# On Kinematic Singularities of Low Dimension

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy

by Catherine Ann Hobbs



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## Abstract

This thesis is an investigation into the types of singularities that can appear on trajectories of rigid motions, *kinematic singularities*, motivated by problems in mechanical engineering of designing mechanisms. Here we consider rigid motions of the plane and space with one and two degrees of freedom.

In order to study these singularities we prove a multi-germ transversality result and also a result about the restrictions on the codimension of the singularity given by the number of degrees of freedom of the motion. Some of the classifications of the singularities we are interested in have already been completed but all the simple singularities of space curves and also most of the multi-germs, both of the plane and of space, are classified here. We also study the unfoldings and bifurcation sets of all the kinematic singularities on our lists.

**Acknowledgements**

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*This thesis is dedicated to my parents.*

"And what is the use of a book," thought Alice,
"without pictures or conversation?"
–Lewis Carroll, *Alice's Adventures in Wonderland*

# Contents

## Chapter 4 – One-Dimensional Motions of the Plane



## Chapter 5 – One-Dimensional Motions of Space



## Chapter 6 – Two-Dimensional Motions of the Plane



## Chapter 7 – Two-Dimensional Motions of Space

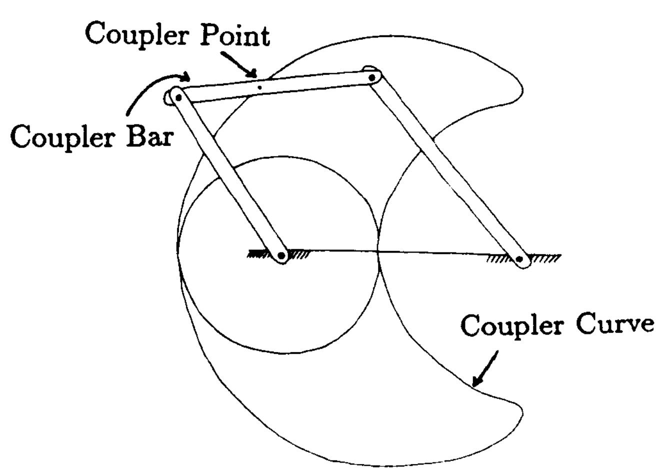
# Introduction

The main purpose of this thesis is to investigate the types of singularities that generically occur on the trajectories of rigid body motions of the plane and of space with one or two degrees of freedom. The problem arises from questions in mechanical engineering: what types of singularities would we expect to see on the trajectory traced out by a robotic arm?

## 0.1 Motivation – Planar and Spatial Kinematics

A rigid body in Euclidean $n$-space is an oriented, connected subset of $E^n$ containing $n+1$ affinely independent points. Displacing these rigidly gives rise to a rigid body motion. For example, if we have $m$ rigid bodies jointed together, where some of the bodies are fixed, we obtain a rigid body (also known as a closed kinematic chain) which moves around to give a rigid body motion. Such a construction is known as a $m$-bar linkage, or mechanism. The study of such mechanisms is known as kinematics.

Perhaps the simplest (though by no means simple) type of such a mechanism is the four-bar linkage. One bar is fixed, while the others can move in the plane. The mechanism has one degree of freedom, that is, a point rigidly attached to the mechanism traces out a curve as the motion progresses. We call this point the coupler point (as it is generally attached to the bar opposite the fixed one, the coupler bar) and the curve it traces out is the coupler curve. This curve is generally of degree 6 and can be quite complicated. An example of a four-bar mechanism is shown in Fig. 1.



Four-bar mechanisms and features of their coupler curves are widely used in engineering. A well-known example is the Watt four-bar mechanism (Fig. 2) which was originally designed by James Watt to guide the end of a piston rod along a fair approximation to a straight line. It is currently used for axle suspension in some cars [HD].

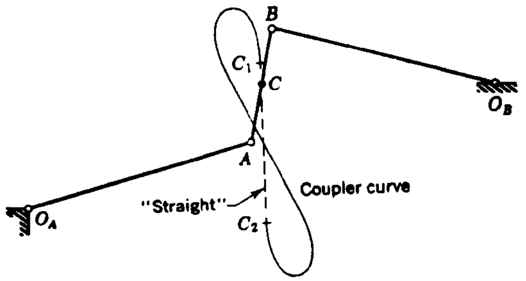



Fig. 2

Another example is the windscreen wiper mechanism shown in Fig. 3 [Hau]. This application of the four-bar linkage transmits motor-driven rotation of the crank to reciprocating motion of the wind-screen wiper. The crank and left rocker arm are pivoted in the vehicle frame at points A and B. The crank coupler is pivoted in the crank at point C and the left rocker arm at point D. The crank, crank coupler, left rocker and vehicle frame constitute a four-bar mechanism. Since the length of BD is greater than that of AC, a full rotation of the crank causes only a partial motion of the left rocker arm, leading to the desired motion of the left windscreen wiper. A second four-bar linkage is formed by the right rocker arm, the rocker coupler, the left rocker (pivoted at B) and the car body. This second linkage transmits reciprocating motion from the left rocker arm to the right rocker arm, driving the right windscreen wiper.

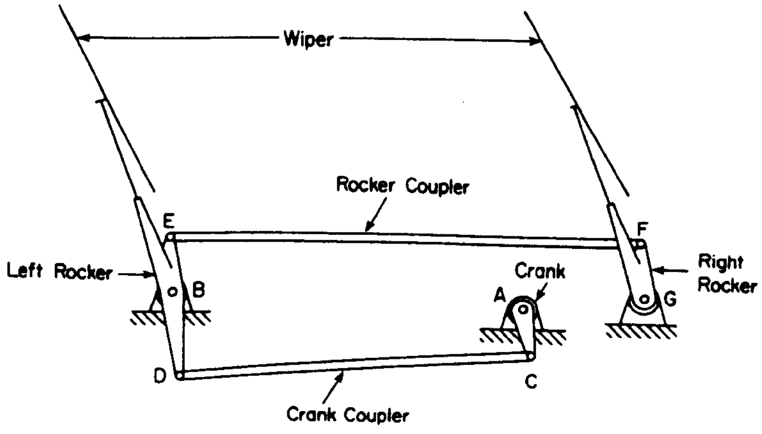
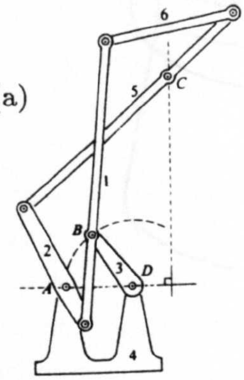



Fig. 3

Planar mechanisms with more than four bars are also used in mechanical engineering. Sometimes in a particular situation it is more convenient to use a six (or more) bar mechanism with one degree of freedom. As an example, consider Hart's "inversor", shown in Fig. 4(a) [H]. This mechanism is a six-bar linkage, drawn in simplified form in Fig. 4(b), which converts rotary motion into true straight line motion.

(a)




(b)

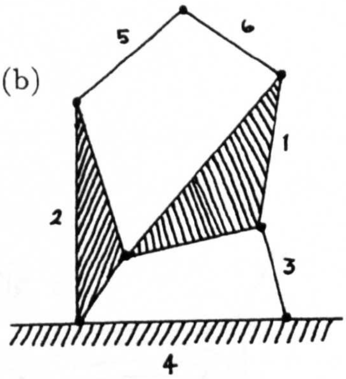



Fig. 4

We can also consider planar mechanisms with two degrees of freedom. In these cases the coupler point will trace out open sets in the plane (coupler sets). The simplest closed kinematic chain with two degrees of freedom is the five-bar mechanism [Hai]. Examples of planar mechanisms with two degrees of freedom are shown in Fig. 5. The first is a seven-link, two-degrees-of-freedom chain and the second is a nine-link two-degrees-of-freedom chain [Hai].


Fig. 5

Spatial rigid body motions are also of great interest, though in the past designers have found them hard to work with as they are not so easily visualized as planar mechansims. With the use of computer graphics this problem can be overcome. The example which comes most immediately to mind is that of the robotic arm. For complete versatility such an arm must have six degrees of freedom. An example is shown in Fig. 6 [Hau].

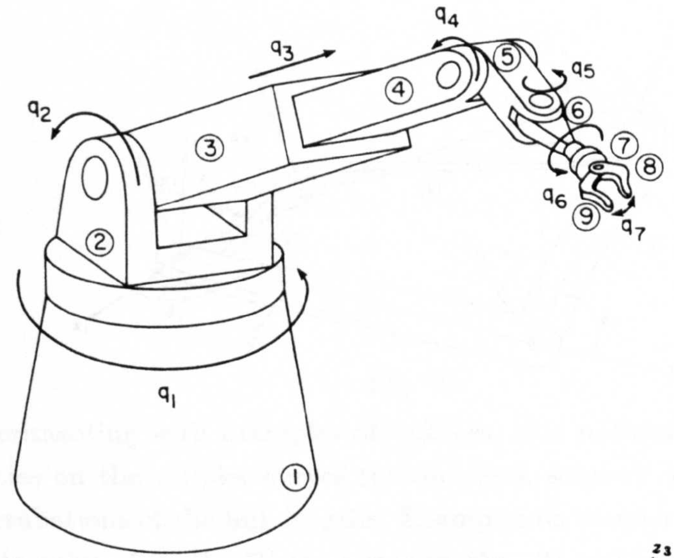



Fig. 6

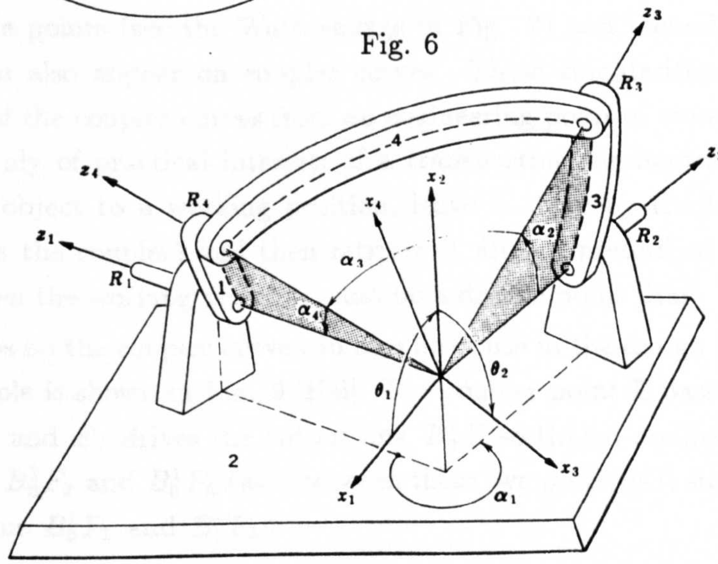



Fig. 7

Another example is the spatial four-bar mechanism, shown in Fig. 7 [Hau]. This has one degree of freedom, like its planar analogue, and so produces a space curve as its coupler point moves around. In fact, it has been proved that most spatial four-bar mechanisms will not move [De]. The only mobile cases are planar four-bars (in 3-space), spherical four-bars (where the joints are on the surface of a sphere, as in Fig. 7) and the Bennett linkage, shown in Fig. 8 [HD]. This linkage has opposite links of equal length and twist, and successive common perpendiculars. However, although it has many interesting mathematical properties (see for example [H], [SE]), it does not appear to have any practical use.

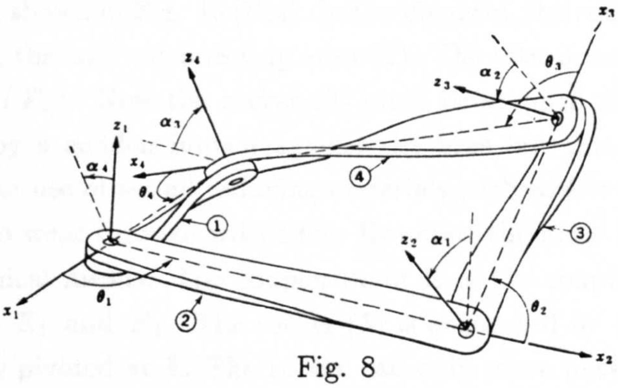


Fig. 8

Experimenting with examples of linkages, it is noticeable that there exist singularities on the coupler curves (or surfaces), some of which remain under small pertubations of the link lengths. Examples on coupler curves in the plane are double points (see the Watt linkage in Fig. 2) and tacnodes (see Fig. 1). Cusps can also appear on coupler curves. These singularities may be useful features of the coupler curves from an engineering point of view. Double points are certainly of practical interest: if a transporting mechanism is required to carry an object to a working position, leave it there by the holder (which is serving as the coupler) and then retrieve it after a prescribed rotation of the crank, then the working position must be a double point [Hai].

Cusps on the coupler curve can also be of use in the design of a mechanism. An example is shown in Fig. 9 [Hai]. The coupler point E passing through the cusps $E_1$ and $E_2$ drives the output link $B_0^1 F$ so that it oscillates between two positions $B_0^1 F_2$ and $B_0^1 F_4$ (also between these two postions it stops momentarily in positions $B_0^1 F_1$ and $B_0^1 F_3$ ).

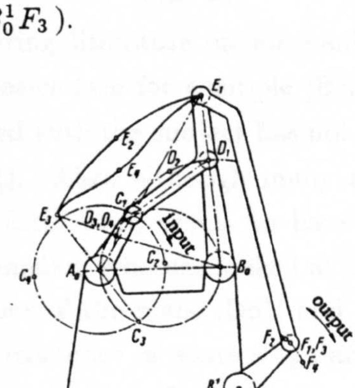



Fig. 9

Another example of the use of cusps is the noise-reducing rocker stop, used in typewriters, shown in Fig. 10 [Hai]. In the diagram, the rocker (3) is activated by the key (1), through a connecting lever (2). The return motion is carried out by the spring ( $F_2$ ). Now the rocker (3) must be stopped at both ends. This can be done by a sudden impact against the base but this produces a harsh noise. Even the use of sound-reducing materials such as felt is not satisfactory as they tend to wear through with time. However, the noise can be reduced by purely mechanical means. The coupler point E of the coupler (5) has a sharp cusp at points $E_1$ and $E_2$. The rocker (3) is connected by 4 with the coupler (5), where 4 is pivoted at E. The rocker can only move between two positions (which correspond to $E_1$ and $E_2$ ). The spring $F_1$, connected by the crank (7), pulls the four-bar linkage $A_0CDB_0$ out of the two positions corresponding to the end positions of the crank and the coupler curve cusps. A loud noise is thus avoided since no sharp impacts occur.

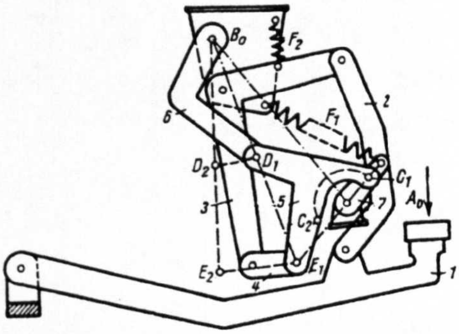



Fig. 10

Considerable engineering literature on mechanisms exists, and indeed on the whole topic of kinematics (see for example [BR], [Hai], [HD], [H]) though the mathematics associated with the subject has not been studied as much until recently (see [GN], [Mar]). Also, although many applications for singularity theory have been found little work seems to have been done to apply it to kinematics (apart from Donelan, who has looked at mono-germs as local models for one-dimensional motions of the plane [Do1] and [Do2]). It is clear from the above examples that the existence of some singularities on the coupler curves and surfaces is already known and exploited (such as the double point and cusp on planar coupler curves). It is possible that other types of singular behaviour, once known to exist, may be of practical value in engineering design and it is the aim of this thesis to find the singularities which may occur in these situations.

It is also interesting to find out what might be seen 'nearby' a singularity, i.e. what we expect to see if we change, very slightly, the parameters that give rise to a particular type. In the language of singularity theory, we want to analyse its *unfolding*, and these analyses are also an aim of this thesis. Here we will study planar and spatial motions with one and two degrees of freedom. Further studies, particularly of spatial motions with higher degrees of freedom, are likely to be of engineering interest.

## 0.2 Formalizing the problem

Let $E(n)$ be the Lie group of rigid motions of $\mathbf{R}^n$ (the semi-direct product of $SO(n)$ and $R^n$). Then consider a map

$$\begin{aligned} \mu : \mathbf{R}^d &\longrightarrow E(n) \\ t &\longmapsto \mu(t) \end{aligned}$$

where $d$ is the number of degrees of freedom of the rigid motion and $\mu(t)$ is itself a rigid motion $\mathbf{R}^n \to \mathbf{R}^n$ (See Fig. 11). Given any point $w \in \mathbf{R}^n$, $\mu(t)$ can be applied to it. Ranging over all $t$ gives the path of $w$ in $\mathbf{R}^n$, or the *trajectory* of $w$. This can be thought of as a map

$$\begin{aligned} \phi_w : \mathbf{R}^d &\longrightarrow \mathbf{R}^n \\ t &\longmapsto (\mu(t))(w) \end{aligned}$$

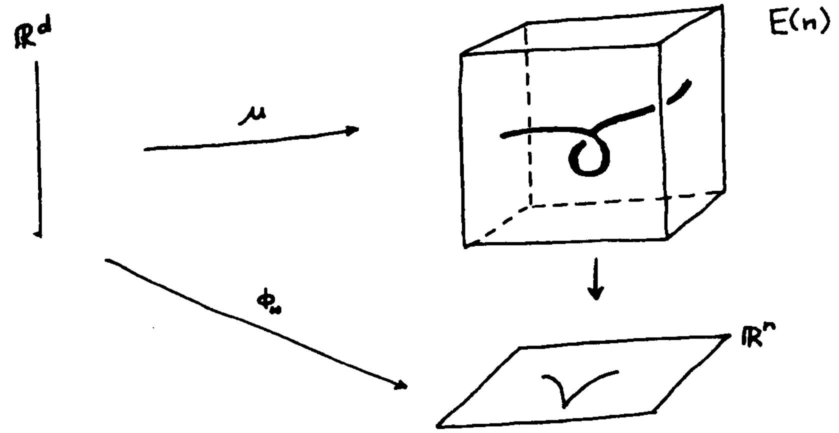



Fig.11

It is the trajectory map $\phi_w : \mathbf{R}^d \to \mathbf{R}^n$ that is to be studied: what singularities does it generically exhibit? We shall call these singularities *kinematic singularities*.

## 0.3 Review of contents

Chapter 1 contains a brief overview of the definitions and results from singularity theory that will be needed in the rest of the thesis, on the whole using the notation of Wall [Wal]. In chapter 2 an important transversality result is proved which allows the original problem to be translated into a finite dimensional one. Further results show that solving the problem is equivalent to classifying singularities of low codimension, up to $\mathcal{A}$-equivalence.

Some of the classifications of singularities that are required to study the problem have already been completed and in these cases the relevant singularities can be extracted straight from these lists and their unfolding spaces can be analysed. However, when such classifications do not exist it is necessary to compute them, in particular the cases of mono-germ singularities $\mathbf{R} \to \mathbf{R}^3$ and various multi-germ singularities. The classification of mono-germs $\mathbf{R} \to \mathbf{R}^3$ up to $\mathcal{A}$-equivalence is described in chapter 3. Multi-germ classifications are carried out as they arise in the next four chapters.

Chapters 4 to 7 contain the investigations of each of the four separate cases that are to be studied here. Firstly, we look at motions with one degree of freedom of the plane (chapter 4) and of space (chapter 5). Secondly, we study rigid motions with two degrees of freedom in the plane (chapter 6) and in space (chapter 7).

(Part of the material in this thesis has appeared in print or in preprints. Chapter 3 appears in shorter form in [GH1], the results of Chapters 2, 4 and 5 correspond to the preprint [GH2] and the multi-germ result in Chapter 6 is included in the preprint [BH].)

# Chapter 1 – Definitions and Techniques

## 1.1 Notation

We will classify singularities of map-germs $f : \mathbf{F}^n, S \to \mathbf{F}^p, 0$ (where $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$ and $S = \{x_1, \ldots, x_r\} \in \mathbf{F}^{rn}$) up to $\mathcal{A}$-equivalence, that is, $f$ is $\mathcal{A}$-equivalent to $g$ ($f \sim g$) if there exist diffeomorphisms $h_1, \ldots, h_r : \mathbf{F}^n, x_i \to \mathbf{F}^n, x_i$ and $k : \mathbf{F}^p, 0 \to \mathbf{F}^p, 0$ such that the following commutes

$$\begin{array}{ccc} \mathbf{F}^n, S & \xrightarrow{f} & \mathbf{F}^p, 0 \\ \downarrow h & & \downarrow k \\ \mathbf{F}^n, S & \xrightarrow{g} & \mathbf{F}^p, 0 \end{array}$$

where $h = h_i$ in a neighbourhood of $x_i$. This equivalence is the natural equivalence for our problem: we want diffeomorphic changes of co-ordinates in the source and target not to affect the classification of the different types of singularity which are possible.

We define $\mathcal{A}$ by

$$\begin{aligned} \mathcal{A} = &\{\text{Diffeomorphism germs } \mathbf{F}^n, S \to \mathbf{F}^n, S\} \\ &\times \{\text{Diffeomorphism germs } \mathbf{F}^p, 0 \to \mathbf{F}^p, 0\} \end{aligned}$$

Write ${}_rj^kf$ for the k-jet of the r-germ $f$ and denote the vector space of k-jets $\mathbf{F}^n, S \to \mathbf{F}^p, 0$ by ${}_rJ^k(n,p)$. This is acted upon smoothly by the Lie group $\mathcal{A}^k$ (the set of k-jets of elements of $\mathcal{A}$), and two k-jets are said to be $\mathcal{A}$-equivalent if they lie in the same $\mathcal{A}^k$-orbit. We also define a subgroup $\mathcal{A}_1$ of $\mathcal{A}$ by

$$\begin{aligned} \mathcal{A}_1 = &\{\text{Diffeomorphism germs } \mathbf{F}^n, S \to \mathbf{F}^n, S \text{ whose } 1-\text{jet is the identity}\} \\ &\times \{\text{Diffeomorphism germs } \mathbf{F}^p, 0 \to \mathbf{F}^p, 0 \text{ whose } 1-\text{jet is the identity}\}. \end{aligned}$$

Define $\mathcal{A}_1^k$ as above. We need to know the tangent spaces to the orbits of the actions of these Lie groups. Following the usual notation, let $\mathcal{E}_n = \mathcal{E}_n(S)$ denote the ring of function germs $\mathbf{F}^n, S \to \mathbf{R}$, and $\mathcal{E}_p$ the ring of function germs $\mathbf{F}^p, 0 \to \mathbf{R}$ and let $m_n$, $m_p$ denote the corresponding maximal ideals. Then let $\mathcal{O}_f$ denote the $\mathcal{E}_n$-module of germs of $C^\infty$ vector fields over $f$ and define

$\mathcal{O}_n = \mathcal{O}_{(1_{\mathbf{F}^n},S)}$, $\mathcal{O}_p = \mathcal{O}_{(1_{\mathbf{F}^p},0)}$. Now we define the following homomorphisms:

$$\begin{aligned} tf : \mathcal{O}_n &\longrightarrow \mathcal{O}_f \\ \phi &\longmapsto df.\phi \end{aligned}$$

$$\begin{aligned} wf : \mathcal{O}_p &\longrightarrow \mathcal{O}_f \\ \psi &\longmapsto \psi \circ f \end{aligned}$$

($df$ is the differential of $f$). Then the tangent space of the $\mathcal{A}^k$-orbit ${}_rj^kf$ in ${}_rJ^k(n,p)$ is given by

$$T\mathcal{A}.f = tf(m_n.\mathcal{O}_n) + wf(m_p.\mathcal{O}_p)$$

and to the $\mathcal{A}_1^k$-orbit:

$$T\mathcal{A}_1.f = tf(m_n^2.\mathcal{O}_n) + wf(m_p^2.\mathcal{O}_p).$$

We also define the 'extended' tangent space:

$$T\mathcal{A}_e.f = tf(\mathcal{O}_n) + wf(\mathcal{O}_p).$$

For mono-germs ($r = 1$) we calculate these tangent spaces by forming the Jacobian ideal of $f$, $\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle$. Then we have

$$T\mathcal{A}.f = m_n\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle + f^*.m_p\langle e_1, \ldots, e_p \rangle$$

(where $e_i = (0,0,\ldots,0,1,0,\ldots,0)$ with 1 in the $i^{th}$ position). Note that $T\mathcal{A}.f$ has a mixed module structure. Similarly,

$$T\mathcal{A}_1.f = m_n^2\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle + f^*m_p^2\langle e_1, \ldots, e_p \rangle$$

and

$$T\mathcal{A}_e.f = \mathcal{E}_n\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle + f^*\mathcal{E}_p\langle e_1, \ldots, e_p \rangle$$

To show how we calculate tangent spaces for multi-germs we introduce the following notation for a multi-germ. Given an $r$-germ $f : \mathbf{F}^n, S \to \mathbf{F}^p, 0$, where $S = \{s_1, \ldots, s_p\}$, we take local co-ordinates at each $s_i$. For ease of calculation we use different letters to represent these co-ordinates, separating them by a semi-colon eg. the tri-germ $f : \mathbf{F}^2, S \to \mathbf{F}^2, 0$, where $S = \{s_1, s_2, s_3\}$, may be written as $f(x,y;X,Y;\tilde{x},\tilde{y}) = (x,y^2;XY,Y^2+X^3;\tilde{x},\tilde{y})$. When we act on $f$ with the Lie group $\mathcal{A}$ we can change co-ordinates, via diffeomorphisms, independently in the source round each $s_i$, but in the target the same vector field must apply to each set of co-ordinates. Thus if we have a multi-germ $f : \mathbf{F}^n, S \to \mathbf{F}^p, 0$ with co-ordinates in $(\mathbf{F}^n, S)$ given by $(x_1, \ldots, x_n; X_1 \ldots, X_n; \ldots; \tilde{x}_1, \ldots, \tilde{x}_n)$ we calculate the tangent spaces as follows:

$$T\mathcal{A}.f = m_n\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle + m_n\langle \partial f/\partial X_1, \ldots, \partial f/\partial X_n\rangle + \cdots$$
$$+ m_n\langle \partial f/\partial \tilde{x}_1, \ldots, \partial f/\partial \tilde{x}_n\rangle + f^*.m_p\langle e_1, \ldots, e_p\rangle.$$

$$T\mathcal{A}_1.f = m_n^2\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle + \cdots + m_n^2\langle \partial f/\partial \tilde{x}_1, \ldots, \partial f/\partial \tilde{x}_n\rangle$$
$$+ f^*.m_p^2\langle e_1, \ldots, e_p\rangle.$$

$$T\mathcal{A}_e.f = \mathcal{E}_n\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle + \cdots + \mathcal{E}_n\langle \partial f/\partial \tilde{x}_1, \ldots, \partial f/\partial \tilde{x}_n\rangle$$
$$+ f^*.\mathcal{E}_p\langle e_1, \ldots, e_p\rangle.$$

## 1.2 Determinacy

We say that $f$ is $k$-$\mathcal{A}$-determined if every map-germ with the same $k$-jet as $f$ is $\mathcal{A}$-equivalent to it (similarly for $\mathcal{A}_1$). A map-germ is $\mathcal{A}$-finite if it is $k$-determined for some $k < \infty$. Then, if we know that a map-germ $f$ is $k$-determined, it is sufficient to work in the $k$-jet space ${}_rJ^k(n,p)$ which is a finite dimensional vector space. The smallest value of $k$ (if it exists) for a given map-germ, is known as the *degree of determinacy* of $f$. We need some criteria for finding this degree of determinacy. From [BdPW] we have the following for mono-germs (where $L(G)$ is the Lie algebra of $G$ – equivalent to the tangent space).

**1.2.1 Theorem** *Let $\mathcal{H}$ be a strongly closed subgroup of $\mathcal{A}$ such that $L(J^1\mathcal{H})$ acts nilpotently on $\mathbf{F}^{n+p}$. Then a $C^r$ map-germ $f : (\mathbf{F}^n, 0) \to (\mathbf{F}^p, 0)$ is $k$-$\mathcal{H}$-determined if and only if*

$$m_n^{k+1}.\mathcal{E}^p \subset L\mathcal{H}.f$$

**1.2.2 Corollary** *A $C^r$ map-germ $f$ is $r$-$\mathcal{A}_1$-determined if and only if*

$$m_n^{r+1}.\mathcal{E}^p \subset T\mathcal{A}_1.f + m_n^{r+1}.(f^*m_p.\mathcal{E}_n + m_n^{r+1})\mathcal{E}_n^p$$

This gives exact values for $\mathcal{A}_1$-determinacy, and good upper estimates for values of $\mathcal{A}$-determinacy. These results generalise in the obvious way to multi-germs.

## 1.3 Codimension

In order to distinguish singularities we need invariants. In particular we use the *codimension* of a map-germ, which is defined in the following way:

### 1.3.1 Definition

(i) *The $\mathcal{A}$-codimension of a map-germ $f$ is given by*

$$\mathcal{A}\text{–codim}(f) = \dim_{\mathbf{F}} \frac{m_n \mathcal{O}_f}{T\mathcal{A}.f}$$

(ii) *The $\mathcal{A}_e$-codimension of $f$ is given by*

$$\mathcal{A}_e\text{–codim}(f) = \dim_{\mathbf{F}} \frac{\mathcal{E}_n \mathcal{O}_f}{T\mathcal{A}_e.f}$$

In fact in our case it is often more useful to calculate the $\mathcal{A}_e$-codimension, but it may be simpler to find the $\mathcal{A}$-codimension. We have the following result which relates the two:

**1.3.2 Theorem** *[Wi] Given an $\mathcal{A}$-finite non-stable map-germ $f : \mathbf{F}^n, S \to \mathbf{F}^p, 0$, where $S = \{s_1, \ldots, s_r\}$ we have*

$$\mathcal{A}_e\text{–codim}(f) = \mathcal{A}\text{–codim}(f) + r(p-n) - p.$$

**Proof** The case $r = 1$ is noted in Wall [Wal]. For a sketch proof, we note that $T\mathcal{A}_e.f$ contains $rn + p$ more things than $T\mathcal{A}.f$, but $\mathcal{E}_n.\mathcal{O}_f$ contains $rp$ more things than $m_n.\mathcal{O}_f$, so

$$\dim_{\mathbf{F}} \frac{\mathcal{E}_n.\mathcal{O}_f}{T\mathcal{A}_e.f} - rp + rn + p = \dim_{\mathbf{F}} \frac{m_n.\mathcal{O}_f}{T\mathcal{A}.f}$$

and the result follows.

More formally, we quote a proof from unpublished notes of Wilson [Wi]. Let $\mathcal{A}_k(f)$ denote the $\mathcal{A}$-orbit of $z = {}_rj^k f(S)$ in ${}_rJ^k(N,P)$ (where $N$ is an $n$-dimensional manifold and $P$ is a $p$-dimensional one). Assume that $f(S) = 0$. We claim that

$$\text{codim}\mathcal{A}_k(f) = \mathcal{A}_e\text{codim}(f) + rn - \dim({}_rj^k f)^{-1}\mathcal{A}_k(f)$$

(where $({}_rj^k f)^{-1}\mathcal{A}_k(f)$ is the manifold of equisingular points of $f$ in $N^{(r)}$).

*Proof of claim.* For simplicity we will give the proof in the case $r = 1$ (it is the same when $r > 1$). We need a result of Mather ([MaIII], Prop. 7.4):

Let $z = j^k f(x)$. We can identify $T_z J^k(N,P)_x$ with $\mathcal{O}_f$. Under this identification,

$$(tf\mathcal{O}_n + wf\mathcal{O}_p)_x = \pi_f T_z \mathcal{A}_k(f)$$

where $\pi_f : T_z J^k(N,P) \to T_z J^k(N,P)_x$ is the projection along $T_z(im\ j^k f)$.

Now we let $x_0$ be our source point, $y_0 = f(x_0)$, $z_0 = j^k f(x_0)$. Let $\alpha_1, \ldots, \alpha_c$ $(c = \text{cod} f)$ project to generators of

$$\left(\frac{\mathcal{O}_f}{tf\mathcal{O}_n + wf\mathcal{O}_p}\right)_{x_0} \cong \frac{T_{z_0} J^k(N,P)}{\pi_f T_{z_0} \mathcal{A}_k(f)}$$

Let $F(x, u_1, \ldots, u_c) = f(x) + \sum u_i.\alpha_i(x)$ ('mini-versal deformation of $f$'). Let $j_1^k F : N \times \mathbf{R}^c \to J^k(N,P)$ denote the map in which we take $k$-jets of $F$ in the $N$ direction only. Thus $j_1^k F \pitchfork \mathcal{A}_k(f)$ at $(x_0, 0)$. It follows that $\text{cod}_k(f) = c + n - \text{dim} M$, where $M = (j_1^k F)^{-1} \mathcal{A}_k(f)$. Now $d(j_1^k F)$ maps $T_{x_0} N \times 0$ into $T_{z_0}(\mathcal{A}_k(f) + \text{im} j^k f)$ and $x_0 \times \mathbf{R}^c$ isomorphically onto a normal space $N_{z_0}(\mathcal{A}_k(f) + \text{im} j^k f)$. It follows that $T_{x_0,0} M \subset T_{x_0} N \times 0$.

At some other point $(x, u) \in M$, $F_u$ is $\mathcal{A}$-equivalent at $x$ to $f$ at $x_0$, so its codimension $c$ is the same. But $j_1^k F \pitchfork \mathcal{A}_k(f)$ holds for $(x,u)$ sufficiently near $(x_0, 0)$ by openness of transversality. Thus we again have that $d(j_1^k F)$ maps $T_x N \times u$ into $T_z(\mathcal{A}_k(F_u) + \text{im} j^k F_u)$ (where $z = j^k F_u(x)$ and $F_u$ is the germ at $x$) and $x \times \mathbf{R}^c$ isomorphically onto $N_z(\mathcal{A}_k(F_u) + \text{im} j^k F_u)$. Thus we again have $T_{(x,u)} M \subset T_x N \times u$. Since $\pi : M \to \mathbf{R}^c$ has rank 0 everywhere, we conclude that $M \subset N \times 0$. Thus $M = (j^k f)^{-1} \mathcal{A}_k(f)$ and the claim is proved.

It remains to note the $\dim({}_r j^k f)^{-1} \mathcal{A}_k(f) = p(r-1)$ to give the desired result. □

## 1.4 Normal Forms and Unfoldings

During the classification process, when we reach a $k$-determined $k$-jet, $j^k f$, this is known as a *normal form* (since by definition all other map-germs with the same $k$-jet are $\mathcal{A}$-equivalent to it). The plan is to find these normal forms and then to distinguish them from each other by finding invariants associated to them.

The next stage is to consider the family of map-germs in the jet space which contains a given map-germ $f$, ie. to see what map-germs are near $f$ in ${}_r J^k(n,p)$. Such a family is called an *unfolding* of $f$. Certain unfoldings contain all functions close to $f$. These are known as *versal unfoldings.*

**1.4.1 Definition** *Given a map-germ* $f : \mathbf{F}^n, S \to \mathbf{F}^p, 0$, *then a map-germ*

$$F : \mathbf{F}^n \times \mathbf{F}^q, S \longrightarrow \mathbf{F}^p \times \mathbf{F}^q, 0$$

*such that* $F(x,0) = (f(x),0)$ *is called a* $q$*-parameter unfolding of* $f$.

*Let* $h : \mathbf{F}^t, 0 \to \mathbf{F}^q, 0$ *be a germ of a diffeomorphism. Then* $F$ *is a versal unfolding of* $f$ *if every unfolding of* $f$ *is induced by* $F$, *ie. it is isomorphic to* $h^*F$ *for some* $h$ *as above.*

If we choose the smallest possible number of unfolding parameters then we have a mini-versal unfolding of $f$. We have the following standard result.

**1.4.2 Lemma** *Given a map-germ* $f : \mathbf{F}^n, S \to \mathbf{F}^p, 0$, *consider a subspace* $T$ *of* ${}_rJ^k(n,p)$ *spanned by elements* $f_1, \ldots, f_q \in {}_rJ^k(n,p)$ *such that*

$${}_rJ^k(T\mathcal{A}_e.f) + T = {}_rJ^k(n,p)$$

*Then*

$$F : \mathbf{F}^n \times \mathbf{F}^q \longrightarrow \mathbf{F}^p$$

$$(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_q) \longmapsto f(x_1, \ldots, x_n) + \sum_{i=1}^{q} \lambda_i f_i(x_1, \ldots, x_n)$$

*is a versal unfolding of* $f$.

So in order to find a versal unfolding for a map-germ $f$ we find $T\mathcal{A}_e.f$ and then look for a suitable set $\{f_1, \ldots, f_q\}$ to make up the whole jet space, ${}_rJ^k(n,p)$. Clearly, $q = \mathcal{A}_e\text{-codim}(f)$ . We can then study the space of parameters $(\lambda_1, \ldots, \lambda_q)$, the *unfolding space.*

## 1.5 Complete Transversals

In some cases the appropriate $\mathcal{A}$-classifications which we need to use have already been carried out (see [B2], [dPT], [Ri2], [Md2]) but, particularly in the case of multi-germs, this is not always so and we will need to do some classification of singularities here. The method which we shall use was first developed by Gibson & Dimca [GD] for contact-equivalence. Bruce & duPlessis [BdP] have subsequently shown that the same method works for $\mathcal{A}_1$-equivalence. This gives a finer classification than $\mathcal{A}$-classification, which is what we would like to do, so when the $\mathcal{A}_1$-classification has been carried out we have to check whether different $\mathcal{A}_1$-normal forms are actually $\mathcal{A}$-equivalent.

The method is an inductive one: given the $k$-jet of a map germ $f$ we look for the $(k+1)$-jets which have that $k$-jet. This could be done by listing all possibilities, but the complete transversal method allows us to cut down on the number of such possibilities dramatically in many cases. When we reach a $k$-determined $k$-jet we stop.

The basis of this method is the following :

**1.5.1 Theorem** [BdP] *Let $f$ be a $k$-jet in ${}_rJ^k(n,p)$ and let $T$ be a vector subspace of the space of map-germs of homogeneous mappings $\mathbf{F}^n, S \to \mathbf{F}^p, 0$ of degree $k+1$, $H^{k+1}(n,p)_r$ (where $S = \{s_1, \dots, s_r\}$). If*

$$ {}_rJ^{k+1}(T\mathcal{A}_1.f) \cap H^{k+1}(n,p)_r + T = H^{k+1}(n,p)_r $$

*then any $(k+1)$-jet $f$ is $\mathcal{A}_1$-equivalent to $f+t$ for some $t \in T$.*

In order to reduce the resulting $\mathcal{A}_1$-classification to an $\mathcal{A}$-classification, we use the following result due to Mather:

**1.5.2 Theorem** (Mather's Lemma) [MaIII]

*Let $G$ be a Lie group acting smoothly on a manifold $U$, and let $V$ be a connected submanifold of $U$. Then $V$ is contained in a single orbit of $G$ if and only if*

(i) *for all $v \in V$, $T_vV \subseteq T_v(G.v)$ and*

(ii) *dim $T_v(G.v)$ is independent of the choice of $v \in V$.*

We use these results extensively in Chapter 3 to classify mono-germs $\mathbf{C}, 0 \to \mathbf{C}^3, 0$ and also in Chapters 4 – 7 to classify multi-germs of various types.

# Chapter 2 – Transversality

## 2.1 Introduction

We want to study one parameter rigid motions of $\mathbf{R}^2$ and $\mathbf{R}^3$ to see which singularities can generically occur on the trajectories of such motions. First we give some definitions (taken from [GG]).

**2.1.1 Definition** *Let $X$ and $Y$ be smooth manifolds and $f : X \to Y$ be a smooth mapping. Let $W$ be a submanifold of $Y$ and $x$ a point in $X$. Then $f$ intersects $W$* transversally *at $x$ (denoted by $f \pitchfork W$ at $x$) if either*

(a) $f(x) \notin W$, *or*

(b) $f(x) \in W$ *and* $T_{f(x)}Y = T_{f(x)}W + (df)_x(T_xX)$. *If $A$ is a subset of $X$ then $f$ intersects $W$ transversally on $A$ (denoted by $f \pitchfork W$ on $A$) if $f \pitchfork W$ at $x$ for all $x \in A$. Finally, $f$ intersects $W$ transversally (denoted by $f \pitchfork W$) if $f \pitchfork W$ on $X$.*

**2.1.2 Definition** *Let $X$ and $Y$ be smooth manifolds as before.*

(i) *Denote by $C^\infty(X,Y)$ the set of smooth mappings from $X$ to $Y$.*

(ii) *Fix a non-negative integer $k$. Let $U$ be a subset of $J^k(X,Y)$. Then denote by $M(U)$ the set*

$$\{f \in C^\infty(X,Y) : \ j^k f \subset U\}$$

*Note that $M(U) \cap M(V) = M(U \cap V)$.*

(iii) *The family of sets $\{M(U)\}$ where $U$ is an open subset of $J^k(X,Y)$ form a basis for a topology on $C^\infty(X,Y)$. This topology is called the* Whitney $C^k$ topology. *Denote by $W_k$ the set of open subsets of $C^\infty(X,Y)$ in the Whitney $C^k$ topology.*

(iv) *The* Whitney $C^\infty$ topology *on $C^\infty(X,Y)$ is the topology whose basis is*

$$W = \cup_{k=0}^{\infty} W_k.$$

Now let $E(p)$ denote the Lie group of rigid motions of $\mathbf{R}^p$. We can identify this with the semi-direct product $SO(p) \times \mathbf{R}^p$, i.e. any element of $E(p)$ is the product of a rotation and a translation. If we have a map $\mu : N \to E(p)$, where $N$ is a smooth manifold of dimension $n$, then $\mu(t)$ is a map $\mathbf{R}^p \to \mathbf{R}^p$ defined by

$$\mu(t)(\omega) = \rho(t)\omega + \tau(t)$$

(where $\rho(t) \in SO(p)$ is a rotation and $\tau(t) \in \mathbf{R}^p$ is a translation.) We can then find the trajectory of a given point $\omega \in \mathbf{R}^p$ under $\mu(t)$,

$$\begin{aligned} \Phi_{\mu,\omega} : N &\longrightarrow \mathbf{R}^p \\ t &\longmapsto \mu(t)(\omega) = \rho(t)\omega + \tau(t) \end{aligned}$$

In section 2.2 we will state and prove a general theorem on transversality which shows that the problem of classifying the singularities of such trajectories is equivalent to stratifying jet-spaces into $\mathcal{A}$-orbits. We then prove some general results on the codimensional restrictions imposed by the given problem of classifying singularities of rigid motions.

## 2.2 Transversality result

The trajectory map $\Phi_{\mu,\omega}$ induces on $k$-jet space the map

$$j^k\Phi_{\mu,\omega} : N \longrightarrow J^k(N, \mathbf{R}^p)$$

Similarly, on the multijet space we have

$$_rj^k\Phi_{\mu,\omega} : N^{(r)} \longrightarrow {}_rJ^k(N, \mathbf{R}^p)$$

Since $\Phi_{\mu,\omega}$ depends smoothly on $\omega$ we can define the map

$$_rj^k\Phi_{\mu} : N^{(r)} \times \mathbf{R}^p \longrightarrow {}_rJ^k(N, \mathbf{R}^p)$$

where, for each point $\omega \in \mathbf{R}^p$ we take $_rj^k\Phi_{\mu,\omega}$.

The following transversality result is based on that of [SD], and the proof modelled on Wall's proof of the same result in [Wa4].

**2.2.1 Theorem** *For any submanifold $S \subseteq {}_rJ^k(N, \mathbf{R}^p)$, the set of curves $\mu : N \to E(p)$ such that $_rj^k\Phi_\mu \pitchfork S$ is open and dense (and so residual) in $C^\infty(N, E(p))$, endowed with the Whitney topology.*

**Proof** We need the following lemma:

**2.2.2 Lemma** *Let $Q$ be any submanifold of the manifold $P$. Then the set $\{f \in C^\infty(N,P) | f \pitchfork Q\}$ is dense in $C^\infty(N,P)$, provided we can always embed $f$ in a family $F : N \times U \to P$ which is a submersion.*

**Proof** Let $F : N \times U \to P$ be a submersion and $Q \subset P$ a submanifold. Then $F^{-1}(Q)$ is a submanifold of $N \times U$, and for each $u \in U$, $f_u = F|_{N\times\{u\}}$ is transverse to $Q$ if and only if $N \times \{u\}$ meets $F^{-1}(Q)$ transversely.

Now changing $F$ for the projection $\pi : N \times U \to U$ in the above we see that equivalently we must have $\pi|F^{-1}(Q)$ transverse to $\{u\}$ (considered as a submanifold of $U$), ie. $u$ needs to be a regular value of $\pi|F^{-1}(Q)$. But by Sard's Theorem, this holds for all $u$ except those in a set of zero measure. □

It follows from this that in order to show density we must embed $\mu$ in a suitable family $\Psi : N \times W \to E(p)$ such that

$$_rj^k\Phi_\Psi : N^{(r)} \times \mathbf{R}^p \times W \longrightarrow {}_rJ^k(N, \mathbf{R}^p)$$

is a submersion.

In fact, in order to show openness, this induced map must be a submersion when restricted to a chosen compact $K = K_1 \times \ldots \times K_r$ in $N^{(r)}$ (where $K_\rho$ is a compact subset of $N$ for $1 \le \rho \le r$). This is follows from:

**2.2.3 Lemma** *If $K$ is a compact subset of $N^{(r)}$ of the form $K_1 \times \ldots \times K_r$ (with $K_\rho$ a compact subset of $N$, $1 \le \rho \le r$) and $S$ is a closed submanifold of $_rJ^k(N,\mathbf{R}^p)$ then the set $\{\mu \in C^\infty(N, E(p)) : {}_rj^k\Phi_\mu|_{K\times\mathbf{R}^p} \pitchfork S\}$ is open in $C^\infty(N, E(p))$.*

**Proof** Write $T_S = \{\mu \in C^\infty(N,E(p)) : {}_rj^k\Phi_\mu|_{K\times\mathbf{R}^p} \pitchfork S\}$. We want to show that $T_S$ is open in $C^\infty(N,E(p))$. We note that if the condition $_rj^k\Phi_\mu \pitchfork S$ defines an open set then the condition $_rj^k\Phi_\mu|_{K\times\mathbf{R}^p} \pitchfork S$ certainly does as well. Recall that we have a map

$$_rj^k\Phi_\mu : N^{(r)} \times \mathbf{R}^p \longrightarrow {}_rJ^k(N, \mathbf{R}^p)$$

Now define

$$\theta : C^\infty(N, E(p)) \longrightarrow C^\infty(N^{(r)} \times \mathbf{R}^p, {}_rJ^k(N, \mathbf{R}^p))$$
$$\mu \longmapsto {}_rj^k\Phi_\mu$$

This is a continuous map (by definition of $\Phi_\mu$ and by properties of the Whitney $C^\infty$ topology).

The set $T = \{f \in C^\infty(N^{(r)} \times \mathbf{R}^p, {}_rJ^k(N, \mathbf{R}^p)) \ : \ f \pitchfork S\}$ is an open subset of $C^\infty(N^{(r)} \times \mathbf{R}^p, {}_rJ^k(N, \mathbf{R}^p))$, since $S$ is a closed submanifold of ${}_rJ^k(N, \mathbf{R}^p)$. So $\theta^{-1}(T)$ is an open subset of $C^\infty(N, E(p))$, $\theta$ being continuous. But $\theta^{-1}(T) = T_S$, so $T_S$ is open. □

Now we want to show that openness holds if $S$ is any submanifold of ${}_rJ^k(N, \mathbf{R}^p)$, not just a closed submanifold. Since $N$ is paracompact, we can write $N^{(r)} = \bigcup_i C_i$, a countable union of sets $C_i$ which are compact and of the form $K_1 \times \ldots \times K_r$, with $C_i^\circ$ open submanifolds of $N^{(r)}$. We can also write $S$ as a countable union $S = \bigcup_j S_j$ where the $S_j$ are compact (and hence closed) and the $S_j^\circ$ are open submanifolds of $S$.

Write

$$T_{i,j} = \{\mu \in C^\infty(N, E(p)) \ : \ {}_rj^k\Phi_\mu|_{C_i \times \mathbf{R}^p} \pitchfork S_j\}$$

Clearly, $T_S = \bigcap_i \bigcap_j T_{i,j}$, a countable intersection of open sets, and so, by Lemma 2.2.3, $T_S$ is open for any submanifold $S \subseteq {}_rJ^k(N, \mathbf{R}^p)$.

So we are looking at sets $K_1 \times \ldots \times K_r$ in $N^{(r)}$. Since we can piece together independent deformations of $\mu$ on $K_1 \times \ldots \times K_r$ using partitions of unity, and since the space of multijets is just a product, it is sufficient to find a family of maps $\Gamma : N \times A \to E(p)$ such that

$$j^k\Phi_\Gamma : N \times A \times \mathbf{R}^p \to J^k(N, \mathbf{R}^p)$$

is a submersion.

But here we are finding a suitable submersion by allowing the point $\omega \in \mathbf{R}^p$ to alter as well as by deforming $\mu$. It is sufficient to consider

$$j : N \times A \times \mathbf{R}^p \to J^k(N, \mathbf{R}^p) \times \mathbf{R}^p$$

$$j(t, a, \omega) = \big(j^k\Phi_{\Gamma_a}(\omega)(t), \omega\big)$$

(where $j^k\Phi_{\Gamma_a}(\omega)(t)$ is the $k$-jet obtained at a fixed $a \in A$ and $\omega \in \mathbf{R}^p$) since the map

$$\pi_1 : J^k(N, \mathbf{R}^p) \times \mathbf{R}^p \longrightarrow J^k(N, \mathbf{R}^p)$$

is a submersion and $j^k\Phi_\Gamma = \pi_1 \circ j$. Then we can concentrate on deforming $\mu$ so that we can get a submersion at any point $\omega \in \mathbf{R}^p$.

Given that $j$ is a submersion, the product

$$N^{(r)} \times A^r \times (\mathbf{R}^p)^r \longrightarrow {}_rJ^k(N, \mathbf{R}^p) \times (\mathbf{R}^p)^r$$

will be one as well, and we can restrict to $K \subset N^{(r)}$, identify $A^r$ with $W$ and restrict $(\mathbf{R}^p)^r$ to the diagonal.

Consider the vector space of translations of $\mathbf{R}^p$, $V$. By the proof of Thom's Theorem [Wa4], we can find a family of maps

$$\Gamma_0 : N \times A \longrightarrow V$$

such that $j^k\Gamma_0 : N \times A \to J^k(N, V)$ is a submersion. Here, $A = Aff(L, V)$ where $N \subset L$ $k$th order non-degenerate. Since V can be identified with $\mathbf{R}^p$, we then have a submersion onto $J^k(N, \mathbf{R}^p)$.

Now define $\Gamma$ by

$$\Gamma(t, a) = \Gamma_0(t, a) \circ \mu(t).$$

So we have

$$\begin{aligned}\Phi_\Gamma(t, a, \omega) &= \Gamma(t, a)(\omega) \\ &= (\Gamma_0(t, a) \circ \mu(t))(\omega) \\ &= \mu(t)(\omega) + \Gamma_0(t, a)\end{aligned}$$

Fixing $t$ and $\omega$ but allowing $a$ to vary, we get something of the form $\Gamma_0(t, a) + \alpha$, with $\alpha$ constant, and so we have a submersion of $A$ onto $J^k_t(N, \mathbf{R}^p)$ and thus the map

$$j : N \times A \times \mathbf{R}^p \longrightarrow J^k(N, \mathbf{R}^p) \times \mathbf{R}^p$$

is a submersion and the theorem is proved.

□

What the theorem tells us that is that if we find a smooth submanifold of ${}_rJ^k(N, \mathbf{R}^p)$ then generically for a given $\mu : N \to E(p)$ the mapping ${}_rj^k\Phi_\mu$ will be transverse to it. In particular, if we stratify ${}_rJ^k(N, \mathbf{R}^p)$ into $\mathcal{A}$-orbits (which are necessarily smooth submanifolds), we get an induced stratification of the domain via $({}_rj^k\Phi_\mu)^{-1}$. Thus if we want to classify the singularity types which occur on the trajectories $\Phi_\mu$ up to $\mathcal{A}$-equivalence it is enough to stratify ${}_rJ^k(N, \mathbf{R}^p)$ into $\mathcal{A}$-orbits. The preimage of each stratum under this map will either be empty or a smooth submanifold of the domain, of the same codimension. This means that we have a restriction on the size of the codimension, and in fact general results can be proved on this matter as follows.

Let smooth manifolds $N, P, Q$ have dimensions $n, p, q$ respectively. Consider a smooth family of maps $f_w : N \to P$, where $w \in Q$, thought of as a single smooth map $F : Q \times N \to P$. Now, for each parameter value $w$ and positive integer $k$ (order of the jet) we have a $r$-multi-jet extension

$${}_rj^kf_w : N^{(r)} \longrightarrow {}_rJ^k(N, P)$$

giving a map

$$_r j^k F : Q \times N^{(r)} \longrightarrow {}_r J^k(N, P)$$

**2.2.4 Proposition** *Given an $\mathcal{A}$-invariant submanifold $X$ of $_rJ^k(n,p)$, giving rise to another $\mathcal{A}$-invariant submanifold $Y$ in $_rJ^k(N,P)$, let $X$ have $\mathcal{A}$-modality $m$. If $_rj^kF$ is transverse to $Y$ then for any $w \in Q$ with $_rj^kf_w \in Y$ the $\mathcal{A}_e$-codimension of $f_w$ is less than or equal to $q+m$.*

**Proof** First we show that in this situation we have $\text{codim}(X) \le p+q+r(n-p)$. $_rJ^k(N,P)$ is the total space of a fibre bundle with fibre $_rJ^k(n,p)$ and base-space an open subset of $(N \times P)^r$. Thus the base-space has dimension $r(p+n)$. $Y$ lies over the subset of the base-space defined by the condition that the $r$ targets coincide: this subset has dimension $rn+p$, and hence codimension $r(p+n)-(rn+p)=(r-1)p$ in the base-space. So, by local triviality,

$$\text{codim of } Y \text{ in } {}_rJ^k(N,P) = \text{codim of } X \text{ in } {}_rJ^k(n,p) + (r-1)p$$

By the transversality hypothesis, the codimension of $Y$ cannot exceed the dimension $q+rn$ of the domain, so we do have $\text{codim}(X) \le p+q+r(n-p)$.

To prove the result, note that from Theorem 1.3.2 we know that for any non-$\mathcal{A}$-stable $f : N \to P$ the following is true:

$$\mathcal{A}_e\text{-codim}(f) = \text{codim of orbit in } {}_rJ^k(n,p) + r(p-n) - p$$

From the above we have

$$\begin{aligned}\text{codim of orbit in } {}_rJ^k(n,p) &= \text{codim of orbit in } X + \text{codim of } X \text{ in } {}_rJ^k(n,p)\\ &\le m+p+q+r(n-p)\end{aligned}$$

So $\mathcal{A}_e$-codim $(f) \le m+p+q+r(n-p)+r(p-n)-p = m+q$. □

Thus if we are considering motions of the plane we have $q=2$ and so we are looking for strata in the jet space with 'codimension' less than 3. If the stratum is an $\mathcal{A}$-simple one then $m=0$ and we require the $\mathcal{A}_e$-codimension to be less than 3, but we have to be more careful when looking at families with moduli. Similarly, for motions of the plane we require the 'codimension' of the strata to be less than 4.

# Chapter 3 – Simple Singularities of Space Curves

## 3.1 Introduction

When considering one dimensional motions of space, what we need to look at are singularities $\mathbf{R}, 0 \to \mathbf{R}^3$ up to $\mathcal{A}$-equivalence. The classification of such singularities had not been carried out, so it seemed natural to begin this classification in order to at least find those singularities with $\mathcal{A}_e$-codimension less than 3. In fact, here we classify all $\mathcal{A}$-simple singularities of space curves (which takes us up to $\mathcal{A}_e$-codimension 11) and give invariants to distinguish the singularities. It turns out that this classification was also carried out, independently, by Farid Tari and Andrew duPlessis [dPT] as part of a wider programme of $\mathcal{A}$-classification.

The classification was done by using the method of complete transversals, as described in section 1.5 of chapter 1. The results are summarised in the two theorems below. The precise definitions of the terms *planar* and *spatial* will be given in section 3.5 of this chapter.

**3.1.1 Theorem** *Every planar $\mathcal{A}$-simple germ of an analytic map $f$ from $\mathbf{C}$ to $\mathbf{C}^3$ is $\mathcal{A}$-equivalent to one of the following normal forms (where $G(f)$ is the set of generators of the semigroup of $f$, $d(f)$ is the degree of $\mathcal{A}_1$-determinacy of $f$ and $c(f)$ is the $\mathcal{A}_e$-codimension):*

| *Normal Form* | | $G(f)$ | $d(f)$ | $c(f)$ |
|---|---|---|---|---|
| $(t,0,0)$ | | $1$ | $1$ | $0$ |
| $(t^2,t^{2m+1},0)$ | | $2,2m+1$ | $2m+1$ | $2m$ |
| $(t^3,t^{3m+1},0)$ | | $3,3m+1$ | $6m-1$ | $6m$ |
| $(t^3,t^{3m+2},0)$ | | $3,3m+2$ | $6m+1$ | $6m+2$ |
| $(t^3,t^{3m+1}+t^{3n+2},0)$ | $m \le n < 2m$ | $3,3m+1$ | $6m-1$ | $4m+n+1$ |
| $(t^3,t^{3m+1}+t^{3n+2},0)$ | $n < m \le 2n$ | $3,3n+2$ | $6n+1$ | $4n+m+2$ |
| $(t^4,t^5,0)$ | | $4,5$ | $11$ | $12$ |
| $(t^4,t^5+t^7,0)$ | | $4,5$ | $11$ | $11$ |
| $(t^4,t^6+t^{2m+1},0)$ | $m \ge 3$ | $4,6,2m+7$ | $2m+9$ | $2m+8$ |

| *Normal Form* | $G(f)$ | $d(f)$ | $c(f)$ |
|---|---|---|---|
| $(t^4, t^7, 0)$ | $4, 7$ | 17 | 18 |
| $(t^4, t^7 + t^9, 0)$ | $4, 7$ | 17 | 16 |
| $(t^4, t^7 + t^{13}, 0)$ | $4, 7$ | 17 | 17 |

**3.1.2 Theorem** *Every spatial $\mathcal{A}$-simple germ of an analytic map from* **C** *to* $\mathbf{C}^3$ *is $\mathcal{A}$-equivalent to one of the following normal forms:*

| *Normal Form* | | $G(f)$ | $d(f)$ | $c(f)$ |
|---|---|---|---|---|
| $(t^3, t^{3m+1}, t^{3n+2})$ | $m \leq n < 2m$ | $3, 3m+1, 3n+2$ | $3n+2$ | $2m+2n+1$ |
| $(t^3, t^{3m+1}, t^{3n+2})$ | $n < m \leq 2n$ | $3, 3n+2, 3m+1$ | $3m+1$ | $2m+2n+1$ |
| $(t^3, t^{3m+1} + t^{3n+2}, t^{3l+2})$ | $m \leq n < l < 2m$ | $3, 3m+1, 3l+2$ | $3l+2$ | $2m+n+l+1$ |
| $(t^3, t^{3m+2} + t^{3n+1}, t^{3l+1})$ | $m < n < l \leq 2m$ | $3, 3m+2, 3l+1$ | $3l+1$ | $2m+n+l+1$ |
| $(t^4, t^5, t^6)$ | | $4, 5, 6$ | 7 | 8 |
| $(t^4, t^5, t^7)$ | | $4, 5, 7$ | 7 | 9 |
| $(t^4, t^5, t^{11})$ | | $4, 5, 11$ | 11 | 11 |
| $(t^4, t^5 + t^7, t^{11})$ | | $4, 5, 11$ | 11 | 10 |
| $(t^4, t^6, t^{2m+1})$ | $3 \leq m$ | $4, 6, 2m+1$ | $2m+3$ | $2m+4$ |
| $(t^4, t^6 + t^{2m+1}, t^{2m+3})$ | $3 \leq m$ | $4, 6, 2m+3$ | $2m+5$ | $2m+5$ |
| $(t^4, t^6 + t^{2m+1}, t^{2m+5})$ | $3 \leq m$ | $4, 6, 2m+5, 2m+7$ | $2m+5$ | $2m+6$ |
| $(t^4, t^6 + t^{2m+1}, t^{2m+9})$ | $3 \leq m$ | $4, 6, 2m+7, 2m+9$ | $2m+9$ | $2m+7$ |
| $(t^4, t^7, t^9)$ | | $4, 7, 9$ | 10 | 13 |
| $(t^4, t^7, t^9 + t^{10})$ | | $4, 7, 9$ | 10 | 12 |
| $(t^4, t^7, t^{10})$ | | $4, 7, 10$ | 13 | 14 |
| $(t^4, t^7 + t^9, t^{10})$ | | $4, 7, 10$ | 13 | 13 |
| $(t^4, t^7, t^{13})$ | | $4, 7, 13$ | 13 | 15 |
| $(t^4, t^7 + t^9, t^{13})$ | | $4, 7, 13$ | 13 | 14 |
| $(t^4, t^7, t^{17})$ | | $4, 7, 17$ | 17 | 17 |
| $(t^4, t^7 + t^9, t^{17})$ | | $4, 7, 17$ | 17 | 15 |
| $(t^4, t^7 + t^{13}, t^{17})$ | | $4, 7, 17$ | 17 | 16 |

### 3.2 Invariant Semigroups

We can associate a value semigroup to the germ of a curve in the following way. There is a natural valuation

$$\begin{aligned} \mathrm{ord} : \mathcal{E}_1 &\longrightarrow \mathbf{Z} \cup \{\infty\} \\ \phi &\longmapsto \mathrm{ord}(\phi) \end{aligned}$$

where $\mathrm{ord}(\phi)$ is the order of the power series expansion of $\phi$ at 0. Now ord is a homomorphism of semigroups, where we take the multiplicative structure on $\mathcal{E}_1$ and the additive one on $\mathbf{Z}$. So to any subring $A \subseteq \mathcal{E}_1$ we can associate a semigroup $\mathrm{ord}(A) \subseteq \mathbf{Z}$. Then we can define the semigroup of a map-germ:

**3.2.1 Definition** *Given a map-germ $f : \mathbf{C} \to \mathbf{C}^3$, and an integer $k \geq 0$, then $f^*(m_3^k) \subseteq \mathcal{E}_1$ is a subalgebra to which we associate the semigroup of integers $S_k(f) = \mathrm{ord}\ f^*(m_3^k)$.*

Immediately we have

**3.2.2 Proposition** *$S_k(f)$ is $\mathcal{A}$-invariant.*

**Proof** Given $f \in J^k(1,3)$, we can use diffeomorphisms in the source and target to change $f$ into an $\mathcal{A}$-equivalent map-germ.

Given a diffeomorphism in the source, $\phi : \mathbf{C} \to \mathbf{C}$, and a polynomial $g : \mathbf{C}^3 \to \mathbf{C}$ we compare the order of $g \circ (f \circ \phi)$ with that of $g \circ f$. We have

$$\begin{aligned} \mathrm{ord}(g \circ (f \circ \phi)) &= \mathrm{ord}((g \circ f) \circ \phi) \\ &= \mathrm{ord}(g \circ f) \cdot \mathrm{ord}(\phi) \end{aligned}$$

(since both $g \circ f$ and $\phi$ are functions of one variable)

Now $\mathrm{ord}(\phi) = 1$ since $\phi$ is a diffeomorphism, so $\mathrm{ord}(g \circ (f \circ \phi)) = \mathrm{ord}(g \circ f)$. Thus the semigroup is $\mathcal{R}$-invariant.

To show that $S_k(f)$ is $\mathcal{L}$-invariant, if we replace $f$ by $h \circ f$, where $h : \mathbf{C}^3, 0 \to \mathbf{C}^3, 0$ is a diffeomorphism, then we have

$$\begin{aligned} (h \circ f)^*(m_3^k) &= h^*(f^*(m_3^k)) \\ &= f^*(m_3^k) \end{aligned}$$

So $S_k(f)$ is $\mathcal{A}$-invariant. □

We now make the following definition:

**3.2.3 Definition** *A semigroup $S \subseteq \mathbf{Z}$ is cofinite when there exists an integer $N \geq 1$ such that every integer $n > N$ is in $S$.*

**3.2.4 Proposition** *Let $f \in J^k(1,3)$ and let $k \geq 0$. Then $f$ is $\mathcal{A}$-finite if and only if $S_k(f)$ is cofinite.*

**Proof** By [Wal] we know that $f$ is $\mathcal{A}$-finite if and only if it is $\mathcal{L}$-finite in this case since $p \geq 2n$, and by [BdPW] we know that a germ is $\mathcal{L}$-finite if and only if it is $\mathcal{L}_k$-finite. Now $f$ is $\mathcal{L}_k$-finite if and only if

$$m_1^{N_k+1}.\mathcal{O}_3 \subseteq f^*(m_3^k)$$

for some $N_k \geq 1$ (from [BGdP]). But $m_1^{N_k+1}$ is generated by $t^{N_k+1}$ and so the proposition is proved. □

Given an $\mathcal{A}$-finite map germ $f$ we would like to know the exact degree of $\mathcal{A}_1$-determinacy, $k$, or at least a maximum value for the degree of $\mathcal{A}_1$-determinacy, so that we can then work in the jet space $J^k(1,3)$ and find normal forms for $f$.

**3.2.5 Proposition** *Let $f \in J^k(1,3)$ be $\mathcal{A}$-finite. Then it is $N_k$-$\mathcal{L}_{k-1}$-determined, where $N_k$ is the maximum number not in $S_k(f)$.*

**Proof** Since $f$ is $\mathcal{A}$-finite, its semigroup $S_k(f)$ must be cofinite and so $N_k$ exists. Now $N_k$ is the least integer $N$ for which

$$m_1^{N+1}.\mathcal{O}_3 \subseteq f^*(m_3^k) \qquad (*)$$

and by [BGdP], $f$ is $N$-$\mathcal{L}_{k-1}$-determined *if and only if* $(*)$ *holds.* □

So, given an $\mathcal{A}$-finite singular germ $f \in J^k(1,3)$ we have a nested sequence of cofinite invariant semigroups

$$S_0(f) \supseteq S_1(f) \supseteq S_2(f) \supseteq \cdots$$

and hence an increasing sequence of integers

$$1 \leq N_0 \leq N_1 \leq N_2 \leq \cdots$$

Now $N_2$ is the degree of $\mathcal{L}_1$-determinacy and also the degree of $\mathcal{L}$-determinacy (by [BGdP]). In particular, $N_2$ is an upper bound for the degree of $\mathcal{A}_1$-determinacy.

A semigroup $S \subset \mathbf{Z}$ can be characterized by its generators and so in Theorems 3.1.1 and 3.1.2 we list the generators of $S_1(f)$ ( in column labelled $G(f)$).

There are some further invariants of singularities $\mathbf{C},0 \to \mathbf{C}^3,0$ which we can look at. First we need the following Lemma (a slightly changed version of a result from [B2]: the proof is a variation on that of the original, suggested by J.W.Bruce).

**3.2.6 Lemma** *A map-germ $f : \mathbf{C} \to \mathbf{C}^3$ is $\mathcal{A}$-finite if and only if it is irreducible, as a parametrization of an algebraic curve.*

**Proof** It is a standard result in curve theory that by a change of co-ordinates in the source one can write $f$ as

$$f(t) = (t^q, \sum_{j=1}^{\infty} a_j t^{m_j}, \sum_{j=1}^{\infty} b_j t^{n_j})$$

and that the germ is irreducible if and only if the highest common factor $h$ of the $m_j$'s, $n_j$'s and $q$ is 1. It suffices therefore to show that $\mathcal{A}$-finiteness is equivalent to this arithmetical condition.

It is shown in [Wal] that, for $p \geq 2n$, a singular holomorphic germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ is $\mathcal{A}$-finite if and only if 0 in $\mathbf{C}^n$ is an isolated singularity of $f$, and the restriction of $f$ to some neighbourhood of 0 in $\mathbf{C}^n$ is injective. Here $n = 1, p = 3$ so certainly $p \geq 2n$. Also, it is clear that any germ of the above form does have an isolated singularity at $0 \in \mathbf{C}$. Thus it remains to check that $f$ is injective on some neighbourhood of 0 if and only if $h = 1$.

Suppose first that $h > 1$. Then we can write $f(t) = (x(t^h), y(t^h), z(t^h))$ for appropriate holomorphic germs $x$, $y$, $z$. In any neighbourhood of $0 \in \mathbf{C}$ we can choose $s,t$ with $s \neq t$ and $s^h = t^h$. Then $f(t) = f(s)$, so $f$ fails to be injective on that neighbourhood.

Conversely, suppose $h = 1$. Let $C = \{(s,t) \in \mathbf{C}^2 : f(s) = f(t)\}$. $C$ is a germ of an analytic variety of $\mathbf{C}^2$, so it has only finitely many irreducible components. $C$ cannot coincide with $\mathbf{C}^2$ else $f$ is constant. Thus $C$ comprises finitely many branches. Certainly, $C$ contains the diagonal $s = t$: we aim to show there are no further branches - which will establish the result. Let $B$ be a branch of $C$ through $0 \in \mathbf{C}^2$, parametrized as $s = u^n$, $t = \alpha u^n + h.o.t.$ (where $\alpha$ may be zero). Now $x(t) = t^q$ so $x(s) = x(t)$ yields $\alpha^q = 1$, $h.o.t. = 0$. Hence $t = \alpha s$ where $\alpha$ is a $q$th root of unity. Now substitute in $y(s) = y(t)$ and $z(s) = z(t)$ and compare coefficients to get $\alpha^m = 1$ for all powers $m$ of $t$ which appear in $f$. Since the highest common factor of these is 1 we must have $\alpha = 1$, so $B$ is the diagonal in $\mathbf{C}^2$, as required. □

**3.2.7 Proposition** *Any $\mathcal{A}$-finite germ in $f : (\mathbf{C},0) \to (\mathbf{C}^3,0)$ is $\mathcal{A}$-equivalent to a germ whose components have the form $x(t) = t^p, y(t) = t^q +$ h.o.t. $, z(t) = t^r +$ h.o.t. where $p < q < r$ (we allow the possibility that $r = \infty$), where no exponent of a power in $y(t)$ lies in the semigroup generated by $p$, and no exponent of a power in $z(t)$ lies in the semigroup generated by $p$ and $q$.*

**Proof** $\mathcal{A}$-finiteness has two consequences. First, we can assume that the components are polynomial: and it is no restriction to suppose these polynomials monic. Second, we can assume (Lemma 3.2.6) that at most one component of the germ is zero. By permuting the co-ordinates we can suppose further that $p \leq q \leq r$. By a change of co-ordinates in the source we can write $x(t) = t^p$. Co-ordinate changes in the target of the form $X = x$, $Y = y - kx^a$, $Z = z$ allow us to get rid of any power of $t$ in $y(t)$ whose exponent is in the semigroup generated by $p$, without altering the first and third components. Then co-ordinate changes in the target of the form $X = x$, $Y = y$, $Z = z - kx^a y^b$ allow us to get rid of any power of $t$ in $z(t)$ whose exponent lies in the semigroup generated by $p$ and $q$, without altering the first and second components. □

A germ in the form described in the above proposition is said to be in a *pre-normal form* . The integers $p$ and $q$ are $\mathcal{A}$-invariants, indeed $p$ is the least positive integer in the value semigroup $S_1(f)$, and $q$ is the least integer in the semigroup which is greater than $p$ and not a multiple of it. We refer to the minimal order $p$ as the *multiplicity* of the germ, and to the pair $(p,q)$ as the ***invariant pair***. Note that $r$ is *not* an $\mathcal{A}$-invariant: for instance the germs $(t^4, t^6 + t^7, t^{13})$, and $(t^4, t^6 + t^7, t^{15})$ are in pre-normal form, yet equivalent under the $\mathcal{L}$-equivalence $X = x$, $Y = y$, $Z = 2z - y^2 + x^3 + x^2 y$.

### 3.3 $\mathcal{A}$-simplicity

Following Lemma 3.1 in [BGa] we have:

**3.3.1 Lemma** *Given any $\mathcal{A}$-finite singular germ $f : (\mathbf{C},0) \longrightarrow (\mathbf{C}^3,0)$ in prenormal form $f(t) = (t^p, t^q + h.o.t, t^r + h.o.t.)$, if $p \geq 5$ or if $p = 4$ and $q > 7$ then $f$ is not $\mathcal{A}$-simple. (See Proposition 3.6.1 for the converse)*

**Proof** (i) For $p \geq 5$: Consider the map

$$\mu : \mathbf{C}^p \times \mathbf{C}^p \times \mathbf{C}^p \longrightarrow J^{2p-1}(1,3)$$
$$(a,b,c) \longmapsto \left(\sum_{i=1}^{p} a_i t^{p+i-1}, \sum_{i=1}^{p} b_i t^{p+i-1}, \sum_{i=1}^{p} c_i t^{p+i-1}\right)$$

Then $\mu(a,b,c)$ is a typical $2p-1$ jet with no terms of order less than $p$ (and $f = \mu(a_0,b_0,c_0)$ for some $a_0, b_0, c_0$).

