ of the hoop-like Nambu–Goto string by the formula $E_{\mathrm{N-G}} = 2\pi E$. Formula (26) for the energy indicates that entire energy of the string is due to the string tension. There is no intrinsic mass attributed to the matter the string is made from. It is easy to see that Eqs (24) imply

$$\dot{z} = \text{const}, \quad \ddot{r} + \frac{\gamma^2}{E^2} r = 0. \tag{27}$$

Thus, the radius of the hoop harmonically oscillates with the frequency $\omega = \gamma/E_0$, which depends on the initial data. In particular, if $\dot{z} = 0$ and $\dot{r} = 0$ at the initial instant of time $t = 0$,

$$\omega = r_0^{-1}, \quad r(t) = r_0 \cos\frac{t}{r_0}, \tag{28}$$

where r_0 is the initial radius of the hoop.

In contradistinction to the Nambu–Goto case, equations of motion for the hoop-like string with rigidity cannot be explicitly solved. Below we show that these equations are equivalent to a system of three ordinary differential equations with derivatives of the second order. We are familiar with such Newton-type systems from classical mechanics of point-particles in three dimensional space. In the case of hoop-like string with rigidity the system turns out to be nonintegrable. The Newton-type system is obtained with the help of canonical formulation of theories with higher derivatives [17,18].

As the first step, we define the canonical momenta

$$p_z \equiv \frac{\partial L}{\partial \ddot{z}}, \quad p_r \equiv \frac{\partial L}{\partial \ddot{r}}, \tag{29}$$

$$P_z \equiv \frac{\partial L}{\partial \dot{z}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{z}}\right), \quad P_r \equiv \frac{\partial L}{\partial \dot{r}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{r}}\right), \tag{30}$$

and the Hamiltonian

$$H \equiv \ddot{r}p_r + \ddot{z}p_z + \dot{r}P_r + \dot{z}P_z - L. \tag{31}$$

In Hamiltonian (31) $\ddot{r}$ and $\ddot{z}$ are regarded as functions of the variables $r, z, \dot{r}, \dot{z}, p_r$ and p_z, obtained by inverting relations (29). At this point a problem of constraints might appear. This problem is not present in our case because assuming Ansatz (10) we have implicitly fixed the gauge. In general, Hamiltonian (31) can depend on the variables $r, z, \dot{r}, \dot{z}, p_r, p_z, P_r$ and P_z which are regarded as independent canonical variables. In our case (Lagrangian (21)) the Hamiltonian does not depend on z, and the dependence on P_r and P_z is explicitly seen on the r.h.s. of formula (31) — it is linear. Hamilton equations of motion have the following form

$$\dot{z} = \frac{\partial H}{\partial P_z}, \quad \dot{r} = \frac{\partial H}{\partial P_r}, \quad \ddot{z} = \frac{\partial H}{\partial p_z}, \quad \ddot{r} = \frac{\partial H}{\partial p_r}, \tag{32}$$

$$\dot{P}_z = -\frac{\partial H}{\partial z}, \; \dot{P}_r = -\frac{\partial H}{\partial r}, \; \dot{p}_z = -\frac{\partial H}{\partial \dot{z}}, \; \dot{p}_r = -\frac{\partial H}{\partial \dot{r}}. \tag{33}$$

This system of equations is equivalent to the Euler–Lagrange equations.

In the case of Lagrangian (21)

$$p_z = -\frac{2\alpha r}{h^5}\left((1-\dot{r}^2)\ddot{z} + \dot{r}\dot{z}\ddot{r}\right), \tag{34}$$

$$p_r = -\frac{2\alpha r}{h^5}\left((1-\dot{z}^2)\ddot{r} + \dot{r}\dot{z}\ddot{z}\right). \tag{35}$$

