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On the Tensor Product of Archimedean Riesz Spaces

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ON THE TENSOR PRODUCT OF ARCHIMEDEAN RIESZ SPACES

DISSERTATION

A Dissertation
presented in partial fulfillment of requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
The University of Mississippi

by

Page Thorn

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ABSTRACT

We characterize when, for any infinite cardinal α, Fremlin's Archimedean Riesz space tensor product of two Archimedean Riesz spaces is Dedekind α-complete. We provide an example of an ideal I in an Archimedean Riesz space E such that the Fremlin tensor product of I with itself is not an ideal in the Fremlin tensor product of E with itself. On the other hand, we present conditions for which the Fremlin tensor product of ideals is an ideal and that the Fremlin tensor product of projection bands is a projection band.

Lastly, we prove that the Carathéodory space of place functions on the free product of two Boolean algebras is Riesz isomorphic with the Fremlin tensor product of their respective Carathéodory spaces of place functions. This result is used to provide a solution to Fremlin's problem 315Y(f) in [16] concerning completeness in the free product of Boolean algebras by applying our results on the Fremlin tensor product to Carathéodory spaces of place functions.

DEDICATION

To my core, Avery and Kit.

ACKNOWLEDGMENTS

Professor Gerard Buskes has guided me through all stages of research: initial learning, asking an interesting question, approaching it from various angles, (often) finding a solution, and framing it to fit into a bigger story. At the same time, he has kindly and patiently grown my confidence. Most importantly, his flexibility and understanding has allowed me to complete my degree while becoming a mother.

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Lastly, I thank God for giving purpose to this work. If the chief end of man is to glorify God by enjoying Him forever, then this work has been a success. It has been a time to rely on God, to see mathematical reasoning as a gift from Him, and to seek out the wisdom of others made in His image.

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LIST OF SYMBOLS

1. INTRODUCTION

The central object of our research is D.H. Fremlin's tensor product of Archimedean Riesz spaces. First, we prove several results about the tensor product of Riesz subspaces as a subset of the tensor product of their respective Riesz spaces. An examination of the Boolean algebra of bands leads to the Riesz space of place functions on a Boolean algebra (in the sense of Carathéodory). The overarching question is which properties are and which are not preserved by taking the tensor product of certain Archimedean Riesz spaces.

Chapter 1 introduces Riesz spaces, tensor products, and several model Riesz spaces. The Riesz space properties that will be examined throughout the work are defined. In the following chapter, those properties are identified in example Riesz spaces and checked for in Fremlin's tensor product of that space with itself. Definitions unless otherwise stated are from [26], [35], and [36].

1.1 ARCHIMEDEAN RIESZ SPACES

Let X be a nonempty set. A *relation*, R, in X is a nonempty subset of the Cartesian product of X with itself. If a pair $(x, y) \in X \times X$ belongs to R, we write xRy. R is an *equivalence relation* in X if for all x, y, $z \in X$, we have that xRx (R is *reflexive*), xRy and yRz imply xRz (R is *transitive*), and xRy implies yRx (R is *symmetric*). If we also have that xRy and yRx imply $x = y$ (R is *anti-symmetric*), then R is a *partial ordering* in X. Whenever R is a partial ordering, we write $x \leq y$ in place of xRy.

Let X be a partially ordered set, and let Y be a nonempty subset of X. Then an element x_0 of X is an *upper bound* of Y if $x_0 \geq y$ for every $y \in Y$. If additionally $x_0 \leq z_0$ for any

other upper bound z_0 of Y, then x_0 is the *least upper bound* or *supremum* of Y. The terms *lower bound*, *greatest lower bound*, and *infimum* are defined analogously.

Definition 1.1.1. *Let X be a partially ordered set. X is called a* lattice *if the supremum $(x \vee y)$ and infimum $(x \wedge y)$ exist for every pair of elements x and y in X.*

Let V be a real vector space. V is an *ordered vector space* if V is partially ordered in such a way that the vector space structure and order structure are compatible. That is, for every x, y, $z \in V$ and $\lambda \geq 0$ in $\mathbb{R}$,

(i) $x \leq y$ implies $x + z \leq y + z$, and

(ii) $x \geq 0$ implies $\lambda x \geq 0$ in V.

Definition 1.1.2. *A* Riesz space *or* vector lattice *is an ordered vector space that is also a lattice with respect to the partial ordering.*

Certainly not every ordered vector space is a Riesz space. For example, consider the set of real-valued polynomials. Let $V = \{f \colon \mathbb{R} \to \mathbb{R} \mid f \text{ is a polynomial}\}$.

1. For every f, $g \in V$ and $\lambda \in \mathbb{R}$, define

$$(\lambda f)(x) = \lambda(f(x)) \quad \text{and} \quad (f + g)(x) = f(x) + g(x)$$

   for every $x \in \mathbb{R}$. The partial ordering is defined by $f \leq g$ for f, $g \in V$ whenever $f(x) \leq g(x)$ for every $x \in \mathbb{R}$. Under this ordering, it is easy to verify that V is an ordered vector space.

2. Consider $f(x) = x$ and $g(x) = -x$ in V. The supremum is defined pointwise; that is, for every $x \in \mathbb{R}$,

$$(f \vee g)(x) = f(x) \vee g(x).$$

Then $f \vee g$ is not an element of V since

$$(f \vee g)(x) = \begin{cases} -x & \text{if } x \leq 0, \\ \\ x & \text{if } x \geq 0, \end{cases}$$

is not a polynomial. However, $f \vee g$ is a so-called piecewise polynomial. Indeed, the set of piecewise polynomials is not only an ordered vector space, but a Riesz space.

Definition 1.1.3. *A function $p : \mathbb{R} \to \mathbb{R}$ is said to be a* piecewise polynomial *if there are $n \in \mathbb{N}$ and $t_1, \cdots, t_n \in (-\infty, \infty)$ such that $t_1 < t_2 < \cdots < t_n$ and p is a polynomial function on $(-\infty, t_1]$, $[t_n, \infty)$ and $[t_i, t_{i+1}]$ for each $i = 2, \cdots, n - 1$. The Riesz space of piecewise polynomials on $\mathbb{R}$ is denoted $PP(\mathbb{R})$.*

For any f in a Riesz space E, we write

(i) $f^+ = f \vee 0$,

(ii) $f^- = (-f) \vee 0$, and

(iii) $|f| = f \vee (-f)$.

Note that $-f^- = f \wedge 0$. The *positive cone* of E, denoted E^+, is the set of all elements $f \in E$ such that $f = f^+$. Two elements f and g are disjoint if $|f| \wedge |g| = 0$, in which case we write $f \perp g$. Of course, $|f^+| \wedge |f^-| = 0$ for every $f \in E$. If $A, B \subseteq E$ such that $f \perp g$ for all $f \in A$ and for all $g \in B$, then we write $A \perp B$.

Theorem 1.1.4. *(5.6 of [36]) If $f = u - v$ with u and v in E^+, then $f^+ \leq u$ and $f^- \leq v$. Hence, the decomposition $f = f^+ - f^-$ as a difference of positive elements is a "minimal" decomposition. In this case, i.e., if $u = f^+$ and $v = f^-$, we have $u \wedge v = 0$. Conversely, if $f = u - v$ with $u \wedge v = 0$, then this is the minimal decomposition, i.e., $u = f^+$ and $v = f^-$.*

Definition 1.1.5. *A Riesz space E is said to be* Archimedean *if*

$$\inf\{n^{-1}u \mid n = 1, 2, ...\} = 0$$

holds for every $u \in E^+$.

An exemplary Archimedean Riesz space is the set of real-valued continuous functions on a topological space X, in short $C(X)$, where scalar multiplication, addition, finite suprema, and finite infima are defined pointwise. It is easy to verify that $C(X)$ is a Riesz space. At the same time,

$$\inf\{n^{-1}f \in C(X) \mid n = 1, 2, \cdots\}(x) = \inf\{n^{-1}f(x) \in \mathbb{R} \mid n = 1, 2, \cdots\} = 0$$

for every $x \in X$, so $C(X)$ is an Archimedean Riesz space. Note that $f \in C(X)^+$ if $f(x) \geq 0$ for all $x \in X$.

Not every Riesz space is Archimedean. Take, for instance, $\mathbb{R}^2$ with a *lexicographical ordering*: $(x_1, y_1) \leq (x_2, y_2)$ whenever either $x_1 < x_2$ or $x_1 = x_2$, $y_1 \leq y_2$. Then $\mathbb{R}^2$ with this ordering forms a Riesz space. However, $(0, 1)$ is a lower bound of $\{n^{-1}(1, 1) \mid n = 1, 2, \cdots\}$, so $\inf\{n^{-1}(1, 1) : n = 1, 2, \cdots\} \neq (0, 0)$.

A Riesz space is "unique" relative to a certain characterization whenever a Riesz isomorphism can be defined between that space and any other Riesz space with the same characterization. In the remainder of this section, we define Riesz homomorpisms, state equivalent criteria, and define Riesz isomorphisms.

Definition 1.1.6. *Let X and Y be vector spaces. A mapping $T\colon X \to Y$ is* linear *if*

$$T(\lambda x + y) = \lambda T(x) + T(y)$$

for every $x \in X$, $y \in Y$, and $\lambda \in \mathbb{R}$.

Definition 1.1.7. *(19.1 of [36]) Let E and F be Riesz spaces. A linear mapping $T\colon E \to F$ is a* Riesz homomorphism *or* lattice homomorphism *if*

$$T(x \vee y) = T(x) \vee T(y)$$

for every $x \in E$ and $y \in E$.

Note that in the literature "Riesz space" and "vector lattice" are used interchangeably. Except for citations, we will use the term "Riesz space."

Theorem 1.1.8. *(19.2 of [36]) Let $T \colon E \to F$ be a linear map from the Riesz space E into the Riesz space F. Then the following conditions for T are equivalent.*

(i) T is a Riesz homomorphism.

(ii) $T(x \wedge y) = T(x) \wedge T(y)$ for all x and y in E.

(iii) $x \wedge y = 0$ implies $T(x) \wedge T(y) = 0$.

(iv) $|Tx| = T(|x|)$ for all $x \in E$.

(v) $Tx^+ = (Tx)^+$ for all $x \in E$.

A Riesz homomorphism T mapping the Riesz space E onto the Riesz space F in a one-to-one way is called a *Riesz isomorphism*. The inverse operator T^{-1} from F onto E is then also a Riesz isomorphism and the spaces E and F are said to be *Riesz isomorphic* (pages 125-126 of [36]).

1.2 THE ALGEBRAIC TENSOR PRODUCT

Definition 1.2.1. *Let X, Y and Z be vector spaces. A map $T \colon X \times Y \to Z$ is* bilinear *if*

$$T(x_1 + x_2, y) = T(x_1, y) + T(x_2, y) \qquad (\forall x_1, \ x_2 \in X, \ y \in Y)$$
$$T(x, y_1 + y_2) = T(x, y_1) + T(x, y_2) \qquad (\forall x \in X, \ y_1, \ y_2 \in Y)$$
$$\lambda T(x, y) = T(\lambda x, y) = T(x, \lambda y) \qquad (\forall \lambda \in \mathbb{R}, \ x \in X, \ y \in Y).$$

$B(X \times Y, Z)$ denotes the set of all bilinear maps $X \times Y \to Z$.

The algebraic tensor product of X and Y, denoted $X \otimes Y$, can be constructed as a subspace of $B(X \times Y)^*$, the set of real-valued linear functions on $B(X \times Y, \mathbb{R})$ (see e.g.

Chapter 1 of [29]). For every $(x, y) \in X \times Y$, set $(x \otimes y)(A) = A(x, y)$ for every $A \in B(X \times Y)$. We refer to $x \otimes y$ as an *elementary tensor*. Then $X \otimes Y$ is the subspace spanned by the elementary tensors in $B(X \times Y)^*$. As a consequence, for every $u \in X \otimes Y$ there exists $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $x_i \in X$, and $y_i \in Y$ such that

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i.$$

This representation need not be unique. Because $\otimes$ is a bilinear map, $\otimes(\lambda x, y) = \otimes(x, \lambda y) = \lambda \otimes (x, y)$. Since X is a vector space, $\lambda x \in X$ for every x. Thus, we more commonly write

$$u = \sum_{i=1}^{n} x_i \otimes y_i.$$

The uniqueness of the algebraic tensor product is guaranteed by its *universal property* which is stated below.

Theorem 1.2.2. *(Proposition 1.5 of [29]) Let X and Y be vector spaces. Suppose there exists a vector space W and a bilinear mapping $B \colon X \times Y \to W$ with the property that, for every vector space Z and every bilinear mapping A from $X \times Y$ into Z, there is a unique linear mapping $L \colon W \to Z$ such that $A = L \circ B$. Then there is an isomorphism J from $X \otimes Y$ into W such that $J(x \otimes y) = B(x, y)$ for every $x \in X$, $y \in Y$.*

$X \otimes Y$ satisfies the conditions for W in Theorem 1.2.2 (e.g. Proposition 1.4 of [29]). Thus, for every bilinear map $A \colon X \times Y \to Z$, there exists a unique linear map $L \colon X \otimes Y \to Z$ such that the diagram below commutes.

$$
\begin{array}{ccc}
X \times Y & \overset{\otimes}{\longrightarrow} & X \otimes Y \\
{\scriptstyle A}\big\downarrow & \swarrow {\scriptstyle L} & \\
Z & &
\end{array}
$$

In other words, every bilinear mapping A on $X \times Y$ factors through the linear mapping

$$(x, y) \in X \times Y \mapsto x \otimes y \in X \otimes Y.$$

See [29] for more on the algebraic tensor product.

1.3 THE ARCHIMEDEAN RIESZ SPACE TENSOR PRODUCT

Let E and F be Riesz spaces. The algebraic tensor product $E \otimes F$ uniquely exists, but in most cases it is not closed under finite suprema and infima. Chapter 2 provides several examples of E and F such that $E \otimes F$ is properly contained in the smallest Archimedean Riesz space containing $E \otimes F$. One motivation for defining an Archimedean Riesz space tensor product is to have a minimal, unique up to isomorphism, Archimedean Riesz space with a universal property enhancing that of the algebraic tensor product. D.H. Fremlin constructs such a product in [14]. In this thesis, we assume E and F to be Archimedean.

The analogue of a linear map on a vector space is a Riesz homomorphism on a Riesz space.

Definition 1.3.1. *Let E, F, and G be Archimedean Riesz spaces. A* Riesz bimorphism *is a bilinear map $T : E \times F \to G$ such that the maps*

$$z \longmapsto T(z, y) : E \to G$$
$$z \longmapsto T(x, z) : F \to G$$

are Riesz homomorphisms for all $x \in E^+$ and all $y \in F^+$.

Definition 1.3.2. *The linear subspace V of E is called a* Riesz subspace *of E if f, $g \in V$ implies $f \vee g$, $f \wedge g \in V$.*

Definition 1.3.3. *(page 30 of [36]) For any nonempty subset D of the Riesz space E, the intersection of all Riesz subspaces containing D is called the* Riesz subspace generated by *D.*

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Theorem 1.3.4 defines the (Fremlin) Archimedean Riesz space tensor product and, for convenience, summarizes all relevant properties from [14].

Theorem 1.3.4. *(4.2 and 4.4 of [14]) Let E and F be Archimedean Riesz spaces. There exists an Archimedean Riesz space G and a Riesz bimorphism $\varphi\colon E \times F \to G$ with the following properties.*

   (i) *Whenever H is an Archimedean Riesz space and $\psi\colon E \times F \to H$ is a Riesz bimorphism, there is a unique Riesz homomorphism $T\colon G \to H$ such that $T \circ \varphi = \psi$. In other words, the following diagram commutes.*

$$
\begin{array}{ccc}
E \times F & \xrightarrow{\ \varphi\ } & G \\
{\scriptstyle\psi}\big\downarrow & \swarrow{\scriptstyle T} & \\
H & &
\end{array}
$$

   (ii) *If $\psi(x, y) > 0$ in H whenever $x > 0$ in E and $y > 0$ in F, then G may be identified with the Riesz subspace of H generated by $\psi[E \times F]$.*

   (iii) *φ induces an embedding of the algebraic tensor product of E and F, denoted $E \otimes F$, in G.*

   (iv) *$E \otimes F$ is uniformly dense in G, i.e., for every $w \in G$ there exist $x_0 \in E$ and $y_0 \in F$ such that for every $\delta > 0$ there is a $v \in E \otimes F$ such that $|w - v| \leq \delta x_0 \otimes y_0$.*

   (v) *If $w > 0$ in G, then there exist $x \in E^+$ and $y \in F^+$ such that $0 < x \otimes y \leq w$.*

G of Theorem 1.3.4 is the *Archimedean Riesz space tensor product of E and F*, denoted $E \bar{\otimes} F$. Any Archimedean Riesz space paired with a Riesz bimorphism satisfying the universal property (i) is Riesz isomorphic to G.

Theorem 1.3.5. *(2.2.11 of [24]) Let D be a linear subspace of a Riesz space E, and let R be the Riesz subspace generated by D in E. Then any element of R can be written in the*

form

$$\sup_{i \in I} \inf_{j \in J} x_{ij}$$

for some finite sets I and J of $\mathbb{N}$, where x_{ij} are elements of D.

