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Research Paper

A review on the MCUSUM Charts in Detecting the Shifts of the Process with Comparison Study

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ABSTRACT

In this paper, we compare the performance of different MCUSUM methods presented in the literature. First, we briefly introduce MCUSUM methods in multivariate normal distribution. In order to evaluate their performance, we present a comparative study with simulation. Furthermore, we compare the average out-of-control run length of MCUSUM methods under different scenarios of mean shifts, standard deviation shifts, and correlation shifts. The results of the simulation study show that MCUSUM methods have different efficiency in detecting process shifts and based on the required application, the appropriate MCUSUM chart should be selected .

1. Introduction

Hotelling (1947) was the first who proposed the control techniques for Multivariate processes (Hotelling, 1947). Multivariate control is divided into two stages. The first stage is the *retrospective* examination of process behavior. The second is *prospective* examination of process (Sullivan and Woodall, 1996). In the first stage, observations are analyzed to determine if the process was in control and the covariance, mean, and control limit are determined. In the second stage, a control chart is applied to control if the parameters determined in the first stage are correct.

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Jackson (1991) stated that any multivariate process control method must satisfy four conditions: they must provide a response to (1) whether the process is in control, (2) whether the specified probability of Type I error is preserved has it been or not and (3) whether they considered relationships between variables, and (4) answering the question, "If the process is out of control, what is the problem?" Must be available (Jackson, 1991).

Bersimis et al. (2007) provide a literature review of multivariable process control chart techniques. They investigated multivariate extensions for a variety of univariate control charts, such as multivariate Shewhart-type control charts, multivariate CUSUM control charts, and multivariate EWMA control charts. In addition, they explore unique methods for constructing multivariate control charts, based on multivariate statistical techniques such as principal component analysis (PCA) and partial least squares (PLS). MCUSUM control charts are divided into two categories. In the former, the direction of shift is known, while in the latter, the direction of change is assumed to be unknown (Bersimis et al., 2007).

A MCUSUM can be designed from CUSUMs based on two methods. One involves reducing each multivariate observation to a scalar then designing a CUSUM of the scalars. The second method is by accumulating the vectors before reducing it to a scalar, which is designing a MCUSUM directly from the observations (Crosier, 1988). Woodall and Ncube (1985) proposed a method for the bi-variate normal distributions, they showed that their MCUSUM method performs better than the Hotelling's T^2 method (Woodall and Ncube, 1985).

Rasay et al. (2018) applied control charts as a condition monitoring technique, and inferences about the operating modes of the system are based on the information collected about the quality of the items produced (Rasay et al., 2018).

Akhavan Niaki and Fallah Nezhad (2009) proposed a new method in this paper to monitor the change of the overall mean and the classification of states of a multivariate quality control system Based on the Bayesian rule (Akhavan Niaki and Fallah Nezhad, 2009) .

Jafarian-Namin et al. (2021) have examined the integration of triple components including statistical process monitoring (SPM), maintenance policy (MP) and economic production quantity (EPQ) (Jafarian-Namin et al., 2021).

Rasay et al. (2019) considered a two-step affiliate process in which each step has a unique qualitative characteristic. According to the regression formula, the qualitative characteristic of the second stage is dependent on the first stage (Rasay et al., 2019). Today multivariate control charts are widely applied in industrial application. Thus selection of the appropriate multivariate control chart is very important in practice. Industrial application of quality control methods are discussed in (Ghahremani, and Mohseni, 2021).

We compare the performance of different MCUSUM methods presented in the literature. For this purpose, we first briefly introduce MCUSUM methods in multivariate normal distribution. In order to evaluate their performance, we present a comparative study with simulation. Furthermore, we compare the average run lengths of in- and out-of-control MCUSUM methods under different scenarios of mean shifts, standard deviation shifts, and correlation shifts. The results of the simulation study show that MCUSUM methods have different efficiency in detecting process shifts and based on the required application, the appropriate MCUSUM chart should be selected.

This paper is the result of a simulation analysis of the recent MCUSUM charts in the area of multivariate statistical process control. Section 2 describes the most significant multivariate cumulative sum (CUSUM) control charts. The simulation analysis is presented in Section 3. Finally, some concluding remarks are offered in Section 4.