By inverting these relations we obtain

$$\ddot{z} = \frac{h^3}{2\alpha r}\left(\dot{r}\dot{z}p_r - (1-\dot{z}^2)p_z\right), \tag{36}$$

$$\ddot{r} = \frac{h^3}{2\alpha r}\left(\dot{r}\dot{z}p_z - (1-\dot{r}^2)p_r\right), \tag{37}$$

and

$$\begin{aligned} H =& \dot{r}P_r + \dot{z}P_z + \gamma r h + \frac{\alpha}{rh}(1-\dot{z}^2) \\ & - \frac{h^3}{4\alpha r}\left((1-\dot{z}^2)p_z^2 + (1-\dot{r}^2)p_r^2 - 2\dot{r}\dot{z}p_r p_z\right). \end{aligned} \tag{38}$$

The first two of Hamilton equations of motion (32) are merely trivial identities, the next two are identical with formulae (36), (37). Four equations (33) can be written in the following form

$$\dot{P}_z = 0, \tag{39}$$

$$\dot{P}_r = -2\gamma h + \frac{1}{r}(H - \dot{z}P_z - \dot{r}P_r), \tag{40}$$

$$\begin{aligned} \dot{p}_z =& - P_z + \frac{3\dot{z}}{h^2}(H - \dot{r}P_r - \dot{z}P_z) - \frac{h^3}{2\alpha r}p_z(\dot{r}p_r + \dot{z}p_z) \\ & - 2\gamma r\frac{\dot{z}}{h} - \frac{2\alpha\dot{z}}{rh} - \frac{4\alpha\dot{z}\dot{r}^2}{rh^3}, \end{aligned} \tag{41}$$

$$\begin{aligned} \dot{p}_r =& - P_r + \frac{3\dot{r}}{h^2}(H - \dot{r}P_r - \dot{z}P_z) - \frac{h^3}{2\alpha r}p_r(\dot{r}p_r + \dot{z}p_z) \\ & - 2\gamma r\frac{\dot{r}}{h} - \frac{4\alpha\dot{r}}{rh} - \frac{4\alpha\dot{r}^3}{rh^3}, \end{aligned} \tag{42}$$

where H is given by formula (38).

The Hamiltonian H is constant during the motion, as well as P_z, see Eq. (39). Multiplying these integrals of the motion by 2π we obtain the energy and the x^3 component of the momentum for the string with rigidity with Lagrangian (1). With the help of the formulae (7)–(9) it is easy to check that the only nonvanishing integrals of motion for the string with rigidity, which follow from Poincaré invariance are $P_0 = 2\pi H$, $P_3 \equiv 2\pi P_z$ and

$$M_{03} = 2\pi\Big(-tP_z + zH + \frac{2\alpha r}{h^3}\ddot{z}\Big). \tag{43}$$

The rather complicated set of Eqs ((36),(37),(40)-(42)) can be significantly simplified. The time evolution of the z variable can be computed from formula (43) regarded as the second order differential equation for $z(t)$:

$$\frac{2r}{h^3}\ddot{z} = \alpha^{-1}\Big(tP_z - zH + \frac{M_{03}}{2\pi}\Big). \tag{44}$$

Relatively simple equation for the function $r(t)$ can be obtained in the following manner. Using equation (37) we obtain

$$\frac{d}{dt}\left(\frac{2\alpha r}{h^3}\ddot{r}\right) = -\dot{p}_r + \ddot{r}(\dot{r}p_r + \dot{z}p_z) + \dot{r}\frac{d}{dt}(\dot{r}p_r + \dot{z}p_z).$$

Next, we can eliminate $\dot{p}_r, \dot{p}_z, \ddot{r}, \ddot{z}$ on the r.h.s. of this formula with the help of Eqs (36,37,41,42). We obtain

$$\frac{d}{dt}\left(\frac{2\alpha r}{h^3}\ddot{r}\right) = P_r - H\dot{r} + \frac{2\alpha\dot{r}}{rh}. \tag{45}$$

Equations (44,45) and (40) form a closed system of equations from which we can in principle determine the functions $r(t)$ and $z(t)$. From this system of equations we shall pass to another system of equations such that all equations have the Newton-type form. To this end we introduce an auxiliary variable y defined by the formula:

$$y = \frac{2r}{h^3}(z\ddot{z} + r\ddot{r}) - \frac{2r}{h} + \frac{H}{2\alpha}(r^2 + z^2). \tag{46}$$

