COMMENTS ON EQUATIONS OF MOTION, BOUNDARY CONDITIONS AND ENERGY-MOMENTUM FOR A CLASSICAL RIGID STRING*

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The correct forms of the equations of motion, of the boundary conditions and of the conserved energy-momentum for a classical rigid string are given. Certain consequences of the equations of motion are presented. We also point out that in Hamiltonian description of the rigid string the usual time evolution equation $\dot{F} = \{H, F\}$ is modified by some boundary terms.

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1. Introduction

Nowadays there exist quite a few string models, *e.g.* Nambu–Goto string [1], several kinds of superstrings [2] and a more recent one — the rigid string [3, 4]. These models are expected to have many different applications. The rigid string, in which we are interested in this paper, is expected to appear in a string interpretation of QCD. Such a hope has been expressed in papers [3], and it is supported by other investigations [5]. The rigid string also appears as an idealization of a thin vortex, even though there is some controversy about this [6, 7]. Such idealization of the vortex is useful for description of cosmic strings [6]. The rigid string (or rather its Euclidean version) has also been considered in a statistical theory of random surfaces, in connection with two-dimensional, quantized gravity [8].

Investigations of the rigid string model are not easy to carry out because equations of motion of the classical string and the corresponding canonical

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structure are rather complicated. In particular, it seems that the classical equations of motion cannot be linearized by an appropriate choice of coordinates on the world-sheet of the string. Nevertheless, several interesting results have already been obtained. Several special solutions of the classical equations of motion have been found [9]. It has been noticed that the energy of the classical rigid string is not bounded from below [10]. Canonical formulation of the model has also been constructed [11, 12]. Formal quantization of the rigid string has been considered in papers [13]. Mean field approximation for the quantized rigid string has been considered in papers [14].

Our main goal in this paper is to rederive the classical equations of motion, boundary conditions and conserved energy–momentum of the rigid string. The first reason to discuss in detail such basics is that the rigid string model is an example of a Lagrangean field theory with higher order derivatives. In such case the seemingly standard derivations contain many interesting points which, in our opinion, have not been sufficiently emphasized. The second reason is that one can find in the literature many misleading or even erroneous statements concerning the equations of motion, the boundary conditions and the energy–momentum.

We also make several observations about properties of the rigid string. We point out that in general it is not possible to have so called orthonormal coordinates on the world-sheet of the rigid string with the additional property that the evolution parameter is the physical time, *i.e.* $x^0(\tau, \sigma) = \tau$. We prove that the ends of an open rigid string move with the velocity of light. Another our observation is that Hamiltonian description of the open, rigid string involves some boundary terms. Because of their presence the physical–time–evolution of the open rigid string is generated by a Hamiltonian which is not equal to the conserved energy of the string. We also notice that the classical rigid string can have tachyonic trajectories. Let us recall that Nambu–Goto string is not tachyonic on the classical level — it becomes tachyonic only after quantization.

The plan of our paper is the following. In Section 2 we present the derivation of the Euler–Lagrange equations of motion, of the boundary conditions and of the conserved energy–momentum in the case of a generic Lagrangian with second order derivatives. In Section 3 we present the corresponding formulae in the case of rigid string, *i.e.* for the specific Lagrangian given at the beginning of Section 3. There we also derive some simple consequences of the equations of motion. In Section 4 we point out the peculiar features of the Hamiltonian formalism appearing in the case of the open rigid string.

More detailed investigations of dynamics of the classical rigid string will be presented in a forthcoming paper [15].

2. Basic formulae in the case of a generic Lagrangian with second order derivatives

We shall consider an open string described by the action

$$S = \int_R d^2u \mathcal{L}\left(x^\mu, x^\mu{}_{,a}, x^\mu{}_{,ab}\right), \tag{1}$$

where $a,\ b = 0,\ 1, (u^0,\ u^1)$ are coordinates on the world-sheet of the string, $x^\mu = x^\mu(u^0, u^1)$ and $x^\mu{}_{,a} \equiv \frac{\partial x^\mu}{\partial u^a}$.

In the following we shall frequently use another notation for the coordinates, namely $\tau \equiv u^0$, $\sigma \equiv u^1$. R denotes the rectangle $\tau_1 \leq \tau \leq \tau_2$, $0 \leq \sigma \leq \pi$. The world-sheet Σ of the string is the image of R by mapping $x^\mu(u^0,\ u^1)$. Thus, Σ is a surface in Minkowski space-time M. The metric in M is $(+,-,-,-)$. We assume that $\tau \equiv u^0$ is a time-like evolution parameter, *i.e.* $\dot{x}^2 > 0$, $\dot{x}^0 > 0$, where $\dot{x}^\mu \equiv \partial_0 x^\mu$, and $\sigma \equiv u^1$ is a space-like parameter, *i.e.* $x'^2 < 0$, where $x'^\mu \equiv \partial_1 x^\mu$. If we additionally assume that for all τ

$$x^\mu(\tau, \sigma = 0) = x^\mu(\tau, \sigma = \pi). \tag{2}$$

we obtain a closed string.