*Claim:* $T_{\mu(a,b,c)} Im(\mu) \not\subseteq T_{\mu(a,b,c)} \mathcal{A}(\mu(a,b,c))$ if $p \geq 5$.

(where $Im(\mu)$ is the image of $\mathbf{C}^p \times \mathbf{C}^p \times \mathbf{C}^p$ under $\mu$).

*Proof of claim:* $T_{\mu(a,b,c)} \mathcal{A}(\mu(a,b,c)) = tf(m_1.\theta_1) + wf(m_3.\theta_3)$ so it only contains the following $p+9$ vectors (the first $p$ are from the $tf$ part of the tangent space and the remaining 9 are from the $wf$ part):

From $tf$:

$$\begin{array}{lll}
(\sum_{i=1}^{p} a_i(p+i-1)t^{p+i-1} & ,\sum_{i=1}^{p} b_i(p+i-1)t^{p+i-1} & ,\sum_{i=1}^{p} c_i(p+i-1)t^{p+i-1}) \\
(\sum_{i=1}^{p-1} a_i(p+i-1)t^{p+i} & ,\sum_{i=1}^{p-1} b_i(p+i-1)t^{p+i} & ,\sum_{i=1}^{p-1} c_i(p+i-1)t^{p+i}) \\
(\sum_{i=1}^{p-2} a_i(p+i-1)t^{p+i+1} & ,\sum_{i=1}^{p-2} b_i(p+i-1)t^{p+i+1} & ,\sum_{i=1}^{p-2} c_i(p+i-1)t^{p+i+1}) \\
\vdots & \vdots & \vdots \\
(a_1 p t^{2p-1} & ,b_1 p t^{2p-1} & ,c_1 p t^{2p-1})
\end{array}$$

From $wf$:

$$\begin{array}{lll}
(\sum_{i=1}^{p} a_i t^{p+i-1},0,0) & (0,\sum_{i=1}^{p} a_i t^{p+i-1},0) & (0,0,\sum_{i=1}^{p} a_i t^{p+i-1}) \\
(\sum_{i=1}^{p} b_i t^{p+i-1},0,0) & (0,\sum_{i=1}^{p} b_i t^{p+i-1},0) & (0,0,\sum_{i=1}^{p} b_i t^{p+i-1}) \\
(\sum_{i=1}^{p} c_i t^{p+i-1},0,0) & (0,\sum_{i=1}^{p} c_i t^{p+i-1},0) & (0,0,\sum_{i=1}^{p} c_i t^{p+i-1})
\end{array}$$

(Since we are working in $J^{2p-1}(1,3)$ we do not need to consider powers of $t$ higher than $2p-1$).

Now $T_{\mu(a,b,c)} Im(\mu) = \left\{(t^{p+i-1},0,0),(0,t^{p+i-1},0),(0,0,t^{p+i-1})\right\}_{i=1}^{p}$ and thus it contains $3p$ independent vectors, so if $9+p < 3p$ (ie. $p \geq 5$) then the claim is proved.

So, when $p \geq 5$, no $\mu(a,b,c)$ can be simple and, in particular, $f$ is not simple.

(ii) For $p = 4$ and $q \geq 8$: Define the map

$$\eta : \mathbf{C}^4 \times \mathbf{C}^4 \longrightarrow J^{11}(1,3)$$
$$(a,b) \longmapsto (t^4, \sum_{i=1}^{4} a_i t^{7+i}, \sum_{i=1}^{4} b_i t^{7+i})$$

Then $\eta(a,b)$ is a typical 11-jet with first component of order 4 and next components of orders greater than 7.

*Claim:* $T_{\eta(a,b)} Im(\eta) \not\subseteq T_{\eta(a,b)} \mathcal{A}(\eta(a,b))$ if $p = 4$ and $q \geq 8$.

*Proof of claim:* $T_{\eta(a,b)} \mathcal{A}(\eta(a,b)) = tf(\theta_1) + wf(\theta_3)$

From $tf$ we get 4 vectors:

$$\begin{array}{lll}
(4t^4, & \sum_{i=1}^4 a_i(7+i)t^{7+i}, & \sum_{i=1}^4 b_i(7+i)t^{7+i}) \\
(4t^5, & \sum_{i=1}^3 a_i(7+i)t^{8+i}, & \sum_{i=1}^3 b_i(7+i)t^{8+i}) \\
(4t^6, & \sum_{i=1}^2 a_i(7+i)t^{9+i}, & \sum_{i=1}^2 b_i(7+i)t^{9+i}) \\
(4t^7, & 8a_1 t^{11}, & 8b_1 t^{11})
\end{array}$$

And from $wf$ we get 12 vectors:

$$\begin{array}{lll}
(t^4, 0, 0) & (0, t^4, 0) & (0, 0, t^4) \\
(t^8, 0, 0) & (0, t^8, 0) & (0, 0, t^8) \\
(\sum_{i=1}^4 a_i t^{7+i}, 0, 0) & (0, \sum_{i=1}^4 a_i t^{7+i}, 0) & (0, 0, \sum_{i=1}^4 a_i t^{7+i}) \\
(\sum_{i=1}^4 b_i t^{7+i}, 0, 0) & (0, \sum_{i=1}^4 b_i t^{7+i}, 0) & (0, 0, \sum_{i=1}^4 b_i t^{7+i})
\end{array}$$

Now $T_{\eta(a,b)} Im(\eta) = \{(0, t^{7+i}, 0), \quad (0, 0, t^{7+i})\}_{i=1}^4$ and so it contains 8 linearly independent vectors. From the above, the vectors $(0, t^8, 0)$ and $(0, 0, t^8)$ are obvious. The remaining relevant vectors are:

$$(0, \sum_{i=1}^4 a_i(7+i)t^{7+i}, \sum_{i=1}^4 b_i(7+i)t^{7+i})$$
$$(0, \sum_{i=1}^4 a_i t^{7+i}, 0) \quad (0, \sum_{i=1}^4 b_i t^{7+i}, 0)$$
$$(0, 0, \sum_{i=1}^4 a_i t^{7+i}) \quad (0, 0, \sum_{i=1}^4 b_i t^{7+i})$$

There are only five of them, so we cannot get everything in $T_{\eta(a,b)} Im(\eta)$. Hence the claim is proved, and as before, $f$ cannot be simple. $\square$

We now know that in order to classify $\mathcal{A}$-simple germs we only have to consider $f(t) = (t^p, t^q + h.o.t., t^r + h.o.t.)$ where $p = 2, 3, 4$ and, if $p = 4$, $q < 8$.

### 3.4 The Classification

We use the complete transversal method, described in chapter 1, to classify the singularities of space curves up to $\mathcal{A}_1$-equivalence and then apply the Mather Lemma (Theorem 1.5.2) to obtain the $\mathcal{A}$-classification from this. In fact, if $f$ has multiplicity 2 or 3 we can obtain an $\mathcal{A}$-classification from the $\mathcal{A}_1$-classification simply by using diffeomorphic changes of co-ordinates in the source and target. It is when $f$ has invariant pair $(4,5)$, $(4,6)$ or $(4,7)$ that we really need to apply the Mather Lemma.

The values of the $\mathcal{A}_e$-codimensions of each of the germs, given in the statements of Theorem 3.1.1 and 3.1.2, are calculated by the methods described in Section 1.3. These calculations are found in Appendix B.

We begin with $f$ having multiplicity 2. Note that Proposition 3.4.1 is essentially given in [W].

**3.4.1 Proposition** *If $f \in J^k(1,3)$ is $\mathcal{A}$-finite and has multiplicity 2 then it is $\mathcal{A}$-equivalent to $(t^2, t^{2m+1}, 0)$ for some integer $m \geq 1$.*

**Proof** Consider $(t^2, 0, 0)$ to be the $m$-jet of $f$. Then $T\mathcal{A}_1 \cdot f$ contains:

| | | |
|---|---|---|
| $(2t^{i+1}, 0, 0)$ | $i \geq 2$ | (from $tf$ part) |
| $(t^{2i}, 0, 0)$ | $i \geq 2$ | (from vector fields) |
| $(0, t^{2i}, 0)$ | $i \geq 2$ | |
| $(0, 0, t^{2i})$ | $i \geq 2$ | |

We now use the method of Theorem 1.5.1. Clearly, if $m$ is odd, $J^{m+1}(T\mathcal{A}_1 \cdot f) \cap H^{m+1}(1,3) = H^{m+1}(1,3)$ and so $T$ is trivial. For $f$ to be $\mathcal{A}$-finite there must be a value of $2m \geq 4$ for which the $(2m+1)$-jet is non-zero. Choose the least such $m$. Then

$$J^{2m+1}(T\mathcal{A}_1 \cdot f) \cap H^{2m+1}(1,3) = \{(t^{2m+1}, 0, 0)\}$$

so $T = \{(0, t^{2m+1}, 0),\ (0, 0, t^{2m+1})\}$ and a complete transversal is

$$(t^2, \alpha t^{2m+1}, \beta t^{2m+1}).$$

So $f$ is $\mathcal{A}_1$-equivalent to one of the following:

(i) $(t^2, t^{2m+1}, 0)$

(ii) $(t^2, 0, t^{2m+1})$

(iii) $(t^2, t^{2m+1}, t^{2m+1})$.

By smooth changes of co-ordinates, $f$ is $\mathcal{A}$-equivalent to $(t^2, t^{2m+1}, 0)$. To find the degree of determinacy of this germ we use Corollary 1.2.2 and look at the $\mathcal{A}_1$-tangent space.

$$T\mathcal{A}_1.f = m_1^2\langle(2t, (2m+1)t^{2m}, 0)\rangle + f^*.m_3^2\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

This contains all even powers of $t$ greater than 2 in each position, and all odd powers of $t$ greater than $2m+1$ in each position (though we do get lower odd powers in the first and second positions). Thus we have

$$T\mathcal{A}_1.f \subseteq m_1^{2m+2}.\mathcal{E}^3$$

and,by Corollary 1.2.2, $f$ is $(2m+1)$-determined. So we get the required normal form and we do not need to consider higher jets. □

**3.4.2 Proposition** *If $f \in J^k(1,3)$ is $\mathcal{A}$-finite and has multiplicity three then it is $\mathcal{A}$-equivalent to one of the following:*

| | | |
|---|---|---|
| (i) | $(t^3, t^{3m+1}, 0)$ | (where $1 \le m$) |
| (ii) | $(t^3, t^{3m+2}, 0)$ | (where $1 \le m$) |
| (iii) | $(t^3, t^{3m+1}, t^{3n+2})$ | (where $1 \le m \le n < 2m$ or $1 \le n < m \le 2n$) |
| (iv) | $(t^3, t^{3m+1} + t^{3n+2}, 0)$ | (where $1 \le m \le n < 2m$ or $1 \le n < m \le 2n$) |
| (v) | $(t^3, t^{3m+1} + t^{3n+2}, t^{3l+2})$ | (where $1 \le m \le n < l < 2m$) |
| (vi) | $(t^3, t^{3m+2} + t^{3n+1}, t^{3l+1})$ | (where $1 \le m < n < l \le 2m$) |

**Proof** Suppose $(t^3, 0, 0)$ is the $q$-jet of $f$. Then we have

$$T = \begin{cases} \{(0, t^{3m+1}, 0), (0, 0, t^{3m+1})\} & \text{if } q = 3m \\ \{(0, t^{3m+2}, 0), (0, 0, t^{3m+2})\} & \text{if } q = 3m+1 \\ \textit{trivial} & \text{if } q = 3m+2 \end{cases}$$

For $f$ to be $\mathcal{A}$-finite there must exist an $m \ge 1$ for which the $(3m+1)$ or the $(3m+2)$-jet is non-zero. So

$$f \sim \begin{cases} (t^3, a_1 t^{3m+1}, a_2 t^{3m+1}) \\ (t^3, b_1 t^{3m+2}, b_2 t^{3m+2}) \end{cases}$$

with $(a_1, a_2) \ne (0,0) \ne (b_1, b_2)$. After some linear changes of co-ordinates,

$$f \sim \begin{cases} (t^3, t^{3m+1}, 0) & \text{(i)} \\ (t^3, t^{3m+2}, 0) & \text{(ii)} \end{cases}$$

Neither of these is completely determined so we must go further.

(i) Suppose the $q$-jet of $f$ is $(t^3, t^{3m+1}, 0)$. Then $T = \{(0, t^{3n+2}, 0), (0, 0, t^{3n+2})\}$ if $q = 3n+1$ and is trivial otherwise. So a complete transversal for $f$ is $(t^3, t^{3m+1} + a_1 t^{3n+2}, a_2 t^{3n+2})$ and

$$f \sim \begin{cases} (t^3, t^{3m+1}, 0) & (6m-1) \text{ determined} \\ (t^3, t^{3m+1} + t^{3n+2}, 0) & (6m-1) \text{ determined} \\ (t^3, t^{3m+1}, t^{3n+2}) & (3n+2) \text{ determined} \end{cases}$$

(Again, the determinacy calculations are carried out using Corollary 1.2.2, as in Proposition 3.4.1 above.) The first and last of these cases are now complete. Let $(t^3, t^{3m+1} + t^{3n+2}, 0)$ be the $q$-jet of $f$. Then $T = \{(0, 0, t^{3l+2})\}$ if $q = 3l+1$ and is trivial otherwise. So a complete transversal is $(t^3, t^{3m+1} + t^{3n+2}, at^{3l+2})$ and we have

$$f \sim \begin{cases} (t^3, t^{3m+1} + t^{3n+2}, 0) & \\ (t^3, t^{3m+1} + t^{3n+2}, t^{3l+2}) & (3l+2) \text{ determined} \end{cases}$$

So we have obtained four of the normal forms listed in the proposition.

(ii) Suppose that the $q$-jet of $f$ is $(t^3, t^{3m+2}, 0)$. Then we have

$$T = \{(0, t^{3n+1}, 0), (0, 0, t^{3n+1})\}$$

if $q = 3n$, but if $q = 3n+1$ or $3n+2$, $T$ is trivial. So a complete transversal is $(t^3, t^{3m+2} + a_1 t^{3n+1}, a_2 t^{3n+1})$ and, by linear algebra,

$$f \sim \begin{cases} (t^3, t^{3m+2}, 0) & (6m+1) \text{ determined} \\ (t^3, t^{3m+2} + t^{3n+1}, 0) & (6m+1) \text{ determined} \\ (t^3, t^{3m+2}, t^{3n+1}) & (3n+1) \text{ determined} \end{cases}$$

Only the middle case is not now completely determined. Suppose that $(t^3, t^{3m+2} + t^{3n+1}, 0)$ is the $q$-jet of $f$. Then $T = \{(0, 0, t^{3l+1})\}$ if $q = 3l$ and is trivial otherwise, so we have

$$f \sim \begin{cases} (t^3, t^{3m+2} + t^{3n+1}, 0) & \\ (t^3, t^{3m+2} + t^{3n+1}, t^{3l+1}) & (3l+1) \text{ determined} \end{cases}.$$

All cases are now computed up to their degree of determinacy. □

**3.4.3 Proposition** *If $f$ is $\mathcal{A}$-finite and has invariant pair $(4,5)$ then it is $\mathcal{A}$-equivalent to one of the following:*

(i) $(t^4, t^5, 0)$

(ii) $(t^4, t^5 + t^7, 0)$

(iii) $(t^4, t^5, t^6)$

(iv) $(t^4, t^5, t^7)$

(v) $(t^4, t^5, t^{11})$

(vi) $(t^4, t^5 + t^7, t^{11})$

**Proof** If the 5-jet of $f$ is $(t^4, t^5, 0)$ then $f$ is at most 11 determined, by Proposition 3.2.5. Suppose that the $p$-jet of $f$ is $(t^4, t^5, 0)$, with $5 \leq p \leq 10$, then we look for the $(p+1)$-jets. We find that

$$T = \begin{cases} \{(0, t^6, 0), (0, 0, t^6)\} & p+1 = 6 \\ \{(0, t^7, 0), (0, 0, t^7)\} & p+1 = 7 \\ \{(0, 0, t^{11})\} & p+1 = 11 \end{cases}$$

and $T$ is trivial otherwise. So $f$ is $\mathcal{A}_1$-equivalent to one of

(i) $(t^4, t^5, 0)$

(ii) $(t^4, t^5, t^6)$

(iii) $(t^4, t^5 + t^6, 0)$

(iv) $(t^4, t^5, t^7)$

(v) $(t^4, t^5 + t^7, 0)$

(vi) $(t^4, t^5, t^{11})$

(i) This case is 11 determined and we do not need to go any further.

(ii) This case is 7 determined. We find that the possible 7-jets are

$$f \sim \begin{cases} (t^4, t^5, t^6) \\ (t^4, t^5 + at^7, t^6) \\ (t^4, t^5, t^6 + bt^7) \\ (t^4, t^5 + at^7, t^6 + bt^7) \end{cases}$$

We can show that all of these jets are $\mathcal{A}$-equivalent, by Mather's Lemma (see Section 1.5). Here $U = J^7(1,3)$, $G = \mathcal{A}^{(7)}$ and $V$ is the affine family of jets $(t^4, t^5 + at^7, t^6 + bt^7)$ in $U$. The tangent space to $V$ is spanned by $(0, t^7, 0)$ and $(0, 0, t^7)$. The $\mathcal{A}$-tangent space is given by

$$T\mathcal{A}.f = m_1\langle(4t^3, 5t^4 + 7at^6, 6t^5 + 7bt^6)\rangle + f^*.m_3\langle(1,0,0), (0,1,0), (0,0,1)\rangle$$

so we get $(0, t^7, 0)$ (from $t^3\frac{df}{dt} + \frac{b}{4}t^4\frac{df}{dt} - (z, 0, 0)$) and $(0, 0, t^7)$ (from $t^2\frac{df}{dt} - 4(y, 0, 0) + at^4\frac{df}{dt} - 5(0, z, 0) + 5b(0, t^7, 0)$). Thus the first condition of Mather's Lemma is satisfied. The second condition (that the dimension of the tangent space is independent of the choice of $a$ and $b$) is also satisfied, by inspection. Thus we are free to choose any $a$ and $b$ for all values represent the same normal form. We choose $(a,b)=(0,0)$ and so the normal form is $f(t)=(t^4,t^5,t^6)$.

(iii) This case is 7 determined and is $\mathcal{A}$-equivalent to $(t^4,t^5+t^7,0)$. For if we consider $f(t)=(t^4,t^5+at^6+bt^7,0)$ then we have

$$T\mathcal{A}.f = m_1\langle(4t^3,5t^4+6at^5+7bt^6,0)\rangle + f^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and we find that unless $6a^2=5b$ we obtain $(0,t^6,0)$ and $(0,t^7,0)$. Thus, by Mather's Lemma, we can choose any values of $a$ and $b$ apart from those on the parabola $6a^2=5b$ and get $\mathcal{A}$-equivalent germs. In particular the choices $(1,0)$ and $(0,1)$ give $\mathcal{A}$-equivalent germs. (We also note that if $6a^2=5b$ then the vector we obtain by substituting for $b$ in $f(t)$ and differentiating by $a$ is also in the $\mathcal{A}_e$-tangent space and so we can apply Mather's Lemma again and choose a representative for $(a,b)$ on $6a^2=5b$, eg. $(0,0)$ to give the normal form $(t^4,t^5,0)$.)

(iv) This case is 7 determined so no further jets need to be considered.

(v) This case is 11 determined so we must look at the 8,9,10 and 11-jets. In fact we find that $T$ is trivial for $p+1=8,9,10$ and $T=\{(0,0,t^{11})\}$ for $p+1=11$. So a complete transversal is $(t^4,t^5+t^7,at^{11})$ and we have the two possiblities of

$$f\sim\begin{cases}(t^4,t^5+t^7,0)\\(t^4,t^5+t^7,t^{11})\end{cases}$$

(vi) This case is 11 determined so we need go no further with calculations. □

**3.4.4 Proposition** *If $f$ is $\mathcal{A}$-finite and has invariant pair $(4,6)$ then it is $\mathcal{A}$-equivalent to one of the following infinite families, where $m\geq 3$:*

(i) $(t^4,t^6+t^{2m+1},0)$

(ii) $(t^4,t^6,t^{2m+1})$

(iii) $(t^4,t^6+t^{2m+1},t^{2m+3})$

(iv) $(t^4,t^6+t^{2m+1},t^{2m+5})$

(v) $(t^4,t^6+t^{2m+1},t^{2m+9})$

**Proof** Suppose $(t^4, t^6, 0)$ is the $q$-jet of $f$. We want to obtain the possible $(q+1)$-jets.

We find that $T = \{0, t^{2m+1}, 0), (0, 0, t^{2m+1})\}$ if $q = 2m$ and is trivial if $q = 2m+1$. So a complete transversal is $(t^4, t^6 + a_1 t^{2m+1}, a_2 t^{2m+1})$ (where $m \geq 3$). Then we have

$$f \sim \begin{cases} (t^4, t^6 + t^{2m+1}, 0) & \text{(i)} \\ (t^4, t^6, t^{2m+1}) & \text{(ii)} \end{cases}$$

(i) Suppose the $q$-jet of $f$ is $(t^4, t^6 + t^{2m+1}, 0)$. Since this germ is at most $(2m+9)$ determined we need only consider $2m+1 \leq q \leq 2m+8$. We find that if $q+1 = 2m+2i$ then $T$ is trivial, and it is also trivial if $q+1 = 2m+7$, since $y^2 - x^3 = t^{2m+7} + h.o.t.$

If $q+1 = 2m+3$ then $T = \{(0, t^{2m+3}, 0), (0, 0, t^{2m+3})\}$ and a complete transversal is

$$(t^4, t^6 + t^{2m+1} + a_1 t^{2m+3}, a_2 t^{2m+3}).$$

When $a_2 \neq 0$ we obtain $f \sim (t^4, t^6 + t^{2m+1}, t^{2m+3})$, which is $(2m+5)$ determined. We find that $T = \{(0, 0, t^{2m+5})\}$ and so $f \sim (t^4, t^6 + t^{2m+1}, t^{2m+3})$ or $f \sim (t^4, t^6 + t^{2m+1}, t^{2m+3} + at^{2m+5})$. When $a_2 = 0$ we get $f \sim (t^4, t^6 + t^{2m+1} + at^{2m+3}, 0)$. This is $(2m+9)$ determined. We find that if $q+1 = 2m+5$ then $T = \{(0, 0, t^{2m+5})\}$, if $q+1 = 2m+7$ then $T$ is trivial and if $q+1 = 2m+9$ then $T = \{(0, 0, t^{2m+9})\}$ so

$$f \sim \begin{cases} (t^4, t^6 + t^{2m+1} + at^{2m+3}, 0) \\ (t^4, t^6 + t^{2m+1} + at^{2m+3}, t^{2m+5}) \\ (t^4, t^6 + t^{2m+1} + at^{2m+3}, t^{2m+9}) \end{cases}$$

Only the middle case is not fully determined, and in fact we find that in $J^6(1,3)$ and $J^7(1,3)$ $T$ is trivial.

Returning to $(t^4, t^6 + t^{2m+1}, 0)$, if $q+1 = 2m+5$ then $T = \{(0, 0, t^{2m+5})\}$ and so a complete transversal is $(t^4, t^6 + t^{2m+1}, at^{2m+5})$. If $a \neq 0$ then the jet is $(2m+5)$ determined. Otherwise we go on to $J^7(1,3)$ where we find that $T$ is trivial, as noted above. However, if $q+1 = 2m+9$ then $T = \{(0, t^{2m+9}, 0), (0, 0, t^{2m+9})\}$ and a complete transversal is $(t^4, t^6 + t^{2m+1} + a_1 t^{2m+9}, a_2 t^{2m+9})$. Then

$$f \sim \begin{cases} (t^4, t^6 + t^{2m+1}, 0) \\ (t^4, t^6 + t^{2m+1}, t^{2m+9}) \\ (t^4, t^6 + t^{2m+1} + a_1 t^{2m+9}, 0) \end{cases}$$

which are all $(2m+9)$ determined.

(ii) Suppose that the $(2m+1)$-jet of $f$ is $(t^4, t^6, t^{2m+1})$. This is $(2m+3)$-determined, so we go on to look at the $(2m+3)$-jets. Now $T = \{(0, 0, t^{2m+3})\}$ so a complete transversal is $(t^4, t^6, t^{2m+1} + at^{2m+3})$ and $f \sim (t^4, t^6, t^{2m+1})$ or $f \sim (t^4, t^6, t^{2m+1} + t^{2m+3})$, which is $(2m+3)$ determined.

So we know that if the 6-jet of $f$ is $(t^4, t^6, 0)$ then $f$ is $\mathcal{A}_1$-equivalent to one of the following:

(i) $(t^4, t^6 + t^{2m+1}, 0)$

(ii) $(t^4, t^6 + t^{2m+1} + at^{2m+3}, 0)$

(iii) $(t^4, t^6 + t^{2m+1} + at^{2m+9}, 0)$

(iv) $(t^4, t^6 + t^{2m+1}, t^{2m+3})$

(v) $(t^4, t^6 + t^{2m+1}, t^{2m+3} + at^{2m+5})$

(vi) $(t^4, t^6 + t^{2m+1}, t^{2m+5})$

(vii) $(t^4, t^6 + t^{2m+1} + at^{2m+3}, t^{2m+5})$

(viii) $(t^4, t^6 + t^{2m+1}, t^{2m+9})$

(ix) $(t^4, t^6 + t^{2m+1} + at^{2m+3}, t^{2m+9})$

(x) $(t^4, t^6, t^{2m+1})$

(xi) $(t^4, t^6, t^{2m+1} + at^{2m+3})$

In fact, using Mather's Lemma, we find that (i),(ii) and (iii) are all $\mathcal{A}$-equivalent. For consider $f(t) = (t^4, t^6 + t^{2m+1} + at^{2m+3} + bt^{2m+9}, 0)$. This is $2m+9$ determined. The $\mathcal{A}$-tangent space is given by

$$\begin{aligned} T\mathcal{A}.f = {} & m_1\langle(4t^3, 6t^5 + (2m+1)t^{2m} + a(2m+3)t^{2m+2} + b(2m+9)t^{2m+8}, 0)\rangle \\ & + f^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \end{aligned}$$

and we get the vectors $(0, t^{2m+9}, 0)$ (from $t^9\frac{df}{dt} - 4(x^3, 0, 0) - 6(0, xy^2, 0)$) and $(0, t^{2m+3}, 0)$. By inspection, the dimension of the tangent space will remain constant whatever the values of $a$ and $b$ and so we can apply Mather's Lemma and choose $(a, b) = (0, 0)$ to get the normal form in (i).

Similarly, (iv) and (v) are $\mathcal{A}$-equivalent. If we look at $f(t) = (t^4, t^6 + t^{2m+1}, t^{2m+3} + at^{2m+5})$ then we find that it is $2m+5$ determined and we have

$$\begin{aligned} T\mathcal{A}.f = {} & m_1\langle(4t^3, 6t^5 + (2m+1)t^{2m}, (2m+3)t^{2m+2} + a(2m+5)t^{2m+4})\rangle \\ & + f^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \end{aligned}$$

Then we find that $(0, 0, t^{2m+5})$ is in the tangent space and the dimension is constant so we can apply Mather's Lemma and choose $a = 0$.

Also, (vi) and (vii) are $\mathcal{A}$-equivalent. For consider $f(t) = (t^4, t^6 + t^{2m+1} + at^{2m+3}, t^{2m+5})$. This is $2m+7$ determined. Then the $\mathcal{A}$-tangent space is given by

$$\begin{aligned} T\mathcal{A}.f = {} & m_1\langle(4t^3, 6t^5 + (2m+1)t^{2m} + a(2m+3)t^{2m+2}, (2m+5)t^{2m+4})\rangle \\ & + f^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \end{aligned}$$

Then we obtain the vector $(0, t^{2m+3}, 0)$ and observe that the dimension of the tangent space remains constant, so Mather's Lemma can be applied.

Now (viii) and (ix) are another $\mathcal{A}$-equivalent pair. If we consider $f(t) = (t^4, t^6 + t^{2m+1} + at^{2m+3}, t^{2m+9})$ then the $\mathcal{A}$-tangent space is given by

$$\begin{aligned} T\mathcal{A}.f = {} & m_1\langle(4t^3, 6t^5 + (2m+1)t^{2m} + a(2m+3)t^{2m+2}, (2m+9)t^{2m+8})\rangle \\ & + f^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \end{aligned}$$

Once again, it is easy to find the vector $(0, t^{2m+3}, 0)$ in the tangent space. The second criterion of Mather's Lemma is also satisfied and so the two germs are $\mathcal{A}$-equivalent.

Finally, (x) and (xi) are $\mathcal{A}$-equivalent, since if we consider $f(t) = (t^4, t^6, t^{2m+1} + at^{2m+3})$ then we have

$$T\mathcal{A}.f = m_1\langle(4t^3, 6t^5, (2m+1)t^{2m} + a(2m+3)t^{2m+2})\rangle + f^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

We have the vector $(0, 0, t^{2m+3})$ (from $t^3\frac{df}{dt} - 4(y,0,0) - 6(0,x^2,0)$) and the dimension of the tangent space is constant.

So the list in the statement of the proposition is obtained. □

**3.4.5 Proposition** *If $f$ is $\mathcal{A}$-finite and has invariant pair $(4,7)$ then it is $\mathcal{A}$-equivalent to one of the following:*

| | | | |
|---|---|---|---|
| (i) | $(t^4, t^7, 0)$ | (vii) | $(t^4, t^7 + t^9, t^{10})$ |
| (ii) | $(t^4, t^7 + t^9, 0)$ | (viii) | $(t^4, t^7, t^{13})$ |
| (iii) | $(t^4, t^7 + t^{13}, 0)$ | (ix) | $(t^4, t^7 + t^9, t^{13})$ |
| (iv) | $(t^4, t^7, t^9)$ | (x) | $(t^4, t^7, t^{17})$ |
| (v) | $(t^4, t^7, t^9 + t^{10})$ | (xi) | $(t^4, t^7 + t^9, t^{17})$ |
| (vi) | $(t^4, t^7, t^{10})$ | (xii) | $(t^4, t^7 + t^{13}, t^{17})$ |

**Proof** Suppose that the $q$-jet of $f$ is $(t^4, t^7, 0)$. Now $f$ must be at most 17 determined, by Proposition 3.2.5, so $7 \leq q \leq 16$.

By considering $T\mathcal{A}_1 \cdot f$ we can see that $T$ is trivial if $q+1 = 8, 11, 12, 14, 15$ or 16. If $q+1 = 17$ then $T = \{(0,0,t^{17})\}$ and a complete transversal is $(t^4, t^7, at^{17})$. So $f \sim (t^4, t^7, 0)$ or $f \sim (t^4, t^7, t^{17})$. We need go no further in this case. If $q+1 = 9$ then $T = \{(0, t^9, 0), (0, 0, t^9)\}$ and so we have the following possibilities for $f$:

$$f \sim \begin{cases} (t^4, t^7, 0) & \\ (t^4, t^7, t^9) & \text{(i)} \\ (t^4, t^7 + t^9, 0) & \text{(ii)} \end{cases}$$

If $q+1 = 10$ then $T = \{(0, t^{10}, 0), (0, 0, t^{10})\}$ and we have

$$f \sim \begin{cases} (t^4, t^7, 0) & \\ (t^4, t^7, t^{10}) & \text{(iii)} \\ (t^4, t^7 + t^{10}, 0) & \text{(iv)} \end{cases}$$

Finally, if $q+1 = 13$ then $T = \{(0, t^{13}, 0), (0, 0, t^{13})\}$ and we have

$$f \sim \begin{cases} (t^4, t^7, 0) & \\ (t^4, t^7, t^{13}) & \\ (t^4, t^7 + t^{13}, 0) & \text{(v)} \end{cases}$$

All of these, bar cases (i) to (v), are now finished as they are all $k$ determined $k$-jets. We now look further at cases (i) to (v).

(i) Suppose that the 9-jet of $f$ is $(t^4, t^7, t^9)$. This is 10 determined so we need to look at the possible 10-jets. We find that $T = \{(0, t^{10}, 0), (0, 0, t^{10})\}$ and so

$$f \sim \begin{cases} (t^4, t^7, t^9) \\ (t^4, t^7 + t^{10}, t^9) \\ (t^4, t^7, t^9 + t^{10}) \\ (t^4, t^7 + t^{10}, t^9 + t^{10}) \end{cases}$$

All of these are 10 determined.

(ii) Suppose that the 9-jet of $f$ is $(t^4, t^7 + t^9, 0)$. This is 17 determined, so if we think of it as the $q$-jet of $f$, where $9 \leq q \leq 16$, we want to look at the $(q+1)$-jets. We find that

$$T = \begin{cases} \{(0, t^{q+1}, 0), (0, 0, t^{q+1})\} & \text{q+1=10} \\ \{(0, 0, t^{q+1})\} & \text{q+1=13,17} \end{cases}$$

but otherwise $T$ is trivial. So if $q+1 = 10$ a complete transversal is $(t^4, t^7 + t^9 + a_1 t^{10}, a_2 t^{10})$ and

$$f \sim \begin{cases} (t^4, t^7 + t^9, 0) \\ (t^4, t^7 + t^9, t^{10}) \\ (t^4, t^7 + t^9 + at^{10}, 0) \end{cases}$$

These are all 10 determined. If $q+1=13$ then a complete transversal is $(t^4, t^7+t^9, at^{13})$ so $f \sim (t^4, t^7+t^9, 0)$ or $f \sim (t^4, t^7+t^9, t^{13})$ (this is 13 determined). If $q+1=17$ then a complete transversal is $(t^4, t^7+t^9, at^{17})$ and so $f \sim (t^4, t^7+t^9, 0)$ or $f \sim (t^4, t^7+t^9, t^{17})$. Both are 17 determined.

(iii) Suppose the 10-jet of $f$ is $(t^4, t^7, t^{10})$. If this is the $q$-jet, for $10 \leq q \leq 12$, we want to look at the $(k+1)$-jets. We find that if $q+1 = 11, 12$ then $T$ is trivial but if $q+1=13$ then $T = \{(0, t^{13}, 0), (0, 0, t^{13})\}$ and so we have

$$f \sim \begin{cases} (t^4, t^7, t^{10}) \\ (t^4, t^7+t^{13}, t^{10}) \\ (t^4, t^7, t^{10}+t^{13}) \\ (t^4, t^7+t^{13}, t^{10}+t^{13}) \end{cases}$$

These are all 13 determined.

(iv) Suppose that the 10-jet of $f$ is $(t^4, t^7+t^{10}, 0)$. We find that unless $q+1=13$ or $q+1=17$, $T$ is trivial. If $q+1=13$ then $T = \{(0, t^{13}, 0), (0, 0, t^{13})\}$ and we have

$$f \sim \begin{cases} (t^4, t^7+t^{10}, 0) \\ (t^4, t^7+t^{10}, t^{13}) \\ (t^4, t^7+t^{10}+at^{13}, 0) \end{cases}$$

In fact, the last of those three is still not completely determined, but we can show that it is $\mathcal{A}$-equivalent to $(t^4, t^7+t^{13}, 0)$, which is investigated further later on. If $q+1=17$ then $T = \{(0, 0, t^{17})\}$ and so $f \sim (t^4, t^7+t^{10}, 0)$ or $f \sim (t^4, t^7+t^{10}, t^{17})$, which is 17 determined.

(v) Suppose that the 13-jet of $f$ is $(t^4, t^7+t^{13}, 0)$. If this is the $q$-jet, for $13 \leq q \leq 16$, we want to find the $(q+1)$-jets. We find that if $q+1 = 14, 15, 16$ then $T$ is trivial. If $q+1=17$ then $T = \{(0, 0, t^{17})\}$ and a complete transversal is $(t^4, t^7+t^{13}, at^{17})$. So $f \sim (t^4, t^7+t^{13}, 0)$ or $f \sim (t^4, t^7+t^{13}, t^{17})$, which are both 17 determined.

We now have the following $\mathcal{A}_1$-classification for $f \sim (t^4, t^7, 0)$:

| | | | |
|---|---|---|---|
| (i) | $(t^4, t^7, 0)$ | (xii) | $(t^4, t^7 + at^{13}, t^{10})$ |
| (ii) | $(t^4, t^7 + t^9, 0)$ | (xiii) | $(t^4, t^7, t^{10} + at^{13})$ |
| (iii) | $(t^4, t^7 + t^9 + at^{10}, 0)$ | (xiv) | $(t^4, t^7 + at^{13}, t^{10} + bt^{13})$ |
| (iv) | $(t^4, t^7 + t^{10}, 0)$ | (xv) | $(t^4, t^7 + at^{10}, t^{13})$ |
| (v) | $(t^4, t^7 + t^{13}, 0)$ | (xvi) | $(t^4, t^7, t^{13})$ |
| (vi) | $(t^4, t^7, t^9)$ | (xvii) | $(t^4, t^7 + t^9, t^{13})$ |
| (vii) | $(t^4, t^7 + at^{10}, t^9)$ | (xviii) | $(t^4, t^7, t^{17})$ |
| (viii) | $(t^4, t^7, t^9 + t^{10})$ | (xix) | $(t^4, t^7 + t^9, t^{17})$ |
| (ix) | $(t^4, t^7 + at^{10}, t^9 + t^{10})$ | (xx) | $(t^4, t^7 + t^{10}, t^{17})$ |
| (x) | $(t^4, t^7 + t^9, t^{10})$ | (xxi) | $(t^4, t^7 + t^{13}, t^{17})$ |
| (xi) | $(t^4, t^7, t^{10})$ | | |

Using Mather's Lemma we are able to show that (ii) and (iii) are $\mathcal{A}$-equivalent. Consider $f(t) = (t^4, t^7 + t^9 + at^{10}, 0)$. This is 17 determined and has $\mathcal{A}$-tangent space

$$T\mathcal{A}.f = m_1\langle(4t^3, 7t^6 + 9t^8 + 10at^9, 0)\rangle + f^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

The vector $(0, t^{10}, 0)$ is easily seen to be in the tangent space and also the dimension of the tangent space is constant, so we can apply Mather's Lemma and set $a = 0$.

Cases (iv) and (v) are another $\mathcal{A}$-equivalent pair. For if $f(t) = (t^4, t^7 + at^{10} + bt^{13}, 0)$ (which is 17 determined) then the $\mathcal{A}$-tangent space is given by

$$T\mathcal{A}.f = m_1\langle(4t^3, 7t^6 + 10at^9 + 13bt^{13}, 0)\rangle + f^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

Computation verifies that for $14b \neq 17a^2$ the tangent space contains the vectors $(0, t^{10}, 0)$ and $(0, t^{13}, 0)$. Since the condition $14b \neq 17a^2$ defines a smooth connected subset of the jet-space we conclude that all jets of form $(t^4, t^7 + at^{10} + bt^{13}, 0)$ satisfying this condition are $\mathcal{A}$-equivalent. In particular the jets $(t^4, t^7 + t^{10}, 0)$ and $(t^4, t^7 + t^{13}, 0)$ are $\mathcal{A}$-equivalent.

Cases (vi) and (vii) are also $\mathcal{A}$-equivalent, since if $f(t) = (t^4, t^7 + at^{10}, t^9)$ (which is 10 determined) then we have

$$T\mathcal{A}.f = m_1\langle(4t^3, 7t^6 + 10at^9, 9t^8)\rangle + f^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and we obtain $(0, t^{10}, 0)$ from $t^4\frac{df}{dt} - 4(y,0,0) + at^7\frac{df}{dt}$. So we can apply Mather's Lemma and conclude that the two germs are $\mathcal{A}$-equivalent.

Also, (viii) and (ix) are $\mathcal{A}$-equivalent. Consider $f(t) = (t^4, t^7 + at^{10}, t^9 + t^{10})$. This is 10 determined. The $\mathcal{A}$-tangent space is given by

$$T\mathcal{A}.f = m_1\langle(4t^3, 7t^6 + 10at^9, 9t^8 + 10t^9)\rangle + f^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

Once again we obtain $(0,t^{10},0)$ from $t^4\frac{df}{dt}-4(y,0,0)+at^7\frac{df}{dt}$ and the dimension of the tangent space remains constant. Thus we can apply Mather's Lemma.

Cases (xv) and (xvi) are $\mathcal{A}$-equivalent too, since if $f(t)=(t^4,t^7+at^{10},t^{13})$ (which is 13 determined) then

$$T\mathcal{A}.f = m_1\langle(4t^3,7t^6+10at^9,13t^{12})\rangle + f^*.m_3\langle(1,0,0),1,0),(0,0,1)\rangle$$

and we obtain the vector $(0,t^{10},0)$ in a similar way. We apply Mather's Lemma to see that the two germs are $\mathcal{A}$-equivalent.

Finally we see that (xi), (xii), (xiii) and (xiv) are all $\mathcal{A}$-equivalent. For consider $f(t)=(t^4,t^7+at^{13},t^{10}+bt^{13})$. This is 13 determined and has $\mathcal{A}$-tangent space

$$T\mathcal{A}.f = m_1\langle(4t^3,7t^6+13at^{12},10t^9+13bt^{12})\rangle + f^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

Then we get the vector $(0,t^{13},0)$ from $t^7\frac{df}{dt}-4(z,0,0)+bt^{10}\frac{df}{dt}$ and the vector $(0,0,t^{13})$ from $t^4\frac{df}{dt}-4(y,0,0)+at^{10}\frac{df}{dt}-7(0,z,0)+7b(0,t^{13},0)$. As the dimension of the tangent space is constant, we can apply Mather's Lemma and choose $(0,0)$ as a representative for $(a,b)$.

Thus the list in the statement of the proposition is obtained. □

### 3.5 Planarity

We now return to the question of planarity, mentioned in the statement of Theorem 3.1.1.

#### 3.5.1 Definition

(i) *A germ $f:(\mathbf{C},0)\to(\mathbf{C}^3,0)$ is planar when it is $\mathcal{A}$-equivalent to a germ having a representative with image contained in a plane in $\mathbf{C}^3$.*

(ii) *A germ $f:(\mathbf{C},0)\to(\mathbf{C}^3,0)$ which is not planar is spatial.*

Clearly, all the germs listed in the statement of Theorem 3.1.1 are planar. We wish to show that all the germs in the statement of Theorem 3.1.2 are spatial. This proceeds via the observation that $f$ is planar if and only if there exists a submersive germ $h:(\mathbf{C}^3,0)\to(\mathbf{C},0)$ for which $h\circ f$ is the germ of the zero function.

**3.5.2 Proposition** *Let $f$ be an $\mathcal{A}$-finite singular germ of a space curve in prenormal form $x(t) = t^p, y(t) = t^q +$ h.o.t. , $z(t) = t^r +$ h.o.t. with $r < \infty$. If $f$ is planar then $r$ lies in the invariant semigroup $S_2(g)$ where $g = \pi \circ f$ and $\pi : (\mathbf{C}^3, 0) \to (\mathbf{C}^3, 0)$ is given by $\pi(x, y, z) = (x, y, 0)$.*

**Proof** Since $f$ is assumed planar, there exists (by the above remark) a submersive germ $h : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ with $h \circ f$ the zero germ. Write

$$h(x, y, z) = Ax + By + Cz + \phi(x, y, 0) + z\psi(x, y, z)$$

where at least one of $A$, $B$, $C$ is non-zero, where $\phi$ has neither constant nor linear terms, and where $\psi$ has no constant terms. Then

$$0 \equiv h(f(t)) = Ax(t) + By(t) + Cz(t) + \phi(x(t), y(t), 0) + \text{ terms of order } > \mathrm{r}\,.$$

Since the germ is in prenormal form the leading terms $t^p$, $t^q$, in the first two components cannot cancel out with any other power of $t$. Thus $A = B = 0$ and $C \neq 0$. But then $r = \operatorname{ord}\phi(x(t), y(t), 0)$, and hence $r$ lies in $S_2(g)$. □

All the germs $f$ listed in the statement of Theorem 3.1.2 are in prenormal form, with order $r$ not in the invariant semigroup $S_2(g)$. It follows immediately from the above Proposition that they are spatial.

The question of planarity of a germ relates to the value semigroup. From [SK] we have

**3.5.3 Definition** *A semigroup $S \subset \mathbf{Z}$ is symmetric when there exists an integer $M$ with the property that if two integers $a$, $b$ satisfy the relation $M = a + b$ then one of them is in $S$, and the other is not.*

It is well-known that the value semigroup associated to a branch of a plane curve is symmetric [SK]: and indeed that is the case for all the germs in Theorem 3.1.1. However, there are also germs in Theorem 3.1.2 whose value semigroup is symmetric. The list of such germs is shown below:

$$(t^4, t^5, t^6)$$
$$(t^4, t^7, t^{10})$$
$$(t^4, t^7 + t^9, t^{10})$$
$$(t^4, t^6, t^{2k+1})$$
$$(t^4, t^6 + t^{2k-1}, t^{2k+1})$$

I would like to thank Phillip Cook [C] for extracting the following from the literature: in any embedding dimension the value semigroup is known to be symmetric if and only if the local ring of the singularity is Gorenstein [Ku]. Moreover it is well-known that a germ of a (local) complete intersection necessarily has a local ring which is Gorenstein [M]. Thus in any embedding dimension a germ of a (local) complete intersection gives rise to a symmetric value semigroup. By a theorem of Serre [Se] all three properties coincide in the special case when the difference between the embedding dimension and the dimension of the local ring is at most 2. Thus the above represent the only $\mathcal{A}$-simple spatial germs of space curves which are (local) complete intersections. Such a germ corresponds to the zero set of a $\mathcal{K}$-finite germ $(\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$.

In order to find which $\mathcal{K}$-finite germ each element $(t^p, t^q + t^s, t^r)$ (where $s$ may be zero) on our list is associated with we find a minimal generating system of equations in $x = t^p$, $y = t^q + t^s$ and $z = t^r$ and then see that, possibly after some cyclic permutations of co-ordinates and rescaling, we have a normal form from the lists of $\mathcal{K}$-finite germs $\mathbf{C}^3, 0 \to \mathbf{C}^2, 0$ given in [Wa2] and [Wa3]. In each case the germ turns out to be $\mathcal{K}$-simple.

Following Kunz [K] we determine a minimal generating system for germs of the type $(t^p, t^q, t^r)$. We know that $p$, $q$ and $r$ have greatest common divisor 1 and we look for integers $c_1$, $c_2$, $c_3$ such that $c_1$ is the least positive integer with $pc_1 \in q\mathbf{N} + r\mathbf{N}$, $c_2$ is the least positive integer such that $qc_2 \in p\mathbf{N} + r\mathbf{N}$ and $c_3$ is the least positive integer with $rc_3 \in p\mathbf{N} + q\mathbf{N}$. Then generating equations are

$$\begin{aligned} x^{c_1} - y^{r_1} z^{r_2} &= 0 \qquad \text{some } r_1, r_2 \in \mathbf{N} \\ y^{c_2} - x^{s_1} z^{s_2} &= 0 \qquad \text{some } s_1, s_2 \in \mathbf{N} \\ z^{c_3} - x^{t_1} y^{t_2} &= 0 \qquad \text{some } t_1, t_2 \in \mathbf{N} \end{aligned}$$

where $x = t^p$, $y = t^q$ and $z = t^r$. If there are two independent equations then we have a complete intersection while if all three are independent then we do not have a complete intersection.

### 3.5.4 Case 1 – $(t^4, t^5, t^6)$

Putting $x = t^4$, $y = t^5$ and $z = t^6$ and following the method above we find that a minimal generating set of equations is $(xz - y^2, x^3 - z^2)$ so we shall consider the germ $\phi : \mathbf{C}^3, 0 \to \mathbf{C}^2, 0$ given by

$$\phi(x, y, z) = (xz - y^2, x^3 - z^2).$$

A cyclic permutation of co-ordinates gives the $\mathcal{K}$-equivalent germ

$$(xy - z^2, y^3 - x^2)$$

Scaling $x$ to $ix$ we obtain

$$(ixy - z^2, y^3 + x^2)$$

Now scaling $z$ to $\lambda z$ ($\lambda \neq 0$) we get

$$(ixy - \lambda^2 z^2, x^2 + y^3)$$

Choosing $\lambda$ so that $i = -\lambda^2$ and then scaling the first co-ordinate we get the normal form

$$(xy + z^2, x^2 + y^3)$$

which is the $\mathcal{K}$-finite germ $K_8$.

### 3.5.5 Case 2 – $(t^4, t^7, t^{10})$

As before, putting $x = t^4$, $y = t^7$ and $z = t^{10}$ we get $(xz - y^2, x^5 - z^2)$ for our minimal generating set and so we look at the germ

$$\psi(x, y, z) = (xz - y^2, x^5 - z^2).$$

We can follow exactly the same steps as before and perform a cyclic permutation of the co-ordinates followed by rescaling $x$ and $z$ to obtain the normal form

$$(xy - z^2, x^2 + y^5)$$

which is $K_{14}$.

### 3.5.6 Case 3 – $(t^4, t^7 + t^9, t^{10})$

In order to get a minimal generating set we cannot use the method of Kunz as it stands, but it turns out that using an obvious variation we obtain the equations $xz - y^2 + x^2 z + 2x^4 = 0$ and $x^5 - z^2 = 0$. So we shall consider the germ $\mathbf{C}^3, 0 \to \mathbf{C}^2, 0$ given by

$$\rho(x, y, z) = (xz - y^2 + x^2 z + 2x^4, x^5 - z^2).$$

We have to work harder to get this into a normal form. First we swap the co-ordinates $x$ and $y$, rescale $x$ to $ix$, multiply the second component by -1 and then swap $y$ and $z$ to get the $\mathcal{K}$-equivalent germ

$$(x^2 + yz + yz^2 + 2z^4, y^2 - z^5)$$

Then if we rescale $y$ to $2y$ and the second component by $\frac{1}{4}$ we have

$$(x^2 + 2yz + 2yz^2 + 2z^4, y^2 - \frac{z^5}{4})$$

which has two-jet $(x^2+2yz, y^2)$. Following [Wa2], we would like to have the germ in pre-normal form

$$(x^2+2yz+b(z), y^2+2xd(z)+c(z))$$

then we could eliminate $y$ to get the function

$$\phi(x,z) = (x^2+b(z))^2 + 4z^2(2xd(x)+c(z))$$

which would give us the Milnor number of $\rho$ by the formula $\mu(\rho) = \mu(\phi) - 4$. But in our case we have an extra $yz^2$ term in the first component. To remove this we set $Z = z + z^2$ so that we get

$$(x^2 + 2yZ + 2Z^4 + h.o.t, y^2 - \frac{Z^5}{4} + h.o.t.)$$

(where the h.o.t. only involve $Z$). Then we have

$$\begin{aligned}\phi(x,Z) =&(x^2+2Z^4+\ldots)^2 + 4Z^2(\frac{-Z^5}{4} + \ldots)\\ =&x^4 + 2x^2Z^4 + 4Z^8 + \ldots - Z^7 + \ldots\end{aligned}$$

Giving $x$ weight $\frac{1}{4}$ and $Z$ weight $\frac{1}{7}$ we get

$$\phi(x,Z) = (x^4 - Z^7) + \{\text{terms of weight} > 1\}$$

which has Milnor number $(4-1)(7-1) = 18$. So $\mu(\rho) = 14$ and we have the $\mathcal{K}$-finite germ $K_{14}$ again.

### 3.5.7 Case 4 – $(t^4, t^6, t^{2k+1})$

Putting $x = t^4$, $y = t^6$ and $z = t^{2k+1}$ we obtain the minimal generating set of equations $(x^3 - y^2, z^2 - x^{k-1}y)$ so that we get the germ

$$\tau(x,y,z) = (x^3 - y^2, z^2 - x^{k-1}y).$$

Permuting the co-ordinates cyclically and rescaling $x$ to $ix$ we get

$$(z^3 + x^2, y^2 - ixz^{k-1})$$

This gives the $\mathcal{K}$-finite germ $G^1_n$, where $n + 9 = \mu(\tau)$. Following [Wa2] again, we find this Milnor number by writing $\tau$ in the form

$$(x^2 + z^3, y^2 + 2xd(z) + c(z))$$

Here $d(z) = -iz^{k-1}/2$ and $c(z) = 0$. Then

$$\begin{aligned}\mu(\tau) =&min(3 + 2\text{ord}(c), 6 + 2\text{ord}(d))\\ =&6 + 2(k-1)\\ =&2k - 4\end{aligned}$$

So $\mu(\tau) = 2k - 5$ and the required $\mathcal{K}$-finite germ is $G^1_{2k-5}$.

### 3.5.8 Case 5 – $(t^4, t^6 + t^{2k-1}, t^{2k+1})$

Once again, the method for finding a minimal generating set of equations cannot be used directly for a germ of this form, and in this case the variation used in Case 3 does not work either. By trial and error we obtain the equations $x^3 - y^2 + 2xz + x^{k-1}y - x^{k-4}yz + x^{2k-4} = 0$ and $z^2 - x^{k-1} + x^{k-3}yz - x^{2k-3} = 0$. These are certainly generators, so we consider the germ

$$\theta(x,y,z) = (x^3 - y^2 + 2xz + x^{k-1}y - x^{k-4}yz + x^{2k-4}, z^2 - x^{k-1} + x^{k-3}yz - x^{2k-3})$$

If we permute the co-ordinates via $x \mapsto z$, $y \mapsto x$ and $z \mapsto y$ and then rescale $y$ to $-y$, the 2-jet of $\theta$ is $(x^2 + 2yz, y^2)$ so we have a member of the $K$ series. Note that

$$\begin{aligned}(x^2 - z^3)^2 =& (t^{12} + 2t^{2k+5} + t^{4k-2} - t^{12})^2 \\ =& 4t^{4k-10} + 4t^{6k+3} + t^{8k-4} \\ =& 4z^{k+1}x + z^{2k-1}\end{aligned}$$

So we can put $\phi(x,z) = (x^2 - z^3)^2 - 4z^{k+1}x - z^{2k-1}$ which is essentially in normal form for $W^{\#1}_{2k-7}$ (see [A2]). So the member of the $K$ series we want is $KW^{\#1}_{2k-7}$.

We summarize these results in the table below.

| Normal Form | Generating Equations | K-Type | |
|---|---|---|---|
| $(t^4, t^5, t^6)$ | $(xz - y^2, x^3 - z^2)$ | $K_8$ | |
| $(t^4, t^7, t^{10})$ | $(xz - y^2, x^5 - z^2)$ | $K_{14}$ | |
| $(t^4, t^7 + t^9, t^{10})$ | $(xz - y^2 + x^2z + 2x^4, x^5 - z^2)$ | $K_{14}$ | |
| $(t^4, t^6, t^{2k+1})$ | $(x^3 - y^2, x^{k-1}y - z^2)$ | $G^1_{2k-5}$ | $(k \geq 3)$ |
| $(t^4, t^6 + t^{2k-1}, t^{2k+1})$ | $(x^3 - y^2 + 2xz + x^{k-1}y - x^{k-4}yz + x^{2k-4},$ $z^2 - x^{k-1} + x^{k-3}yz - x^{2k-3})$ | $KW^{\#1}_{2k-7}$ | $(k \geq 4)$ |

### 3.6 A Sufficiency Result for $\mathcal{A}$-Simplicity

Finally, we have a sufficiency result for $\mathcal{A}$-simplicity of $\mathcal{A}$-finite germs of space curves.

**3.6.1 Proposition** *All $\mathcal{A}$-finite map-germs $f : \mathbf{C} \to \mathbf{C}^3$ with codimension less than 12 are $\mathcal{A}$-simple.*

**Proof** Given an integer $p$, $1 \leq p \leq k$, define the $\mathcal{A}$-invariant set $X_p \subset J^k(1,3)$ by

$$X_p = \{j^k f : j^k f \text{ has multiplicity } p\}$$

Also, given another integer $q > p$, where $p$ does not divide $q$, define the $\mathcal{A}$-invariant set $X_{p,q} \subset X_p$ by

$$X_{p,q} = \{j^k f : j^k f \text{ has invariant pair } (p, q'),\ q' \geq q\}$$

The classification carried out in this chapter shows that $X_1$, $X_2$ and $X_3$ are all finite unions of orbits, as are $X_{4,5}$, $X_{4,6}$ and $X_{4,7}$, i.e. everything in them is $\mathcal{A}$-simple.

We want to consider the codimensions of the sets $X_{4,9}$ and $X_5$, the sets in which we would first expect to find non-$\mathcal{A}$-simple germs (from Proposition 3.3.1). Clearly, $X_p$ is smooth for all $p$ and has codimension $3(p-1)$ and so $X_5$ has codimension 12. Consider $X_{4,9}$. Given an element in $X_4$, we need to know what further conditions must be placed on it in order for it to be an element of $X_{4,9}$. An element of $X_4$ is written

$$(a_1t^4+a_2t^5+a_3t^6+a_4t^7+\ldots, b_1t^4+b_2t^5+b_3t^6+b_4t^7+\ldots, c_1t^4+c_2t^5+c_3t^6+c_4t^7+\ldots)$$

(Note that $(a_1, b_1, c_1) \neq (0,0,0)$).

Given an arbitrary polynomial, $g(x,y,z) = Ax+By+Cz+Dx^2+Exy+\ldots$, we want it to have no $t^5$, $t^6$ or $t^7$ terms. So we need $Aa_1 + Bb_1 + Cc_1 = 0$ to imply

$$Aa_2 + Bb_2 + Cc_2 = 0 \tag{1}$$

$$Aa_3 + Bb_3 + Cc_3 = 0 \tag{2}$$

$$Aa_4 + Bb_4 + Cc_4 = 0 \tag{3}$$

Condition (1) is satisfied if $(a_2, b_2, c_2)$ is a linear combination of $(a_1, b_1, c_1)$ ie.

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

has rank 1. This gives two quadratic conditions to be satisfied.

Similarly, conditions (2) and (3) are satisfied if

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

and

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_4 & b_4 & c_4 \end{pmatrix}$$

each have rank 1, giving four more quadratic conditions.

Since $X_{4,9}$ is specified by the rank of a matrix being 1 it must be a smooth manifold. It has codimension $3 \times 3 + 2 + 2 + 2 = 15$. We can then take the union of these two smooth manifolds and thus get another smooth manifold with the least codimension of any element in the union being 12. There are only $\mathcal{A}$-simple germs in the sets $X_{p,q}$ up to these, and so all germs up to codimension 11 are simple. □

It is in fact possible to find examples of non-$\mathcal{A}$-simple germs with $\mathcal{A}_e$-codimension 12. Consider $f_\lambda(t) = (t^5, t^6 + \lambda t^9, t^7 + t^9)$. Calculation shows that the vector $(0, t^9, 0)$ does not lie in the tangent space to the $\mathcal{A}^{(9)}$-orbit for any choice of $\lambda$ and so all the germs $f_\lambda$ are non-simple. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(5t^4, 6t^5 + 9\lambda t^8, 7t^6 + 9t^8)\rangle + f^*\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and computation shows that the $\mathcal{A}_e$-codimension is 12.

We now give a table of all the $\mathcal{A}$-simple singular germs $(\mathbf{C}, 0) \to (\mathbf{C}^3, 0)$ with $\mathcal{A}_e$-codimension less that 12. The number in the lefthand column is the $\mathcal{A}_e$-codimension.

| | | | | | |
|---|---|---|---|---|---|
| 2 | $(t^2, t^3, 0)$ | | | | |
| 3 | none | | | | |
| 4 | $(t^2, t^5, 0)$ | | | | |
| 5 | $(t^3, t^4, t^5)$ | | | | |
| 6 | $(t^2, t^7, 0)$ | $(t^3, t^4, 0)$ | $(t^3, t^4 + t^5, 0)$ | | |
| 7 | $(t^3, t^5, t^7)$ | | | | |
| 8 | $(t^2, t^9, 0)$ | $(t^3, t^5, 0)$ | $(t^4, t^5, t^6)$ | $(t^3, t^5 + t^7, 0)$ | |
| 9 | $(t^3, t^7, t^8)$ | $(t^4, t^5, t^7)$ | | | |
| 10 | $(t^2, t^{11}, 0)$ | $(t^3, t^7 + t^8, t^{11})$ | $(t^4, t^5 + t^7, t^{11})$ | $(t^4, t^6, t^7)$ | |
| 11 | $(t^3, t^7 + t^8, 0)$ | $(t^4, t^5 + t^7, 0)$ | $(t^3, t^7, t^{11})$ | $(t^3, t^8, t^{10})$ | $(t^4, t^5, t^{11})$ |

# Chapter 4 – One-Dimensional Motions of the Plane

## 4.1 Introduction

For the first step in the programme of studying local models of kinematic singularities we will look at motions of the plane with one degree of freedom – this must be the simplest possible case. The results of this chapter should be applicable to planar mechanisms such as the four-bar linkage, mentioned in the introduction to the thesis. The following theorem summarizes the results of this chapter.

**4.1.1 Theorem** *On the trajectory of a generic motion of the plane with one degree of freedom we locally expect only to see multi-germs $\mathcal{A}$-equivalent to those in the table below.*

| Normal Form | $\mathcal{A}_e$-Codim | Name |
|---|---|---|
| $(t,0)$ | 0 | $A_0$ |
| $(t^2,t^3)$ | 1 | $A_2$ |
| $(t^2,t^5)$ | 2 | $A_4$ |
| $(t,0;0,s)$ | 0 | $A_1$ |
| $(t,0;s,s^2)$ | 1 | $A_3$ |
| $(t,0;s,s^3)$ | 2 | $A_5$ |
| $(t,0;s^3,s^2)$ | 2 | $D_5$ |
| $(t,0;0,s;u,u)$ | 1 | $D_4$ |
| $(t,0;0,s;u,u^2)$ | 2 | $D_6$ |
| $(t,0;0,s;u,u;v,\lambda v)$ | 3 | $\tilde{E}_7$ |

See Fig. 4.1.1 for pictures of all of these normal forms. The names of the singularities are those associated to the $\mathcal{K}$-types of germs at 0 of smooth functions on the plane whose zero sets give the image of the multi-germs.

We first give a result which gives codimensional restrictions on the number of branches possible in this situation and also gives limiting conditions on these branches.

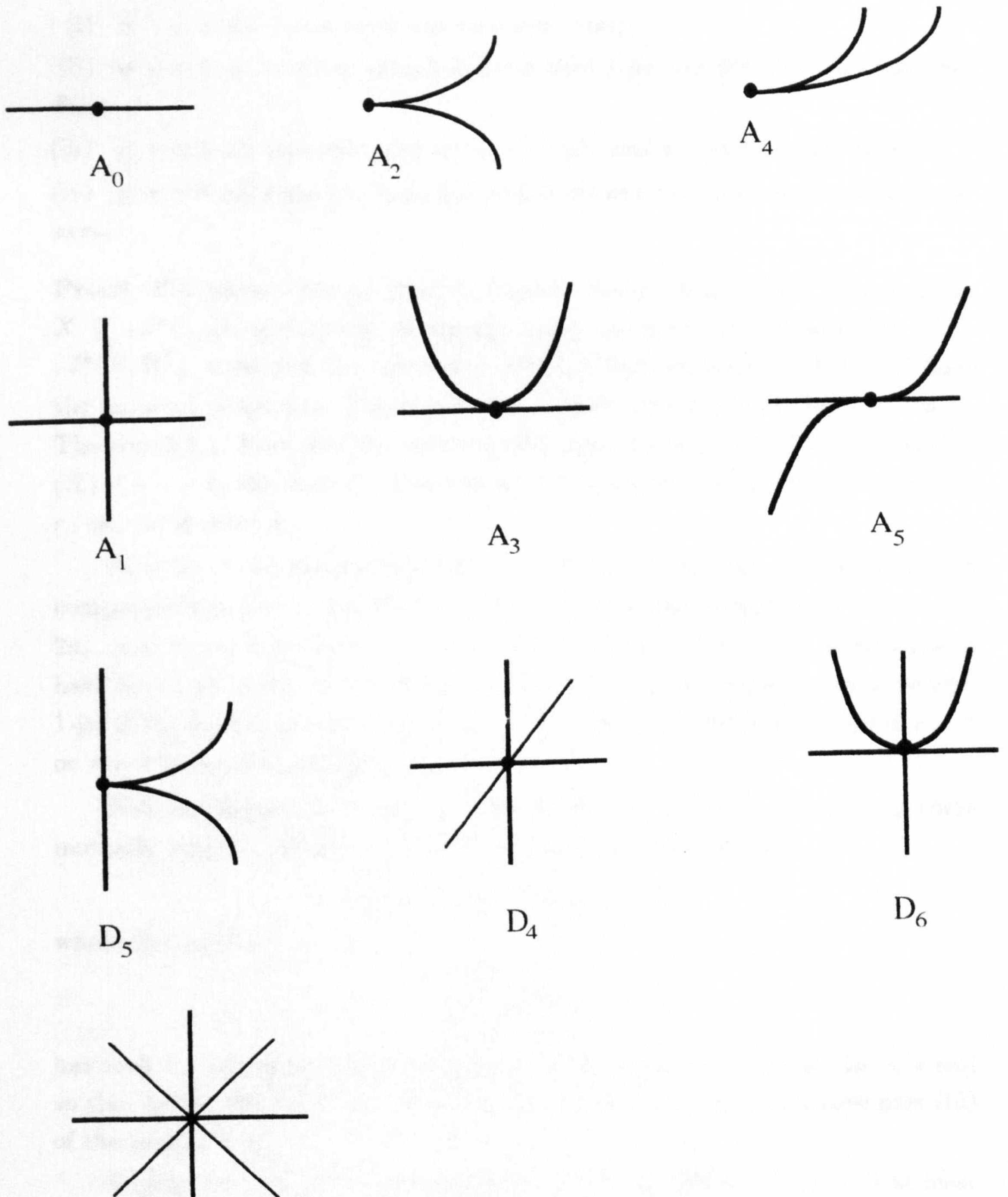



Fig. 4.1.1

**4.1.2 Proposition** *A multi-germ singularity* $\mathbf{R}, 0 \to \mathbf{R}^2, 0$ *with codimension less than 3 can have at most* $r = 4$ *branches. Also*

(i) *if* $r = 1$ *the mono-germ has non-zero 2-jet;*

(ii) *if* $r = 2$ *at least one branch has non-zero 1-jet and the other has non-zero 2-jet;*

(iii) *if* $r = 3$ *all branches have non-zero 1-jet, and at most two are tangent;*

(iv) *if* $r = 4$ *all branches have non-zero 1-jet and they are all mutually transverse.*

**Proof** We prove this by finding finitely many $\mathcal{A}$-invariant submanifolds $X \subseteq {}_rJ^k(1,2)$, giving rise to finitely many $\mathcal{A}$-invariant submanifolds $Y \subseteq {}_rJ^k(N, \mathbf{R}^2)$, such that the motions $\mu$ with ${}_rj^k\Phi_\mu$ transverse to all the $Y$ have the required properties. The result then follows from the transversality result, Theorem 2.2.1. Note that the transversality hypothesis implies that codimension $(X) \leq 4 - r$ in this case (by Proposition 2.2.3) and so the number of branches, $r$, can be at most 4.

Now let $X$ be the submanifold of ${}_rJ^k(1,2)$ comprising $k$-jets whose $i$th component has zero $a_i$-jet, for $1 \leq i \leq r$. Then $X$ has codimension $2a_1 + \ldots + 2a_r$, and so we have $2a_1 + \ldots + 2a_r \leq 4 - r$. Thus if $r = 1$ it is possible to have zero 1-jet if the 2-jet is non-zero. If $r = 2$ then one branch may have zero 1-jet if the 2-jet is non-zero whilst the other branch is immersive, and if $r = 3$ or $r = 4$ then all branches must be immersive.

Now we choose $X$ to be the submanifold of ${}_3J^k(1,2)$ comprising three mutually tangent one-jets. An element of $X$ will have the form

$$(a_1t, b_1t; a_2s, b_2s; a_3u, b_3u)$$

where the matrix

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

has rank 1. This gives two polynomial conditions on the $a_i$'s and the $b_i$'s and so the codimension of $X$ is 2. This is larger than $4 - r$ and so we have part (iii) of the proposition.

Finally, let $V$ to be the subvariety of ${}_4J^k(1,2)$ of four 1-jets with at least one tangency. This can be divided into four smooth manifolds according to the multiplicity of the lines. If $X$ is the submanifold of four mutually tangent one-jets then, as above the codimension is given by the rank of a matrix being one and gives codimension of X is 3 which is too large. If $X$ is the submanifold comprising four 1-jets with three mutual tangencies then two $2 \times 2$ minors must be zero and so the codimension of $X$ is 2, and if $X$ consists of four 1-jets with two of them being tangent then the codimension is 1 as one $2 \times 2$ minor is zero. Thus the codimension of $X$ is greater than $4 - r$ in each case and part (iv) of the proposition is proved. □

### 4.2 Mono-germs $(\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$

The mono-germs were classified by Bruce [B2]; he also calculated the $\mathcal{A}_e$-codimensions of such germs and showed that all those of codimension less than 8 are $\mathcal{A}$-simple. So we can read directly from his list to find which singularities have codimension $\leq 2$ and we know that these are all $\mathcal{A}$-simple. Thus we have:

| Germ | $\mathcal{A}_e$-codimension |
|---|---|
| $f(t) = (t, 0)$ | 0 |
| $f(t) = (t^2, t^3)$ | 1 |
| $f(t) = (t^2, t^5)$ | 2 |

### 4.3 Bi-germs $(\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$

Given a bi-germ, $F$, we know that for the codimension to be sufficiently small for our purposes at least one of the two branches is an immersion (Proposition 4.1.2). We have the following Proposition (which has also been proved in [dPT]).

**4.3.1 Proposition** *An $\mathcal{A}$-finite bi-germ* $\mathbf{R}, 0 \to \mathbf{R}^2, 0$ *with at least one immersive branch and the other branch with non-zero 2-jet is $\mathcal{A}$-equivalent to one of the following $\mathcal{A}$-simple normal forms.*

| Bi-germ | | $\mathcal{A}_e$-codimension |
|---|---|---|
| $(t,0;0,s)$ | | $0$ |
| $(t,0;s,s^k)$ | | $k-1$ |
| $(t,0;s^{2k+1},s^2)$ | | $k+1$ |
| $(t,0;s^2,s^{2k+1})$ | | $3k$ |
| $(t,0;s^2,s^{2k+1}+s^{2l})$ | $k<l\le 2k+1$ | $l+k$ |
| $(t,0;s^2,s^{2k}+s^{2l+1})$ | $k\le l$ | $l+k$ |

**Proof** Assume that the first branch is immersive. Then we may write it as $(t,0)$. Using a method similar to that used in [BGi], we will then work with a subgroup of $\mathcal{A}_1$ which fixes the $x$-axis, i.e. we will use deformations which do not affect $y=0$. In practise this means that when we are calculating the tangent space to the orbit of a germ under this group we must use vector fields preserving $y=0$. So instead of using vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ we shall use $\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$. Let us call this subgroup $\mathcal{B}_1$.

If both branches of the bi-germ are immersive the two possible 1-jets are

(i) (t,0;0,s) and

(ii) (t,0;s,0).

(i) $j^1F=(t,0;0,s)$. This is clearly 1-determined so we need go no further, and we obtain the first bi-germ in table 1. The $\mathcal{A}_e$-codimension can be calculated from the $\mathcal{A}_e$-tangent space:

$$\begin{aligned} T\mathcal{A}_e.F &= \mathcal{E}_1\langle(1,0;0,0)\rangle+\mathcal{E}_1\langle(0,0;0,1)\rangle+F^*.\mathcal{E}_2\langle(1,0),(0,1)\rangle \\ &= \mathcal{E}_1.\mathcal{O}_{2,2} \end{aligned}$$

So the $\mathcal{A}_e$-codimension is 0.

(ii) $j^1F=(t,0;s,0)$. We have

$$T\mathcal{B}_1.f_2=m_1^2\langle(1,0)\rangle+f_2^*.m_2^2\langle(1,0)\rangle.$$

Thus, using the method of Theorem 1.5.1, $T=\{(0,s^k)\}$ and so a complete transversal is $(t,0;s,as^k)$. In order for $F$ to be $\mathcal{A}$-finite we must have $a\neq 0$ for some $k>1$ (if $k=1$ we get case (i)). Then the $\mathcal{A}_1$-tangent space is

$$\begin{aligned} T\mathcal{A}_1.F &= m_1^2\langle(1,0;0,0)\rangle+m_1^2\langle(0,0;1,ks^{k-1})\rangle+F^*.m_2^2\langle(1,0),(0,1)\rangle \\ &\supseteq m_2^{k+1}.\mathcal{E}_{2,2} \end{aligned}$$

so $F$ is $k$ determined. Calculating the $\mathcal{A}_e$-tangent space in a similar way we see that the $\mathcal{A}_e$-codimension is $k-1$. We now have the second entry in the table.

Now let us consider bi-germs with only one immersive branch. Since the second branch is to have non-zero 2-jet (Propostion 4.1.2) there are two possibilities,

(i) $(t,0;0,s^2)$ and

(ii) $(t,0;s^2,0)$.

Again we work with the group $\mathcal{B}_1$.