The Riesz bimorphism $\otimes \colon E \times F \to E \bar{\otimes} F$ embeds the algebraic tensor product $E \otimes F$ into $E \bar{\otimes} F$ via $\otimes(e, f) = e \otimes f$ for all $e \in E$, $f \in F$. According to Theorem 1.3.4 (ii), $E \bar{\otimes} F$ may be identified with the Riesz space generated by the elementary tensors. It follows from Theorem 1.3.5 that for every element h of $E \bar{\otimes} F$, there exist finite sets I, J of $\mathbb{N}$ and $g_{ij} \in E \otimes F$ such that

$$h = \sup_{i \in I} \inf_{j \in J} \{g_{ij}\},$$

where $g_{ij} = \sum_{k=1}^{n} e_k \otimes f_k$ for some $n \in \mathbb{N}$, $e_k \in E$, and $f_k \in F$.

1.4 RIESZ SUBSPACES AND THE MAIN INCLUSION THEOREM

The so-called "main inclusion theorem" organizes the most important Riesz space properties. It was originally named after the diagram in 25.1 of [26] but has since been expanded. The contents vary in the literature, so we choose those which are examined in this thesis. First, we define specific types of Riesz subspaces. Then Riesz space properties of interest, along with the the main inclusion theorem diagram, are given and serve as a reference throughout this thesis.

Definition 1.4.1. *Let E be a Riesz space.*

(i) *The Riesz subspace I of E is called an* ideal *in E if $f \in I$, $g \in E$, and $|g| \leq |f|$ imply $g \in I$.*

(ii) *The ideal B of E is called a* band *in E if, whenever $D \subseteq B$ and $\sup(D)$ exists in E, then $\sup(D) \in B$.*

(iii) *For any nonempty subset D of E the intersection of all Riesz subspaces containing D is called the* Riesz subspace generated by D. *In the particular case that D consists of one*

element f, the Riesz subspace generated by D is called a principal band *and denoted by* $[f]$.

(iv) *The band B of E is called a* projection band *of E if $E = B \oplus B^d$, where*

$$B^d = \{f \in E \mid |f| \wedge |g| = 0 \; \forall g \in B\}.$$

(v) *The ideal I of E is called a* maximal ideal *of E if, whenever $I \subseteq J$ for some ideal J of E, it follows that $I = J$ or $J = E$.*

(vi) *Let $f \in E$. The* principal ideal *generated by f, denoted E_f, is the smallest ideal of E containing f. More concretely,*

$$E_f = \{g \in E \mid |g| \leq |\lambda f| \text{ for some } \lambda \in \mathbb{R}\}.$$

Definition 1.4.2. *Let E be a Riesz space.*

(i) *(DC) E is called* Dedekind complete *if every nonempty subset of E that is bounded above (below) has a supremum (infimum).*

(ii) *(DαC) Let α be an infinite cardinal. E is called* Dedekind α-complete *if every nonempty subset A of E that is bounded above (bounded below) has a supremum (infimum) whenever A has cardinality no more than α (i.e., $|A| \leq \alpha$). If α is countable, we say that E is* Dedekind σ-complete.

(iii) *(PP) E is said to have the* projection property *if every band in E is a projection band.*

(iv) *(PPP) E is said to have the* principal projection property *if every principal band in E is a projection band.*

(v) *(UC) Let E be a Riesz space and $0 < u \in E$. A sequence $\{f_n\}_{n \in \mathbb{N}}$ is said to be a u-uniform Cauchy sequence if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_m - f_n| < \epsilon u$*

for all m, n ≥ N. A sequence $\{f_n\}_{n \in \mathbb{N}}$ converges u-uniformly to a u-uniform limit f if there exists a sequence of numbers $\epsilon_n \downarrow 0$ such that $|f - f_n| \le \epsilon_n u$ for all n. E is uniformly complete if, for every u > 0 in E, every u-uniform Cauchy sequence has a u-uniform limit.

Theorem 1.4.3. *The following implications hold for any Riesz space, with "and" represented by " + " and the abbreviations referring to Theorem 1.4.2.*

<div align="center">

The Main Inclusion Theorem

</div>

$$D\alpha C \Longrightarrow D\sigma C$$

$$(PP + UC) \cong DC \qquad\qquad Archimedean$$

$$PP \Longrightarrow PPP$$
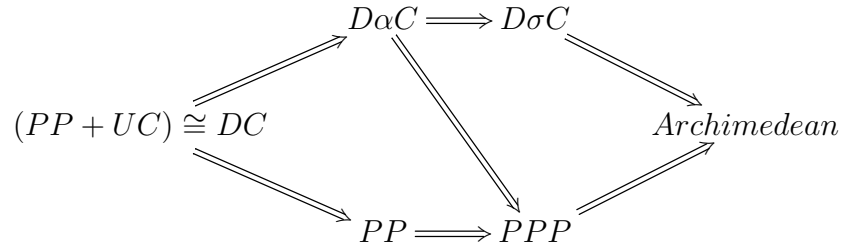
No implication in the converse direction holds. For counterexamples, see Section 25 of [26]. Additional examples are provided as needed.

Theorem 1.4.4. *(12.4 of [36]) If the Riesz space E has one of the properties in the main inclusion theorem and A is an ideal in E, then A has the same property.*

Since $C(X)$ is an iconic example of an Archimedean Riesz space, we identify a few relationships between Riesz space properties of $C(X)$ and topological properties of X. The corresponding definitions are included.

Definition 1.4.5. *Let X be a topological space.*

(i) *X is said to be* completely regular *provided that it is a Hausdorff space such that, whenever F is a closed set and x is a point in its complement, there exists a function $f \in C(X)$ such that $f(x) = 1$ and $f[F] = \{0\}$.*

(ii) *Two subsets A and B of X are said to be* completely separated *(from one another) in X if there exists a function $f \in C(X)$ such that $0 \le f(x) \le 1$ for every $x \in X$, $f(x) = 0$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$.*

Definition 1.4.6. *Let X be a completely regular topological space and α an infinite cardinal.*

(i) *X is* extremally disconnected *if every open set has open closure (1H of [17]).*

(ii) *Every set of the form $\{x \mid f(x) = 0\}$ for some $f \in C(X)$ is called a* zero-set *of X; a* cozero-set *is the complement of zero-set (1.10, 1.11 of [17]). X is* basically disconnected *if every cozero-set has an open closure (1H of [17]).*

(iii) *A subset $V \subseteq X$ is said to be an α-cozero set if $V = \bigcup \mathcal{U}$, where $\mathcal{U}$ is a set of cozero-sets in X with $|\mathcal{U}| \leq \alpha$. X is α-disconnected if every α-cozero set has open closure ([27]).*

Theorem 1.4.7. *([31] and 3N of [17]) Let X be a completely regular topological space and α an infinite cardinal.*

(i) *$C(X)$ is Dedekind complete if and only if X is extremally disconnected.*

(ii) *$C(X)$ is Dedekind σ-complete if and only if X is basically disconnected.*

(iii) *$C(X)$ is Dedekind α-complete if and only if X is α-disconnected.*

Definition 1.4.8. *A completely regular topological space X is an* F-space *provided that disjoint cozero sets are completely separated.*

2. MOTIVATING EXAMPLES

Chapter 2 examines three Riesz spaces—namely $c(\mathbb{N})$, $l^\infty(\mathbb{N})$, and $\mathbb{R}^X$—in order to gain intuition for the behavior of the tensor product of each space with itself. Both their algebraic and Fremlin tensor products are studied. Many of the proof-methods are inspired by Anthony Hager in [22].

2.1 EXAMPLE RIESZ SPACES AND THEIR TENSOR PRODUCTS

We begin by defining a few Riesz spaces that will serve as motivating examples.

Definition 2.1.1.

1. *The set of all convergent sequences is denoted $c(\mathbb{N})$, i.e,*

$$c(\mathbb{N}) = \{f\colon \mathbb{N} \to \mathbb{R} \mid \exists L \in \mathbb{R} \ s.t. \ \forall \epsilon > 0, \exists N \ s.t. \ n \geq N \implies |f(n) - L| \leq \epsilon\}.$$

   The set of all sequences convergent to zero is denoted $c_0(\mathbb{N})$, i.e.,

$$c_0(\mathbb{N}) = \{f\colon \mathbb{N} \to \mathbb{R} \mid \forall \epsilon > 0, \exists N \ s.t. \ n \geq N \implies |f(n)| \leq \epsilon\}.$$

   More generally, let X be a locally compact Hausdorff space. Then $c(X)$ is the set of all convergent functions on X, i.e. $f\colon X \to \mathbb{R}$ such that for some $L \in \mathbb{R}$,

$$\forall \epsilon > 0, \ \exists Z \subseteq X \ compact \ where \ u \in X \backslash Z \implies |f(u) - L| \leq \epsilon\}.$$

   Similarly, $c_0(X)$ is the set of all functions on X convergent to zero.

2. *The set of bounded sequences is denoted $l^\infty(\mathbb{N})$, i.e.,*

$$l^\infty(\mathbb{N}) = \{f \colon \mathbb{N} \to \mathbb{R} \mid \exists M \in \mathbb{R} \ s.t. \ \forall n \in \mathbb{N}, \ |f(n)| \le M\}.$$

Similarly, $l^\infty(\mathbb{N} \times \mathbb{N})$ is the set of all bounded sequences on $\mathbb{N} \times \mathbb{N}$.

3. *Let X be a set. $\mathbb{R}^X$ denotes the set of all functions from X to $\mathbb{R}$.*

As Fremlin notes in [14], for topological spaces X and Y we have the following:

$$C(X) \otimes C(Y) \subseteq C(X)\bar{\otimes}C(Y) \subseteq C(X \times Y).$$

We will prove in Examples 2.1.4 and 2.1.8 and Corollary 2.3.2 that the inclusions are proper whenever $C(X)$ and $C(Y)$ are replaced with $c(\mathbb{N})$.

Hager's Proposition 1 of [22] "tests" elements of $C(X \times Y)$ to determine if they are elements of $C(X) \otimes C(Y)$. His proposition holds specifically for $c(\mathbb{N}) \otimes c(\mathbb{N})$ inside $c(\mathbb{N} \times \mathbb{N})$ and $c_0(\mathbb{N}) \otimes c_0(\mathbb{N})$ inside $c_0(\mathbb{N} \times \mathbb{N})$. In the remainder of the section, we assume X and Y to be locally compact.

Theorem 2.1.2. *(Hager, [22]) Suppose $F \in C(X \times Y)$. Then $F \in C(X) \otimes C(Y)$ if and only if the dimension of the vector subspace of $C(X)$ generated by $\{F(\cdot, y) : y \in Y\}$ is finite.*

Example 2.1.3. $c_0(\mathbb{N}) \otimes c_0(\mathbb{N}) \ne c_0(\mathbb{N} \times \mathbb{N})$.

Proof. Let $F \in c_0(\mathbb{N} \times \mathbb{N})$ such that

$$F(x, y) = \frac{1}{x^y}, \qquad ((x, y) \in \mathbb{N} \times \mathbb{N}).$$

The functions $\{f_n(y) = \frac{1}{n^y}\}_{n \in \mathbb{N}}$ are linearly independent. Thus by Theorem 2.1.2, $F \notin c_0(\mathbb{N}) \otimes c_0(\mathbb{N})$, and $c_0(\mathbb{N}) \otimes c_0(\mathbb{N}) \ne c_0(\mathbb{N} \times \mathbb{N})$.

$\square$

Note that $F(x,y) = \frac{1}{x^y}$ is an element of $c(\mathbb{N} \times \mathbb{N})$ and of $l^\infty(\mathbb{N} \times \mathbb{N})$. By the same argument as in Example 2.1.3, we obtain the following two results.

Example 2.1.4. $c(\mathbb{N}) \otimes c(\mathbb{N}) \neq c(\mathbb{N} \times \mathbb{N})$.

Example 2.1.5. $l^\infty(\mathbb{N}) \otimes l^\infty(\mathbb{N}) \neq l^\infty(\mathbb{N} \times \mathbb{N})$.

Example 2.1.3 proves that $c_0(X) \otimes c_0(Y) \neq c_0(X \times Y)$ when $X = Y = \mathbb{N}$. We follow the ideas in 2.1.6 to characterize when $c_0(X) \otimes c_0(Y) = c_0(X \times Y)$.

Theorem 2.1.6. *(Hager, 2 of [22])* $C(X) \otimes C(Y) = C(X \times Y)$ *if and only if X or Y is finite.*

Theorem 2.1.7. $c_0(X) \otimes c_0(Y) = c_0(X \times Y)$ *if and only if X or Y is finite.*

Proof. If X is finite and $f \in c_0(X \times Y)$, then $\{f(x, \cdot) : x \in X\}$ is finite and $f \in c_0(X) \otimes c_0(Y)$ by Theorem 2.1.2.

We prove the contrapositive. Suppose X and Y are infinite. Since X is Hausdorff, there are distinct elements $x_1, x_2, \cdots$ in X and $U_1, U_2, \ldots$ open subsets of X that are pairwise disjoint with $x_n \in U_n$ for every $n \in \mathbb{N}$. Since X is locally compact, for every i there exists a compact neighborhood V_i of x_i such that $\bar{V}_i \subseteq U_i$. Hence there exist $V_1, V_2, \ldots \subseteq X$ open neighborhoods of $x_1, x_2, \ldots \in X$ respectively such that $V_i \cap V_j = \emptyset$ for every $i \neq j$.

Let $f_n \in C(X)$ satisfy $f_n(x_n) = 1, f_n(x) = 0$ if $x \notin V_n$, and $0 \leq f_n \leq 1$. Indeed, by definition f_n is an element of $c_0(X)$ since $f_n \equiv 0$ outside a compact set. Likewise, select distinct $y_1, y_2, \cdots \in Y$ and $W_1, W_2, \ldots$ open pairwise disjoint subsets of Y such that $y_n \in W_n$. Let $g_n \in c_0(Y)$ satisfy $g_n(y_n) = 1, g_n(y) = 0$ if $y \notin W_n$, and $0 \leq g_n \leq 1$. Set

$$F(x,y) = \sum_{n=1}^\infty \frac{1}{2^n} f_n(x) g_n(y).$$

Then $F \in c_0(X \times Y)$ and $F(x, y_k) = \sum_{n=1}^\infty \frac{1}{2^n} f_n(x) g_n(y_k) = \frac{1}{2^k} f_k(x)$. Since $\{f_k : k \in \mathbb{N}\}$ are independent, $\{F(\cdot, y_k) : k \in \mathbb{N}\}$ are independent. Hence, $F(x,y) \notin c_0(X) \otimes c_0(Y)$ by Theorem 2.1.2. $\qquad \square$

The following example compares the algebraic tensor product of $c(\mathbb{N})$ with itself to the Fremlin tensor product of $c(\mathbb{N})$ with itself. For a subset A of a set X, we use 1_A to denote the characteristic function of A. In the case that E and F are spaces of real-valued functions on X and Y respectively, for $f \in E$, $g \in F$, $x \in X$, and $y \in Y$ we simply write $f(x)g(y)$ rather than $(f \otimes g)(x, y)$.

Example 2.1.8. $c(\mathbb{N}) \otimes c(\mathbb{N}) \neq c(\mathbb{N})\bar{\otimes}c(\mathbb{N})$.

Proof. Let $F(x, y) = (\frac{1}{x}1_{\mathbb{N}}(y)) \vee (1_{\mathbb{N}}(x)\frac{1}{y})$ for every $(x, y) \in \mathbb{N} \times \mathbb{N}$. Then $F \in c(\mathbb{N})\bar{\otimes}c(\mathbb{N})$. Fix $n \in \mathbb{N}$ and consider the matrix determined by $(a_{ij}) = F(i, j)$ for all $i, j \in \{1, \cdots, n\}$. By the use of elementary row operations, the determinant is nonzero. Indeed,

$$
\begin{vmatrix}
F(1,1) & F(1,2) & F(1,3) & \cdots & F(1,n) \\
F(2,1) & F(2,2) & F(2,3) & \cdots & F(2,n) \\
F(3,1) & F(3,2) & F(3,3) & \cdots & F(3,n) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
F(n,1) & F(n,2) & F(n,3) & \cdots & F(n,n)
\end{vmatrix}
$$

$$
=
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n}
\end{vmatrix}
$$

$$
=
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & \frac{1}{2}-1 & \frac{1}{2}-1 & \cdots & \frac{1}{2}-1 \\
0 & 0 & \frac{1}{3}-\frac{1}{2} & \cdots & \frac{1}{3}-\frac{1}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{1}{n}-\frac{1}{n-1}
\end{vmatrix}
$$

$$
= (1)\left(\frac{1}{2}-1\right)\left(\frac{1}{3}-\frac{1}{2}\right)\cdots\left(\frac{1}{n}-\frac{1}{n-1}\right) \neq 0.
$$

Hence, $F(1, \cdot), F(2, \cdot), ..., F(n, \cdot)$ are linearly independent elements of $c(\mathbb{N})$ and the dimension of the vector space generated by $\{F(\cdot, y) \mid y \in \mathbb{N}\}$ is infinite. By Theorem 2.1.2, $F \notin c(\mathbb{N}) \otimes c(\mathbb{N})$. As a result, $c(\mathbb{N}) \otimes c(\mathbb{N}) \neq c(\mathbb{N}) \bar{\otimes} c(\mathbb{N})$. $\qquad\square$

We highlight the handiness of the "determinant trick" that tests whether an element of $C(X \times Y)$ is an element of $C(X) \otimes C(Y)$. A similar test to determine exactly which elements of $C(X \times Y)$ belong to $C(X) \bar{\otimes} C(Y)$ is not yet in the literature.