With the aid of Eqs (44,45) it is easy to check that

$$\frac{dy}{dt} = \alpha^{-1}(rP_r + zP_z). \tag{47}$$

On the other hand, Eq. (40) can be written in the form

$$\frac{d}{dt}(rP_r + zP_z) = -2\gamma hr + H.$$

Thus

$$\ddot{y} = -2\frac{\gamma}{\alpha}hr + \frac{H}{\alpha}. \tag{48}$$

Formula (46) can be written as

$$\frac{2r}{h^3}\ddot{r} = \frac{y}{r} - \frac{H}{2\alpha r}(r^2 - z^2) + \frac{2}{h} - \frac{z}{\alpha r}\left(tP_z + \frac{M_{03}}{2\pi}\right), \tag{49}$$

where we have used Eq. (44) to eliminate $\ddot{z}$.

Equations (44), (48) and (49) form the Newton-type set of equations for the hoop-like string with rigidity. The constants H, P_z and M_{03}, as well as the initial data $y(0), \dot{y}(0)$ are determined by the initial data for the original equations of motion (22), *i.e.*, $r(0), z(0), \dot{r}(0), \dot{z}(0), \ddot{r}(0), \ddot{z}(0), \dddot{r}(0), \dddot{z}(0)$. The value of H can be computed from formula

$$H = \left(\gamma r + \frac{\alpha}{r}\right)\frac{1}{h} - \frac{\alpha \dot{r}^2}{r h^3} + \frac{2\alpha \dot{r}}{h^5}\left(\dot{r}\ddot{r} + \dot{z}\ddot{z}\right) - \frac{\alpha r}{h^5}(\ddot{r}^2 + \ddot{z}^2)$$
$$+ \frac{2\alpha r}{h^5}(\dot{r}\dddot{r} + \dot{z}\dddot{z}) + \frac{5\alpha r}{h^7}\left(\dot{r}\ddot{r} + \dot{z}\ddot{z}\right)^2, \tag{50}$$

which follows from the definitions (29,30,31) and formula (21). The value of P_z is given by the formula

$$P_z = 2\alpha \frac{d}{dt}\left(\frac{r\ddot{z}}{h^3}\right) + \dot{z}H, \tag{51}$$

obtained by differentiating formula (44) with respect to the time. $y(0)$ and $\dot{y}(0)$ can be computed from formulae (46) and (47), with P_r determined from formula (45). The constant M_{03} is given by formula (43). We see that in the case $\gamma = 0$ H, P_z and M_{03} are proportional to α. Therefore, in this case, Eqs (44), (48), (49) in fact do not contain the parameter α. On the other hand, if $\gamma \neq 0$ then the r.h.s. of Eq (48) suggests nonanalyticity of the function $y(t)$ in the parameter α at $\alpha = 0$ if the function $y(t)$ has nonvanishing the second derivative $\ddot{y}(t)$. The static solution (55), which is presented in the next Section, does not obey this condition — for that solution y is constant.

4. Radial motions of the string with rigidity

In this section we consider motions during which z is constant, *e.g.* $z(t) = 0$. For such trajectories

$$P_z = 0, \quad \text{and} \quad h = (1 - \dot{r}^2)^{\frac{1}{2}}.$$

The Hamiltonian H given by the formula (50) reduces in this case to

$$H = \left(\gamma r + \frac{\alpha}{r}\right)\frac{1}{h} - \frac{\alpha \dot{r}^2}{r h^3} - \frac{6\alpha r \ddot{r}^2}{h^5} + \frac{2\alpha r \dot{r}\dddot{r}}{h^5}$$
$$+ \frac{2\alpha \dot{r}^2 \ddot{r}}{h^5} + \frac{5\alpha r \ddot{r}^2}{h^7}. \tag{52}$$

Instead of equations of motion (48,49) we may use the condition

$$H = \text{const}, \tag{53}$$

except for solutions with $\dot{r} = 0$. The point is that if $\dot{r} \neq 0$ we can compute from formula (52) the third derivative $\dddot{r}$ thus obtaining the equation of the form

$$\dddot{r} = F(r, \dot{r}, \ddot{r}, H), \tag{54}$$

from which we can determine the trajectory $r(t)$. Here F is a function to be determined by actually performing the calculation.