(i) Suppose that $j^{q-1}F=(t,0;0,s^2)$. We have

$$T\mathcal{B}_1.f_2 = m_1^2\langle(0,2s)\rangle + m_2^2\langle(1,0)\rangle + f_2^*.m_2^1\langle(0,s^2)\rangle$$

So if $q$ is odd, ie $q=2k+1$ then $T=\{(s^{2k+1},0)\}$; otherwise $T$ is trivial. Thus when $q=2k+1$ a complete transversal is $(t,0;as^{2k+1},s^2)$. If $F$ is to be $\mathcal{A}$-finite then we need $a\neq 0$ for some $k\geq 1$ so $j^{2k+1}F=(t,0;s^{2k+1},s^2)$. Then

$$\begin{aligned} T\mathcal{A}_1.F =& m_2^2\langle(1,0;0,0)\rangle + m_2^2\langle(0,0;(2k+1)s^{2k},2s)\rangle + F^*.m_2^2\langle(1,0),(0,1)\rangle \\ \supseteq& m_2^{2k+2}.\mathcal{E}_{2,2} \end{aligned}$$

and so $F$ is $2k+1$ determined. Similarly, the $\mathcal{A}_e$-codimension is $k+1$. We have the third entry on the list.

(ii) Suppose that $j^{k-1}F=(t,0;s^2,0)$. We have

$$T\mathcal{B}_1.f_2 = m_1^2\langle(2s,0)\rangle + m_2^2\langle(1,0)\rangle.$$

So $T=\{(0,s^k)\}$ and a complete transversal is $(t,0;s^2,as^k)$. As before, since $F$ is $\mathcal{A}$-finite we must have $a\neq 0$ for some $k\geq 1$. So $j^kF=(t,0;s^2,s^k)$. Since this is not $k$ determined we continue with the complete transversal method. Suppose that $j^{m-1}F=(t,0;s,s^k)$. Now we have

$$T\mathcal{B}_1.f_2 = m_1^2\langle(2s,ks^{k-1})\rangle + f_2^*.m_2^2\langle(1,0)\rangle + f_2^*.m_2^1\langle(0,s^k)\rangle$$

Now if $k=2n+1$

$$J^m(T\mathcal{B}_1.f_2)\cap H^m(1,2) = \begin{cases} \{(s^m,0),(0,s^m)\} \\ \{s^m,0)\} \end{cases}$$

where the first case occurs if $m=2n+2p+1$ for any $p\geq 1$, or for $m>4n+1$, and the second case occurs if $m=2n+2p$ for $0\leq p\leq n$. So either $T$ is trivial or $T=\{(0,s^m)\}$ where $m$ is even and less than $4n+2$. Then we have

$$F\sim\begin{cases} (t,0;s^2,s^{2n+1}) \\ (t,0;s^2,s^{2n+1}+s^{2p}) & n<p<2n+1 \end{cases}$$

The latter is completely determined so we can now look at $k = 2n$. Then we have

$$J^m(T\mathcal{B}_1.f_2) \cap H^m(1,2) = \begin{cases} \{(s^m,0),(0,s^m)\} \\ \{s^m,0)\} \end{cases}$$

with the first case if $m$ is even and the second case if $m$ is odd. Thus if $m = 2l+1$ we have $T = \{(0, s^{2l+1})\}$ and otherwise $T$ is trivial. So the cases are

$$F \sim \begin{cases} (t,0;s^2,s^{2n}) \\ (t,0;s^2,s^{2n}+s^{2l+1}) & n<l \end{cases}$$

The latter is $(2l+1)$-determined so we have now completed case (ii) and the table in the statement of the proposition. The codimension calculations are straightforward. □

From this it is easy to see that the bi-germs we are interested in, those with codimension $\leq 2$, are as follows:

| Bi-germ | Name | $\mathcal{A}_e$-codimension |
|---|---|---|
| $(t,0;0,s)$ | $A_1$ | 0 |
| $(t,0;s,s^2)$ | $A_2$ | 1 |
| $(t,0;s,s^3)$ | $A_3$ | 2 |
| $(t,0;s^3,s^2)$ | $D_3$ | 2 . |

## 4.4 Tri-germs $(\mathbf{R},0) \to (\mathbf{R}^2,0)$

By Proposition 4.1.2, all branches of a tri-germ with codimension less than 3 must be immersive, and also at most two of the branches can be tangent.

**4.4.1 Proposition** *An $\mathcal{A}$-finite tri-germ $F$ with all branches immersions and with at most two branches tangent is $\mathcal{A}$-equivalent to $(t,0;0,s;u,u^k)$ (which is $\mathcal{A}$-simple) for some $k \geq 1$.*

**Proof** Given a tri-germ with three immersive branches and no more than two tangent to each other, we can choose the two non-tangent ones and change coordinates to get them into the form $(t,0;0,s)$. As before we use a geometric subgroup of $\mathcal{A}_1$ which preserves these two which we shall denote $\mathcal{C}_1$. The vector fields that we shall use instead of the standard ones are $x\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$.

By linear algebra, since the third branch has non-zero 1-jet, the possible 1-jets are

(i) $(t,0;0,s;u,u)$ or

(ii) $(t,0;0,s;u,0)$

(i) If $j^1F = (t,0;0,s;u,u)$ then we find that

$$\begin{aligned} T\mathcal{A}_1.F &= m_2^2\langle(1,0;0,0;0,0)\rangle + m_2^2\langle(0,0;0,1;0,0)\rangle \\ &\quad + m_2^2\langle(0,0;0,0;1,1)\rangle + F^*.m_2^2\langle(1,0),(0,1)\rangle \\ &\supseteq m_2^2.\mathcal{E}_{2,3} \end{aligned}$$

so $F$ is 1-determined. The $\mathcal{A}_e$-codimension is 1.

(ii) $j^1F = (t,0;0,s;u,0)$ and so

$$T\mathcal{C}_1.f_3 = m_2^2\langle(1,0)\rangle + f_3^*.m_2^2\langle(u,0)\rangle$$

Then a complete transversal is $(t,0;0,s;u,au^k)$, and since $F$ is $\mathcal{A}$-finite we h $a \neq 0$ for some $k > 1$ (if $k = 1$ we go to case (i)). So $j^kF = ((t,0;0,s;u,$ and in fact this is $k$ determined, with $\mathcal{A}_e$-codimension $k$.

Thus the relevant possibilities are

| Trigerm | $\mathcal{A}_e$-codimension |
|---|---|
| $(t,0;0,s;u;u)$ | 0 |
| $(t,0;0,s;u,u^2)$ | 2 |

## 4.5 Four-germs $(\mathbf{R},0) \to (\mathbf{R}^2,0)$

From Proposition 4.1.2 we see that the only possible four-germ that co occur is $\mathcal{A}$-equivalent to one with four immersive branches, $(t,0;0,s;u,u;v,$ (where $\lambda \neq 0,1$). Such a multi-germ has a modulus, $\lambda$, attached to it (the cro ratio) and so is not simple. Each individual four-germ has $\mathcal{A}_e$-codimensior however, the stratum consisting of all such four-germs has $\mathcal{A}_e$-codimensio and thus could occur in our situation.

## 4.6 Unfoldings of One Dimensional Motions of the Plane

As we have now listed the possible singularities of 1 dimensional motions the plane, it is interesting to go on to look at the unfoldings of these singularit i.e. to look at all the germs which are close to them. Using the methods describ in section 1.4, to calculate versal unfoldings we look at our calculations of $T\mathcal{A}$ (from the codimension calculations) and then find a suitable set $\{f_1,\ldots,f_n\}$ make up the whole jet space. We then study the unfoldings to get a picture the parameter space.

(i) $f(t) = (t, 0)$ has trivial unfolding since the $\mathcal{A}_e$-codimension is zero.

(ii) $f(t) = (t^2, t^3)$. The $\mathcal{A}_e$-codimension is 1, $f_1 = (0, t)$ and so the unfolding is $F_a(t) = (t^2, t^3 + at)$. The picture in parameter space is shown in Fig. 4.6.1. The bifurcation set in this case is $\{a = 0\}$. When $a > 0$ there is no singular point on the trajectory but when $a < 0$ there is a transverse crossing on $F_a(t)$. At $a = 0$ we have a cusp.

(iii) $f(t) = (t^2, t^5)$. The $\mathcal{A}_e$-codimension is 2, $f_1 = (0, t^3)$ and $f_2 = (0, t)$ so an unfolding is $F_{a,b}(t) = (t^2, t^5 + at^3 + bt)$. The picture in parameter space is shown in Fig. 4.6.2. Here the bifurcation set consists of the half parabola, $a^2 = 4b$, where $a \leq 0$, and the $x$-axis. The half parabola is a curve of tacnodes.

(iv) $f(t; s) = (t, 0; 0, s)$. The $\mathcal{A}_e$-codimension is 0 so the unfolding is trivial.

(v) $f(t; s) = (t, 0; s, s^2)$. The $\mathcal{A}_e$-codimension is 1 and $f_1 = (0, 1; 0, 0)$, so an unfolding is $F_a(t; s) = (t, a; s, s^2)$ and in parameter space we see the picture shown in Fig. 4.6.3. Again, the bifurcation set is just the point $a = 0$. When $a < 0$ there are no singularities on $F_a(t)$; when $a = 0$ there is a tacnode, and when $a > 0$, $F_a(t)$ has two distinct transverse crossings.

(vi) $f(t; s) = (t, 0; s, s^3)$. The $\mathcal{A}_e$-codimension is 2 and $f_1 = (0, 1; 0, 0)$, $f_2 = (0, 0; 0, s)$. So an unfolding is $F_{a,b}(t; s) = (t, a; s, s^3 + bs)$ and the parameter space is shown in Fig. 4.6.4. The bifurcation set consists of the cuspidal cubic $27a^2 = -4b^3$. On this curve all germs have a double point and a tacnode (which coalesce at $(0, 0)$ to give a crossing with order of contact 3). Inside the cusp each germ has three distinct double points and outside each has only one double point.

(vii) $f(t; s) = (t, 0; s^3, s^2)$. The $\mathcal{A}_e$-codimension is 2 and we can choose $f_1 = (0, 1; 0, 0)$, $f_2 = (0, 0; s, 0)$. So the unfolding is $F_{a,b}(t; s) = (t, a; s^3 + bs, s^2)$ and in the parameter space is shown in Fig. 4.6.5. Here the bifurcation set is made up of the $a$- and $b$-axes and the half-line $a = -b$, where $a < 0$. Along this half-line every germ has a triple point. Germs along the $x$-axis all have a tacnode (except for $F_{0,0}$) and all those on the $y$-axis have a cusp.

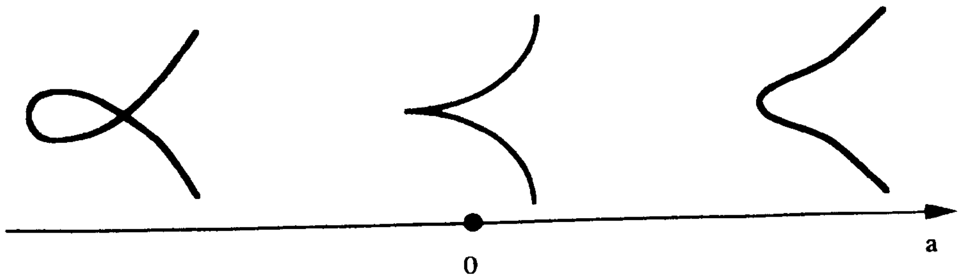



Fig. 4.6.1

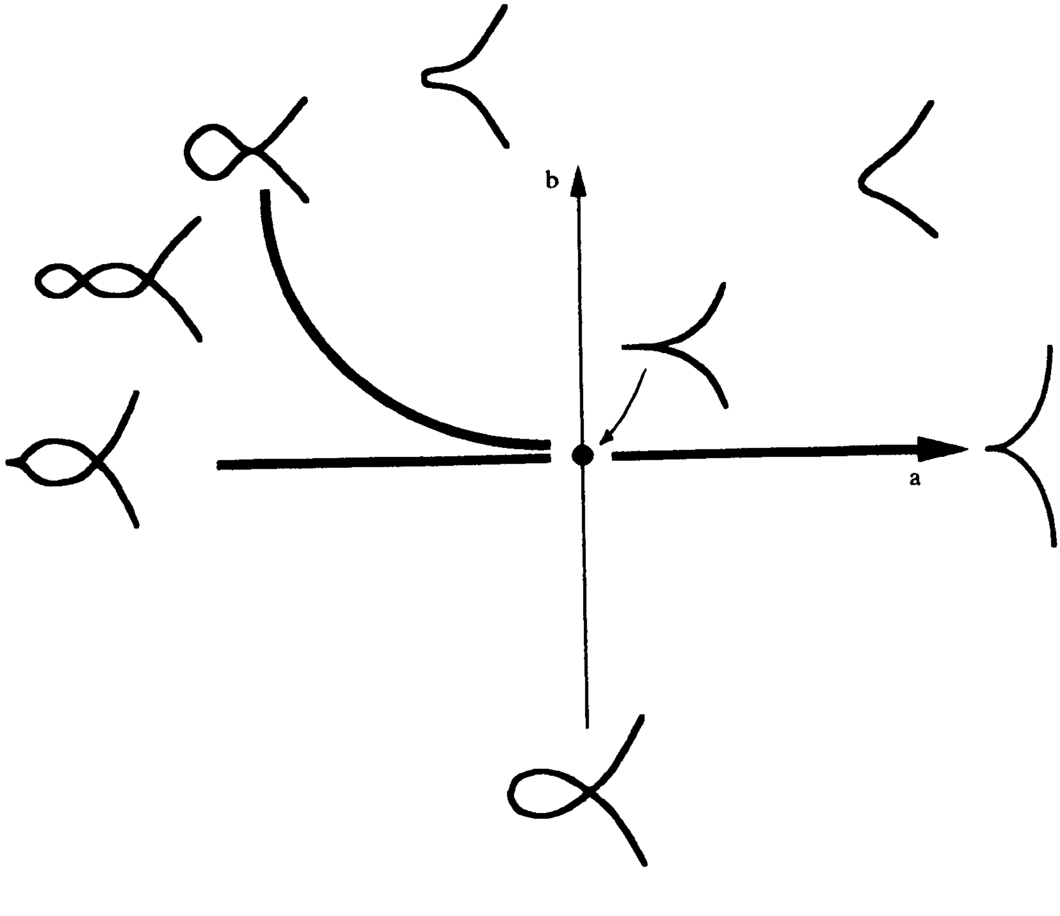



Fig. 4.6.2

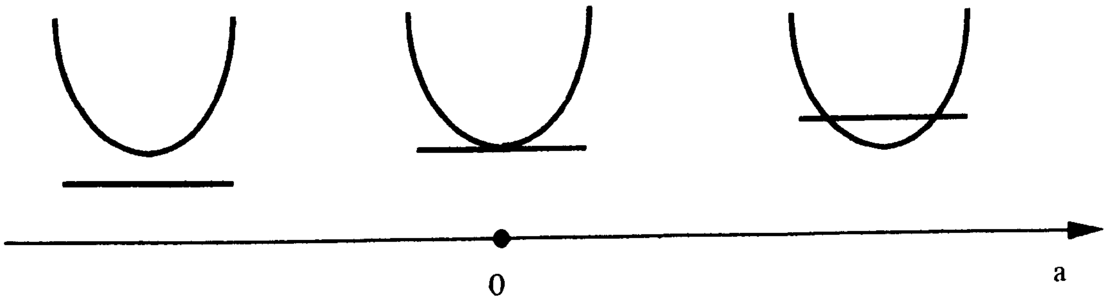



Fig. 4.6.3

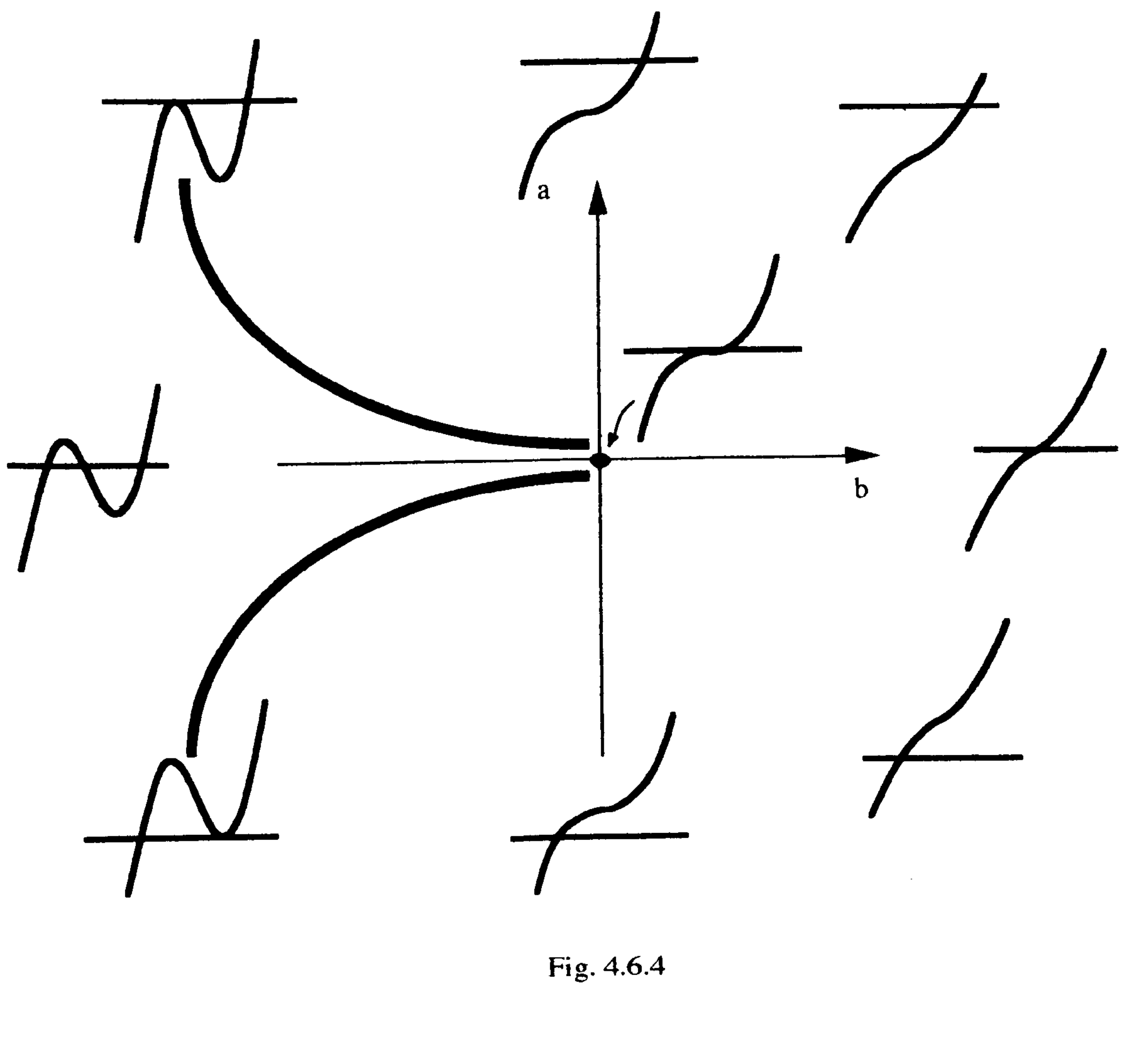



Fig. 4.6.4

(viii) $f(t;s;u) = (t,0;0,s;u,u)$. The $\mathcal{A}_e$-codimension is 1 and $f_1 = (0,0;0,0;0,1)$, so an unfolding is $F_a(t;s;u) = (t,0;0,s;u,u+a)$ and the picture in the unfolding space is shown in Fig. 4.6.6. The bifurcation set is the point $a = 0$. Here the trajectory has a triple point; otherwise there are three double points on $F_a(t;s;u)$.

(ix) $f(t;s;u) = (t,0;0,s;u,u^2)$. The $\mathcal{A}_e$-codimension is 2. Choose $f_1 = (0,t;0,0;0,0)$ and $f_2 = (0,1;0,0;0,0)$, then the unfolding is $F_{a,b}(t;s;u) = (t,at+b;0,s;u,u^2)$. Parameter space is shown in Fig. 4.6.7. The bifucation set in this case is the parabola $a^2 = -4b$ together with the $y$-axis.

Apart from at the origin, along the parabola each germ has a tacnode and two double points, while inside the parabola the germs have only two double points. Along the $y$-axis all germs have one triple and one double point, excluding the origin.

(x) $f(t;s;u;v) = (t,0;0,s;u,u;v,\lambda v)$ (where $\lambda \neq 0,1$). The $\mathcal{A}_e$-codimension of the stratum is 2 and if we choose $f_1 = (0,0;0,0;1,0;0,0)$ and $f_2 = (0,0;0,0;0,0;1,0)$ then the unfolding is $F_{a,b}(t;s;u;v) = (t,0;0,s;u+a,u;v+b,\lambda v)$. The picture, when $\lambda = 1$, is shown in Fig. 4.6.8. The bifurcation set consists of four lines through the origin. On these lines (apart from at the origin where there is a quadruple point) the trajectory has a triple point and three distinct double points, otherwise it has six distinct double points.

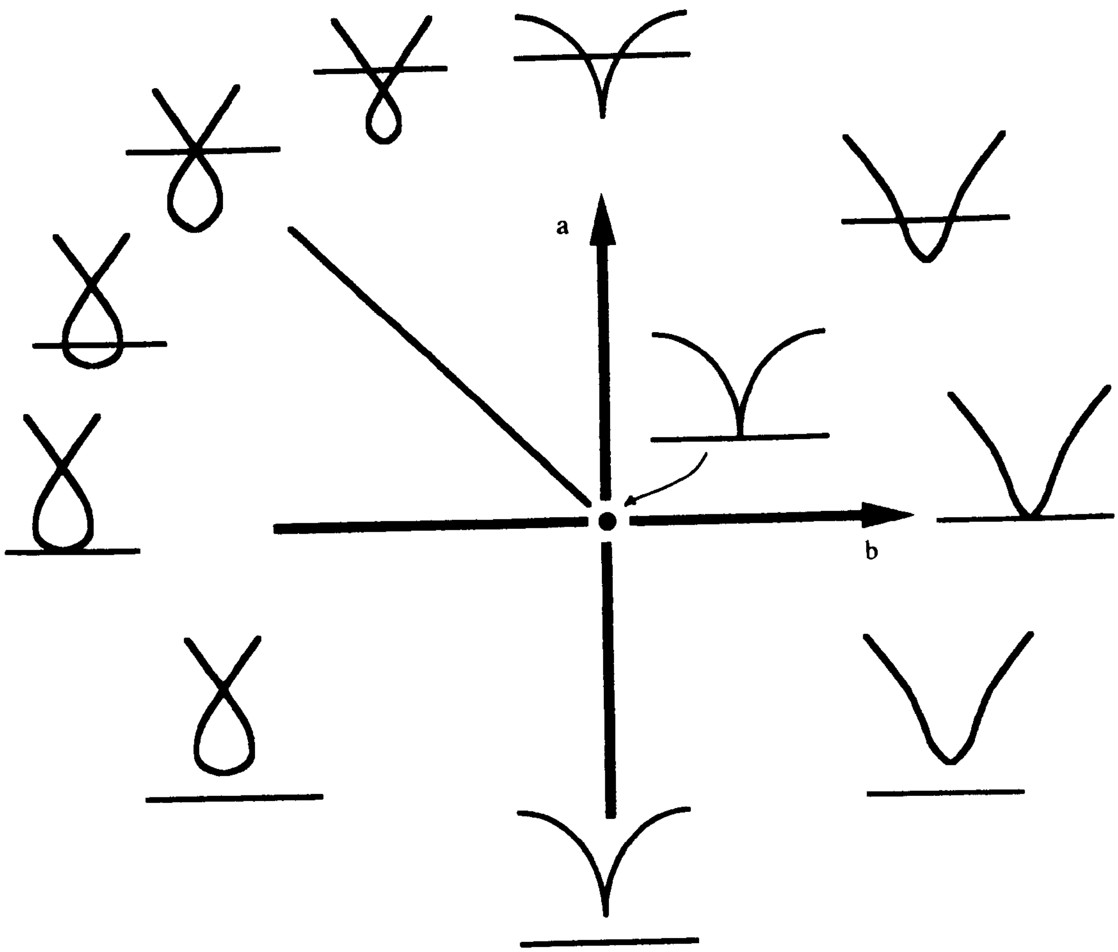



Fig. 4.6.5

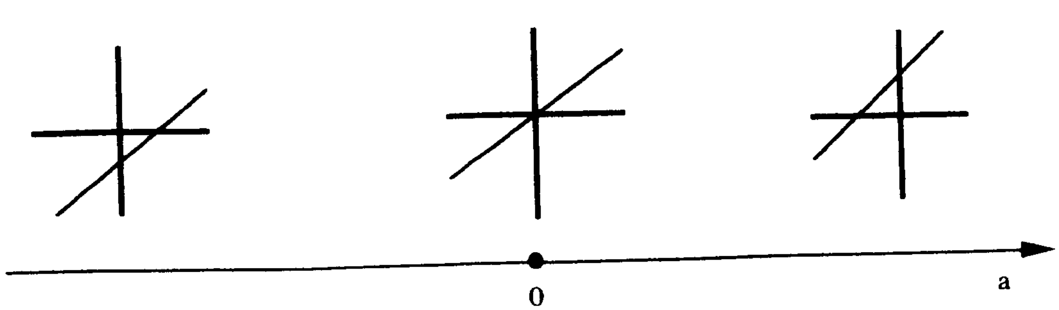



Fig. 4.6.6

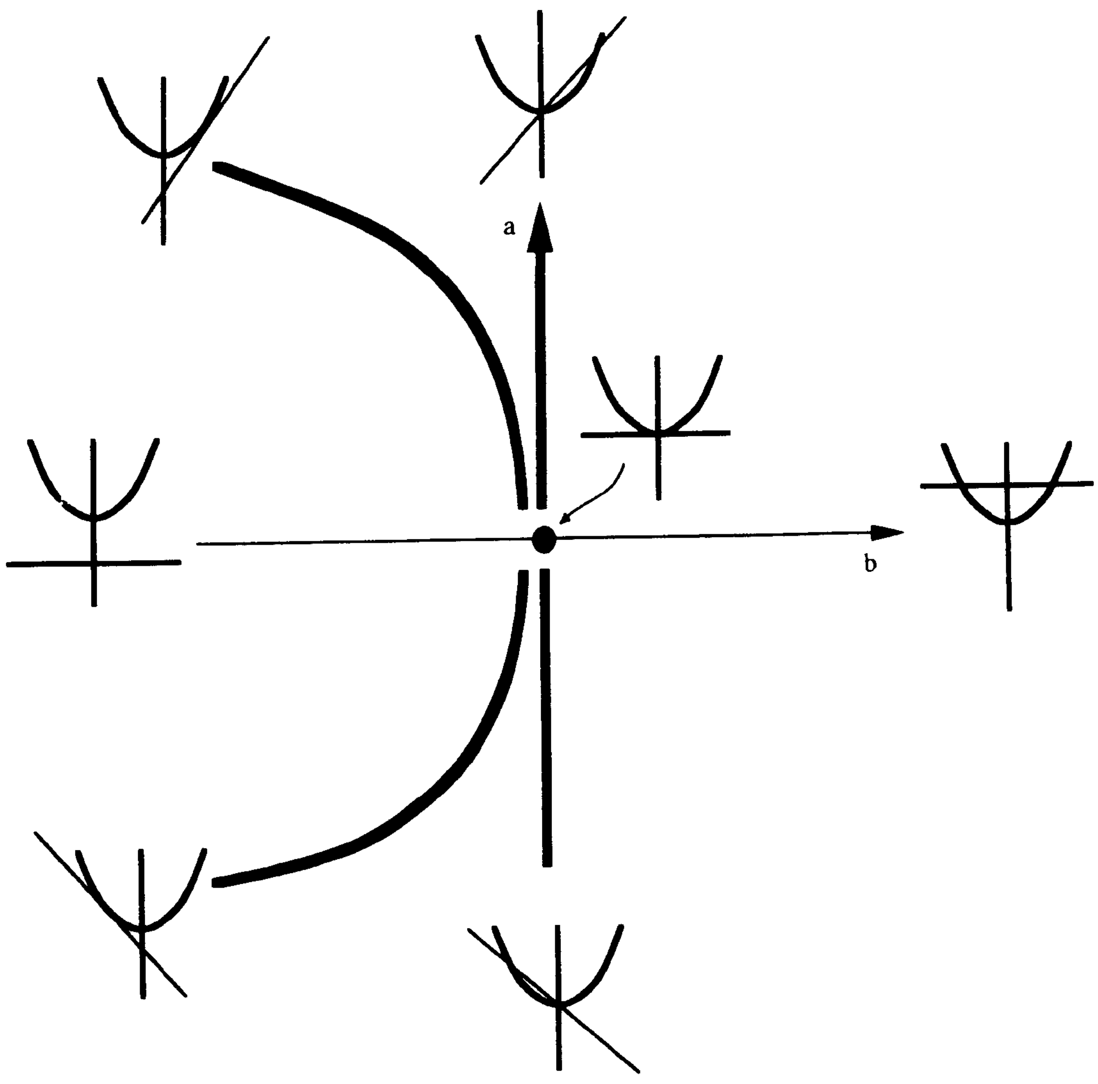



Fig. 4.6.7

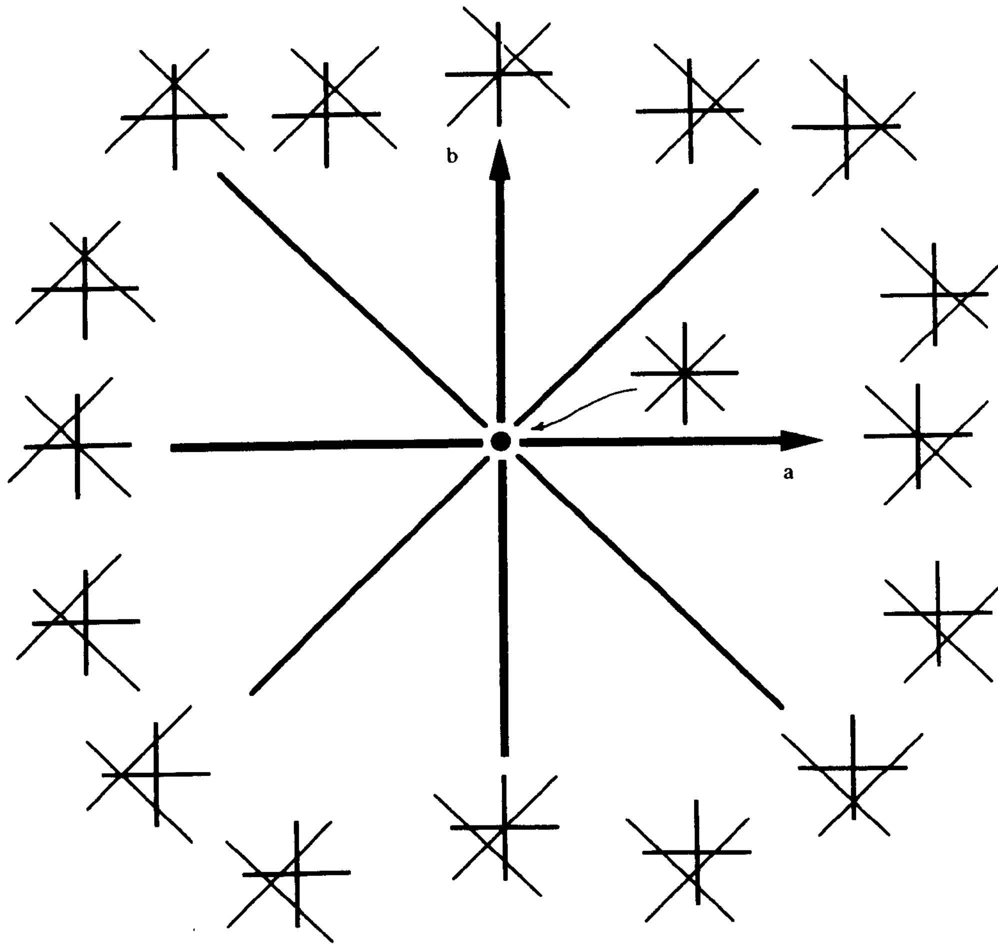



Fig. 4.6.8

# Chapter 5 – One-Dimensional Motions of Space

## 5.1 Introduction

We now go on to look at motions of space with one degree of freedom. We find that there are fewer types of singularities which can generically occur on the trajectories of one dimensional motions of space than of those in the plane. The different possibilities are summarized in the next theorem.

**5.1.1 Theorem** *On the trajectory of a generic motion of space with one degree of freedom we expect, locally, to see only multi-germs $\mathcal{A}$-equivalent to those in the following table.*

| Normal Form | $\mathcal{A}_e$-codim | Name |
|---|---|---|
| $(t,0,0)$ | 0 | $A_0$ |
| $(t^2,t^3,0)$ | 2 | $A_2$ |
| $(t,0,0;0,s,0)$ | 1 | $A_1$ |
| $(t,0,0;s,s^2,0)$ | 3 | $A_3$ |
| $(t,0,0;0,s,0;0,0,u)$ | 2 | $D_4$ |

The pictures of these normal forms are shown in Fig. 5.1.1. Again, the names refer to the $\mathcal{K}$-types of the images of the multi-germs.

As in chapter 4 we now give a result limiting the number of branches a multi-germ $\mathbf{R},0 \to \mathbf{R}^3,0$ can have in order for the codimension to be less than 4.

**5.1.2 Proposition** *A multi-germ singularity $\mathbf{R},0 \to \mathbf{R}^3,0$ with codimension less than 4 can have at most $r = 3$ branches. Also*

(i) *If $r = 1$ the germ must have non-zero two jet;*

(ii) *If $r = 2$ both branches must be immersions (i.e. both must have non-zero one jet);*

(iii) *If $r = 3$ all three branches must be immersions and they must all be mutually transverse.*

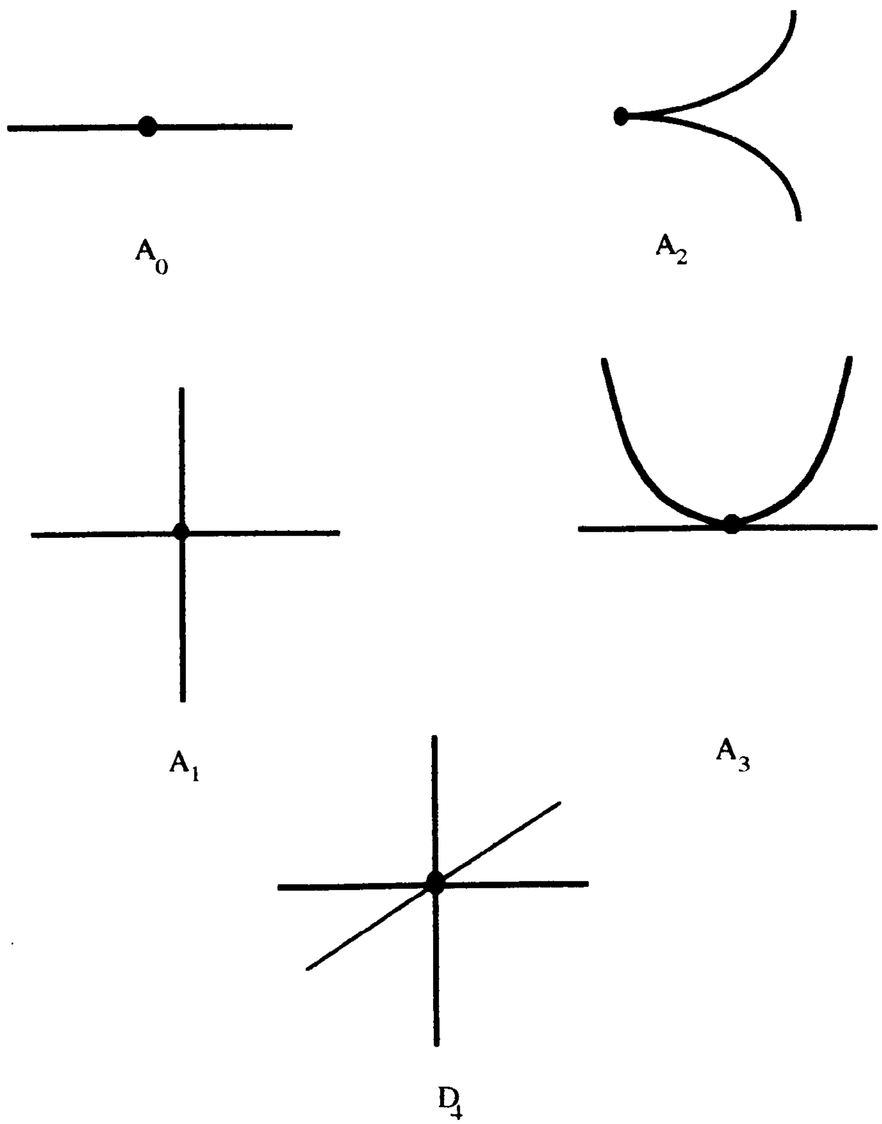



Fig. 5.1.1

**Proof** As in Proposition 4.1.2, we will prove this by finding finitely many $\mathcal{A}$-invariant submanifolds of ${}_rJ^k(1,3)$ such that the motions with trajectories transverse to the corresponding submanifolds of ${}_rJ^k(N,\mathbf{R}^3)$ have the stated properties. By Proposition 2.2.4 we know that the codimension of $X$ is $\leq 6-2r$ and so we have $r \leq 3$, giving the first part of the proposition.

Now if $X$ is the submanifold of ${}_rJ^k(1,3)$ consisting of $r$-germs with $i$th component having zero $a_i$-jet then $X$ has codimension $3a_1 + \ldots + 3a_r$. So if $r = 1$ we must have codim $X \leq 4$ so at most the two-jet must be non-zero. If $r = 2$ then codim $X \leq 2$ so both branches must be immersive, and if $r = 3$ then codim $X = 0$ so again all branches must be immersive.

Suppose that $X$ is the submanifold of ${}_3J^k(1,3)$ comprising three mutually tangent 1-jets. Then an element of $X$ has form

$$(a_1t, a_2t, a_3t; b_1s, b_2s, b_3s; c_1u, c_2u, c_3u)$$

where the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

has rank 1. So the codimension of $X$ is 2, which is too large. Similarly, if $X$ comprises three 1-jets with two being tangent then the matrix must have rank 2 and codim $X = 1$. □

## 5.2 Monogerms $(\mathbf{R},0) \to (\mathbf{R}^3,0)$

Mono-germs $\mathbf{R},0 \to \mathbf{R}^3,0$ were classified in chapter 3. In order to find all those germs of codimension less than or equal to 3 we simply need to read from the list in section 3.6. Clearly, all the singular germs we are interested in will be $\mathcal{A}$-simple, by Proposition 3.6.1. The possibilities are either an immersive germ, which will be $\mathcal{A}$-equivalent to the normal form $(t,0,0)$, or the singular mono-germ $(t^2,t^3,0)$, giving the first two entries in the table above.

## 5.3 Bi-germs $(\mathbf{R},0) \to (\mathbf{R}^3,0)$

As before, in order to classify the bi-germs we fix one branch and consider the possibilities for the other one. Both branches are immersions (Proposition 5.1.2) so let the first branch be $\mathcal{A}$-equivalent to $(t,0,0)$ and consider the subgroup, $\mathcal{B}_1$, of $\mathcal{A}_1$ which keeps $y = z = 0$ fixed. When calculating tangent spaces with respect to this group we shall use the following vector fields which preserve $y = z = 0$:

$$\frac{\partial}{\partial x} \qquad y\frac{\partial}{\partial y} \qquad z\frac{\partial}{\partial y} \qquad y\frac{\partial}{\partial z} \qquad z\frac{\partial}{\partial z}$$

rather than the standard vector fields.

**5.3.1 Proposition** *If $F$ is an $\mathcal{A}$-finite bi-germ $(\mathbf{R}, 0) \to (\mathbf{R}^3, 0)$ with two immersive branches then it is $\mathcal{A}$-equivalent to either $(t, 0, 0; 0, s, 0)$ or to $(t, 0, 0; s, s^k, 0)$ for some $k > 1$. In the first case the $\mathcal{A}_e$-codimension is 1 and in the second it is $2k - 1$. All possibilities are $\mathcal{A}$-simple.*

**Proof** We know that both branches of the bi-germ have non-zero 1-jet. By linear algebra, the possibilities are

(1) $F \sim (t, 0, 0; 0, s, 0)$

(2) $F \sim (t, 0, 0; s, 0, 0)$

**Case(1)** Consider $F \sim (t, 0, 0; 0, s, 0)$. If we calculate $T\mathcal{B}_1.f_2$ we see that the bi-germ is 1-determined. To find the $\mathcal{A}_e$-codimension we calculate

$$\begin{aligned} T\mathcal{A}_e.F =& \mathcal{E}_1\langle(1,0,0;0,0,0)\rangle + \mathcal{E}_1\langle(0,0,0;0,1,0)\rangle + F^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \\ =& (\mathcal{E}_1, \mathcal{E}_1, \mathcal{E}_1 - \{1\}; \mathcal{E}_1, \mathcal{E}_1, \mathcal{E}_1 - \{1\}) + \mathbf{R}\langle(0,0,1;0,0,1)\rangle \end{aligned}$$

and so the $\mathcal{A}_e$ codimension is 1.

**Case(2)** Suppose that the $(k-1)$-jet of $F$ is $(t, 0, 0; s, 0, 0)$. We want to look at the possible $k$-jets over this. We have

$$T\mathcal{B}_1.f_2 = m_1^2\langle(1,0,0)\rangle + f_2^*.m_2^2\langle(1,0,0)\rangle$$

Thus, by the method of Theorem 1.5.1, $T = \{(0, s^k, 0), (0, 0, s^k)\}$ and a complete transversal is $(t, 0, 0; s, as^k, bs^k)$. If $F$ is to be $\mathcal{A}$-finite then $(a, b) \neq (0, 0)$ and, by changes of co-ordinates, $F \sim (t, 0, 0; s, s^k, 0)$. Now

$$\begin{aligned} T\mathcal{B}_1.f_2 &= m_1^2\langle(1, ks^{k-1}, 0)\rangle + f_2^*.m_2^2\langle(1,0,0)\rangle + f_2^*.m_2^1\langle(0, s^k, 0), (0, 0, s^k)\rangle \\ &\supset m_1^{k+1}.\mathcal{E}_{3,1} \end{aligned}$$

So the bi-germ is $k$-determined and we need continue no further.

To find the $\mathcal{A}_e$-codimension we calculate the $\mathcal{A}_e$-tangent space:

$$\begin{aligned} T\mathcal{A}_e.F &= \mathcal{E}_1\langle(1,0,0;0,0,0)\rangle + \mathcal{E}_1\langle(0,0,0;1,ks^{k-1},0)\rangle \\ &\quad + F^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \\ &= (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2 - \{1, s, s^2, \ldots, s^{k-2}\}, \mathcal{E}_2 - \{1, s, s^2, \ldots, s^{k-1}\}) \end{aligned}$$

and so the $\mathcal{A}_e$-codimension is $2k-1$. □

### 5.4 Tri-germs $(\mathbf{R},0) \to (\mathbf{R}^3,0)$

By Proposition 5.1.2, a tri-germ $(\mathbf{R},0) \to (\mathbf{R}^3,0)$ with codimension less than 4 must have three immersive branches which are all transverse so the one jet of the tri-germ must be $(t,0,0;0,s,0;0,0,u)$. On calculating the $\mathcal{A}_1$-tangent space we see that this tri-germ is one determined (and $\mathcal{A}$-simple) and so we get the final entry in the table of Theorem 5.1.1. To work out the $\mathcal{A}_e$-codimension we simply find the $\mathcal{A}_e$-tangent space:

$$\begin{aligned} T\mathcal{A}_e.F =& \mathcal{E}_1\langle(1,0,0;0,0,0;0,0,0)\rangle + \mathcal{E}_1\langle(0,0,0;0,1,0;0,0,0)\rangle \\ &+ \mathcal{E}_1\langle(0,0,0;0,0,0;0,0,1)\rangle + F^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \\ =& (\mathcal{E}_1,\mathcal{E}_1-\{1\},\mathcal{E}_1-\{1\};\mathcal{E}_1-\{1\},\mathcal{E}_1,\mathcal{E}_1-\{1\};\mathcal{E}_1-\{1\},\mathcal{E}_1-\{1\},\mathcal{E}_1) \\ &+ \mathbf{R}\langle(1,0,0;1,0,0;1,0,0),(0,1,0;0,1,0;0,1,0),(0,0,1;0,0,1;0,0,1)\rangle \end{aligned}$$

so clearly the $\mathcal{A}_e$-codimension is 3.

### 5.5 Unfoldings of One-Dimensional Motions of Space

We use the methods of section 1.4 to calculate unfoldings and then analyse the unfolding spaces.

(i) $f(t)=(t,0,0)$. The $\mathcal{A}_e$-codimension is zero and so the unfolding is trivial.

(ii) $f(t)=(t^2,t^3,0)$. The $\mathcal{A}_e$-codimension is 2 and we choose $f_1=(0,t,0)$ and $f_2=(0,0,t)$, so the unfolding is $F_{a,b}(t)=(t^2,t^3+at,bt)$. Parameter space is shown in Fig. 5.5.1.

The bifurcation set consists of the $a$-axis. Along this axis $F_{a,0}$ is planar, while above and below the axis it is a twisted cubic.

(iii) $f(t;s)=(t,0,0;0,s,0)$. The $\mathcal{A}_e$-codimension is 1, $f_1=(0,0,0;0,0,1)$. So an unfolding is $F_a(t;s)=(t,0,0;0,s,a)$. Parameter space is shown in Fig. 5.5.2. The bifurcation set contains only the point $a=0$. At this point we have a transverse double point, but when $a\neq 0$ there is no singularity on $F_a(t;s)$.

(iv) $f(t;s)=(t,0,0;s,s^2,0)$. The $\mathcal{A}_e$-codimension is 3 and we choose $f_1=(0,1,0;0,0,0)$, $f_2=(0,0,1;0,0,0)$ and $f_3=(0,0,0;0,0,s)$. Then an unfolding is given by $F_{a,b,c}(t;s)=(t,a,b;s,s^2,cs)$. Parameter space is shown in Fig.

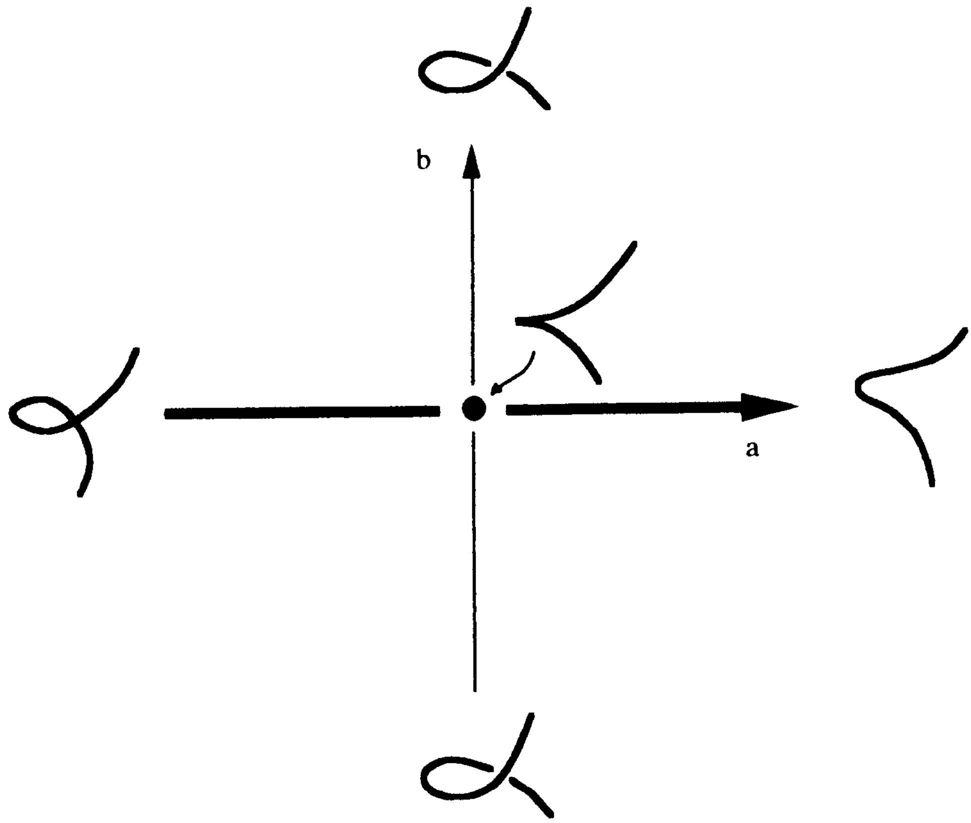



Fig. 5.3.1

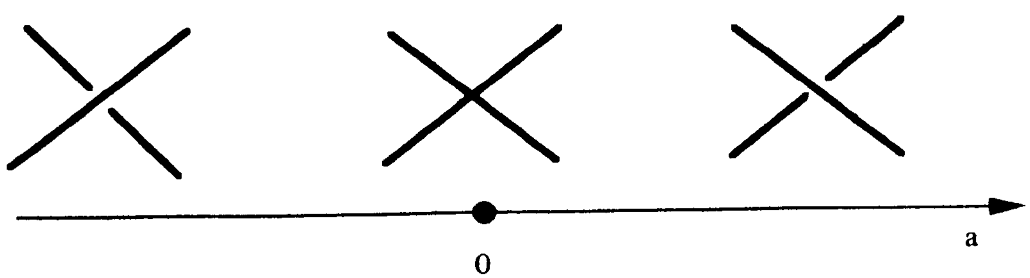



Fig. 5.3.2

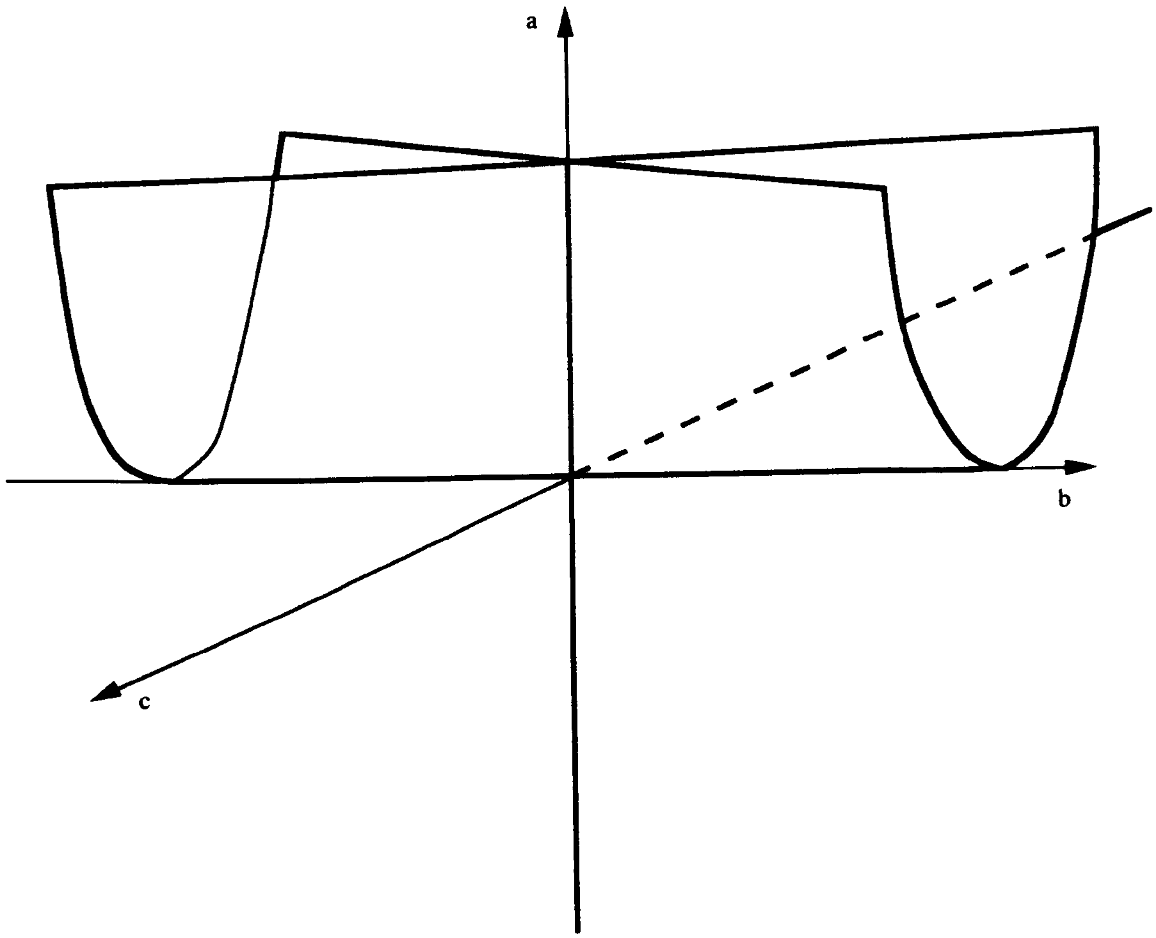



Fig. 5.3.3

5.5.3. In this case the bifurcation set is the Whitney umbrella, $ac^2 = b^2$. Off the surface, $F_{a,b,c}(t;s)$ is non-singular; on the surface itself (apart from the $a$-axis) the trajectory has one transverse crossing point; on the positive $a$-axis it has two transverse crossing points and at $(0,0,0)$ there is a tacnode.

(v) $f(t;s;u) = (t,0,0;0,s,0;0,0,u)$. The $\mathcal{A}_e$-codimension is 3 and $f_1 = (0,1,0;0,0,0;0,0,0)$, $f_2 = (0,0,0;1,0,0;0,0,0)$, $f_3 = (0,0,0;0,0,1;0,0,0)$. So an unfolding is $F_{a,b,c}(t;s;u) = (t,a,0;b,s,c;0,0,u)$. Parameter space is shown in Fig. 5.5.4. The bifurcation set consists of three orthogonal planes. At the origin there is a triple point on the trajectory, on the axes $F_{a,b,c}$ has two transverse crossing points and on the planes it has one such point. Otherwise the trajectory is non-singular.

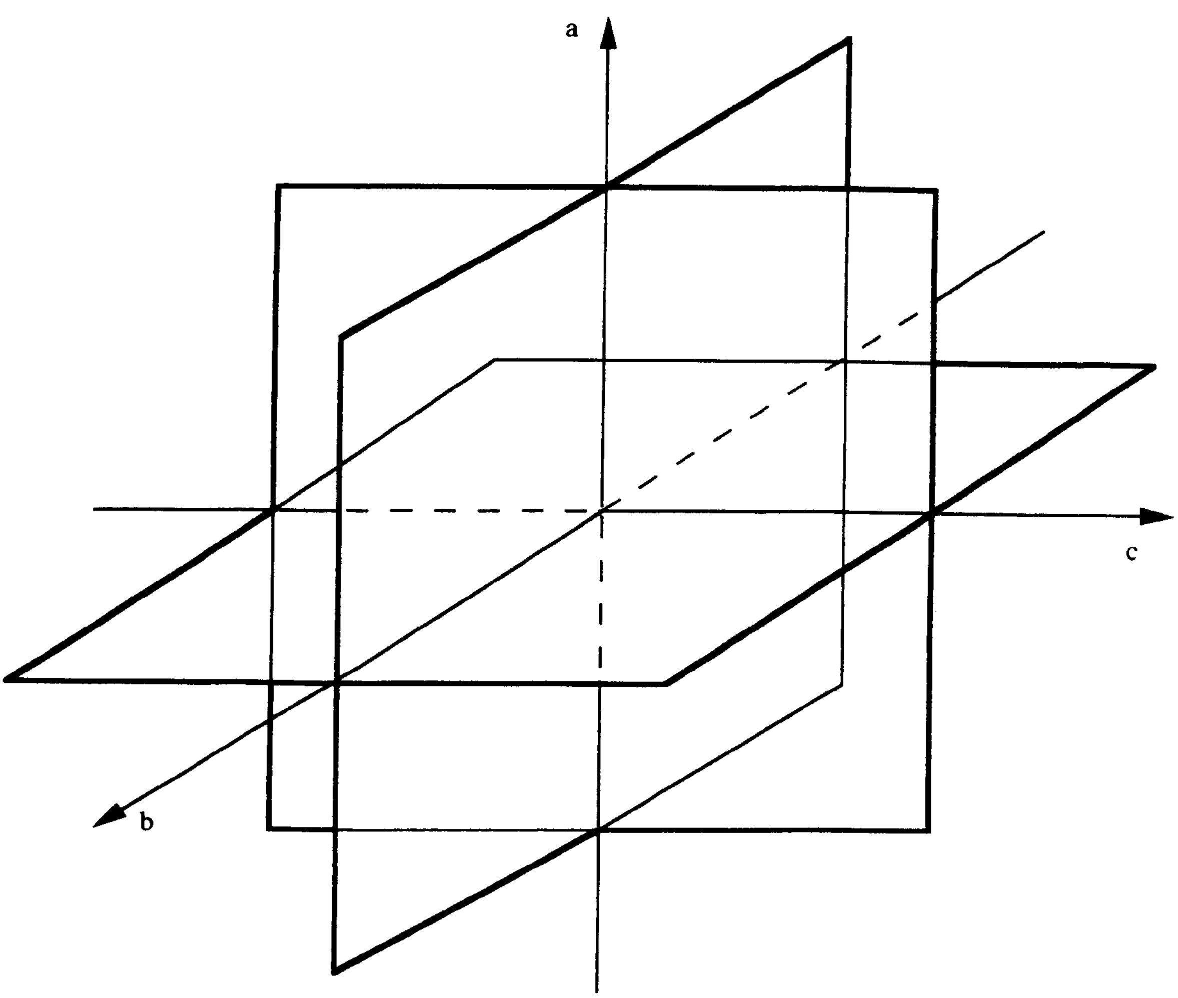



Fig. 5.3.4

# Chapter 6 – Two-Dimensional Motions of the Plane

## 6.1 Introduction

We now study motions of the plane with two degrees of freedom and look for the local models that we would expect to see on the trajectories of such motions. Examples of such motions are five-bar linkages, as discussed in the Introduction. By the transversality result, classifying motions of the plane with two degrees of freedom is equivalent to classifying singularities $\mathbf{R}^2, 0 \to \mathbf{R}^2, 0$. The trajectory of a generic motion of the plane with two degrees of freedom is a union of open sets in the plane and we expect the boundaries of these sets to exhibit, locally, the singularities listed in Theorem 6.1.1. Whitney [Wh] in 1955 showed that stable maps (here he refers to mono-germs) from the plane to the plane must be immersions, folds or cusps. These all have $\mathcal{A}_e$-codimension 0. As for non-stable map-germs $f$, we know that

$$\mathcal{A}_e\text{−codim}(f) = \mathcal{A}\text{−codim}(f) + r(p-n) - p$$

and so in this case we have

$$\mathcal{A}_e\text{−codim}(f) = \mathcal{A}\text{−codim}(f) - 2$$

for all values of $r$. So in the $\mathcal{A}$-simple case we need to look for multi-germ singularities with $\mathcal{A}$-codimension less than 5.

**6.1.1 Theorem** *For a generic motion of the plane with two degrees of freedom, any multi-germ of a trajectory is $\mathcal{A}$-equivalent to one of the normal forms listed below.*

| Type | Normal Form | $\mathcal{A}$-codim |
|---|---|---|
| 1 | $(x, y)$ | 0 |
| 2 | $(x, y^2)$ | 1 |
| 3 | $(x, xy + y^3)$ | 2 |
| $4_2$ | $(x, y^3 \pm x^2 y)$ | 3 |
| 5 | $(x, xy + y^4)$ | 3 |
| $4_3$ | $(x, y^3 + x^3 y)$ | 4 |

| Type | Normal Form | $\mathcal{A}$-codim |
|---|---|---|
| 6 | $(x, xy + y^5 \pm y^7)$ | 4 |
| $11_5$ | $(x, xy^2 + y^4 + y^5)$ | 4 |
| $\mathrm{I}_{2,2}^{1,1}$ | $(x^2 + y^3, y^2 + x^3)$ | 4 |
| $\mathrm{II}_{2,2}^{1}$ | $(x^2 - y^2 + x^3, xy)$ | 4 |
| | $(x, y^2; X^2, Y)$ | 2 |
| | $(x, y^2; X, XY + Y^2)$ | 3 |
| | $(x, y^2; X, XY + Y^3)$ | 4 |
| | $(x, y^2; XY + X^3, Y)$ | 3 |
| | $(x, y^2; X, Y^2 + X^3)$ | 4 |
| | $(x, y^2; XY^2 \pm X^3, Y)$ | 4 |
| | $(x, y^2; XY + X^4, Y)$ | 4 |
| | $(x, xy + y^3; XY + X^3, Y)$ | 4 |
| | $(x, y^2; X^2, Y; \tilde{x}, \tilde{x} + \tilde{y}^2)$ | 3 |
| | $(x, y^2; X^2, Y; \tilde{x}, \tilde{x}\tilde{y} + \tilde{y}^2)$ | 4 |
| | $(x, y^2; X^2, Y; \tilde{x}, \tilde{x} + \tilde{x}\tilde{y} + \tilde{y}^3)$ | 4 |
| | $(x, y^2; X^2, Y; \tilde{x}, \tilde{x} + \tilde{y}^2; \tilde{X}, \lambda\tilde{X} + \tilde{Y}^2)$ | 5 |

In the last case we have $\lambda \neq 0, 1$. The $\mathcal{A}$-codimension of the orbit is 5 and that of the whole stratum is 4. The type-numbers in the first column come from Rieger's list in [Ri2] and Rieger and Ruas [RiR].

In the case of singularities $\mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ it is not possible to limit the number of branches of the multi-germ in the same way as before. However, we have a result which gives codimensional restrictions on the possible types of behaviour.

**6.1.2 Proposition** *There exists a residual set of motions of the plane with two degrees of freedom such that an $r$-germ $f$ of a trajectory of such a motion has at most 4 branches (where we restrict to branches which are not $\mathcal{A}$-equivalent to $(x, y)$) and the following hold:*

(i) *if $r = 1$ then the 2-jet is non-zero;*

(ii) *if $r = 2$ then both branches have non-zero 1-jets;*

(iii) *if* $r = 3$ *then all branches have non-zero 1-jets and at most two of these are tangent;*

(iv) *if* $r = 4$ *then all branches have non-zero 1-jet and none of them are tangent.*

**Proof** As in Proposition 4.1.2 we show this by exhibiting $\mathcal{A}$-invariant submanifolds $X$ of ${}_rJ^k(2,2)$, giving rise to $\mathcal{A}$-invariant submanifolds $Y$ of ${}_rJ^k(\mathbf{R}^2,\mathbf{R}^2)$, such that the motions $\mu : \mathbf{R}^2 \to E(2)$ with ${}_rj^k\Phi_\mu$ transverse to $Y$ have the stated properties. The result then follows from Theorem 2.2.1.

From Proposition 2.2.4 we see that the transversality hypothesis implies that the codimension of $X$ is less than 5. This gives no immediate restriction on the number of branches, $r$. However, we observe that if any branch is $\mathcal{A}$-equivalent to $(x,y)$ (which has codimension 0) then it will not affect the remaining branches either visually or by increasing the codimension. Any number of such planes could be added to an existing multi-germ and not produce anything new, so we will assume that in an $r$-germ none of the branches are $\mathcal{A}$-equivalent to $(x,y)$.

Now let $X$ be the submanifold of ${}_rJ^k(2,2)$ consisting of $k$-jets whose $i$th component jet has zero $a_i$-jet for $1 \leq i \leq r$. Now the dimension of $J^0(2,2)$ is 0 and that of $J^1(2,2)$ is 4. Thus if $r = 1$ we can allow the germ to have non-zero 2-jet. It would seem that for $r > 1$ we should be able to have one non-zero 2-jet if the other branches have non-zero 1-jets but in fact if one of the branches has zero 1-jet and non-zero 2-jet then the only possibility for the other $r-1$ branches is $(x,y)$ (see Proposition 6.4.1), which we have excluded. Thus we allow only non-zero 1-jets.

For the other parts of the proposition, consider the general form of a 1-jet

$$(a_1x + b_1y, a_2x + b_2y)$$

(where this could be any of the $r$ branches of an $r$-germ). Now since we are not allowing this to be $\mathcal{A}$-equivalent to $(x,y)$ the following matrix must have rank 1:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

This gives one condition on the multi-germ. Thus if $X$ is the submanifold of ${}_5J^k(2,2)$ comprising 5-jets which are not $\mathcal{A}$-equivalent to $(x,y)$ then it has codimension 5, which is too large. So we have $r \leq 4$. If $X$ is the submanifold of ${}_4J^k(2,2)$ comprising 4-jets which are not $\mathcal{A}$-equivalent to $(x,y)$ then it has codimension 4, which is possible. However, if we impose any further conditions on $X$, such as two of the branches being tangent this will take the codimension over 4. Finally, if $X$ is the submanifold of ${}_3J^k(2,2)$ comprising 3-jets which are not $\mathcal{A}$-equivalent to $(x,y)$ then $X$ has codimension 3. Now if two of the branches are tangent one further condition is imposed (another $2 \times 2$ matrix must have rank 1) giving codimension 4. No further conditions can be added. Thus we have proved the Proposition. □

## 6.2 Mono-germs $\mathbf{R}^2, 0 \to \mathbf{R}^2, 0$

Rieger [Ri2] has classified all the corank 1 mono-germs up to codimension 6 and Rieger & Ruas [RiR] have studied the corank 2 mono-germs. From Proposition 6.1.2 these are the only two cases to consider. We find that in both of these cases all germs of up to codimension 4 are $\mathcal{A}$-simple.

The corank 1 list is as follows:

| | Normal Form | $\mathcal{A}$-codim | $c(f)$ | $d(f)$ |
|---|---|---|---|---|
| 1 | $(x,y)$ | 0 | | |
| 2 | $(x,y^2)$ | 1 | | |
| 3 | $(x,xy+y^3)$ | 2 | 1 | 0 |
| $4_2$ | $(x,y^3 \pm x^2y)$ | 3 | 2 | 0 |
| 5 | $(x,xy+y^4)$ | 3 | 2 | 1 |
| $4_3$ | $(x,y^3+x^3y)$ | 4 | 3 | 0 |
| 6 | $(x,xy+y^5 \pm y^7)$ | 4 | 3 | 3 |
| $11_5$ | $(x,xy^2+y^4+y^5)$ | 4 | 3 | 2 |

Here $c(f)$ and $d(f)$ are geometrical invariants associated with the map-germ $f$. In the complex case, $c$ is the number of cusps and $d$ is the number of transverse fold crossings which are exhibited on the discriminant of the map-germ under a generic deformation. However, in the real case $c$ and $d$ are only upper bounds for these numbers.

In fact, Rieger [Ri2] has given an $\mathcal{A}$-invariant stratification of $\Sigma^1 J^k(2,2)$, the corank one $k$-jets of maps from $\mathbf{C}^2$ to $\mathbf{C}^2$, such that for $k \geq 2$, all strata of codimension at most 6 are the $\mathcal{A}^r$-orbits, or unions of orbits where moduli figure, of germs whose determinacy degree is less than or equal to 11. Also, the

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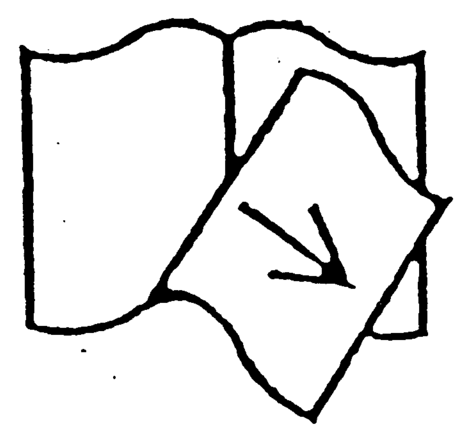


We first note that one branch of the bi-germ will be one of the two following forms:

(i) $(x, y^2)$

(ii) $(x, xy + y^3)$ (discounting the possibility of $(x, y)$.)

In order to stratify the jet space $\Sigma^1{}_2 J^k(2,2)$ we will use subgroups $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ which preserve the fold and the cusp respectively (c.f. Propostion 4.3.1 and [BGi1]) and start with each of the two cases above.

### 6.3.1 Bi-germs with at least one fold

To preserve $(x, y^2)$ we can use the vector fields $\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$. We define a subgroup of $\mathcal{A}_1$ which uses these vector fields instead of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Call this subgroup $\mathcal{B}_1$. Now we consider the other branch. Since it has non-zero one-jet it is of form $j^1 f_2 \sim (aX + bY, cX + dY)$. By linear algebra we see that the two cases to consider here are

(I) $j^1 f_2 \sim (0, Y)$ and

(II) $j^1 f_2 \sim (X, 0)$

We start with case (I).Now

$$T\mathcal{B}_1.f_2 = m_2^2\langle(0,1)\rangle + f_2^*.m_2^2\langle(1,0)\rangle + f_2^*.m_2\langle(0,Y)\rangle$$

and so, by the method of Theorem 1.5.1, $T = \{(X^2, 0), (XY, 0)\}$. Thus the possible cases are

$$j^2 f_2 = \begin{cases} (X^2, Y) & \text{(i)} \\ (0, Y) & \text{(ii)} \\ (XY, Y) & \text{(iii)} \end{cases}$$

(Note that $(x, y^2; XY + X^2, Y)$ is $\mathcal{A}$-equivalent to $(x, y^2; X^2, Y)$. For

$$(x, y^2; XY + X^2, Y) = (x, y^2; (X + Y/2)^2 - Y^2/4, Y)$$

Then the change of co-ordinates in the target $u \mapsto u - v^2/4$ gives

$$(x, y^2; (X + Y/2)^2 - Y^2/4, Y) \sim (x - y^4/4, y^2; (X + Y/2)^2, Y)$$

But this germ is 2 determined since

$$\begin{aligned} T\mathcal{A}_1.f &= m_2^2\langle(1,0;0,0),(0,2y;0,0)\rangle + m_2^2\langle(0,0;Y+2X,0),(0,0;Y,1)\rangle \\ &\quad + f^*.m_2^2\langle(1,0),(0,1)\rangle \\ &\supseteq m_2^3.\mathcal{E}_{2,2} \end{aligned}$$

so we can ignore the $y^4$ in the first component. A final change of co-ordinates in the source, given by $X \mapsto X + Y/2$, gives

$$(x, y^2; (X + Y/2)^2, Y) \sim (x, y^2; X^2, Y)$$

and so the two are $\mathcal{A}$-equivalent.)

Case (i) is 2 determined (by similar calculations to the above) and has $\mathcal{A}$-codimension 2.

For case (ii) we find $T\mathcal{B}_1.f_2$ and see that $T = \{(X^3, 0), (X^2Y, 0), (XY^2, 0)\}$. So a complete transversal is $(x, y^2; aX^3 + bX^2Y + cXY^2, Y)$. After lengthy calculations involving considering each combination of $(a, b, c)$ separately we find that the only possibility for $f_2$ which is finitely determined and has $\mathcal{A}$-codimension $< 4$ is

$$F = (x, y^2; XY^2 \pm X^3, Y)$$

This is 3 determined and has $\mathcal{A}$-codimension 4. All other 3-jets have codimension greater than 3 in ${}_2J^3(2,2)$.

In case (iii) we find that $T = \{(X^3, 0)\}$ and so we have

$$j^3 f_2 = \begin{cases} (XY + X^3, Y) \\ (XY, Y) \end{cases}$$

The first of these is 3 determined and has $\mathcal{A}$-codimension 3. For the second we calculate $T\mathcal{B}_1.f_2$ and find that $T = \{(X^4, 0)\}$ so either $F = (x, y^2; XY + X^4, Y)$, which is 4 determined and has $\mathcal{A}$-codimension 4, or we get the non-sufficient stratum $\mathcal{A}^4(x, y^2; XY, Y)$. We shall show that this has $\mathcal{A}$-codimension 5 in the jet-space ${}_2J^4(2,2)$. Consider $T\mathcal{A}.f \cap {}_2J^4(2,2)$ (where $f = \mathcal{A}^4(x, y^2; XY, Y)$):

$$\begin{aligned} T\mathcal{A}.f \cap {}_2J^3(2,2) =& m_2\langle(1,0;0,0),(0,2y;0,0)\rangle + m_2\langle(0,0;Y,0),(0,0;X,1)\rangle \\ &+ f^*.m_2\langle(1,0),(0,1)\rangle \\ =&(m_2, m_2 - \{y\}; m_2 - \{X\}, m_2 - \{X, X^2, X^3\}) \end{aligned}$$

So there are 5 elements missing from ${}_2J^4(2,2)$.