2.2 THE UNIFORM CLOSURE OF THE ALGEBRAIC TENSOR PRODUCT

Example 2.1.4 shows that $c(\mathbb{N}) \otimes c(\mathbb{N})$ is a proper subset of $c(\mathbb{N} \times \mathbb{N})$. Since $c(\mathbb{N} \times \mathbb{N})$ is uniformly complete (43.1 of [26]), we compare the set of Cauchy sequences in $c(\mathbb{N}) \otimes c(\mathbb{N})$ with their relative uniform limits to the set $c(\mathbb{N} \times \mathbb{N})$. Let $\overline{c_0(\mathbb{N}) \otimes c_0(\mathbb{N})}^{u.c}$ be the set of all equivalence classes of u-Cauchy sequences of $c_0(\mathbb{N}) \otimes c_0(\mathbb{N})$ together with their u-uniform limits in $c_0(\mathbb{N} \times \mathbb{N})$.

Theorem 2.2.1. $c_0(\mathbb{N} \times \mathbb{N}) = \overline{c_0(\mathbb{N}) \otimes c_0(\mathbb{N})}^{u.c}$.

Proof. Let $F \in c_0(\mathbb{N} \times \mathbb{N})$. To show $c_0(\mathbb{N} \times \mathbb{N}) \subseteq \overline{c_0(\mathbb{N}) \otimes c_0(\mathbb{N})}^{u.c}$, there must exist a Cauchy sequence of functions in $c_0(\mathbb{N}) \otimes c_0(\mathbb{N})$ that converges to F relative to some u in $c_0(\mathbb{N}) \otimes c_0(\mathbb{N})$. Since $F \in c_0(\mathbb{N} \times \mathbb{N})$, for every $n \in \mathbb{N}$ there exists $X_n \subseteq \mathbb{N}$ such that $F(x, y) \leq \frac{1}{n^3}$ for all $(x, y) \notin (X_n \times X_n)$. Define

$$F_n(x, y) = F(x, y) \cdot \mathbf{1}_{\{X_n \times X_n\}}(x, y).$$

Note that for every $n \in \mathbb{N}$, the set $\{F(x, y) : (x, y) \in X_n \times X_n\}$ is finite, so $F_n(x, y)$ is an element of $c_0(\mathbb{N}) \otimes c_0(\mathbb{N})$. For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|F_n(x, y) - F_m(x, y)| \leq \epsilon \frac{1}{xy}$$

for every $(x, y) \in X \times Y$ whenever $n, m \geq N$. Thus, $(F_n)_{n \in \mathbb{N}}$ is a u-Cauchy sequence for $u(x, y) = \frac{1}{xy}$.

Let $\epsilon_n = \frac{1}{n}$ for every $n \in \mathbb{N}$. Then there exists $M \in \mathbb{N}$ such that $\forall n \geq M$,

$$|F - F_n|(x, y) \begin{cases} = 0 & \text{if } (x, y) \in X_n \times X_n \\ \leq \epsilon_n u(x, y) & otherwise. \end{cases}$$

By definition 1.4.2 (v), F_n converges u-uniformly to F in $c_0(\mathbb{N} \times \mathbb{N})$. Since $\overline{c_0(\mathbb{N}) \otimes c_0(\mathbb{N})}^{u.c} \subseteq c_0(\mathbb{N} \times \mathbb{N})$, the spaces are equal. $\qquad \square$

2.3 THE FREMLIN TENSOR PRODUCT AND CERTAIN RIESZ SPACE PROPER-TIES

$c(\mathbb{N})$ holds numerous Riesz space properties from the main inclusion theorem in Section 1. Namely, $c(\mathbb{N})$ is Dedekind complete. In particular, $c(\mathbb{N})$ is uniformly complete and has the projection property. The following is an example of a Dedekind complete Riesz space whose Fremlin tensor product with itself is not uniformly complete and thus not Dedekind complete.

Theorem 2.3.1. $c(\mathbb{N}) \bar{\otimes} c(\mathbb{N})$ *is not uniformly complete.*

Proof. Define the sequences of functions

$$a_n(x, y) = 1_{\{1, \cdots, n\}}(x) 1_{\{1, \cdots, n\}}(y), \qquad ((x, y) \in \mathbb{N} \times \mathbb{N})$$

$$b_n(x, y) = \left| \left(\frac{1}{y} \sin x \right) (a_n(x, y)) \right|, \qquad ((x, y) \in \mathbb{N} \times \mathbb{N}).$$

Note that for every n, the function $b_n(x, y)$ vanishes outside a finite collection of points. Thus, $b_n \in c(\mathbb{N}) \bar{\otimes} c(\mathbb{N})$ for all $n \in \mathbb{N}$. At the same time, $|\frac{1}{y} \sin x| \leq 1_X \frac{1}{y}$. For all $\epsilon > 0$, there exists N such that $|b_n - b_m| \leq \epsilon(1_X \frac{1}{y})$ whenever $n, m \geq N$, so $\{b_n\}_{n \in \mathbb{N}}$ is a u-uniform Cauchy sequence for $u(x, y) = \frac{1}{y}$. However, the u-uniform limit of $\{b_n\}_{n \in \mathbb{N}}$ is $|\frac{1}{y} \sin x|$. Since $\sin(x)$

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does not converge to any real number, the u-uniform limit does not exist in $c(\mathbb{N})\bar{\otimes}c(\mathbb{N})$. Thus, $c(\mathbb{N})\bar{\otimes}c(\mathbb{N})$ is not uniformly complete. $\square$

Corollary 2.3.2. $c(\mathbb{N})\bar{\otimes}c(\mathbb{N}) \neq c(\mathbb{N} \times \mathbb{N})$.

Proof. $c(\mathbb{N} \times \mathbb{N})$ is uniformly complete. By Theorem 2.3.1, $c(\mathbb{N})\bar{\otimes}c(\mathbb{N})$ is not uniformly complete. Thus, the two cannot be isomorphic as Riesz spaces. $\square$

Next, we check whether the Fremlin tensor product of $c_0(\mathbb{N})$ with itself has the projection property. We make use of $c_0(\mathbb{N})\bar{\otimes}c_0(\mathbb{N})$ as a Riesz subspace of $c_0(\mathbb{N} \times \mathbb{N})$ in order to apply the following two theorems.

Definition 2.3.3. *(23.1 of [36]) A Riesz subspace V of the Riesz space E is said to be order dense in E if for each $u > 0$ in E (i.e., $u \geq 0$, $u \neq 0$) there exists an element $v \in V$ such that $0 < v \leq u$.*

Theorem 2.3.4. *(5.12 of [32]) Let Y be a Riesz space, X an order dense subspace of Y, and I a band in X. Then there is a band J in Y such that $I = J \cap X$.*

Theorem 2.3.5. *(5.2 of [25]) Let X be a locally compact Hausdorff space and J a vector subspace in $c_0(X)$.*

   (i) J is a band if and only if $J = \{f \in c_0(X) : f|_F \equiv 0\}$ for some closed set F in X with $F = \overline{Int(F)}$.

   (ii) J is a projection band if and only if $J = \{f \in c_0(X) : f|_F \equiv 0\}$ for some clopen set F in X.

Theorem 2.3.6. *(Riesz decomposition property) Let the elements f, g_1, g_2 in E^+ satisfy $f \leq g_1 + g_2$. Then there exist elements f_1, f_2 in E^+ such that $f_1 \leq g_1$, $f_2 \leq g_2$ and $f = f_1 + f_2$.*

Theorem 2.3.7. *$c_0(\mathbb{N})\bar{\otimes}c_0(\mathbb{N})$ has the projection property.*

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Proof. Let B be a band of $c_0(\mathbb{N})\bar{\otimes}c_0(\mathbb{N})$. Since $c_0(\mathbb{N})\bar{\otimes}c_0(\mathbb{N})$ is order dense in $c_0(\mathbb{N}\times\mathbb{N})$ (Theorem 1.3.4 (v)), there exists a band D in $c_0(\mathbb{N}\times\mathbb{N})$ such that $B = c_0(\mathbb{N})\bar{\otimes}c_0(\mathbb{N})\cap D$ by Theorem 2.3.4. Because $\mathbb{N}\times\mathbb{N}$ is a locally compact Hausdorff space, by Theorem 5.2 (i) of 2.3.5 there exists a regularly closed set $F\subseteq\mathbb{N}\times\mathbb{N}$ such that $D = \{f\in c_0(\mathbb{N}\times\mathbb{N}) : f|_F \equiv 0\}$. Since $\mathbb{N}\times\mathbb{N}$ is discrete, F is clopen. Consequently, D is a projection band in $c_0(\mathbb{N}\times\mathbb{N})$ by Theorem 2.3.5 (ii). Thus, $D\oplus D^d = c_0(\mathbb{N}\times\mathbb{N})$.

Let $f\in(c_0(\mathbb{N})\bar{\otimes}c_0(\mathbb{N}))^+\subseteq c_0(\mathbb{N}\times\mathbb{N})^+$. Then there exist positive elements $h_1\in D$ and $h_2\in D^d$ such that $f = h_1+h_2$. Since B is order dense in D, there exists positive $g_1\in B$ such that $h_1\leq g_1$. Likewise, there exists $g_2\in B^d$ such that $h_2\leq g_2$. Then $f = h_1 + h_2 \leq g_1 + g_2$. By the Riesz decomposition property, there exist elements $f_1,\ f_2 \in (c_0(\mathbb{N})\bar{\otimes}c_0(\mathbb{N}))^+$ such that $f_1\leq g_1$, $f_2\leq g_2$ and $f = f_1 + f_2$. Since B and B^d are, in particular, ideals, $f_1\in B$ and $f_2\in B^d$. Every element of $c_0(\mathbb{N})\bar{\otimes}c_0(\mathbb{N})$ is a difference of two positive elements, and thus we have shown that $c_0(\mathbb{N})\bar{\otimes}c_0(\mathbb{N})\subseteq B\oplus B^d$. It is clear that $B\oplus B^d\subseteq c_0(\mathbb{N})\bar{\otimes}c_0(\mathbb{N})$, so $c_0(\mathbb{N})\bar{\otimes}c_0(\mathbb{N})$ has the projection property. $\qquad\square$

Throughout this work, we combine the fact that the algebraic tensor product is uniformly dense in the Fremlin tensor product (Theorem 1.3.4 (iv)) with the existence of a unique uniform completion of any Archimedean Riesz space. Here, this pattern is introduced to prove Theorem 2.3.12. For the existence and uniqueness of the uniform completion of Archimedean Riesz spaces, see [3] and [33].

Definition 2.3.8. *(see [3]) A* uniform completion *of an Archimedean Riesz space E is a pair $(\tilde{E},\, i)$ consisting of a uniformly complete Riesz space $\tilde{E}$ and a Riesz homomorphism $i\colon E\to\tilde{E}$ such that for every uniformly complete Riesz space F and every Riesz homomorphism $\varphi\colon E\to F$, there is a unique Riesz homomorphism $\tilde{\varphi}\colon\tilde{E}\to F$ with $\varphi = \tilde{\varphi}\circ i$. In other*

words, the following diagram commutes.

$$E \xrightarrow{\;i\;} \tilde{E}$$
$$\varphi \downarrow \quad \swarrow \tilde{\varphi}$$
$$F$$

Theorem 2.3.9. *(3 of [3]) For an Archimedean Riesz space E, a uniform completion of E exists and is unique up to Riesz isomorphisms.*

Definition 2.3.10. *Let E be an Archimedean Riesz space. A subset V of E is* uniformly dense *in E if for every $w \in E$ there exists $u \in V$ such that for every $\epsilon > 0$ there is a $v \in V$ such that $|w - v| \leq \epsilon u$.*

Definition 2.3.11. *Let E be an Archimedean Riesz space. An element $u > 0$ for which E is the principal ideal generated by u in E is called a* strong order unit *of E.*

Excepting $c_0(\mathbb{N})$, all of the example Riesz spaces in Chapter 2 have $1_{\mathbb{N}}$ or 1_X as their strong order units.

Theorem 2.3.12. *Let E be a uniformly complete Riesz space with a strong order unit u. If V is a uniformly dense Riesz subspace of E containing u, then E is a uniform completion of V.*

Proof. Let $\iota \colon V \to E$ be an embedding of V into E as a Riesz subspace of E. Suppose F is a uniformly complete Riesz space and $\varphi \colon V \to F$ is a Riesz homomorphism. For every $x \in E$, there exists a u-uniform Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ of V that converges to x in E. Since $(x_n)_{n \in \mathbb{N}}$ is a u-uniform Cauchy sequence, for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon u$ whenever $n, m \geq N$. Then for all $\epsilon > 0$, there exists N such that $\forall n, m \geq N$,

$$|\varphi(x_n) - \varphi(x_m)| = |\varphi(x_n - x_m)| = \varphi(|x_n - x_m|) < \epsilon\varphi(u).$$

Thus, $(\varphi(x_n))_{n \in \mathbb{N}}$ is a $\varphi(u)$-uniform Cauchy sequence of F and converges $\varphi(u)$-uniformly in F. Define $\tilde{\varphi}(x)$ to be the $\varphi(u)$-uniform limit of $(\varphi(x_n))_{n \in \mathbb{N}}$ in F. By Theorem 10.3 (ii) of

[36], $\tilde{\varphi}$ is a Riesz homomorphism that is uniquely determined by φ. Then $\varphi = \tilde{\varphi} \circ \iota$ and (E, ι) is a uniform completion of V. $\qquad\square$

To conclude this section, we check for the Dedekind completeness of the Fremlin tensor product of $l^{\infty}(\mathbb{N})$ with itself and of R^X with R^Y for discrete, infinite dimensional topological spaces. In order to do so, we will use the Stone-Weierstrass Theorem and the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ (e.g. [17]).

Theorem 2.3.13. *(Stone-Weierstrass Theorem) Let X be a compact space and A an algebra of continuous real-valued functions on X that separates the points of X and contains the constant functions. Then A is a dense subset of $C(X)$.*

Example 2.3.14. $l^{\infty}(\mathbb{N})\bar{\otimes}l^{\infty}(\mathbb{N})$ *is not Dedekind complete.*

Proof. Suppose $l^{\infty}(\mathbb{N})\bar{\otimes}l^{\infty}(\mathbb{N})$ is Dedekind complete. Note that $l^{\infty}(\mathbb{N}) \cong C(\beta\mathbb{N})$. Then $C(\beta\mathbb{N})\bar{\otimes}C(\beta\mathbb{N})$ is in particular uniformly complete.

By Fremlin's Theorem 1.3.4, $C(\beta\mathbb{N}) \otimes C(\beta\mathbb{N})$ is uniformly dense in $C(\beta\mathbb{N})\bar{\otimes}C(\beta\mathbb{N})$. By the Stone-Weierstrass theorem, $C(\beta\mathbb{N}) \otimes C(\beta\mathbb{N})$ is uniformly dense in $C(\beta\mathbb{N} \times \beta\mathbb{N})$. Then $C(\beta\mathbb{N})\bar{\otimes}C(\beta\mathbb{N})$ is a uniformly dense Riesz subspace of $C(\beta\mathbb{N} \times \beta\mathbb{N})$ with unit $1_{\beta\mathbb{N}\times\beta\mathbb{N}}$. By Theorem 2.3.12, $C(\beta\mathbb{N})\bar{\otimes}C(\beta\mathbb{N})$ is a uniform completion of $C(\beta\mathbb{N} \times \beta\mathbb{N})$. However, $C(\beta\mathbb{N})\bar{\otimes}C(\beta\mathbb{N})$ is uniformly complete by assumption, so $C(\beta\mathbb{N})\bar{\otimes}C(\beta\mathbb{N}) = C(\beta\mathbb{N} \times \beta\mathbb{N})$. Then $C(\beta\mathbb{N} \times \beta\mathbb{N})$ is also Dedekind complete which implies $\beta\mathbb{N} \times \beta\mathbb{N}$ is extremally disconnected, a contradiction to Glicksberg's Theorem 1 of [18]. $\qquad\square$

Example 2.3.15. *Let X, Y be infinite.*

1. $\mathbb{R}^X\bar{\otimes}\mathbb{R}^Y$ *is not Riesz isomorphic to $\mathbb{R}^{X\times Y}$.*

2. $\mathbb{R}^X\bar{\otimes}\mathbb{R}^Y$ *is not Dedekind complete.*

Proof. Note that $\mathbb{R}^{X\times Y}$ is Dedekind complete. Suppose that $\mathbb{R}^X\bar{\otimes}\mathbb{R}^Y$ is Dedekind complete.