The static solutions, *i.e.*, the ones with $\dot{r} = 0$, have to be found from equations (48,49). It is easy to see that there is only one static solution

$$r(t) = r_0 \equiv \sqrt{\frac{\alpha}{\gamma}}. \tag{55}$$

This solution was already found in paper [20]. It exists only if $\alpha > 0$. Notice that this solution is analytic in the parameter $\beta = \sqrt{\alpha}$.

It is not possible to find a general form of the trajectory $r(t)$ because the condition (53) is rather complicated. For this reason, we shall only present a particular solution, which nevertheless gives interesting insight into the dynamics of the string with rigidity. Namely we shall consider the motion of the string which started at $t = 0$ with the initial velocity $\dot{r}(0) = 0$. The initial radius of the hoop is equal to r_0. The corresponding solution for the Nambu–Goto string is given by formula (28). For small t

$$r(t) \cong r_0 \left(1 - \frac{1}{2}\left(\frac{t}{r_0}\right)^2 + \frac{1}{24}\left(\frac{t}{r_0}\right)^4\right). \tag{56}$$

The trajectory (28) obeys also the equations of motion for the string with rigidity. However, the string with rigidity, being the higher derivative dynamical system, possesses twice as much degrees of freedom as the Nambu–Goto string. By this we mean the fact that the initial data now consist of four numbers

$$r(0) = r_0, \;\; \dot{r}(0) = 0, \;\; \ddot{r}(0), \;\; \dddot{r}(0). \tag{57}$$

The Nambu–Goto solution (28) corresponds to the following particular initial data for the string with rigidity

$$r(0) = r_0, \;\; \dot{r}(0) = 0, \;\; \ddot{r}(0) = -\frac{1}{r_0}, \;\; \dddot{r}(0) = 0. \tag{58}$$

It is clear from (57) that in the case of string with rigidity the Nambu–Goto solution (28) is embedded into a much wider set of trajectories parametrized by the initial values of $\ddot{r}$ and $\dddot{r}$. Such new trajectories lie also in a close vicinity of the Nambu–Goto solution — it is sufficient to choose the initial data close to the ones specified by (58). As an example, let us consider trajectories with the following initial data

$$r(0) = r_0, \ \dot{r}(0) = 0, \ \ddot{r}(0) = \frac{-1+\varepsilon}{r_0}, \ \dddot{r}(0) = 0, \tag{59}$$

where ε is arbitrary. For sufficiently small times t we may seek the solution in the approximate form

$$r(t) \cong r_0 + \frac{-1+\varepsilon}{2r_0}t^2 + \frac{c}{24}\frac{t^4}{r_0^3}. \tag{60}$$

Inserting this function in formula (52) and requiring that H is constant up to the order t^2 we obtain the following value of the coefficient c

$$c = 1 - 4\varepsilon + 6\varepsilon^2 - \frac{5}{2}\varepsilon^3 - \frac{\gamma r_0^2}{2\alpha}\varepsilon. \tag{61}$$

For $\varepsilon = 0$ we obtain from formula (61)

$$c = 1,$$

in agreement with (56).