We now look at case (II), where $j^1 f_2 = (X, 0)$. Now

$$T\mathcal{B}_1.f_2 = m_2^2\langle(1,0)\rangle + f_2^*.m_2^2\langle(1,0)\rangle$$

and then $T = \{(0, X^2), (0, XY), (0, Y^2)\}$. So a complete transversal is

$$(x, y^2; X, aX^2 + bXY + cY^2)$$

and the $\mathcal{A}$-tangent space is

$$\begin{aligned} T\mathcal{A}.f = & m_1\langle(1,0;0,0),(0,2y;0,0)\rangle + m_1\langle(0,0;1,2aX+bY),(0,0;0,bX+2cY)\rangle \\ & + f^*.m_3\langle(1,0),(0,1)\rangle \end{aligned}$$

We find that if $c(4ac-b^2) \neq 0$ then we get the vectors $(0,0;0,X^2)$, $(0,0;0,XY)$ and $(0,0;0,Y^2)$ which we need in order to apply Mather's Lemma. This condition defines a smooth submanifold of $a,b,c$-space and so we can choose any representative for $(a,b,c)$ where $c(4ac-b^2) \neq 0$ to give the normal form, eg. $(0,1,1)$.

If $c(4ac-b^2) = 0$ then either $c = 0$ or $4ac = b^2$. Suppose $c = 0$. Then we get all the required vectors provided $b \neq 0$. So we choose the representative $(0,1,0)$. If $b = c = 0$ then $4ac = b^2$ also. Suppose $4ac = b^2$. Substituting for $a$ in $(x,y^2;X,aX^2+bXY+cY^2)$ and then differentiating with respect to $b$ and $c$ we get the vectors

$$(0,0;0,\frac{b}{2c}X^2+XY)$$
$$(0,0;0,\frac{-b}{4c^2}X^2+Y^2)$$

Both of these are in the $\mathcal{A}$-tangent space so we can apply Mather's Lemma once more and choose a representative for the submanifold $4ac = b^2$, eg. $(0,0,1)$. We now have the following alternatives for the bi-germ $F = (x,y^2;f_2)$:

$$j^2 f_2 = \begin{cases} (X, XY+Y^2) & \text{(i)} \\ (X, XY) & \text{(ii)} \\ (X, Y^2) & \text{(iii)} \end{cases}$$

In case (i) we find that $(x,y^2;X,XY+Y^2)$ is 2 determined with $\mathcal{A}$-codimension 3.

In case (ii), $F \sim (x,y^2;X,XY)$. We calculate $T\mathcal{B}_1.f_2$ and find that $T = \{(0,Y^3)\}$. Then either $j^3F = (x,y^2;X,XY+Y^3)$, which is 3 determined and has $\mathcal{A}$-codimension 4, or we have the non-sufficient stratum $\mathcal{A}^3(x,y^2;X,XY)$ which has $\mathcal{A}$-codimension 5 in the jet-space (the calculations are similar to those given in the case of $\mathcal{A}^4(x,y^2;XY,Y)$).

In case (iii) we find $T\mathcal{B}_1.f_2$ and see that $T = \{(0,X^3)\}$. So either $j^3F = (x,y^2;X,Y^2+X^3)$, which is 3 determined and has $\mathcal{A}$-codimension 4, or we have the non-sufficient stratum $\mathcal{A}^3(x,y^2,X,Y^2)$ which has $\mathcal{A}$-codimension 5 in the jet-space.

Stratification diagrams representing the above are shown in fig. 6.3.1(i)-(iv).

Fig. 6.3.1 (i)

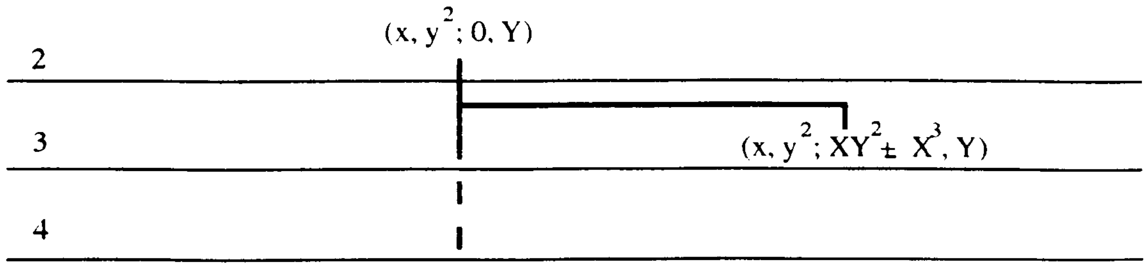



Fig. 6.3.1(ii)

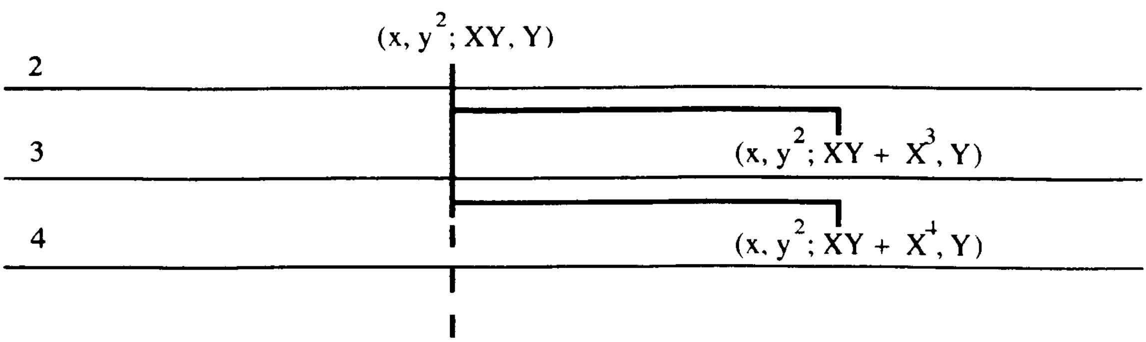



Fig. 6.3.1(iii)

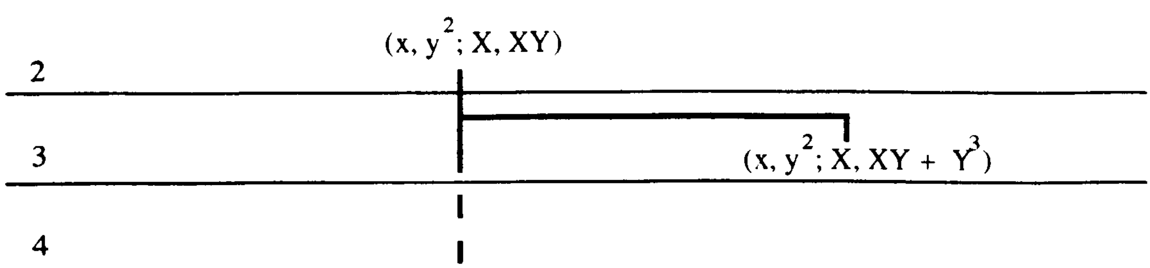
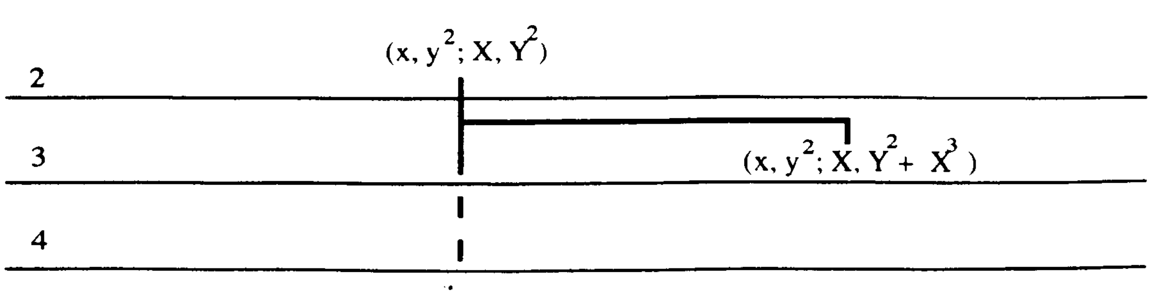



Fig. 6.3.1(iv)



### 6.3.2 Bi-germs with at least one cusp

We need vector fields which will preserve $(x, xy+y^3)$. The discriminant of the map-germ is given by $F^{-1}(0)$, where $F(X,Y) = Y^2 + \frac{4}{9}X^3$. This function has Jacobian $(\ \frac{4}{3}X^2 \quad 2Y\ )$, and the kernel of this gives vector fields preserving $F^{-1}(0)$. So we need to find $f(X,Y)$ and $g(X,Y)$ such that

$$\frac{4}{3}X^2 f(X,Y) + 2Y g(X,Y) = 0$$

(with the additional information that $Y^2 + \frac{4}{9}X^3 = 0$). Here we choose the (independent) vector fields

$$3y\frac{\partial}{\partial x} - 2x^2\frac{\partial}{\partial y}$$
$$2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}$$

We define another subgroup of $\mathcal{A}_1$ using these vector fields and denote it $\mathcal{C}_1$. Again, we have two possibilities for the other mono-germ:

(I) $j^1 f_2 \sim (0,Y)$ and

(II) $j^1 f_2 \sim (X,0)$

First consider case (I). Then

$$T\mathcal{C}_1.f_2 = m_2^2\langle(0,1)\rangle + f_2^* m_2\langle(3Y,0),(0,3Y)\rangle$$

So $T = \{(X^2,0),(XY,0)\}$. The three possible cases are

$$j^2 f_2 = \begin{cases} (X^2,Y) & \text{(i)} \\ (XY,Y) & \text{(ii)} \\ (0,Y) & \text{(iii)} \end{cases}$$

The first of these has already been studied above. For case (ii) we calculate $T\mathcal{C}_1.f_2$ and find that $T = \{(X^3,0)\}$. So either $F = (x, xy+y^3; XY+X^3, Y)$, which is 3 determined and has $\mathcal{A}$-codimension 4, or we have $\mathcal{A}^3(x, xy+y^3; XY, Y)$ which has $\mathcal{A}$-codimension 6 in the jet-space. In case (iii), we get the non-sufficient stratum $\mathcal{C}^2(x, xy+y^3; 0, Y)$ which has $\mathcal{A}$-codimension 5 in the jet-space.

Now for case (II), where $j^1 f_2 = (X,0)$. The $\mathcal{C}_1$-tangent space is given by

$$T\mathcal{C}_1.F = m_2^2\langle(1,0)\rangle + f_2^*.m_2\langle(0,-2X^2),(2X,0)\rangle$$

and so $T = \{(0,X^2),(0,XY),(0,Y^2)\}$ and a complete tranversal is $j^3F = (x, xy+y^3; X, aX^2+bXY+cY^2)$. Unless $a=b=c=0$ all cases have been studied in the subsection above. If all the coefficients are zero we have the non-sufficient stratum $\mathcal{A}^3(x, xy+y^3; X, 0)$ which has codimension 9 in ${}_2J^3(2,2)$.

### 6.4 A Result on the Codimensions of Multi-germs

Consider the list of all bigerms which have $\mathcal{A}$-codimension less than 5 (in order of codimension):

| | Bigerm | $\mathcal{A}$-codim | Separate Codims | |
|---|---|---|---|---|
| (1) | $(x, y^2; X^2, Y)$ | 2 | $1+1$ | T |
| (2) | $(x, y^2; X, XY + Y^2)$ | 3 | $1+1$ | NT |
| (3) | $(x, y^2; XY + X^3, Y)$ | 3 | $2+1$ | NT |
| (4) | $(x, y^2; X, XY + Y^3)$ | 4 | $1+2$ | NT |
| (5) | $(x, y^2; X, Y^2 + X^3)$ | 4 | $1+1$ | NT |
| (6) | $(x, y^2; XY^2 \pm X^3, Y)$ | 4 | $3+1$ | T |
| (7) | $(x, y^2; XY + X^4, Y)$ | 4 | $1+3$ | T |
| (8) | $(x, xy + y^3; XY + X^3, Y)$ | 4 | $2+2$ | T |

Here the letters T and NT stand for 'Transverse' and 'Non-Transverse', where 'Transverse' means that the tangents to the discriminant sets of the two monogerms in a bigerm are mutually transverse. We see that in the T cases, the codimension is the sum of the two mono-codimensions, while in the NT cases the codimension is greater than this sum. This suggests, firstly, the following proposition:

**6.4.1 Proposition** *If $F = (f_2, f_2)$ is a bigerm then*

$$\mathcal{A}\text{–codim}(F) \geq \mathcal{A}\text{–codim}(f_1) + \mathcal{A}\text{–codim}(f_2) \qquad (*)$$

**Proof** We prove this by showing that there is an injective inclusion

$$T\mathcal{A}.F \longrightarrow T\mathcal{A}.f_1 \oplus T\mathcal{A}.f_2$$

(and hence a natural surjection $\frac{\mathcal{O}_f}{T\mathcal{A}.F} = \frac{(\oplus \mathcal{O}_{f_i})}{T\mathcal{A}.F} \longrightarrow \oplus \frac{\mathcal{O}_{f_i}}{T\mathcal{A}.f_i}$ ).

Suppose this were not the case. Consider $g \in {}_2J^k(2,2)$ with $g \notin T\mathcal{A}.f_1 \oplus T\mathcal{A}.f_2$ but $g \in T\mathcal{A}.F$. Then there exist some vector fields $\phi_1$, $\phi_2$ and $\psi$ in the appropriate maximal ideals with $g = tf_1(\phi_1) + tf_2(\phi_2) + wF(\psi)$.

Now $wF(\psi) = (f_1^*(\psi(u,v)); f_2^*(\psi(u,v))$ ( where $(u,v)$ are the co-ordinates in the target). Since $tf_1(\phi_1)$ and $f_1^*(\psi(u,v))$ must be in $T\mathcal{A}.f_1$ and $tf_2(\phi_2)$ and

$f_2^*(\psi(u,v)$ must be in $T\mathcal{A}.f_2$ we must have $g \in T\mathcal{A}.f_1 \oplus T\mathcal{A}.f_2$, contradicting the hypothesis. So certainly

$$\mathcal{A}\text{–codim}(F) = \dim_{\mathbf{R}} \frac{\mathcal{O}_F}{T\mathcal{A}.F} \geq \dim_{\mathbf{R}} \frac{\mathcal{O}_{f_1}}{T\mathcal{A}.f_1} + \dim_{\mathbf{R}} \frac{\mathcal{O}_{f_2}}{T\mathcal{A}.f_2}$$
$$= \mathcal{A}\text{–codim}(f_1) + \mathcal{A}\text{–codim}(f_2)$$

□

Secondly, we would like to know what equality in $(*)$ implies geometrically. From the table above it would seem that there is a connection between this equality and the discriminants being tangent. First we make a definition:

**6.4.2 Definition** *Let $\Delta_1$, $\Delta_2$ be germs at $0 \in \mathbf{C}^2$ of reduced analytic sets. We say that $(\Delta_1, \Delta_2)$ is trivial if the following is true. Given 2 germs of diffeomorphisms $\Phi_i : \mathbf{C}^2 \times \mathbf{C}, 0 \to \mathbf{C}^2 \times \mathbf{C}, 0$, $\Phi_i(y,t) = (\phi_i(y,t),t)$ with $\phi_i(y,0) = y$ and $\phi_1(0,t) = 0$, $i = 1,2$, there is a germ of a diffeomorphism $\Phi : \mathbf{C}^2 \times \mathbf{C}, 0 \to \mathbf{C}^2 \times \mathbf{C}, 0$ of the same form with $\Phi(\Delta_i, t) = \phi_i(\Delta_i, t)$, $i = 1,2$.*

Intuitively, what this means is that if we can move the $\Delta_i$ about in a one-parameter family independently then they can be straightened out uniformly. In particular, when $\Delta_1$, $\Delta_2$ are smooth and transverse then we have a trivial pair.

Then we have the following result, due to Bruce [BH],

**6.4.3 Theorem** *If $F = (f_1, f_2)$ is a bi-germ, where $f_i : \mathbf{C}^2, x_i \to \mathbf{C}^2, 0$, then*

$$\mathcal{A}\text{–codim}(F) = \mathcal{A}\text{–codim}(f_1) + \mathcal{A}\text{–codim}(f_2)$$

*if and only if the discriminant pair $(\Delta(f_1), \Delta(f_2))$ is trivial.*

**Proof** Suppose that $(\Delta(f_1), \Delta(f_2))$ is a trivial pair, and let $F_i : \mathbf{C}^2, x_i \to \mathbf{C}^2, 0$ be smooth $\mathcal{A}$-trivial families with $F_i(x,0) = f_i(x)$ and $F_i(0,t) = 0$.Then there are diffeomorphisms $\Phi_i : \mathbf{C}^2 \times \mathbf{C}, 0 \to \mathbf{C}^2 \times \mathbf{C}, 0$ with $\Phi_i(y,t) = (\phi_i(y,t),t)$, $\phi_i(y,0) = y$, $\phi_i(0,t) = 0$ and $\phi_{i,t} = \phi_i(-,t)$ taking $\Delta(F_{i,t})$ to $\Delta(f_i = F_{i,0})$. Since the pair of discriminants is trivial we can find a diffeomorphism $\Phi : \mathbf{C}^2 \times \mathbf{C}, 0 \to \mathbf{C}^2 \times \mathbf{C}, 0$ with $\Phi(y,t) = (\phi(y,t),t)$ and $\phi_t$ taking $\Delta(F_{i,t})$ to $\Delta(f_i)$ for all small $t$.

By a result of [BdPW] we now see that the bi-germ

$$F : \mathbf{C}^2 \times \mathbf{C}, S \times \{0\} \longrightarrow \mathbf{C}^2, 0$$

(where $S = \{x_1, x_2\}$ ) is $\mathcal{A}$-trivial. But the $F_i$ were arbitrary (based) $\mathcal{A}$-trivial families so $\frac{\partial F}{\partial t}(x,0)$ is an arbitrary element of $\oplus T\mathcal{A}(f_i)$. This argument shows that $\frac{\partial F}{\partial t}(x,0)$ lies in $T\mathcal{A}(f)$.

Now for the converse. Suppose that equality holds and consider maps $\Phi_i$ as in Definition 6.4.2, where the $\Delta_i = \Delta(f_i)$. Consider the families $f_{i,t} = \phi_i(f,t)$. These are obviously $\mathcal{A}$-trivial. For small $t$ they define a bi-germ $F_t : \mathbf{C}^2, S \to \mathbf{C}^2, 0$. We claim that equality also holds for this bi-germ. First consider the natural surjection

$$\frac{V(F_t)}{T\mathcal{A}(F_t)} \longrightarrow \frac{V(f_{1,t})}{T\mathcal{A}(f_{1,t})} \oplus \frac{V(f_{2,t})}{T\mathcal{A}(f_{2,t})}$$

When $t = 0$ this map is, by hypothesis, an isomorphism. Moreover, the dimension of the vector space on the right-hand side remains constant for small $t$. So

$$\begin{aligned} \dim\frac{V(F_t)}{T\mathcal{A}(F_t)} &\geq \dim\left(\frac{V(f_{1,t})}{T\mathcal{A}(f_{1,t})} \oplus \frac{V(f_{2,t})}{T\mathcal{A}(f_{2,t})}\right) \\ &= \dim\left(\frac{V(f_1)}{T\mathcal{A}(f_1)} \oplus \frac{V(f_2)}{T\mathcal{A}(f_2)}\right) \\ &= \dim\left(\frac{V(F_0)}{T\mathcal{A}(F_0)}\right) \end{aligned}$$

On the other hand, in a deformation, codimension can only drop so all the above inequalities are equalities. It follows that $T\mathcal{A}(F_t) = T\mathcal{A}(f_{1,t}) \oplus T\mathcal{A}(F_{2,t})$ for small $t$. Since the germs and the bi-germs are all finitely determined we can work in some sufficiently large jet-space $(J^k(2,2))^r$. On this we have the action of the jet group product $\Pi_{i=1}^r J^k(\mathcal{A})$ which we denote by $\mathcal{A}^k(r)$, and also $(\Pi_{i=1}^r J^k(\mathcal{R})) \times J^k(\mathcal{L}) = \mathcal{B}^k(r)$. The latter is the natural group for multi-germs since it allows only a common change of co-ordinates in the target. Clearly the orbit $\mathcal{B}^k(r).F_t$ is a subset of the orbit $\mathcal{A}^k(r).F_t$. However we have just shown that they are of the same dimension, so they coincide on some neighbourhood $U_t$ of $F_t$. In particular for some neighbourhood $U$ of $F_0$ we have

$$U \cap \mathcal{A}^k(r).F_0 = U \cap (\cup_t(\mathcal{A}^k(r).F_t)) \supset U \cap (\cup_t(\mathcal{B}^k(r).F_t)) \supset U \cap \mathcal{B}^k(r).F_0.$$

Since the first and last sets in this chain of inclusions coincide it follows that the $F_t$ lie in the orbit $\mathcal{B}^k(r).F_0$ for small $t$. Since we are working in a sufficient jet-space this means that $F_t$ is $\mathcal{A}$-trivial, and consequently the change of co-ordinates in the target $\Phi$ which makes it trivial has the properties required in Definition 6.4.2. □

(In fact this result generalises to r-germs - see [BH].)

### 6.5 Tri-germs and 4-germs $\mathbf{R}^2, 0 \to \mathbf{R}^2, 0$

From Proposition 6.1.2 we know that any tri-germ will have non-zero 1-jets for each branch and that at most two of these can be tangent. In this section we seek to show that the only tri-germs with $\mathcal{A}_e$-codimension less than or equal to 2 are shown in the table below. All are $\mathcal{A}$-simple.

| Normal Form | $\mathcal{A}$-codim |
|---|---|
| $(x, y^2; X^2, Y; \tilde{x}, \tilde{x} + \tilde{y}^2)$ | 3 |
| $(x, y^2; X^2, Y; \tilde{x}, \tilde{x}\tilde{y} + \tilde{y}^2)$ | 4 |
| $(x, y^2; X^2, Y; \tilde{x}, \tilde{x} + \tilde{x}\tilde{y} + \tilde{y}^3)$ | 4 |

Again, we need to stratify $\Sigma^1{}_3J^k(2,2)$ in an $\mathcal{A}$-invariant way and then show that the complement of the union of orbits which we want has codimension greater than 4 in the jet-space.

We start by considering the simplest form for the first two branches of the tri-germ: transverse folds $(x, y^2)$ and $(X^2, Y)$. We shall apply the complete transversal method to the third branch, and so we use a subgroup of $\mathcal{A}_1$ which preserves the discriminants of these two. Thus instead of the standard vector fields we use $x\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$. We denote the subgroup $\mathcal{B}_1$. The third branch is either transverse to the first two or tangent to one of them.

Suppose the third branch is transverse to the other two. We can change co-ordinates so that $j^1 f_3 = (\tilde{x}, \tilde{x})$. Then the $\mathcal{B}_1$-tangent space is given by

$$T\mathcal{B}_1 \cdot f_3 = m_1^2\langle(1,1)\rangle + f_3^* . m_2\langle(\tilde{x}, 0), (0, \tilde{x})\rangle$$

and so a complete transversal $j^2 f_3 = (\tilde{x}, \tilde{x} + a\tilde{x}\tilde{y} + b\tilde{y}^2)$. Thus the possibilities for $f_3$ are

$$j^3 f_3 = \begin{cases} (\tilde{x}, \tilde{x}) \\ (\tilde{x}, \tilde{x} + \tilde{x}\tilde{y}) \\ (\tilde{x}, \tilde{x} + \tilde{y}^2) \end{cases}$$

The first case is not finitely determined and is a non-sufficient stratum with codimension 5 in the jet-space ${}_3J^2(2,2)$.

For the second case, $j^2 f_3 = (\tilde{x}, \tilde{x} + \tilde{x}\tilde{y})$ and so

$$T\mathcal{B}_1 \cdot f_3 = m_2^2\langle(1, 1 + \tilde{y}), (0, \tilde{x})\rangle + f_3^* . m_2\langle(\tilde{x}, 0), (0, \tilde{x} + \tilde{x}\tilde{y})\rangle$$

So a complete transversal is $j^3f_3 = (\tilde{x}, \tilde{x} + \tilde{x}\tilde{y} + a\tilde{y}^3)$. If $a \neq 0$ then this gives a 3-determined tri-germ which has $\mathcal{A}$-codimension 4, while if $a = 0$ we have a non-sufficient stratum $\mathcal{A}^2(x, y^2; X^2, Y; \tilde{x}, \tilde{x} + \tilde{x}\tilde{y})$ with codimension 5 in ${}_3J^2(2,2)$.

The final case, $(x, y^2; X^2, Y; \tilde{x}, \tilde{x} + \tilde{y}^2)$, is 2 determined and has $\mathcal{A}$-codimension 3.

The other case we have to look at is that of two transverse immersions with a third branch which has one-jet tangent to one of them. Then $j^1f_3$ is $\mathcal{A}$-equivalent to $(\tilde{x}, 0)$. The $\mathcal{B}_1$-tangent space is given by

$$T\mathcal{B}_1.f_3 = m_1^2\langle(1,0)\rangle + f_3^*.m_2\langle(\tilde{x},0)\rangle$$

and so, going up to 2-jets, $T = \{(0,\tilde{x}^2),(0,\tilde{x}\tilde{y}),(0,\tilde{y}^2)\}$ and a complete transversal is $j^2f_3 = (\tilde{x}, a\tilde{x}^2 + b\tilde{x}\tilde{y} + c\tilde{y}^2)$. Calculating the $\mathcal{A}$-tangent space tells us that if $c(4ac - b^2) \neq 0$ then the first condition of Mather's Lemma is satisfied. The second condition is easily seen to be satisfied and so we can choose a representative for $(a,b,c)$ where $c(4ac - b^2) \neq 0$, eg $(0,1,1)$. If $c = 0$, then Mather's Lemma can be applied to the submanifold provided $b \neq 0$. So we choose $(0,1,0)$. If $c = b = 0$ then $4ac - b^2 = 0$ too. Consider this case. Substituting for $a$ in $(x, y^2; X^2, Y; \tilde{x}, a\tilde{x}^2 + b\tilde{x}\tilde{y} + c\tilde{y}^2)$ and differentiating with respect to $b$ and $c$ gives vectors which are in the $\mathcal{A}$-tangent space, so the first criterion of Mather's Lemma is satisfied. The second one is also satisfied and so we can apply the Lemma. We choose the representative $(0,0,1)$. Then the possible cases are:

$$j^2f_3 = \begin{cases} (\tilde{x}, \tilde{x}\tilde{y} + \tilde{y}^2) \\ (\tilde{x}, \tilde{x}\tilde{y}) \\ (\tilde{x}, \tilde{y}^2) \end{cases}$$

The first of these cases is 2 determined and has $\mathcal{A}$-codimension 4. The other two are not finitely detemined. Consider $(x, y^2; X^2, Y; \tilde{x}, \tilde{x}\tilde{y})$. Then

$$T\mathcal{B}_1.f_3 = m_2^2\langle(1,\tilde{y}),(0,\tilde{x})\rangle + f_3^*.m_2\langle(\tilde{x},0)\rangle + f_3^*.m_2\langle(0,\tilde{x}\tilde{y})\rangle$$

So a complete 3-jet transversal is $(x, y^2; X^2, Y; \tilde{x}, \tilde{x}\tilde{y} + a\tilde{y}^3)$. If $a \neq 0$ then the tri-germ is 3 determined but has $\mathcal{A}$-codimension 5, and if $a = 0$ then we have a non-sufficient stratum with codimension 6 in ${}_3J^3(2,2)$.

The final case, $(x, y^2; X^2, Y; \tilde{x}, \tilde{y}^2)$, has $\mathcal{B}_1$ tangent space

$$T\mathcal{B}_1.f_3 = m_2^2\langle(1,0),(0,2\tilde{y})\rangle + f_3^*.m_2\langle(\tilde{x},0)(0,\tilde{y}^2)\rangle$$

So $T = \{(0,\tilde{x}^3)\}$ and we have complete transversal $(x, y^2; X^2, Y; \tilde{x}, \tilde{y}^2 + a\tilde{x}^3)$.

If $a \neq 0$ the tri-germ is 3-determined but the $\mathcal{A}$-codimension is 5, and if $a = 0$ we have a non-sufficient stratum with codimension 6 in ${}_3J^3(2,2)$.

If the trigerm is any worse than the above cases then $j^3 f$ will be of the form

$$j^3 f = (x, xy + y^3; X^2, Y; \tilde{x}, a\tilde{x}\tilde{y} + b\tilde{y}^3)$$

For all values of $a$ and $b$ we have non-sufficient strata with codimensions $\geq 5$ in the jet-space ${}_3J^3(2,2)$.

Finally we look at 4-germs $\mathbf{R}^2, 0 \to \mathbf{R}^2, 0$. By Proposition 6.1.2, each branch of a 4-germ must have corank 1 and there cannot be any tangencies amongst the branches. Thus the only 4-germ with $\mathcal{A}_e$-codimension (stratum) $\leq 2$ is $\mathcal{A}$-equivalent to $(x, y^2; X^2, Y; \tilde{x}, \tilde{x} + \tilde{y}^2; \tilde{X}, \tilde{X}\tilde{Y} + \lambda\tilde{Y}^2)$. Each orbit has $\mathcal{A}_e$-codimension 3 but the stratum has $\mathcal{A}_e$-codimension 2.

## 6.6 Unfoldings of Two-Dimensional Motions of the Plane

In order to look at the bifurcation sets of the singularities on our list we first make some definitions.

### 6.6.1 Definition

(i) *Given a map-germ* $f : \mathbf{F}^n, 0 \to \mathbf{F}^p, 0$ *(where* $\mathbf{F}$ *is* $\mathbf{C}$ *or* $\mathbf{R}$*), the critical set of* $f$*,* $\Sigma f$*, is the set of points in* $\mathbf{F}^n$ *where* $df$ *is not surjective (i.e. where the Jacobian of* $f$ *is zero).*

(ii) *With* $f$ *as before, the discriminant of* $f$ *is the set of critical values* $f(\Sigma f)$.

Clearly $\mathcal{A}$-equivalent maps have diffeomorphic discriminants. In what follows we shall be looking at the discriminants of the map-germs on our list and seeing how they vary with changes of parameters. We note that the geometric invariants, $c$ and $d$, described in section 6.2 give us some information about these discriminants.

Some of these bifurcation sets have been analysed in [Ri2] and [BGi2] but here we try to give clearer pictures of how the discriminants vary under deformation.

### 6.6.2 Unfoldings of Mono-germs

From Rieger [Ri2] we have the following adjancency diagram for corank 1 singularities (the numbers are the type-numbers from the tables in the statement of Theorem 6.1.1):

$$\begin{tikzcd}
1 & 2 \arrow[l] & 3 \arrow[l] & 4_2 \arrow[l] & 4_3 \arrow[l] & \cdots \arrow[l] \\
 & & & 5 \arrow[ul] & 11_5 \arrow[l] \arrow[ul] & \cdots \arrow[l] \\
 & & & & & 6 \arrow[ul] & \cdots \arrow[l]
\end{tikzcd}$$

For corank 2 singularities we have the following adjacency diagram from Rieger and Ruas [RiR]:

$$\begin{tikzcd}
2 & 3 \arrow[l] & 4_2^- \arrow[l] & \cdots \arrow[l] \\
 & II_{2,2}^1 \arrow[u] & I_{2,2}^{1,1} \arrow[u] &
\end{tikzcd}$$

These diagrams tell us what types of singularity we expect to see when we unfold.

(i) $f(x, y) = (x, y)$. This is an immersion; it is stable so the unfolding is trivial.

(ii) $f(x, y) = (x, y^2)$. This is a fold and is also stable. The critical set is given by $2y = 0$ and the discriminant is $(x, 0)$. See Fig. 6.6.1.

(iii) $f(x, y) = (x, xy + y^3)$. This is a cusp, which is stable. The critical set is $x + 3y^2 = 0$ and the discriminant is $(-3y^2, -2y^3)$. See Fig. 6.6.2.

(iv) $f(x, y) = (x, y^3 \pm x^2 y)$. This is the lips/beaks map and has $\mathcal{A}_e$-codimension 1. Then

$$T\mathcal{A}_e.f = (\mathcal{E}_2, \mathcal{E}_2 - \{y\})$$

and the unfolding is given by $f_a(x, y) = (x, y^3 \pm x^2 y + ay)$. From the adjancency diagram we expect only cusps in the unfolding. In the positive case the critical set is defined by $3y^2 + x^2 + a = 0$, and the discriminant is $(\sqrt{-3y^2 - a}, -2y^3)$. Thus the bifurcation set is $a = 0$ and the unfoldings are shown in Fig. 6.6.3. In the negative case the discriminant is $(\sqrt{3y^2 - a}, 4y^3)$. The bifurcation set is still $a = 0$ and the unfoldings are shown in Fig. 6.6.4.

(v) $f(x, y) = (x, xy + y^4)$. This is a swallowtail map. We have

$$T\mathcal{A}_e.f = (\mathcal{E}_2, \mathcal{E}_2 - \{y^2\})$$

Fig.6.6.1

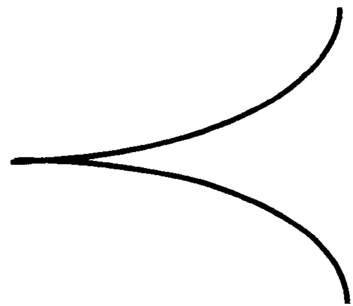


Fig.6.6.2

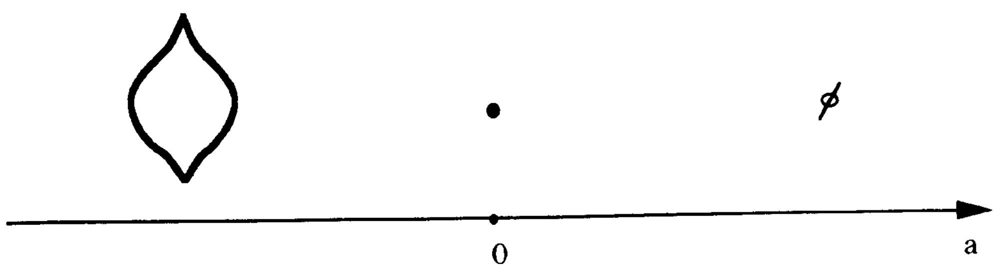



Fig.6.6.3

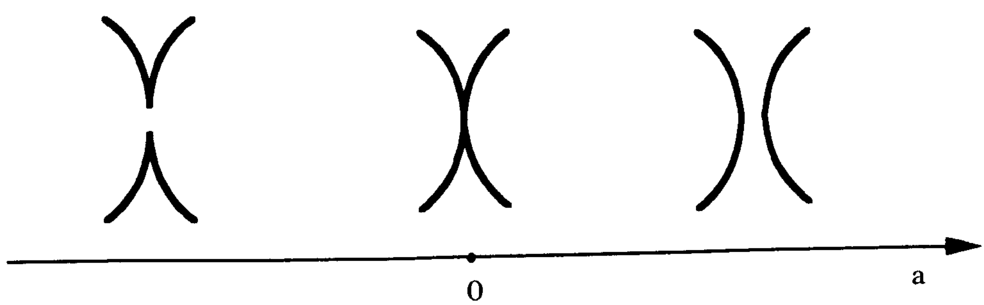



Fig.6.6.4

and so the unfolding is $f_a(x,y) = (x, xy+y^4+ay^2)$. Again, from the adjancency diagram we expect only cusps in the unfolding. The critical set is given by $x+4y^3+2ay=0$ and the discriminant is $(-4y^3-2ay, -3y^4-ay^2)$. Then the bifurcation set is $a=0$ and parameter space is shown in Fig. 6.6.5.

(vi) $f(x,y) = (x, y^3+x^3y)$. This is known as the goose singularity. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = (\mathcal{E}_2, \mathcal{E}_2 - \{y, xy\})$$

and so an unfolding is given by $f_{a,b}(x,y) = (x, y^3+x^3y+ay+bxy)$. From the adjacancy diagram we know that the only singularities we expect to see when we unfold the goose are lips/beaks singularities. These occur when $\Sigma$ is singular. If we write $f(x,y) = (x, g(x,y))$ with $g(x,y) = y^3+x^3y+ay+bxy$ then this condition is equivalent to

$$g_y = g_{yy} = g_{xy} = 0$$

(where $g_y$ is the partial derivative of $g$ with respect to $y$ and $g_{xy}$ is the partial derivative of $g$ with respect to $x$ and then $y$.) These give $g_y = 3y^2+x^3+a+bx = 0$, $g_{yy} = 6y = 0$ and $g_{xy} = 3x^2+b = 0$, yielding $a = 2x^3$ and $b = -3x^2$. So the lips/beaks stratum is parametrized by $(2x^3, -3x^2)$ The picture in unfolding space is shown in Fig. 6.6.6.

(vii) $f(x,y) = (x, xy+y^5 \pm y^7)$. This is the butterfly map. We have

$$T\mathcal{A}_e.f = (\mathcal{E}_2, \mathcal{E}_2 - \{y^2, y^3\})$$

and so the unfolding is $f_{a,b}(x,y) = (x, xy+y^5 \pm y^7 + ay^2 + by^3)$. From the adjacency diagram we expect to see swallowtail singularities and possibly cusp-and-fold singularities and tacnode singularities. Write $f(x,y) = (x, g(x,y))$ where $g(x,y) = xy+y^5 \pm y^7 + ay^2 + by^3$.

The swallowtail stratum is given by the conditions

$$g_y = g_{yy} = g_{yyy} = 0$$

So we have

$$\begin{aligned} g_y &= \pm\, 7y^6 + 5y^4 + 3by^2 + 2ay + x \\ g_{yy} &= \pm\, 42y^5 + 20y^3 + 6by + 2a \\ g_{yyy} &= \pm\, 210y^4 + 60y^2 + 6b \end{aligned}$$

These yield $a = 20y^3 \pm 84y^5$ and $b = -10y^2 \mp 35y^4$ so the swallowtail stratum is parametrized by $(20y^3 \pm 84y^5, -10y^2 \mp 35y^4)$, which is a cusp.

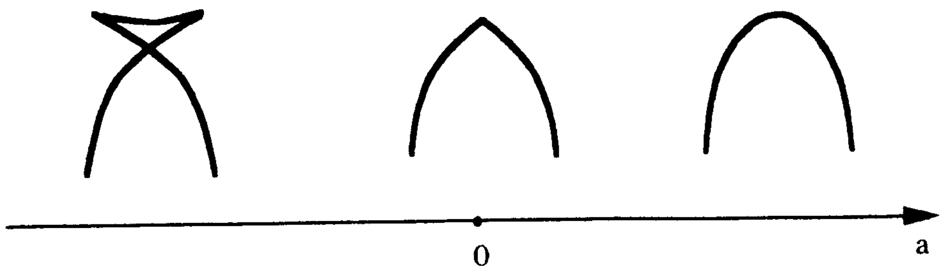



Fig.6.6.5

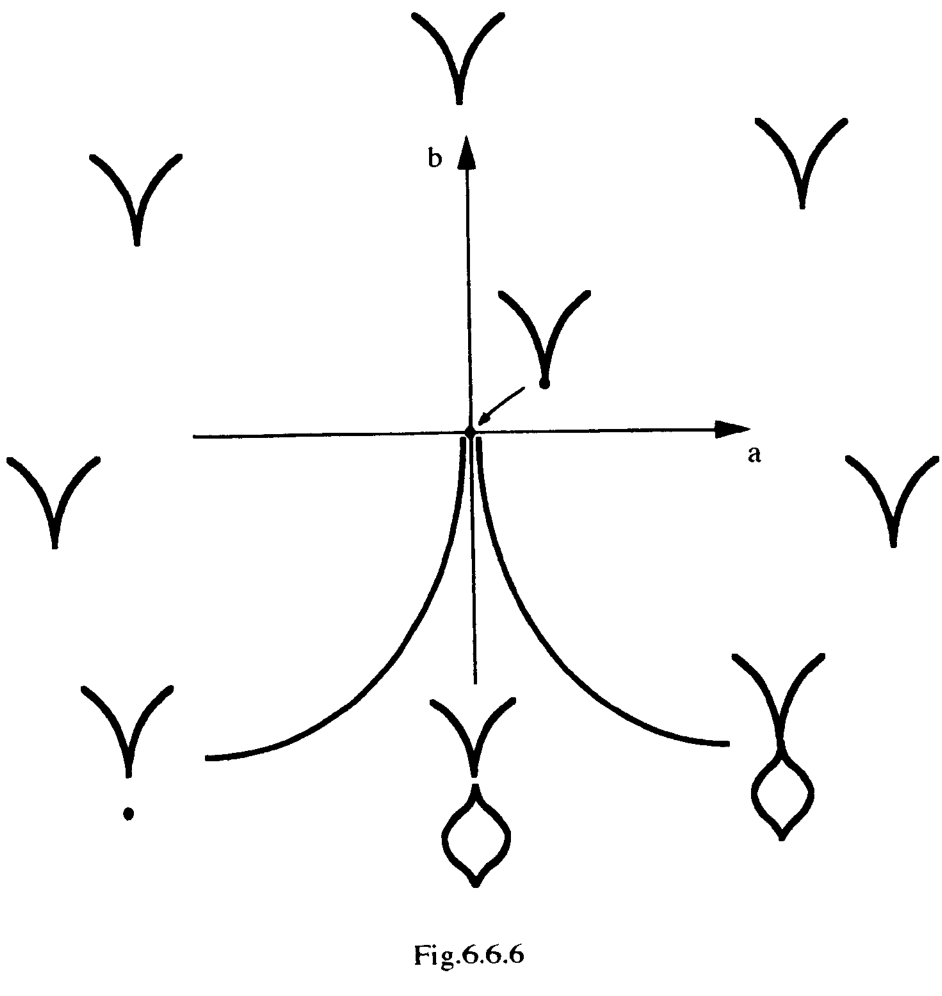



Fig.6.6.6

The cusp-and-fold stratum occurs when $f$ has a cusp singularity at $(x_1, y_1)$ and a fold singularity at $(x_2, y_2)$ with $f(x_1, y_1) = f(x_2, y_2)$ (and $(x_1, y_1) \neq (x_2, y_2)$). Now $f(x_1, y_1) = f(x_2, y_2)$ if and only if $x_1 = x_2$ and $g(x_1, y_1) = g(x_2, y_2)$ (so $y_1 \neq y_2$). The condition for a fold at $(x_1, y_2)$ is that $g_y(x_1, y_2) = 0$ and the condition for a cusp at $(x_1, y_1)$ is that $g_y(x_1, y_1) = g_{yy}(x_1, y_1) = 0$. So there is a real $u$ such that $g(x_1, y) - u$ has one real repeated root and one triple root. Hence

$$g(x_1, y) - u = (y - y_2)^2 (y - y_1)^3 (\pm y^2 + Ay + B)$$

(since $g$ is of degree 7). Comparing coefficients with $g(x, y)$ gives $a = \frac{-10}{27} y_2^3 + \ldots$ and $b = \frac{-5}{3} y_2^2 + \ldots$ so the cusp-and-fold stratum is parametrized by

$$\left(\frac{-10}{27} y_2^3 + \ldots, \frac{-5}{3} y_2^2 + \ldots\right)$$

As for a tacnode stratum, the conditions for a tacnode are that $f$ has a fold at $(x_1, y_1)$ and another one at $(x_2, y_2)$ with $f(x_1, y_1) = f(x_2, y_2)$ (but $(x_1, y_1) \neq (x_2, y_2)$) and that the folds are tangent. But then $x_1 = x_2$ and since we must have $g_x(x_1, y_1) = g_x(x_1, y_2)$ then $y_1 = y_2$, contradicting $(x_1, y_1) \neq (x_2, y_2)$. So no tacnode singularities occur.

The full bifurcation set is shown in Fig. 6.6.7.

(viii) $f(x, y) = (x, xy^2 + y^4 + y^5)$. This is the gull singularity. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = (\mathcal{E}_2, \mathcal{E}_2 - \{y, y^3\})$$

and so a versal unfolding is $f_{a,b}(x, y) = (x, xy^2 + y^4 + y^5 + ay + by^3)$. From the adjancency diagram we expect to see swallowtail, lips/beaks and tacnode strata in the bifurcation space. Write $g(x, y) = xy^2 + y^4 + y^5 + ay + by^3$.

For the swallowtail stratum the conditions are $g_y = g_{yy} = g_{yyy} = 0$. So we get

$$\begin{aligned} g_y &= 2xy + 4y^3 + 5y^4 + 3by^2 + a = 0 \\ g_{yy} &= 2x + 12y^2 + 20y^3 + 6by = 0 \\ g_{yyy} &= 24y + 60y^2 + 6b = 0 \end{aligned}$$

Then $a = -4y^3 - 15y^4$ and $b = -4y - 10y^2$ so a parametrization of the swallowtail stratum is given by

$$(-4y^3 - 15y^4, -4y - 10y^2)$$

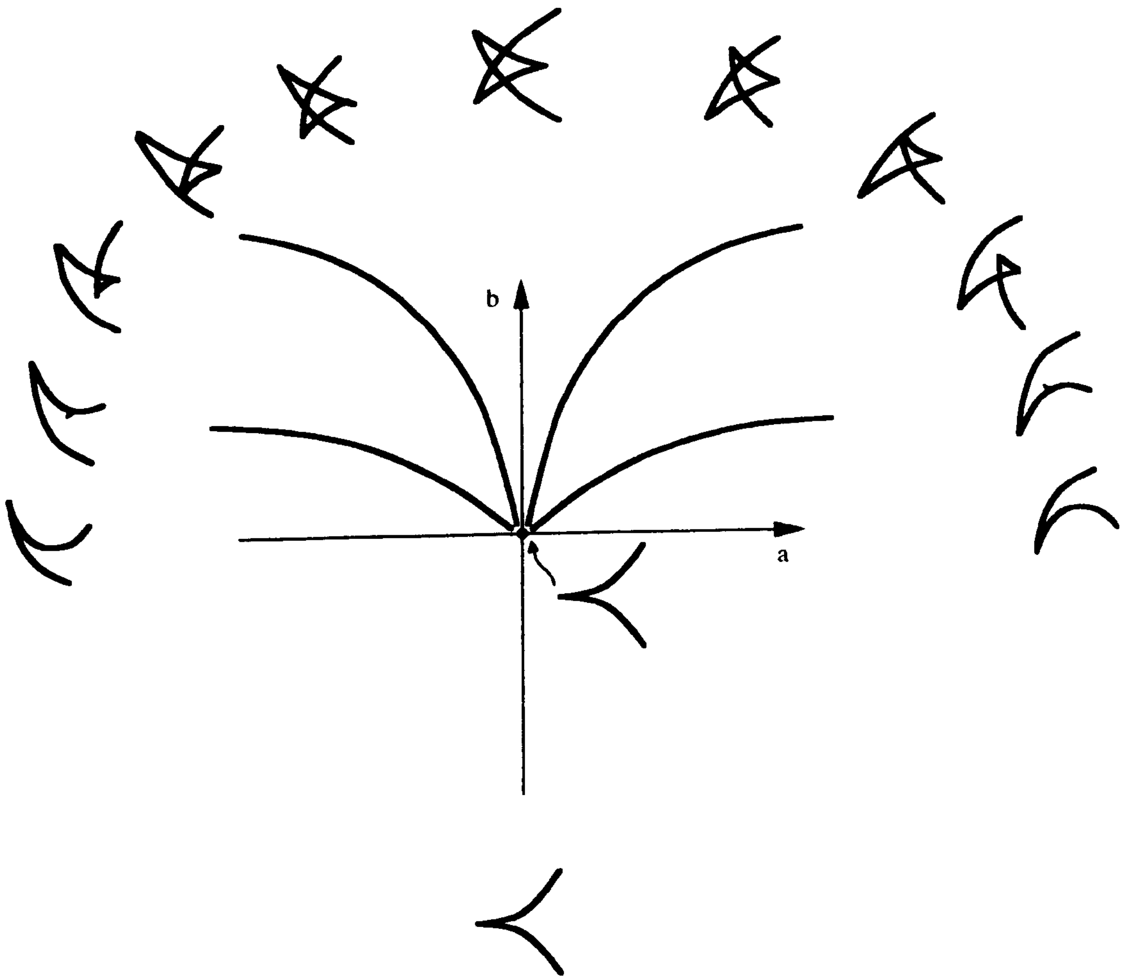



Fig. 6.6.7

The lips/beaks stratum occurs when $\Sigma$ is singular, so $g_y = 0$ and $g_{xy} = g_{yy} = 0$. These conditions yield

$$g_y = 2xy + 4y^3 + 5y^4 + 3by^2 + a = 0$$
$$g_{xy} = 2y = 0$$
$$g_{yy} = 2x + 12y^2 + 20y^3 + 6by = 0$$

So $x = y = 0$ and $a = 0$. Then the lips/beaks stratum is $a = 0$. In fact we see beaks here rather than lips.

The tacnode stratum occurs when $f$ has two folds $(x_1, y_1)$ and $(x_2, y_2)$ with $f(x_1, y_1) = f(x_2, y_2)$ (but $(x_1, y_1) \neq (x_2, y_2)$) and the folds are tangential. The tangent to a fold is given by the image of the tangent of the critical set, $\Sigma$, by $Df$. Since $\Sigma$ is given by $g_y = 0$, its tangent is given by $(g_{yy}, -g_{xy})$. Hence the tangent to the fold is given by

$$\begin{pmatrix} 1 & 0 \\ g_x & g_y \end{pmatrix} \begin{pmatrix} g_{yy} \\ -g_{xy} \end{pmatrix} = g_{yy}(1, g_x)$$

(Note that $g_{yy} \neq 0$). So the tangent has direction $(1, g_x)$ at $(x_1, y_1)$ and at $(x_2, y_2)$. These are parallel if and only if $y_1 = \pm y_2$. Since $f(x_1, y_1) = f(x_2, y_2)$ we have $x_1 = x_2$. So $y_1 = -y_2$. Now $g(x_1, y_1) = g(x_1, -y_1)$ and $g_y(x_1, y_1) = g_y(x_1, -y_1) = 0$ so $g(x_1, y) - g(x_1, y_1)$ has two repeated roots, $y_1$ and $-y_1$. Hence

$$g(x_1, y) - g(x_1, y_1) = (y - y_1)^2 (y + y_1)^2 (y - y_3)$$

Comparing coefficients we get $a = y_1^4$ and $b = -2y_1^2$. So the tacnode stratum is parametrized by $(y_1^4, -2y_1^2)$. The full bifurcation set is shown in Fig. 6.6.8.

(ix) $f(x, y) = (x^2 + y^3, y^2 + x^3)$. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = (\mathcal{E}_2 - \{y\}, \mathcal{E}_2 - \{x\})$$

and so a versal unfolding is $f_{a,b}(x, y) = (x^2 + y^3 + ay, y^2 + x^3 + bx)$. From the adjancency diagram we expect to see lips/beaks singularities in the unfolding and the condition for this is that the critical set, $\Sigma$, is singular. $\Sigma$ is given by $4xy - (3x^2 + b)(3y^2 + a) = 0$ and it is singular if the following hold:

$$4y - 6x(3y^2 + a) = 0$$
$$4x - 6y(3x^2 + b) = 0$$

These give $16xy = 4(3x^2 + b)(3y^2 + a) = 36xy(3x^2 + b)(3y^2 + a)$ so either $3y^2 + a = 0$ or $3x^2 + b = 0$ or $9xy = 1$. The latter can be ignored since we are working locally. Then if $3x^2 + b = 0$, either $x = 0$ or $y = 0$ so either $b = 0$ or

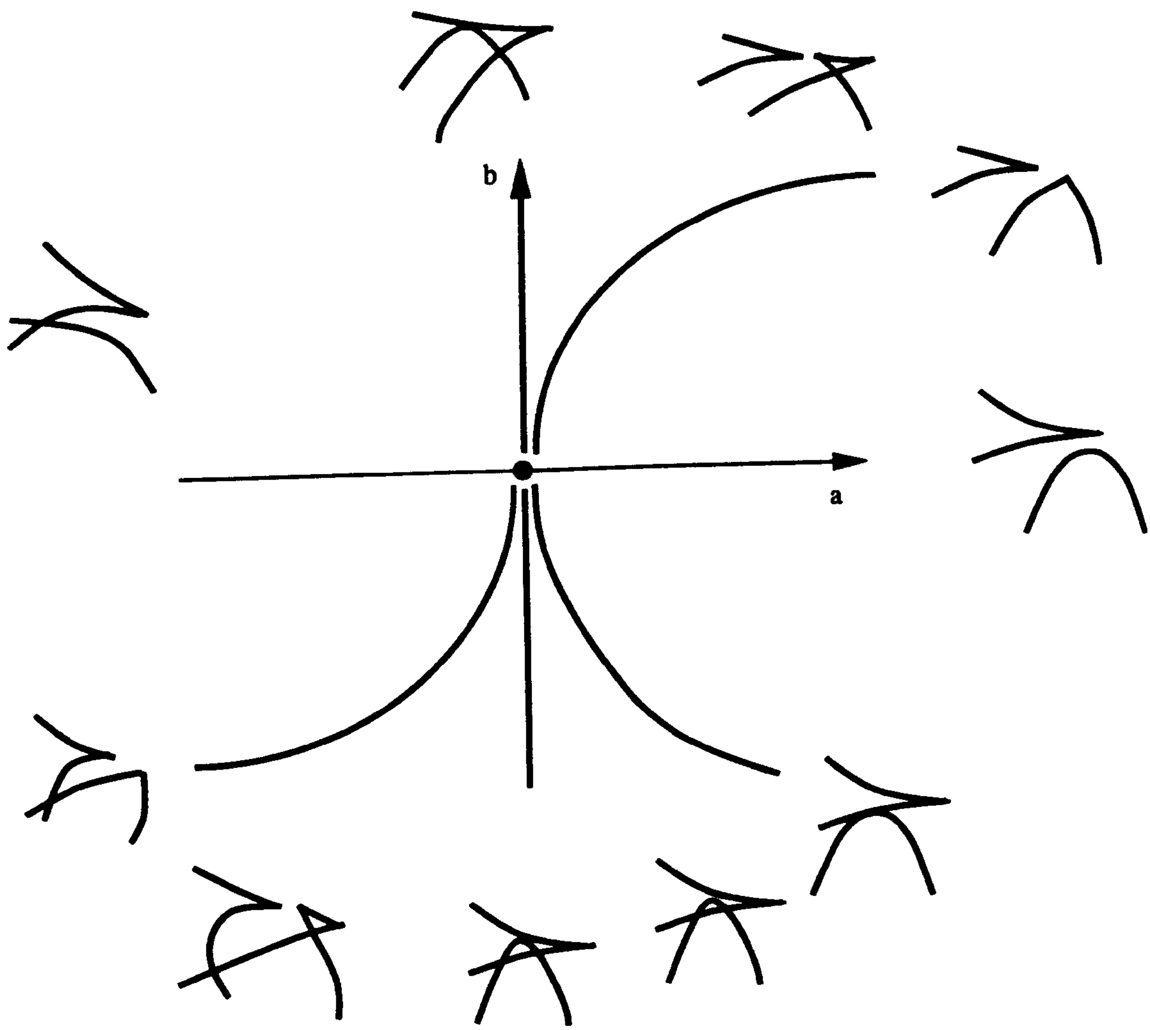



Fig. 6.6.8

$a = 0$, and if $3y^2 + a = 0$ then either $y = 0$ or $x = 0$ and so either $a = 0$ or $b = 0$. Hence the lips/beaks stratum is $\{a = 0\} \cup \{b = 0\}$.

We now look for a tacnode stratum. For this we need two points $(x_1, y_1)$ and $(x_2, y_2)$ which are both singular, have the same image and have tangent fold lines. In fact, if we follow [B3] we see that in this case the tacnode stratum is empty.

Computer generated pictures of various unfoldings are shown in Fig. 6.6.9.

(x) $f(x,y) = (x^2 - y^2 + x^3, xy)$. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = (\mathcal{E}_2 - \{x,y\}, \mathcal{E}_2 - \{x,y\}) + \{(2x,y),(-2y,x)\}$$

so we can choose an unfolding. Consider unfolding $f_{a,b}(x,y) = (x^2 - y^2 + x^3 + ax, xy + bx)$. From the adjacency diagram, the only mono-germ singularities we expect are cusps.

Computer generated pictures of various unfoldings are shown in Fig. 6.6.10.

### 6.6.3 Unfoldings of Bi-germs $\mathbf{R}^2, 0 \to \mathbf{R}^2, 0$

We now want to look at unfoldings of the bi-germs on our list. As with mono-germs, we shall be looking at the discriminants of the bi-germs. Again we can find the geometric invariants $c$ and $d$ to give upper bounds for the number of cusps and transverse fold crossings we expect to see on the discriminant of each map-germ. These bifurcation sets, as before, appear in Kergosien [K] and Rieger [Ri1] but here we try to give a clearer explanation of their origin and also clearer diagrams.

(i) $f(x,y;X,Y) = (x, y^2; X^2, Y)$. This is a stable bi-germ, the transverse fold crossing. See Fig. 6.6.11.

(ii) $f(x,y;X,Y) = (x, y^2; X, XY + Y^2)$. This bi-germ has $\mathcal{A}_e$-codimension 1. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f = (\mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2 - \{1\})$$

and so the unfolding is

$$f_a(x,y;X,Y) = (x, y^2; X, XY + Y^2 + a).$$

The two discriminants are defined by $(X, 0)$ and $(-2Y, -Y^2 + a)$ and the bifurcation set is $a = 0$. See Fig. 6.6.12.

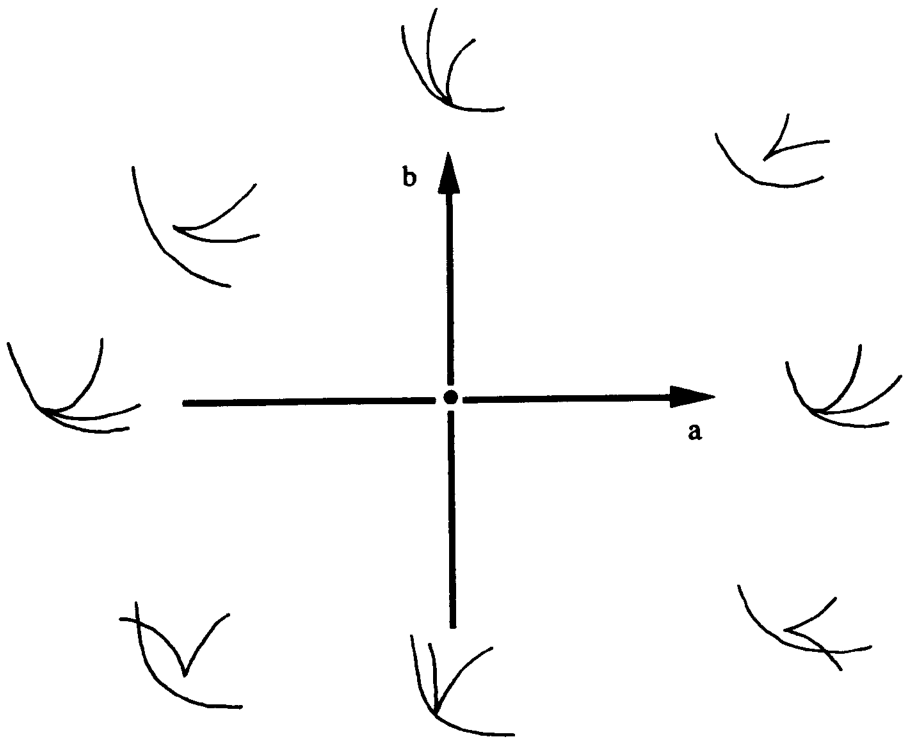



Fig.6.6.9

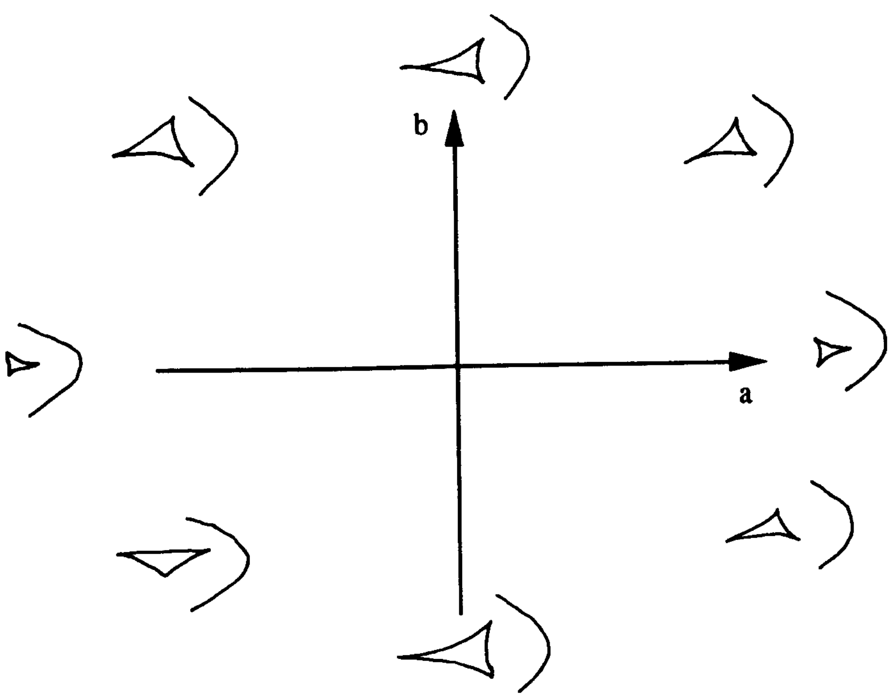



Fig.6.6.10

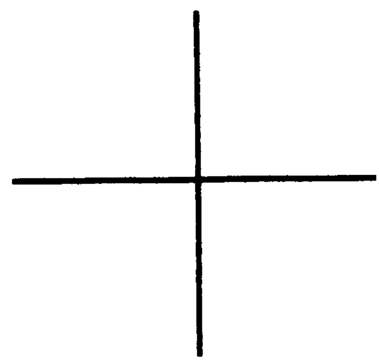


Fig.6.6.11

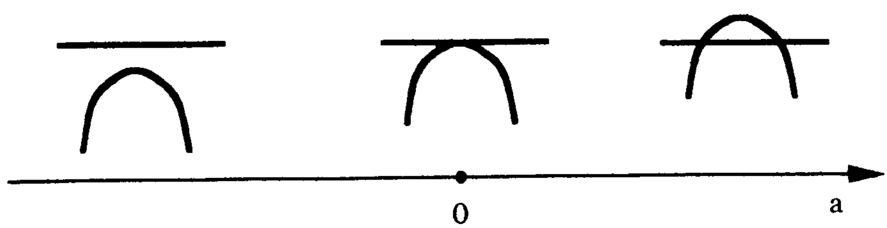



Fig.6.6.12

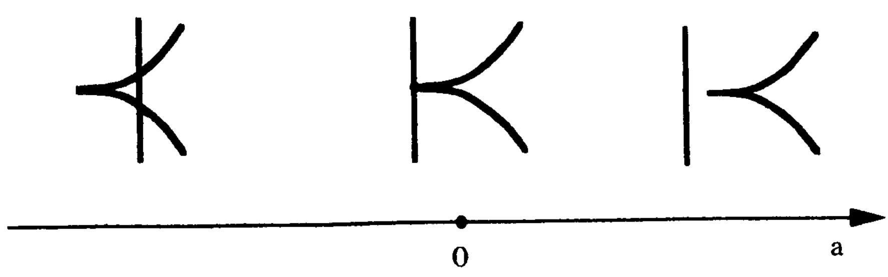



Fig.6.6.13

(iii) $f(x,y;X,Y) = (x,y^2;XY+X^3,Y)$. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = (\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2 - \{1\},\mathcal{E}_2)$$

so the versal unfolding is

$$f_a(x,y;X,Y) = (x,y^2;XY+X^3+a,Y)$$

Thus the bifurcation set is $a=0$. See Fig. 6.6.13.

(iv) $f(x,y;X,Y) = (x,y^2;X,XY+Y^3)$. The $\mathcal{A}_e$-codimension is 2 and we have

$$T\mathcal{A}_e.f = (\mathcal{E}_2,\mathcal{E}_2 - \{1,x\};\mathcal{E}_2,\mathcal{E}_2 - \{1,X\}) + \mathbf{R}\langle(0,1;0,1),(0,x;0,X)\rangle$$

So an unfolding is

$$f_{a,b} = (x,y^2+a+bx;X,XY+Y^3)$$

The two discriminants are $(x,a+bx)$ and $(-3Y^2,-2Y^3)$. They will intersect if

$$2Y^3 - 3bY^2 + a = 0$$

The discriminant of this cubic is $ab^3 - a^2$ and so the bifurcation set consists of the union of $a=0$ and $a=b^3$. See Fig. 6.6.14.

(v) $f(x,y;X,Y) = (x,y^2;X,Y^2+X^3)$. The $\mathcal{A}_e$-codimension is 2 and the tangent space is

$$T\mathcal{A}_e.f = (\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2 - \{1,X\})$$

so the versal unfolding is

$$f_{a,b}(x,y;X,Y) = (x,y^2;X,Y^2+X^3+a+bX)$$

The discriminants are given by $(x,0)$ and $(X,X^3+bX+a)$, so the bifurcation set is the cuspidal cubic $-4b^3 = 27a^2$ and the pictures in parameter space are shown in Fig. 6.6.15.

(vi) $f(x,y;X,Y) = (x,y^2;XY^2 \pm X^3,Y)$. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = (\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2 - \{X,XY\},\mathcal{E}_2)$$

and so a versal unfolding is

$$f_{a,b}(x,y;X,Y) = (x,y^2;XY^2 \pm X^3 + aX + bXY,Y)$$

We know that this bi-germ is a lips/beaks map combined (transversely) with a fold. We expect to see 'cusp-and-fold' singularities and birth (or death) of lips/beaks singularities.

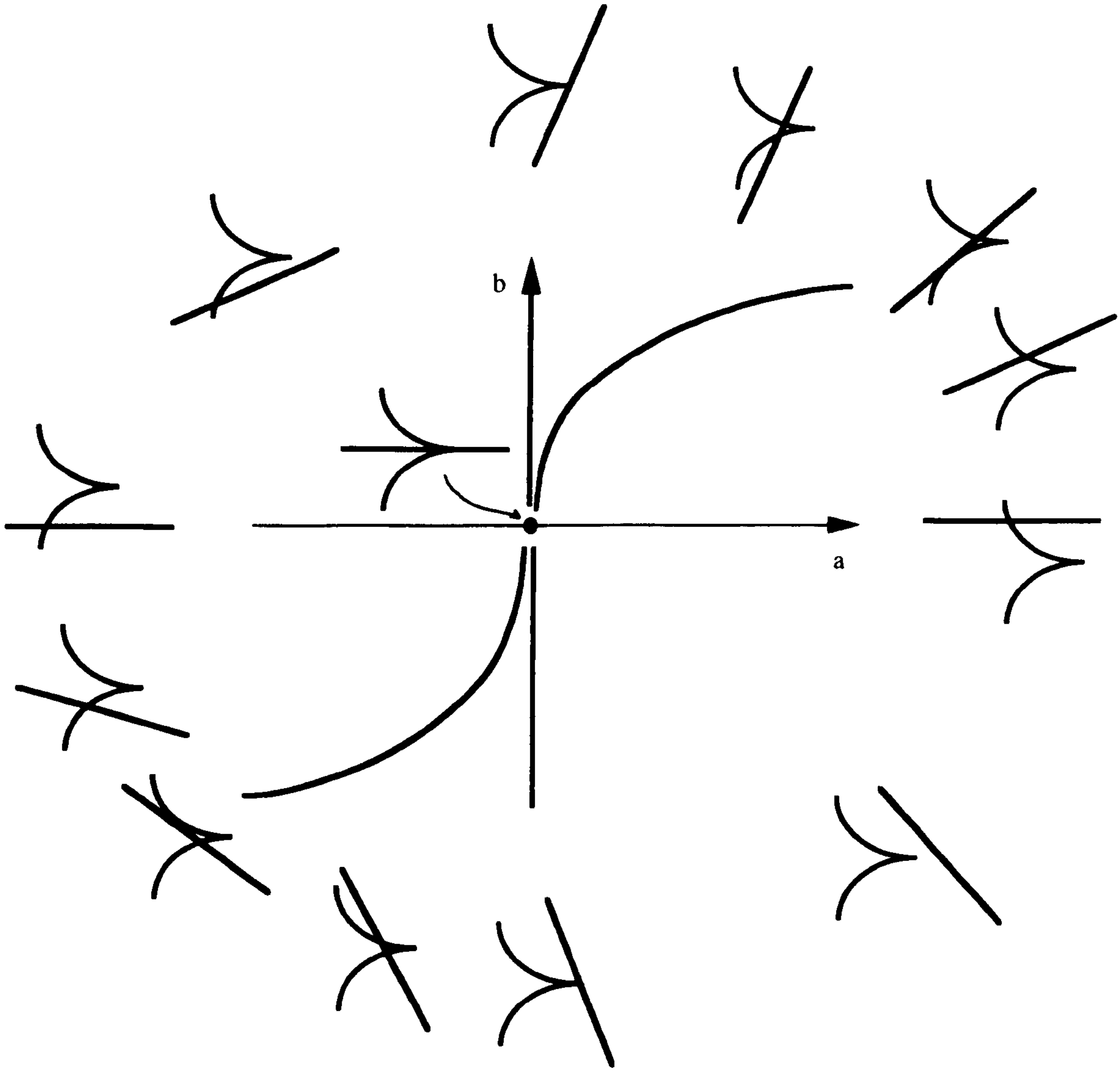



Fig. 6.6.14

The discriminant of the first branch of the bi-germ is just $y = 0$. If we write the second branch as $f_2(X,Y) = (g(X,Y),Y)$, where $g(X,Y) = XY^2 \pm X^3 + aX + bXY$, then there will be a cusp singularity when $g_X = g_{XX} = 0$ which gives $Y^2 \pm 3X^2 + a + bY = \pm 6X = 0$. Thus cusps appear when $Y^2 + bY + a = 0$, and meet the discriminant of the first branch (transversely) when $Y = 0$, which gives $a = 0$ as the stratum of cusp-and-fold singularities in the parameter space.

Birth of lips/beaks occurs when $g_X = g_{XX} = g_{XY} = 0$, i.e. when $Y^2 \pm 3X^2 + a + bY = \pm 6X = 2Y + b = 0$. This gives $2b^2 + a = 0$ as the stratum of lips/beaks in the parameter space. In the lips case the unfolding diagram is shown in Fig. 6.6.16. For the beaks case see Fig. 6.6.17.

(vii) $f(x,y;X,Y) = (x,y^2;XY + X^4,Y)$. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = (\mathcal{E}_2, \mathcal{E}_2 - \{1\}; \mathcal{E}_2 - \{X^2\}, \mathcal{E}_2)$$

so the unfolding is

$$f_{a,b}(x,y;X,Y) = (x, y^2 + a; XY + X^4 + bX^2, Y)$$

The discriminant of the first branch is $(x,a)$ and of the second branch it is $(-3X^4 - bX^2, -4X^3 - 2bX)$. Clearly, if $a = 0$ and $b < 0$ there will be a triple point in the image. There will also be cusp-and-fold singularities if the line $(x,a)$ meets the swallowtail with a repeated root i.e. if $a = -4X^3 - 2bX$ has a repeated root. This gives the cuspidal cubic $-8b^3 = 27a^2$. See Fig. 6.6.18.

(viii) $f(x,y;X,Y) = (x, xy + y^3; XY + X^3, Y)$. The $\mathcal{A}_e$-codimension is 2 and the $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = (\mathcal{E}_2, \mathcal{E}_2 - \{1\}; \mathcal{E}_2 - \{1\}, \mathcal{E}_2)$$

so the versal unfolding is

$$f_{a,b}(x,y;X,Y) = (x, xy + y^3 + a; XY + X^3 + b, Y)$$

The first branch has discriminant $(-3y^2, -2y^3 + a)$, which is a cuspidal cubic with cusp at $(0,a)$. The second branch has discriminant $(-2X^3 + b, -3X^2)$, which again is a cuspidal cubic, this time with cusp at $(b,0)$.

The cusp of the first branch meets the other discriminant in a cusp-and-fold singularity if $(0,a) = (-2X^3 + b, -3X^2)$, which gives the condition $-4a^3 = 27b^2$. Similarly, the cusp of the second branch meets the first branch in a cusp-and-fold singularity if $(b,0) = (-3y^2, -2y^3 + a)$ which gives the condition $-27a^2 = 4b^3$. Thus the bifurcation set consists of these two cuspidal cubics. See Fig. 6.6.19.