Let us perform the variation

$$x^\mu(u) \to x^\mu(u) + \delta x^\mu(u). \tag{3}$$

In the case of the closed string we additionally assume that

$$\delta x^\mu(\tau, \sigma = 0) = \delta x^\mu(\tau, \sigma = \pi). \tag{4}$$

Straightforward computation gives the following formula for the variation of the action S corresponding to the variation (3).

$$\begin{aligned}\delta S \equiv \int_R d^2u &\left[\frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial x^\mu{}_{,a}}\partial_a \delta x^\mu + \frac{\partial \mathcal{L}}{\partial x^\mu{}_{,00}}\delta x^\mu{}_{,00}\right.\\ &\left.+\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,01}}\delta x^\mu{}_{,01} + \frac{\partial \mathcal{L}}{\partial x^\mu{}_{,11}}\delta x^\mu{}_{,11}\right]\\ = \int_R &d^2u\left(R_\mu \delta x^\mu + \epsilon^{ab}\partial_a Y_b + \partial_0\partial_1 Z\right),\end{aligned} \tag{5}$$

where

$$\begin{aligned}R_\mu(\tau,\sigma) \equiv\ & \partial_0^2\left(\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,00}}\right) + \partial_0\partial_1\left(\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,01}}\right) + \partial_1^2\left(\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,11}}\right)\\ &- \partial_a\left(\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,a}}\right) + \frac{\partial \mathcal{L}}{\partial x^\mu},\end{aligned} \tag{6}$$

$$Y_0(\tau,\sigma) \equiv \left[-\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,1}} + \partial_a \left(\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,a1}}\right)\right] \delta x^\mu - \frac{\partial \mathcal{L}}{\partial x^\mu{}_{,11}} \delta x^\mu{}_{,1}\,, \tag{7}$$

$$Y_1(\tau,\sigma) \equiv \left[\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,0}} - \partial_a \left(\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,0a}}\right)\right] \delta x^\mu + \frac{\partial \mathcal{L}}{\partial x^\mu{}_{,00}} \delta x^\mu{}_{,0}\,, \tag{8}$$

$$Z(\tau,\sigma) \equiv \frac{\partial \mathcal{L}}{\partial x^\mu{}_{,01}} \delta x^\mu\,, \tag{9}$$

and ϵ^{ab} is the totally antisymmetric symbol, $\epsilon^{01} = +1$.

Using Stokes theorem we can write δS in the following form

$$\delta S = \int\limits_R d^2u R_\mu \delta x^\mu + \int\limits_{\partial R} Y_a du^a + Z(\tau_2,\pi) - Z(\tau_2,0) - \left(Z(\tau_1,\pi) - Z(\tau_1,0)\right), \tag{10}$$

where δR denotes the boundary of the rectangle R. The advantage of this form of the variation δS is that it involves the least possible number of derivatives of the variations δx^μ. The remaining derivatives of δx in formula (10) cannot be removed by any partial integrations.

The Z-terms present on the r.h.s. of formula (10) for δS can be regarded as a contribution from the corner points of the rectangle R. For the closed string they cancel each other. However, for the open string they give a nonvanishing contribution if the Lagrangian $\mathcal{L}$ depends on $x^\mu{}_{,01}$. The Z-terms have appeared because in the case of open rigid string we encounter a coincidence of the following two mathematical obstacles: the presence of higher derivatives in the Lagrangian, and the fact that the field $x^\mu(\tau,\sigma)$ is defined on the finite-width strip $0 \le \sigma \le \pi$, $-\infty < \tau < +\infty$, which has boundaries.