If the rigidity term in the Lagrangian (1) is regarded as a small correction to the Nambu–Goto Lagrangian, then it is natural to suppose that the parameter ε should differ only slightly from its value for the Nambu–Goto solution. For instance, we may take

$$\varepsilon = \alpha^2 b, \tag{62}$$

where α is regarded as a small constant and b is a dimensionless parameter which is not too large. Then

$$c \cong 1 - \frac{\gamma r_0^2}{2}\alpha b + O(\alpha^2). \tag{63}$$

The energy H for this solution is equal to

$$H = \gamma r_0 + \frac{2\alpha}{r_0}\varepsilon - \frac{\alpha}{r_0}\varepsilon^2 = \gamma r_0 + \frac{2\alpha^3}{r_0}b + O(\alpha^5). \tag{64}$$

The radial motions of the hoop-like string with rigidity seem to be nonintegrable. The phase space is four-dimensional $\big((r, \dot{r}, p_r, P_r)\big)$ while we have only one integral of the motion, namely the energy H. For the integrability a second integral of motion is necessary. Therefore we expect that the trajectories $r(t)$ will be unstable in general. On the other hand it is possible that the trajectory $r(t)$ will not show a chaotic behavior in spite of the nonintegrability, because the hypersurfaces of constant energy in the phase space are not compact. These questions require further investigation. The Nambu–Goto trajectories are periodic. In nonintegrable systems trajectories initially close to periodic trajectories typically diverge from them. Therefore we expect that the trajectories (60) will stay close to the Nambu–Goto trajectories (28) only for times sufficiently small in comparison with r_0.

In the case $\gamma = 0$ we can give an exact solution of the equations of motion — it is the so called hyperbolic motion,

$$r(t) = (a^2 + t^2)^{\frac{1}{2}},$$

where a is an arbitrary constant. For such trajectories $H = 0$.

Finally let us note that the classical mass spectrum for the radial motions of the hoop is non-negative, *i.e.*, $M^2 > 0$. This follows from the fact that for the radial motions $P_z = 0, P_x = P_y = 0$. Therefore

$$M^2 \equiv H^2 - \vec{P}^2 = H^2 \geq 0.$$

The classical energy spectrum for the radial motions is unbounded from below, as it is clear from formula (52).

5. Tachyonic trajectories of the string with rigidity

In the previous Section we have shown that due to the presence of the degrees of freedom related to the higher derivatives there exist solutions which are different from the Nambu–Goto solutions but they stay close to them for some time. In the present Section we point out that there exist solutions which cannot be regarded as close to Nambu–Goto solutions. These solutions may be called tachyonic, because they have negative value of the classical mass squared, *i.e.*,

$$M^2 = H^2 - \vec{P}^2 < 0,$$

where for the hoop-like configurations $\vec{P} = (0, 0, P_z)$. Similarly as the solutions of the previous Section, the tachyonic solutions exist because of the presence of the degrees of freedom related to the higher derivatives. The tachyonic solutions are of "run-away" type, and we think that it is unlikely that they have some relevance for physical applications of the string with rigidity. Rather the presence of these solutions signals shortcomings of the model.

The presence of the tachyonic solutions is obvious if we recall formulae (50), (51) for H and P_z. H depends linearly on $\ddot{z}$ and $\ddot{r}$, while P_z depends linearly on $\dddot{z}$. It is clear that we can choose initial data in such a way that M^2 is negative. The trajectory determined by these initial data is the tachyonic trajectory. Of course, we take the initial data such that the motion of the string is subluminal, *i.e.*,

$$h^2 = 1 - \dot{z}^2 - \dot{r}^2 > 0.$$

It is clear that the existence of the tachyonic solutions for the hoop-like string is tied up with nonvanishing of the first term on the r.h.s. of formula (51) for P_z. Otherwise, $P_z = \dot{z}H$ and $M^2 = (1-\dot{z}^2)H^2 > 0$ because $\dot{z}^2 < 1$.

One may wonder whether a tachyonic trajectory which has started from a subluminal initial data can reach the light-cone (*i.e.*, $h = 0$) after a finite time, and later on continue as a superluminal trajectory. Formulae (50), (51) imply that this cannot happen. For the superluminal velocities h is imaginary. Therefore after the crossing of the light-cone H is imaginary, while before the crossing it was real. Because H is constant during the motion, it must vanish for such trajectories, $H = 0$. Similarly, from formula (51) follows that $P_z = 0$. Thus $M^2 = 0$ - the trajectory is not tachyonic. Let us mention here that is not possible to cross the light-cone even if we choose subluminal initial data such that $H = P_z = 0$, but we will not discuss this case here.