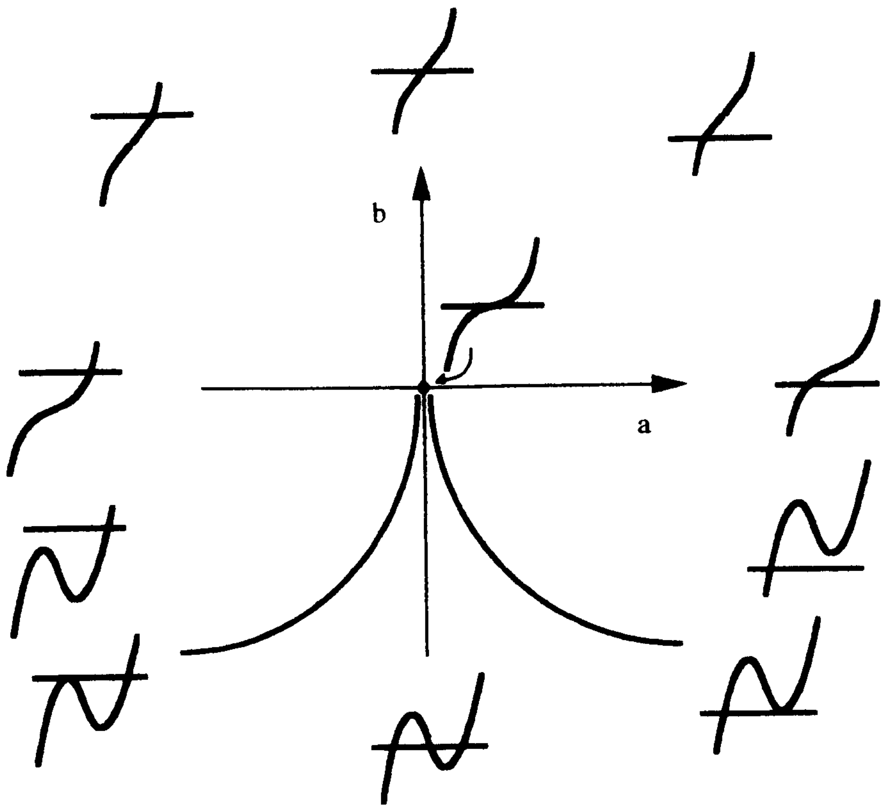



Fig.6.6.15

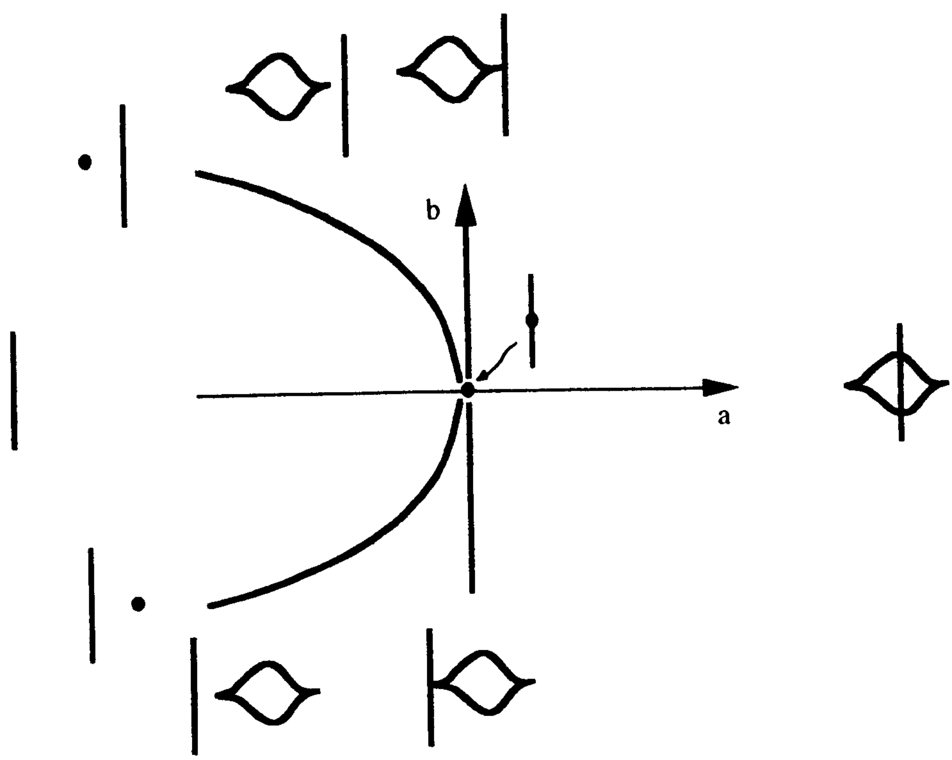



Fig.6.6.16

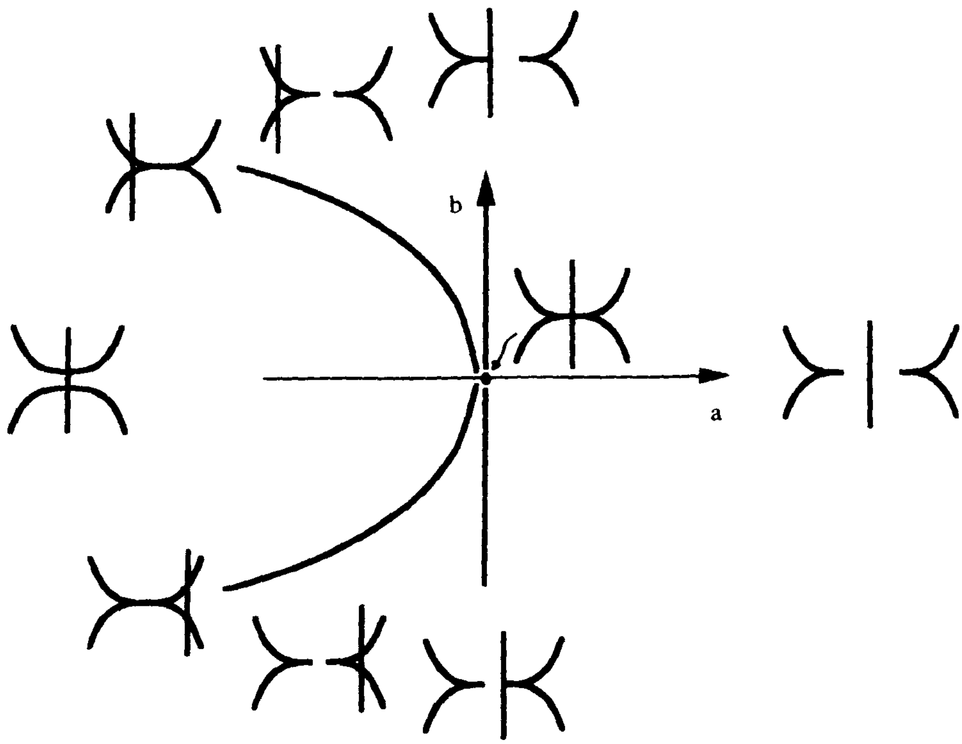



Fig. 6.6.17

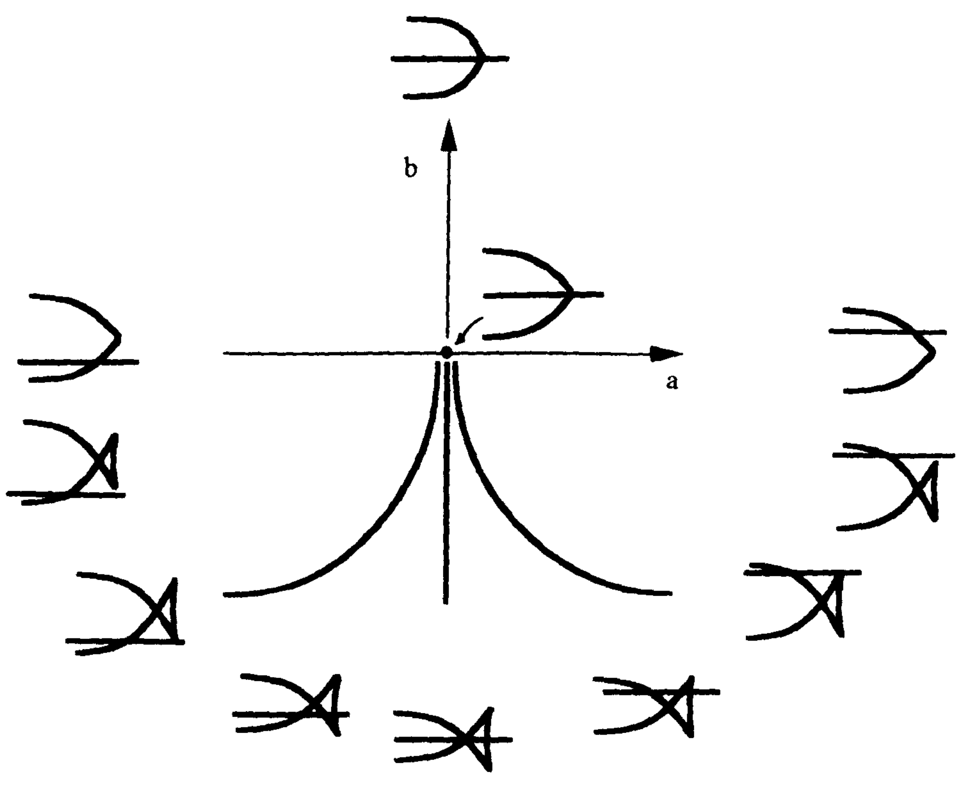



Fig. 6.6.18

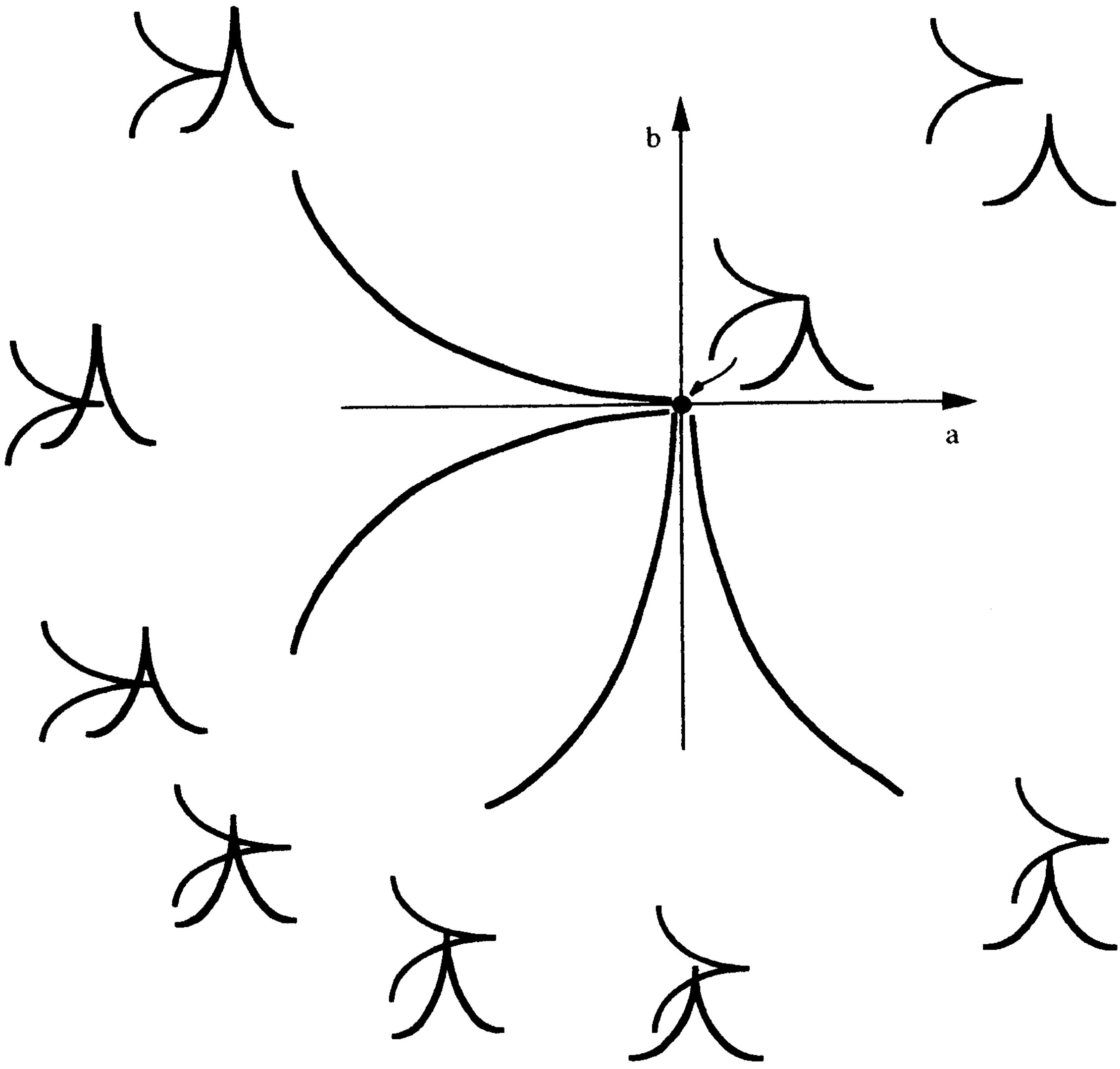



Fig. 6.6.19

### 6.6.4 Unfoldings of Tri-germs and 4-germs $\mathbf{R}^2 \to \mathbf{R}^2$

(i) $f(x,y;X,Y;\tilde{x},\tilde{y}) = (x,y^2;X^2,Y;\tilde{x},\tilde{x}+\tilde{y}^2)$. The $\mathcal{A}_e$-codimension is 1 and the $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = (\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2-\{1\},\mathcal{E}_2)$$

So a versal unfolding is

$$f_a = (x,y^2;X^2,Y;\tilde{x}+a,\tilde{x}+\tilde{y}^2)$$

Then the three discriminants are $(x,0)$, $(0,Y)$ and $(\tilde{x}+a,\tilde{x})$ and the bifurcation set is clearly given by $a=0$. See Fig. 6.6.20.

(ii) $f(x,y;X,Y;\tilde{x},\tilde{y}) = (x,y^2;X^2,Y;\tilde{x},\tilde{x}\tilde{y}+\tilde{y}^2)$. This tri-germ has $\mathcal{A}_e$-codimension 2 and the tangent space is

$$T\mathcal{A}_e.f = (\mathcal{E}_2,\mathcal{E}_2-\{1,x\};\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2-\{1,\tilde{x}\},\mathcal{E}_2)+\mathbf{R}\langle(0,1;0,0;0,1),(0,x;0,0;0,\tilde{x})\rangle$$

So we can choose the unfolding

$$f_{a,b} = (x,y^2+a+bx;X^2,Y;\tilde{x},\tilde{x}\tilde{y}+\tilde{y}^2)$$

The discriminant of $f_1$ is then $(x,a+bx)$, and that of $f_3$ is $(\tilde{x},\tilde{x}^2/4)$. These intersect if

$$x^2+4bx+4a=0$$

which has two distinct solutions if $b^2>a$, one repeated root (and hence a tacnode) if $b^2=a$ and no solutions otherwise. There will also be a triple point in the image if $a=0$ so the bifurcation set is the union of this with $b^2=a$. See Fig. 6.6.21.

(iii) $f(x,y;X,Y;\tilde{x},\tilde{y}) = (x,y^2;X^2,Y;\tilde{x},\tilde{x}+\tilde{x}\tilde{y}+\tilde{y}^3)$. In this case the $\mathcal{A}_e$-codimension is 2 and we have

$$T\mathcal{A}_e.f = (\mathcal{E}_2,\mathcal{E}_2-\{1\};\mathcal{E}_2-\{1\},\mathcal{E}-2;\mathcal{E}_2,\mathcal{E}_2)$$

So a versal unfolding is

$$f_{a,b} = (x,y^2+a;X^2+b,Y;\tilde{x},\tilde{x}+\tilde{x}\tilde{y}+\tilde{y}^3)$$

The discriminants are $(x,a)$, $(b,Y)$ and $(-3\tilde{y}^2,-3\tilde{y}^2-2\tilde{y}^3)$. The first two must intersect in $(a,b)$, and they all intersect, to give a triple point, if $(a-b)^2+4b^3/9=0$. Clearly, if $a=0$ or $b=0$ then there will be a cusp-and-fold singularity. Thus the bifurcation set is shown in Fig. 6.6.22.

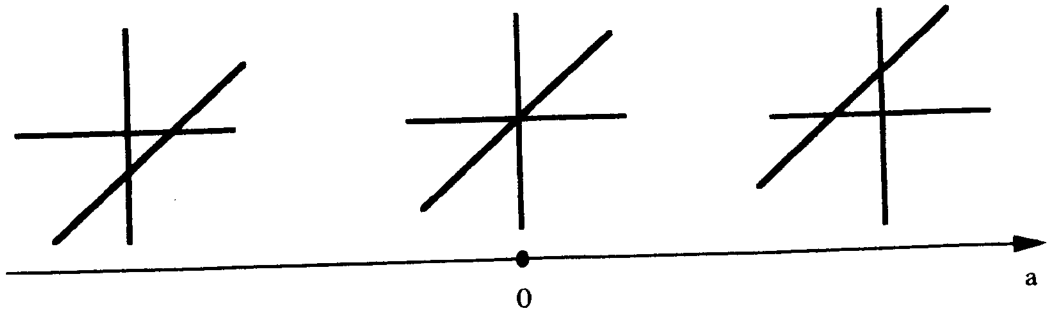



Fig. 6.6.20

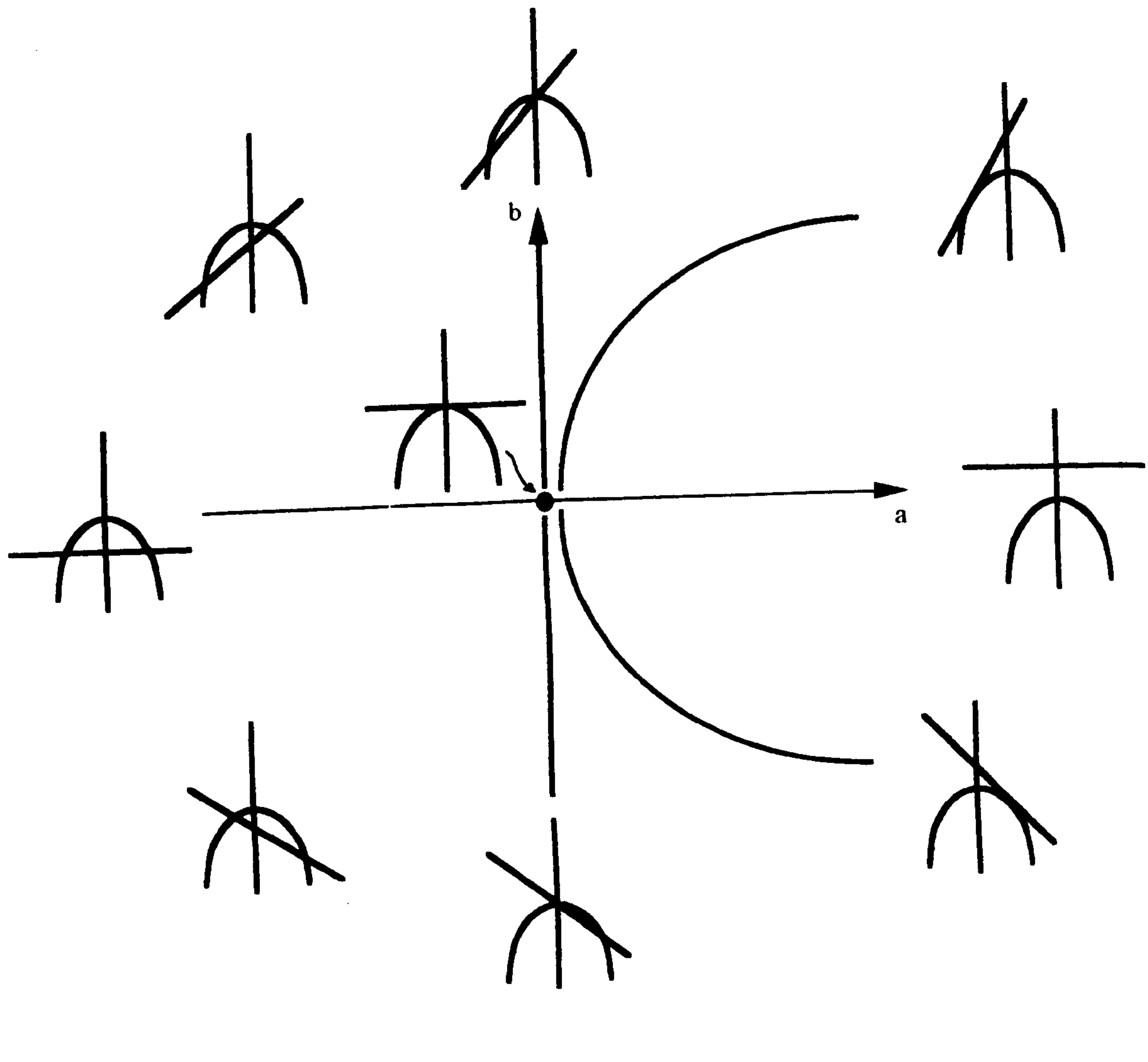



Fig. 6.6.21

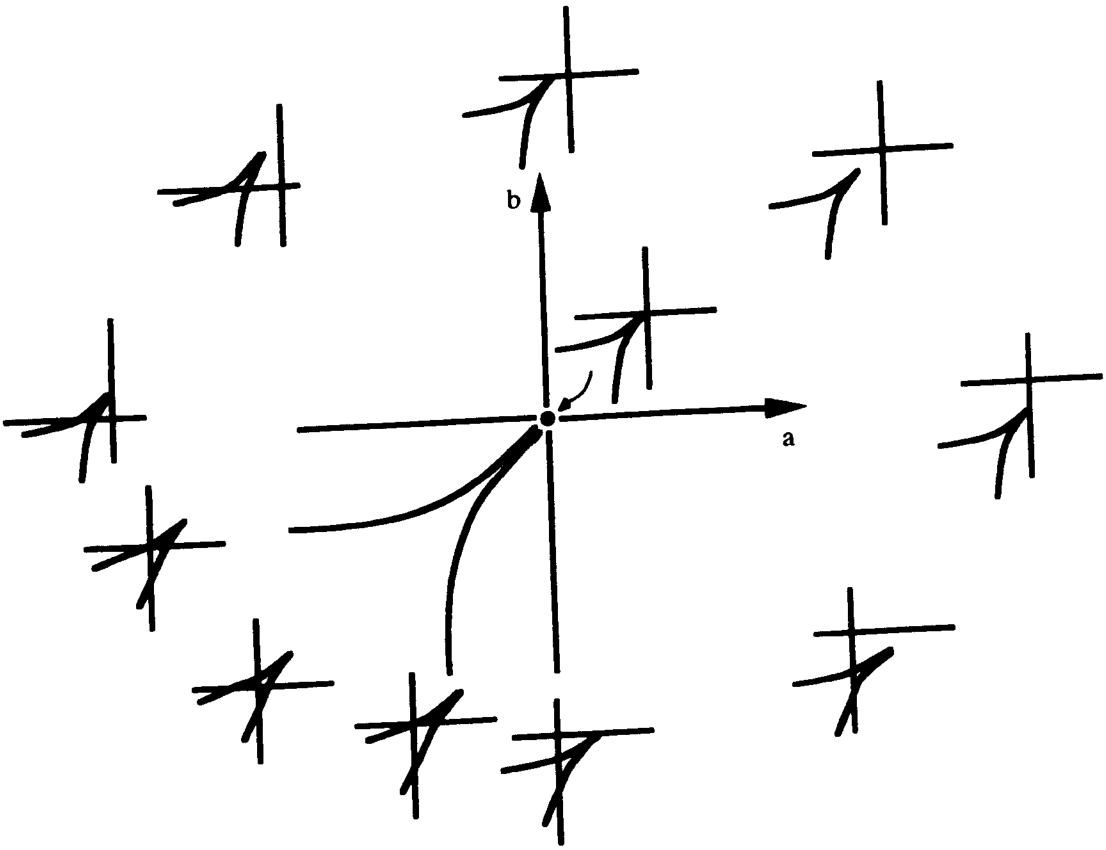



Fig. 6.6.22

(iv) Finally we come to the 4-germ,

$$f(x,y;X,Y;\tilde{x},\tilde{y};\tilde{X},\tilde{Y}) = (x,y^2;X^2,Y;\tilde{x},\tilde{x}+\tilde{y}^2;\tilde{X},\lambda\tilde{X}+\tilde{Y}^2)$$

where $\lambda \neq 0,1$. The $\mathcal{A}_e$-codimension of the stratum is 2 and the $\mathcal{A}_e$-tangent space to the stratum is given by

$$T\mathcal{A}_e.f = (\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2-\{1\},\mathcal{E}_2;\mathcal{E}_2-\{1\},\mathcal{E}_2)$$

and the bifurcation set consists of the four lines $a=0$, $b=0$, $a=b$ and $a=\lambda b$. On these lines the image of the unfolding has a triple point. See Fig. 6.6.23 (which gives the case $\lambda=-1$).

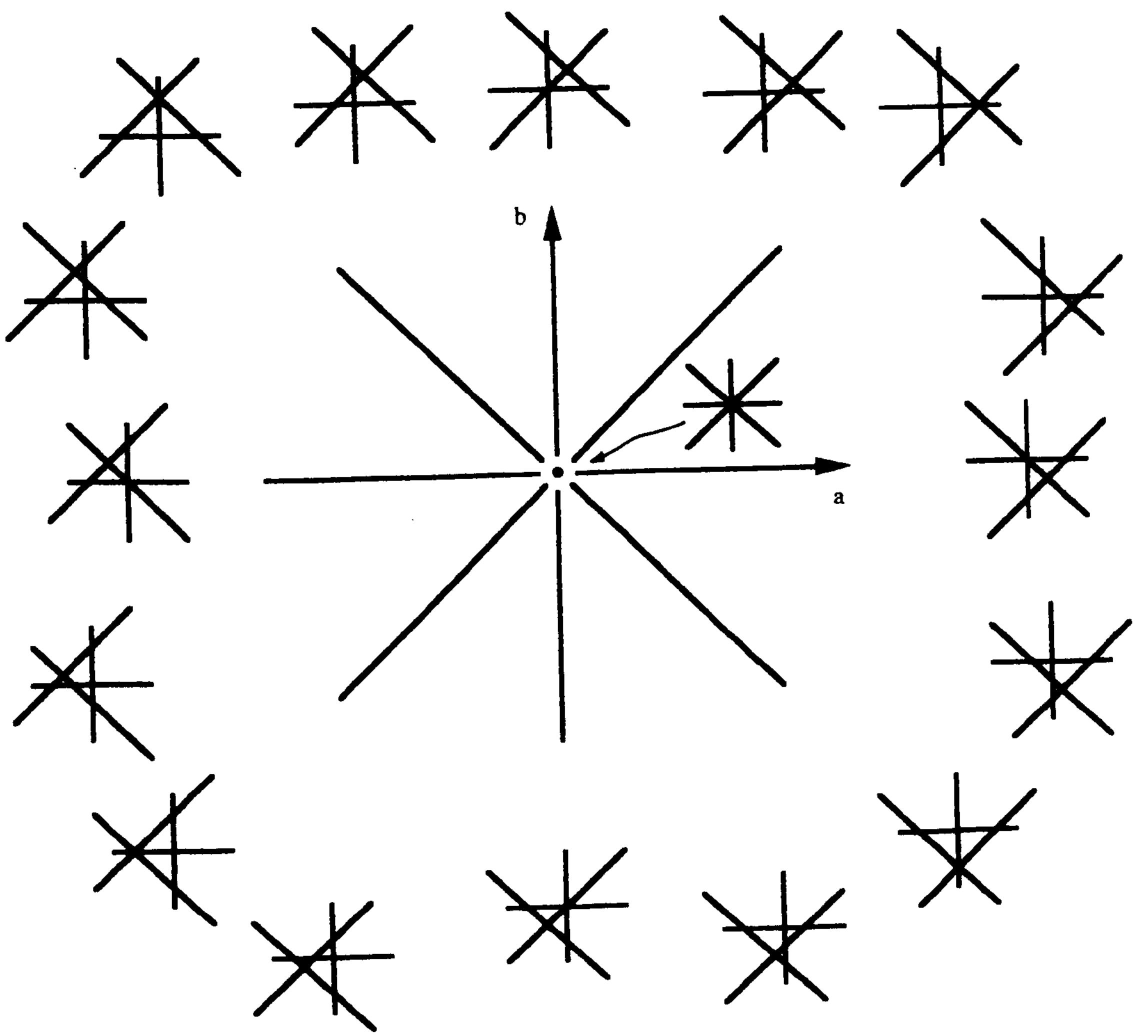



Fig. 6.6.23

# Chapter 7 – Two-Dimensional Motions of Space

## 7.1 Introduction

Looking at motions of 3-space which have two degrees of freedom is equivalent (by the Transversality Theorem) to classifying singularities $\mathbf{R}^2, 0 \to \mathbf{R}^3, 0$ up to $\mathcal{A}$-equivalence. We are looking for singularities of up to codimension 3, by Proposition 2.2.4. In the mono-germ case almost all the relevant possibilities are simple, with the exception being $P_3$.

**7.1.1 Theorem** *For a generic motion of space with two degrees of freedom, any mono-germ of a trajectory is $\mathcal{A}$-equivalent to one of the following normal forms.*

| Name | Normal Form | $\mathcal{A}_e$-codimension |
|---|---|---|
| $S_0$ | $(x, y^2, xy)$ | 0 |
| $S_1^\pm$ | $(x, y^2, y^3 \pm x^2y)$ | 1 |
| $S_2$ | $(x, y^2, y^3 + x^3y)$ | 2 |
| $S_3^\pm$ | $(x, y^2, y^3 \pm x^4y)$ | 3 |
| $B_2^\pm$ | $(x, y^2, x^2y \pm y^5)$ | 2 |
| $B_3^\pm$ | $(x, y^2, x^2y \pm y^7)$ | 3 |
| $C_3^\pm$ | $(x, y^2, xy^3 \pm x^3y)$ | 3 |
| $H_2$ | $(x, y^3, xy + y^5)$ | 2 |
| $H_3$ | $(x, y^3, xy + y^8)$ | 3 |
| $P_3$ | $(x, xy + y^3, xy^2 + cy^4)$ | 4 |

In the case of $P_3$ the $\mathcal{A}_e$-codimension of the stratum is 3 although the $\mathcal{A}_e$-codimension of each orbit is 4. The names of the singularities which are given in these tables are taken from David Mond's notation [Md1]. These singularities have been classified in [Md1] (also see [B]).

**7.1.2 Theorem** *For a generic motion of space with two degrees of freedom, any bi-germ of a trajectory is $\mathcal{A}$-equivalent to one of the following normal forms.*

| Name | Normal Form | $\mathcal{A}_e$-codimension |
|---|---|---|
| | $(x, y, 0; 0, X, Y)$ | 0 |
| $[A_1^{\pm}]$ | $(x, y, 0; X, Y, X^2 \pm Y^2)$ | 1 |
| $[A_2]$ | $(x, y, 0; X, Y, X^2 + Y^3)$ | 2 |
| $[A_3^{\pm}]$ | $(x, y, 0; X, Y, X^2 \pm Y^4)$ | 3 |
| | $(x, y, 0; Y^2, XY + Y^3, X)$ | 1 |
| | $(x, y, 0; Y^2, XY + Y^5, X)$ | 2 |
| | $(x, y, 0; Y^2, XY + Y^7, X)$ | 3 |
| $[S_1^{\pm}]$ | $(x, y, 0; Y^3 \pm X^2Y, Y^2, X)$ | 2 |
| $[S_2]$ | $(x, y, 0; Y^3 + X^3Y, Y^2, X)$ | 3 |
| | $(x, y, 0; X, XY, Y^2 + X^3)$ | 3 |
| | $(x, y, 0; X, Y^2, XY + Y^4)$ | 3 |

**7.1.3 Theorem** *For a generic motion of space with two degrees of freedom, higher multi-germs of trajectories are $\mathcal{A}$-equivalent to one of the following normal forms.*

| Normal Form | $\mathcal{A}_e$-codimension |
|---|---|
| $(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y)$ | 0 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^2)$ | 1 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^3)$ | 2 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, X^2 \pm Y^2)$ | 2 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^4)$ | 3 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, XY + X^3)$ | 3 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, X^2 + Y^3)$ | 3 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X}, \bar{Y}, \bar{X} + \bar{Y})$ | 1 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X}, \bar{Y}, \bar{X} + \lambda\bar{Y}; \mathrm{x}, \mathrm{y}, \mathrm{x} + \mu\mathrm{y})$ | 2 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X}, \bar{Y}, \bar{X} + \lambda\bar{Y}; \mathrm{x} + \mu\mathrm{y}, \mathrm{x}, \mathrm{y}; \bar{\mathrm{x}}, \rho\bar{\mathrm{x}} + \gamma\bar{\mathrm{y}}, \bar{\mathrm{y}})$ | 2 |

(where the last two multi-germs on the list are strata and the $\mathcal{A}_e$-codimension listed is that of the whole stratum.)

We will prove these theorems, as before, by listing all multi-germ singularities with codimension less than 4. In order to begin this listing we have:

**7.1.4 Proposition** *There exists a residual set of motions of space with two degrees of freedom such that an $r$-germ $f$ of a trajectory of such a motion has $r \leq 6$ and such that the one jets of all the constituent mono-germs of $f$ are all non-zero. Also if $r = 3$ or $r = 4$ at most one branch of $f$ is not $\mathcal{A}$-equivalent to $(x, y, 0)$. If $r = 5$ or $6$ then all branches are immersive.*

**Proof** Following the pattern of Proposition 4.1.2, we look for a finite number of $\mathcal{A}$-invariant submanifolds $X$ of ${}_rJ^k(2,3)$ giving rise to a finite number of $\mathcal{A}$-invariant submanifolds $Y$ of ${}_rJ^k(\mathbf{R}^2, \mathbf{R}^3)$ so that motions $\mu$ with trajectories ${}_rj^k\Phi_\mu$ transverse to the $Y$ satisfy the above properties. Then the result follows by Theorem 2.2.1.

Firstly, by Proposition 2.2.4, we see that codimension $X \leq 6 - r$ and so $r$ can be at most 6. If $X$ is the submanifold of ${}_rJ^k(2,3)$ with any of the $r$ branches having zero 1-jet then codimension $X = 6$, so all branches must have rank 1.

Now if any branch is not an immersion, ie. its 1-jet is not $\mathcal{A}$-equivalent to $(x, y, 0)$, then it must have 1-jet $\mathcal{A}$-equivalent to $(x, 0, 0)$. A general 1-jet has form

$$(ax + by, cx + dy, ex + fy)$$

So in order to have $j^1 f \sim (x, 0, 0)$ the matrix

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$$

must have rank 1. This gives 2 conditions on the jet. So if $r = 3$ let $X$ be the submanifold consisting of tri-germs with two branches having 1-jet $\mathcal{A}$-equivalent to $(x, 0, 0)$. Then codimension $X \geq 4$ but we must have codimension $X \leq 6 - 3 = 3$, so this is not possible. Hence we get the third part of the proposition. Similarly, if we put $r = 4$, 5 and 6 we get the other parts. $\square$

## 7.2 Mono-germs

From Proposition 7.1.4 we know that we need only look at corank 1 maps from the plane to space. David Mond has classified these in his thesis [Md1] (also given in [Md2]). We need to know that everything of codimension less than 4 (including strata) is contained in these lists. From Theorem 1.2 of [Md2] we see that the paper contains an $\mathcal{A}$-invariant stratification of $\Sigma^1 J^k(2,3)$ with the following properties:

(a) For $k \geq 11$, all strata of codimension less than or equal to 6 are the $\mathcal{A}^k$-orbits, or unions of orbits where moduli figure, of germs whose determinacy degree is less than or equal to 11;

(b) the complement in $\Sigma^1 J^k(2,3)$ of the union of these strata has codimension 7 (for $k \geq 11$).

Thus all strata of codimension $\leq 3$ are contained in this stratification and so the complete list of possibilities is as shown in the table in the statement of Theorem 7.1.1.

## 7.3 Bi-germs $\mathbf{R}^2, 0 \to \mathbf{R}^3, 0$

In this section we shall show that the only bi-germs from $\mathbf{R}^2$ to $\mathbf{R}^3$ with $\mathcal{A}_e$-codimension less than or equal to 3 are those given in the table below.

| Name | Normal Form | $\mathcal{A}_e$-codimension |
|---|---|---|
| | $(x,y,0;0,X,Y)$ | 0 |
| $[A_1^{\pm}]$ | $(x,y,0;X,Y,X^2 \pm Y^2)$ | 1 |
| $[A_2]$ | $(x,y,0;X,Y,X^2 + Y^3)$ | 2 |
| $[A_3^{\pm}]$ | $(x,y,0;X,Y,X^2 \pm Y^4)$ | 3 |
| | $(x,y,0;Y^2,XY+Y^3,X)$ | 1 |
| | $(x,y,0;Y^2,XY+Y^5,X)$ | 2 |
| | $(x,y,0;Y^2,XY+Y^7,X)$ | 3 |
| $[S_1^{\pm}]$ | $(x,y,0;Y^3 \pm X^2Y,Y^2,X)$ | 2 |
| $[S_2]$ | $(x,y,0;Y^3 + X^3Y,Y^2,X)$ | 3 |
| | $(x,y,0;X,XY,Y^2+X^3)$ | 3 |
| | $(x,y,0;X,Y^2,XY+Y^4)$ | 3 |

We do this by giving an $\mathcal{A}$-invariant stratification of ${}_2J^k(2,3)$ for $k \geq 2$ and, following Mond [Md2, Theorem 1.2], showing that for $k \geq 7$ all orbits (note that all germs on the list are $\mathcal{A}$-simple) of $\mathcal{A}$-codimension less than or equal to 4 are $\mathcal{A}^k$-orbits of germs whose determinacy degree is less than or equal to 7 and that the complement in ${}_2J^k(2,3)$ of the union of these orbits has codimension 5 in the jet-space.

If both branches are immersive the bi-germ can be written in the form $(x, y, 0; X, Y, \phi(X,Y))$ (unless it is the bi-germ consisting of two planes, which we can write $(x, y, 0; 0, X, Y)$, which is one-determined with $\mathcal{A}_e$-codimension 0). For such bi-germs we have the following result from [Md1].

**7.3.1 Theorem** (Mond) *Bi-germs of immersions are classified for $\mathcal{A}$ by the $\mathcal{K}$-classes of the separation function $\phi(X,Y)$.*

**Proof** See [Md1], I.10:2. □

Using this we find that the bi-germs of immersions with $\mathcal{A}_e$-codimension $\leq 3$ are $[A_1^{\pm}]$, $[A_2]$ and $[A_3^{\pm}]$ where these symbols come from the $\mathcal{K}$-classes of the separation functions. So the first part of the stratification of ${}_2J^k(2,3)$ is given in fig. 7.3.1. The only two non-sufficient strata are then $\mathcal{A}^2(x, y, 0; X, Y, 0)$ and $\mathcal{A}^4(x, y, 0; X, Y, X^2)$ which both have codimension 5 (greater than $6-2$) in their respective jet-spaces, since for the first case if we let $F = (x, y, 0; X, Y, 0)$ then

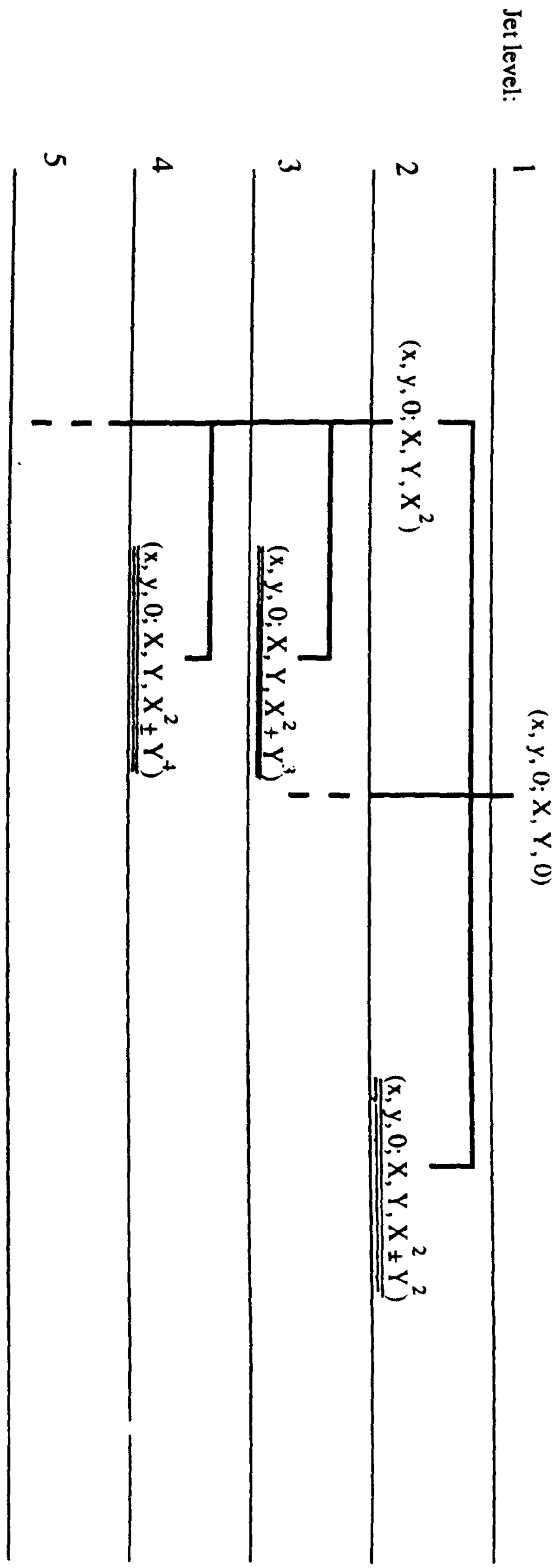
$$\begin{aligned} T\mathcal{A}.F =& m_2\langle(1,0,0;0,0,0),(0,1,0;0,0,0)\rangle + m_2\langle(0,0,0;1,0,0),(0,0,0;0,1,0)\rangle \\ &+ F^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \end{aligned}$$

and so $T\mathcal{A}.F \cap {}_2J^2(2,3) = (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2 - \{X, Y, X^2, XY, Y^2\})$. In the second case, if we let $G = (x, y, 0; X, Y, X^2)$ then

$$\begin{aligned} T\mathcal{A}.G =& m_2\langle(1,0,0;0,0,0),(0,1,0;0,0,0)\rangle + m_2\langle(0,0,0;1,0,2X),(0,0,0;0,1,0)\rangle \\ &+ G^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \end{aligned}$$

So $T\mathcal{A}.G \cap {}_2J^4(2,3) = (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2 - \{X, Y, X^2, X^3, X^4\})$.

Now we need to consider bi-germs consisting of one immersion and one branch with 1-jet equivalent to $(x, 0, 0)$. To begin the classification we have the following theorem on bi-germs consisting of an immersion and a cross-cap meeting transversely from [Md1].



Jet level:
1
2
3
4
5
(x, y, 0; X, Y, 0)
(x, y, 0; X, Y, X²)
(x, y, 0; X, Y, X² ± Y²)
(x, y, 0; X, Y, X² + Y³)
(x, y, 0; X, Y, X² ± Y⁴)


Fig. 7.3.1

**7.3.2 Theorem** *(Mond)*

(i) *The bi-germ* $(x,y,0;Y^2,XY+Y^{2k+1},X)$ *is* $(2k+1)$*-determined.*

(ii) *Every finitely determined germ of an immersion and a cross-cap, meeting transversely, is equivalent to one of the germs defined in (i).*

**Proof** See [Md1], I.10.6. □

We then have the following:

**7.3.3 Proposition** *A bi-germ of form* $F(x,y;X,Y) = (x,y,0;Y^2,XY+Y^{2k+1},X)$ *has* $\mathcal{A}_e$*-codimension* $k$.

**Proof** From Mond, Thm.I.10.6, we have an expression for $T\mathcal{A}_e.F$:

$$\begin{aligned} T\mathcal{A}_e.F =&(\mathcal{E}_2,\mathcal{E}_2,m_2-\{x,x^2,\ldots,x^{k-1}\};\\ &\mathcal{E}_2,\mathcal{E}_2-\{Y,Y^3,\ldots,Y^{2k-1}\},m_2-\{Y^2,Y^4,\ldots,Y^{2k-1}\})\\ &+\mathbf{R}\{(0,0,x^i;0,0,Y^{2i}),(0,0,0;0,Y^{2i+1},Y^{2i})\}_{0\leq i\leq k-1}\end{aligned}$$

So we can choose $(0,0,0;0,Y^{2i+1},0)$, $0 \leq i \leq k-1$ as a minimal basis for $\mathcal{O}_F/T\mathcal{A}_e.F$ and the $\mathcal{A}_e$-codimension is $k$. □

Thus we get the next three entries in the table above.

Now the two possible situations for bi-germs consisting of one immersion and one non-zero 1-jet are

(1) $j^1f = (x,y,0;0,0,X)$ or

(2) $j^1f = (x,y,0;X,0,0)$.

We shall consider the first case. If $j^1f = (x,y,0;0,0,X)$ then the $\mathcal{A}_1$-tangent space is given by

$$\begin{aligned} T\mathcal{A}_1.f =&m_2^2\langle(1,0,0;0,0,0),(0,1,0;0,0,0)\rangle\\ &+m_2^2\langle(0,0,0;0,0,1)\rangle+f^*.m_3^2\langle(1,0,0),(0,1,0),(0,0,1)\rangle\end{aligned}$$

Following the complete transversal method of Theorem 1.5.1 we look for those elements of $H^2(2,3)_2$ which are not contained in $J^2(T\mathcal{A}_1.f)\cap H^2(2,3)_2$. We find that

$$T = \{(0,0,0;Y^2,0,0),(0,0,0;XY,0,0),(0,0,0;0,Y^2,0),(0,0,0;0,XY,0)\}$$

We have already studied the cross-cap case so it remains to look at

$$j^2f \sim \begin{cases}(x,y,0;0,0,X)\\(x,y,0;0,Y^2,X)\end{cases}$$

In the first of these $\mathcal{A}^2(x,y,0;0,0,X)$ is a non-sufficient stratum and has codimension 6 in the jet-space ${}_2J^2(2,3)$ (with calculations as above).

In the second case we have

$$\begin{aligned} T\mathcal{A}_1.f =& m_2^2\langle(1,0,0;0,0,0),(0,1,0;0,0,0)\rangle \\ &+ m_2^2\langle(0,0,0;0,0,1),(0,0,0;0,2Y,0)\rangle \\ &+ f^*.m_3^2\langle(1,0,0),(0,1,0),(0,0,1)\rangle \end{aligned}$$

and $T = \{(0,0,0;Y^3,0,0),(0,0,0;X^2Y,0,0)\}$. Thus the possibilities for $j^3f$ are:

$$j^3f = \begin{cases} (x,y,0;0,Y^2,X) & \textbf{(i)} \\ (x,y,0;Y^3 \pm X^2Y,Y^2,X) & \textbf{(ii)} \\ (x,y,0;Y^3,Y^2,X) & \textbf{(iii)} \\ (x,y,0;X^2Y,Y^2,X) & \textbf{(iv)} \end{cases}$$

Case **(i)** leads to a non-sufficient stratum, $\mathcal{A}^3(x,y,0;0,Y^2,X)$, which has codimension 5 in the jet-space ${}_2J^3(2,3)$.

Case **(ii)** Here $j^3f = (x,y,0;Y^3 \pm X^2Y,Y^2,X)$ which is 3 determined with $\mathcal{A}_e$-codimension 2. This gives the eighth entry in the table above.

Case **(iii)** Here $j^3f = (x,y,0;Y^3,Y^2,X)$ and we apply the complete transversal method to get $T = \{(0,0,0;X^3Y,0,0)\}$ so the possible 4-jets are

$$j^4f = \begin{cases} (x,y,0;Y^3,Y^2,X) \\ (x,y,0;Y^3+X^3Y,Y^2,X) \end{cases}$$

The first of these two is not 3 determined and the codimension of $\mathcal{A}^4(x,y,0;Y^3,Y^2,X)$ in the jet-space ${}_2J^4(2,3)$ is 5 since if $F = (x,y,0;Y^3,Y^2,X)$ then

$$\begin{aligned} T\mathcal{A}.F =& m_2\langle(1,0,0;0,0,0),(0,1,0;0,0,0)\rangle + m_2\langle(0,0,0;0,0,1),(0,0,0;3Y^2,2Y,0)\rangle \\ &+ F^*.m_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \end{aligned}$$

so $T\mathcal{A}.F \cap {}_2J^4(2,3) = (\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2 - \{Y,XY,X^2Y,X^3Y\},\mathcal{E}_2 - \{Y\},\mathcal{E}_2)$.

The second case, $j^4f = (x,y,0;Y^3+X^3Y,Y^2,X)$, is 4 determined and has $\mathcal{A}_e$-codimension 3, providing the ninth entry in the table above.

Case **(iv)** If $j^3f = (x,y,0;X^2Y,Y^2,X)$ we can change co-ordinates in the target via $\bar{u} = u + w^3/3 + vw$ to get

$$\begin{aligned} (x,y,0;X^2Y,Y^2,X) &\sim (x,y,0;-(Y^3+3XY^2+X^3)/3,Y^2,X) \\ &\sim (x,y,0;-y^3/3,Y^2,X) \end{aligned}$$

so we go to case (iii).

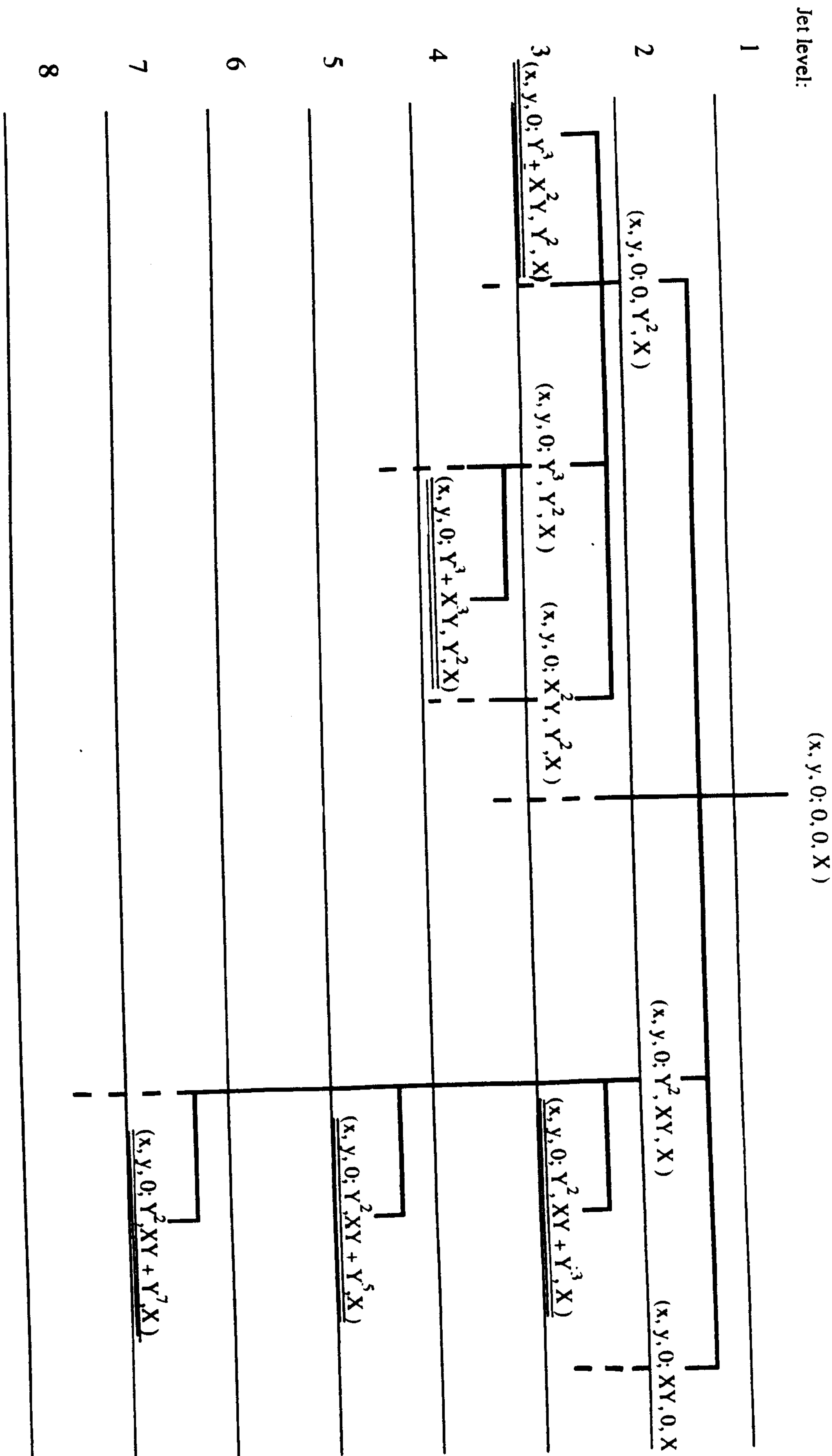

Jet level:
1
2
3
4
5
6
7
8
(x, y, 0; 0, 0, X)
(x, y, 0; 0, Y², X)
(x, y, 0; Y³ + X²Y, Y², X)
(x, y, 0; Y³, Y², X)
(x, y, 0; Y³ + X³Y, Y², X)
(x, y, 0; X²Y, Y², X)
(x, y, 0; Y², XY, X)
(x, y, 0; Y², XY + Y³, X)
(x, y, 0; Y², XY + Y⁵, X)
(x, y, 0; Y², XY + Y⁷, X)
(x, y, 0; XY, 0, X)

Fig. 7.3.2

We have now exhausted all possibilities stemming from the one-jet $(x, y, 0; 0, 0, X)$ and so we obtain the stratification shown in fig. 7.3.2.

**Case (2)** When the one-jet is $(x, y, 0; X, 0, 0)$, we see that the $\mathcal{A}_1$-tangent space is

$$\begin{aligned} T\mathcal{A}_1.f =& m_2^2\langle(1,0,0;0,0,0),(0,1,0;0,0,0)\rangle + m_2^2\langle(0,0,0;1,0,0)\rangle \\ &+ f^*.m_3^2\langle(1,0,0),(0,1,0),(0,0,1)\rangle \end{aligned}$$

and so the complete transversal in ${}_2J^2(2,3)$ is

$$j^2f = (x, y, 0; X, \alpha XY + \beta Y^2, \gamma X^2 + \delta XY + \rho Y^2)$$

Considering all the combinations of $\alpha$, $\beta$, $\gamma$, $\delta$ and $\rho$, allowing each to be zero or non-zero (yielding 32 possible two-jets), we are left with 3 possibilities with codimension less than 5 in the jet space ${}_2J^2(2,3)$. All the others are non-sufficient strata in ${}_2J^2(2,3)$ with codimension 5 or more. The cases are:

(2)(i) $j^2f = (x, y, 0; X, XY, Y^2)$

(2)(ii) $j^2f = (x, y, 0; X, Y^2, XY)$

(2)(iii) $j^2f = (x, y, 0; X, Y^2, X^2 + XY)$

We shall follow each of these, looking for the complete transversal in ${}_2J^3(3,2)$ and computing determinacy degree and codimension where necessary.

**(2)(i)** If $j^2f = (x, y, 0; X, XY, Y^2)$ then the complete transversal is $T = \{(0,0,0;0,Y^3,0),(0,0,0;0,0,X^3)\}$. The three-jets $j^3f = (x, y, 0; X, XY, Y^2)$ and $j^3f = (x, y, 0; X, XY \pm Y^3, Y^2)$ are both non-sufficient strata with codimension 5 in ${}_2J^3(3,2)$. This leaves $j^3f = (x, y, 0; X, XY, Y^2 + X^3)$ and $j^3f = (x, y, 0; X, XY \pm Y^3, Y^2 + X^3)$ which both have codimension 4 in ${}_2J^3(3,2)$ so it is possible that they will be completely determined bi-germs with $\mathcal{A}_e$-codimension 3. In both cases we use the criterion for determinacy in Corollary 1.2.2 and we are able to show that

$$T\mathcal{A}_1.f + m_2^4.(f^*m_3.\mathcal{E}_2 + m_2^4)\mathcal{E}_2^3 \supset m_2^4.\mathcal{E}_2^3$$

so that we see that both bi-germs are 3 determined. They also both have $\mathcal{A}_e$-codimension 3. We now use Mather's Lemma (Theorem 1.5.2) to show that in fact these two bi-germs are $\mathcal{A}$-equivalent. Consider $f = (x, y, 0; X, XY \pm \alpha Y^3, Y^2 + X^3)$. Then $T\mathcal{A}.f$ contains the vectors $(0,0,0;Y^2 + X^3, 0, 0)$, $(x^3, 0, 0; X^3, 0, 0)$, $(x^3, 0, 0; 0, 0, 0)$ and $Y^2\frac{\partial f}{\partial X} = (0,0,0;Y^2,Y^3,0)$, so

$(0,0,0;0,Y^3,0)$ is in the tangent space. Since the $\mathcal{A}_e$-codimension of both bi-germs is the same the criteria of Mather's Lemma are satisfied and the two are $\mathcal{A}$-equivalent. We use the normal form $f = (x,y,0;X,XY,Y^2+X^3)$.

**(2)(ii)** If $j^2f = (x,y,0;X,Y^2,XY)$ then a complete transversal is $T = \{(0,0,0;0,0,XY^2),(0,0,0;0,0,Y^3)\}$. First we look at $j^3f = (x,y,0;X,Y^2,XY)$. This has codimension 4 in ${}_2J^3(2,3)$ so we consider the complete transversal in ${}_2J^4(2,3)$. We find that it is $T = \{(0,0,0;0,0,Y^4)\}$. Then $j^4f = (x,y,0;X,Y^2,XY)$ has codimension 5 in ${}_2J^4(2,3)$ so we need not follow it any further, but $j^4f = (x,y,0;X,Y^2,XY+Y^4)$ is 4 determined (using Corollary 1.2.2) and has $\mathcal{A}_e$-codimension 3. Thus we have another bi-germ for our list.

Now we consider $j^3f = (x,y,0;X,Y^2,XY+XY^2)$. Again, this has codimension 4 in the jet space ${}_2J^3(2,3)$ so we must consider the complete transversal in ${}_2J^4(2,3)$. This yields $T = \{(0,0,0;0,0,Y^4)\}$. The four jet $j^4f = (x,y,0;X,Y^2,XY+XY^2)$ has codimension 5 in ${}_2J^4(2,3)$ but $j^4f = (x,y,0;X,Y^2,XY+XY^2+Y^4)$ is 4 determined and has $\mathcal{A}_e$-codimension 3. We shall return to this case later.

The third possibility here is $j^3f = (x,y,0;X,Y^2,XY \pm Y^3)$. Just as above we find that the complete transversal in ${}_2J^4(2,3)$ is $T = \{(0,0,0;0,0,Y^4)\}$ and that $j^4f = (x,y,0;X,Y^2,XY \pm Y^3)$ has codimension 5 in ${}_2J^4(2,3)$ but that $j^4f = (x,y,0;X,Y^2,XY \pm Y^3 \pm Y^4)$ is 4 determined and has $\mathcal{A}_e$-codimension 3.

The final case is that of $j^3f = (x,y,0;X,Y^2,XY \pm XY^2 \pm Y^3)$. Once again, we have to go to the four jet level to show that $j^4f = (x,y,0;X,Y^2,XY \pm XY^2 \pm Y^3)$ has codimension 5 in ${}_2J^4(2,3)$ but that $j^4f = (x,y,0;X,Y^2,XY \pm XY^2 \pm Y^3 \pm Y^4)$ is 4 determined and has $\mathcal{A}_e$-codimension 3.

Now we shall apply Mather's Lemma to show that in fact these four possibilities are $\mathcal{A}$-equivalent. Consider $f = (x,y,0;X,Y^2,XY+Y^4+\alpha XY^2+\beta Y^3)$. Just as in case (2)(i) we find that the vectors $(0,0,0;0,0,XY^2)$ and $(0,0,0;0,0,Y^3)$ are in $T\mathcal{A}.f$. The dimension of the tangent space remains constant whatever the values of $\alpha$ and $\beta$ so we can apply Mather's Lemma. All cases are $\mathcal{A}$-equivalent to $(x,y,0;X,Y^2,XY+Y^4)$.

**(2)(iii)** If $j^2f = (x,y,0;X,Y^2,X^2+XY)$ then the complete transversal in

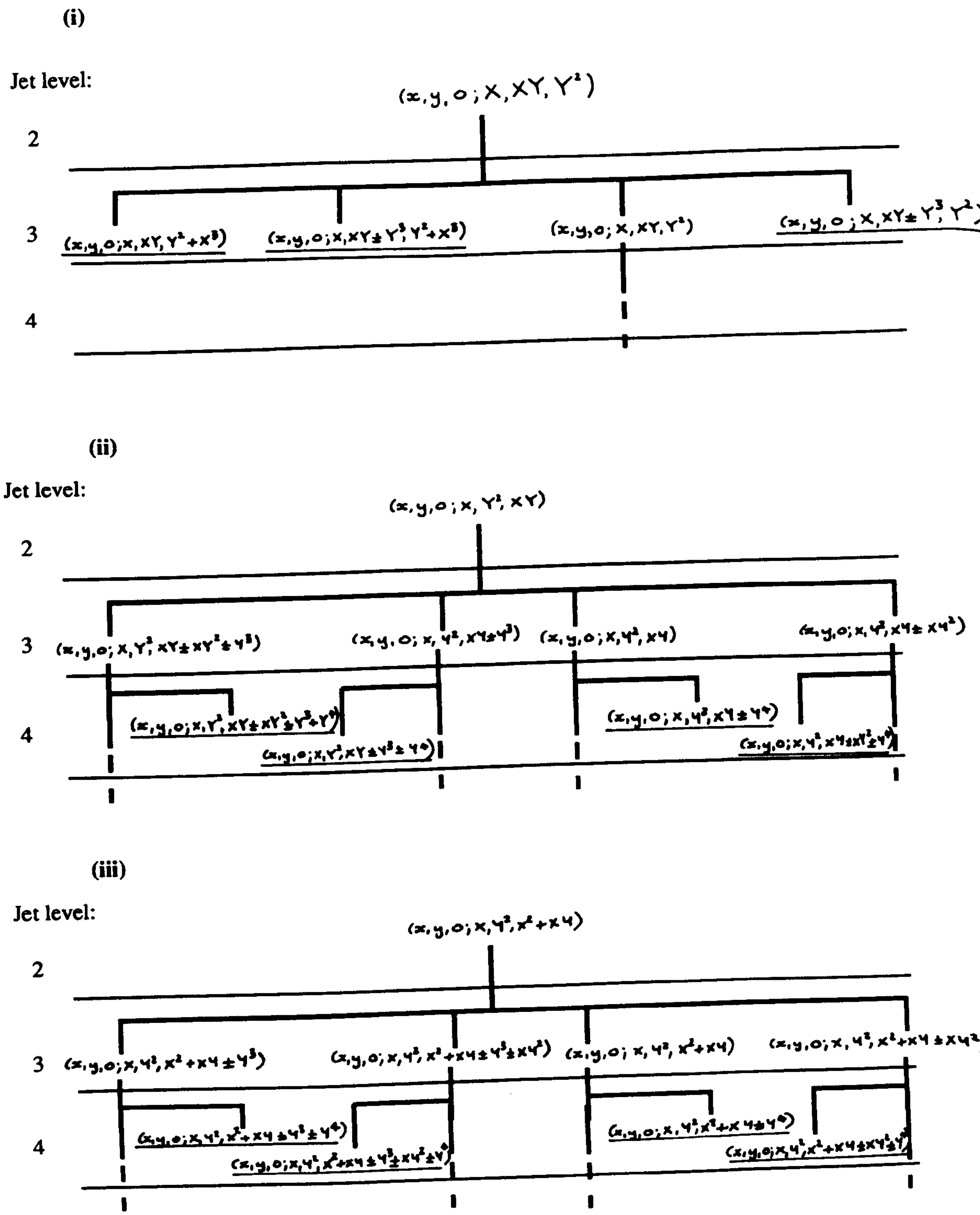
${}_2J^3(2,3)$ is $T = \{(0,0,0;0,0,XY^2),(0,0,0;0,0,Y^3)\}$. Again there are four cases to study.

First we look at $j^3f = (x,y,0;X,Y^2,X^2+XY)$. This has codimension 4 in ${}_2J^3(2,3)$ so we go up to the four-jet level. Then the complete transversal is $T = \{(0,0,0;0,0,Y^4)\}$ and $j^4f = (x,y,0;X,Y^2,X^2+XY)$ has codimension 5 in ${}_2J^4(2,3)$ while $j^4f = (x,y,0;X,Y^2,X^2+XY+Y^4)$ is four determined (by Corollary 1.2.2) with $\mathcal{A}_e$-codimension 3. Now we shall show that this $\mathcal{A}$-equivalent to $(x,y,0;X,Y^2,XY+Y^4)$. Consider $f = (x,y,0;X,Y^2,aX^2+XY+Y^4)$. It is easy to show that the vector $(0,0,0;0,0,X^2)$ is in $T\mathcal{A}.f$ since the vectors $X\frac{\partial f}{\partial Y} = (0,0,0;0,2XY,X^2+4XY^3)$, $Y^3\frac{\partial f}{\partial Y} = (0,0,0;0,2Y^4,XY^3)$, $(0,0,0;0,Y^4,0)$, $(0,0,0;0,X^2,0)$ and $(0,0,0;0,aX^2+XY+Y^4,0)$ are in $T\mathcal{A}.f$. Thus we can apply Mather's Lemma to get the required result.

Next, we have $j^3f = (x,y,0;X,Y^2,X^2+XY\pm XY^2)$. Similar calculations to the above show that the only bi-germ with $\mathcal{A}_e$-codimension less than 4 to follow from this stratum is $(x,y,0;X,Y^2,X^2+XY\pm XY^2+Y^4)$ which is 4 determined and has $\mathcal{A}_e$-codimension 3. We want to show that it is $\mathcal{A}$-equivalent to $(x,y,0;X,Y^2,XY+Y^4)$, and once again consideration of the bi-germ $f = (x,y,0;X,Y^2,aX^2+XY+bXY^2+Y^4)$ shows that the vectors $(0,0,0;0,0,X^2)$ and $(0,0,0;0,0,XY^2)$ are in $T\mathcal{A}.f$, independent of the values of $a$ and $b$.

The third and fourth cases follow similarly, and we get the two 4 determined bi-germs $(x,y,0;X,Y^2,X^2+XY\pm Y^3\pm Y^4)$ and $(x,y,0;X,Y^2,X^2+XY\pm Y^3\pm XY^2\pm Y^4)$ which both have $\mathcal{A}_e$-codimension 3. Now we want to show that these two bi-germs are $\mathcal{A}$-equivalent to the bi-germ $(x,y,0;X,Y^2,XY+Y^4)$. This time we have to work harder. Putting $f = (x,y,0;X,Y^2,aX^2+XY\pm bY^3\pm cXY^2\pm Y^4)$ and calculating $T\mathcal{A}.f$ gives the vectors $(0,0,0;0,0,X^2)$ and $(0,0,0;0,0,XY^2)$ in $T\mathcal{A}.f$ provided $b\neq 0$. As in Proposition 3.4.3 (iii), we now consider the smooth connected subset of the jet-space defined by $b=0$. The vector $(0,0,0;0,0,Y^3)$ is in the tangent space to $(x,y,0;X,Y^2,aX^2+XY\pm cXY^2+Y^4)$ so we can apply Mather's Lemma again and we see that the bi-germ is equivalent to $(x,y,0;X,Y^2,XY+Y^4)$. See Fig. 7.3.3.

Finally, we have to look at the cases where both branches of the bi-germ are non-immersions. The two possibilities are $(x,0,0;0,X,0)$ and $(x,0,0;X,0,0)$. Consideration of the non-sufficient strata $\mathcal{A}^1(x,0,0;0,X,0)$ and $\mathcal{A}^1(x,0,0;X,0,0)$ shows that they have codimensions 4 and 6 in the jet-space ${}_2J^1(2,3)$ respectively so we need not pursue stratifications from these.



(i)
Jet level:
2
3
4
(x, y, 0 ; X, XY, Y²)
(x,y,0;x,XY,Y²+X³)
(x,y,0;x,XY±Y³,Y²+X³)
(x,y,0;X,XY,Y²)
(ii)
Jet level:
2
3
4
(x,y,0;x,Y²,XY)
(x,y,0;x,Y²,xy±y⁴)
(x,y,0;x,Y²,xy)
(iii)
Jet level:
2
3
4
(x,y,0;x,y²,x²+xy)
(x,y,0;x,y²,x²+xy)


Fig. 7.3.3

## 7.4 Tri-germs $\mathbf{R}^2, 0 \to \mathbf{R}^3, 0$

The list of tri-germs from $\mathbf{R}^2$ to $\mathbf{R}^3$ with $\mathcal{A}_e$-codimension less than 4 will be shown in this section to consist of the following:

| Normal Form | $\mathcal{A}_e$-codimension |
|---|---|
| $(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y)$ | 0 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^2)$ | 1 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^3)$ | 2 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, X^2 \pm Y^2)$ | 2 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^4)$ | 3 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, XY + X^3)$ | 3 |
| $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, X^2 + Y^3)$ | 3 |

Once again, we seek to stratify the jet-space ${}_3J^k(2, 3)$ for large enough $k$ and show that the complement in ${}_3J^k(2, 3)$ of all the orbits with $\mathcal{A}$-codimension $\leq 3$ has codimension 4 in the jet-space.

From Proposition 7.1.4 we know that a tri-germ must have at least two immersive branches. Certainly we can have three transverse immersions:

$$(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y)$$

This is one-determined and has $\mathcal{A}_e$-codimension 0. Now let us suppose that two of the mono-germs are immersions (meeting transversely). Then we can change co-ordinates to get:

$$(x, y, 0; \bar{x}, 0, \bar{y}; \phi(X, Y), \psi(X, Y), \theta(X, Y))$$

Then the possible 1-jets we have to consider are

(i) $j^1 f = (x, y, 0; \bar{x}, 0, \bar{y}; X, Y, 0)$

(ii) $j^1 f = (x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y)$

(iii) $j^1 f = (x, y, 0; \bar{x}, 0, \bar{y}; X, 0, 0)$

(iv) $j^1 f = (x, y, 0; \bar{x}, 0, \bar{y}; 0, X, 0)$

We shall consider each case in turn.

Case (i) If $j^1 f = (x, y, 0; \bar{x}, 0, \bar{y}; X, Y, 0)$ then we calculate the $\mathcal{A}_1$-tangent space and find the complete transversal in ${}_3J^2(2,3)$. We get

$$H^2(2,3)_3 - J^2(T\mathcal{A}_1.f) \cap H^2(2,3)_3 = \{(0,0,0;0,0,0;0,0,X^2)\ (0,0,0;0,0,0;0,0,XY),(0,0,0;0,0,0;0,0,Y^2)\}$$

Taking these possibilities one by one we find that they reduce to

(i)(a) $j^2f = (x,y,0;\bar{x},0,\bar{y};X,Y,X^2)$

(i)(b) $j^2f = (x,y,0;\bar{x},0,\bar{y};X,Y,Y^2)$

(i)(c) $j^2f = (x,y,0;\bar{x},0,\bar{y};X,Y,XY)$

(i)(d) $j^2f = (x,y,0;\bar{x},0,\bar{y};X,Y,X^2 \pm Y^2)$

(i)(e) $j^2f = (x,y,0;\bar{x},0,\bar{y};X,Y,0)$

(i)(a) If $j^2f = (x,y,0;\bar{x},0,\bar{y};X,Y,X^2)$ then the complete transversal in ${}_3J^3(2,3)$ is $T = \{(0,0,0;0,0,0;0,0,Y^3)\}$. The 3-jet

$$j^3f = (x,y,0;\bar{x},0,\bar{y};X,Y,X^2+Y^3)$$

is three determined and has $\mathcal{A}_e$-codimension 3. Otherwise $\mathcal{A}^2(x,y,0;\bar{x},0,\bar{y};X,Y,X^2)$ is a non-sufficient stratum with codimension 4 in ${}_3J^2(2,3)$.

(i)(b) $\mathcal{A}^2(x,y,0;\bar{x},0,\bar{y};X,Y,Y^2)$ is a non-sufficient stratum with codimension 5 in ${}_3J^2(2,3)$.

(i)(c) If $j^2f = (x,y,0;\bar{x},0,\bar{y};X,Y,XY)$ then the complete transversal in ${}_3J^3(2,3)$ is $T = \{(0,0,0;0,0,0;0,0,X^3)\}$. The 3-jet

$$j^3f = (x,y,0;\bar{x},0,\bar{y};X,Y,XY+X^3)$$

is 3-determined and has $\mathcal{A}_e$-codimension 3 while $\mathcal{A}^2(x,y,0;\bar{x},0,\bar{y};X,Y,XY)$ is a non-sufficient stratum with codimension 4 in ${}_3J^2(2,3)$.

(i)(d) The 2-jet $j^2f = (x,y,0;\bar{x},0,\bar{y};X,Y,X^2 \pm Y^2)$ is 2-determined and has $\mathcal{A}_e$-codimension 2.

(i)(e) In this case we have the non-sufficient stratum $\mathcal{A}^2(x,y,0;\bar{x},0,\bar{y};X,Y,0)$ which has codimension 5 in ${}_3J^2(2,3)$.