Let A be a bounded subset of $l^{\infty}(\mathbb{N})\bar{\otimes}l^{\infty}(\mathbb{N})$. Since X and Y are infinite, let D_X and D_Y represent homeomorphic copies of $\mathbb{N}$ in X and Y respectively. For every $f \in A$, there exists

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an extension $f' \colon X \times Y \to \mathbb{R}$ such that $f'(x, y) = f(x, y)$ for every $(x, y) \in D_X \times D_Y$, and $f'(x, y) = 0$ otherwise. Then

$$g(x, y) = \sup_{f \in A} f'(x, y)$$

exists in $\mathbb{R}^X \bar{\otimes} \mathbb{R}^Y$. Since the supremum is taken pointwise, $g(x, y) = 0$ for all $(x, y) \in (X \times Y) \backslash (D_X \times D_Y)$. Thus, $g|_{D_X \times D_Y}$ is an element of $l^\infty(\mathbb{N}) \bar{\otimes} l^\infty(\mathbb{N})$ and the supremum of A. Our choice of a bounded subset of $l^\infty(\mathbb{N}) \bar{\otimes} l^\infty(\mathbb{N})$ was arbitrary, so we have shown that $l^\infty(\mathbb{N}) \bar{\otimes} l^\infty(\mathbb{N})$ is Dedekind complete, a contradiction to Example 2.3.14. Therefore, $\mathbb{R}^X \bar{\otimes} \mathbb{R}^Y$ is not Dedekind complete and $\mathbb{R}^X \bar{\otimes} \mathbb{R}^Y$ is not Riesz isomorphic to $\mathbb{R}^{X \times Y}$. $\qquad\square$

3. TENSOR PRODUCT OF RIESZ SUBSPACES

Chapter 2 gives examples of two Riesz spaces having a certain Riesz space property but their Fremlin tensor product not having that property. Chapter 3 discusses Riesz subspaces in arbitrary Archimedean Riesz spaces E, F. In particular, many questions are of the following nature:

"If $A \subseteq E$ and $B \subseteq F$ are of subspace type P, is $A \bar{\otimes} B$ also of subspace type P in $E \bar{\otimes} F$?"

We refer the reader to definition 1.4.1 for each subspace type.

3.1 POSITIVE RESULTS

A "positive result" refers to instances where the answer to the question proposed above is "yes." Fremlin gave the first positive result in his paper defining the Archimedean Riesz space tensor product (see Theorem 3.1.1). We prove two additional positive instances which will be provided in Theorems 3.1.9 and 3.1.13.

Theorem 3.1.1. *(4.5 of [14]) Let E and F be Archimedean Riesz spaces with Riesz subspaces A and B respectively. Then $A \bar{\otimes} B$ can be identified with the Riesz subspace of $E \bar{\otimes} F$ generated by $A \otimes B$.*

In order to prove results for principal ideals and projection bands, we use the Kakutani representation theorem and a description of positive elements of the Fremlin tensor product given by Allenby and Labuschagne in [4].

Theorem 3.1.2. *(2.2 of [4]) Let E and F be Archimedean Riesz spaces. If $h \in (E \bar{\otimes} F)^+$, then there exists $(x, y) \in E^+ \times F^+$ such that for every $\epsilon > 0$ there exist $u, v \in E^+ \otimes F^+$ where*

$$E^+ \otimes F^+ := \left\{ \sum_{i=1}^{n} f_i \otimes g_i \middle| f_i \in E^+, \ g_i \in F^+, \ n \in \mathbb{N} \right\},$$

such that

$$0 \le h - u \le \epsilon x \otimes y \ and \ 0 \le v - h \le \epsilon x \otimes y.$$

Definition 3.1.3. *(page 181 of [2]) A norm $|| \cdot ||$ on a Riesz space is said to be a* lattice norm *whenever $|x| \le |y|$ implies $||x|| \le ||y||$. A Riesz space equipped with a lattice norm is known as a* normed Riesz space. *If a normed Riesz space is norm complete, then it is referred to as a* Banach lattice.

Definition 3.1.4. *(4.20 of [2]) A Banach lattice E is said to be an* abstract M-space *(or AM-space) whenever its norm is an M-norm, i.e., if $x \wedge y = 0$ in E implies*

$$||x \vee y|| = \max\{||x||, ||y||\}.$$

Theorem 3.1.5. *(4.21 of [2]) Let E be a Banach lattice, and let $x \in E$. Then the principal ideal E_x generated by x in E under the norm $|| \cdot ||_\infty$ defined by*

$$||y||_\infty = \inf\{\lambda > 0 \mid |y| \le \lambda |x|\} \qquad (y \in E_x),$$

is an AM-space, whose closed unit ball is the order interval $[-|x|, |x|]$.

Definition 3.1.6. *Let E and F be Banach lattices. E and F are* lattice isometric *if there exists a Riesz isomorphism $\varphi \colon E \to F$ such that $||x||_E = ||\varphi(x)||_F$ for all $x \in E$.*

Theorem 3.1.7. *(Kakutani-Bohnenblust and M.Krein-S.Krein, 4.29 of [2]) A Banach lattice E is an AM-space with an order unit if and only if it is lattice isometric to some $C(X)$ for a (unique up to homeomorphism) Hausdorff compact topological space X.*

In particular, a Banach lattice is an AM-space if and only if it is lattice isometric to a closed Riesz subspace of some $C(X)$-space.

If a Riesz space E is a uniformly complete, then $(E_x, ||\cdot||_\infty)$ is a Banach lattice for any $x \in E^+$. Indeed, let E be uniformly complete and fix $x \in E^+$. Then E_x is uniformly complete (Theorem 1.4.4). E_x can be equipped with the norm defined in Theorem 3.1.5 so that every Cauchy sequence in E_x converges and, in particular, converges relative to the norm. Therefore, E_x is a Banach lattice that by Theorem 3.1.5 is an AM-space with unit. As a consequence of Theorem 3.1.7, E_x is Riesz isomorphic to $C(X)$ for some compact Hausdorff space X.

Lemma 3.1.8. *Let E and F be uniformly complete Archimedean Riesz spaces. Then $E_x \bar{\otimes} F_y$ is uniformly dense in $(E \bar{\otimes} F)_{x \otimes y}$ for every $x \in E^+$ and $y \in F^+$.*

Proof. There exist compact Hausdorff spaces X and Y such that E_x is Riesz isomorphic to $C(X)$ and F_y is Riesz isomorphic to $C(Y)$. Then $E_x \bar{\otimes} F_y \cong C(X) \bar{\otimes} C(Y)$. Also, $C(X) \bar{\otimes} C(Y)$ is uniformly dense in $C(X \times Y)$ where $x \in E^+$ and $y \in F^+$ correspond to the unit functions 1_X and 1_Y respectively (2.2 of [4]). Finally, $(E \bar{\otimes} F)_{x \otimes y}$ has a unique uniform completion that is Riesz isomorphic to a Riesz subspace of $C(X \times Y)$. Since $C(X) \bar{\otimes} C(Y)$ is uniformly dense in $C(X \times Y)$, it follows that $E_x \bar{\otimes} F_y$ is uniformly dense in $(E \bar{\otimes} F)_{x \otimes y}$. $\qquad\square$

Theorem 3.1.9. *Let E and F be Dedekind α-complete for an infinite cardinal α. If $E \bar{\otimes} F$ is Dedekind α-complete, then $E_x \bar{\otimes} F_y$ is a principal ideal for every $x \in E^+$ and $y \in F^+$. In particular,*

$$E_x \bar{\otimes} F_y = (E \bar{\otimes} F)_{x \otimes y}.$$

Proof. Since E and F are in particular uniformly complete, there exist compact Hausdorff spaces X, Y such that $E_x \cong C(X)$ and $F_y \cong C(Y)$. By Lemma 3.1.8, $E_x \bar{\otimes} F_y$ contains $x \otimes y$ and is uniformly dense in $(E \bar{\otimes} F)_{x \otimes y}$. As an ideal of a Dedekind α-complete space, $(E \bar{\otimes} F)_{x \otimes y}$ is Dedekind α-complete (Theorem 1.4.4). Thus, $(E \bar{\otimes} F)_{x \otimes y}$ is a uniform completion of $E_x \bar{\otimes} F_y$ by Theorem 2.3.12. On the other hand, $E_x \bar{\otimes} F_y$ is Riesz isomorphic to $C(X) \bar{\otimes} C(Y)$ which

has $C(X \times Y)$ as a uniform completion. Since the uniform completion is unique up to isomorphism, $C(X \times Y) \cong (E \bar{\otimes} F)_{x \otimes y}$. Hence, $C(X \times Y)$ is Dedekind α-complete. Therefore, $X \times Y$ is, in particular, basically disconnected which implies $X \times Y$ is an F-space (e.g. [12]). The product of two infinite compact spaces cannot be an F-space (14Q of [17]), so X or Y is finite dimensional. By Theorem 2.1.6, $C(X) \bar{\otimes} C(Y) = C(X \times Y)$. Therefore, $E_x \bar{\otimes} F_y \cong (E \bar{\otimes} F)_{x \otimes y}$. $\qquad \square$

The following inequalities and equalities are quite useful and listed below for convenience. They will be referenced in Lemma 3.1.11.

Theorem 3.1.10. *(6.5 of [36]) Let u, v, w be elements in E^+ and let f, g be arbitrary elements in a Riesz space E. Then*

$$(u + v) \wedge w \leq (u \wedge w) + (v \wedge w),$$

$$(f + g) \vee w \leq (f \vee w) + (g \vee w).$$

Furthermore, $v \wedge w = 0$ implies

$$(u + v) \wedge w = (u \wedge w) + (v \wedge w),$$

$$(f + g) \vee w = (f \vee w) + (g \vee w).$$

Lemma 3.1.11. *Let E and F be Archimedean Riesz spaces with B_1, B_2 bands of E. If $B_1 \perp B_2$, then $(B_1 \bar{\otimes} F) \perp (B_2 \bar{\otimes} F)$ in $B_1 \bar{\otimes} B_2$.*

Proof. Let $h_1 \in B_1 \bar{\otimes} F$ and $h_2 \in B_2 \bar{\otimes} F$. We will show that $|h_1| \wedge |h_2| = 0$. Since $|h_1|$, $|h_2| \geq 0$, we need only prove $(B_1 \bar{\otimes} F)^+ \perp (B_2 \bar{\otimes} F)^+$.

Step 1 Consider elementary tensors. Let $b_1 \in B_1^+$ and $b_2 \in B_2^+$. Since $B_1 \perp B_2$, we have for all $f_1, f_2 \in F^+$,

$$(b_1 \otimes f_1) \wedge (b_2 \otimes f_2) \leq (b_1 \otimes (f_1 + f_2)) \wedge (b_2 \otimes (f_1 + f_2))$$
$$= (b_1 \wedge b_2) \otimes (f_1 + f_2)$$
$$= 0 \otimes (f_1 + f_2)$$
$$= 0.$$

Step 2 We prove that $(B_1 \otimes F)^+ \perp (B_2 \otimes F)^+$. Let $h_1 = \sum_{i=1}^n b_i^1 \otimes f_i$ and $h_2 = \sum_{j=1}^m b_j^2 \otimes f_j$ for $b_i^1 \in B_1^+$, $b_j^2 \in B_2^+$, and $f_i, f_j \in F^+$. Then by Step 1 and Theorem 3.1.10,

$$h_1 \wedge h_2 = \left(\sum_{i=1}^n b_i^1 \otimes f_i \right) \wedge \left(\sum_{j=1}^m b_j^2 \otimes f_j \right)$$
$$\leq \sum_{j=1}^m \left(\left(\sum_{i=1}^n b_i^1 \otimes f_i \right) \wedge b_j^2 \otimes f_j \right)$$
$$\leq \sum_{j=1}^m \sum_{i=1}^n (b_i^1 \otimes f_i) \wedge (b_j^2 \otimes f_j)$$
$$= 0.$$

Step 3 We show that the positive cone of the Fremlin tensor product of B_1 and B_2 is disjoint with the positive cone of the algebraic tensor product of B_1 and B_2, i.e.

$$(B_1 \bar{\otimes} F)^+ \perp (B_2 \otimes F)^+.$$

Let $f \in (B_1 \bar{\otimes} F)^+$ and $g \in (B_2 \otimes F)^+$. Since $f \in B_1 \bar{\otimes} F$, there exists $\{f_n\}_{n=1}^\infty \in B_1 \otimes F$ such that $\{f_n\}_{n=1}^\infty$ converges uniformly relative to some $h \in B_1 \otimes F$. Then there exists $\{\epsilon_n\} \downarrow 0$

such that

$$|f| \wedge |g| \leq (|f - f_n| + |f_n|) \wedge |g| \leq (|f - f_n| \wedge |g|) + (|f_n| \wedge |g|) \leq \epsilon_n |h| \wedge |g| + 0$$

since $f_n \in B_1 \otimes F$ and $g \in B_2 \otimes F$ implies $|f_n| \wedge |g| = 0$ for every n by Step 2. Since $\{\epsilon_n\} \downarrow 0$, $|f| \wedge |g| = 0$.

Step 4 It remains to show that $(B_1 \bar{\otimes} F)^+ \perp (B_2 \bar{\otimes} F)^+$. Let $k \in (B_1 \bar{\otimes} F)^+$, $i \in (B_2 \bar{\otimes} F)^+$. Since $i \in B_2 \bar{\otimes} F$, there exists $\{i_n\}_{n=1}^{\infty} \in B_2 \otimes F$ such that $\{i_n\}_{n=1}^{\infty}$ converges uniformly relative to some $j \in B_2 \otimes F$. Then there exists $\{\hat{\epsilon}_n\} \downarrow 0$ such that

$$|i| \wedge |k| \leq (|i - i_n| + |i_n|) \wedge |k| \leq (|i - i_n| \wedge |k|) + (|i_n| \wedge |k|) \leq \hat{\epsilon}_n |j| \wedge |k|$$

since $|i_n| \wedge |k| = 0$ for every n by Step 3. Since $\{\hat{\epsilon}_n\} \downarrow 0$, we have $|i| \wedge |k| = 0$. Therefore, $(B_1 \bar{\otimes} F) \perp (B_2 \bar{\otimes} F)$. $\qquad \square$

Theorem 3.1.12. *(8.4 of [36]) Let D be a nonempty subset of E. Then the disjoint complement D^d is a band in E.*

Theorem 3.1.13. *Let E, F be Archimedean Riesz spaces with B a projection band of E and C a projection band of F. Then $B \bar{\otimes} C \subseteq E \bar{\otimes} F$ is a projection band.*

Proof. B is a projection band of E implies $E = B \oplus B^d$. Lemma 3.1.11 proves that $B \bar{\otimes} F \perp B^d \bar{\otimes} F$. Hence,

$$E \bar{\otimes} F = (B \oplus B^d) \bar{\otimes} F = (B \bar{\otimes} F) \oplus (B^d \bar{\otimes} F)$$

by substitution and the fact that $\otimes$ is a Riesz bimorphism. Likewise, $F = C \oplus C^d$ so

$$
\begin{aligned}
E \bar{\otimes} F &= (B \bar{\otimes} F) \oplus (B^d \bar{\otimes} F) \\
&= (B \bar{\otimes} (C \oplus C^d)) \oplus (B^d \bar{\otimes} (C \oplus C^d)) \\
&= (B \bar{\otimes} C) \oplus (B \bar{\otimes} C^d) \oplus (B^d \bar{\otimes} C) \oplus (B^d \bar{\otimes} C^d)
\end{aligned}
$$

with each of the four tensor products disjoint from one another.

It follows that $(B\bar{\otimes}C)^d = (B\bar{\otimes}C^d) \oplus (B^d\bar{\otimes}C) \oplus (B^d\bar{\otimes}C^d)$. By Theorem 3.1.12, $(B\bar{\otimes}C)^d$ is a band. Set $D = (B\bar{\otimes}C^d) \oplus (B^d\bar{\otimes}C) \oplus (B^d\bar{\otimes}C^d)$. Then likewise, D^d is a band. Then

$$D^d = \{h \in E\bar{\otimes}F \mid h \perp f \text{ for all } f \in D\}$$

and $B\bar{\otimes}C \perp D$, so $B\bar{\otimes}C \subseteq D^d$. On the other hand, $D \subseteq E\bar{\otimes}F\backslash(B\bar{\otimes}C)$ implies $D^d \subseteq B\bar{\otimes}C$. Hence, $D^d = B\bar{\otimes}C$ is a band. Therefore, $E\bar{\otimes}F \cong (B\bar{\otimes}C) \oplus (B\bar{\otimes}C)^d$ so that $B\bar{\otimes}C$ is a projection band. $\qquad\square$

3.2 NEGATIVE RESULTS

A "negative result" refers to instances where the answer to the question proposed in the introduction of Chapter 3 is "no."

Theorem 3.2.1. *(5.3 of [32]) Let Y be a Riesz space, X an order dense subspace of Y, and J an ideal of Y. Then the set $J \cap X$ is an ideal in X.*

Example 3.2.2. *Let X and Y be compact Hausdorff spaces. If I and J are maximal ideals in $C(X)$ and $C(Y)$ respectively, then $I\bar{\otimes}J$ is not a maximal ideal in $C(X)\bar{\otimes}C(Y)$.*

Proof. For $a \in X$, consider $M_a = \{f \in C(X) \mid f(a) = 0\}$. M_a is a Riesz subspace of $C(X)$. Suppose $f \in C(X)$ and $g \in M_a$ such that $0 \le |f| \le |g|$. Then $g(a) = 0$ so $f(a) = 0$. Thus, $f \in M_a$, and M_a is an ideal in $C(X)$. Furthermore, if I is an ideal such that $M_a \subsetneq I$, then there exists $f \in I$ such that $f(x) \neq 0$ for all $x \in X$. Since I is an ideal, $1_X \le |\lambda f|$ for some $\lambda \in \mathbb{R}$ implies $1_X \in I$. Thus, $I = C(X)$, and M_a is a maximal ideal. In fact, all maximal ideals in $C(X)$ are of the form $\{f \in C(X) \mid f(a) = 0\}$ for some $a \in X$ (4.6 of [17]).