Equations (44),(48) and (49) contain no singularities for $r > 0$. Therefore, there is no problem with existence of the solution for a given choice of initial data, at least in a finite interval of time. We have not found the analytic form of the tachyonic solution. However, we have performed a numerical study of it. As expected, there were no problems with generating such solution. Details of the numerical solutions are not so important — for this reason we do not present here this numerical solution.

The question whether the tachyonic solutions exist also for arbitrarily large time t is much more difficult. Numerical methods cannot help us in this problem. A constructive way to solve it is to find an asymptotic form of the tachyonic solutions for large time t. We have found such asymptotics in the particular case of $\gamma = 0$, *i.e.*, when the Nambu–Goto term is absent. In this case there exists the following particular asymptotic solution for large t,

$$\begin{aligned} r(t) &= t + a_1 t^{-1} + a_2 t^{-3} + O(t^{-5}), \\ z(t) &= b_1 t^{-1} + b_2 t^{-3} + O(t^{-5}), \end{aligned} \tag{65}$$

where $a_1 > 0$ is an arbitrary positive constant, while b_1, a_2 and b_2 are expressed by a_1 and the corresponding values of H and P_z:

$$b_1 = (4\alpha)^{-1}(2a_1)^{\frac{3}{2}} P_z, \tag{66}$$

$$a_2 = -\frac{1}{2}a_1^2\left(1 + \frac{5}{6}\alpha^{-2}a_1P_z^2 + \frac{2}{3}\alpha^{-1}(2a_1)^{\frac{1}{2}}H\right), \tag{67}$$

$$b_2 = -(3\alpha)^{-1}\sqrt{2}a_1^{\frac{5}{2}}P_z\left(1 + 9(4\alpha)^{-2}a_1P_z^2 + (2\alpha)^{-1}(2a_1)^{\frac{1}{2}}H\right). \tag{68}$$

Formula (66) follows from Eq. (44), formula (67) follows from formulae (66) and (50) in the limit $t \to \infty$, and formula (68) follows from Eq. (44) and formulae (66), (67). It is clear that formulae (66-68) make sense for any values of P_z and H, in particular for the tachyonic ones (*i.e.*, $H^2 - P_z^2 < 0$). Solution (65) is not the most general one, nevertheless it provides the example of the tachyonic solution.

Solution (65) is of the "run-away" type in the sense that the radius $r(t)$ of the hoop increases with the velocity $\dot{r}$ approaching the velocity of light. Notice also that this solution does not depend on α because H and P_z are proportional to α.

For $P_z = 0$ the solution (65) reduces to the hyperbolic motion mentioned in the Section 4.

6. Comments and remarks

a) Numerous shortcomings of the string with rigidity model, like the indefiniteness of energy or the tachyonic mass spectrum indicate that this model is not a good candidate for a fundamental theory. This is certainly true for the unquantized version of the model. On the other hand, we cannot a priori exclude the possibility that after an appropriate quantization the model will be free of the unpleasant features. The example of free Dirac field shows that such thing can happen. The trouble is that at present there are no hints how to find the appropriate quantization procedure. This problem is a part of a wider problem how to quantize theories with higher derivatives, see *e.g.*[21] for a discussion. In the present paper we regard the string with rigidity as an effective model which arises in a string-like limit of vortices in classical string theory [6].

b) We regard the string with rigidity as an approximate description of a vortex, *e.g.*of the one present in the Abelian Higgs model. The Nambu–Goto string can also be interpreted in this manner [1,2]. The main difference between the two kinds of strings lies in the fact that the string with rigidity has more degrees of freedom than the Nambu–Goto string. In this sense, the string with rigidity is an intermediate model, which is placed in between the extreme simplification of the vortex dynamics provided by the Nambu–Goto string on one side, and the full field-theoretical complexity of the vortex dynamics on the other side.