This completes the stratification from the 1-jet $j^1f = (x,y,0;\bar{x},0,\bar{y};X,Y,0)$. A diagram of this is shown in fig. 7.4.1

Case (ii) If $j^1f = (x,y,0;\bar{x},0,\bar{y};X,Y,Y)$ then the complete transversal in ${}_3J^2(2,3)$ is $T = \{(0,0,0;0,0,0;0,0,X^2)\}$. Then the two jet is

$$j^2f = (x,y,0;\bar{x},0,\bar{y};X,Y,Y+X^2)$$

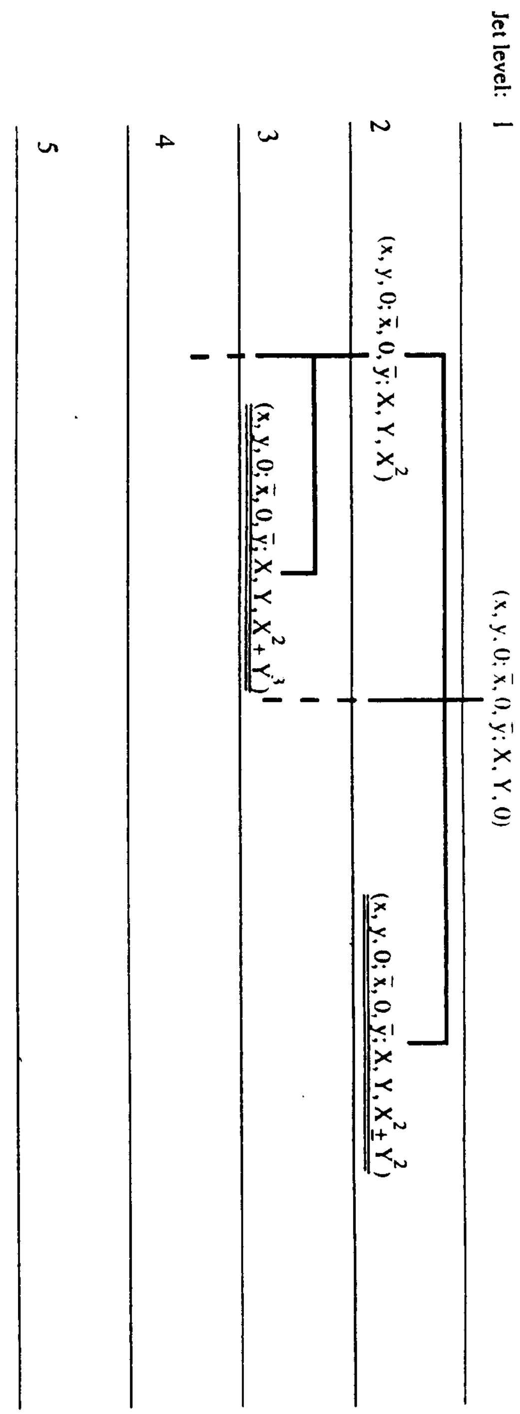



Fig. 7.4.1

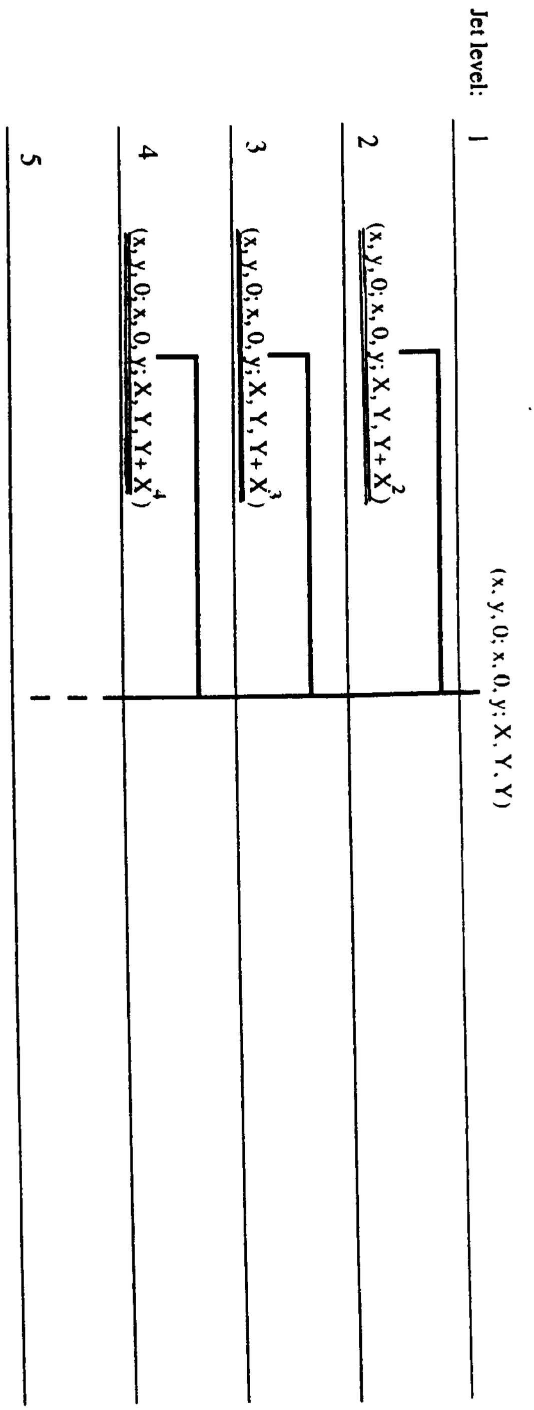



Fig. 7.4.2

which is 2 determined and has $\mathcal{A}_e$-codimension 1. Otherwise, we go on to the 3-jet level. The complete transversal in ${}_3J^3(2,3)$ is $T = \{(0,0,0;0,0,0;0,0,X^3)\}$. The three jet $j^3f = (x,y,0;\bar{x},0,\bar{y};X,Y,Y+X^3)$ is 3 determined and has $\mathcal{A}_e$-codimension 2.

At the four jet level the complete transversal is $T = \{(0,0,0;0,0,0;0,0,X^4)\}$ and the four jet $j^4f = (x,y,0;\bar{x},0,\bar{y};X,Y,Y+X^4)$ is 4 determined and has $\mathcal{A}_e$-codimension 3. Finally, $\mathcal{A}^4(x,y,0;\bar{x},0,\bar{y};X,Y,Y)$ is a non-sufficient stratum with codimension 4 in ${}_3J^4(2,3)$. The stratification diagram is shown in fig. 7.4.2.

Case (iii) We have a non-sufficient stratum $\mathcal{A}^1(x,y,0;\bar{x},0,\bar{y};X,0,0)$ which has $\mathcal{A}$-codimension 4 in the jet-space ${}_3J^1(2,3)$ so there is no need to pursue a stratification from this one-jet.

Case (iv) The non-sufficient stratum $\mathcal{A}^1(x,y,0;\bar{x},0,\bar{y};0,X,0)$ has $\mathcal{A}$-codimension 3 in the jet-space ${}_3J^1(2,3)$, so it is possible that there are tri-germs which lie above this, are finitely determined and have $\mathcal{A}$-codimension 3. The $\mathcal{A}_1$-tangent space is given by:

$$\begin{aligned} T\mathcal{A}_1.f =& m_2^2\langle(1,0,0;0,0,0;0,0,0),(0,1,0;0,0,0;0,0,0)\rangle \\ &+ m_2^2\langle(0,0,0;1,0,0;0,0,0),(0,0,0;0,0,1;0,0,0)\rangle \\ &+ m_2^2\langle(0,0,0;0,0,0;0,1,0)\rangle + f^*.m_3^2\langle(1,0,0),(0,1,0),(0,0,1)\rangle \end{aligned}$$

and the complete transversal is

$$j^2f = (x,y,0;\bar{x},0,\bar{y};\alpha XY+\beta Y^2, X, \gamma X^2+\delta XY+\rho Y^2)$$

A lengthy calculation involving considering all the possible combinations of $\alpha$, $\beta$, $\gamma$, $\delta$ and $\rho$ (allowing each one to be either zero or non-zero) yields eighteen possible cases. All except one of these are non-sufficient strata in ${}_3J^2(2,3)$ with codimensions greater than 3 in this jet space. The remaining case has two jet $\mathcal{A}$-equivalent to

$$j^2f = (x,y,0;\bar{x},0,\bar{y};XY,X,XY\pm Y^2)$$

A complete transversal for this in ${}_3J^3(2,3)$ is $T = \{(0,0,0;0,0,0;Y^3,0,0)\}$. Now $\mathcal{A}^3(x,y,0;\bar{x},0,\bar{y};XY,X,XY\pm Y^2)$ is a non-sufficient stratum with codimension 4 in ${}_3J^3(2,3)$, but consideration of $\mathcal{A}^3(x,y,0;\bar{x},0,\bar{y};XY\pm Y^3,X,XY\pm Y^2)$ shows that

$$T\mathcal{A}_1.f + m_2^4.(f^*m_3.\mathcal{E}_2 + m_2^4)\mathcal{E}_2^3 \supset m_2^4$$

so, using the criterion for $\mathcal{A}_1$-determinacy in Corollary 1.2.2, we see that the tri-germ is 3 determined. However, it turns out that the $\mathcal{A}_e$-tangent space is

$$\begin{aligned} T\mathcal{A}_e.f =&\mathcal{E}_2\langle(1,0,0;0,0,0;0,0,0),(0,1,0;0,0,0;0,0,0)\rangle \\ &+\mathcal{E}_2\langle(0,0,0;1,0,0;0,0,0),(0,0,0;0,0,1;0,0,0)\rangle \\ &+\mathcal{E}_2\langle(0,0,0;0,0,0;Y,1,Y),(0,0,0;0,0,0;X\pm 3Y^2,0,X\pm 2Y)\rangle \\ &+f^*\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \\ =&(\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2-\{1,Y^2\},\mathcal{E}_2-\{1,X\}) \end{aligned}$$

so the $\mathcal{A}_e$-codimension is 4.

Thus our classification of tri-germs with $\mathcal{A}_e$-codimension less than or equal to 3 is complete. We note that it seems that in order to have low $\mathcal{A}_e$-codimension we must have three immersive mono-germs in the tri-germ.

### 7.5 4-germs and Higher Multi-germs $\mathbf{R}^2,0 \to \mathbf{R}^3,0$

We shall show that list of higher multi-germs $\mathbf{R}^2,0 \to \mathbf{R}^3,0$ contains the following:

| Normal Form | $\mathcal{A}_e$-codimension |
|---|---|
| $(x,y,0;\bar{x},0,\bar{y};0,X,Y;\bar{X},\bar{Y},\bar{X}+\bar{Y})$ | 1 |
| $(x,y,0;\bar{x},0,\bar{y};0,X,Y;\bar{X},\bar{Y},\bar{X}+\lambda\bar{Y};\mathrm{x},\mathrm{y},\mathrm{x}+\mu\mathrm{y})$ | 2 |
| $(x,y,0;\bar{x},0,\bar{y};0,X,Y;\bar{X},\bar{Y},\bar{X}+\lambda\bar{Y};\mathrm{x}+\mu\mathrm{y},\mathrm{x},\mathrm{y};\bar{\mathrm{x}},\rho\bar{\mathrm{x}}+\gamma\bar{\mathrm{y}},\bar{\mathrm{y}})$ | 2 |

First we look at 4-germs. By Proposition 7.1.4, we know that any 4-germ must have at least three immersive branches. The 4-germ with four transverse immersions, which is $\mathcal{A}$-equivalent to four planes meeting transversely, can be written in the following way:

$$(x,y,0;\bar{x},0,\bar{y};0,X,Y;\bar{X},\bar{Y},\bar{X}+\bar{Y})$$

This is one-determined and

$$T\mathcal{A}_e.f=(\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2-\{1\},\mathcal{E}_2,\mathcal{E}_2)$$

so the $\mathcal{A}_e$-codimension is 1.

Now suppose that the fourth branch is still an immersion but is tangent to one of the other branches. Thus $j^1 f = (x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X}, \bar{Y}, 0)$ and the complete transversal is

$$T = \{(0,0,0;0,0,0;0,0,0;0,0,\bar{X}^2), (0,0,0;0,0,0;0,0,0;0,0,\bar{X}\bar{Y}), (0,0,0;0,0,0;0,0,0;0,0,\bar{Y}^2)\}$$

So we have $j^2 f = (x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X}, \bar{Y}, \alpha\bar{X}^2 + \beta\bar{X}\bar{Y} + \gamma\bar{Y}^2)$. Consideration of all possible combinations of $\alpha$, $\beta$ and $\gamma$, with each being either zero or non-zero, yields only non-sufficient strata with codimension greater than 3 in the jet-space ${}_4J^2(2,3)$.

Finally, if the fourth branch is not an immersion then the 4-germ must be of the form

$$(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X}, 0, 0)$$

This is a non-sufficient stratum with codimension 4 in ${}_4J^1(2,3)$. So the only 4-germ which has $\mathcal{A}_e$-codimension less than 4 is the four transverse planes.

We now look at 5-germs. By Proposition 7.1.4 we know that any 5-germ must have five immersive branches. Consider the case of five transverse planes in $\mathbf{R}^3$. There will be two moduli associated with such a 5-germ. We can write it in the following way:

$$(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X}, \bar{Y}, \bar{X} + \lambda\bar{Y}; \mathrm{x}, \mathrm{y}, \mathrm{x} + \mu\mathrm{y})$$

Then the $\mathcal{A}_e$-tangent space of a single orbit is

$$T\mathcal{A}_e.f = (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2 - \{1\}, \mathcal{E}_2, \mathcal{E}_2 - \{\bar{X}\}; \mathcal{E}_2 - \{1\}, \mathcal{E}_2, \mathcal{E}_2 - \{\mathrm{y}\})$$

so the $\mathcal{A}_e$-codimension of an orbit is 4, but the $\mathcal{A}_e$-codimension of the stratum is 2 (since we can also differentiate with respect to $\lambda$ and $\mu$). Thus this stratum does appear on our list of multi-germs of $\mathcal{A}_e$-codimension less than 4.

If we go on to look at 5-germs with five immersive branches, two of which are tangent, we have a one-jet of the form:

$$(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X}, \bar{Y}, \bar{X} + \lambda\bar{Y}; 0, \mathrm{x}, \mathrm{y})$$

This is a non-sufficient stratum in ${}_5J^1(2,3)$ with codimension 2 in the jet-space. Since we are only interested in 5-germs with codimension less than 2 in the jet-space ( by Proposition 2.2.4) this case is ruled out.

Finally we consider 6-germs from $\mathbf{R}^2$ to $\mathbf{R}^3$. These must consist of six immersive branches, by Proposition 7.1.4, and since the codimension in the multi-jet space must be zero, by Proposition 2.2.4, we would expect no tangencies between these immersions. Thus we look at six transverse planes:

$$(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X}, \bar{Y}, \bar{X} + \lambda\bar{Y}; \mathrm{x} + \mu\mathrm{y}, \mathrm{x}, \mathrm{y}; \bar{\mathrm{x}}, \rho\bar{\mathrm{x}} + \gamma\bar{\mathrm{y}}, \bar{\mathrm{y}})$$

(The letters $\lambda$, $\mu$, $\rho$ and $\gamma$ represent the four moduli associated with such a germ.) The tangent space to each orbit is then

$$T\mathcal{A}_e.f = (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2 - \{\bar{Y}\};$$
$$\mathcal{E}_2 - \{\mathrm{y}\}, \mathcal{E}_2 - \{1\}, \mathcal{E}_2; \mathcal{E}_2 - \{1\}, \mathcal{E}_2 - \{\bar{\mathrm{x}}, \bar{\mathrm{y}}\}, \mathcal{E}_2)$$

and so the $\mathcal{A}_e$-codimension of an individual orbit is 6 but the $\mathcal{A}_e$-codimension of the stratum is 2. Thus we have all the multi-germs on our list.

### 7.6 Unfoldings of Two-Dimensional Motions of Space – Mono-germs

We now want to look at the unfoldings and bifurcation sets of the mono-germs on our list. As usual, we use the methods of section 1.4 but we also have some further techniques for analysing the unfolding spaces of mono-germ singularities in this case and these are described below in 7.6.3.

We first give a table summarising the unfoldings. The calculations for these are routine and are given in Appendix B.2.

| Name | Unfolding | $\mathcal{A}_e$-codim | $C(f)$ | $T(f)$ | $\mu(\widetilde{D}^2/\mathbf{Z}_2)$ |
|---|---|---|---|---|---|
| $S_0$ | $(x, y^2, xy)$ | 0 | 1 | 0 | 0 |
| $S_1^+$ | $(x, y^2, y^3 + x^2y + ay)$ | 1 | 2 | 0 | 0 |
| $S_1^-$ | $(x, y^2, y^3 - x^2y + ay)$ | 1 | 2 | 0 | 0 |
| $S_2$ | $(x, y^2, y^3 + x^3y + ay + bxy)$ | 2 | 3 | 0 | 0 |
| $S_3^+$ | $(x, y^2, y^3 + x^4y + ay + bxy + cx^2y)$ | 3 | 4 | 0 | 0 |
| $S_3^-$ | $(x, y^2, y^3 - x^4y + ay + bxy + cx^2y)$ | 3 | 4 | 0 | 0 |
| $B_2^+$ | $(x, y^2, x^2y + y^5 + ay + by^3)$ | 2 | 2 | 0 | 2 |
| $B_2^-$ | $(x, y^2, x^2y - y^5 + ay + by^3)$ | 2 | 2 | 0 | 2 |
| $B_3^+$ | $(x, y^2, x^2y + y^7 + ay + by^3 + cy^5)$ | 3 | 2 | 0 | 4 |
| $B_3^-$ | $(x, y^2, x^2y - y^7 + ay + by^3 + cy^5)$ | 3 | 2 | 0 | 4 |
| $C_3^+$ | $(x, y^2, xy^3 + x^3y + ay + by^3 + cxy)$ | 3 | 3 | 0 | 2 |
| $C_3^-$ | $(x, y^2, xy^3 - x^3y + ay + by^3 + cxy)$ | 3 | 3 | 0 | 2 |
| $H_2$ | $(x, y^3 + ay, xy + y^5 + by^2)$ | 2 | 2 | 1 | 0 |
| $H_3$ | $(x, y^3 + ay, xy + y^8 + by^2 + cy^5)$ | 3 | 2 | 2 | 0 |
| $P_3$ | $(x, xy + y^3 + ay, xy^2 + cy^4 + by + dy^3 + ey^4)$ | 4 | 3 | 1 | 2 |

Table 7.6.1

The columns headed $C(f)$, $T(f)$ and $\mu(\widetilde{D}^2/\mathbf{Z}_2)$ give invariants and come from the table in [Md2]. In the complex case, C gives the number of cross-caps the appear in a generic unfolding of the map-germ; it is defined in the following way [Md2]:

$$C(f) = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^2,0}}{\mathcal{R}_f}$$

where $\mathcal{R}_f$ is the ideal generated by the $2 \times 2$ minors of $df$. T gives the number of triple points in the complex case and is defined as follows [Md2]:

$$T(f) = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^2,0}}{\mathcal{F}_2(f_*\mathcal{O}_{\mathbf{C}^2,0})}$$

where $\mathcal{F}_2(f_*\mathcal{O}_{\mathbf{C}^2,0})$ is the second Fitting ideal (see section 7.6.3). The third invariant, $\mu(\widetilde{D}^2/\mathbf{Z}_2)$ is a geometric invariant which is the Milnor number of the double point curve $D^2(f)$ when it is acted on by $\mathbf{Z}_2$ (see Marar & Mond [MdM1]). It can be calculated using the formula [MdM1]

$$\mu(\widetilde{D}^2/\mathbf{Z}_2) = \mathcal{A}_e\text{-codim}(f) - C - T + 1.$$

Here, we would like to know what happens when we unfold the real germ. From Mond [Md2] we have the following theorem:

**7.6.1 Theorem** *Let $f : \mathbf{R}^2, 0 \to \mathbf{R}^3, 0$ be a finitely determined map-germ with corank 1 at 0. Then there exist arbitrary small real deformations of $f$ exhibiting $C(f)$ real cross-caps.*

In the case of triple points, from Marar & Mond [MdM2] we have

**7.6.2 Remark:** Let $f : \mathbf{R}^2, 0 \to \mathbf{R}^3, 0$ be a finitely determined map-germ with corank 1 at 0. Then if $f$ is from the $H_k$ series of singularities, $T(f)$ real triple points will be exhibited under some small real deformation.

In particular, we will be able to find 1 real triple point when we deform $H_2$ and 2 real triple points when we deform $H_3$. However in the case of $P_3$ no such result exists. So, except in this case, we know at least that if $F : \mathbf{R}^2 \times \mathbf{R}^d, 0 \to \mathbf{R}^3 \times \mathbf{R}^d, 0$, $F(x,t) = (f_t(x), t)$ is the unfolding of $f : \mathbf{R}^2 \to \mathbf{R}^3$ then there will be some values of $t \in \mathbf{R}^d$ such that the image of $f_t$ is a singular surface with $C(f)$ cross-caps and $T(f)$ triple points.

### 7.6.3 The conditions for cross-caps and triple points

Given $f : X \to Y$ finite and analytic, we want to look at the set $M_k(f)$ of points in $Y$ whose preimage consists of $k$ or more points, counting multiplicity. Following the method in Mond & Pellikaan [MdP], this can be done by looking at the scheme structure of $M_k(f)$ using the Fitting ideals of the coherent $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$. The $M_k(f)$ are the varieties of zeros of these Fitting ideals, $\mathcal{F}_k(\mathcal{O}_X)$, so we first construct these ideals, from a presentation of $f_*\mathcal{O}_X$, using the algorithm below. The notation is due to Mond; we state the algorithm for the case $f : \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$.

Given a map-germ $f : \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ which is finite we follow the steps below in order to find the double point curve and the triple points (if they exist).

(1) Choose a projection $\pi : \mathbf{C}^{n+1} \to \mathbf{C}^n$ such that $\widetilde{f} = \pi \circ f$ is still finite (this is Noether normalization and is always possible). We may suppose that $f(x) = (\widetilde{f}(x), f_{n+1}(x))$.

(2) Find generators $g_1 = 1$, $g_2, \ldots, g_h$ for $\mathcal{O}_{\mathbf{C}^n,0} = \mathcal{O}_n$ (source) over $\mathcal{O}_n$ (target) via $\widetilde{f}^*$ ( by the Weierstrass Preparation Theorem these can be found).

(3) Find the (unique) $\alpha^i_j \in \mathcal{O}_n$, $1 \leq i, j \leq h$ such that

$$g_j f_{n+1} = \sum_i (\alpha^i_j \circ \widetilde{f}) g_i$$

(4) Now note that $f_{n+1} = x_{n+1} \circ f$ and let $\lambda^i_j$ be such that

$$\lambda^i_j = \begin{cases} \alpha^i_j \circ \pi & i \neq j \\ \alpha^i_i \circ \pi - x_{n+1} & i = j \end{cases}$$

then we can view the equations as relations among the $g_i$ over $\mathcal{O}_{n+1}$,

$$\lambda^1_j . g_1 + \ldots + \lambda^h_j . g_h = 0$$

(5) Then the matrix $\lambda$ is the presentation matrix of $f_* \mathcal{O}_n$ over $\mathcal{O}_{n+1}$

$$\mathcal{O}^h_{n+1} \xrightarrow{\lambda} \mathcal{O}^h_{n+1} \longrightarrow f_* \mathcal{O}_n \longrightarrow 0$$

We define the kth Fitting ideal of $f_* \mathcal{O}_n$, $\mathcal{F}_k(f_* \mathcal{O}_n)$ to be the ideal in $\mathcal{O}_{n+1}$ generated by all $(h-k) \times (h-k)$ minors of $\lambda$ for $0 \leq k < h$. Denote the variety of zeros of $\mathcal{F}_k(f_* \mathcal{O}_n)$ by $M_k(f)$. Then:

$f^* M_0(f)$ gives the preimage

$f^* M_1(f)$ gives the double point curve in the source

$f^* M_2(f)$ gives the preimages of triple points in the source.

In fact, if $f^{-1}(\mathbf{C}^{n+1})$ is Gorenstein it can be shown that $M_1(f)$ may be found by deleting the first column and $(n+1)$st row and considering only that $n \times n$ determinant – all others are in the ideal generated by this.

In particular we have:

**7.6.4 Proposition** *If* $f : \mathbf{C}^2, 0 \to \mathbf{C}^3, 0$ *is given by* $f(x,y) = (x, y^2, yp(x,y^2))$ *then* $p(x,y^2) = 0$ *is the double point curve.*

**Proof** In this case we can assume Gorenstein [MdP]. We take $\widetilde{f}(x,y) = (x, y^2)$. The generators of $\mathcal{O}_2$ (source, co-ordinates $(x,y)$) over $\mathcal{O}_2$ (target, co-ordinates $(X,Y)$) are $1$, $y$. Then we have

$$Z.1 = yp(X,Y)$$
$$Z.y = Yp(X,Y)$$

So the presentation matrix is

$$\lambda = \begin{pmatrix} -Z & p(X,Y) \\ Yp(X,Y) & -Z \end{pmatrix}$$

and $M_1(f) = \{p(X,Y) = 0\} = \{p(x,y^2) = 0\}$. □

Note that this situation occurs for all the mono-germ singularities we wish to analyse, apart from $H_2$, $H_3$ and $P_3$. Excluding these for now, our approach is to consider the double point curve in the source for each singularity and then to take its image under the fold map $(x,y) \to (x,y^2)$. This gives the curve of self-intersection of the singular surface which is the image of the stable mapping $f_t : \mathbf{R}^2 \to \mathbf{R}^3$. We know that if $f(x,y) = (x, r(x,y), s(x,y))$ then the co-ordinates of the cross-caps are obtained as a solution of the system of equations

$$\begin{cases} \frac{\partial r}{\partial y}(x,y) = 0 \\ \frac{\partial s}{\partial y}(x,y) = 0 \end{cases}$$

since this is the condition for the Jacobian of $f$ to drop rank. So, when $f$ is of the form $f(x,y) = (x, y^2, yp(x,y^2))$ then the intersection points of $p(x,y^2) = 0$ with the $x$-axis give the cross-caps in the image.In order to get the maximum number of cross-caps we need to look at the conditions on the parameters which give the expected number of real solutions of $p(x,0) = 0$. We also want to know what the transition states are and to look at other interesting features of the double point curve, such as the geometric implications of $\mu(\widetilde{D}^2/\mathbf{Z}_2) \neq 0$.

Insight into the geometry of the double point curve was given in many cases by using the program *Algcurve*, developed by Ralph Martin and Richard Morris, on a Silicon Graphics Iris Workstation, and invaluable help with the problem was given by looking at rotatable images of the surfaces involved on the same workstation using Richard Morris's program *Compsurf*. Some of these beautiful pictures are reproduced in Appendix A.

### 7.6.5 Analysing the Unfoldings

We now look at the unfolding spaces of each germ and use the methods above to find out what conditions must hold on the parameters to give the maximum number of cross-caps and triple points, and what the possible transition states are. From the adjacancy table in Mond [Md1] (see Fig. 7.6.1) we know what specialisations we expect to see when we perturb a particular singularity, but further study is required to see exactly when these transitions occur in terms of the parameter space, and how they relate to the image.

$$\begin{tikzcd}
 & & C_3 \arrow[d] & \\
 & B_3 \arrow[r] & B_2 \arrow[d] & \\
S_3 \arrow[r] & S_2 \arrow[r] & S_1 \arrow[r] & S_0 \\
 & & H_2 \arrow[u] & \\
 & & H_3 \arrow[u] &
\end{tikzcd}$$

Fig. 7.6.1

The reader will note that work of this nature, though in the setting of focal surfaces, has been done by Bill Bruce and Tim Wilkinson [BW], who have analysed the bifurcation sets of the $S_k$ (for $k = 2, 3$), $B_k$ (for $k = 1, 2, 3$) and $C_3$ (also see Wilkinson [W], Arnold [A]). However, here we follow a different method for determining the bifurcation sets which can be applied to all examples, including those which do not have 2-jet $(x, y^2, 0)$.

### 7.6.6 $S_1^+$

The unfolding is given by $F_a(x, y) = (x, y^2, y^3 + x^2y + ay)$. From Table 7.6.1 we can expect to find at most 2 cross-caps and no triple points when we unfold. The double point curve in the source is

$$p(x, y^2) = y^2 + x^2 + a.$$

Then $p(x,0) = x^2 + a$ so we can only have two distinct solutions if $a < 0$. The bifurcation set is $a = 0$. So we either get no cross-caps or 2 cross-caps, with birth occuring at $a = 0$. The picture in the source is shown in Fig. 7.6.2.

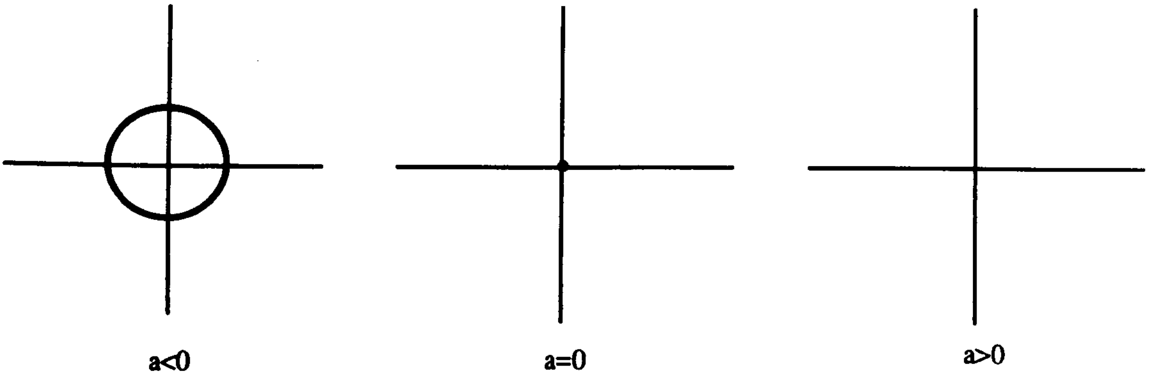



Fig. 7.6.2

and the image of the double point curve is shown in Fig. 7.6.3.

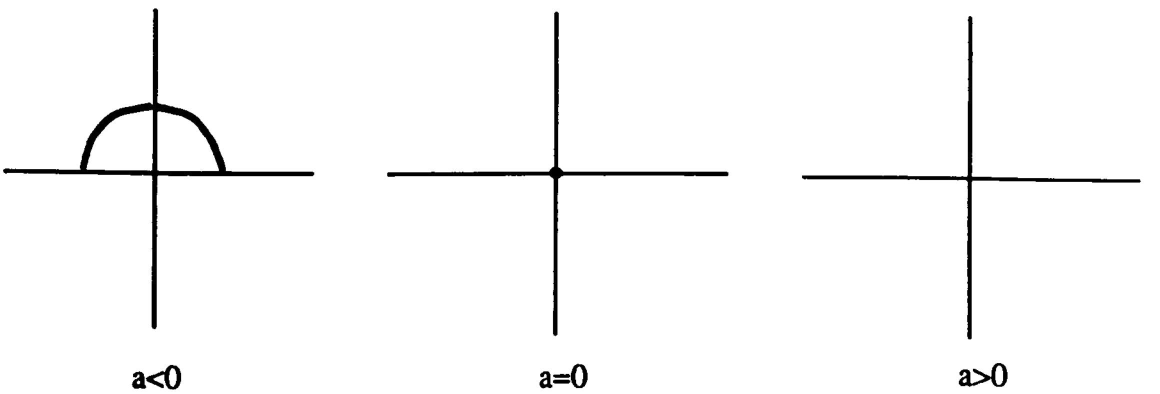



Fig. 7.6.3

At the point $a = 0$ the cross-section of the surface is a cusp. We will call such a point a pinch-point. Parameter space is shown in Fig. 7.6.4.

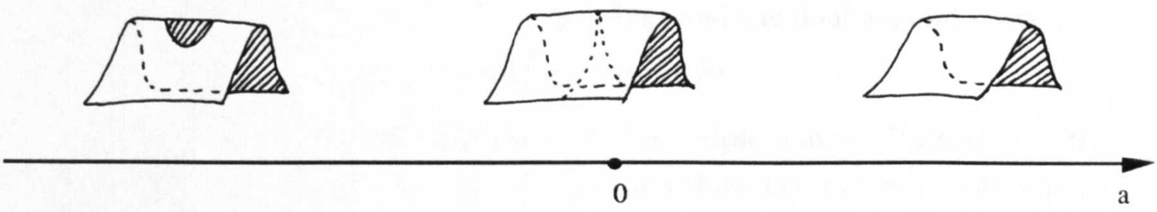



Fig. 7.6.4

### 7.6.7 $S_1^-$

The unfolding is $f_a(x,y) = (x, y^2, y^3 - x^2y + ay)$. Clearly the double point curve is

$$p(x, y^2) = y^2 - x^2 + a.$$

Again, we would expect at most two cross-caps and no triple points. Exactly the same reasoning shows that the bifurcation set is $a = 0$, but here we get two cross-caps when $a > 0$ and none when $a < 0$. The double-point curve in the source is shown in Fig. 7.6.5,

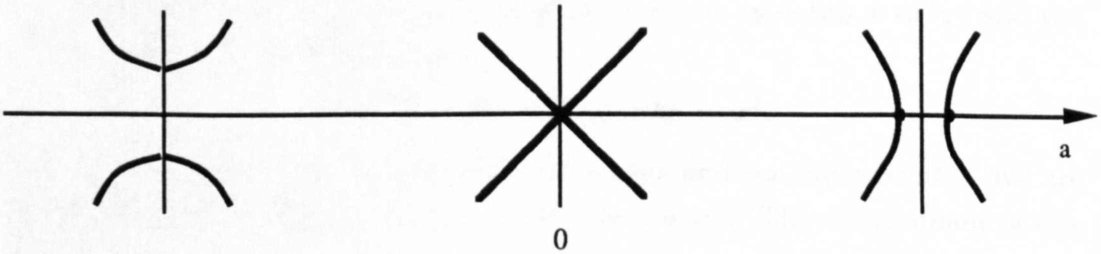



Fig. 7.6.5

while in the target we have (Fig. 7.6.6):

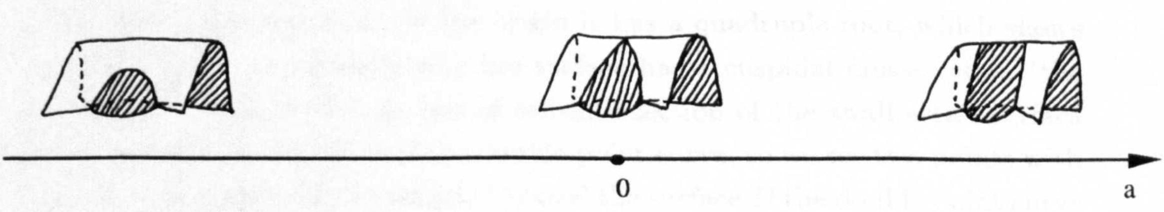



Fig. 7.6.6

See Fig 2, Appendix A.

### 7.6.8 $S_2$

In this case the positive and negative versions coincide. The unfolding is given by $f_{a,b}(x,y) = (x, y^2, y^3 + x^3y + ay + bxy)$ and the double-point curve is

$$p(x, y^2) = y^2 + x^3 + a + bx.$$

We expect at most three cross-caps and no triple points. Putting $y = 0$ into $p(x,y^2) = 0$ gives $x^3 + bx + a = 0$. This has three distinct real roots when the discriminant, $D$, is greater than zero. Here $D = -b^3/27 - a^2/4$, so the bifurcation set is $D = 0$, ie the cusp $27a^2 = -4b^3$. When $D > 0$ we get three real roots and so three cross-caps, while when $D < 0$ we only have one real root and so have one-cross-cap. So we have one cross-cap above the cusp, and three below it. On the critical set itself we either get three cross-caps with two meeting at the same point (if $a < 0$) or one cross-cap and a pinch point (if $a > 0$), and only one cross-cap at the origin. The double point curve in the source is shown in Fig. 7.6.7. The picture in the target is shown in Fig. 7.6.8. Also see Fig 3, Appendix A.

### 7.6.9 $S_3^+$

The unfolding is $f_{a,b,c}(x,y) = (x, y^2, y^3 + x^4y + ay + bxy + cx^2y)$ and the double point curve in the source is

$$p(x, y^2) = y^2 + x^4 + a + bx + cx^2.$$

We would expect to see at most four cross-caps and no triple points. We are looking for solutions of $p(x,0) = x^4 + cx^2 + bx + a = 0$. The discriminant of this quartic is

$$D = -4c^3b^2 - 27b^4 + 16ac^4 - 128a^2c^2 + 144ab^2c + 256a^3$$

The bifurcation set is given by $D = 0$ which is the algebraic representation of the swallowtail surface (see Teissier [Te]). On this surface the double point curve always has a repeated root. At the origin it has a quadruple root, which shows up in the target as a point where the surface has a cuspidal cross-section (the birth of a cross-cap). On the line of self-intersection of the swallowtail surface there are two repeated roots of the double point curve, so we see two points with cuspidal cross-section in the target. 'Above' the surface $D$ the double point curve does not intersect with $y = 0$ and so the surface is an immersion, and 'below' $D$ we have two real roots of the double point curve so the surface has two distinct

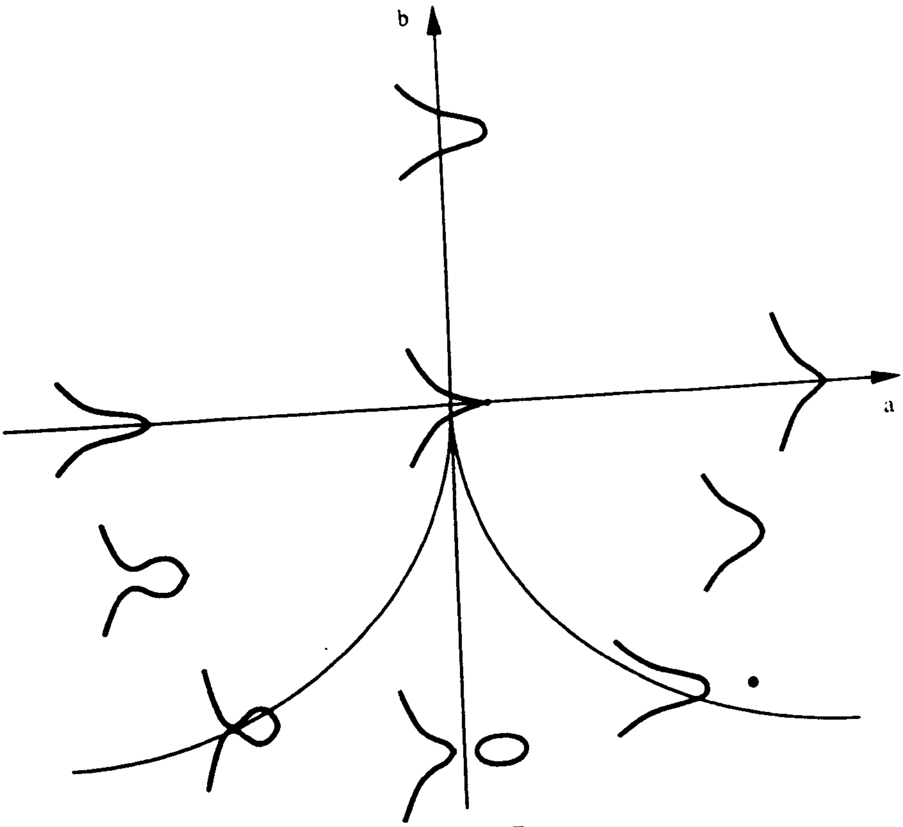



Fig.7.6.7

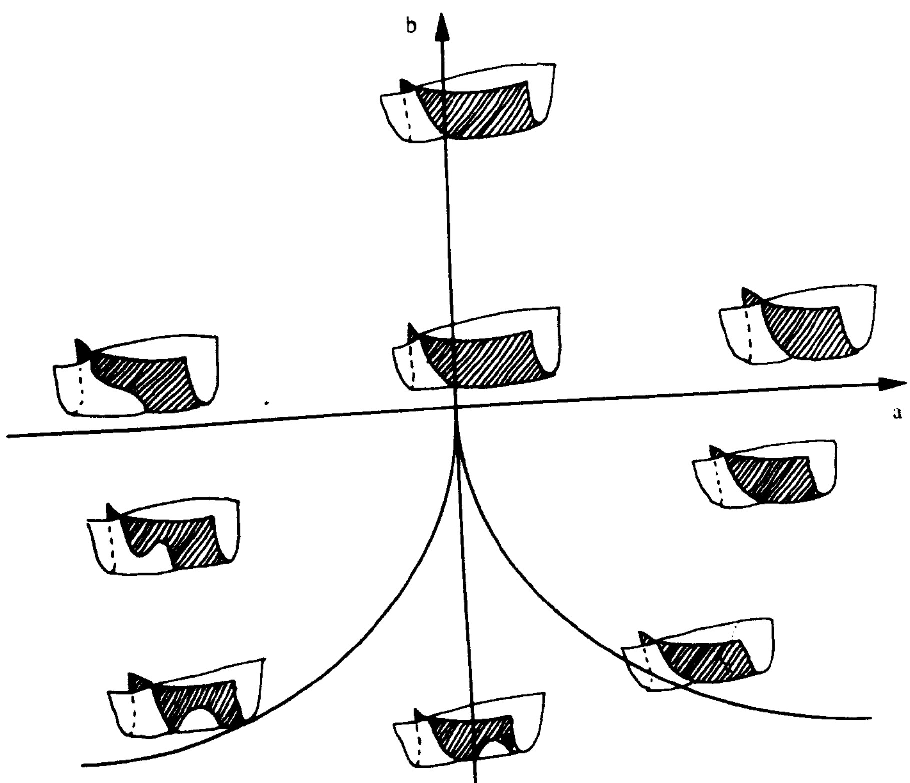



Fig.7.6.8

cross-caps, while inside the swallowtail we see the maximum number of cross-caps since the double point curve has four real roots here. These transitions are shown in parameter space in the source in Fig. 7.6.9 and in the target in Fig. 7.6.10. See Fig 4, Appendix A.

### 7.6.10 $S_3^-$

The unfolding is $f_{a,b,c} = (x, y^2, y^3 - x^4y + ay + bxy + cx^2y)$ and the double point curve is

$$p(x, y^2) = y^2 - x^4 + a + bx + cx^2.$$

Again, we expect at most four cross-caps and the bifurcation set is given by the discriminant of the quartic $x^4 - cx^2 - bx - a$ being equal to zero: the swallowtail surface. As in $S_3^+$, on this surface the double point always has at least one repeated root, which in this case corresponds to two cross-caps coalescing. As above, the maximum number of cross-caps is realised 'inside' $D = 0$ and below $D = 0$ we see two cross-caps while above we see none (though in this case $S_3^-$ is only an immersion close to the origin when we are above $D = 0$). Parameter space in the source is shown in Fig. 7.6.11, while in the target it is shown in Fig. 7.6.12. See Fig 5, Appendix A.

### 7.6.11 $B_2^+$

The unfolding is $f_{a,b}(x,y) = (x, y^2, x^2y + y^5 + ay + by^3)$ and the double point curve is

$$p(x, y^2) = x^2 + y^4 + a + by^2$$

We would expect to see at most two cross-caps and no triple points. We also note that $\mu(\widetilde{D}^2/\mathbf{Z}_2) = 2$ for this case. That this number is non-zero leads us to expect that we may see non-transverse self-tangencies on the double point curve.

To find the condition for the maximum number of cross-caps to appear we look for solutions of $p(x, 0) = 0$, so here we consider $x^2 + a = 0$. In order to get two cross-caps we need $a < 0$, and so the bifurcation set is the line $a = 0$.

We can think of the double point curve as a (symmetrical) quartic $X = -y^4 - by^2 - a$, where $X = x^2$. This is plotted for various values of $a$ and $b$ in Fig.7.6.13. Self-tangencies of the double point curve occur when this curve is

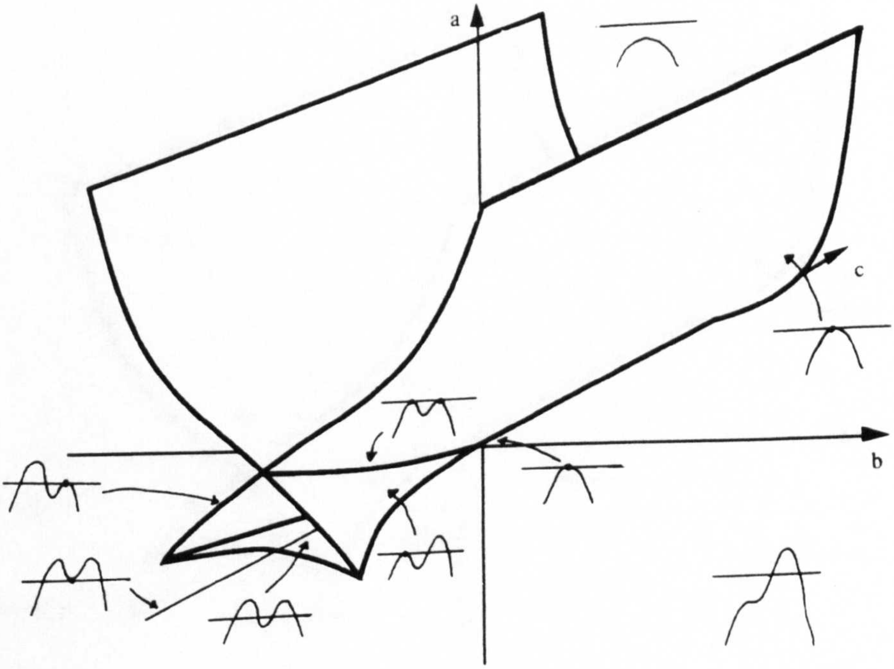



Fig.7.6.9

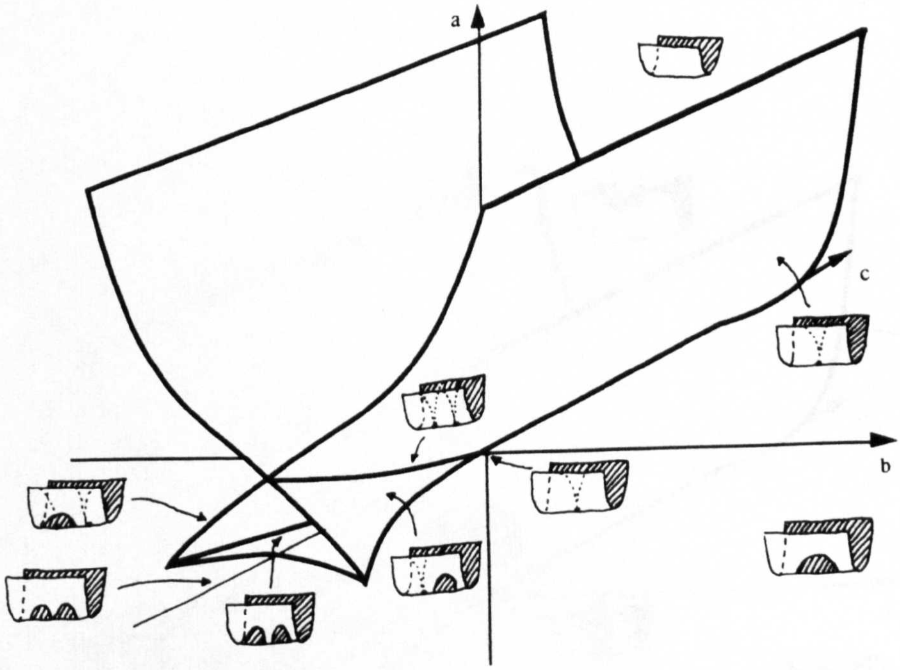



Fig.7.6.10

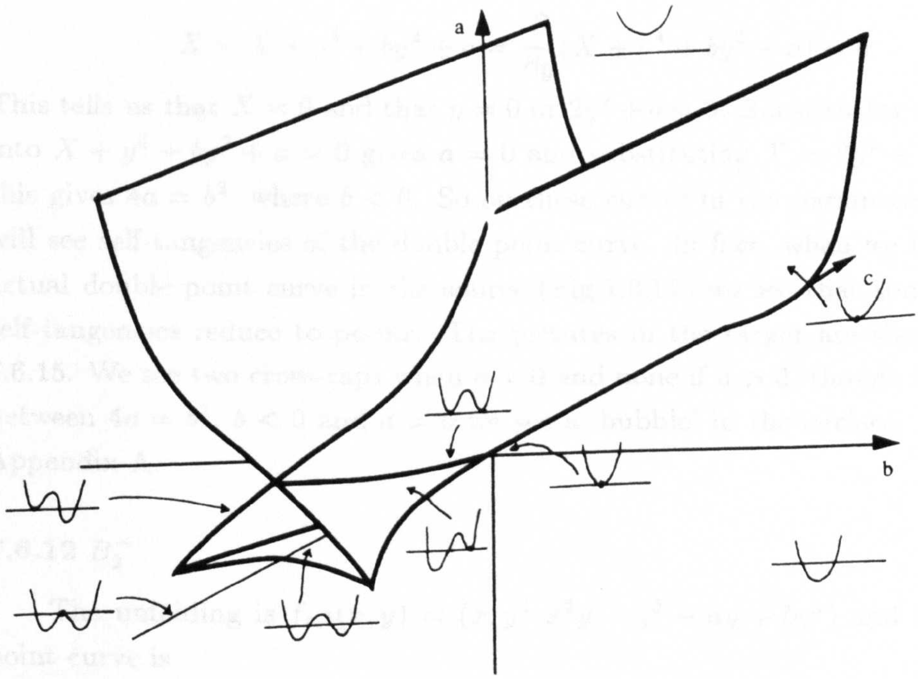



Fig.7.6.11

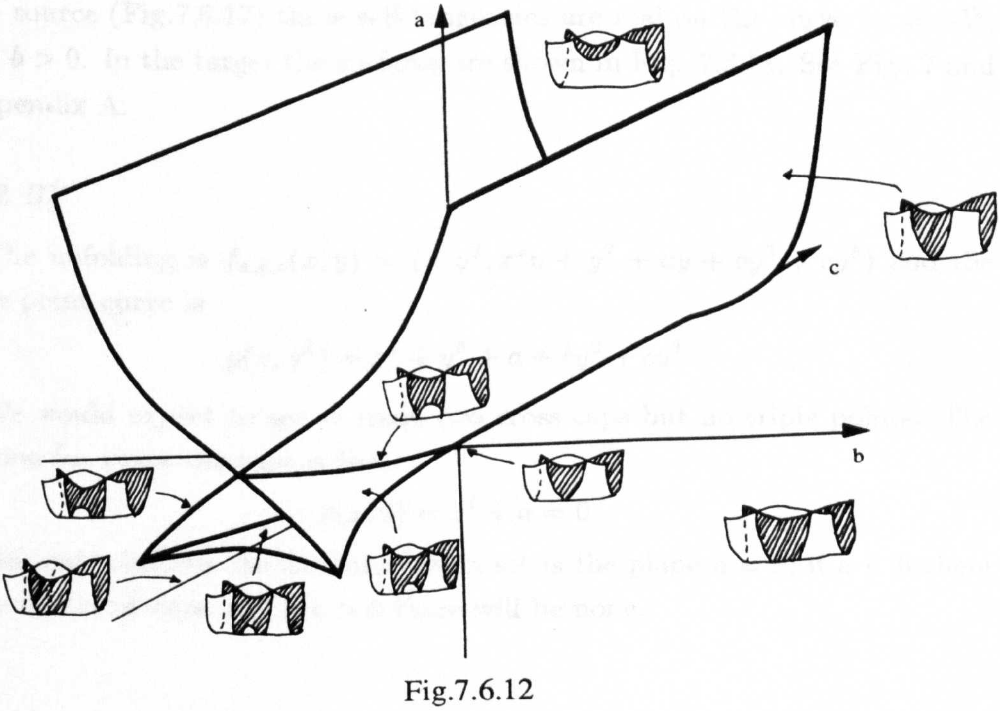



Fig.7.6.12

tangent to $X = 0$. The conditions for this are that

$$X = X + y^4 + by^2 + a = \frac{\partial}{\partial y}(X + y^4 + by^2 + a) = 0$$

This tells us that $X = 0$ and that $y = 0$ or $2y^2 + b = 0$. Substituting $X = y = 0$ into $X + y^4 + by^2 + a = 0$ gives $a = 0$ and substituting $X = 2y^2 + b = 0$ into this gives $4a = b^2$, where $b < 0$. So on these curves in the parameter space we will see self-tangencies of the double point curve. In fact, when we look at the actual double point curve in the source (Fig.7.6.14) we see that some of these self-tangencies reduce to points. The pictures in the target are shown in Fig. 7.6.15. We see two cross-caps when $a < 0$ and none if $a > 0$, though in the area between $4a = b^2$, $b < 0$ and $a > 0$ we see a 'bubble' in the surface. See Fig 6, Appendix A.

### 7.6.12 $B_2^-$

The unfolding is $f_{a,b}(x,y) = (x, y^2, x^2y - y^5 + ay + by^3)$ and the double point curve is

$$p(x, y^2) = x^2 - y^4 + a + by^2$$

Again, we expect to see at most two cross-caps and the condition for this to occur is $a < 0$. The conditions for self-tangencies of the double point curve are given by $4a = -b^2$, with $b > 0$, or $a = 0$. Again, we plot the curve $X = y^4 - by^2 - a$ in the source (Fig.7.6.16). In this case, when we look at the actual double point curve in the source (Fig.7.6.17) these self-tangencies are real on the curve $4a = -b^2$, where $b > 0$. In the target the surfaces are shown in Fig. 7.6.18. See Figs 7 and 8, Appendix A.

### 7.6.13 $B_3^+$

The unfolding is $f_{a,b,c}(x,y) = (x, y^2, x^2y + y^7 + ay + by^3 + cy^5)$ and the double point curve is

$$p(x, y^2) = x^2 + y^6 + a + by^2 + cy^4$$

We would expect to see at most two cross-caps but no triple points. The condition for two cross-caps is that

$$p(x, 0) = x^2 + a = 0$$

has two real solutions. So the bifurcation set is the plane $a = 0$; if $a < 0$ there will be two cross-caps and if $a > 0$ there will be none.

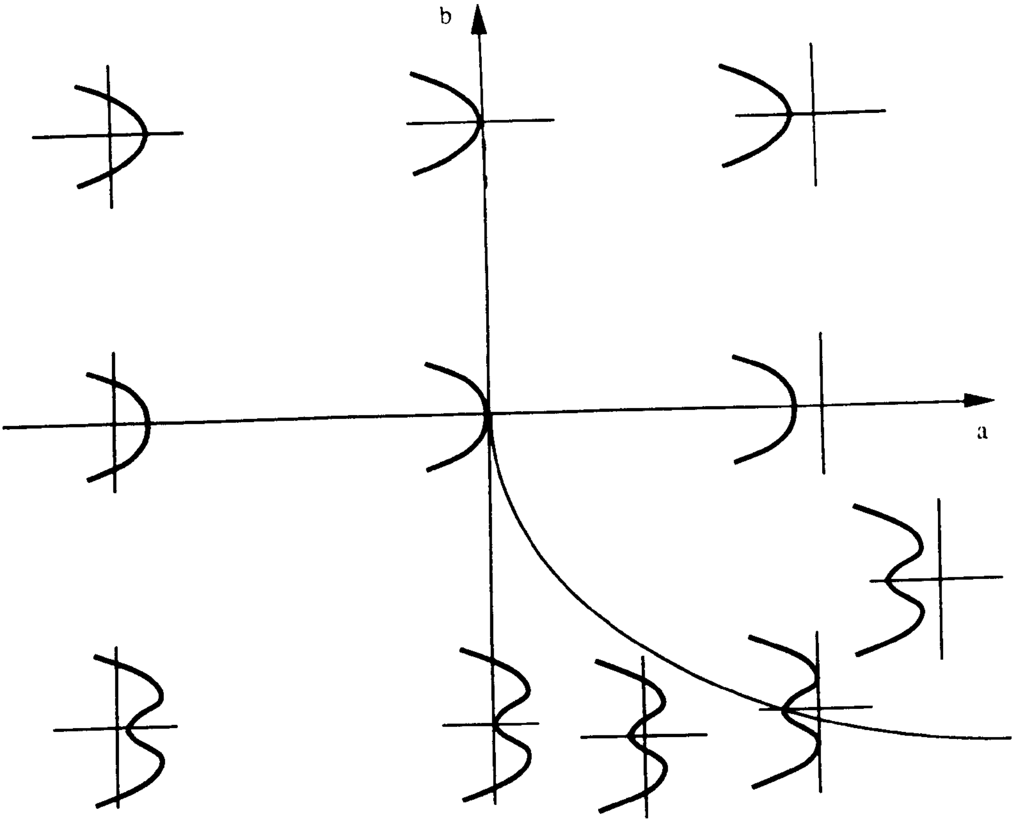



Fig.7.6.13

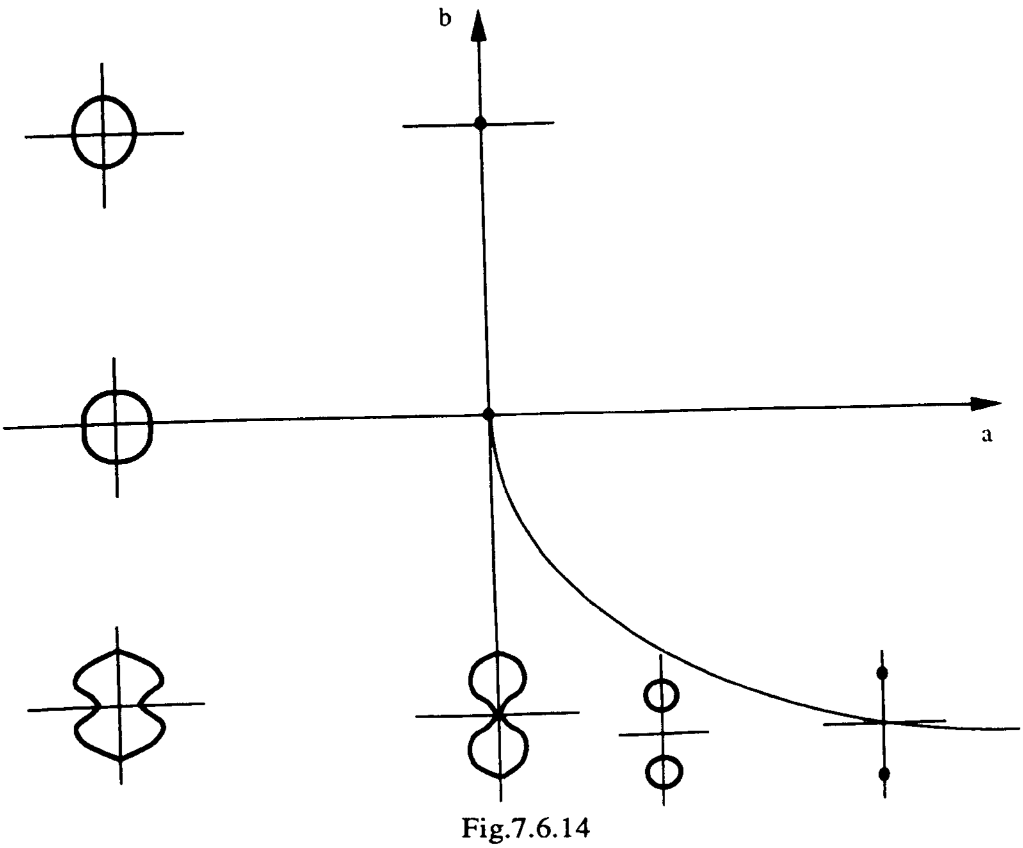



Fig.7.6.14

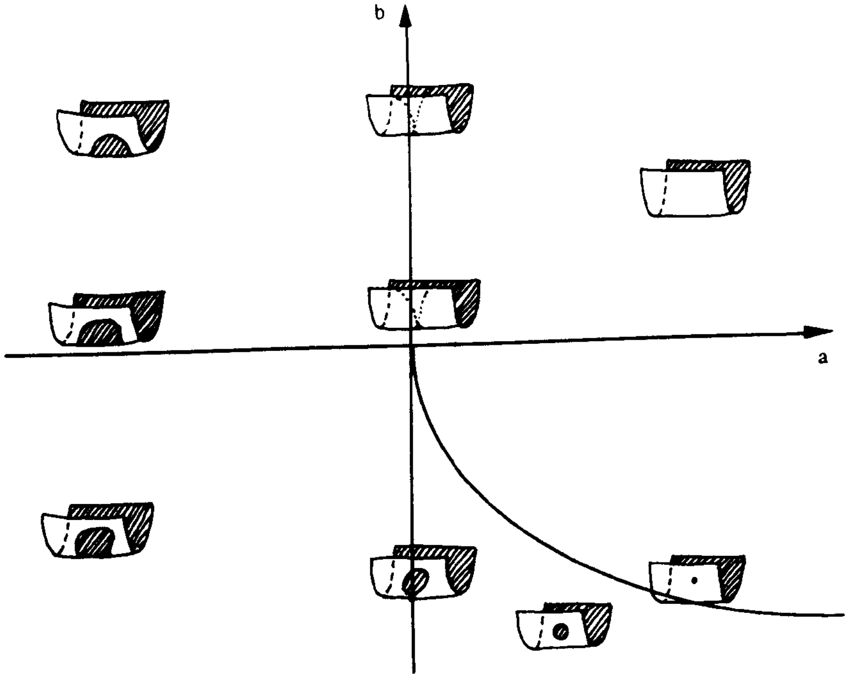



Fig.7.6.15

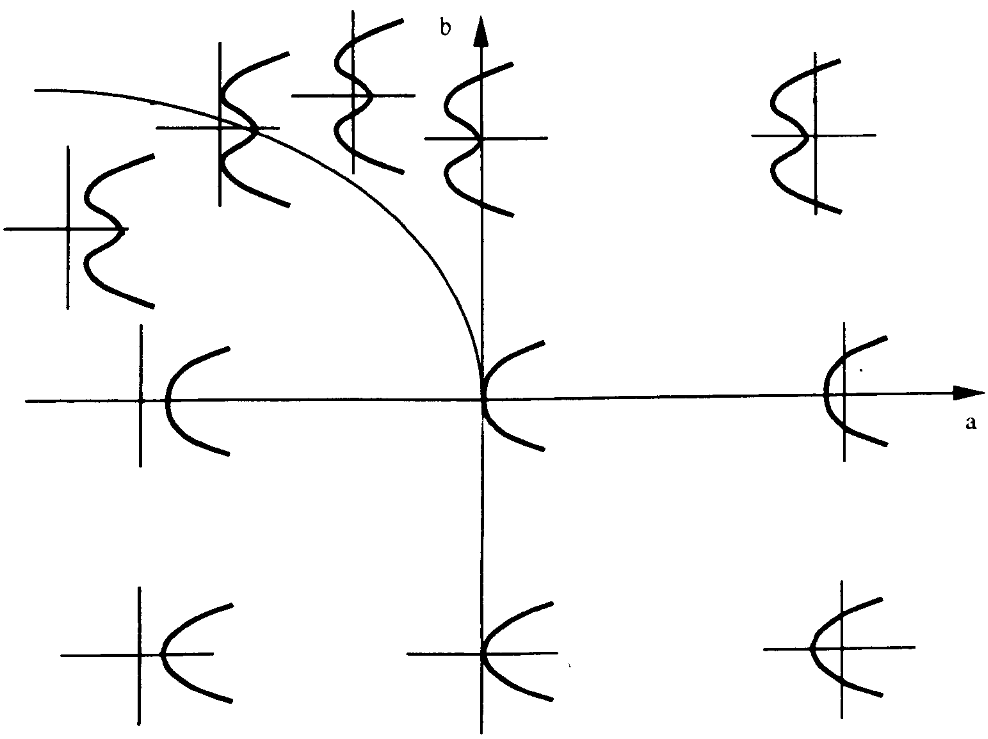



Fig.7.6.16

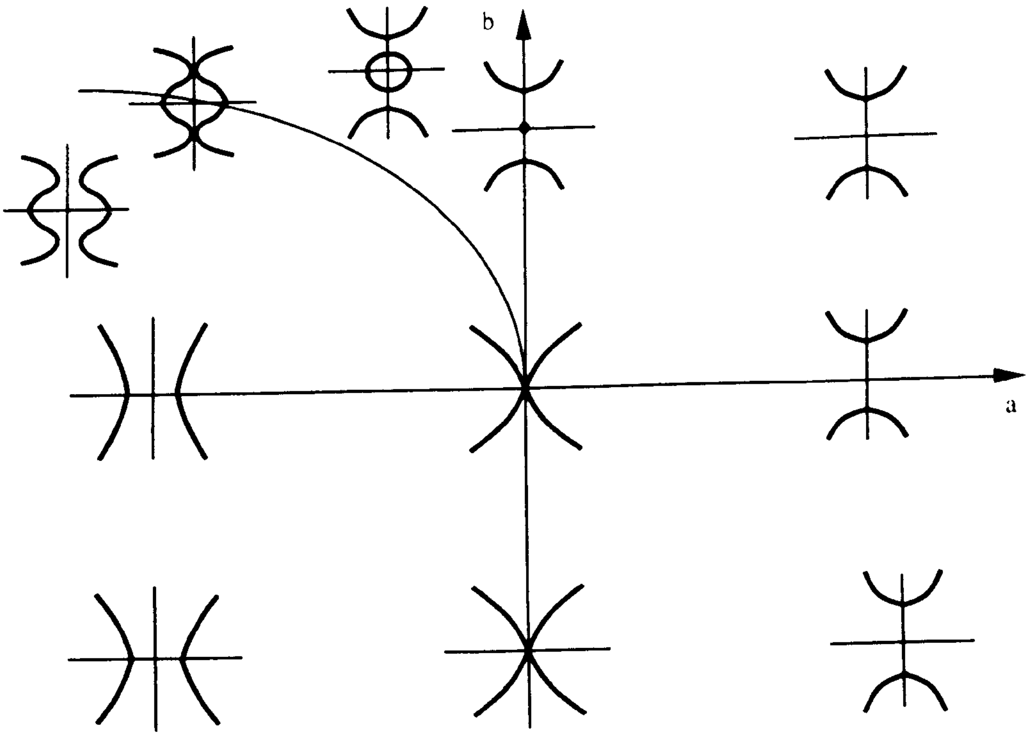



Fig.7.6.17

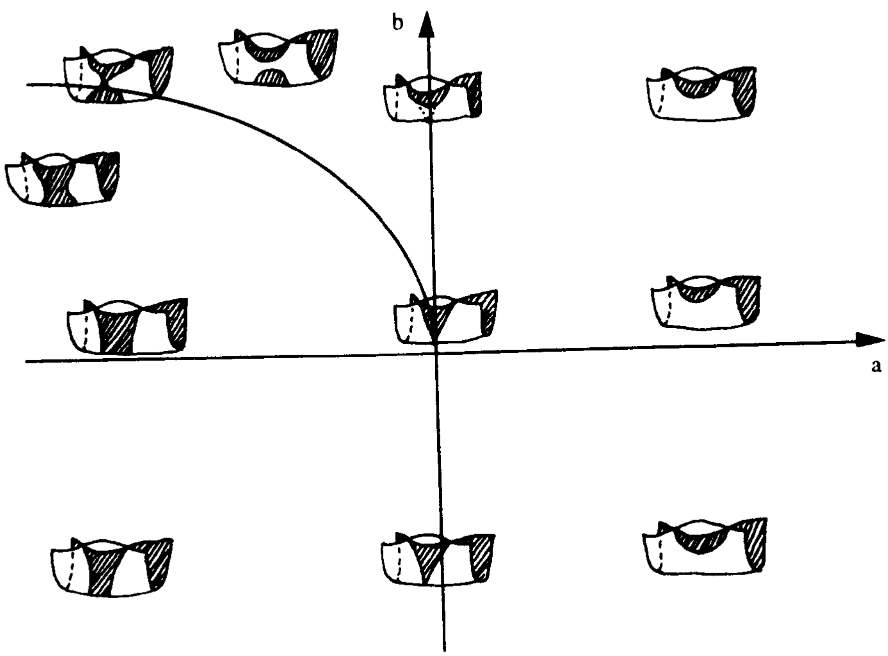



Fig.7.6.18

We also note that in this case $\mu(\widetilde{D}^2/\mathbf{Z}_2) = 4$ so we expect to find self-tangencies of the double point curve. We shall again consider the double point curve as a sextic curve $X = -y^6 - cy^4 - by^2 - a$ (where $X = x^2$). The self-tangencies will occur when the curve is tangent to $X = 0$ i.e. when

$$X = X + y^6 + a + by^2 + cy^4 = \frac{\partial}{\partial y}(X + y^6 + a + by^2 + cy^4) = 0.$$

This gives us $X = 0$ and either $y = 0$ or $3y^2 = -c \pm \sqrt{c^2 - 3b}$. Substituting $X = y = 0$ back into $X = -y^6 - cy^4 - by^2 - a$ gives $a = 0$. The latter case can only have real solutions if $c^2 - 3b > 0$ and $-c \pm \sqrt{c^2 - 3b} \geq 0$. If $b < 0$ then there will be one real solution for $y^2$, while if $b \geq 0$, $c < 0$ and $c^2 - 3b > 0$ there will be two. Otherwise, there will be no solution for $y^2$. Substituting these values of $y^2$ into $X = -y^6 - cy^4 - by^2 - a$ (where $X = 0$) gives the surface of self-tangencies. In parameter space we have

$$D = 27a^2 - 18abc + 4ac^3 - b^2c^2 + 4b^3$$

where $b < 0$ or $b \geq 0$, $c < 0$ and $c^2 - 3b > 0$. The surface $D = 0$ is 'half' of a cuspidal edge with non-zero torsion. We can study the transitions of the double point curve in the source (see Fig.7.6.19) and of the image $f_{a,b,c}$ (see Fig.7.6.20) by taking sections across this surface. When $a < 0$ there are two cross-caps. 'Inside' the surface (between $D$ and $a < 0$) we see two cross-caps and a bubble. As we approach $a = 0$ the cross-caps coalesce and the bubble remains. Between $a > 0$ and $D$ the image still has a bubble, which vanishes on $D$. 'Behind' both surfaces the image is just an immersion.

### 7.6.14 $B_3^-$

The unfolding is $f_{a,b,c}(x,y) = (x, y^2, x^2y - y^7 + ay + by^3 + cy^5)$ and the double point curve is

$$p(x, y^2) = x^2 - y^6 + cy^4 + by^2 + a$$

Again we expect at most two cross-caps and the condition for this is that $a < 0$.

The conditions for self-tangencies of the double point curve are that $a = 0$ or $D = 0$ where $D$ is given by

$$D = 27a^2 + 18abc + 4ac^3 - b^2c^2 - 4b^3 = 0$$

where $b > 0$ or $b \leq 0$, $c > 0$ and $c^2 + 3b > 0$. In the source we see the curves shown in Fig. 7.6.21 and, in the target, we see the surfaces shown in Fig. 7.6.22.