The classical equations of motion and the boundary conditions for the open rigid string follow from the requirement

$$\delta S = 0 \tag{11}$$

for any variation δx^μ obeying the following conditions

$$\delta x^\mu(\tau,\sigma) = 0 \quad \text{for} \quad \tau = \tau_1, \tau_2, \quad \sigma \in [0,\pi]\,, \tag{12a}$$

$$\delta x^\mu{}_{,0}(\tau,\sigma) = 0 \quad \text{for} \quad \tau = \tau_1, \tau_2, \quad \sigma \in [0,\pi]\,. \tag{12b}$$

The condition (12b) is due to the fact that Lagrangian (1) contains the second order derivatives with respect to the evolution parameter τ. From (12a) it follows that

$$\delta x^\mu{}_{,1}(\tau,\sigma) = 0 \quad \text{for} \quad \tau = \tau_1, \tau_2 \quad \sigma \in [0,\pi]\,. \tag{13}$$

On the other hand, neither δx^μ nor $\delta x^\mu{}_{,1}$ are fixed for $\sigma = 0$, $\sigma = \pi$, $\tau \in (\tau_1, \tau_2)$. Now, it is clear that the requirement (11) implies the following equations of motion

$$R_\mu(\tau, \sigma) = 0\,, \tag{14}$$

and the following boundary conditions

$$B_\mu(\tau, \sigma = 0) = 0, \qquad B_\mu(\tau, \sigma = \pi) = 0\,, \tag{15a}$$

$$C_\mu(\tau, \sigma = 0) = 0, \qquad C_\mu(\tau, \sigma = \pi) = 0\,, \tag{15b}$$

where

$$B_\mu(\tau, \sigma) \equiv -\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,1}} + \partial_a \left(\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,1a}} \right)\,, \tag{16}$$

and

$$C_\mu(\tau, \sigma) \equiv \frac{\partial \mathcal{L}}{\partial x^\mu{}_{,11}}\,. \tag{17}$$

In the case of the closed string $\delta x^\mu(\tau, \sigma)$ obeys condition (4). Then, the variational principle (11) implies only the equations of motion (14).

Now, let us pass to the derivation of the energy-momentum four-vector corresponding to the action (1). We again use the formula (10), but now

$$\delta x^\mu \equiv \epsilon^\mu = \text{const.}\,, \tag{18}$$

what corresponds to an infinitesimal translation. We also assume that the Lagrangian is translationally invariant, *i.e.*

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0\,.$$

In this case it follows directly from definition (1) that

$$\delta S = 0\,. \tag{19}$$

We assume that $x^\mu(\tau, \sigma)$ obeys equations of motion (14) and boundary conditions (15). Then, it follows from formulae (8) and (10) that

$$P_\mu \equiv \int_0^\pi d\sigma \left[-\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,0}} + \partial_a \left(\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,0a}} \right) \right] - \left.\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,01}}\right|_{\sigma=\pi} + \left.\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,1}}\right|_{\sigma=0} \tag{20}$$

is constant during the τ-evolution. We notice that the last two terms on the r.h.s. of formula (20) cancel with the term $\int_0^\pi d\sigma \partial_1 \left(\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,01}} \right)$. Therefore, the final formula for the energy-momentum four-vector has the form

$$P_\mu = \int_0^\pi d\sigma p_\mu\,, \tag{21a}$$

where

$$p_\mu \equiv -\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,0}} + \partial_0 \left(\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,00}} \right) . \tag{21b}$$

The equations of motion (14) can be written in the following form

$$\partial_1 B_\mu + \partial_0 p_\mu = 0 . \tag{22}$$

Integrating formula (22) over σ, and taking into account boundary conditions (15a), we again obtain that

$$\partial_0 P_\mu = 0 .$$

This is a check that our formulae (21a), (21b) are correct.

By a similar reasoning we obtain a conserved angular-momentum tensor $M_{\mu\nu}$ for the rigid string. The only difference is that now

$$\delta x^\mu = \omega^\mu{}_\nu x^\nu , \tag{23}$$

instead of formula (18). Here $\omega^{\mu\nu} = -\omega^{\nu\mu}$ are the six infinitesimal parameters of Lorentz transformations. After a partial integration, contribution of the Z-terms is cancelled by other terms. The final formula for $M_{\mu\nu}$ has the following form

$$M_{\mu\nu} = \int_0^\pi d\sigma (x_\mu p_\nu - x_\nu p_\mu) + \int_0^\pi d\sigma \left(\frac{\partial \mathcal{L}}{\partial x^\mu{}_{,0a}} x_{\nu,a} - \frac{\partial \mathcal{L}}{\partial x^\nu{}_{,0a}} x_{\mu,a} \right) , \tag{24}$$

where p_μ is the momentum density given by formula (21b).