2. MCUSUM Methods

Healy (1987) used the sequential probability ratio tests, in order to develop a MCUSUM chart (Healy, 1987). Let x_i be the i_{th} observation, that follows a multi normal distribution $N_p(\mu_0, \Sigma_0)$ with an in-control mean vector μ_0 and a known covariance matrix Σ_0. Let μ_1 be the out-of-control mean vector. The CUSUM for detecting the out-of-control mean μ_1 may be written as

$$S_i = \max[(S_{i-1} + a_t(x_i - \mu_0) - 0.5\lambda(\mu_1)), 0] \quad i = 1, 2, 3...$$

Where $\lambda(\mu_1)$ is determined as follows,

$$\lambda^2(\mu_1) = (\mu_1 - \mu_0)^t \Sigma_0^{-1} (\mu_1 - \mu_0)$$

$\lambda^2(\mu_1)$ is the non-centrality parameter, and,

$$a_t = {(\mu_1 - \mu_0)^t \Sigma_0^{-1}} / {\lambda(\mu_1)}$$

When $S_i \geq H$ then an out-of-control signal is observed.

Crosier developed two multivariate CUSUM schemes. The first CUSUM proposed by Crosier is a CUSUM of the D_i that is given by

$$D_i^2 = (x_i - \mu_0)^t \Sigma_0^{-1} (x_i - \mu_0)$$

$$S_i = \max[(S_{i-1} + D_i - k), 0] \quad i = 1, 2, 3...$$

where $S_0 \geq 0$ and $k \geq 0$. When $S_i \geq H$ then an out-of-control signal is observed, Crosier proposed the optimal value of k is the square root of the number of variables (Bersimis et al., 2007).

The second CUSUM proposed by Crosier is a CUSUM of vectors that is given by

$$\gamma_i = \left[S_i{}^t \Sigma_0^{-1} S_i\right]^{1/2} \quad i = 1, 2, 3...$$

where $S_i = (S_{i-1} + x_i - \mu_0)(1 - kC_i^{-1})$, if $C_i > k$ and $S_i = 0$ otherwise and

$$C_i = \left[(S_{i-1} + x_i - \mu_0)^t \Sigma_0^{-1} (S_{i-1} + x_i - \mu_0)\right]^{1/2}$$

when $\gamma_i > h$ then an out-of-control signal is observed , h is selected to achieve an assumed in-control ARL by simulation. Crosier (1988) proposed that $k = 0.5\lambda(\mu_1)$ and $S_0 = 0$ (Crosier, 1988).

Moreover, Pignatiello and Runger (1990) introduced two MCUSUM schemes (Pignatiello and Runger, 1990). The first CUSUM was proposed by Pignatiello and Runger , defined as

$$S_i = \max[(S_{i-1} + D_i^2 - k), 0] \quad i = 1, 2, 3...$$

where $S_0 = 0$, and *k* is $0.5\lambda^2(\mu_1) + p$. The process is out of control if S_i was more than a control limit *H*.

The second CUSUM was proposed by Pignatiello and Runger (1990) can be constructed by defining MC_i as

$$MC_i = Max\left\{\left(D_i^t \Sigma_0^{-1} D_i\right)^{1/2} - kn_i, 0\right\}, \quad i = 1, 2, 3, ...$$

Where $MC_0 = 0$ and *k* is $0.5\lambda(\mu_1)$ and

$$D_i = \sum_{l=i-n_i+1}^{i} (x_l - \mu_0)$$

Where n_i is the number of subgroups since the most recent renewal (i.e. zero value) of the CUSUM chart, formally defined as

$$n_i = \begin{cases} n_{i-1} + 1 & \text{if } MC_{i-1} > 0 \\ 0 & \text{otherwise} \end{cases}$$

This chart operates by plotting MC_i on a control chart with an upper control limit of H (H is investigated by simulation). If MC_i exceeds H, then the process is out of control.

3. Simulation Results for Bi-variates Normal Case

3.1. Shift in the Process Mean

First we generate pairs of independent uniform random variates (R_{1i}, R_{2i}) ; $i = 1, 2, ..., k, k+1, k+2, ...$ and use $Z_i = \sqrt{-2\log(R_{1i})}\cos(2\pi R_{2i})$ to generate standard normal observations. If we define the quality characteristics to be *X* and *Y* random variables, assuming $\rho = 0.5$, at stage i of the data gathering process we generate $X_i = Z_i$; $i = 1, 2, ...$ with mean zero and variance one and Y_i by use of

$$E(Y_i | X_i) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X_i - \mu_X) \text{ and } \sigma_X^2 = (1-\rho^2)\sigma_Y^2 \text{ , where } \mu_Y = 0 \text{ and } \sigma_Y = 1.$$

In 10000 independent replications, for an intended ARL_0 of 320, we assumed $\mu_1 = (1,1)$ is the out-of-control mean vector thus $\lambda(\mu_1)$ is calculated equal to $\sqrt{\frac{4}{3}}$ and the parameters of each MCUSUM method is determined. We pick the H parameters of the methods such that ARL_0 of the methods becomes 320. For the comparison study, we estimate the ARL_1 values of the MCUSUM methods by 10000 independent replications in each of the different scenarios of mean shifts.