Let I and J be maximal ideals of $C(X)$ and $C(Y)$ respectively. Then there exist $a \in X$ and $b \in Y$ such that $I = \{f \in C(X) \mid f(a) = 0\}$ and $J = \{f \in C(Y) \mid f(b) = 0\}$. Let

$$M = \{f \in C(X \times Y) \mid f(x,y) = 0 \text{ if } x = a \text{ or } y = b\}.$$

Since $C(X)\bar{\otimes}C(Y)$ is order dense in $C(X \times Y)$, we have that $M \cap (C(X)\bar{\otimes}C(Y))$ is an ideal in $C(X)\bar{\otimes}C(Y)$ by Theorem 3.2.1. Note that $I\bar{\otimes}J \subseteq M \cap C(X)\bar{\otimes}C(Y)$. If $I\bar{\otimes}J$ is not an ideal, we are done. If $I\bar{\otimes}J$ is an ideal, it is properly contained in the ideal

$$\{f \in C(X)\bar{\otimes}C(Y) \mid f(x) = 0 \text{ if } x = a\}.$$

Thus, $I\bar{\otimes}J$ is not a maximal ideal in $C(X)\bar{\otimes}C(Y)$. $\qquad\square$

Recall from definition 1.1.3 that $PP([0,\infty))$ is the Archimedean Riesz space of piecewise polynomials on $[0,\infty)$. For $\epsilon > 0$ and $a, b \in [0,\infty)$, we define

$$B((a,b),\epsilon) = \{(x,y) \in [0,\infty) \times [0,\infty) \mid \sqrt{(x-a)^2 + (y-b)^2} < \epsilon\}.$$

We thank Samuel Lisi for his contributions to Lemma 3.2.3 and Theorem 3.2.4 which sharpen the main result of Chapter 4, namely Theorem 4.1.7.

Lemma 3.2.3. *If $p(x)$ is a piecewise polynomial on $[0,\infty)$ and $|p(x)| \leq Cx$ for some $C \in \mathbb{R}^+$, then there exists $k \in \mathbb{R}^+$ such that $p''(x) = 0$ whenever $x > k$.*

Proof. By definition, there exist $n \in \mathbb{N}$ and $t_1, \cdots, t_n \in [0,\infty)$ such that $t_1 < t_2 < \cdots < t_n$ and p is a polynomial function on $[t_n, \infty)$ and $[t_i, t_{i+1}]$ for each $i = 1, \cdots, n-1$. In particular, there exist a polynomial p_n on $[0,\infty)$ such that $p(x) = p_n(x)$ for all $x \in [t_n, \infty)$. Since by assumption $|p(x)| \leq Cx$ for some $C \in \mathbb{R}^+$, the degree of p_n is less than or equal to 1. Therefore, $p_n''(x) = 0$ for all $x \in [0,\infty)$. Hence, $x > t_n$ implies that $p''(x) = 0$. $\qquad\square$

Theorem 3.2.4. *Let E and F be $PP([0,\infty))$. Let $p(x) = x$ and $q(y) = y$. Then $E_p\bar{\otimes}F_q$ is not an ideal in $E\bar{\otimes}F$.*

Proof. Let $h(x,y) = 1 \otimes y^2 \wedge x^2 \otimes 1 \in E\bar{\otimes}F$. If $h \in E_p\bar{\otimes}F_q$, there exist finite subsets I, J of $\mathbb{N}$ such that

$$h(x,y) = \sup_{i \in I} \inf_{j \in J} \{g_{ij}(x,y)\}$$

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for some $g_{ij} \in E_p \otimes F_q$. For each g_{ij}, there exist $n \in \mathbb{N}$, $p_r \in E_p$, and $q_r \in F_q$ such that

$$g_{ij}(x, y) = \sum_{r=1}^{n} p_r(x) \otimes q_r(y).$$

Since $p_r \in E_p$ and $q_r \in F_q$, there exist $\lambda_r \in \mathbb{R}^+$ such that

$$|g_{ij}(x, y)| \leq \sum_{r=1}^{n} |p_r(x) \otimes q_r(y)| \leq \sum_{r=1}^{n} \lambda_r (x \otimes y) = \left(\sum_{r=1}^{n} \lambda_r \right) x \otimes y.$$

By Lemma 3.2.3, for every $i \in I$ and $j \in J$ there exists $k_{ij} \in \mathbb{R}^+$ such that

$$x > k_{ij} \text{ and } y > k_{ij} \implies (g_{ij})_{xx} = (g_{ij})_{yy} = 0, \tag{3.1}$$

where $(g_{ij})_{xx}$, $(g_{ij})_{yy}$ are two of the second order partial derivatives of g_{ij} with respect to x and y, respectively.

Let $k = \sup_{ij}\{k_{ij}\}$, $D = \{[k, \infty) \times [k, \infty)\}$, and $S_{ij} = \{(x, y) \in D \mid h(x, y) = g_{ij}(x, y)\}$. Note from the definition of $h(x, y)$ that $D = \bigcup_{i \in I, j \in J} S_{ij}$. It follows that $int(\bigcup_{i \in I, j \in J} S_{ij})$ is nonempty, and there exist $i \in I$, $j \in J$ such that $int(S_{ij}) \neq \emptyset$. Therefore, there exists $(c, d) \in D$ and $\epsilon > 0$ such that $B((c, d), \epsilon) \subseteq int(S_{ij})$. That is, $h(x, y) = g_{ij}(x, y)$ for every $(x, y) \in B((c, d), \epsilon)$ and for such $i \in I$, $j \in J$. It follows from (3.1) that

$$h_{xx}(x, y) = h_{yy}(x, y) = 0 \qquad \Big((x, y) \in B((c, d), \epsilon) \Big). \tag{3.2}$$

On the other hand, by the very definition of h,

$$\text{if } a > b, \text{ then } h_{yy}(a, b) = 2 \neq 0;$$
$$\text{if } a < b, \text{ then } h_{xx}(a, b) = 2 \neq 0.$$

This contradicts (3.2). Then $h \notin E_p \bar{\otimes} F_q$. Since $h \leq |p \otimes q|$, it follows that $E_p \bar{\otimes} F_q$ is not an ideal in $E \bar{\otimes} F$. $\qquad \square$

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Notice that Theorem 3.1.9 shows that the Fremlin tensor product of ideals is an ideal under very extreme conditions. However, Theorem 3.2.4 proves that for ideals I and J in Archimedean Riesz spaces E and F, $I \bar{\otimes} J$ need not be an ideal in $E \bar{\otimes} F$.

4. THE FREMLIN TENSOR PRODUCT AND DEDEKIND COMPLETENESS

We return to a Riesz space property from the main inclusion theorem in Section 1.4, namely, Dedekind completeness. Chapter 4 characterizes when, for any infinite cardinal α, the Fremlin tensor product of two Archimedean Riesz spaces is Dedekind α-complete. We thank Anton Schep for providing us with [34] and for the slides of his talk [30] where he proved Theorem 4.1.1 for the Banach lattice tensor product.

4.1 DEDEKIND α-COMPLETENESS

While the terms Dedekind complete and Dedekind σ-complete are common in the literature, we prove this chapter's results more generally for Dedekind "α-completeness," where α is any infinite cardinal. Our conclusion is negative in the sense that the preservation of even Dedekind σ-completeness "rarely" happens.

Theorem 4.1.1. *Let X and Y be compact, Hausdorff spaces and α an infinite cardinal. Let $C(X)$ and $C(Y)$ be Dedekind α-complete. $C(X)\bar{\otimes}C(Y)$ is Dedekind α-complete if and only if X or Y is finite.*

Proof. Assume $C(X)\bar{\otimes}C(Y)$ is Dedekind α-complete. $C(X \times Y)$ is a uniform completion of $C(X)\bar{\otimes}C(Y)$ by Theorem 2.3.12. Since $C(X)\bar{\otimes}C(Y)$ is uniformly complete by assumption, $C(X)\bar{\otimes}C(Y)$ is Riesz isomorphic to $C(X \times Y)$. Consequently, $X \times Y$ is α-disconnected which implies $X \times Y$ is an F-space (e.g. [12]). The product of two infinite compact spaces cannot be an F-space (14Q of [17]), so X or Y is finite.

Assume that Y is finite. Then $C(Y) \cong C(\{1, \cdots, n\})$ for some $n \in \mathbb{N}$. Let B be a bounded subset of $C(X)\bar{\otimes}C(Y)$ with $|B| \leq \alpha$. For every $h \in B$, there is $\{h_i\}_{i=1}^n \in C(X)$

such that $h_i(x) = h(x, i)$ for $x \in X$ and $i \in \{1, \cdots, n\}$. For each $i \in \{1, \cdots, n\}$, note that $\{h_i(x)\}_{h \in B}$ is a bounded subset of $C(X)$ with cardinality no greater than α . Since $C(X)$ is Dedekind α-complete,

$$\sup_{h \in B} h(x, y) = \begin{cases} \sup_{h \in B} h_1(x) & \text{if } y = 1 \\ \vdots & \vdots \\ \sup_{h \in B} h_n(x) & \text{if } y = n \end{cases}$$

exists for every $(x, y) \in X \times Y$. Define $g_i(x) = \sup_{h \in B} h_i(x)$. Since $1_{\{i\}} \in C(\{1, \cdots, n\})$, we have

$$\sup_{h \in B} h = \sum_{i=1}^{n} g_i \otimes 1_{\{i\}}$$

is an element of $C(X) \bar{\otimes} C(\{1, \cdots, n\})$. Thus, $C(X) \bar{\otimes} C(Y)$ is Dedekind α-complete. $\square$

Note that the second half of Theorem 4.1.1 follows from Theorem 2.1.6. Indeed, if X or Y is finite, then $C(X) \otimes C(Y) = C(X \times Y)$. Thus, $C(X) \bar{\otimes} C(Y) = C(X \times Y)$. Since the product of an α-disconnected space with a finite space is α-disconnected, $C(X \times Y)$ is Dedekind α-complete.

Definition 4.1.2. *Let E be an Archimedean Riesz space and let I be a nonempty set. $c_{00}(I, E)$ is the set of all maps $f \colon I \to E$ such that*

$$S(f) = \{x \in I : f(x) \neq 0\}$$

is finite. We write $c_{00}(I)$ in place of $c_{00}(I, \mathbb{R})$.

For $f, g \in c_{00}(I, E)$, $f \leq g$ if and only if $f(x) \leq g(x)$ in E for every $x \in I$. With this pointwise ordering, $c_{00}(I, E)$ is an Archimedean Riesz space.

Lemma 4.1.3. *Let I be a nonempty set. Then $c_{00}(I) \bar{\otimes} E$ and $c_{00}(I, E)$ are Riesz isomorphic.*

Proof. We show that $c_{00}(I, E)$ has the universal property for the Fremlin tensor product of $c_{00}(I)$ and E.

Define $\varphi\colon c_{00}(I) \times E \to c_{00}(I, E)$ by

$$\varphi(f, e) = f_e \qquad (f \in c_{00}(I), \ e \in E),$$

where $f_e(x) = f(x)e$ for every $x \in X$. Let $f_1, f_2 \in c_{00}(I)$, $e \in E$, and $x \in X$. Since E is a vector space,

$$\begin{aligned}
\varphi(f_1 + f_2, e) &= (f_1 + f_2)_e \\
&= (f_1 + f_2)(x)e \\
&= f_1(x)e + f_2(x)e \\
&= \varphi(f_1, e) + \varphi(f_2, e).
\end{aligned}$$

Let $e_1, e_2 \in E$, $f \in c_{00}(I)$, and $x \in X$. Since E is a vector space,

$$\begin{aligned}
\varphi(f, e_1 + e_2)(x) &= f_{(e_1 + e_2)} \\
&= f(x)(e_1 + e_2) \\
&= f(x)e_1 + f(x)e_2 \\
&= \varphi(f, e_1) + \varphi(f, e_2).
\end{aligned}$$

Thus, φ is bilinear. For $f_1, f_2 \in c_{00}(I)$, $e \in E^+$, and $x \in X$,

$$\begin{aligned}
\varphi(f_1 \vee f_2, e)(x) &= (f_1 \vee f_2)_e \\
&= (f_1 \vee f_2)(x)e \\
&= f_1(x)e \vee f_2(x)e \\
&= \varphi(f_1, e) \vee \varphi(f_2, e).
\end{aligned}$$

For $f \in c_{00}(I)^+$, e_1, $e_2 \in E$, and $x \in X$,

$$\varphi(f, e_1 \vee e_2)(x) = f(x)(e_1 \vee e_2)$$
$$= f(x)e_1 \vee f(x)e_2$$
$$= \varphi(f, e_1) \vee \varphi(f, e_2).$$

Thus, φ is a Riesz bimorphism.

Let F be an Archimedean Riesz space, and suppose $\psi \colon c_{00}(I) \times E \to F$ is a Riesz bimorphism. If $g \in c_{00}(I, E)$, then g is uniquely represented by $g = \sum_{y \in S(g)} g(y) 1_{\{y\}}$. Then

$$g(x) = \sum_{y \in S(g)} \varphi(1_{\{y\}}, g(y))(x) \qquad (x \in I)$$

and g is in the Riesz space generated by the range of φ. Define $T \colon c_{00}(I, E) \to F$ by $T(g) = \sum_{y \in S(g)} \psi(1_{\{y\}}, g(y))$. Let $\lambda \in \mathbb{R}$ and f, $g \in c_{00}(I, E)$. Then

$$T(\lambda f + g) = T\left(\lambda \sum_{y \in S(f)} f(y) 1_{\{y\}} + \sum_{y \in S(g)} g(y) 1_{\{y\}} \right)$$
$$= T\left(\sum_{y \in S(f) \cup S(g)} (\lambda f + g)(y) 1_{\{y\}} \right)$$
$$= \sum_{y \in S(f) \cup S(g)} \psi(1_{\{y\}}, (\lambda f + g)(y))$$
$$= \sum_{y \in S(f) \cup S(g)} \left(\psi(1_{\{y\}}, \lambda f(y)) + \psi(1_{\{y\}}, g(y)) \right)$$
$$= \lambda \sum_{y \in S(f)} \psi(1_{\{y\}}, f(y)) + \sum_{y \in S(g)} \psi(1_{\{y\}}, g(y))$$
$$= \lambda T(f) + T(g).$$

Thus, T is linear.

Let $f, g \in c_{00}(I, E)$ with $f \wedge g = 0$. Then $S(f) \cap S(g) = \emptyset$. Since ψ is a Riesz bimorphism,

$$
\begin{aligned}
T(f) \wedge T(g) &= T\big(\sum_{y \in S(f)} f(y) 1_{\{y\}} \big) \wedge T\big(\sum_{y \in S(g)} g(y) 1_{\{y\}} \big) \\
&= \sum_{y \in S(f)} T(f(y) 1_{\{y\}}) \wedge \sum_{y \in S(g)} T(g(y) 1_{\{y\}}) \\
&= \sum_{y \in S(f)} \psi(1_{\{y\}}, f(y)) \wedge \sum_{y \in S(g)} \psi(1_{\{y\}}, g(y)) \\
&= 0.
\end{aligned}
$$

Then T is a Riesz homomorphism by Theorem 1.1.8 (iii), and $\psi = T \circ \varphi$. Consequently, $c_{00}(I, E) \cong c_{00}(I) \bar{\otimes} E$. $\qquad \square$

Theorem 4.1.4. *Let I be a nonempty set and let E be an Archimedean Riesz space. If E is Dedekind α-complete for an infinite cardinal α, then $c_{00}(I) \bar{\otimes} E$ is Dedekind α-complete.*

Proof. Let B be a bounded subset of $c_{00}(I, E)$ such that $|B|$ is less than α. Then there exists an $f \in c_{00}(I, E)$ such that $S(h) \subseteq S(f)$ for every $h \in B$. From the Dedekind α-completeness of E it follows that $\sup_{h \in B} h(x)$ exists for each $x \in I$. Define

$$
g(x) = \sup_{h \in B} h(x).
$$

$S(g)$ is finite since $S(g) \subseteq S(f)$ and $S(f)$ is finite. Then $g \in c_{00}(I, E)$. By Lemma 4.1.3, $c_{00}(I) \bar{\otimes} E \cong c_{00}(I, E)$ is Dedekind α-complete. $\qquad \square$

Theorem 4.1.5. *Let I be a nonempty set and let E be an Archimedean Riesz space. If E is Dedekind complete, then $c_{00}(I) \bar{\otimes} E$ is Dedekind complete.*

Proof. Let B be a bounded subset of $c_{00}(I, E)$. Then B has some cardinality, say α. E is in particular Dedekind α-complete, so $\sup B$ exists in $c_{00}(I) \bar{\otimes} E$ by Theorem 4.1.4. $\qquad \square$

Theorem 4.1.6. *(61.4 of [26]) Let E be a Riesz space, not consisting exclusively of the null element. $E \cong c_{00}(X)$ for some nonempty point set X if and only if every principal ideal in E is a finite dimensional Archimedean Riesz space.*

In Theorem 4.1.7, we combine our results on ideals and Dedekind α-completeness to characterize exactly when the Fremlin tensor product of two Dedekind α-complete Riesz spaces is Dedekind α-complete.