3. The case of the rigid string

The Lagrangian for the rigid string has the form [3]

$$\mathcal{L} = \sqrt{-g}(-\gamma + \alpha \Delta x^\mu \Delta x_\mu) , \tag{25}$$

where $\alpha \neq 0$, $\gamma > 0$ are constants. The sign of α is not decided upon. For $\alpha = 0$ we would obtain the usual Nambu-Goto string. The Laplacian Δx^μ is given by the formula

$$\Delta x^\mu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial u^a} \left(\sqrt{-g} g^{ab} \frac{\partial x^\mu}{\partial u^b} \right) , \tag{26}$$

where

$$g_{ab} = \partial_a x^\mu \partial_b x_\mu \tag{27}$$

is the metric tensor on the world-sheet Σ of the string, g^{ab} denote components of the inverse metric tensor, and $g = \det(g_{ab})$.

In the case of Lagrangian (25), equations of motion (14) have the form

$$\begin{aligned}(\gamma - \alpha \Delta x_\sigma \Delta x^\sigma)\Delta x_\mu + 2\alpha\Big[\Delta(\Delta x_\mu) - g^{ab} x^\sigma{}_{,a} x^\mu{}_{,b} \Delta(\Delta x_\sigma)\Big] \\ - 4\alpha g^{ab} g^{cd} (\Delta x_\sigma)_{,b} x^\sigma{}_{,c} \nabla_a x_{\mu,d} = 0\,,\end{aligned} \tag{28}$$

where $\nabla_a x_{\mu,d}$ denotes the covariant derivative of the covariant vector $x_{\mu,d}$, $d = 0, 1$, defined with the use of Christoffel symbols for the metric given by (27). Equations (28) are explicitly invariant under reparametrization of the world-sheet Σ.

The l.h.s. of Eqs (28) has vanishing projections on the directions tangent to the world-sheet Σ, *i.e.*

$$R_\mu x^\mu{}_{,a} = 0\,. \tag{29}$$

In order to check this, it is convenient to use the identities (valid for any $a, b, c = 0, 1$)

$$x^\mu{}_{,c} \nabla_a x_{\mu,b} = 0\,, \tag{30}$$

from which it also follows that

$$x^\mu{}_{,c} \Delta x_\mu = 0\,, \tag{31}$$

because

$$\Delta x_\mu = g^{ab} \nabla_a x_{\mu,b}\,.$$

Identities (30) are due to the fact that we use the particular metric g_{ab} on the world-sheet Σ—the one given by formula (27). Christoffel symbols Γ^a_{bc} corresponding to this metric have the following form

$$\Gamma^c_{ab} = g^{cd} x^\mu{}_{,d} x_{\mu,ab}\,,$$

and the covariant derivative $\nabla_a x_{\mu,d}$ is equal to

$$\nabla_a x_{\mu,b} \equiv \partial_a x_{\mu,b} - \Gamma^c_{ab} x_{\mu,c} = x_{\sigma,ab}\left(\delta^\sigma_\mu - g^{cd} x^\sigma{}_{,d} x_{\mu,c}\right).$$

In fact, identities (29) are Noether identities corresponding to the invariance of the action S with respect to reparametrizations of the world-sheet Σ. Identities (29) follow directly from formula (10) if we consider reparametrizations

$$u'^a = u^a + \epsilon^a(u)\,, \tag{32}$$

with infinitesimal functions $\epsilon^a(u)$ which vanish together with their derivatives on ∂R. For such reparametrizations

$$\delta S = 0\,, \qquad \delta x^\mu(u) = x^\mu{}_{,a}\epsilon^a(u)\,,$$

and the Y-term and the Z-terms vanish. Hence, we have identities (29).

Equations (28) are very complicated. They contain fourth-order partial derivatives and nonlinearities. For $\alpha = 0$ they reduce to equations of motion for the Nambu–Goto string

$$\Delta x_\mu = 0\,. \tag{33}$$

Equations (33) are also nonlinear. However, it is a well-known fact that they can be locally linearized by choosing so called orthonormal coordinates on the world-sheet Σ [16]. For a generic world-sheet Σ such coordinates exist only locally. They are defined by the conditions

$$\dot{x}x' = 0\,, \qquad \dot{x}^2 > 0\,, \qquad x'^2 < 0\,, \tag{34a}$$

$$\dot{x}^2 = -x'^2\,. \tag{34b}$$

In such coordinates Eqs (33) have the form

$$(\partial_0^2 - \partial_1^2)x^\mu(\tau,\sigma) = 0\,. \tag{35}$$

Coordinates τ, σ which would linearize equations (28) are not known.