The shifts are given as multiples of the process standard deviations and are shown in the first column of Table (1). The second up to the tenth column of Table (1) show the ARL_1 values of the methods under consideration and their standard deviations.

Table 1: The results of ARL_1 study for mean shifts (bi-variate normal)

Mean Shifts	**In-control and Out-of-control Average Run Lengths (ARL_1)**									
	Healy's MCUSUM	SD	Crosier 's First MCUSUM	SD	Crosier 's Second MCUSUM	SD	Pignatiello' s First MCUSUM	SD	Pignatiello' s Second MCUSUM	SD
$(0,0)$	326.28	321.6	324.77	322.0	319.6	316.9	326.93	324.0	319.13	316.3
$(1.0\sigma_X,.0)$	10.81	7.25	41.79	40.49	13.30	9.88	58.13	57.73	10.63	6.26
$(2.0\sigma_X,0)$	3.79	1.55	5.41	4.04	3.72	1.58	7.75	7.12	3.85	1.31
$(3.0\sigma_X,0)$	2.39	0.75	2.14	1.09	2.22	0.69	2.27	1.57	2.49	0.64
$(0,1.0\sigma_Y)$	21.98	17.99	27.87	26.08	9.70	6.49	40.91	40.68	8.35	4.31
$(1.0\sigma_X,1.0\sigma_Y)$	4.86	2.28	11.82	10.47	5.68	2.98	18.27	17.50	5.46	2.25
$(2.0\sigma_X,1.0\sigma_Y)$	2.70	0.91	3.71	2.41	3.03	1.14	4.87	4.12	3.27	1.00
$(3.0\sigma_X,1.0\sigma_Y)$	1.97	0.57	1.87	0.88	2.05	0.62	1.92	1.21	2.32	0.57
$(0,2.0\sigma_Y)$	6.72	3.75	3.64	2.34	3.05	1.16	4.85	4.19	3.27	1.01
$(1.0\sigma_X,2.0\sigma_Y)$	3.15	1.14	3.03	1.79	2.73	0.97	3.67	2.94	2.98	0.86
$(2.0\sigma_X,2.0\sigma_Y)$	2.15	0.63	2.07	1.02	2.17	0.67	2.16	1.43	2.44	0.63
$(3.0\sigma_X,2.0\sigma_Y)$	1.67	0.52	1.47	0.60	1.75	0.53	1.41	0.70	2.03	0.43
$(0,3.0\sigma_Y)$	3.80	1.56	1.67	0.73	1.91	0.57	1.65	0.94	2.18	0.50
$(1.0\sigma_X,3.0\sigma_Y)$	2.38	0.74	1.58	0.67	1.83	0.55	1.52	0.82	2.11	0.46
$(2.0\sigma_X,3.0\sigma_Y)$	1.81	0.52	1.35	0.53	1.64	0.53	1.29	0.56	1.95	0.41
$(3.0\sigma_X,3.0\sigma_Y)$	1.42	0.50	1.16	0.38	1.40	0.50	1.11	0.33	1.75	0.45
Parameters	$k=\sqrt{\frac{1}{3}}$ H=3.55		$k=\sqrt{2}$ H=0.51		$k=\sqrt{\frac{1}{3}}$ H=1.79		$k=\frac{8}{3}$ H=0.76		$k=\sqrt{\frac{1}{3}}$	

If we use the average of ranks as the performance criteria, we see from Table 2 that First MCUSUM method proposed by Crosier, (1988). is the best methods. Second MCUSUM method proposed by Crosier, (1988) is the second method in performance based on the assumed performance criteria. First MCUSUM method proposed by Pignatiello and Runger (1990) is preferred to the other remained control charts. In general, we see that first MCUSUM proposed by Pignatiello and Runger (1990) is the best chart in detecting the large mean shifts. The methods proposed by Healy, (1987) and Crosier, (1988) are the best charts for detecting medium mean shifts, and second MCUSUM proposed by Pignatiello and Runger (1990) is the best chart for detecting small mean shifts.