We have seen that the presence of the additional degrees of freedom makes the dynamics of the string with rigidity much richer. In general, for the hoop-like string with rigidity the dynamics becomes non-integrable. The periodic trajectories of the Nambu–Goto type (28) are still present, but they are surrounded by infinitely many other solutions. Some of them stay close to a Nambu–Goto solution during certain time interval. The example was given in Section 4. There are also solutions for which there is no close Nambu–Goto solution. The tachyonic solution is the example. Also the static ring solution mentioned in Section 4 cannot be regarded as close to a Nambu–Goto solution.

The existence of the static ring solution is sometimes regarded as an argument that the string with rigidity exhibits intrinsic stiffness, in contradistinction with the Nambu–Goto string. However, the fact that all Nambu–Goto trajectories are viable also in the case of string with rigidity points to the lack of the intrinsic stiffness.

We have seen in Section 3 that the hoop-like string with rigidity can be regarded as a family of 3-dimensional Newton-type dynamical systems parameterized by values of H, P_z and M_{03}. Similar observation was made in [22] for a point-particle model with action functional involving second order derivatives $\ddot{x}^\mu$ of the trajectory $x^\mu(\tau)$. Characteristic feature of these Newton-type systems is the lack of integrals of motion which would depend only on dynamical variables r, z, y and their first derivatives $\dot{r}, \dot{z}$ and $\dot{y}$. Therefore these systems are likely non-integrable. This explains why it is not easy to find analytic form of trajectories, except in very special cases.

c) Because of the presence of the additional degrees of freedom one might hope that the string with rigidity will approximate the vortex in a more accurate way than the Nambu–Goto string. However, the indefiniteness of energy and the presence of tachyonic trajectories which are absent in the field-theoretical description of the vortex, indicate that the string with rigidity can be a good approximation to the vortex in a restricted region of phase space in the best case. This problem we are leaving for another investigation. For the time being we will be satisfied with the following rough estimate. Lagrangian (1) can be regarded as the beginning of an infinite series of terms which arises if we try to describe the full vortex dynamics in terms of the string variables $x^\mu(\tau, \sigma)$ [6]. The further terms in this series can contain still higher derivatives, *e.g.*$\triangle(\triangle x_\mu)\triangle x^\mu$ or higher powers of $\triangle x_\mu$, *e.g.*$(\triangle x_\mu \triangle x^\mu)^2$. For the hoop Ansatz (10) such terms are relatively small if the following conditions are simultaneously obeyed:

(*i*) γr^2 is sufficiently large,
(*ii*) $\dot{r}, \dot{z}$ are not too close to 1,
(*iii*) $\gamma^{-1}\ddot{r}^2, \gamma^{-1}\ddot{z}^2$ are small.

We expect that the string with rigidity is a good approximation to the vortex if these conditions are satisfied. The trajectories of the radial motion discussed in Section 4 obey these conditions if the time interval is not too large. For the tachyonic trajectories these conditions are not satisfied.

Notice that our conditions (*i*) – (*iii*) are not satisfied by the Nambu–Goto trajectories (28) if r is small. We think that dynamics of small vortex rings is not described correctly neither by the Nambu–Goto string nor by the string with rigidity.

d) We would like to close our investigations of the string with rigidity with the following remark. The exact description of dynamics of the vortex requires infinitely more degrees of freedom in addition to the degrees of freedom the simple Nambu–Goto string possesses. Better approximation to the vortex we wish to have, more degrees we have to add to the Nambu–Goto string. One may try to partially account for those additional degrees of freedom by passing from Nambu–Goto string to a string with higher derivatives in the Lagrangian, *e.g.* to the string with rigidity. However, the resulting model has many formal shortcomings, like the indefiniteness of energy or the tachyonic trajectories. Moreover, there is no transparent relation between the degrees of freedom introduced by the higher derivatives and some degrees of freedom of the vortex. Due to this shortcoming, the degrees of freedom introduced with the help of the higher derivatives have no clear physical interpretation. Therefore we think that it is necessary to search for a way to approximate the vortex by a string without invoking the theories with higher derivatives.

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