At this point we would like to remark that in general it is not possible to have the orthonormal coordinates which would have the additional property that the evolution parameters τ is the physical time, *i.e.*

$$x^0 = \tau\,. \tag{36}$$

For the Nambu–Goto string such coordinates always can be introduced [16]. The point is that the definition (34a), (34b) does not specify the orthonormal coordinates τ, σ uniquely — there remains a freedom of transformations

$$\tau \to \tau' = f_1(\tau,\sigma)\,, \qquad \sigma \to \sigma' = f_2(\tau,\sigma)\,, \tag{37}$$

with any f_1, f_2 obeying the condition

$$\Box f_i = 0\,, \qquad i = 1,2\,, \tag{38}$$

where $\Box \equiv \partial_0^2 - \partial_1^2$. Comparing Eq. (38) with Eq. (35) we see that we can take

$$\tau' = x^0(\tau,\sigma)\,. \tag{39}$$

In the case of the rigid string, equations of motion in the orthonormal coordinates do not have the simple form of equations (35). Therefore, in this case the transformation (39) does not preserve the conditions (34a), (34b), in general. Of course, there is a subclass of world-sheet Σ of the rigid string for which coordinates obeying (34a), (34b) and (36) can be introduced. For instance, each solution of equations (33) also obeys equations (28). Thus, the set of world-sheets of the Nambu–Goto string is a subclass of the set of all world-sheets of the rigid string.

The functions $B_\mu(\tau,\sigma)$, $C_\mu(\tau,\sigma)$ which appear in boundary conditions (15a), (15b) in the case of Lagrangian given by formula (25) have the following form

$$\begin{aligned} B_\mu(\tau,\sigma) = {} & \sqrt{-g}(\gamma - \alpha\Delta x^\lambda \Delta x_\lambda) g^{1a} x_{\mu,a} + 2\alpha\sqrt{-g} g^{1a} g^{bc} x^\lambda{}_{,a} x_{\lambda,bc} \Delta x_\mu \\ & + 4\alpha\sqrt{-g}\Delta x_\sigma x^\sigma{}_{,ab} g^{1b} g^{ac} x_{\mu,c} + 2\alpha\partial_0\left(\sqrt{-g} g^{01}\Delta x_\mu\right) \\ & + 2\alpha\partial_b\left(\sqrt{-g} g^{1b}\Delta x_\mu\right), \end{aligned} \tag{40}$$

$$C_\mu(\tau,\sigma) = 2\alpha\sqrt{-g} g^{11}\Delta x_\mu . \tag{41}$$

In the orthonormal coordinates (34a), (34b), condition (15b) implies that

$$\Box x_\mu = 0 \ \text{ for } \ \sigma = 0, \pi . \tag{42}$$

Condition (15a), after taking into account condition (42) has the form

$$\gamma x'_\mu + 2\alpha\partial_1(\Box x_\mu) = 0 \ \text{ for } \ \sigma = 0, \pi . \tag{43}$$

Multiplying both sides of condition (43) by x'^μ, using the identity

$$x'^\mu \partial_1(\Box x_\mu) = -x''_\mu \Box x^\mu \tag{44}$$

and condition (42), we obtain that

$$x'^2 = 0 \ \text{ for } \ \sigma = 0, \pi . \tag{45}$$

Identity (44) follows from another identity,

$$x'_\mu \Box x^\mu = 0 ,$$

which follows from identity (31) written in the orthonormal coordinates. Because of relation (34b), we also have

$$\dot{x}^2 = 0 \ \text{ for } \ \sigma = 0, \pi . \tag{46}$$

Condition (46) means that the ends of the open rigid string always move with the velocity of light. In this respect the rigid string does not differ from the Nambu–Goto string.

The energy-momentum density p_μ corresponding to Lagrangian (25) has the following form

$$p_\mu = \sqrt{-g}g^{0b}(\gamma - \alpha\Delta x_\sigma \Delta x^\sigma)x_{\mu,b} + 2\alpha\partial_0\Big(\sqrt{-g}g^{00}\Delta x_\mu\Big)$$
$$+ 2\alpha\sqrt{-g}g^{0a}g^{bc}\Big(2\Delta x_\sigma x^\sigma{}_{,ab}x_{\mu,c} + x^\lambda{}_{,bc}x_{\lambda,a}\Delta x_\mu\Big)\,. \tag{47}$$

In the orthonormal coordinates this formula is simplified to

$$p_\mu = \dot{x}_\mu\left(\gamma - \alpha\frac{\Box x_\sigma \Box x^\sigma}{(\dot{x}^2)^2} + 4\alpha\frac{\Box x_\sigma \ddot{x}^\sigma}{(\dot{x}^2)^2}\right) + 2\alpha\partial_0\Big(\frac{1}{\dot{x}^2}\Box x_\mu\Big)$$
$$- \frac{4\alpha}{(\dot{x}^2)^2}\Box x_\sigma \dot{x}'^\sigma x'_\mu\,. \tag{48}$$

In the Nambu–Goto case ($\alpha = 0$)

$$p_\mu = \gamma\dot{x}_\mu\,. \tag{48'}$$

More detailed study of dynamics of the rigid string will be presented in the forthcoming paper [15].