Table 2: Ranking of the different methods in detecting mean shifts

Mean Shifts	In-control and Out-of-control Average Run Lengths (ARL_1)									
	Healy's MCUSUM	Rank	Crosier 's First MCUSUM	Rank	Crosier 's Second MCUSUM	Rank	Pignatiello' s First MCUSUM	R	Pignatiello' s Second MCUSUM	Ra
$(1.0\sigma_X,0)$	10.81	2	41.79	4	13.30	3	58.13	5	10.63	1
$(2.0\sigma_X,0)$	3.79	2	5.41	4	3.72	1	7.75	5	3.85	3
$(3.0\sigma_X,0)$	2.39	4	2.14	1	2.22	2	2.27	3	2.49	5
$(0,1.0\sigma_Y)$	21.98	3	27.87	4	9.70	2	40.91	5	8.35	1
$(1.0\sigma_X,1.0\sigma_Y)$	4.86	1	11.82	4	5.68	3	18.27	5	5.46	2
$(2.0\sigma_X,1.0\sigma_Y)$	2.70	1	3.71	4	3.03	2	4.87	5	3.27	3
$(3.0\sigma_X,1.0\sigma_Y)$	1.97	3	1.87	1	2.05	4	1.92	2	2.32	5
$(0,2.0\sigma_Y)$	6.72	5	3.64	3	3.05	1	4.85	4	3.27	2
$(1.0\sigma_X,2.0\sigma_Y)$	3.15	4	3.03	3	2.73	1	3.67	5	2.98	2
$(2.0\sigma_X,2.0\sigma_Y)$	2.15	2	2.07	1	2.17	4	2.16	3	2.44	5
$(3.0\sigma_X,2.0\sigma_Y)$	1.67	3	1.47	2	1.75	4	1.41	1	2.03	5
$(0,3.0\sigma_Y)$	3.80	5	1.67	2	1.91	3	1.65	1	2.18	4
$(1.0\sigma_X,3.0\sigma_Y)$	2.38	5	1.58	2	1.83	3	1.52	1	2.11	4
$(2.0\sigma_X,3.0\sigma_Y)$	1.81	4	1.35	2	1.64	3	1.29	1	1.95	5
$(3.0\sigma_X,3.0\sigma_Y)$	1.42	4	1.16	2	1.40	3	1.11	1	1.75	5
Average		3.20		2.60		2.67		3.13		3.47

Also by applying a randomized complete block design on the data from five methods in Table1, following result is concluded,

Source	DF	SS	MS	F	P
Methods	4	496	123.9	3.15	0.020
Shifts	15	476511	31767.4	808.28	0.000
Error	60	2358	39.3		
Total	79	479364			

From above analysis, it is concluded that hypothesis H_0 that is the equality of performance of different method is rejected in α level 0.05.

3.2. Shifts in the Process Standard Deviation

The results of Table (3) show that the first MCUSUM method proposed by Crosier, (1988) is the best method in detecting the shifts of the standard deviation. Also first MCUSUM proposed by Pignatiello and Runger (1990) is the second best chart in detecting the standard deviation shifts. Since this method coincides with a similar procedure that is proposed by Healy, (1987) for controlling process dispersion, it was expected that this method denotes the good performance in detecting standard deviation shifts

Table 3: The results of ARL_1 study for standard deviation shifts (bi-variate normal)