Theorem 4.1.7. *Let α be an infinite cardinal. Suppose E and F are Dedekind α-complete. The following are equivalent.*

1. *$E_x \bar{\otimes} F_y$ is Dedekind α-complete for every $x \in E^+$ and $y \in F^+$.*

2. *$[E_x$ is finite dimensional for every $x \in E^+]$ or $[F_y$ is finite dimensional for every $y \in F^+]$.*

3. *$E \cong c_{00}(I)$ for a set $I \subseteq E$ or $F \cong c_{00}(J)$ for a set $J \subseteq F$.*

4. *$E \bar{\otimes} F \cong c_{00}(I, F)$ for a set $I \subseteq E$ or $E \bar{\otimes} F \cong c_{00}(J, E)$ for a set $J \subseteq F$.*

5. *$E \bar{\otimes} F$ is Dedekind α-complete.*

Proof. (1) $\implies$ (2) By the Kakutani representation theorem, there exist compact, Hausdorff topological spaces X, Y such that $E_x \cong C(X)$ and $F_y \cong C(Y)$. If $E_x \bar{\otimes} F_y$ is Dedekind α-complete, then X or Y is finite by Theorem 4.1.1. Then E_x or F_y is finite dimensional. We claim that this holds either for every $x \in E^+$ or for every $y \in F^+$. Indeed, if there exists $u \in E^+$, $v \in F^+$ such that E_u, F_v are infinite dimensional, then $E_u \bar{\otimes} F_v$ cannot be Dedekind α-complete by Theorem 4.1.1. Thus, either E_x is finite dimensional for every $x \in E^+$ or F_y is finite dimensional for every $y \in F^+$.

(2) $\implies$ (3) Follows from Theorem 4.1.6.

(3) $\implies$ (4) Follows from Lemma 4.1.3.

(4) $\implies$ (5) Follows from Theorem 4.1.4.

(5) $\implies$ (1) By Theorem 3.1.9, the fact that $E\bar{\otimes}F$ is Dedekind α-complete implies that $E_x\bar{\otimes}F_y$ is an ideal for every $x \in E^+$, $y \in F^+$. Thus, $E_x\bar{\otimes}F_y$ is Dedekind α-complete. $\quad\square$

Theorem 4.1.7 can be generalized to a Dedekind β-complete space E and a Dedekind γ-complete space F for $\beta \neq \gamma$. In this case, $min\{\beta, \gamma\}$ replaces α in statements (1) and (5). We emphasize the result for α a countable cardinal, in which case we use the term Dedekind σ-complete. Since Theorem 4.1.7 holds for any infinite cardinal, we have the concluding corollary.

Corollary 4.1.8. *Suppose E and F are Dedekind complete. The following are equivalent.*

1. *$E_x\bar{\otimes}F_y$ is Dedekind complete for every $x \in E^+$ and $y \in F^+$.*

2. *[E_x is finite dimensional for every $x \in E^+$] or [F_y is finite dimensional for every $y \in F^+$].*

3. *$E \cong c_{00}(I)$ for a set $I \subseteq E$ or $F \cong c_{00}(J)$ for a set $J \subseteq F$.*

4. *$E\bar{\otimes}F \cong c_{00}(I, F)$ for a set $I \subseteq E$ or $E\bar{\otimes}F \cong c_{00}(J, E)$ for a set $J \subseteq F$.*

5. *$E\bar{\otimes}F$ is Dedekind complete.*

5. BOOLEAN ALGEBRAS

5.1 BOOLEAN ALGEBRA PRELIMINARY MATERIAL

Fremlin asserts in exercise 315Y(f) of [16] that the Boolean algebra tensor product of two nontrivial Boolean algebras is complete if and only if one is finite and the other is complete. In Theorem 4.1.7, we proved that the Fremlin tensor product of two Dedekind complete Riesz spaces rarely is Dedekind complete. In fact, if the tensor product is Dedekind complete, then one of the two spaces is Riesz isomorphic to the set of all finite-valued functions on a subset of that space. To connect 315Y(f) of [16] with Theorem 4.1.7, we employ Carathéodory spaces of place functions on a Boolean algebra. The main result in this chapter is Theorem 5.2.1 with applications given in Section 5.3.

The necessary terms for Boolean algebras, their free product, and Carathéodory spaces of place functions are provided. As a general reference, we reserve $\mathcal{A}$, $\mathcal{B}$ for Boolean algebras and E, F, G for Archimedean Riesz spaces throughout. For Boolean algebras, see Chapter 31 of [16].

BOOLEAN ALGEBRAS AND THEIR FREE PRODUCT

A lattice X is called *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all x, y, z in X. Equivalently, X is distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all x, y, and z in X (Theorem 2.2 of [36]). An element $y \in X$ is a *complement* of an element $x \in X$ if $x \wedge y = 0$ and $x \vee y = 1$. Thereupon we write $y = x'$.

Definition 5.1.1. *A* Boolean algebra *is a distributive lattice with zero* 0 *and unit* 1 *having the property that every element has a complement.*

Two elements x and y of a Boolean algebra are called *disjoint* if $x \wedge y = 0$, in which case we write $x \perp y$. Two subsets A and B of a Boolean algebra are disjoint if $x \perp y$ for every $x \in A$ and $y \in B$, in which case we write $A \perp B$. We define the *disjoint sum* of two elements x and y in a Boolean algebra by

$$x \oplus y = (x \wedge y') \vee (x' \wedge y).$$

A Boolean algebra is *complete* if every nonempty subset has a supremum.

Definition 5.1.2. *(312F of [16]) Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras. A function $\chi \colon \mathcal{A} \to \mathcal{B}$ is said to be a* Boolean homomorphism *if for all x, $y \in \mathcal{A}$,*

(i) $\chi(x \wedge y) = \chi(x) \wedge \chi(y)$;

(ii) $\chi(x \oplus y) = \chi(x) \oplus \chi(y)$;

(iii) $\chi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

A bijective Boolean homomorphism is called a Boolean isomorphism. *If there exists a Boolean isomorphism $\chi \colon \mathcal{A} \to \mathcal{B}$, then the Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be* Boolean isomorphic.

Proposition 312H of [16] proves additionally that every Boolean homomorphism preserves finite suprema, that is $\chi(x \vee y) = \chi(x) \vee \chi(y)$ for every x, $y \in \mathcal{A}$.

The *Stone space* of a Boolean algebra $\mathcal{A}$ is the set Z of nonzero ring homomorphisms from $\mathcal{A}$ to $\mathbb{Z}_2$. Set

$$\hat{a} = \{z \mid z \in Z, z(a) = 1\}.$$

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By Stone's Theorem (see, for instance, 311E of [16]), the canonical map

$$a \mapsto \hat{a} \mid \mathcal{A} \to \mathcal{P}(Z), \quad (a \in \mathcal{A})$$

is an injective ring homomorphism which we call the *Stone representation.* For more on Stone spaces, see [16] where Fremlin uses them in order to define a Boolean algebra tensor product, called the *free product.*

Definition 5.1.3. *(Fremlin, 315I of [16])*

(i) *Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of Boolean algebras. For each $i \in I$, let Z_i be the Stone space of $\mathcal{A}_i$. Set $Z = \prod_{i \in I} Z_i$, with the product topology. Then the* free product *of $\{\mathcal{A}_i\}_{i \in I}$ is the algebra of open-and-closed sets in Z, denoted $\bigotimes_{i \in I} \mathcal{A}_i$.*

(ii) *For $i \in I$ and $a \in \mathcal{A}_i$, the set $\hat{a} \subseteq Z_i$ representing a is an open-and-closed subset of Z_i; because $z \mapsto z(i) \colon Z \to Z_i$ is continuous,*

$$\epsilon_i(a) = \{z \mid z(i) \in \hat{a}\}$$

is open-and-closed, so belongs to $\mathcal{A}$. In this context, $\epsilon_i \colon \mathcal{A}_i \to \mathcal{A}$ is called the canonical map.

In the following theorem, we list the material from 315J and 315K of [16] that will be used in later proofs.

Theorem 5.1.4. *Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of Boolean algebras, with free product $\mathcal{A}$.*

(i) *The canonical map $\epsilon_i \colon \mathcal{A}_i \to \mathcal{A}$ is a Boolean homomorphism for every $i \in I$.*

(ii) *For any Boolean algebra $\mathcal{B}$ and any family $\{\varphi_i\}_{i \in I}$ such that φ_i is a Boolean homomorphism from $\mathcal{A}_i$ to $\mathcal{B}$ for every i, there is a unique Boolean homomorphism $\varphi \colon \mathcal{A} \to \mathcal{B}$ such that $\varphi_i = \varphi \circ \epsilon_i$ for each i.*

(iii) Write C for the set of those members of $\mathcal{A}$ expressible in the form $\inf_{j \in J} \epsilon_j(a_j)$, where $J \subseteq I$ is finite and $a_j \in \mathcal{A}_j$ for every j. Then every member of $\mathcal{A}$ is expressible as the supremum of a disjoint finite subset of C.

(iv) $\mathcal{A} = \{0_\mathcal{A}\}$ if and only if there is some $i \in I$ such that $\mathcal{A}_i = \{0_{\mathcal{A}_i}\}$.

(v) If $\mathcal{A}_i \neq \{0_\mathcal{A}\}$ for every $i \in I$, then ϵ_i is injective for every $i \in I$.

(vi) Let $\mathcal{A}_i \neq \{0_\mathcal{A}\}$ for every $i \in I$. If $J \subseteq I$ is finite and a_j is a nonzero member of $\mathcal{A}_j$ for each $j \in J$, then $\inf_{j \in J} \epsilon_j(a_j) \neq 0$.

BOOLEAN ALGEBRA OF BANDS

As an intermediary between Archimedean Riesz spaces and Boolean algebras, we consider Boolean algebras of bands.

Theorem 5.1.5. *Let E be a Riesz space. Define*

$$\mathcal{B}(E) = \{B \subseteq E \mid B \text{ is a band}\}.$$

(i) $\{0\}$ *and E are elements of $\mathcal{B}(E)$ (2.2 (i) of [13]);*

(ii) *intersections of bands are bands (2.2 (ii) of [13]);*

(iii) *for any subset D of E, the disjoint complement of D, which is*

$$D^d = \{f \in E \mid |f| \wedge |g| = 0 \text{ for all } g \in D\},$$

is an element of $\mathcal{B}(E)$ (3.3 of [13]).

Theorem 5.1.6. *(22.6, 22.8 of [26]) Let E be a Riesz space. $\mathcal{B}(E)$ is an order complete distributive lattice. $\mathcal{B}(E)$, partially ordered by inclusion, is a Boolean algebra if and only if E is Archimedean.*

In order to introduce $\mathcal{B}(E)$ as a Boolean algebra, we outline the structure used in the proof of the preceding theorem. $\mathcal{B}(E)$ is partially ordered by

"$B_1 \leq B_2$ whenever B_1 is contained in B_2."

For every pair of bands B_1 and B_2 in E, the infimum $B_1 \wedge B_2$, which is defined to be $B_1 \cap B_2$, exists in $\mathcal{B}(E)$. There are instances when $B_1 + B_2$ is not a band (page 14 of [13]). Thus, $B_1 \vee B_2$ is defined to be $[B_1 + B_2]$, the band generated by $B_1 + B_2$. In $\mathcal{B}(E)$, the complement of B is B^d. The zero is $\{0\}$ and the unit is E.

Theorem 5.1.7. *(7.8 of [36]) The band $[A]$ generated by the ideal A in the Riesz space E consists of all $f \in E$ satisfying*

$$|f| = \sup\{u \mid u \in A, \ 0 \leq u \leq |f|\}.$$

Theorem 5.1.8. *(Infinite distributive laws, 6.1 of [36]) Let D be a subset of a Riesz space E possessing a supremum, i.e., $f_0 = \sup D = \sup\{f \mid f \in D\}$ exists. Then, for any $g \in E$, we have*

$$f_0 \wedge g = \sup\{f \wedge g \mid f \in D\}.$$

Similarly, if $f_1 = \inf D$ exists, then

$$f_1 \vee g = \inf\{f \vee g \mid g \in D\}.$$

Lemma 5.1.9. *Let E be a Riesz space and $f, g \in E$. Then $|f| \wedge |g| = 0$ implies $[f] \perp [g]$.*

Proof. Assume $|f| \wedge |g| = 0$ in E. For $h_1 \in E_f$ and $h_2 \in E_g$, find λ_1 and $\lambda_2 \in \mathbb{R}^+$ such that $|h_1| \leq \lambda_1 |f|$ and $|h_2| \leq \lambda_2 |g|$. Then

$$|h_1| \wedge |h_2| \leq \max\{\lambda_1, \lambda_2\}(|f| \wedge |g|) = 0.$$

Consequently, $E_f \perp E_g$.

Let $h_3 \in [f]$ and $h_4 \in [g]$. By Theorem 5.1.7,

$$|h_3| = \sup\{u \mid u \in E_f,\, 0 \le u \le |h_3|\},$$

$$|h_4| = \sup\{v \mid v \in E_g,\, 0 \le v \le |h_4|\}.$$

Then by the infinite distributive laws,

$$
\begin{aligned}
|h_3| \wedge |h_4| &= \sup\{u \mid u \in E_f,\, 0 \le u \le |h_3|\} \wedge \sup\{v \mid v \in E_g,\, 0 \le v \le |h_4|\}\\
&= \sup\{u \wedge [\sup\{v \mid v \in E_g,\, 0 \le v \le |h_4|\}] \mid u \in E_f,\, 0 \le u \le |h_3|\}\\
&= \sup\{\sup\{u \wedge v \mid v \in E_g,\, 0 \le v \le |h_4|\} \mid u \in E_f,\, 0 \le u \le |h_3|\}\\
&= 0.
\end{aligned}
$$

Thus, $[f] \perp [g]$. $\qquad\square$

Theorem 5.1.10. *(26.10 of [26]) If the Archimedean Riesz space E has the property that any set of mutually disjoint nonzero elements is finite, then E is of finite dimension.*

Lemma 5.1.11. *If E is an infinite dimensional Archimedean Riesz space, then $\mathcal{B}(E)$ is not finite.*

Proof. By the contrapositive of Theorem 5.1.10, there is an infinite set of mutually disjoint nonzero elements in E. Thus, there exists an infinite number of mutually disjoint bands in E by Lemma 5.1.9. $\qquad\square$

CARATHÉODORY SPACES OF PLACE FUNCTIONS

Definition 5.1.12. *(page 40 of [2]) Let E be a Riesz space and $e \in E^+$. Then $x \in E^+$ is said to be a* component *of e whenever $x \wedge (e - x) = 0$.*

The collection of all components of e, denoted $\mathcal{C}(e)$, is a Boolean algebra under the partial ordering induced by E (page 40 of [2]). With e as an order unit, a connection between Archimedean Riesz spaces and Boolean algebras is described explicitly in the following theorem.

Theorem 5.1.13. *(Buskes, de Pagter, van Rooij, 4.1 of [7]) Let $\mathcal{A}$ be a Boolean algebra. There exists an Archimedean Riesz space E with an order unit e with the following properties.*

(i) There exists a Boolean isomorphism $\chi \colon \mathcal{A} \to \mathcal{C}(e)$;

(ii) E is the linear span of $\mathcal{C}(e)$.

(E, χ) is unique up to isomorphism. It is denoted by $\mathcal{C}(\mathcal{A})$ and is called the Carathéodory *space of place functions on $\mathcal{A}$.*

Let λ_i, $\gamma_j \in \mathbb{R}$ be nonzero; n, $m \in \mathbb{N}$; $x_i \in \mathcal{A}$ be pairwise disjoint; and $y_j \in \mathcal{A}$ be pairwise disjoint. Two elements

$$f = \sum_{i=1}^{n} \lambda_i \chi(x_i) \quad \text{and} \quad g = \sum_{j=1}^{m} \gamma_j \chi(y_j)$$

are equivalent if $\bigvee_{i=1}^{n} x_i = \bigvee_{j=1}^{m} y_j$ and if $\lambda_i = \gamma_j$ whenever $x_i \wedge y_j \neq 0$. We call the set of all equivalence classes $\mathcal{C}(\mathcal{A})$. Henceforth, we take $f = \sum_{i=1}^{n} \lambda_i \chi(x_i)$ to represent all elements of $\mathcal{C}(\mathcal{A})$ that are equivalent to f.

We define addition in $C(\mathcal{A})$ in the style of Goffman in [19] and Jakubik in [23]. For a different approach, see [7]. For x, $y \in \mathcal{A}$, let $x -_1 y$ be the complement of $x \wedge y$ relative to x, that is, $x \wedge (x \wedge y)'$. Then addition in $C(\mathcal{A})$ is defined by

$$f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (\lambda_i + \gamma_j) \chi(x_i \wedge y_j) + \sum_{i=1}^{n} \lambda_i \chi(x_i -_1 \bigvee_{j=1}^{m} y_j) + \sum_{j=1}^{m} \gamma_j \chi(y_j -_1 \bigvee_{i=1}^{n} x_i)$$

where in the summation only those terms are taken into account in which $\lambda_i + \gamma_j \neq 0$ and the elements $x_i \wedge y_j$, $x_i -_1 \bigvee y_j$, and $y_j -_1 \bigvee x_i$ are nonzero.