4. Remarks on Hamiltonian description of the open rigid string

Discussion of Hamiltonian formulation of dynamics of systems with reparametrization invariance, which is a special case of local gauge invariance, is complicated by a problem of constraints. In order to avoid this obstacle, we shall discuss the Hamiltonian description of the rigid string in the physical gauge, which is defined by the requirement that the evolution parameter τ is equal to the physical time x^0, *i.e.*

$$x^0(\tau,\sigma) = \tau\,.$$

In this gauge, the independent dynamical variables are $x^i(t,\sigma), i = 1,2,3$, $t \equiv x^0$. Variations (3) are now replaced by

$$\vec{x}(t,\sigma) \to \vec{x}(t,\sigma) + \delta\vec{x}(t,\sigma)\,, \tag{49}$$

where $\vec{x} = (x^i)$. The considerations of Section 2 can be repeated with the only difference that the index $\mu = 0,1,2,3$ is now replaced by the index $i = 1,2,3$. In particular, the equations of motion and the boundary conditions have the form given by formulae (14), (15) with the replacement $\mu \to i$. From the invariance under the spatial translations,

$$\delta\vec{x} = \vec{\epsilon} = \text{const}\,,$$

we obtain the conserved momentum — it is given by the spatial components of formula (21). As for formula for the conserved energy P_0, it is derived in a standard manner from the invariance of the action

$$S = \int_{t_1}^{t_2} dt \int_0^{\pi} d\sigma \mathcal{L}(\vec{x}, \dot{\vec{x}}, \vec{x}', \ddot{\vec{x}}, \dot{\vec{x}}', \vec{x}'')$$

with respect to the time translations

$$t \to t + \epsilon\,.$$

The result is

$$P_0 = \int_0^{\pi} d\sigma \left\{ \ddot{\vec{x}} \frac{\partial \mathcal{L}}{\partial \ddot{\vec{x}}} + \dot{\vec{x}} \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} - \partial_0 \left(\frac{\partial \mathcal{L}}{\partial \ddot{\vec{x}}} \right) \right) + \dot{\vec{x}}' \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}'} - \mathcal{L} \right\}\,. \tag{50}$$

In order to obtain this formula, the equations of motion and the boundary conditions have been used. Also some partial integrations over σ have been performed.

Now, let us pass to the discussion of the Hamiltonian formulation in the physical gauge. We shall follow the approach of Ref. [12]. Our goal is to point out the presence of the boundary terms. In the case of Lagrangian $\mathcal{L}$ with second order derivatives there are two independent "configuration-space-type" variables

$$q_1^i \equiv x^i\,, \qquad q_2^i \equiv \dot{x}^i\,, \tag{51}$$

and the corresponding canonical momenta

$$p_{1i} \equiv -\frac{\partial \mathcal{L}}{\partial q_2^i} + \partial_\tau \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2^i} \right) + \partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial q_2'^{\,i}} \right)\,. \tag{52}$$

$$P_{2i} \equiv -\frac{\partial \mathcal{L}}{\partial \dot{q}_2^i}\,. \tag{53}$$

The Lagrangian $\mathcal{L}$ is regarded as a function of the variables $q_1, q_1', q_1'', q_2, \dot{q}_2, q_2'$. The Hamiltonian $\mathcal{H}$ is defined by the formula

$$\mathcal{H} \equiv -p_{1i} q_2^i - p_{2i} \dot{q}_2^i - \mathcal{L}(q_1, q_1', q_1'', q_2, q_2', \dot{q}_2)\,, \tag{54}$$

where $\dot{q}_2$ is the unique function of p_2 and of the other variables obtained by solving for $\dot{q}_2$ formula (53). The function $\dot{q}_2$ is unique because we have fixed the gauge. The equations of motion (14) are equivalent to the following set of Hamilton equations of motion:

$$\begin{aligned} \dot{q}_1 &= -\frac{\delta H}{\delta p_1}, \qquad \dot{q}_2 = -\frac{\delta H}{\delta p_2}, \\ \dot{p}_1 &= \frac{\delta H}{\delta q_1}, \qquad \dot{p}_2 = \frac{\delta H}{\delta q_2}, \end{aligned} \tag{55}$$