Standard Deviation Shifts	In-control and Out-of-control Average Run Lengths (ARL_1)									
	Healy's MCUSUM	SD	Crosier 's First MCUSUM	SD	Crosier 's Second MCUSUM	SD	Pignatiello' s First MCUSUM	SD	Pignatiello' s Second MCUSUM	SD
(1,1)	321.52	319.6	324.7	322.0	319.6	316.9	326.93	324.0	319.13	316.34
(1,1.5)	125.61	121.2	24.51	23.44	37.33	35.10	28.10	27.62	51.17	48.22
(1,2)	58.08	55.84	8.24	7.31	13.03	11.79	8.95	8.43	19.83	18.17
(1,2.5)	32.86	31.22	4.73	3.82	7.46	6.48	4.87	4.33	11.19	10.04
(1.5,1)	46.45	44.75	24.86	23.51	37.10	35.40	28.26	27.90	50.33	47.13
(1.5,1.5)	34.34	32.45	9.96	8.80	17.61	16.03	12.25	11.67	26.69	24.70
(1.5,2)	25.49	23.49	5.39	4.31	9.23	8.06	6.06	5.48	14.92	13.37
(1.5,2.5)	19.39	18.07	3.68	2.79	5.96	4.99	3.91	3.24	9.32	8.32
(2,1)	19.40	17.88	8.24	7.38	13.10	11.99	8.85	8.34	19.49	17.82
(2,1.5)	16.73	15.38	5.37	4.38	9.13	7.84	6.07	5.42	14.67	13.35
(2,2)	14.63	13.46	3.71	2.78	6.27	5.21	4.03	3.43	10.01	8.77
(2,2.5)	12.71	11.52	2.94	2.05	4.60	3.73	2.99	2.35	7.25	6.33
(2.5,1)	11.66	10.41	4.83	4.00	7.30	6.35	4.93	4.28	11.24	10.43
(2.5,1.5)	10.99	9.85	3.65	2.76	5.92	5.06	3.87	3.32	9.53	8.50
(2.5,2)	10.26	9.05	2.90	2.03	4.60	3.72	2.99	2.38	7.36	6.35
(2.5,2.5)	9.42	8.42	2.41	1.59	3.65	2.84	2.45	1.81	5.64	4.84

3.3. Shifts in the Process Correlation

The results of Table (4) show that the MCUSUM proposed by Healy, (1987) is sensitive to the positive shifts in the correlation. As can be seen in Table (4), other methods are not sensitive to the shifts in the correlation.

Table 4: The results of ARL_1 study for correlation shifts (bi-variate normal)

Correlation Shifts	In-control and Out-of-control Average Run Lengths (ARL_1)									
	Healy's MCUSUM	SD	Crosier 's First MCUSUM	SD	Crosier 's Second MCUSUM	SD	Pignatiello' s First MCUSUM	SD	Pignatiello' s Second MCUSUM	SD
0	2489.48	2571.46	305.8	308.9	324.09	324.9	322.43	317.4	322.81	333.46
0.1	1319.08	1337.83	317.9	315.7	323.84	311.7	322.90	313.0	318.68	309.73
0.2	887.44	877.87	316.2	309.3	316.68	314.5	317.13	299.4	319.18	309.01
0.3	614.86	614.68	328.2	313.9	317.35	327.3	317.05	306.6	307.35	295.65
0.4	430.48	438.93	328.8	340.2	324.20	315.4	325.23	327.6	319.97	308.96
0.5	338.96	347.82	312.7	312.2	318.05	322.7	322.93	352.1	328.89	337.19
0.6	264.38	262.57	349.8	341.2	326.28	321.3	349.56	340.9	317.52	315.71
0.7	209.33	198.74	337.0	325.1	313.26	305.0	332.47	335.7	332.71	334.65
0.8	170.95	168.67	339.5	342.6	309.72	321.9	340.32	327.9	308.40	316.53
0.9	141.61	143.01	311.4	307.1	322.72	303.9	319.26	315.8	315.94	299.91

4. Conclusion

In this paper, we compared the performance of the different MCUSUM methods. To do this, first, we introduced the different MCUSUM methods proposed in the literature. Then we compared the performance of the MCUSUM methods via simulation and we concluded that first MCUSUM proposed by Pignatiello and Runger (1990) is the best chart in detecting the large mean shifts. Healy's method (Healy, 1987) and Crosier's Methods (Crosier, 1988)are the best charts for detecting medium mean shifts, and second MCUSUM proposed by Pignatiello and Runger (1990) is the best chart in detecting small mean shifts. Also first MCUSUM method proposed by Crosier, (1988) is the best method in detecting the shifts of the standard deviation. Also first MCUSUM proposed by Pignatiello and Runger (1990) is the second best chart in detecting the standard deviation shifts. Since this method coincides with a similar procedure that is proposed by Healy, (1987) for controlling process dispersion, it was expected that this method denotes the good performance in detecting standard deviation shifts. The MCUSUM proposed by Healy, (1987) is sensitive to the positive shifts in the correlation and other methods are not sensitive to the shifts in the correlation.

Authors Contributions

Mohammad Saber Fallahnezhad: Conceptualization, Methodology, Software, Investigation, Writing-Original draft. *Amir Ghalichehbaf*: Writing - Review & Editing.

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Conflicts of Interest

The authors declare no conflict of interest related to this publication.

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