Addition is well-defined in $\mathcal{C}(\mathcal{A})$, for if f and g are equivalent, $\bigvee_i x_i = \bigvee_j y_j$ and $\lambda_i = \gamma_j$ whenever $x_i \wedge y_j \neq 0$. Thus,

$$f - g = \sum_{i,j}(\lambda_i - \gamma_j)\chi(x_i \wedge y_j) + \sum_i \lambda_i \chi(x_i -_1 \bigvee_{j=1}^{m} y_j) - \sum_j \gamma_j \chi(y_j -_1 \bigvee_{i=1}^{n} x_i)$$

$$= 0.$$

Definition 5.1.14. *A Boolean algebra is* complete *if every subset has a supremum.*

Jakubik proves in [23] that the completeness of a Boolean algebra is equivalent to the Dedekind completeness of its Carathéodory space of place functions. However, his propositions assume "complete distributivity." Since this work has no need for a Boolean algebra to be completely distributive, we prove Theorem 5.1.16. The proof is adapted from Jakubik's outline in Propositions 5.6 and 5.3 (a2) of [23].

Definition 5.1.15. *(Jakubik, page 231 of [23]) Let Y be a sublattice of a lattice X. Y is said to be a* regular sublattice *of X if:*

(i) whenever $x_0 \in Y$ and $\emptyset \neq X \subseteq Y$ such that $x_0 = \sup_Y X$, then $x_0 = \sup_X X$; and

(ii) whenever $x_1 \in Y$ and $\emptyset \neq X \subseteq Y$ such that $x_1 = \inf_Y X$, then $x_1 = \inf_X X$.

Theorem 5.1.16. *Let $\mathcal{A}$ be a Boolean algebra. $\mathcal{A}$ is complete if and only if $\mathcal{C}(\mathcal{A})$ is Dedekind complete.*

Proof. Assume that $\mathcal{A}$ is complete. Let D be a bounded subset of $\mathcal{C}(\mathcal{A})$. Then there exists $g \in \mathcal{C}(\mathcal{A})$ such that $g \geq f$ for every $f \in \mathcal{C}(\mathcal{A})$. Find $\lambda_i \in \mathbb{R}$, $n \in \mathbb{N}$, and $x_i \in \mathcal{A}$ such that $g = \sum_{i=1}^{n} \lambda_i \chi(x_i)$. Set

$$x = x_1 \vee \cdots \vee x_n \qquad \text{and} \qquad \lambda = max\{\lambda_1, \cdots, \lambda_n\}.$$

Then $D \subseteq [0, \lambda\chi(x)]$. By assumption, the interval $[0, x]$ is complete in $\mathcal{A}$. It follows from Corollary 4.4 of [23] that $\mathcal{A}$ is a regular subset of $\mathcal{C}(\mathcal{A})$. Then the interval $[0, \chi(x)]$ is complete as a subset of $\mathcal{C}(\mathcal{A})$. In particular, $[0, \lambda\chi(x)]$ is complete, so $\sup(D)$ exists in $\mathcal{C}(\mathcal{A})$.

To prove sufficiency, assume that $\mathcal{C}(\mathcal{A})$ is complete. Let $\chi \colon \mathcal{A} \to \mathcal{C}(e)$ be the Boolean isomorphism from (i) of Theorem 5.1.13. Let D be a subset of $\mathcal{A}$. Then $\chi(D)$ is a bounded subset of $\mathcal{C}(e)$. Since $\mathcal{C}(\mathcal{A})$ is Dedekind complete, $\sup \chi(D)$ exists in $\mathcal{C}(\mathcal{A})^+$. For every $x \in D$, $\chi(x)$ is a component of $\chi(1_\mathcal{A})$. Thus, $\sup \chi(D) = 2 \sup \chi(D) \wedge \chi(1_\mathcal{A})$ so that $0 = \sup \chi(D) \wedge (\chi(1_\mathcal{A}) - \sup \chi(D))$. By definition, $\sup \chi(D)$ is a component of e.

Let $y = \chi^{-1}(\sup \chi(D))$. Since χ is a Boolean homomorphism, y is an upper bound for D. Suppose there exists y' such that $x \le y' < y$ for every $x \in D$. Then $\chi(y') \ge \sup_{x \in D} \chi(x) = \chi(y)$. Thus, $\chi(y') = \chi(y)$. Since χ is one-to-one, $y' = y$. Therefore, $y = \sup(D)$. $\qquad\square$

5.2 THE FREMLIN TENSOR PRODUCT OF CARATHÉODORY SPACES OF PLACE FUNCTIONS

In this section, we relate Boolean algebras $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A}\otimes\mathcal{B}$ to their respective Carathéodory spaces of place functions $\mathcal{C}(\mathcal{A})$, $\mathcal{C}(\mathcal{B})$, and $\mathcal{C}(\mathcal{A}\otimes\mathcal{B})$. The notation of Theorem 5.1.13 is used with the addition of subscripts to indicate which Boolean algebra is at work. The symbols in (1), (2), and (3) below will be used freely.

(1) $\chi_A \colon \mathcal{A} \to C(\mathcal{A})$, $\chi_B \colon \mathcal{B} \to C(\mathcal{B})$, and $\hat{\chi} \colon \mathcal{A}\otimes\mathcal{B} \to C(\mathcal{A}\otimes\mathcal{B})$ are the Boolean isomorphisms from Theorem 5.1.13.

(2) $C(\mathcal{A})$, $C(\mathcal{B})$, and $C(\mathcal{A}\otimes\mathcal{B})$ have units $\chi_A(1_\mathcal{A})$, $\chi_B(1_\mathcal{B})$, and $\hat{\chi}(1_{\mathcal{A}\otimes\mathcal{B}})$ respectively.

(3) $\epsilon_A \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ and $\epsilon_B \colon \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ are the canonical Boolean homomorphisms in definition 5.1.3.

Theorem 5.2.1. $C(\mathcal{A})\bar{\otimes}C(\mathcal{B})$ *and* $C(\mathcal{A} \otimes \mathcal{B})$ *are Riesz isomorphic.*

Proof. Assume that $\mathcal{A}$ and $\mathcal{B}$ are nontrivial Boolean algebras. For $f \in \mathcal{C}(\mathcal{A})$, there exist $n \in \mathbb{N}$, pairwise disjoint $x_i \in \mathcal{A}$, and nonzero $\lambda_i \in \mathbb{R}$ such that $f = \sum_{i=1}^n \lambda_i \chi_A(x_i)$. For $g \in \mathcal{C}(\mathcal{B})$,

there exist $m \in \mathbb{N}$, pairwise disjoint $u_j \in \mathcal{B}$, and nonzero $\gamma_j \in \mathbb{R}$ so $g = \sum_{j=1}^{m} \gamma_j \chi_B(u_j)$.

Define $\psi \colon \mathcal{C}(\mathcal{A}) \times \mathcal{C}(\mathcal{B}) \to \mathcal{C}(\mathcal{A} \otimes \mathcal{B})$ by

$$\psi(f, g) = \psi \left(\sum_{i=1}^{n} \lambda_i \chi_A(x_i), \sum_{j=1}^{m} \gamma_j \chi_B(u_j) \right)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \gamma_j \hat{\chi}(\epsilon_A(x_i) \wedge \epsilon_B(u_j)).$$

Claim: The definition of ψ is independent of the representations chosen for f and g.

Suppose that there exists $n_0 \in \mathbb{N}$, pairwise disjoint $x_{i_0} \in \mathcal{A}$, and nonzero $\lambda_{i_0} \in \mathbb{R}$ such that $f_0 = \sum_{i_0=1}^{n_0} \lambda_{i_0} \chi_A(x_{i_0})$ is equivalent to f in $\mathcal{C}(\mathcal{A})$. Then $\psi(f, g)$ and $\psi(f_0, g)$ are respectively

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \gamma_j \hat{\chi}(\epsilon_A(x_i) \wedge \epsilon_B(u_j)) \quad \text{and} \quad \sum_{i_0=1}^{n_0} \sum_{j=1}^{m} \lambda_{i_0} \gamma_j \hat{\chi}(\epsilon_A(x_{i_0}) \wedge \epsilon_B(u_j)).$$

Fix j. It follows from the equivalence of f and f_0 that

$$\bigvee_{i=1}^{n} (\epsilon_A(x_i) \wedge \epsilon_B(u_j)) = \left(\epsilon_A(\bigvee_{i=1}^{n} x_i) \right) \wedge \epsilon_B(u_j)$$
$$= \left(\epsilon_A(\bigvee_{i_0=1}^{n_0} x_{i_0}) \right) \wedge \epsilon_B(u_j)$$
$$= \bigvee_{i_0=1}^{n_0} (\epsilon_A(x_{i_0}) \wedge \epsilon_B(u_j)).$$

Now suppose $[\epsilon_A(x_i) \wedge \epsilon_B(u_j)] \wedge [\epsilon_A(x_{i_0}) \wedge \epsilon_B(u_j)] \neq 0$. Then it follows from Theorem 5.1.4 (iv) that $\epsilon_A(x_i) \wedge \epsilon_A(x_{i_0}) = \epsilon_A(x_i \wedge x_{i_0}) \neq 0$. By (v) of the same theorem, ϵ_A is injective. Then $x_i \wedge x_{i_0} \neq 0$ in which case $\lambda_i = \lambda_{i_0}$. Thus, $\lambda_i \gamma_j = \lambda_{i_0} \gamma_j$ whenever $[\epsilon_A(x_i) \wedge \epsilon_B(u_j)] \wedge [\epsilon_A(x_{i_0}) \wedge \epsilon_B(u_j)] \neq 0$.

We have shown that $\sum_i \lambda_i \gamma_j \hat{\chi}(x_i \otimes u_j)$ and $\sum_{i_0} \lambda_{i_0} \gamma_j \hat{\chi}(x_{i_0} \otimes u_j)$ are equivalent in $\mathcal{C}(\mathcal{A} \otimes \mathcal{B})$ for every j, and thus $\psi(f, g) = \psi(f_0, g)$. Symmetrically, the map ψ does not depend on the representation in the second variable.

Let $f_1 = f$ and $f_2 = \sum_{k=1}^{p} \delta_k \chi_A(y_k)$ for nonzero $\delta_k \in \mathbb{R}$, $p \in \mathbb{N}$ and pairwise disjoint $y_k \in \mathcal{A}$. Recall that $f_1 + f_2$ is defined to be

$$\sum_i \sum_k (\lambda_i + \delta_k)\chi_A(x_i \wedge y_k) + \sum_i \lambda_i \chi_A\left(x_{i-1} \bigvee_k y_k\right) + \sum_k \delta_k \chi_A\left(y_{k-1} \bigvee_i x_i\right).$$

Claim: ψ is bilinear.

$$\psi(f_1 + f_2, g)$$

$$=\psi\left(\sum_i \lambda_i \chi_A(x_i) + \sum_k \delta_k \chi_A(y_k), \sum_j \gamma_j \chi_B(u_j)\right)$$

$$=\sum_{i,k,j} (\lambda_i + \delta_k)\gamma_j \hat{\chi}\left(\epsilon_A(x_i \wedge y_k) \wedge \epsilon_B(u_j)\right) + \sum_{i,j} \lambda_i \gamma_j \hat{\chi}\left(\epsilon_A\left(x_{i-1} \bigvee_k y_k\right) \wedge \epsilon_B(u_j)\right)$$

$$+ \sum_{k,j} \delta_k \gamma_j \hat{\chi}\left(\epsilon_A\left(y_{k-1} \bigvee_i x_i\right) \wedge \epsilon_B(u_j)\right)$$

$$=\sum_{i,j}\left[\sum_k \lambda_i \gamma_j \hat{\chi}\left(\epsilon_A(x_i \wedge y_k) \wedge \epsilon_B(u_j)\right) + \lambda_i \gamma_j \hat{\chi}\left(\epsilon_A\left(x_{i-1} \bigvee_k y_k\right) \wedge \epsilon_B(u_j)\right)\right]$$

$$+ \sum_{k,j}\left[\sum_i \delta_k \gamma_j \hat{\chi}\left(\epsilon_A(x_i \wedge y_k) \wedge \epsilon_B(u_j)\right) + \delta_k \gamma_j \hat{\chi}\left(\epsilon_A\left(y_{k-1} \bigvee_i x_i\right) \wedge \epsilon_B(u_j)\right)\right]$$

$$=\sum_{i,j} \lambda_i \gamma_j \left[\hat{\chi}\left(\bigvee_k \epsilon_A(x_i \wedge y_k) \wedge \epsilon_B(u_j)\right) + \hat{\chi}\left(\epsilon_A\left(x_{i-1} \bigvee_k y_k\right) \wedge \epsilon_B(u_j)\right)\right]$$

$$+ \sum_{k,j} \delta_k \gamma_j \left[\hat{\chi}\left(\bigvee_i \epsilon_A(x_i \wedge y_k) \wedge \epsilon_B(u_j)\right) + \hat{\chi}\left(\epsilon_A\left(y_{k-1} \bigvee_i x_i\right) \wedge \epsilon_B(u_j)\right)\right] \quad (*)$$

$$=\sum_{i,j} \lambda_i \gamma_j \hat{\chi}(\epsilon_A(x_i) \wedge \epsilon_B(u_j)) + \sum_{k,j} \delta_k \gamma_j \hat{\chi}(\epsilon_A(y_k) \wedge \epsilon_B(u_j))$$

$$=\psi(f_1, g) + \psi(f_2, g),$$

where $(*)$ is justified because $y_k \perp y_{k'}$ for all $k \neq k'$. Symmetrically, $\psi(f, g_1 + g_2) = \psi(f, g_1) + \psi(f, g_2)$ for $f \in \mathcal{C}(\mathcal{A})$ and $g_1, g_2 \in \mathcal{C}(\mathcal{B})$. It follows from the definition of ψ that

$$\psi(\lambda f, g) = \psi(f, \lambda g) = \lambda \psi(f, g)$$

for every $\lambda \in \mathbb{R}$.

Claim: ψ is a Riesz bimorphism.

Assume $f_1 \wedge f_2 = 0$ and $g \in \mathcal{C}(\mathcal{B})^+$. Using the same representations as above, $x_i \perp y_k$ for all i and k. Then since the maps $\hat{\chi}$ and ϵ_A are Boolean homomorphisms and $x_i \wedge y_k = 0$ for all i, k,

$$
\begin{aligned}
\psi(f_1, g) \wedge \psi(f_2, g) &= \psi\left(\sum_i \lambda_i \chi_A(x_i), \sum_j \gamma_j \chi_B(u_j)\right) \wedge \psi\left(\sum_k \delta_k \chi_A(y_k), \sum_j \gamma_j \chi_B(u_j)\right) \\
&= \left(\sum_{i,j} (\lambda_i \gamma_j) \hat{\chi}(\epsilon_A(x_i) \wedge \epsilon_B(u_j))\right) \wedge \left(\sum_{k,j} (\delta_k \gamma_j) \hat{\chi}(\epsilon_A(y_k) \wedge \epsilon_B(u_j))\right) \\
&= 0.
\end{aligned}
$$

Likewise if $f \in \mathcal{C}(\mathcal{A})^+$ and $g_1 \wedge g_2 = 0$ in $\mathcal{C}(\mathcal{B})$, then $\psi(f, g_1) \wedge \psi(f, g_2) = 0$. By Theorem 1.1.8, ψ is a Riesz bimorphism.

It follows from the universal property of the Fremlin tensor product that there exists a unique Riesz homomorphism $T \colon C(\mathcal{A}) \bar{\otimes} C(\mathcal{B}) \to C(\mathcal{A} \otimes \mathcal{B})$ such that $\psi = T \circ \otimes$.

$$
\begin{array}{ccc}
C(\mathcal{A}) \times C(\mathcal{B}) & \xrightarrow{\otimes} & C(\mathcal{A}) \bar{\otimes} C(\mathcal{B}) \\
{\scriptstyle \psi} \downarrow & \swarrow {\scriptstyle T} & \\
C(\mathcal{A} \otimes \mathcal{B}) & &
\end{array}
$$

Step 1: T is onto.

Let $h \in C(\mathcal{A} \otimes \mathcal{B})$. Then $h = \sum_{i=1}^n \lambda_i \hat{\chi}(e_i)$ for some $e_i \in \mathcal{A} \otimes \mathcal{B}$, $n \in \mathbb{N}$, and nonzero $\lambda_i \in \mathbb{R}$. Fix $i \in \{1, \cdots, n\}$. Since $e_i \in \mathcal{A} \otimes \mathcal{B}$, by Theorem 5.1.4 (iii) there exist a finite disjoint subset $\{\epsilon_A(a_k) \wedge \epsilon_B(b_k)\}_{k=1}^m$ $(m \in \mathbb{N})$ of $\mathcal{A} \otimes \mathcal{B}$ such that

$$
e_i = \bigvee_{k=1}^m \epsilon_A(a_k) \wedge \epsilon_B(b_k).
$$

Then it follows from the definition of ψ that

$$
\begin{aligned}
\hat{\chi}(e_i) =& \hat{\chi}\left(\bigvee_{k=1}^{m} \epsilon_A(a_k) \wedge \epsilon_B(b_k)\right) \\
=& \bigvee_{k=1}^{m} \hat{\chi}(\epsilon_A(a_k) \wedge \epsilon_B(b_k)) \\
=& \bigvee_{k=1}^{m} \psi(\chi_A(a_k), \chi_B(b_k)) \\
=& \bigvee_{k=1}^{m} T \circ \otimes(\chi_A(a_k), \chi_B(b_k)).
\end{aligned}
$$

Since T preserves finite suprema, $\hat{\chi}(e_i)$ is in the image of T for every i. It follows from the linearity of T that h is in the image of T.

Step 2: T is one-to-one.