where $H = H(q_1, q_1', q_1'', q_2, q_2', p_1, p_2)$ is Hamilton functional,

$$H = \int_0^\pi d\sigma \mathcal{H} = \int_0^\pi d\sigma \left\{ \ddot{\vec{x}} \frac{\partial \mathcal{L}}{\partial \ddot{\vec{x}}} + \dot{\vec{x}} \left[\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} - \partial_0 \left(\frac{\partial \mathcal{L}}{\partial \ddot{\vec{x}}} \right) - \partial_1 \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}'} \right) \right] - \mathcal{L} \right\}, \tag{56}$$

and

$$\begin{gathered} \frac{\delta H}{\delta q_1} = \frac{\partial \mathcal{H}}{\partial q_1} - \partial_\sigma \left(\frac{\partial \mathcal{H}}{\partial q_1'} \right) + \partial_\sigma^2 \left(\frac{\partial \mathcal{H}}{\partial q_1''} \right), \\ \frac{\delta H}{\delta q_2} = \frac{\partial \mathcal{H}}{\partial q_2} - \partial_\sigma \left(\frac{\partial \mathcal{H}}{\partial q_2'} \right), \qquad \frac{\delta H}{\delta p_1} = \frac{\partial \mathcal{H}}{\partial p_1}, \qquad \frac{\delta H}{\delta p_2} = \frac{\partial \mathcal{H}}{\partial p_2}, \end{gathered} \tag{57}$$

are variational derivatives of the functional H. Comparing H with the energy P_0 we see that

$$H = P_0 - \int_0^\pi d\sigma \partial_1 \left(\dot{\vec{x}} \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}'} \right) = P_0 - \dot{\vec{x}} \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}'} \Big|_{\sigma=0}^{\pi}.$$

Thus, in the case of the open rigid string H differs from P_0.

Let us consider the time-evolution of a functional

$$F = \int_0^\pi d\sigma \mathcal{F}(q_1, q_1', q_1'', q_2, q_2', p_1, p_2).$$

Applying integration by parts and formulae (57) for variational derivatives with H replaced by F, we obtain

$$\begin{aligned} \frac{dF}{dt} &= \int_0^\pi d\sigma \left(\frac{\partial \mathcal{F}}{\partial q_1} \dot{q}_1 + \frac{\partial \mathcal{F}}{\partial q_1'} \dot{q}_1' + \frac{\partial \mathcal{F}}{\partial q_1''} \dot{q}_1'' + \frac{\partial \mathcal{F}}{\partial q_2} \dot{q}_2 + \frac{\partial \mathcal{F}}{\partial q_2'} \dot{q}_2' + \frac{\partial \mathcal{F}}{\partial p_1} \dot{p}_1 + \frac{\partial \mathcal{F}}{\partial p_2} \dot{p}_2 \right) \\ &= \int_0^\pi d\sigma \left(\frac{\delta F}{\delta q_1} \dot{q}_1 + \frac{\delta F}{\delta q_2} \dot{q}_2 + \frac{\delta F}{\delta p_1} \dot{p}_1 + \frac{\delta F}{\delta p_2} \dot{p}_2 \right) \\ &+ \left(\frac{\partial \mathcal{F}}{\partial q_1'} - \partial_\sigma \left(\frac{\partial \mathcal{F}}{\partial q_1''} \right) \right) \dot{q}_1 \Big|_{\sigma=0}^{\pi} + \frac{\partial \mathcal{F}}{\partial q_1''} \dot{q}_1' \Big|_{\sigma=0}^{\pi} + \frac{\partial \mathcal{F}}{\partial q_2'} \dot{q}_2 \Big|_{\sigma=0}^{\pi}. \end{aligned} \tag{58}$$

Using Hamilton equations of motion (55) we may write that

$$\frac{dF}{dt} = \{F, H\} + \text{“the boundary terms”}, \tag{59}$$

where Poisson bracket $\{F, H\}$ is, by definition, [12]

$$\{F, H\} \equiv \int_0^\pi d\sigma \Big(\frac{\delta F}{\delta p_1} \frac{\delta H}{\delta q_1} - \frac{\delta H}{\delta p_1} \frac{\delta F}{\delta q_1} + \frac{\delta F}{\delta p_2} \frac{\delta H}{\delta q_2} - \frac{\delta H}{\delta p_2} \frac{\delta F}{\delta q_2} \Big).$$

The boundary terms (the last three terms on the r.h.s. of formula (58)) vanish in the case of closed string. In the case of open string they give a non-vanishing contribution even in the case of Nambu–Goto string.