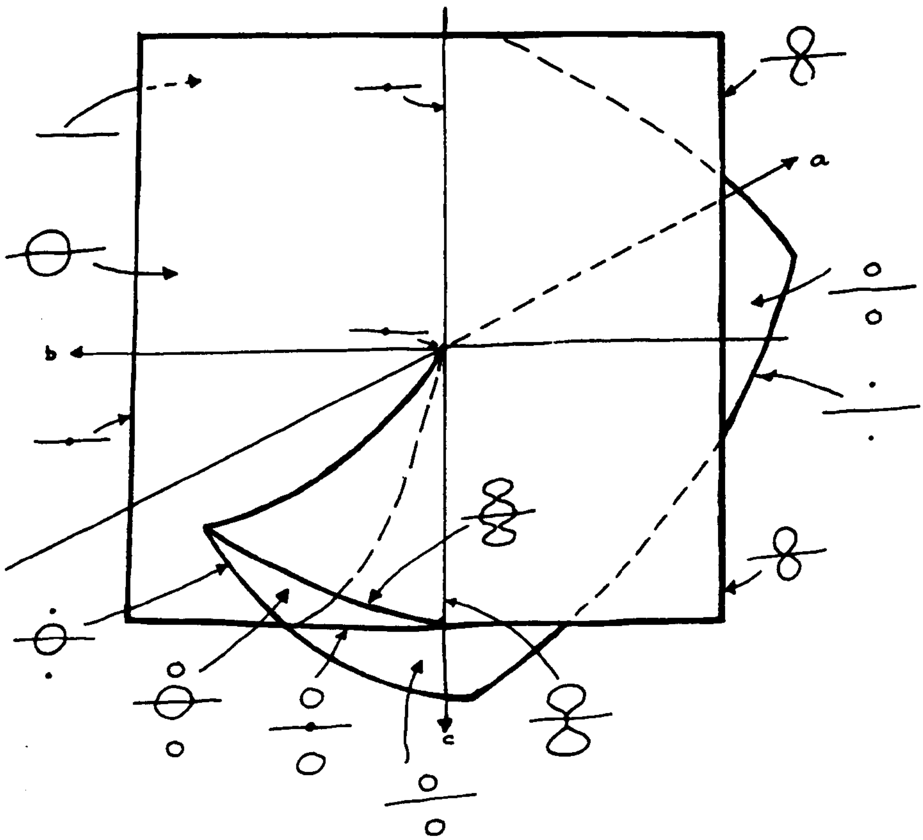



Fig. 7.6.19

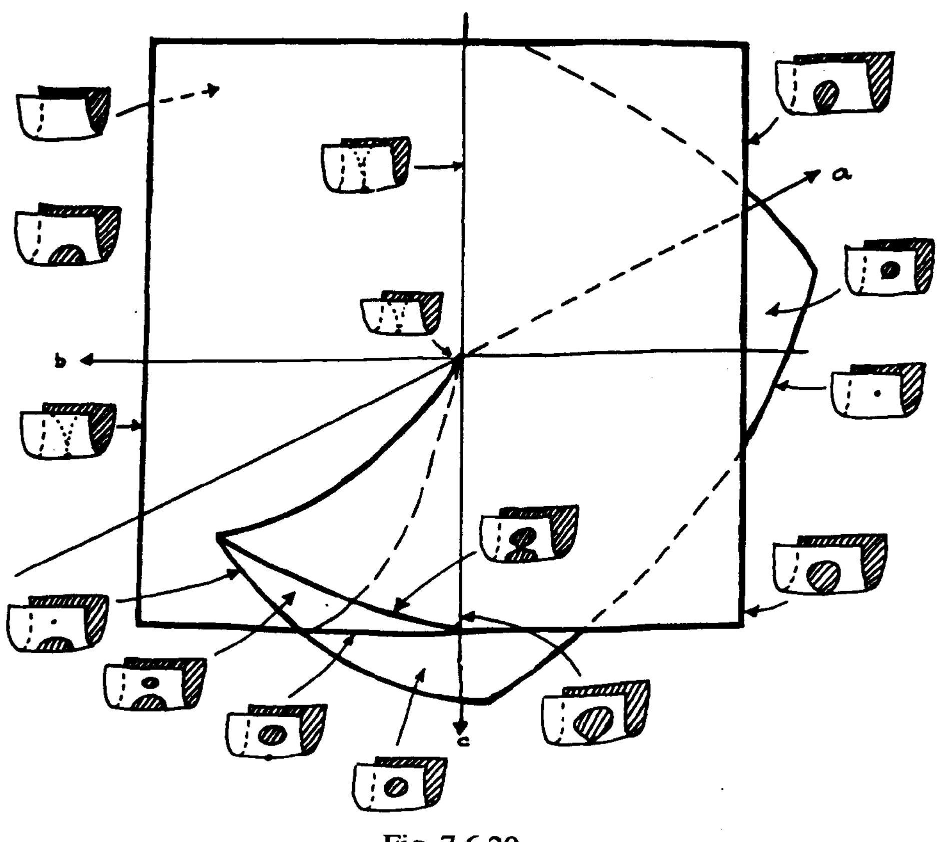



Fig. 7.6.20

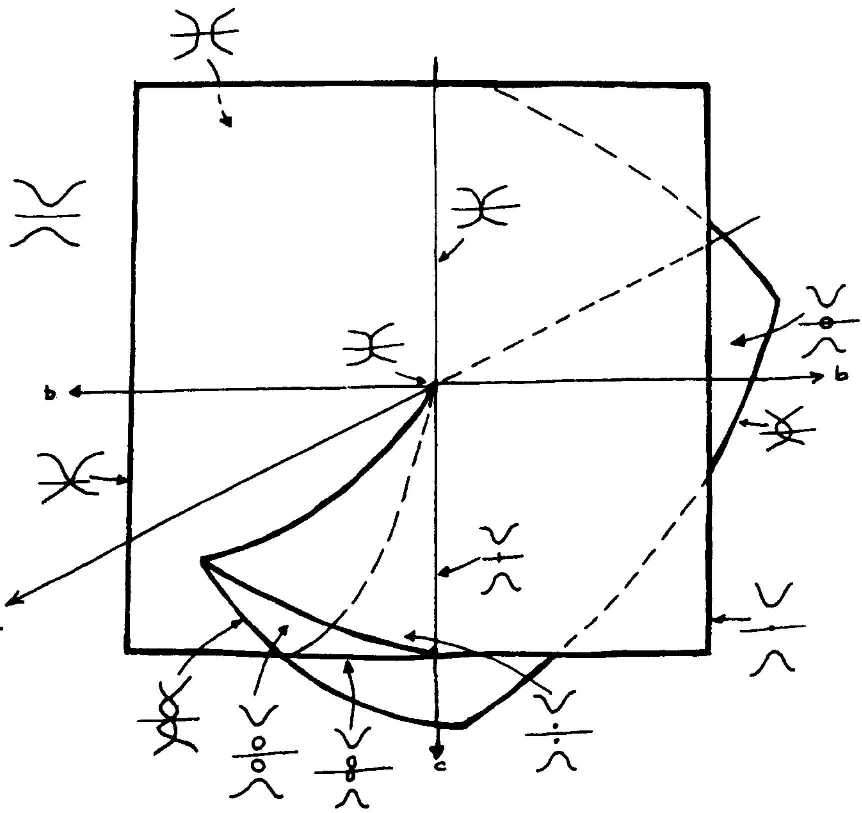



Fig.7.6.21

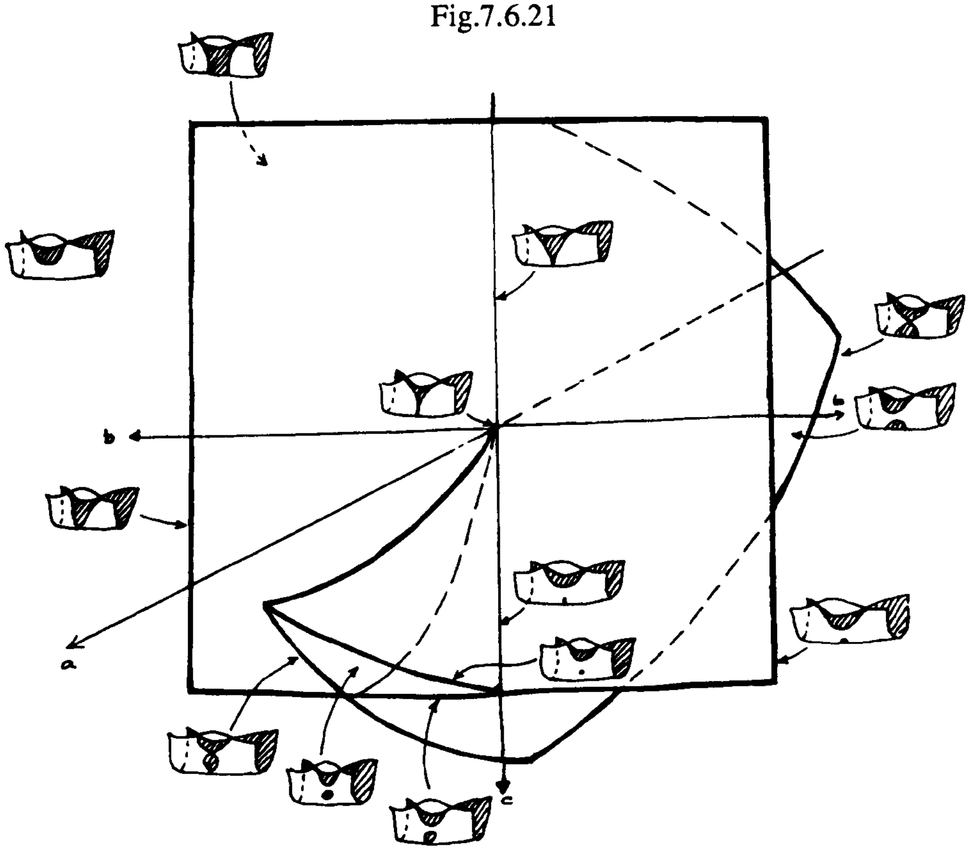



Fig.7.6.22

Again, when $a > 0$ we see no cross-caps, though the image is not an immersion. Between the surface $D$ and $a > 0$ we see a bubble in the image which gets closer to the origin as $a$ gets closer to 0. This becomes two cross-caps when $a < 0$

### 7.6.15 $C_3^+$

The unfolding is $f_{a,b,c}(x, y) = (x, y^2, xy^3 + x^3y + ay + by^3 + cxy)$ and so the double point curve is

$$p(x, y^2) = xy^2 + x^3 + a + by^2 + cx.$$

We would expect at most three cross-caps and no triple points. The condition for the maximum number of cross-caps is that $p(x, 0) = x^3 + cx + a = 0$ has three real solutions, which tells us that we must have $27a^2 < -4c^3$. Thus the bifurcation set is the cuspidal edge, $27a^2 = -4c^3$. We note that in this case $\mu(\widetilde{D}^2/\mathbf{Z}_2) = 2$ so we conjecture that we expect to see a *curve* of self-tangencies of the double point curve in the parameter space. If we use the same method as for $B_2$ and $B_3$, we must solve

$$x = xy^2 + x^3 + a + by^2 + cx = \frac{\partial}{\partial y}(xy^2 + x^3 + a + by^2 + cx) = 0$$

which tells us that $a = b = 0$. So on this line we should see self-tangencies.

We also note that $p(x, y^2) = 0$ can be written $y^2(x+b)+x^3+cx+a = 0$. Then if $x = -b$ the coefficient of $y^2$ is zero and we are left with $b^3 - bc + a = 0$.This is a surface in $a, b, c$-space. It is the surface known as the cusp catastrophe surface, and in this case it is tangent to the cuspidal edge since maxima and minima for the cubic $a = b(c + b^2)$ at a fixed value of $c$ are given by $a = \pm\frac{2}{3}\sqrt{\frac{-c}{3}}$, and these are also the two values of $a$ on the cuspidal edge for a fixed value of $c$.

In the source we have the curves shown in Fig. 7.6.23 and in the target we have the surfaces shown in Fig. 7.6.24. We see that inside the cuspidal edge there are three cross-caps. Two of these coalesce and disappear on the cuspidal edge so that only one cross-cap remains. On the cusp catastrophe surface the double point curve becomes reducible. Note the pitchfork transition which occurs inside the cuspidal edge as $b$ increases. When we consider $b = 0$ the double point curve is of the form $xy^2 + x^3 + cx + a = 0$, with $c < 0$. This compares with the standard pitchfork $x^3 - \lambda x + \epsilon$. In the $(c, b)$-plane in the source we see the curves shown in Fig.7.6.25.

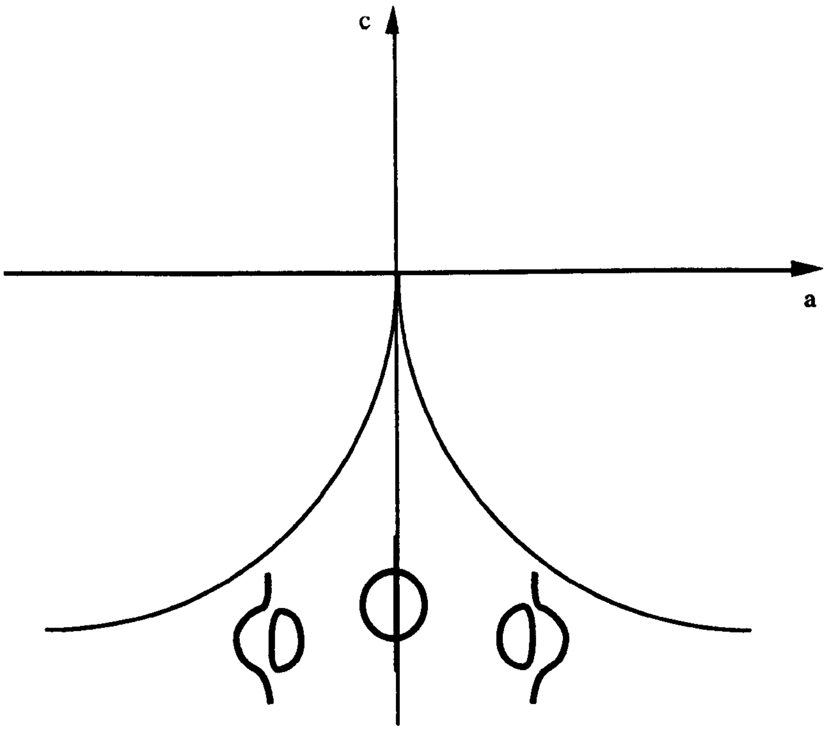



Fig. 7.6.25

See Fig 9, Appendix A. We observe that in this case our picture of the bifurcation set is not the same as Wilkinson's [W]: he has used another unfolding. In his unfolding the cusp catastrophe surface has been flattened, forcing the cuspidal edge to become twisted.

### 7.6.16 $C_3^-$

The unfolding is $f_{a,b,c}(x,y) = (x, y^2, xy^3 - x^3y + ay + by^3 + cxy)$, and then the double point curve is

$$p(x,y^2) = xy^2 - x^3 + a + by^2 + cx.$$

As above, we expect at most three cross-caps, and the bifurcation set is the cuspidal edge $27a^2 = 4c^3$. Again, using the methods from the calculations of $B_2$ and $B_3$, the condition for self-tangencies of the double point curve seems to be $a = b = 0$. By writing the double point curve in the form $y^2(x+b)-x^3+cx-a = 0$ we get the exceptional surface $a = b(c - b^2)$ when the coefficient of $y^2$ is zero. In the source we have the curves shown in Fig. 7.6.26 and in the target we have the surfaces shown in Fig. 7.6.27. Again, on the cusp catastrophe surface the

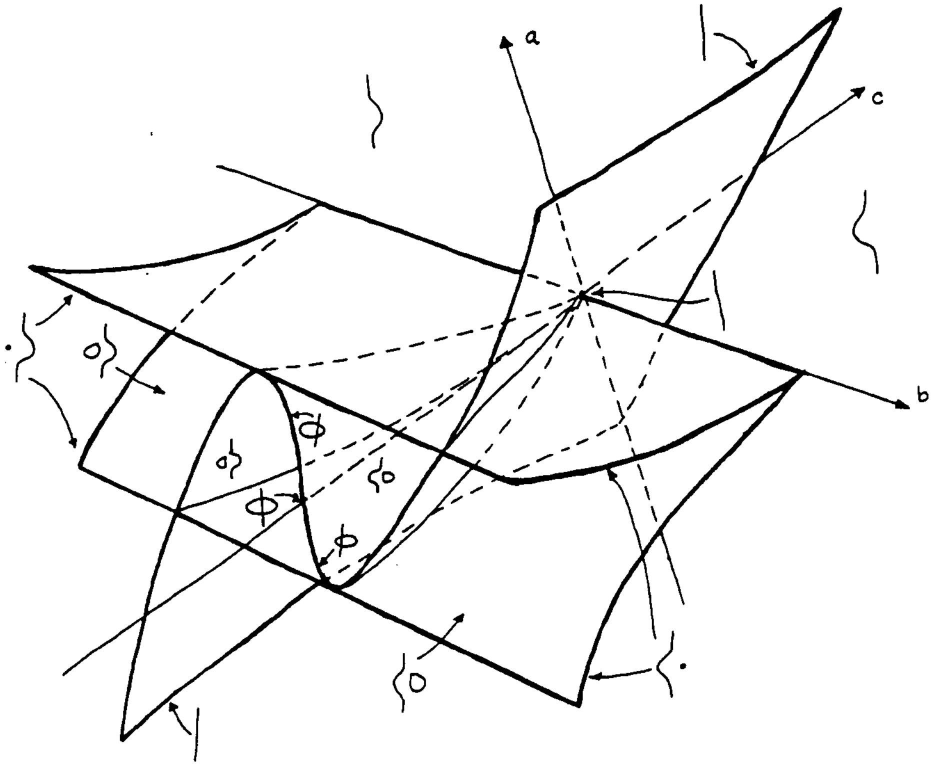



Fig.7.6.23

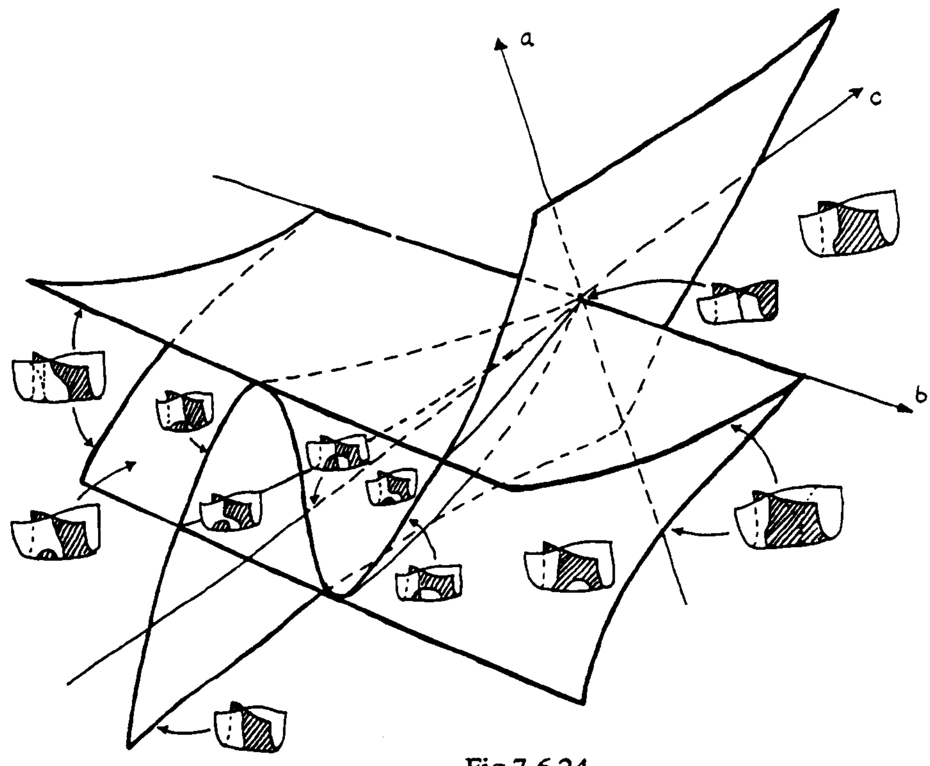



Fig.7.6.24

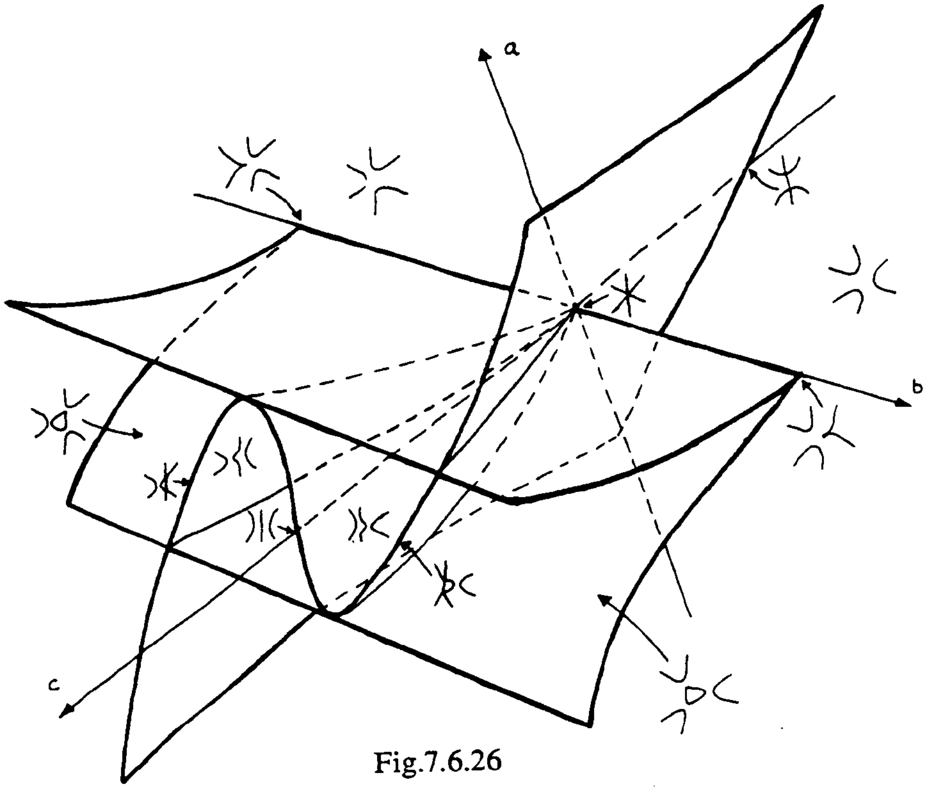



Fig.7.6.26

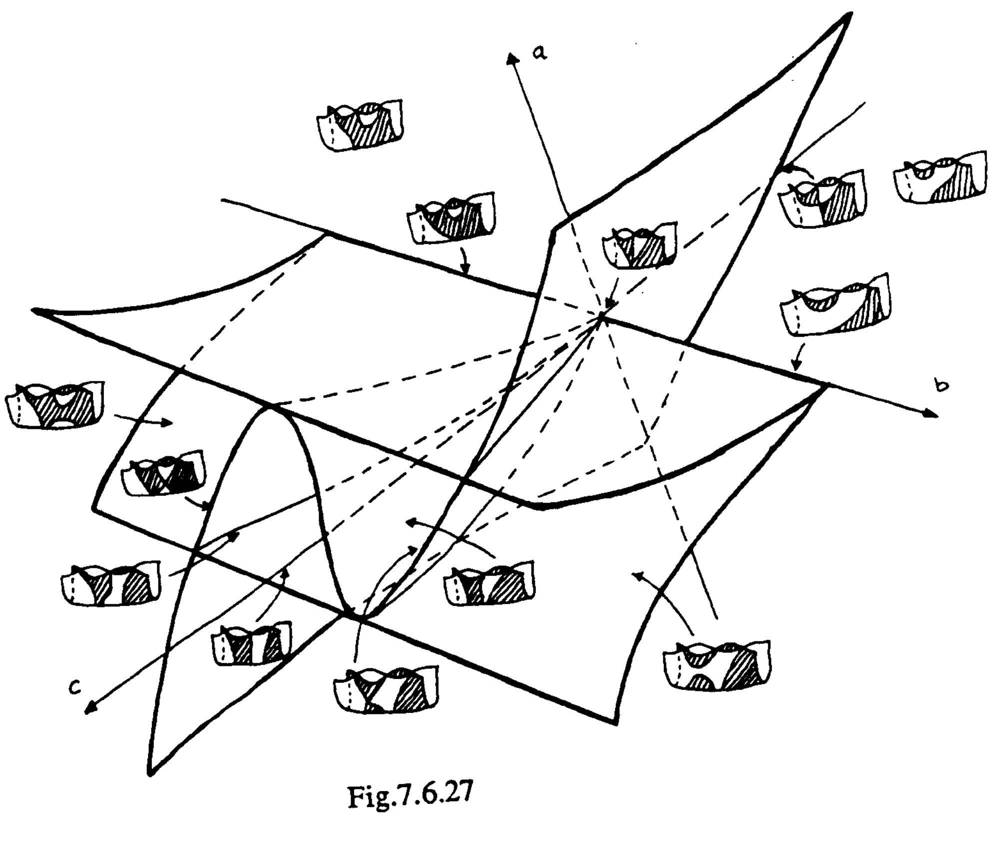



Fig.7.6.27

double point curve is reducible and inside the cuspidal edge we see the maximum number of cross-caps. See Fig 10, Appendix A.

### 7.6.17 $H_2$

The is unfolding $f_{a,b}(x,y) = (x, y^3 + ay, xy + y^5 + by^2)$. We now follow the algorithm in section 7.6.3. Choose $\widetilde{f}(x,y) = (x, y^3 + ay)$. Then the generators for $\mathcal{O}_3$ over $\mathcal{O}_2$ are 1, $y$, $y^2$. Now

$$\begin{aligned} Z.1 &= -aY + (X + a^2)y + (Y + b)y^2 \\ Z.y &= Y(Y+b) + (-2aY - ab)y + (X + a^2)y^2 \\ Z.y^2 &= Y(X + a^2) + (Y^2 + bY - aX - a^3)y + (-2aY - ab)y^2 \end{aligned}$$

and the presentation matrix is

$$\lambda = \begin{pmatrix} -Z - aY & X + a^2 & Y + b \\ Y(Y+b) & -Z - 2aY - ab & X + a^2 \\ Y(X + a^2) & Y^2 + bY - aX - a^3 & -Z - 2aY - ab \end{pmatrix}$$

We can obtain the defining equation of the double point curve by letting the top right $2 \times 2$ determinant, via $f^*$, be equal to zero. This gives:

$$\begin{aligned} &x^2 + 2a^2x + a^4 + xy^4 + y^8 + 2by^5 + 3ay^6 + 4a^2y^4 \\ &+ 4aby^3 + axy^2 + 2a^3y^2 + 3a^2by + bxy + b^2y^2 + ab^2 = 0 \end{aligned}$$

which can be plotted by computer (see Figs. 7.6.29(i)-(iv)). More simply, to find the co-ordinates of cross-caps in the image of an unfolding we solve

$$\frac{\partial}{\partial y}(y^3 + ay) = \frac{\partial}{\partial y}(xy + y^5 + by^2) = 0$$

This yields

$$y = \pm\sqrt{-a/3}$$

so in order to get two cross-caps, as expected from Table 7.6.1, we must have $a < 0$. If $a = 0$ we get one solution to this system of equations, the point $(0,0)$. This just gives a distinguished point in the image $f_{a,b}$, the birth of two cross-caps. Otherwise there are no solutions. So if $a \geq 0$ there are no cross-caps in the image.

To find the condition on the parameters in order to get a triple point in the image of $f_{a,b}$ we take the variety of zeros of the $1 \times 1$ determinants of $\lambda$. Some of these are multiples of each other. The independent ones are:

$$Z + aY = 0 \longrightarrow xy + y^5 + by^2 + ay^3 + a^2y = 0$$
$$X + a^2 = 0 \longrightarrow x = -a^2$$
$$Y + b = 0 \longrightarrow y^3 + ay + b = 0$$

The latter gives us a condition on $a$ and $b$, since if we want three preimages of a point in the image we must have three real roots for this equation. This gives the condition

$$27b^2 < -4a^3.$$

Parameter space is shown in Fig. 7.6.28.

We would like to know what happens to the image $f_{a,b}$ as we cross the cusp. When $27b^2 = -4a^3$, the cubic

$$y^3 + ay + b = 0$$

has one repeated root and one other distinct root (if $a \neq 0 \neq b$). This means that we no longer have a triple point but we get a self-tangency of the double point curve, which corresponds to the birth (or death) of a triple point. If we return to the computer image of the double point curve, for chosen values of $a$ and $b$ we see the transition from three crossing points (Figs. 7.6.29(i),(ii)) to none (Fig. 7.6.29(iv)) via the unstable situation of a self-tangency and a cusp point (Fig. 7.6.29(iii)).

In the source, when $27b^2 < -4a^3$ the three crossing points will come together to give a triple point (see Marar [Ma]). The unstable position, when $27b^2 = -4a^3$, is when the triple point vanishes. From the adjacancy table of Fig. 7.6.1, we know that $H_2$ can only perturb to $S_1$, so on this curve the image must be a deformed version of $S_1$, possibly with some distinguished point. When $27b^2 > -4a^3$ and $a < 0$, we just see two cross-caps, i.e. $S_1$. The isolated point does not seem to have any physical significance.

See Fig 11, Appendix A, for a computer generated image of $H_2$ showing one visible cross-cap and the triple point.

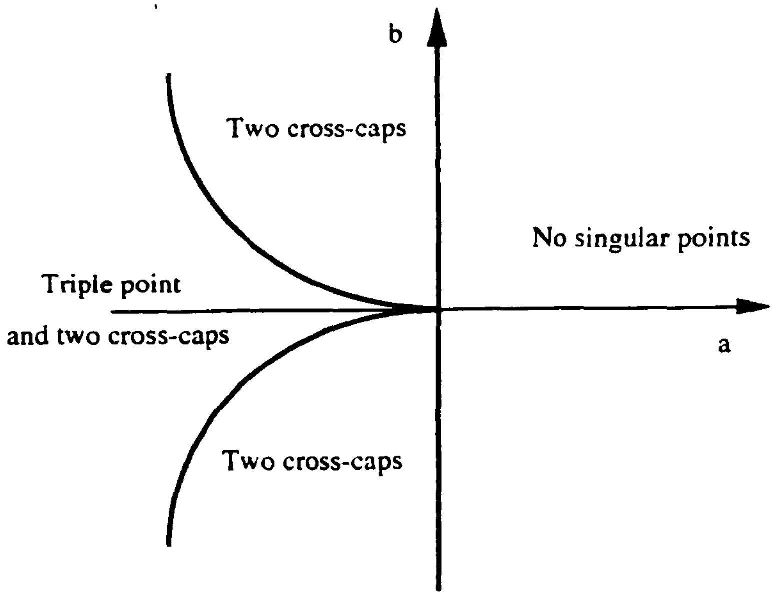



Fig.7.6.28

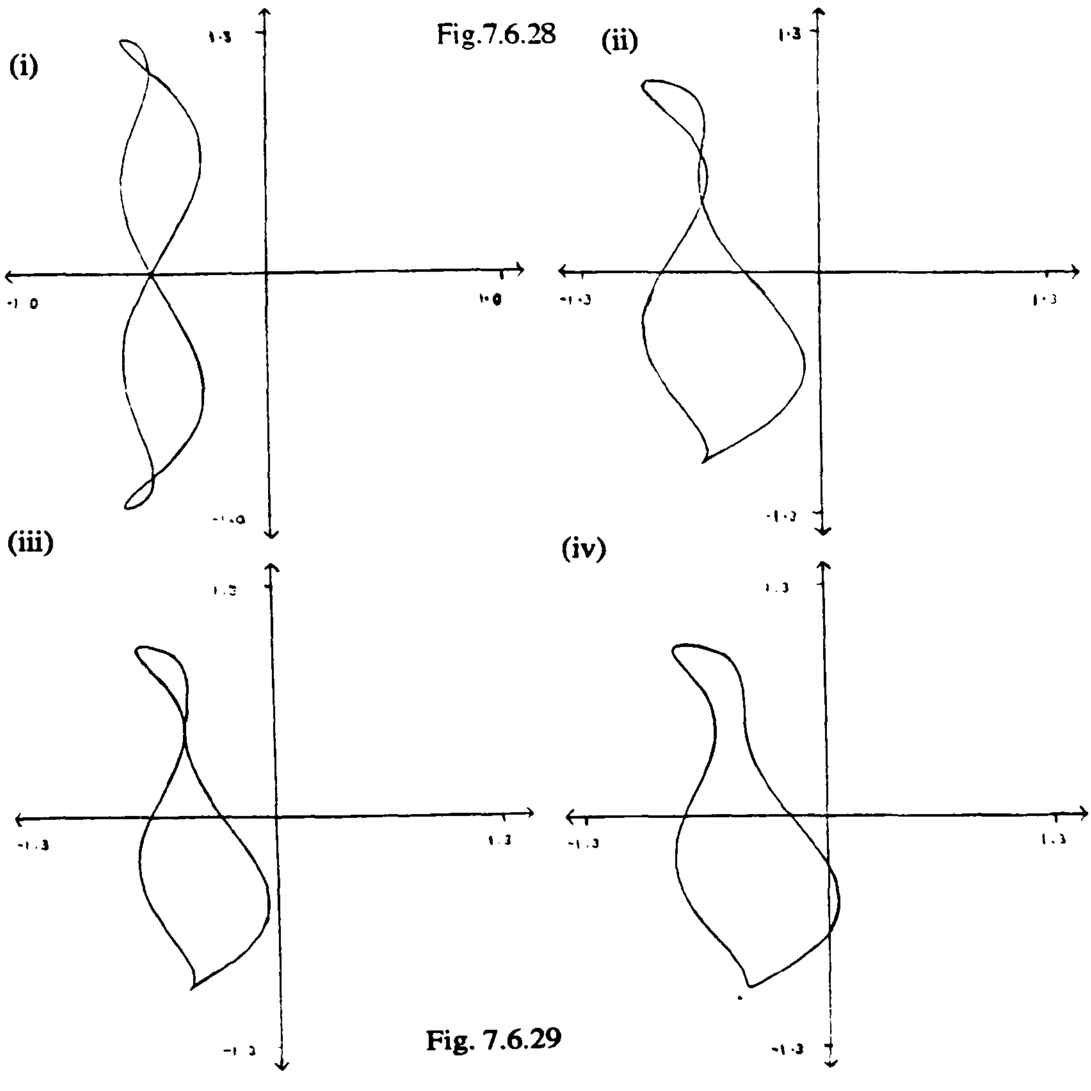



Fig. 7.6.29

**7.6.18 $H_3$**

The unfolding is $f_{a,b,c}(x,y) = (x, y^3 + ay, xy + y^8 + by^2 + cy^5)$. Again taking $\widetilde{f}(x,y) = (x, y^3 + ay)$ we find a presentation matrix with respect to $1, y, y^2$:

$$\lambda = \begin{pmatrix} -Z - 2aY^2 - caY & 3a^2Y + X + ca^2 & Y^2 - a^3 + b + cY \\ Y^3 - Ya^3 + Yb + Y^2c & -Z - 3aY^2 - 2acY + a^4 - ab & 3a^2Y + X + a^2c \\ 3a^2Y^2 + a^2cY + XY & Y(Y^2 - 4a^3 + b + cY) & -Z + a^4 - 2acY - 3aY \end{pmatrix}$$

The double point curve is:

$$\begin{aligned} &y^{14} + 5ay^{12} + 2cy^{11} + y^{10}(7a^2 + a^3) + 8acy^9 + y^8(c^2 + 5a^3 + 2a^4 + 2b) \\ &+ y^7(a^3c + x + 10a^2c) + y^6(3ac^2 + a^5 + 6ab + 11a^4) + y^5(8a^3c + 2a + a^4c + 2bc) \\ &+ y^4(a^3b + 4a^2c^2 + cx - a^6 + 3a^2b + 19a^5) + y^3(8a^4 + 7a^2x + 4abc) \\ &+ y^2(acx + b^2 + 3a^3b + 2a^3 + 2a^3c^2 + 5a^6) + y(3a^2bc + 3a^5c + 5a^3x + bx) \\ &+ x^2 + 2a^2cx + ab^2 + a^7 - 2a^4b + a^4c^2 = 0 \end{aligned}$$

Computer plots of this are not very accurate: in particular they tend to miss the singularities.

Once again, the co-ordinates of the cross-caps in the image are give by

$$\frac{\partial}{\partial y}(y^3 + ay) = \frac{\partial}{\partial y}(xy + y^8 + by^2 + cy^5) = 0$$

and we get $y = \pm\sqrt{-a/3}$. So there are two cross-caps in the image if $a < 0$.

Conditions for triple points (of which we would expect to see two) are given by the $1 \times 1$ determinants of $\lambda$. We have:

$$\begin{aligned} -Z - 2ay^2 - acY = 0 &\longrightarrow -xy - y^8 - 2ay^6 - cy^5 - 4a^2y^4 - y^2(b + 2a^3) - a^2cy = 0 \\ 3a^2Y + X + a^2c = 0 &\longrightarrow 3a^2y^3 + 3a^3y + x + a^2c = 0 \\ Y^2 - a^3 + b + cY = 0 &\longrightarrow y^6 + 2ay^4 + cy^3 + a^2y^2 + acy - a^3 + b = 0 \qquad (*) \end{aligned}$$

Consider the latter. In order to see two triple points we need this equation to have six distinct roots, so we must have the discriminant, $D$, greater than zero. $D$ is given by solving

$$y^6 + 2ay^4 + cy^3 + a^2y^2 + acy - a^3 + b = \frac{\partial}{\partial y}(y^6 + 2ay^4 + cy^3 + a^2y^2 + acy - a^3 + b) = 0$$

i.e. $y^6 + 2ay^4 + cy^3 + a^2y^2 + acy - a^3 + b = 6y^5 + 8ay^3 + 3cy^2 + 2a^2y + ac = 0$.
We get

$$D = -((31a^3 - 27b)^2 + (3a)^3(2c)^2)(c^2 + 4a^3 - 4b)^3$$

The first component of $D = 0$ is the folded umbrella, $B^2 = A^3C^2$, (see Arnold [A]) where we have changed co-ordinates as follows:

$$A = -3a$$
$$B = 31a^3 - 27b$$
$$C = 2c$$

Now $D = (B^2 - A^3C^2)(C^2/4 + 16/729A^3 + 4/27B)$. The surface $D = 0$ is shown in Fig. 7.6.30.

In order to find out what the image $f_{A,B,C}$ looks like in the different connected regions of the parameter space we take sections through the surface. If we encounter a particular kind of behaviour in a connected region of such a section we can assume that this behaviour occurs in the whole of that region of the parameter space.

Consider $C = 0$. In the $(A, B)$-plane we have $B = 0$ and $-4A^3 = 27B$ (see Fig. 7.6.32). We would like to know what happens to the image $f_{A,B,C}$ inside the various regions of the plane. When $C = 0$ the sextic $(*)$ reduces to

$$y^6 + 2ay^4 + a^2y^2 + b - a^3 = 0$$

Substituting $Y = y^2$, $A = -3a$ and $B = 31a^2 - 27b$ gives

$$Y(Y - \frac{1}{3}A)^2 - \frac{1}{27}(\frac{4}{27}A^3 + B) = 0$$

If we want to have roots for the original sextic we must have positive roots of this equation. We find that if

$$0 < \frac{1}{27}(\frac{4}{27}A^3 + B) < \frac{4}{729}A^3$$

(ie. the region between $B = 0$ and $-4A^3 = 27B$, with $A > 0$) then there are six real roots of the sextic which come together in the image as two triple points. Thus in $\mathbf{R}^3$, we will see two triple points in the image $f_{A,B,C}$ whenever we are in the small connected region between the folded umbrella and the other surface. If $A < 0$ or $A > 0$ and $4A^3 < -27B$ then the equation has no positive roots for $Y$, so there are no real roots for $y$, but if $A > 0$ and $4A^3 = -27B$ then we have roots $Y = 0$ and $Y = A/3$ and so we get three real roots for $y$. We might expect to see a single triple point in the image here. As the transition from six roots, through three roots to no roots occurs we would expect the double point curve to vary as shown in Fig. 7.6.31.

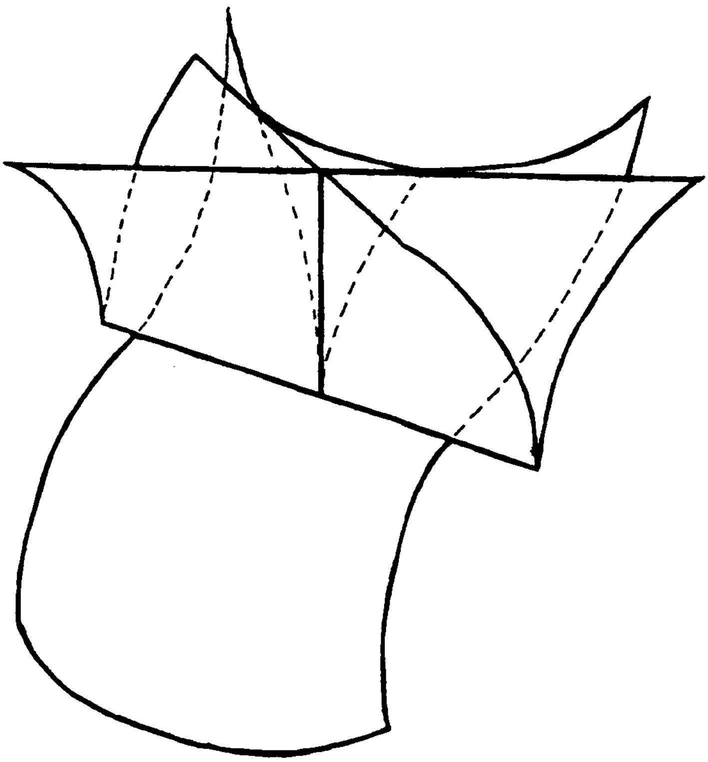


Fig.7.6.30

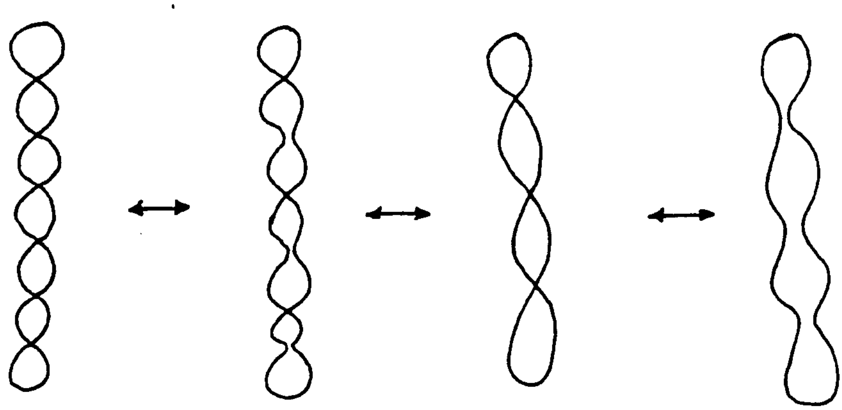


Fig.7.6.31

The only other parts of this section $C = 0$ to investigate are $A > 0$ with $B = 0$ and $A > 0$ with $B > 0$. If $B = 0$ then the equation has a repeated root at $Y = A/9$ and another positive root, so there are four roots for $y$. If $B > 0$ then there is only a single positive root for $Y$ and so two roots for $y$. So we will see no triple points, but there should be two distinguished points in the image.

Now take the section $A = 0$. In the $(B, C)$-plane we see $B = 0$ and $16B = -27C^2$ ( see Fig. 7.6.33). The sextic $(*)$ reduces to $y^6 + \frac{C}{2}y^3 - \frac{B}{27} = 0$, so it will have two real roots if $16B > -27C^2$, one if $16B = -27C^2$ and otherwise no roots.

Finally we take the section $A = 3$. In the $(B, C)$-plane we see $B^2 = 27C^2$ and $B = \frac{-27}{16}C^2 - 4$ (see Fig. 7.6.34). They are tangent at the points $(-8, 8/3\sqrt{3})$ and $(-8, -8/3\sqrt{3})$. We already know that in the top region we expect two roots for the sextic (and thus two distinguished points), in the small central region we expect six roots (and so two triple points), on the parabola below this three roots, and in the bottom region no roots. We want to look at the side regions, and since these are the same on either side of the $B$-axis we may as well consider $C > 0$. Choose the point $B = 0$, $C = 2$. Then $a = -1$, $b = -\frac{31}{27}$ and $c = 1$, and the sextic is

$$(y^3 - y)^2 + (y^3 - y) - \frac{4}{27} = 0$$

This gives two roots for $y^3 - y$, and since in either case the discriminant of this cubic is negative, there are two roots for $y$.

Bringing these results together, we know that if $a < 0$ and hence $A > 0$ then there will be two cross-caps in the image, and otherwise none. If we are in the small region between the two parts of the surface $D = 0$ then we will have two triple points and two cross-caps in the image, passing to one triple point and two cross-caps ($H_2$) and then no triples and two cross-caps ($S_1$), or to two distinguished points and two cross-caps (also $S_1$). This accords with the adjacency diagram Fig. 7.6.1, which tells us that we would expect $H_3$ to deform to $H_2$, $S_1$, $S_0$ or an immersion.

A picture of an unfolding of $H_3$, as drawn on the Iris workstation, is reproduced in Appendix A (Fig. 12), but it will be observed that this is quite difficult to interpret. It is easier to understand the pictures in Fig.7.6.35 (drawn by Ton Marar), showing the steps in the construction of the image $f_{a,b,c}$, to have an idea of the relationship between the triple points and the cross-caps in the image.

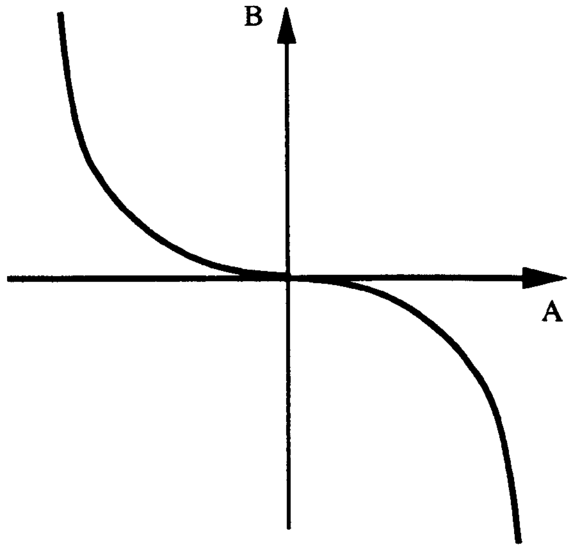



Fig. 7.6.32

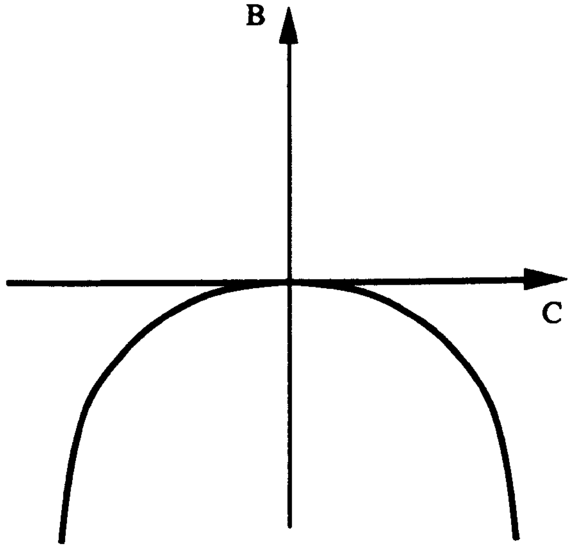



Fig. 7.6.33

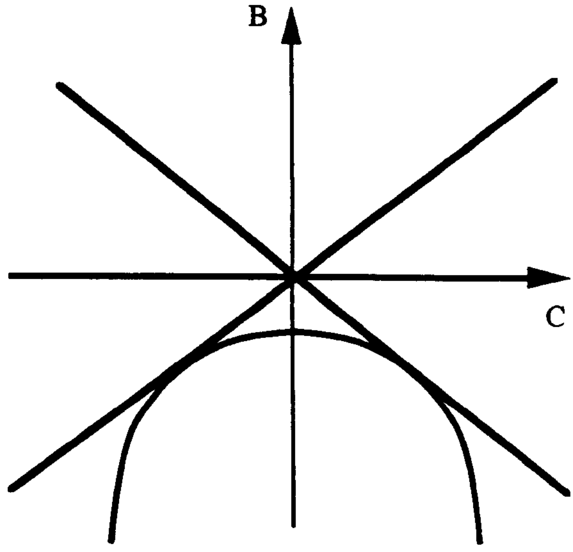



Fig. 7.6.34

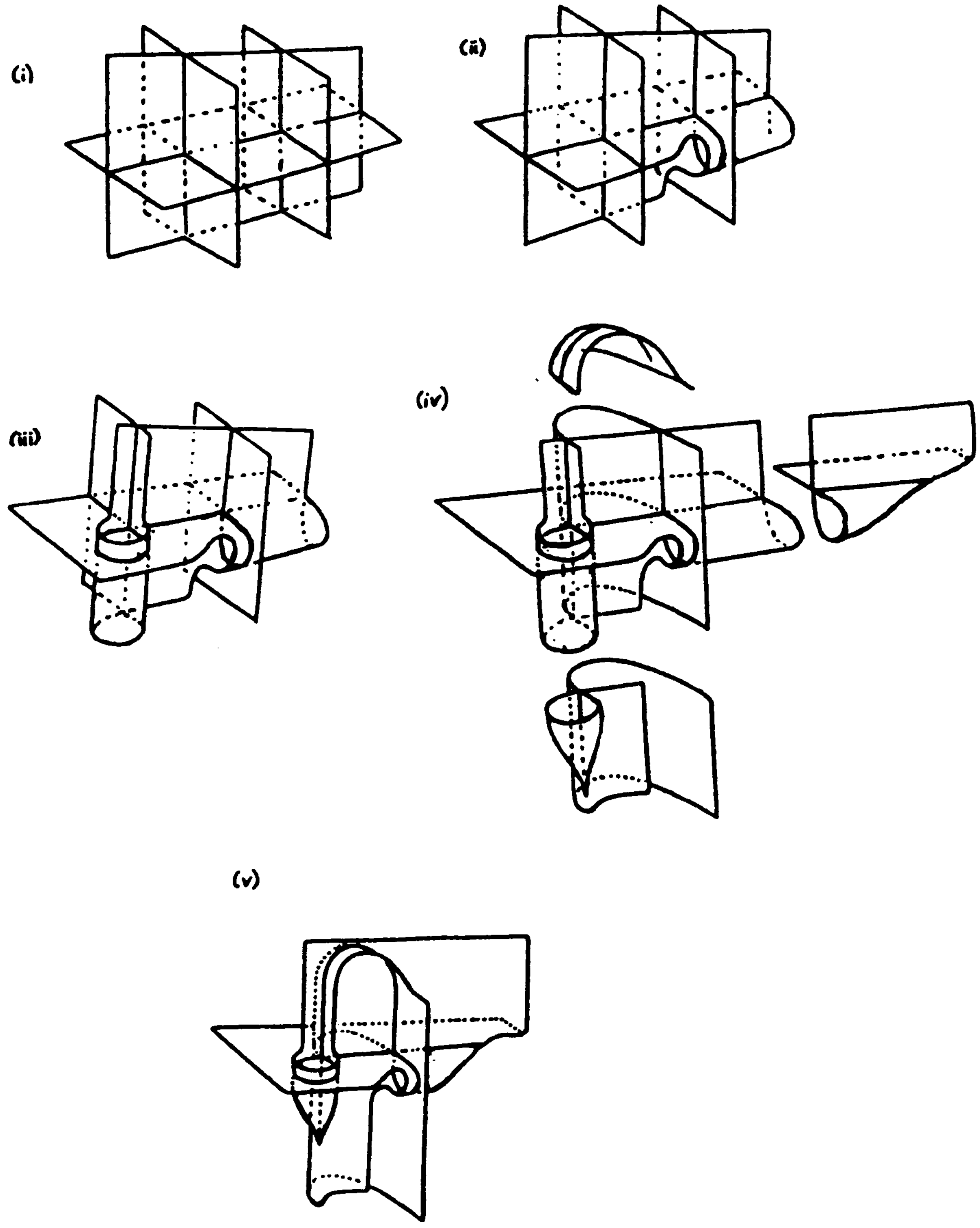



Fig.7.6.35

### 7.6.19 $P_3$

$P_3$ is a unimodular family of germs given by

$$f(x,y) = (x, xy + y^3, xy^2 + cy^4)$$

It has been shown by Mond [Md1] that unless $c = 0$, $1/2$, $1$ or $3/2$ this germ is $\mathcal{A}$-finite. The unfolding (of the stratum) is $f_{a,b,d}(x,y) = (x, xy + y^3 + ay, xy^2 + cy^4 + by + dy^3)$, where $c \neq 0, 1/2, 1, 3/2$. We may see three cross-caps and a triple point in the image of the unfolding, though this is not necessarily the case (see Remark 7.6.2). We also note that the invariant $\mu(\widetilde{D}^2/\mathbf{Z}_2)$ is non-zero in this case.

The presentation matrix, with respect to the generators $1, y, y^2$, is

$$\begin{pmatrix} -Z + dY & H(X,Y) & G(X) \\ YG(X) & -Z + dY - (a+X)G(X) & H(X,Y) \\ YH(X,Y) & YG(X) - (a+X)H(X,Y) & -Z + dY - (a+X)G(X) \end{pmatrix}$$

where $G(X) = X(1-c) - ac$ and $H(X,Y) = cY + b - d(a+X)$.

Then the double point curve in the source is given by

$$\begin{aligned} &c^2y^6 + cxy^4(3c-1) + 3ac^2y^4 + x^2y^2(c^2+c-1) - 2cdxy^3 \\ &+ 2cy^3(b-ad) + acxy^2(1+2c) + x^2y(1-2cd-c) + x^3(c-1)^2 \\ &+ a^2c^2y^2 + xy(b(3c-1) + a(d-c-5cd))x^2((3c-1)(c-1)+d^2) \\ &+ 3acy(b-ad) + x(a^2c(3c-2) + 2d(ad-b)) + (ad-b)^2 + a^3c^2 = 0 \end{aligned}$$

This can be plotted by computer but again it does not provide us with very useful information.

To find the triple point in the image we look at the independent entries in the matrix. These are

$$\begin{aligned} -Z + dY &= 0 \\ cY + b - d(a+X) &= 0 \\ X(1-c) - ac &= 0 \end{aligned}$$

and they yield the cubic

$$c(1-c)y^3 + acy + b(1-c) - ad = 0$$

For a triple point, we want this to have three real roots. The surface in $(a,b,d)$-space which separates three real roots from one is given by the discriminant, which is

$$27(1-c)(b(1-c)-ad)^2 - 4a^3c^2 = 0$$

We shall refer to this as the triple-point surface. If we rescale $a$ and $b$ we see that it is a cuspidal edge (which does not depend on $d$):

$$27B^2 = -4A^3$$

So the cuspidal edge is the $d$-axis.

Now we look for the conditions for cross-caps. We must solve

$$x + 3y^2 + a = 0$$
$$2xy + 4cy^3 + b + 3dy^2 = 0$$

These give the cubic

$$y^3(4c-6) + 3dy^2 - 2ay + b = 0$$

This cubic has three real roots (and hence gives rise to three cross-caps) if the discriminant, $D$, is greater than zero. Noting that $c \neq 3/2$, we find that $D = 0$ is given by

$$3(3(2c-3)b + ad)^2 + (4a(2c-3) + 3d^2)(9bd - 4a^2) = 0$$

This defines another surface in $(a, b, d)$-space which we shall call the cross-cap surface. Again, it has a cuspidal edge. Recall that for a general cubic $Ay^3 + 3By^2 + 3Cy + D$ the discriminant surface is defined by $(AD - BC)^2 = 4(AC - B^2)(BD - C^2)$ and its singular set is the twisted cubic defined by $AD - BC = 0$, $AC - B^2 = 0$ and $BD - C^2 = 0$. Here we have $A = 2(2c-3)$, $B = d$, $C = -2a/3$ and $D = b$ and the cuspidal edge is defined by

$$0 = -\frac{4}{3}a(2c-3) - d^2$$
$$0 = 2b(2c-3) + \frac{2}{3}ad$$
$$0 = bd - \frac{4}{9}a^2$$

Now we would like to know how the triple point surface and the cross-cap surface intersect in $(a, b, d)$-space. It is clear that the cross-cap cuspidal edge meets the cuspidal edge of the triple point surface (the $d$-axis) if and only if $d = 0$. So the two edges intersect only at the origin in $(a, b, d)$-space independent of $c$.

We can also easily investigate how the cuspidal edge of the cross-cap surface intersects the triple point surface itself. If we parametrize the cuspidal edge of the cross-cap surface by $(sy - t)^3$ then

$$s^3 = 2(2c-3), \qquad 3s^2t = 3d, \qquad 3st^2 = -2a, \qquad t^3 = b$$

So we have defined $s$ uniquely in terms of $c$ and $a, b, d$ in terms of $t$. Substitution into the equation for the triple point surface yields

$$(1-c)(5c-8)^2 = c^2(2c-3)$$

(provided $t \neq 0$). This simplifies to $(3c-4)^3 = 0$. So if $c = \frac{4}{3}$ then the whole cuspidal edge of the cross-cap surface is contained in the triple point surface. Otherwise they simply intersect at the origin, as mentioned above. In fact the cuspidal edges are tangent here with the tangent line to both edges being the $d$-axis.

Now we come to the question of how the two whole surfaces intersect. First we note that both surfaces are invariant under the following $\mathbf{R}^*$-action on $(a, b, d)$-space

$$(a, b, d; \lambda) \longmapsto (\lambda^2 a, \lambda^3 b, \lambda d)$$

(where $\lambda \neq 0$). This can be checked by direct substitution in the equations. Thus both surfaces are foliated by twisted cubics (which may degenerate). In particular, if the two surfaces intersect at a point, then they intersect along a whole $\mathbf{R}^*$-orbit. So if we want to see how the two surfaces intersect in $(a, b, d)$-space it is equivalent to look at how they intersect in a general plane, then we just have to count the real intersections of two algebraic curves. We choose the plane $d = 1$ and obtain the curves:

$$3(3(2c-3)b+a)^2 = -(4a(2c-3)+3)(9b-4a^2) \qquad (CC)$$

$$-27(1-c)((1-c)b-a)^2 = 4c^2a^3 \qquad (TP)$$

(where $CC$ denotes the cross-cap curve and $TP$ denotes the triple point curve).

As they are cubics we expect that at worst they will intersect in 9 real points. Now TP is a cuspidal cubic with cusp at the point $(0,0)$ and cuspidal tangent $(1-c)b - a = 0$. Since $(0,0)$ is also on CC but is non-singular with tangent $b = 0$ there are 2 intersections at the point $(0,0)$. Both curves meet the line at infinity in the point $(0;1;0)$. Since this is a simple point of both curves they must each have a flex at this point, so the intersection multiplicity is at least 3. Thus there are 4 or less real finite intersections of the two curves away from $(0,0)$. The exact number, 0, 2 or 4, will depend on the value of $c$. In order to find the number of roots we note that TP is rational so we can parametrize it and then substitute back into the equation for CC. Consider the pencil of lines $b = \lambda a$ through the cusp $(0,0)$. Substitution into TP gives

$$-27(1-c)((1-c)\lambda - 1)^2 a^2 = 4c^2a^3$$

so either $a^2 = 0$ or $a = -27(1-c)((1-c)\lambda - 1)^2/4c^2$. Then $(a, b)$ lies on CC if $\lambda$ satisfies the following quartic

$$(3(c-1)(c-2)\lambda + (5c-6))^2(P\lambda^2 + Q\lambda + R) = 0$$

where $P = 3(c-1)(2c-3)(2c-1)$, $Q = 2(2c-3)(5c-3)$ and $R = 9(c-1)$. Clearly this always has a repeated root. The quadratic part has discriminant $\alpha(2c-3)c^3$, where $\alpha$ is a constant. So there will always be a repeated root of the quartic and if $0 < c < 3/2$ there will be two other real roots. One of these roots coincides with the repeated root if $c = 0$ or $c = 4/3$.

We also note that CC is a cuspidal cubic, for non-exceptional values of $c$. Pictures of the two curves for values of $c$ in the different ranges are shown in Fig. 7.6.36. Computer generated pictures of the two cuspidal edges intersecting, drawn on a Silicon Graphics Iris workstation using Richard Morris' program ***algsurf***, do confirm these calculations and pictures.

We observe that the value $c = 4/3$ appears to have a geometrical significance not previously observed; it is not one of the exceptional values listed in [Md1] but it is the one value of $c$ for which the whole cuspidal edge of the cross-cap surface intersects the triple point surface and it is the only value of $c$ for which TP is cuspidally tangent to CC.

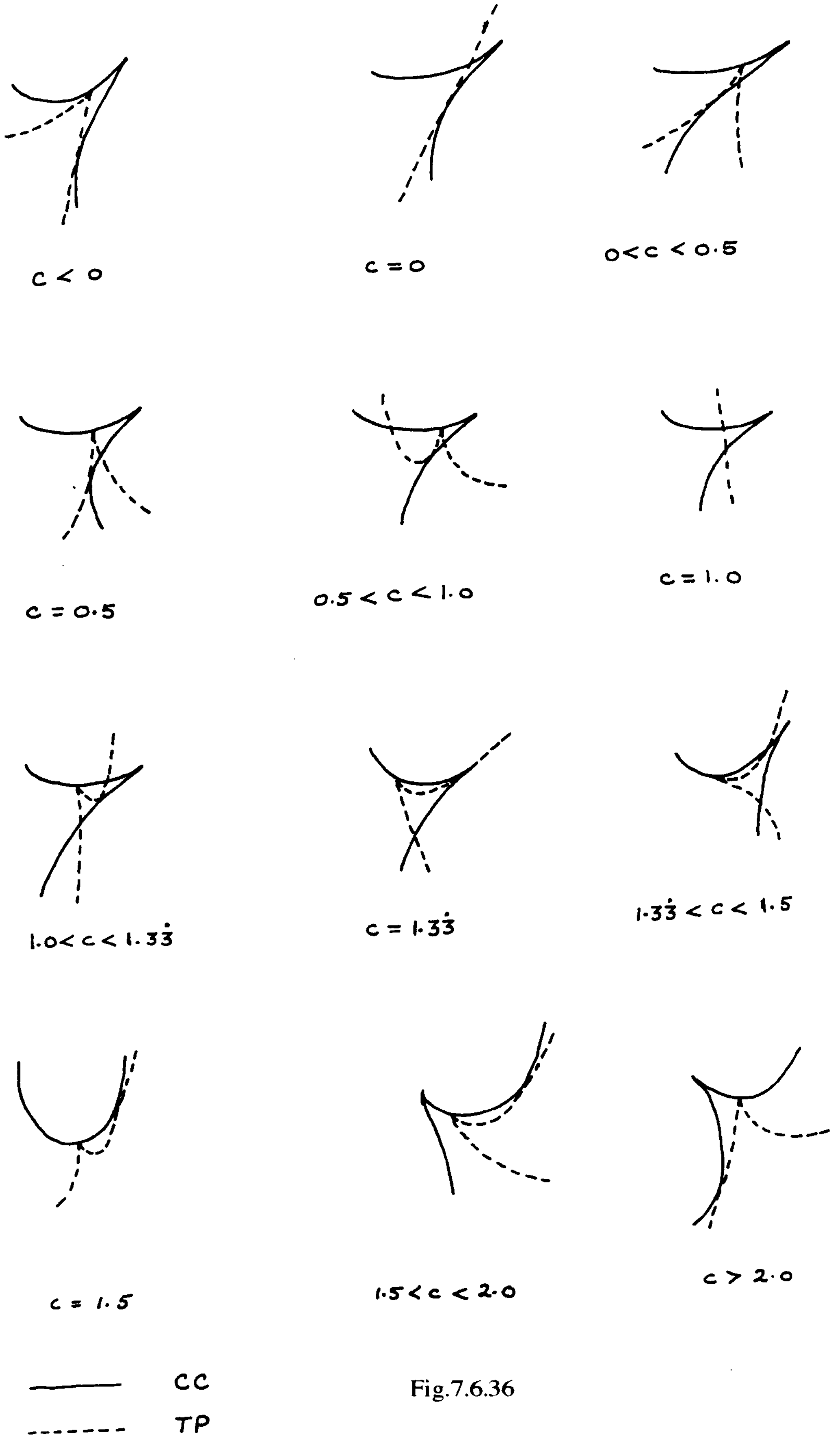

c < 0
c = 0
0 < c < 0.5
c = 0.5
0.5 < c < 1.0
c = 1.0
1.0 < c < 1.3$\dot{3}$
c = 1.3$\dot{3}$
1.3$\dot{3}$ < c < 1.5
c = 1.5
1.5 < c < 2.0
c > 2.0
CC
TP

Fig.7.6.36

## 7.7 Unfoldings of Bi-germs $\mathbf{R}^2, 0 \to \mathbf{R}^3, 0$

We give a table of the unfoldings of the bigerms on our list; again the detailed calculations may be found in Appendix B.2.

| | Normal Form | Unfolding |
|---|---|---|
| | $(x,y,0;0,X,Y)$ | stable |
| $(I)$ | $(x,y,0;X,Y,X^2 \pm Y^2)$ | $(x,y,0;X,Y,X^2 \pm Y^2 + a)$ |
| $(II)$ | $(x,y,0;X,Y,X^2 + Y^3)$ | $(x,y,0;X,Y,X^2 + Y^3 + a + bY)$ |
| $(III)$ | $(x,y,0;X,Y,X^2 \pm Y^4)$ | $(x,y,0;X,Y,X^2 \pm Y^4 + a + bY + cY^2)$ |
| $(IV)$ | $(x,y,0;Y^2,XY + Y^3,X)$ | $(x,y,0;Y^2,XY + Y^3 + aY,X)$ |
| $(V)$ | $(x,y,0;Y^2,XY + Y^5,X)$ | $(x,y,0;Y^2,XY + Y^5 + aY + bY^3,X)$ |
| $(VI)$ | $(x,y,0;Y^2,XY + Y^7,X)$ | $(x,y,0;Y^2,XY + Y^7 + aY + bY^3 + cY^5,X)$ |
| $(VII)$ | $(x,y,0;Y^3 \pm X^2Y,Y^2,X)$ | $(x,y,a;Y^3 \pm X^2Y + bY,Y^2,X)$ |
| $(VIII)$ | $(x,y,0;Y^3 + X^3Y,Y^2,X)$ | $(x,y,a;Y^3 + X^3Y + bY + cXY,Y^2,X)$ |
| $(IX)$ | $(x,y,0;X,XY,Y^2 + X^3)$ | $(x,y,0;X,XY,Y^2 + X^3 + a + bX + cX^2)$ |
| $(X)$ | $(x,y,0;X,Y^2,XY + Y^4)$ | $(x,y,0;X,Y^2,XY + Y^4 + a + bX + cY^2)$ |

$(I)^+$ The unfolding is $F_a(x,y;X,Y) = (x,y,0;X,Y,X^2 + Y^2 + a)$. The first branch is the $x,y$-plane and the second is an immersion with second order contact with this plane. The intersection curve of the plane with the other surface is $X^2 + Y^2 + a = 0$. The bifurcation set is $a = 0$. If $a > 0$ then the two branches do not intersect and if $a < 0$ they intersect in a circle. See Fig.7.7.1.

$(I)^-$ The unfolding is $F_a(x,y;X,Y) = (x,y,0;X,Y,X^2 - Y^2 + a)$. Again we have the $x,y$-plane and an immersion with second order contact. The intersection curve of the plane with the other surface is $X^2 - Y^2 + a = 0$. As in $(I)^+$ the bifurcation set is $a = 0$. See Fig. 7.7.2.

(II) The unfolding is $F_{a,b}(x,y;X,Y) = (x,y,0;X,Y,X^2 + Y^3 + a + bY)$. This time the two immersive branches have third order contact. The intersection curve is $X^2 + Y^3 + a + bY = 0$ and the bifurcation set is given by the discriminant of the cubic $Y^3 + a + bY = 0$ being zero, as in the case of $S_2$. This gives the

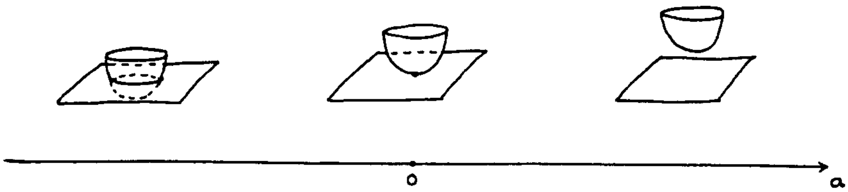



Fig. 7.7.1

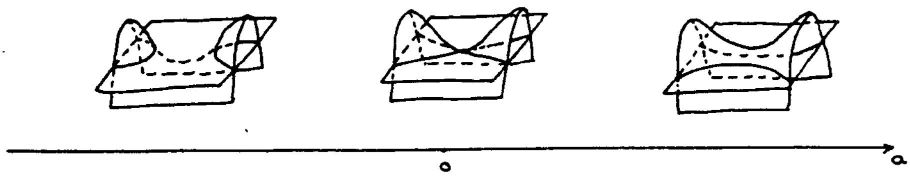



Fig. 7.7.2

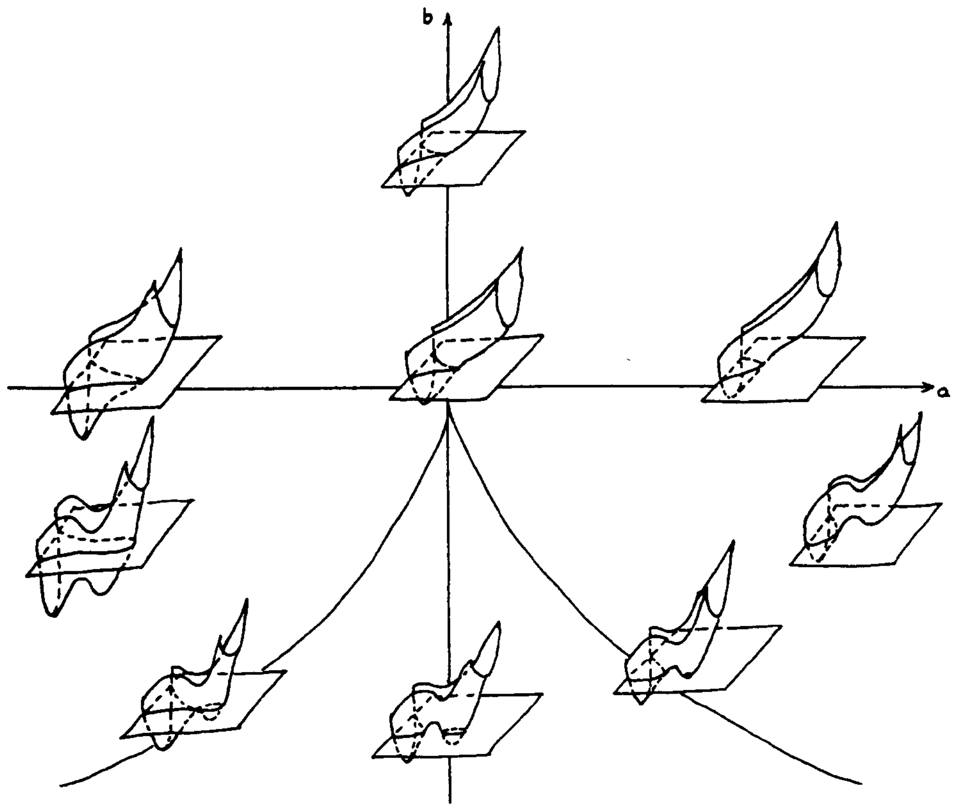



Fig. 7.7.3

cuspidal cubic $27a^2 = -4b^3$ as the bifurcation set, and the intersection curve is a cubic which varies in just the same way as the double point curve of $S_2$. See Fig. 7.7.3.

$(III)^+$ The bi-germ consists of two immersions with fourth order contact and the unfolding is $F_{a,b,c}(x,y;X,Y) = (x,y,0;X,Y,X^2+Y^4+a+bY+cY^2)$. Then the intersection curve is $X^2 + Y^4 + cY^2 + bY + a = 0$ which will have four real distinct roots (yielding two ellipses) when the discriminant, $D$, of the quartic $Y^4 + cY^2 + bY + a = 0$ is positive. Now $D = 0$ gives the swallowtail surface (as in the case of $S_3$), drawn in Fig. 7.7.4. 'Inside' the surface we get $D > 0$ and so four distinct roots; when $D = 0$ we get repeated roots and when $D < 0$ we either have two distinct roots (giving an ellipse as the intersection curve) or no roots (so no intersection curve). See Fig. 7.7.4.

$(III)^-$ The unfolding is $F_{a,b,c}(x,y;X,Y) = (x,y,0;X,Y,X^2 - Y^4 + a + bY + cY^2)$. Again, we have two immersion meeting with fourth order contact. The intersection curve is $X^2-Y^4+cY^2+bY+a = 0$. As above, this will have four real distinct roots when the discriminant, $D$, of the quartic $-Y^4+cY^2+bY+a = 0$ is positive. $D = 0$ gives the swallowtail surface. See Fig. 7.7.5.

(IV) The unfolding is $F_a(x,y;X,Y) = (x,y,0;Y^2,XY+Y^3+aY,X)$. This time the two branches are the $x,y$-plane and a cross-cap meeting transversely, where the double point curve of the cross-cap does not lie in the plane. The intersection curve is given by

$$\begin{aligned} x &= Y^2 \\ y &= XY + Y^3 + aY \\ 0 &= X \end{aligned}$$

so the curve is $(Y^2, Y^3 + aY, 0)$, a cuspidal cubic when $a = 0$. The bifurcation set is $a = 0$. We see the usual cusp transitions as $a$ varies. See Fig. 7.7.6.