Suppose $f \in C(\mathcal{A}) \otimes C(\mathcal{B})$, the algebraic tensor product of $C(\mathcal{A})$ and $C(\mathcal{B})$, such that $f \neq 0$. Then there exist $n \in \mathbb{N}$, nonzero $\lambda_k \in \mathbb{R}$, and nontrivial $x_k \in \mathcal{A}$, $u_k \in \mathcal{B}$ such that

$$
f = \sum_{k=1}^{n} \lambda_k \chi_A(x_k) \otimes \chi_B(u_k).
$$

Since ϵ_A, ϵ_B, and $\hat{\chi}$ are injective Boolean isomorphisms,

$$
\begin{aligned}
T(f) =& T\left(\sum_{k=1}^{n} \lambda_k \chi_A(x_k) \otimes \chi_B(u_k)\right) \\
=& \sum_{k=1}^{n} \lambda_k \psi\left(\chi_A(x_k), \chi_B(u_k)\right) \\
=& \sum_{k=1}^{n} \lambda_k \hat{\chi}(\epsilon_A(x_k) \wedge \epsilon_B(u_j)) \\
&\neq 0.
\end{aligned}
$$

If $g \in C(\mathcal{A}) \bar{\otimes} C(\mathcal{B})$, the Riesz space tensor product of $C(\mathcal{A})$ and $C(\mathcal{B})$, such that $g \neq 0$, then $|g| \neq 0$. By Theorem 3.1.2, for all $\delta > 0$ there exists $f \in \mathcal{C}(\mathcal{A})^+ \otimes \mathcal{C}(\mathcal{B})^+$ such that

$$0 \leq |g| - f \leq \delta \hat{\chi}(1_{\mathcal{A} \otimes \mathcal{B}}).$$

Since $C(\mathcal{A}) \bar{\otimes} C(\mathcal{B})$ is Archimedean, choose $\delta > 0$ such that $|g| \wedge \delta \hat{\chi}(1_{\mathcal{A} \otimes \mathcal{B}}) \neq |g|$. Then f is nonzero. We have shown that $T(f) \neq 0$ when $0 \neq f \in \mathcal{C}(\mathcal{A}) \otimes \mathcal{C}(\mathcal{B})$. Since T is a Riesz homomorphism, $0 < T(f) \leq |T(g)|$. Therefore, $T(g) \neq 0$, and T is a Riesz isomorphism. Consequently, $C(\mathcal{A}) \bar{\otimes} C(\mathcal{B})$ is Riesz isomorphic to $C(\mathcal{A} \otimes \mathcal{B})$. $\qquad\square$

5.3 APPLICATIONS

In this section, we provide a solution for Fremlin's exercise 315Y(f) in [16]. The exercise leads to an observation on Dedekind completeness in the Fremlin tensor product of place functions and a statement on bands in the Fremlin tensor product of infinite dimensional Archimedean Riesz spaces.

Theorem 5.3.1. *(Grobler, 3.6 of [20]) Let E and F be Archimedean Riesz spaces. If the space $E \bar{\otimes} F$ is Dedekind complete, then both E and F are Dedekind complete.*

Theorem 5.3.2. *(Fremlin, 315Y(f) of [16]) Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras. $\mathcal{A} \otimes \mathcal{B}$ is complete if and only if either $\mathcal{A} = \{0\}$ or $\mathcal{B} = \{0\}$ or $\mathcal{A}$ is finite and $\mathcal{B}$ is complete or $\mathcal{B}$ is finite and $\mathcal{A}$ is complete.*

Proof. If $\mathcal{A} = \{0\}$ or $\mathcal{B} = \{0\}$, the result is trivial. Assume $\mathcal{A}$ and $\mathcal{B}$ are nontrivial Boolean algebras.

Suppose $\mathcal{A} \otimes \mathcal{B}$ is complete. It follows from Theorems 5.2.1 and 5.1.16 that $\mathcal{C}(\mathcal{A} \otimes \mathcal{B}) \cong \mathcal{C}(\mathcal{A}) \bar{\otimes} \mathcal{C}(\mathcal{B})$ is Dedekind complete. By Theorem 5.3.1, $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$ are Dedekind complete. From Theorem 5.1.16, $\mathcal{A}$ and $\mathcal{B}$ are complete. It remains to show that one of the Boolean algebras is finite.

By Theorem 4.1.7, the Dedekind completeness of $\mathcal{C}(\mathcal{A})\bar{\otimes}\mathcal{C}(\mathcal{B})$ implies that $\mathcal{C}(\mathcal{A}) \cong c_{00}(I)$ for a set $I \subseteq \mathcal{C}(\mathcal{A})$ or $\mathcal{C}(\mathcal{B}) \cong c_{00}(J)$ for a set $J \subseteq \mathcal{C}(\mathcal{B})$. Since each Carathéodory space of place functions contains a unit, $\mathcal{C}(\mathcal{A})$ or $\mathcal{C}(\mathcal{B})$ is finite dimensional. Thus, $\mathcal{A}$ is finite or $\mathcal{B}$ is finite.

The sufficiency is proven similarly via Theorem 4.1.7. $\qquad\square$

Corollary 5.3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be nontrivial Boolean algebras. $\mathcal{C}(\mathcal{A})\bar{\otimes}\mathcal{C}(\mathcal{B})$ is Dedekind complete if and only if one of $\mathcal{A}$ or $\mathcal{B}$ is finite and the other is complete.*

Recall that for an Archimedean Riesz space E, its collection of bands, denoted $\mathcal{B}(E)$, forms a complete Boolean algebra. Our last application shows that for Archimedean Riesz spaces E and F, the set of bands in $E\bar{\otimes}F$ is rarely Boolean isomorphic to $\mathcal{B}(E) \otimes \mathcal{B}(F)$. That is, if E and F are infinite dimensional, not every band B of $E\bar{\otimes}F$ can be "decomposed" into the Fremlin tensor product of a band in E and a band in F.

Corollary 5.3.4. *Let E and F be infinite dimensional Archimedean Riesz spaces. Then $\mathcal{B}(E) \otimes \mathcal{B}(F)$ is not Boolean isomorphic to $\mathcal{B}(E\bar{\otimes}F)$.*

Proof. By Lemma 5.1.11, neither $\mathcal{B}(E)$ nor $\mathcal{B}(F)$ is finite. Then $\mathcal{B}(E)\otimes\mathcal{B}(F)$ is not complete by Theorem 5.3.2. However, by Theorem 5.1.6 the Boolean algebra of bands is complete for any Archimedean Riesz space, so $\mathcal{B}(E\bar{\otimes}F)$ is complete. $\qquad\square$

6. CONCLUSION

In this final chapter, we provide connections with what has been proven in this thesis to recent results related to Archimedean Riesz spaces and tensor products.

6.1   DEDEKIND COMPLETENESS IN FREMLIN TENSOR PRODUCTS

The absence of Dedekind completeness in the Banach lattice tensor product of Dedekind complete Banach lattices was first exemplified by Fremlin himself in [15].

Definition 6.1.1. *(1C of [15]) Let E and F be Banach lattices.*

(i) *Let G be another Banach lattice and $\varphi\colon E \times F \to G$ be a bilinear map. Then φ induces a map $\hat{\varphi}\colon E \otimes F \to G$ such that $\hat{\varphi}(x \otimes y) = \varphi(x,y)$ for all $x \in E$ and $y \in F$. For each φ,*

$$||\varphi|| = \sup\{||\varphi(x,y)||\colon ||x|| \leq 1, ||y|| \leq 1\}.$$

The positive-projective norm *on $E \otimes F$ is defined by*

$$||u||_{|\pi|} = \sup ||\hat{\varphi}(u)||,$$

where the supremum is taken over all Banach lattices G and all positive bilinear maps φ from $E \times F$ to G with $||\varphi|| \leq 1$.

(ii) *The* Banach lattice tensor product of E and F *is the completion of $E \otimes F$ under $||\ ||_{|\pi|}$ and is denoted by $E \,\hat{\bar{\otimes}}\, F$.*

Example 6.1.2. *(4C of [15]) If $E = F = L^2([0,1])$, then $E \,\hat{\bar{\otimes}}\, F$ is not Dedekind complete.*

In other words, we observe that the Dedekind completeness of $L^2([0,1])$ is not "preserved" even in $L^2([0,1]) \; \widehat{\otimes} \; L^2([0,1])$ which has been completed relative to the positive-projective norm. This phenomenon was the initial motivation for questioning when the (Fremlin) Archimedean Riesz space tensor product of Dedekind complete Riesz spaces is Dedekind complete, which led to Theorem 4.1.7. Indeed, by applying Theorem 4.1.7, we can immediately conclude that $L^2([0,1]) \bar{\otimes} L^2([0,1])$ is not Dedekind complete since $L^2([0,1])$ is not Riesz isomorphic to $c_{00}(I)$ for any $I \subseteq L^2([0,1])$. It is to our knowledge unknown when exactly $E \; \widehat{\otimes} \; F$ is Dedekind complete for Dedekind complete Banach lattices E and F.

6.2   THE TENSOR PRODUCT OF IDEALS

In [11], Buskes and Van Rooij introduce bilinear maps of order bounded variation, bounded semivariation and norm bounded variation in order to broaden the understanding of the projective tensor product of Banach lattices. Of interest to this work, the introduction of [11] states "it is known that the tensor product of two Riesz subspaces is a Riesz subspace of the tensor product; it is unknown, whether the tensor product of two ideals is an ideal in the tensor product." Theorem 3.1.9 answers the question positively under the condition that the Fremlin tensor product is Dedekind σ-complete, but Theorem 3.2.4 contradicts the statement in general.

Additionally, in [5] Ben Amor, Gok, and Yaman cite Theorem 3.2.4 to demonstrate that the Fremlin tensor product of ideals need not be an ideal in the tensor product. Their paper proves that the Dedekind completion of the Fremlin tensor product of two ideals is again an ideal. They go further to prove that the Dedekind completion of the Fremlin tensor product of principal bands is again a principal band in the Dedekind completion of the Fremlin tensor product.

6.3   THE ORDER CONTINUITY OF THE TENSOR MAP

In [21], Grobler and Labuschagne present two approaches to constructing the Fremlin tensor product of Archimedean Riesz spaces. In [20], Grobler continues their work by defining various lattice tensor products in different categories of Riesz spaces. In particular, he defines the Dedekind complete Fremlin tensor product, denoted by $E\bar{\otimes}_\delta F$, which satisfies a universal property with respect to order continuous maps.

Definition 6.3.1. *(1.2 of [1]) A net $\{f_\alpha\}_{\alpha \in A}$ of a Riesz space is said to be* order convergent *to f (in symbols $f_\alpha \xrightarrow{o} f$), if there exists a net $\{z_\beta\}_{\beta \in B}$ such that*

1. $z_\beta \downarrow 0$, and

2. for each $\beta \in B$, there exists some $\alpha_0 \in A$ satisfying $|f_\alpha - f| \le z_\beta$ for all $\alpha \ge \alpha_0$.

Definition 6.3.2. *(1.53 of [2]) A linear map $T\colon E \to F$ between two Riesz spaces is said to be* order continuous *if $f_\alpha \xrightarrow{o} f$ in E implies $T(f_\alpha) \xrightarrow{o} T(f)$ in F.*

Aliprantis and Burkinshaw note that a positive linear map $T\colon E \to F$ between two Riesz spaces is order continuous if and only if $x_\alpha \downarrow 0$ in E implies $T(x_\alpha) \downarrow 0$ in F (or equivalently, if and only if $0 \le x_\alpha \uparrow x$ in E implies $T(x_\alpha) \uparrow T(x)$ in F). Since every Riesz homomorphism is a positive linear map, we apply this equivalent criteria freely.

Definition 6.3.3. *(page 403 of [6]) Let E, F, and G be Riesz spaces. A bilinear map $\varphi\colon E \times F \to G$ is called* separately order continuous *if*

$$e \mapsto \varphi(e, y) \text{ and } f \mapsto \varphi(x, f) \qquad (e \in E, \ f \in F)$$

are order continuous for each $x \in E^+$, and each $y \in F^+$.

Theorem 4.1.7 shows precisely how rarely the Fremlin tensor product is Dedekind complete. $E\bar{\otimes}_\delta F$ is a natural solution whenever Dedekind completeness is desired in the tensor product.

Theorem 6.3.4. *(Grobler, 5.1 of [20]) For Archimedean Riesz spaces E and F, the lattice tensor product $E \bar{\otimes}_\delta F$ is the unique Dedekind complete vector lattice R (up to Riesz isomorphism) with the properties that there exists a Riesz bimorphism $\sigma \colon E \times F \to R$ such that*

(D1) *The map $\sigma \colon E \times F \to R$ defined by $\sigma(x, y) = x \otimes y$ is an order continuous embedding of $E \otimes F$ into R.*

(D2) *If G is a Dedekind complete vector lattice and if $\psi \colon E \times F \to G$ is an order continuous Riesz bimorphism, then there exists a unique order continuous Riesz homomorphism $\tau \colon R \to G$ satisfying $\tau \circ \sigma = \psi$.*

Proposition 3.3 and Corollary 3.4 of [20] claim that the map $\otimes \colon E \times F \to E \bar{\otimes} F$ is separately order continuous for any Archimedean Riesz spaces E and F. Together with Stephan Roberts (see [28], [8]), we provide a proof in Theorem 6.3.8. We begin by utilizing Kakutani's representation theorem once again to get Lemma 6.3.5.

Lemma 6.3.5. *Let X, Y be compact, Hausdorff spaces. Then $\otimes \colon C(X) \times C(Y) \to C(X) \bar{\otimes} C(Y)$ is separately order continuous.*

Proof. Let $g \in C(Y)^+$ be nonzero and $\{f_\alpha\}_{\alpha \in A}$ be a net in $C(X)$ such that $f_\alpha \xrightarrow{o} f \in C(X)$. Since Y is compact, choose $M \in \mathbb{N}$ such that $g \le M 1_Y$. There exists $\{z_\beta\}_{\beta \in B} \downarrow 0$ in $C(X)$ such that for every $\beta \in B$, there is an $\alpha_0 \in A$ satisfying $|f_\alpha - f| \le z_\beta$ whenever $\alpha \ge \alpha_0$. Fix $\beta \in B$. Then for some $\alpha_0 \in A$,

$$
\begin{aligned}
|\otimes(f_\alpha, g) - \otimes(f, g)|(x, y) &= |\otimes(f_\alpha - f, g)|(x, y) \\
&\le |\otimes(z_\beta, M 1_Y)|(x, y) \\
&= (z_\beta \otimes M 1_Y)(x, y) \\
&= M \cdot z_\beta(x) 1_Y(y) \\
&= M z_\beta(x) \downarrow 0
\end{aligned}
$$

for all $\alpha \geq \alpha_0$. By definition 6.3.3, $\otimes$ is separately order continuous on $C(X) \times C(Y)$. $\square$

Theorem 6.3.6. *(1.35 of [2]) Let G be an order dense Riesz subspace of a Riesz space E, and let $D \subseteq G^+$ satisfying $D \downarrow$. Then $D \downarrow 0$ holds in G if and only if $D \downarrow 0$ in E.*

Recall that if E is a uniformly complete Riesz space, E_x is Riesz isomorphic to $C(X)$ for some compact Hausdorff space X (Theorem 3.1.7).

Lemma 6.3.7. *Let E and F be uniformly complete Archimedean Riesz spaces. The tensor map $\otimes \colon E \times F \to E \bar{\otimes} F$ is separately order continuous.*

Proof. Let $g \in F^+$ be nonzero and $f_\alpha \uparrow f$ in E. E_f is Riesz isomorphic to $C(X)$ for some compact Hausdorff space X. Likewise, F_g is Riesz isomorphic to $C(Y)$ for some compact Hausdorff space Y. Then by Lemma 6.3.5, we have that $f_\alpha \otimes g \uparrow f \otimes g$ in $C(X) \bar{\otimes} C(Y)$. It follows from $f \otimes g \in E_f \bar{\otimes} F_g$ that $f_\alpha \otimes g \uparrow f \otimes g$ in $E_f \bar{\otimes} F_g$.

Since E and F are uniformly complete, $E_f \bar{\otimes} F_g$ is order dense in $(E \bar{\otimes} F)_{f \otimes g}$ (Lemma 3.1.8). It follows from Theorem 6.3.6 that $f_\alpha \otimes g \uparrow f \otimes g$ in $(E \bar{\otimes} F)_{f \otimes g}$. If there exists $h \in E \bar{\otimes} F$ such that $h < f \otimes g$ and $f_\alpha \otimes g \uparrow h$, then $h \in (E \bar{\otimes} F)_{f \otimes g}$, a contradiction. Thus, $f_\alpha \otimes g \uparrow f \otimes g$ in $E \bar{\otimes} F$. $\square$

Theorem 6.3.8. *Let E and F be Archimedean Riesz spaces. The tensor map $\otimes \colon E \times F \to E \bar{\otimes} F$ is separately order continuous.*

Proof. Let $g \in F^+$ be nonzero and $f_\alpha \uparrow f$ in E. Let $\tilde{E}$, $\tilde{F}$ be the unique up to Riesz isomorphism uniform completions of E, F respectively. Then since E is order dense in $\tilde{E}$, $f_\alpha \otimes g \uparrow f \otimes g$ in $\tilde{E} \bar{\otimes} \tilde{F}$ by Lemma 6.3.7. Since $E \bar{\otimes} F \subseteq \tilde{E} \bar{\otimes} \tilde{F}$, it follows that $f_\alpha \otimes g \uparrow f \otimes g$ in $E \bar{\otimes} F$. $\square$

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