Equation (59) has a rather unusual implication that the Hamilton H might not be a constant at the motion. From Eq. (59) it follows that

$$\frac{dH}{dt} = \text{“the boundary terms”}. \tag{60}$$

In the case of Nambu–Goto string the boundary terms in Eq. (60) reduce to

$$\frac{\partial \mathcal{H}}{\partial \vec{x}'}\bigg|_{\sigma=0}^{\pi} = -\frac{\partial \mathcal{L}}{\partial \vec{x}'}\bigg|_{\sigma=0}^{\pi} = 0\,,$$

because of boundary condition (15a), which in this case reduces to $\frac{\partial \mathcal{L}}{\partial \vec{x}'} = 0$ for $\sigma = 0, \pi$. In the case of Lagrangian $\mathcal{L}$ with second order derivatives, boundary conditions (15a) allow us to transform formula (60) to the form

$$\frac{dH}{dt} = -\partial_0 \left(\int_0^\pi d\sigma \partial_1 \Big(\dot{\vec{x}} \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}'} \Big) \right). \tag{61}$$

The r.h.s. of Eq. (61) does not vanish, in general. Therefore, $\frac{dH}{dt} \neq 0$. From formula (61) it follows that

$$H + \int_0^\pi d\sigma \partial_1 \left(\dot{\vec{x}} \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}'} \right)$$

is constant during the motion, but this is just the energy P_0 given by formula (50).

In general, the boundary terms will also be present in other gauges, because their appearance is due to the facts that the Lagrangian contains second order derivatives and that the range of the parameter σ is finite.

However, in some particular cases the boundary terms can vanish. For example, in papers [9] a gauge is used which is physical, *i.e.* $x^0(\tau,\sigma)=\tau$, and orthogonal, *i.e.* $\dot{\vec{x}}\vec{x}'=0$ (but condition (34b) is not satisfied). In this gauge Lagrangian (25) can be written in the form which does not contain $\dot{\vec{x}}'$. Thus,

$$\frac{\partial\mathcal{L}}{\partial\dot{\vec{x}}'}=0\,,$$

and the r.h.s. of formula (61) vanishes.

5. Ending remarks

On the preceding pages we have pointed out several consequences of the fact that the Lagrangian of the rigid string contains the second order derivatives of the function $x^\mu(\tau,\sigma)$. Especially interesting is the case of open rigid string, where the boundary terms appear. In this paper we have been interested mainly in Lagrangian and Hamiltonian formalism for the rigid string. An investigation of dynamical properties of the rigid string we shall present in paper [15].

Particularly interesting problem from dynamics of the rigid string is to investigate tachyonic trajectories of the string. It is easy to see that such trajectories are possible. From formulae (50) and (21b) (with $\mu=1,2,3$) one can compute the conserved energy and momentum in the case of Lagrangian (25) in the physical gauge. We shall not quote here these formulae because they are quite complicated. However, in particular case of the following initial data (specified at $t=0$) for a motion of the closed rigid string

$$\vec{x}(0,\sigma)=\vec{f}(\sigma)\,,\quad \dot{\vec{x}}(0,\sigma)=\ddot{\vec{x}}(0,\sigma)=0\,,\quad \dddot{\vec{x}}(0,\sigma)=\vec{h}(\sigma)\,,$$

the conserved energy and momentum are given by relatively simple formulae:

$$P_0=\int\limits_0^\pi d\sigma\sqrt{(\vec{f}')^2}\left\{\gamma+\frac{\alpha}{(\vec{f}')^2}\left[(\vec{f}'')^2-\frac{(\vec{f}''\vec{f}')^2}{(\vec{f}')^2}\right]\right\},$$

$$\vec{P}=2\alpha\int\limits_0^\pi d\sigma\sqrt{(\vec{f}')^2}\left[\vec{h}-\frac{(\vec{f}'\vec{h})\vec{f}'}{(\vec{f}')^2}\right].$$

It follows from these formulae that for sufficiently large $|\vec{h}|$ we can have arbitrarily large $|\vec{P}|$ while the energy P_0 has the fixed value. Thus, we can have negative values of

$$P_\mu P^\mu\equiv P_0^2-\vec{P}^2\,,$$

what corresponds to tachyons. For the Nambu–Goto string ($\alpha = 0$) this is not possible. One can see this from formulae (21a) and (48'). The conserved four-momentum

$$P_\mu = \gamma \int_0^\pi d\sigma \dot{x}_\mu$$

cannot be a space-like four-vector because it is a sum of time-like vectors $\dot{x}_\mu$ with positive zeroth component, $\dot{x}_0 > 0$.

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