(V) The unfolding is $F_{a,b}(x,y;X,Y) = (x,y,0;Y^2,XY + Y^5 + aY + bY^3,X)$. The second branch is a cross-cap whose double point curve is a cubic curve. The intersection curve with the $x,y$-plane is given by

$$\begin{aligned} x &= Y^2 \\ y &= XY + Y^5 + aY + bY^3 \\ 0 &= X \end{aligned}$$

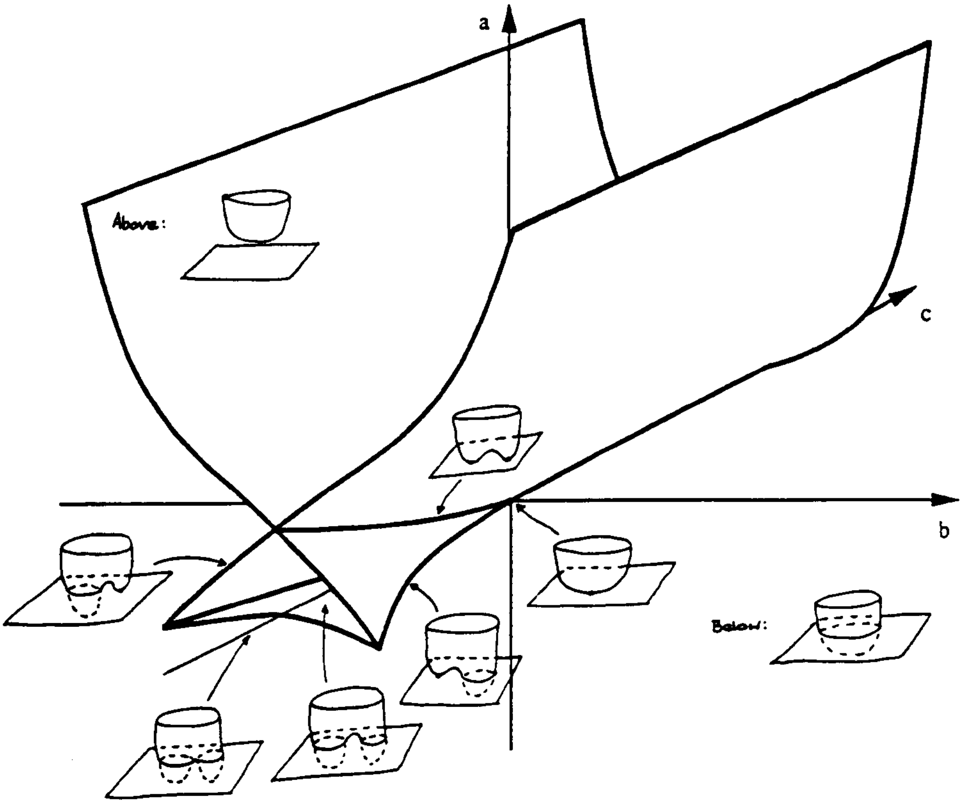



Fig.7.7.4

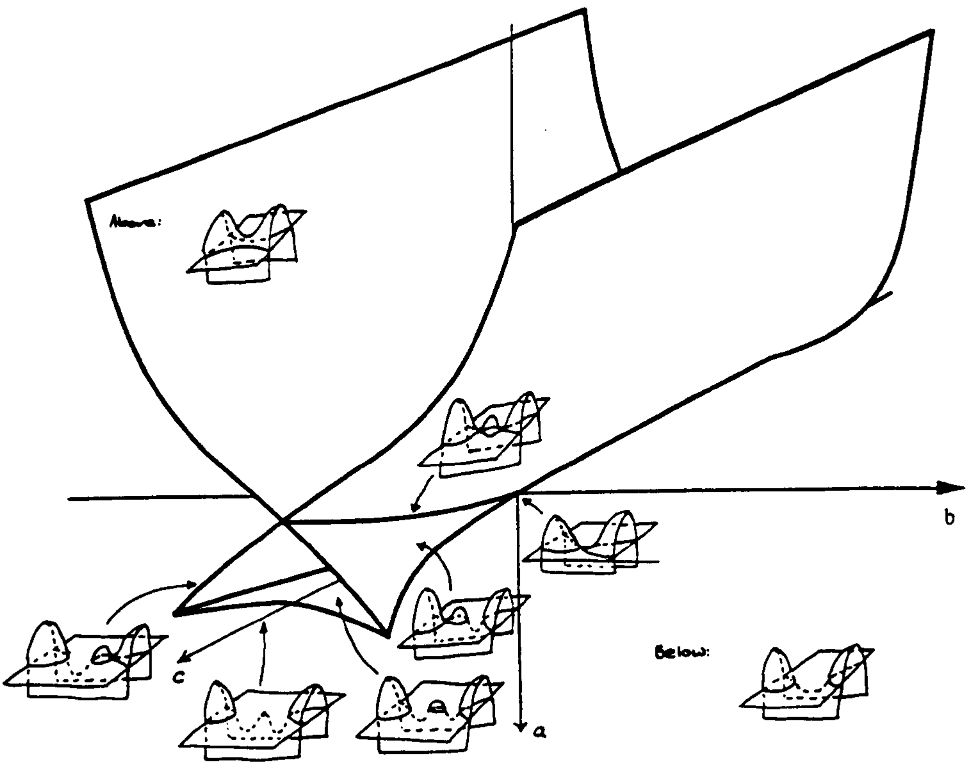



Fig.7.7.5

so the curve is $(Y^2, Y(Y^4 + a + bY^2), 0)$, a rhamphoid cusp when $a = b = 0$. The maximum number of distinct real roots occurs when $Y(Y^4 + bY^2 + a)$ has five real roots. So we need $b^2 - 4a > 0$ and also positive roots for $Y^2$, ie. $b < 0$. Thus the bifurcation set is $b^2 = a$ where $b < 0$, and we see the usual rhamphoid unfolding. See Fig. 7.7.7.

(VI) The unfolding is $F_{a,b,c}(x, y; X, Y) = (x, y, 0; Y^2, XY + Y^7 + aY + bY^3 + cY^5, X)$. This time the cross-cap has an even more complicated double point curve. The intersection curve with the $x, y$-plane is given by

$$x = Y^2$$
$$y = XY + Y^7 + aY + bY^3 + cY^5$$
$$0 = X$$

so the curve is $(Y^2, Y(Y^6 + a + bY^2 + cY^4), 0)$. In order to see the maximum number of roots of the sextic $Y^6 + a + bY^2 + cY^4$ we need the discriminant of the cubic $Z^3 + cZ^2 + bZ + a$ (put $Z = Y^2$) greater than zero, and we also require positive roots for $Z$. These yield the 'half cuspidal-edge' (c.f. $B_3$)

$$D = 27a^2 - 18abc + 4ac^3 - b^2c^2 + 4b^3$$

where $b < 0$ or $b \geq 0$, $c < 0$ and $c^2 - 3b > 0$. We also note that when $a = 0$ the curve $y = Y(Y^6 + a + bY^2 + cY^4)$ has a factor $Y^3$. So $y = 0$ will have a cusp when $a = 0$ and this is another part of the bifurcation set. See Fig. 7.7.8. 'Inside' the smallest region we actually see three loops of the curve - a further degeneration from the rhamphoid cusp unfolding. On the surface $D = 0$ we always see tacnodes and on the plane $a = 0$ we always see cusps, as previously noted. When $a < 0$ and $D > 0$ the curve is a crunodal cubic, and when $a > 0$ and $D > 0$ we see an acnodal cubic.

$(VII)^+$ The unfolding is $F_{a,b}(x, y; X, Y) = (x, y, a; Y^3 + X^2Y + bY, Y^2, X)$. The second branch is the mono-germ $S_1^+$. This develops two cross-caps if $b < 0$ and has none if $b > 0$ so the line $b = 0$ is part of the bifurcation set. The intersection curve with the plane is given by

$$x = Y^3 + X^2Y + bY$$
$$y = Y^2$$
$$a = X$$

so the curve is $(Y^3 + Y(b + a^2), Y^2, a)$. We have the usual cusp transition, and the rest of the bifurcation set will be given by $b + a^2 = 0$. On this curve we will

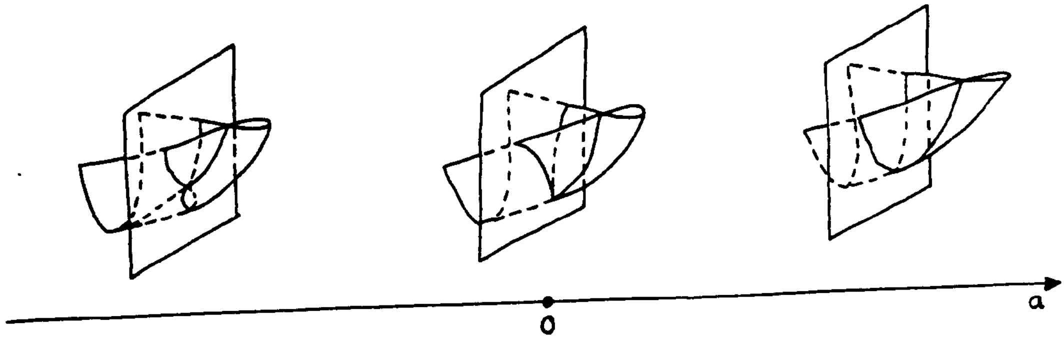



Fig. 7.7.6

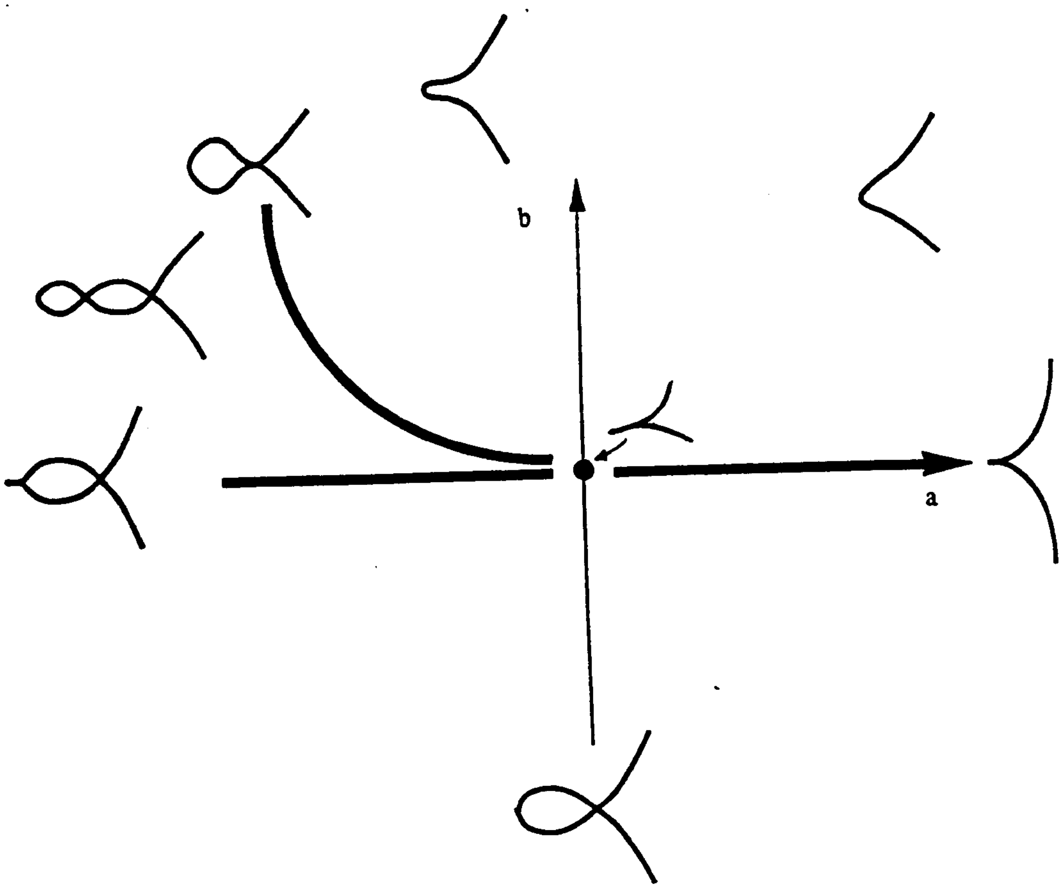



Fig. 7.7.7

see a cusp; when $b + a^2 > 0$ we will have a crunodal cubic and when $b + a^2 < 0$ we will have an acnodal cubic. When $b > 0$ the bi-germ will just consist of two transverse immersions. See Fig. 7.7.9.

$(VII)^-$ The unfolding is $F_{a,b}(x, y; X, Y) = (x, y, a; Y^3 - X^2Y + bY, Y^2, X)$. The second branch of this bi-germ is $S_1^-$ and so, as above, the line $b = 0$ will be part of the bifurcation set. The intersection curve with the plane is given by

$$x = Y^3 - X^2Y + bY$$
$$y = Y^2$$
$$a = X$$

so the curve is $(Y^3 + Y(b - a^2), Y^2, a)$. Again, we have the cusp transition and the rest of the bifurcation set will be given by $b - a^2 = 0$. See Fig. 7.7.10.

(VIII) The unfolding is $F_{a,b,c}(x, y; X, Y) = (x, y, a; Y^3 + X^3Y + bY + cXY, Y^2, X)$ The second branch is $S_2$ which has bifurcation set the cuspidal edge $27b^2 = 4c^3$ (see earlier in this chapter). The intersection curve with the plane is given by

$$x = Y^3 + X^3Y + bY + cXY$$
$$y = Y^2$$
$$a = X$$

so the curve is $(Y^3 + Y(a^3 + b + ac), Y^2, a)$. Once again, the only transitions of this curve we expect to see are the ordinary cusp transitions. The rest of the bifurcation surface is clearly $a^3 + b + ac = 0$, the cusp catastrophe surface. The two parts of the bifurcation set are tangent (c.f. $C_3^{\pm}$). On the cusp catastrophe surface the two branches always meet in a cusp, but this coincides with different cross-caps in the unfolding of $S_2$. Inside the cuspidal edge $S_2$ has three cross-caps so there are three positions for the plane to meet $S_2$ in a cusp, while outside the cuspidal edge there is only one cross-cap on $S_2$, so only one chance for a cusp as the plane varies with the value of $a$. See Fig. 7.7.11.

(IX) The unfolding is $F_{a,b,c}(x, y; X, Y) = (x, y, 0; X, XY, Y^2 + X^3 + a + bX + cX^2)$. The second branch is a cross-cap which has third order contact with the $x, y$-plane. The intersection of the two surfaces in

$$x = X$$
$$y = XY$$
$$0 = Y^2 + X^3 + a + bX + cX^2$$

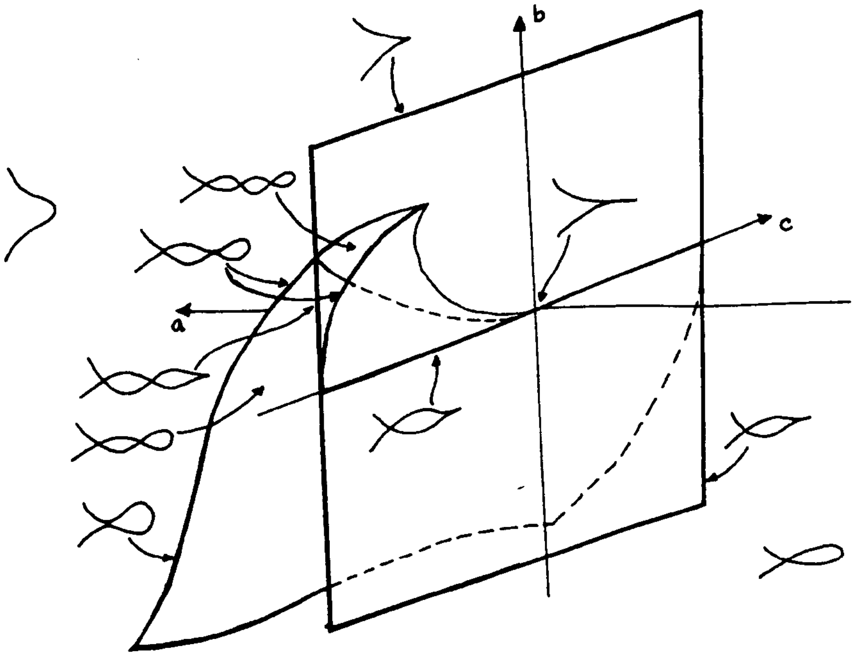



Fig.7.7.8

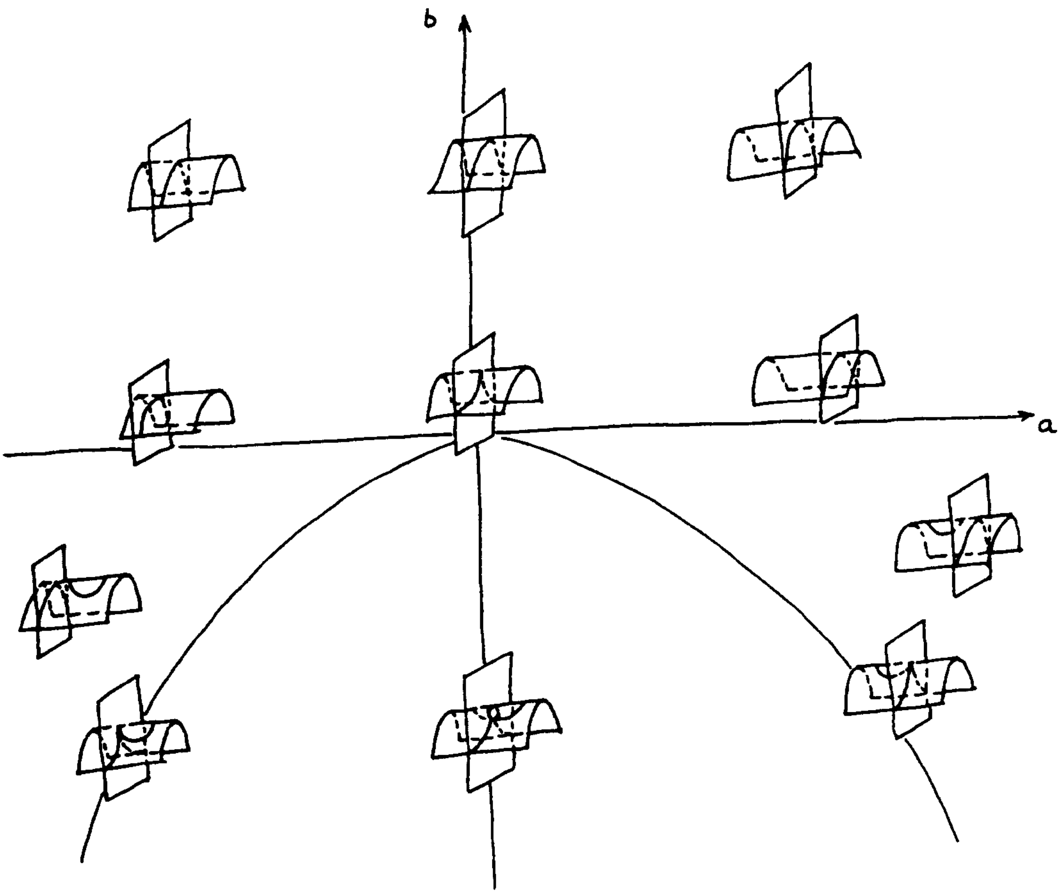



Fig.7.7.9

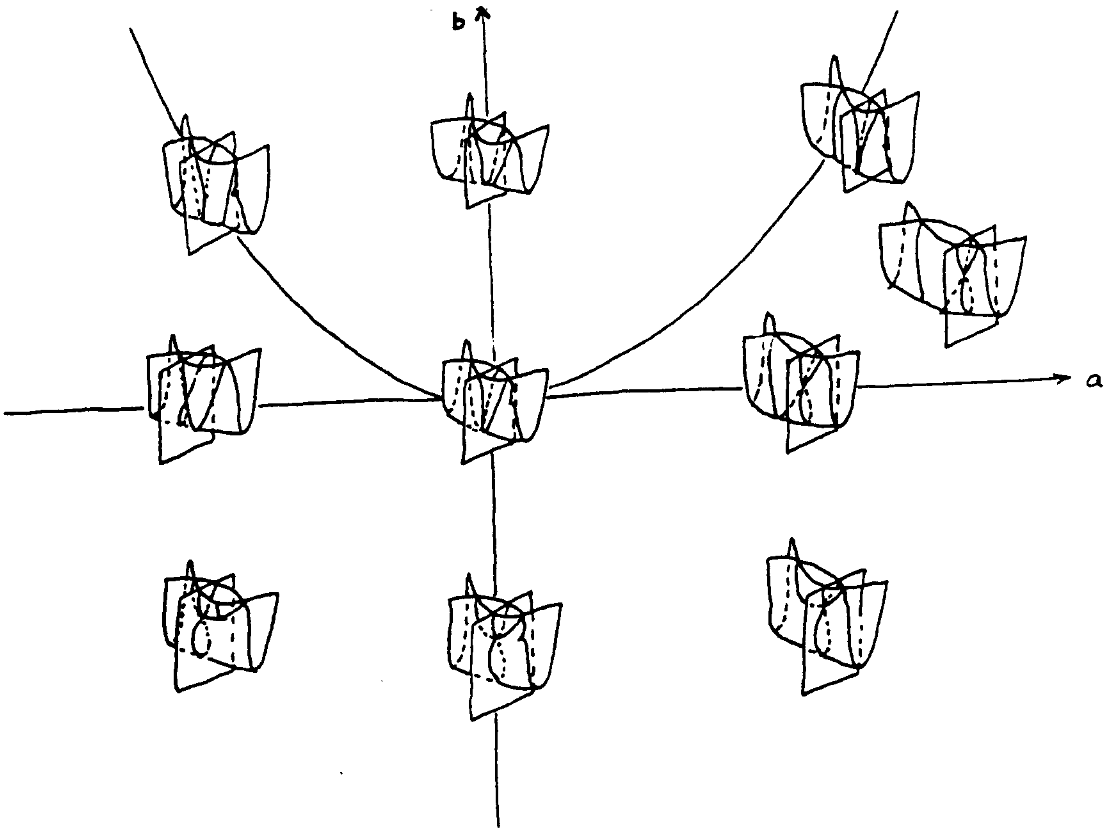



Fig.7.7.10

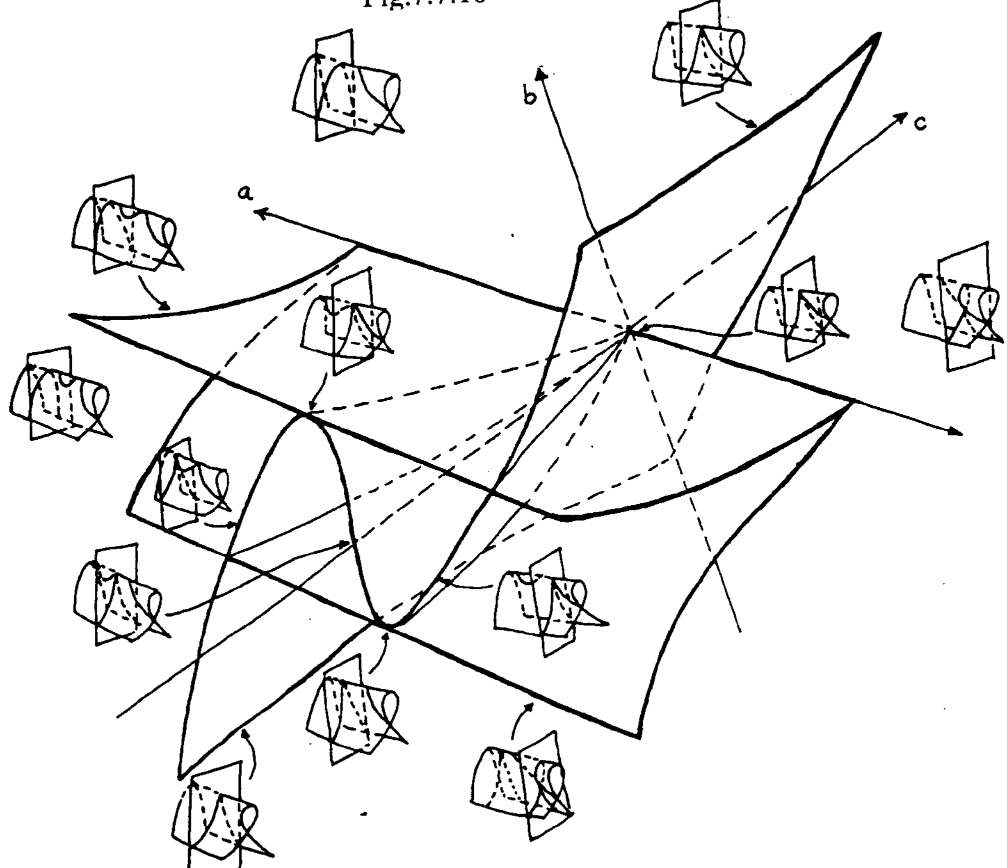



Fig.7.7.11

so the curve of intersection is the quintic

$$y^2 = -x^2(x^3 + cx^2 + bx + a)$$

unless $x = 0$. Note that if $x = 0$ then $y^2 = -a$, so the plane $a = 0$ is part of the bifurcation set. The discriminant of the cubic factor, $x^3 + cx^2 + bx + a$, of the quintic above is given by

$$D = -18abc + 4ac^3 + 4b^3 - b^2c^2 + 27a^2.$$

$D = 0$ is the twisted cuspidal edge surface. The bifurcation set is thus the union of $D = 0$ and the plane $a = 0$. At the origin there is a sharp cusp. On $D = 0$ the curve has at least one tacnode (though if $a > 0$ these tacnodes appear as points). On the plane $a = 0$ the intersection curve always has a cusp, though it may have more than one component. The bifurcation set and transitions in the intersection curve are shown in Fig.7.7.12.

(X) The bi-germ has unfolding $F_{a,b,c}(x, y; X, Y) = (x, y, 0; X, Y^2, XY + Y^4 + a + bX + cY^2)$ and the second branch is a cross-cap which is tangent to the $x, y$-plane. The double point curve of the cross cap is tangent to the $x, y$-plane at the origin. The two surfaces meet in

$$\begin{aligned} x &= X \\ y &= Y^2 \\ 0 &= XY + Y^4 + a + bX + cY^2 \end{aligned}$$

These yield

$$x^2y = (bx + a + y(y + c))^2$$

which can plotted by computer. Clearly if $a = b = c = 0$ then the curve is $x^2y = y^4$ so $y = 0$ or $x^2 = y^3$. Note that if $a = 0$ then the curve is always singular so the plane $\{a = 0\}$ is part of the bifurcation set. There must be other parts to this bifurcation set but so far analysis has not revealed them.

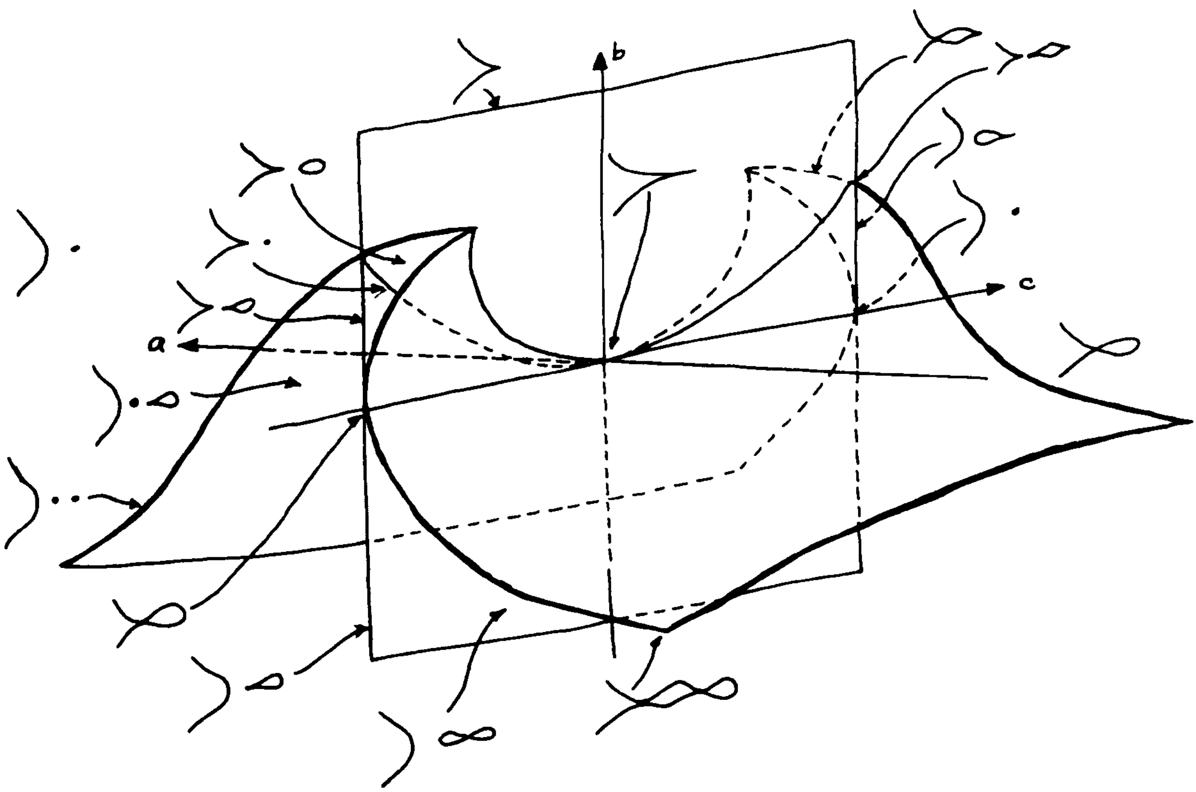



Fig. 7.7.12

## 7.8 Unfoldings of Tri-germs $\mathbf{R}^2, 0 \to \mathbf{R}^3, 0$

We now look at the unfoldings and bifurcation sets for these. The unfoldings are given in the following table. Calculations of these are routine.

| | Normal Form | Unfolding |
|---|---|---|
| | $(x, y, 0; \tilde{x}, 0, \tilde{y}; 0, X, Y)$ | stable |
| $(I)$ | $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^2)$ | $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^2 + a)$ |
| $(II)$ | $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^3)$ | $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^3 + a + bX)$ |
| $(III)$ | $(x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y, X^2 \pm Y^2)$ | $(x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y + a, X^2 \pm Y^2 + b)$ |
| $(IV)$ | $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^4)$ | $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^4 + a + bX + cX^2)$ |
| $(V)$ | $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, XY + Y^3)$ | $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y + a + bX, XY + X^3 + c)$ |
| $(VI)$ | $(x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y, X^2 + Y^3)$ | $(x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y + a, X^2 + Y^3 + b + cY)$ |

**Case** $(I)$ This tri-germ consists of three immersions where one is tangent to the intersection line of the other two. An unfolding is $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y+X^2+a)$. Let the co-ordinates in the target be $(u, v, w)$. Then the first and second branches meet in $(u, 0, 0)$. The intersection of the third branch with the plane $(x, y, 0)$ is given parametrically by $(u, -u^2 - a, 0)$, so we have a parabola with 0,1 or 2 real roots. The bifurcation set is $\{a = 0\}$.

The third branch intersects the plane $(X, 0, Y)$ in the parabola given parametrically by $(u, 0, u^2 + a)$. Again,the bifurcation set is $\{a = 0\}$. Pictures of the unfolding are shown in Fig. 7.8.1.

**Case** $(II)$ This tri-germ consists of three immersions again, where one has third order contact with the intersection line of the other two. An unfolding is $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^3 + a + bX)$. Let the co-ordinates in the target be $(u, v, w)$. Then the first and second branches meet in $(u, 0, 0)$. The intersection of the third branch with the plane $(x, y, 0)$ is given parametrically by $(u, -u^3 - bx - a, 0)$, so we have a cubic with 1, 2 or 3 real roots. The bifurcation set is $\{-4b^3 = 27a^2\}$.

The third branch also intersects the plane $(X, 0, Y)$ in a cubic with 1, 2 or 3 real roots, given parametrically by $(u, 0, u^3 + bx + a)$, and so the bifurcation

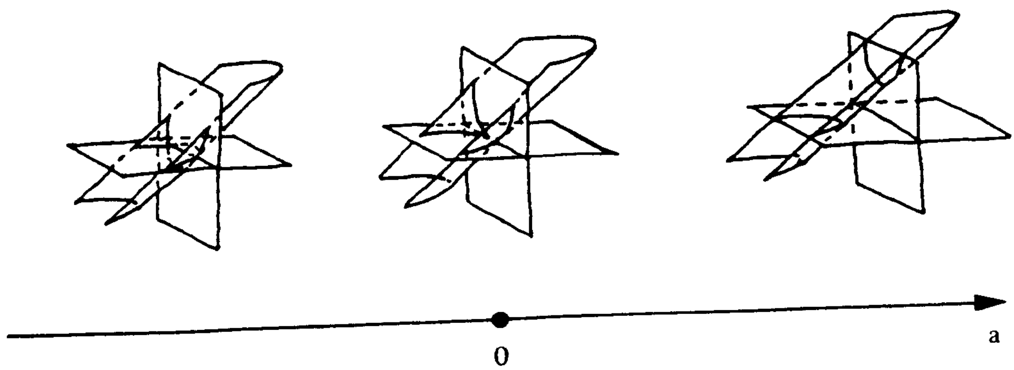



Fig.7.8.1

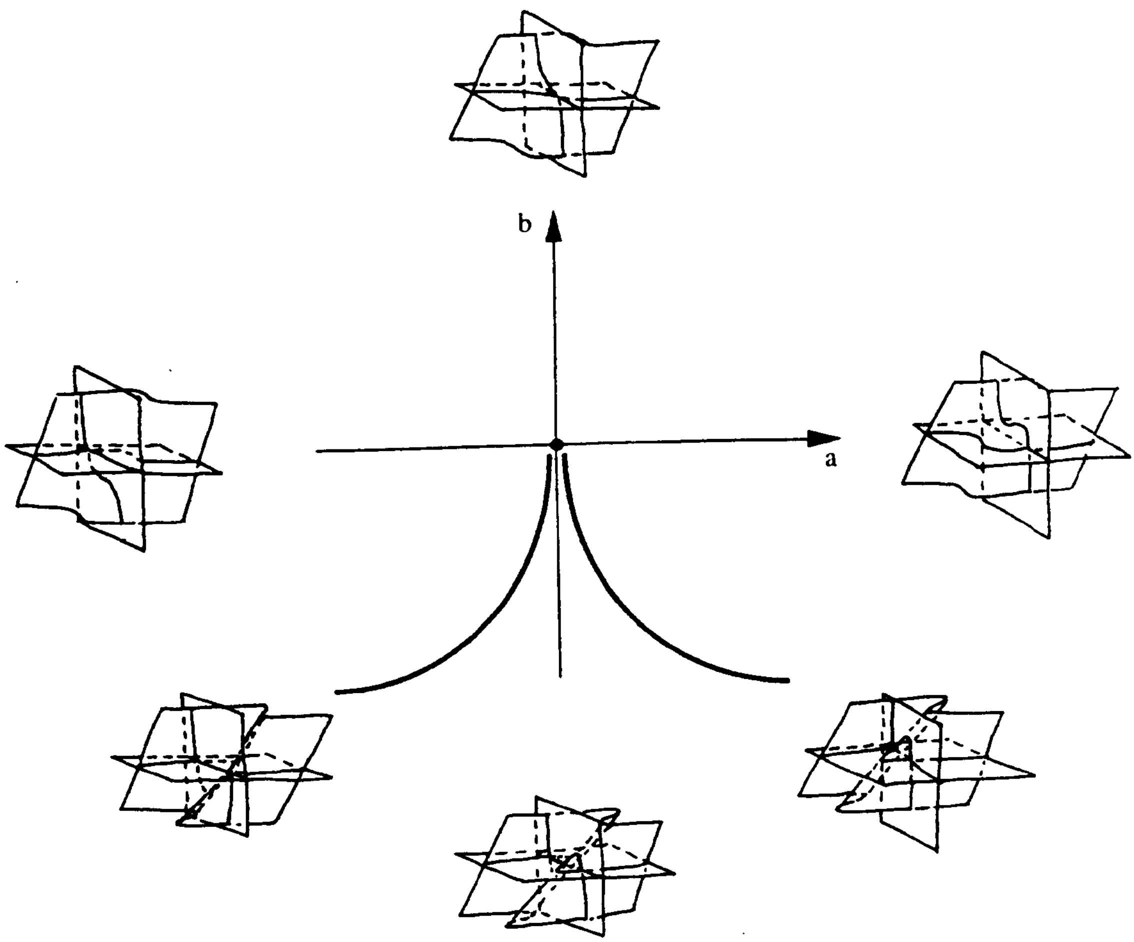



Fig.7.8.2

set is also $\{-4b^3 = 27a^2\}$. At the origin the two cubics are tangent to $(u, 0, 0)$. Inside the cusp of the bifurcation set the tri-germ has two 'bubbles', which degenerate to one bubble on the bifurcation set and then to none. Pictures of the unfolding are shown in Fig. 7.8.2.

**Case** $(III)^+$ The tri-germ consists of three immersions, two of which are tangent. An unfolding is given by $f_{a,b} = (x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y + a, X^2 + Y^2 + b)$. Let the co-ordinates in the target be $(u, v, w)$. The first and second branches meet in $(u, 0, 0)$. Then the third mono-germ can be written $w = u^2 + (v - a)^2 + b$. The intersection of this with the first plane $(x, y, 0)$ is then

$$-b = u^2 + (v - a)^2$$

which gives, as curve of intersection,

(i) a circle, centre $(0, a)$, radius $\sqrt{-b}$ if $b < 0$;

(ii) the point $(0, a)$ if $b = 0$;

(iii) the empty set if $b > 0$. Thus $b = 0$ is part of the bifurcation set.

The intersection with the second plane is

$$w = u^2 + a^2 + b$$

so we always get a parabola, with

(i) no real roots if $a^2 + b > 0$;

(ii) one real root if $a^2 + b = 0$;

(iii) two real roots if $a^2 + b < 0$. So the other part of the bifurcation set is $a^2 + b = 0$. See Fig. 7.8.3.

**Case** $(III)^-$ Again we have three immersions where two of them are tangent. An unfolding is given by $f_{a,b} = (x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y + a, X^2 - Y^2 + b)$.

If we denote the target co-ordinates by $(u, v, w)$ then the third mono-germ can be written $w = u^2 - (v - a)^2 + b$. So the intersection with the first plane is

$$-b = u^2 - (v - a)^2.$$

This gives different hyperbolas depending on whether $b$ is less than, equal to or greater than zero so part of the bifurcation set is the line $b = 0$.

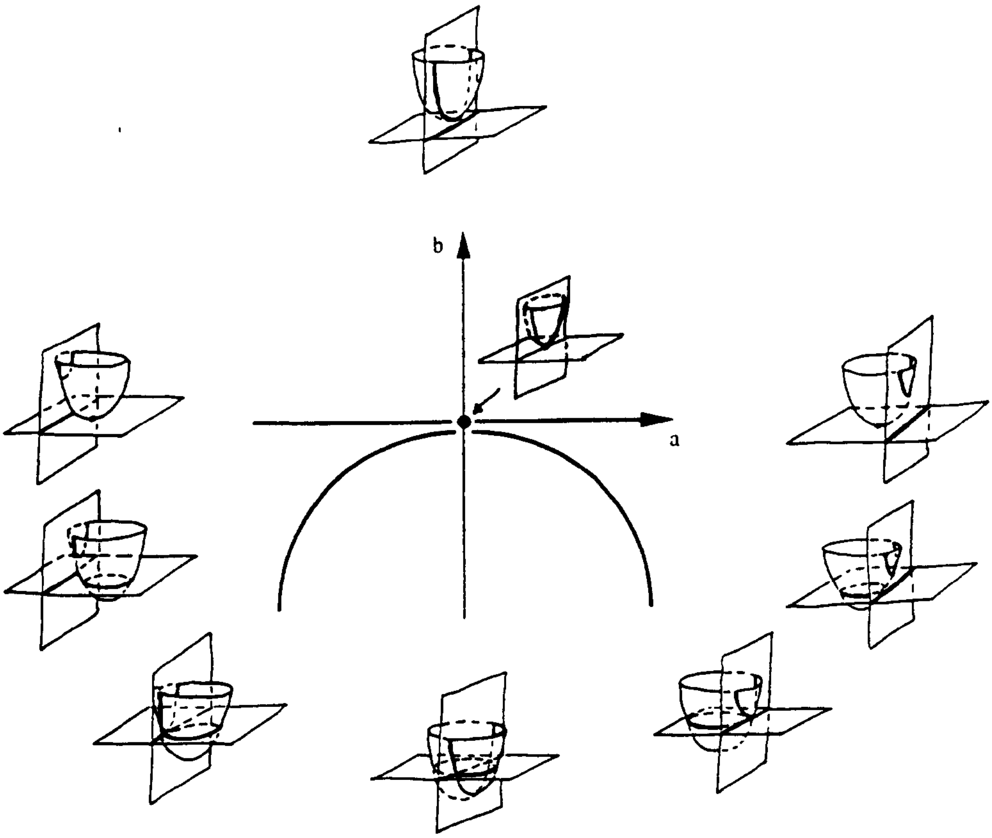



Fig.7.8.3

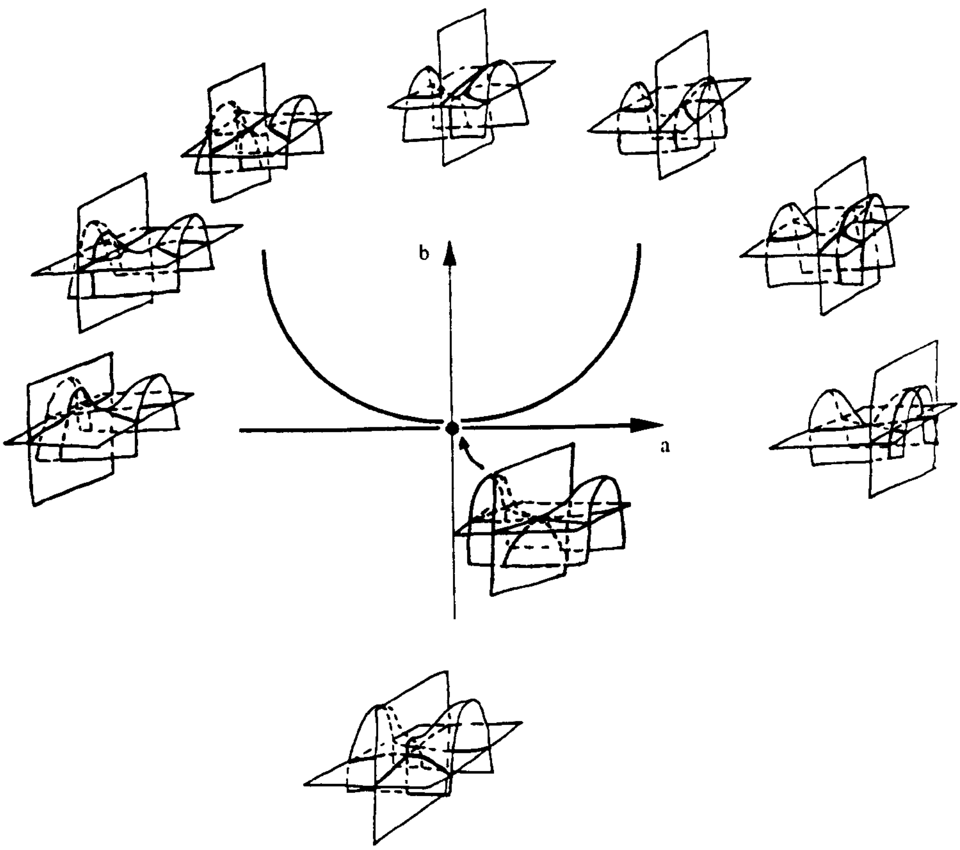



Fig.7.8.4

The intersection with the second plane is

$$w = u^2 - a^2 + b$$

so it is always a parabola, with

(i) no real roots if $-a^2 + b > 0$;

(ii) one real root if $-a^2 + b = 0$;

(iii) two real roots if $-a^2 + b < 0$. So the rest of the bifurcation set is $-a^2 + b = 0$. See Fig. 7.8.4.

**Case** $(IV)$ As in cases (I) and (II), we have three immersions where one has higher order of contact with the intersection line of the other two - in this case fourth order contact. An unfolding is $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y, Y + X^4 + a + bX + cX^2)$. Let the co-ordinates in the target be $(u, v, w)$. Then the first and second branches meet, as usual, in $(u, 0, 0)$. The intersection of the third branch with the plane $(x, y, 0)$ is given parametrically by $(u, -u^4 - cu^2 - bu - a, 0)$. This is a quartic with 0,1,2,3 or 4 real roots. As we would expect, the bifurcation set is the swallowtail surface.

The third branch meets the plane $(X, 0, Y)$ in $(u, 0, u^4 + cu^2 + bu + a)$ - again the bifurcation set is given by the swallowtail surface. The greatest number of distinct roots of the quartic occurs 'inside' this swallowtail, as usual. In this case this corresponds to three 'bubbles' caught between the branches. On the surface the quartic has a repeated root and the third branch is tangent to the intersection line of the two planes at one point at least (two points on the self-intersection curve of the swallowtail). Above the surface there are no points common to all three branches while below the surfaces have two points in common and a single 'bubble' is trapped between them. The various unfoldings are shown in Fig.7.8.5.

**Case** $(V)$The tri-germ consists of three immersions where two are planes and the third is the cusp catastrophe surface. An unfolding is $(x, y, 0; \bar{x}, 0, \bar{y}; X, Y + aX, XY + X^3 + b + cX)$. We again denote the co-ordinates in the target by $(u, v, w)$. The first two branches meet in $(u, 0, 0)$. The intersection of the first and third branches is given by $u^3 + uv - au^2 + cu + b = 0$. This gives a 'pitchfork' intersection curve. Solving for $v$ gives

$$v = (-u^3 + au^2 - cu - b)u^{-1}$$

(provided $u \neq 0$). We expect the curve to have 1, 2 or 3 roots, according as the cubic $-u^3 + au^2 - cu - b = 0$ has 1,2 or 3 solutions. The discriminant of this cubic is

$$27b^2 + 18abc - a^2c^2 - 4a^3b + 4c^3 = 0$$

This is a twisted cuspidal edge. We note that the intersection of the second and third branches is given by $(u, 0, u^3 - au^2 + cu + b)$, i.e. we have a cubic curve and the bifurcation set is again the twisted cuspidal edge $D = 0$, where $D$ is given by $D = 27b^2 + 18abc - a^2c^2 - 4a^3b + 4c^3$. Above $D = 0$ the pitchfork and cubic are distinct. On the surface the pitchfork and the cubic are tangent to the intersection line of the two surfaces at the same point and below both the cubic and the pitchfork have three points in common, all tranverse intersections with the line $v = w = 0$. See Fig.7.8.6 for these transitions in the tri-germ.

Case (VI) This tri-germ consists of three immersions, two of which have third order contact. An unfolding is $f_{a,b,c} = (x, y, 0; \tilde{x}, 0, \tilde{y}; X, Y + a, X^2 + Y^3 + b + cY)$. If we again denote the co-ordinates in the target by $(u, v, w)$ then, as usual, the first and second branches meet in the line $(u, 0, 0)$. Using the target co-ordinates, the third mono-germ can be written $w = u^2 + (v - a)^3 + b + c(v - a)$. Then the intersection with the first plane is

$$ac - b = u^2 + (v - a)^3 + cv$$

which is a cubic which varies in the $b, c$-plane as shown in Fig. 7.8.7. We can write $u = \sqrt{(a - v)^3 - cv + ac - b}$. This has solutions when the cubic in $v$ is positive. The discriminant of the cubic is $27b^2 = -4c^3$, i.e. a cuspidal edge with the $a$-axis as the edge, so varying $a$ does not affect the curve.

The intersection of the second and third branches is $(u, 0, u^2 - a^3 + b - ac)$, which is a parabola with 0, 1 or 2 roots according as $-a^3 + b - ac$ is greater than, equal to or less than zero. So the bifurcation set is the surface $a^3 + ac = b$, the cusp catastrophe surface, together with the cospidal edge, $27b^2 = -4c^3$. The two parts of the bifurcation set are tangent (c.f. the bifurcation set of $C_3^{\pm}$).

See Fig.7.8.8 for the possible transitions in parameter space.

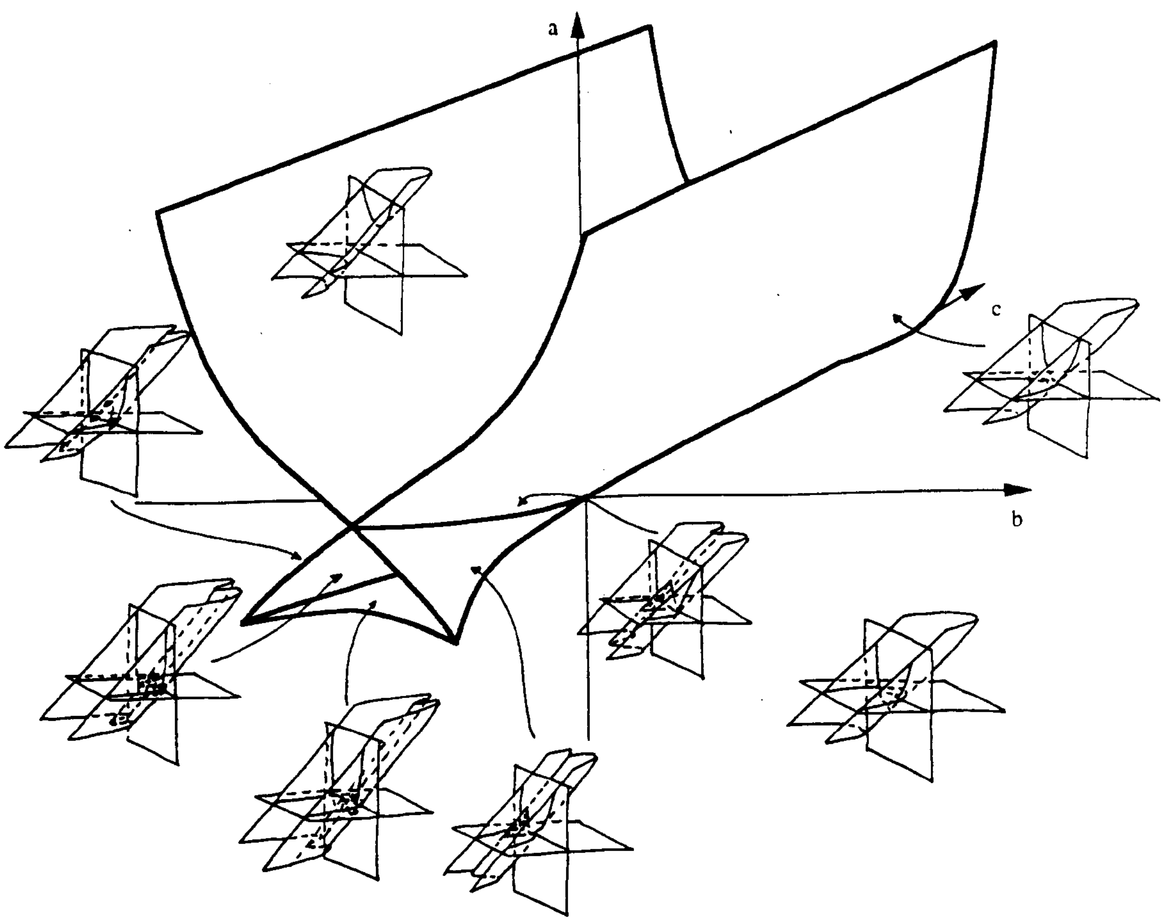



Fig.7.8.5

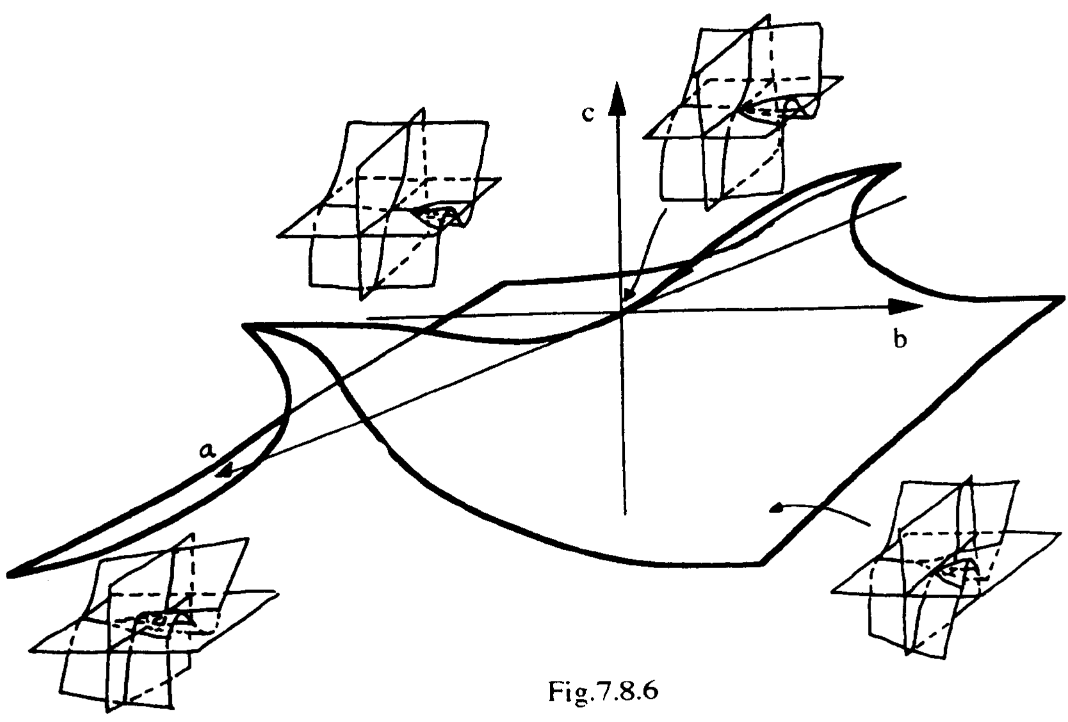



Fig.7.8.6

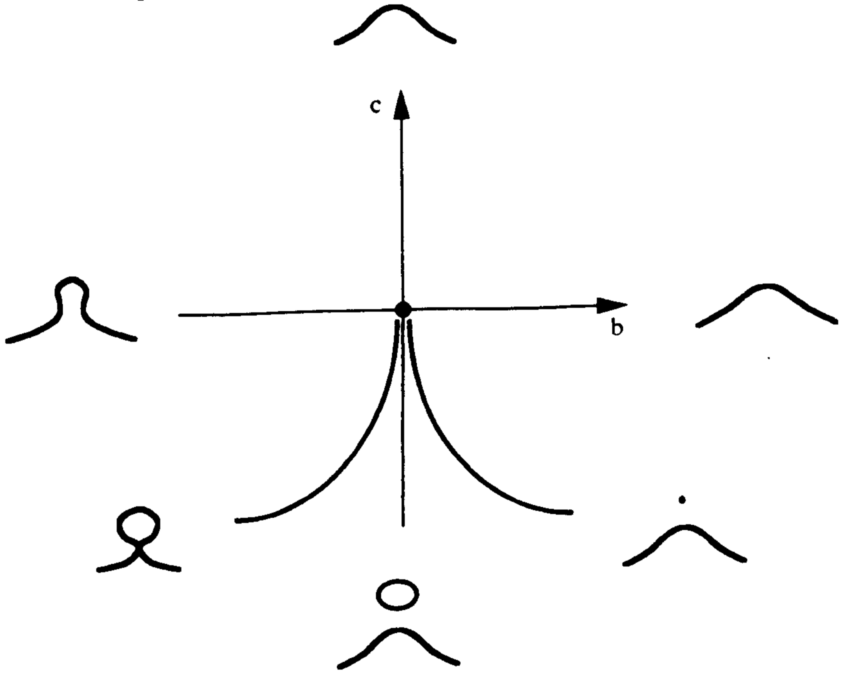



Fig.7.8.7

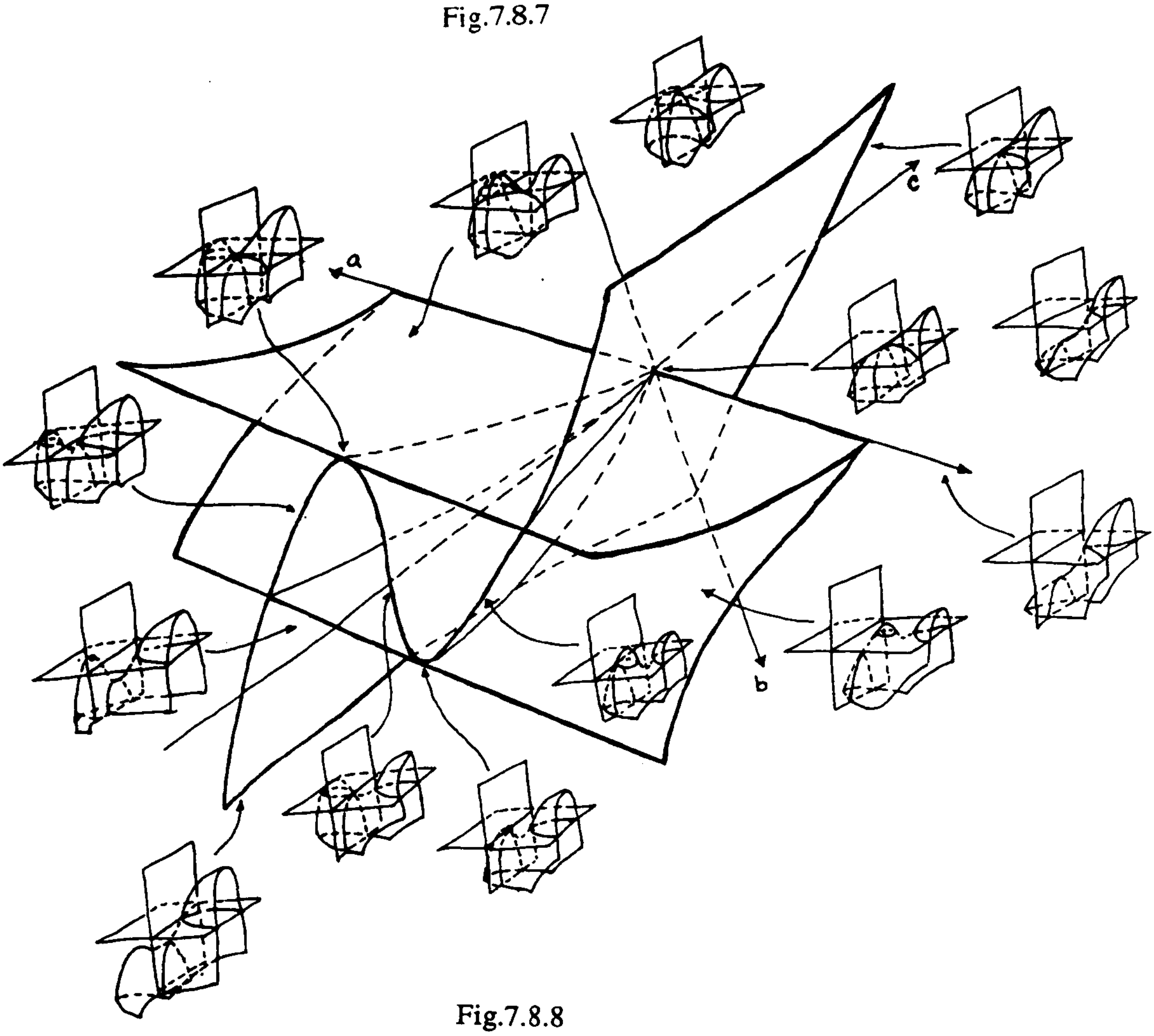



Fig.7.8.8

### 7.9 Unfolding Higher Multi-germs

(I) Given the 4-germ consisting of four transverse planes, the unfolding is

$$(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X} + a, \bar{Y}, \bar{X} + \bar{Y})$$

Then the bifurcation set is the point $\{a = 0\}$, since if $a < 0$ or $a > 0$ only three of the planes go through a point.

(II) The stratum of 5-germs consisting of five planes through the origin has unfolding

$$(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X} + a, \bar{Y}, \bar{X} + \lambda\bar{Y}; \mathrm{x} + b, \mathrm{y}, \mathrm{x} + \mu\mathrm{y})$$

(where $\lambda \neq 0, \mu$ and $\mu \neq 0$). Clearly $a = 0$ and $b = 0$ form the bifurcation set since on these lines four planes will go through the same point. Otherwise only three of the planes go through the same point.

(III) The stratum of 6-germs which has six immersions going through the same point has unfolding

$$(x, y, 0; \bar{x}, 0, \bar{y}; 0, X, Y; \bar{X}, \bar{Y}, \bar{X} + \lambda\bar{Y}; \mathrm{x} + \mu\mathrm{y}, \mathrm{x} + a, \mathrm{y}; \bar{\mathrm{x}} + b, \rho\bar{\mathrm{x}} + \gamma\bar{\mathrm{y}}, \mathrm{y})$$

(where none of the four moduli are equal to each other or 0). The bifurcation set includes the lines $a = 0$ and $b = 0$, where five of the planes will go through the same point.

# Appendix A – Computer Generated Pictures from Chapter 7

Fig. 1 $S_1^+$

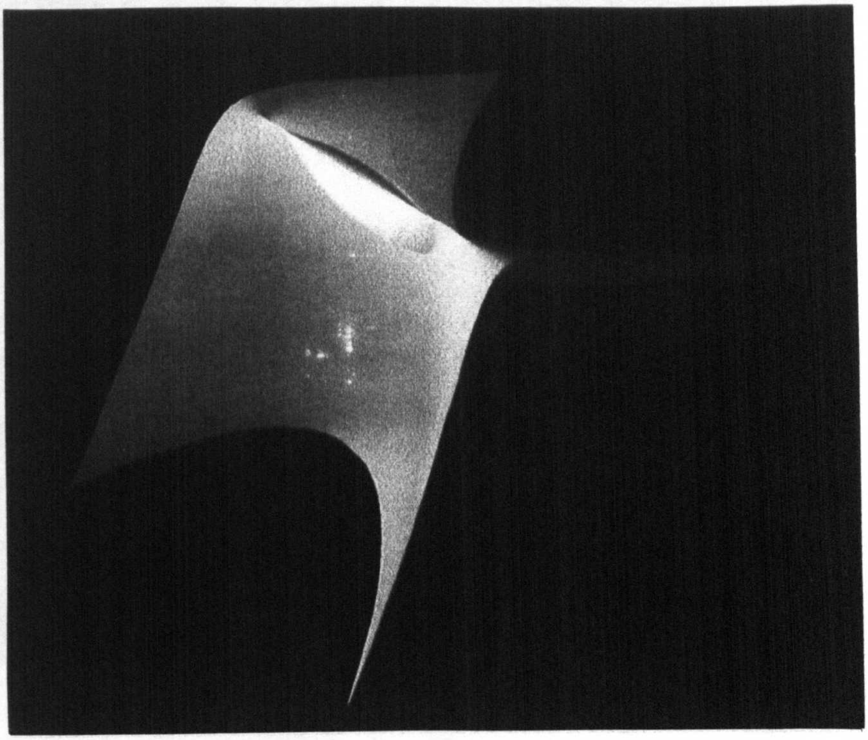
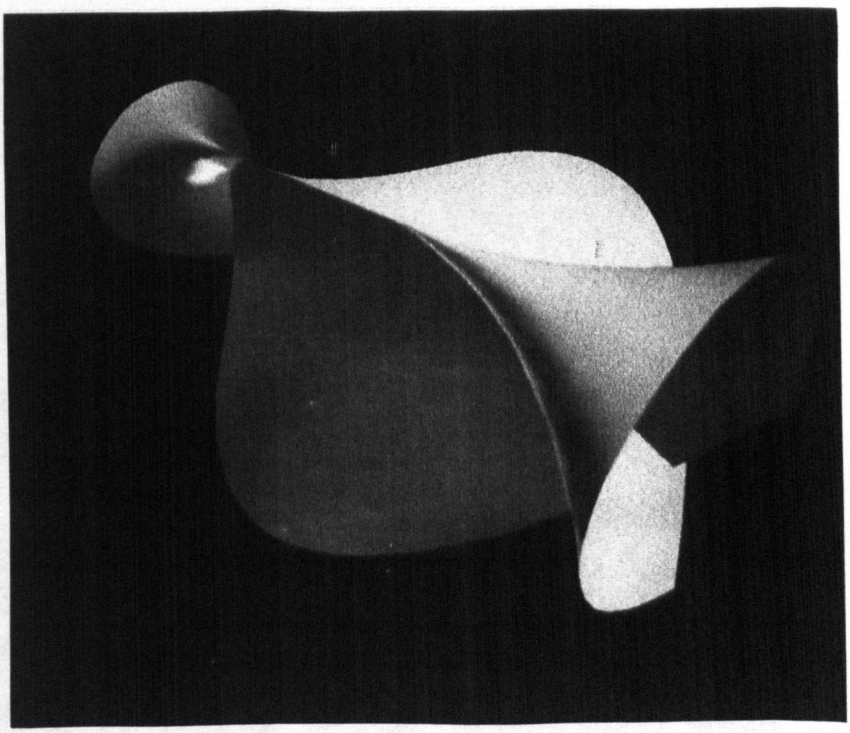


Fig. 2 $S_1^-$



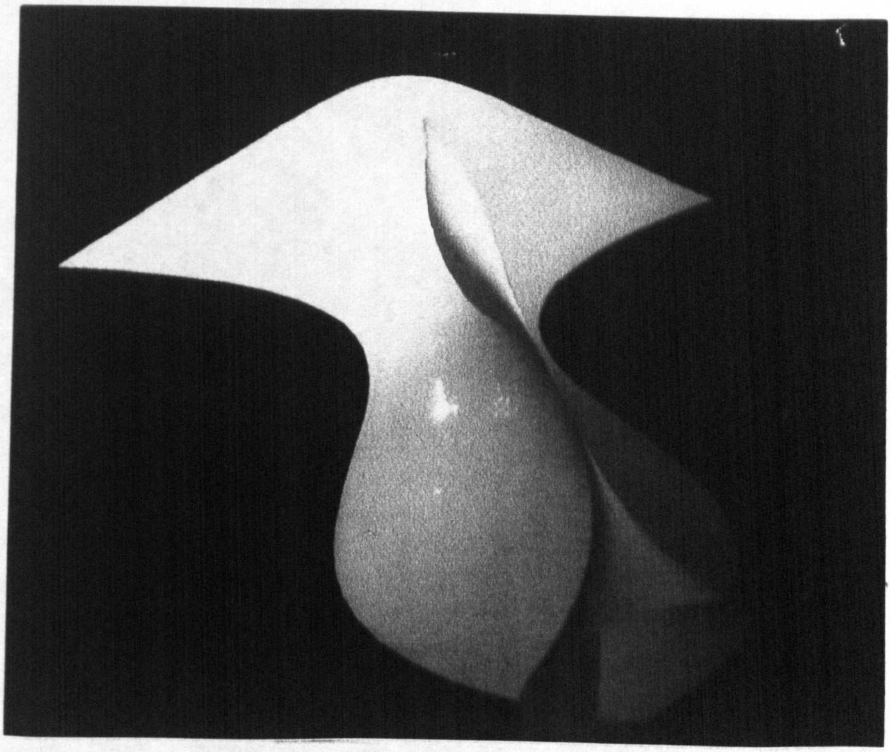


Fig. 3 $S_2$

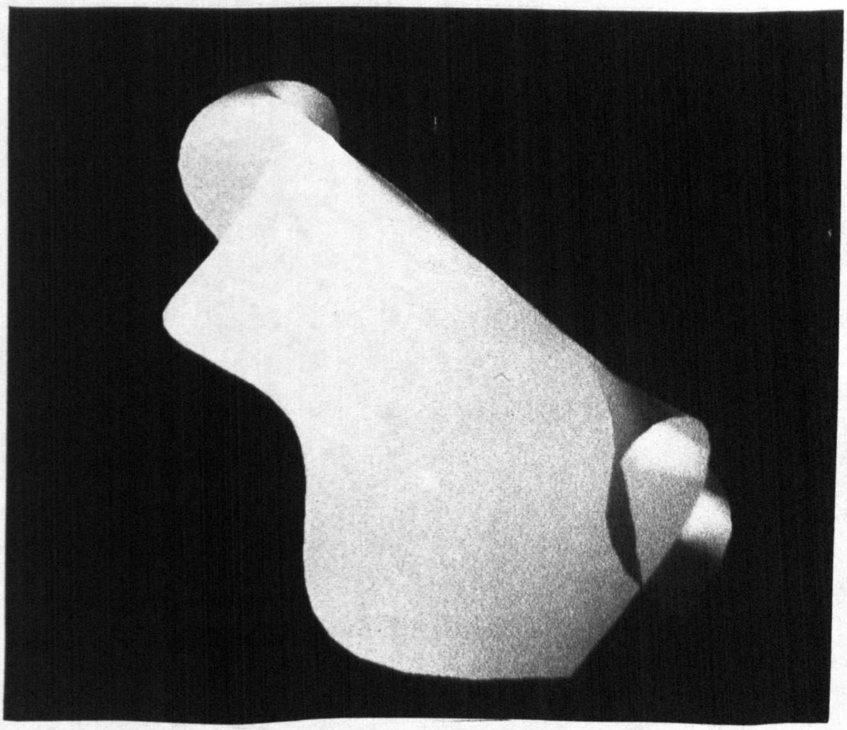


Fig. 4 $S_3^+$

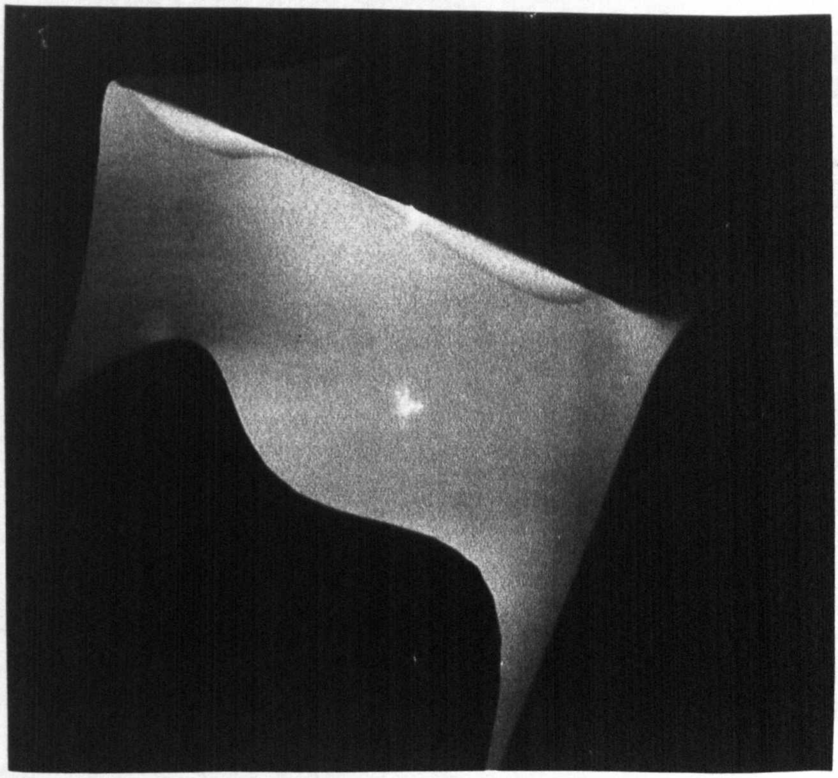


Fig. 5 $S_3^-$

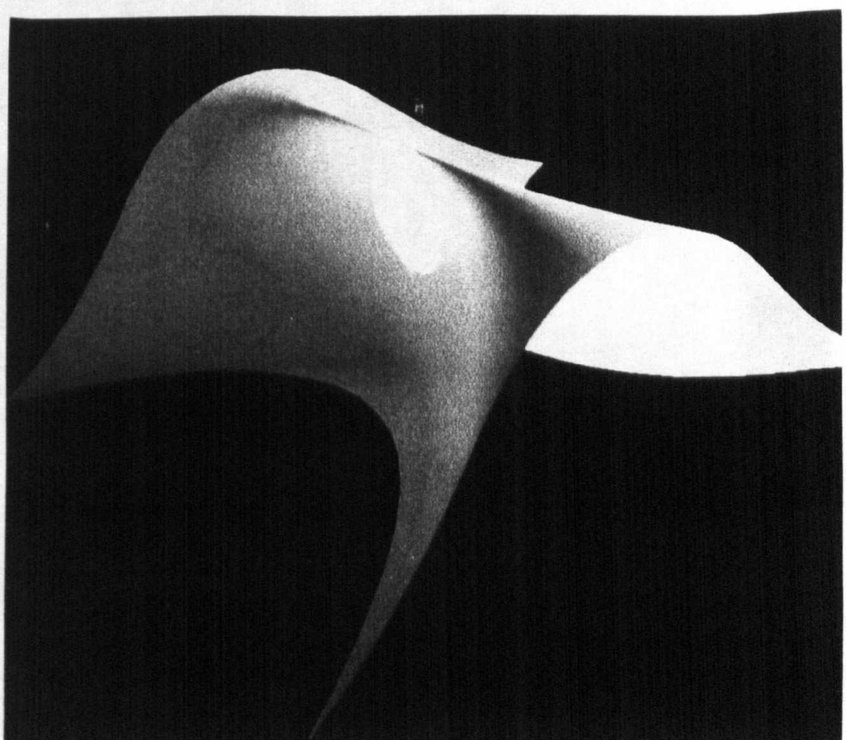


Fig. 6 $B_2^+$

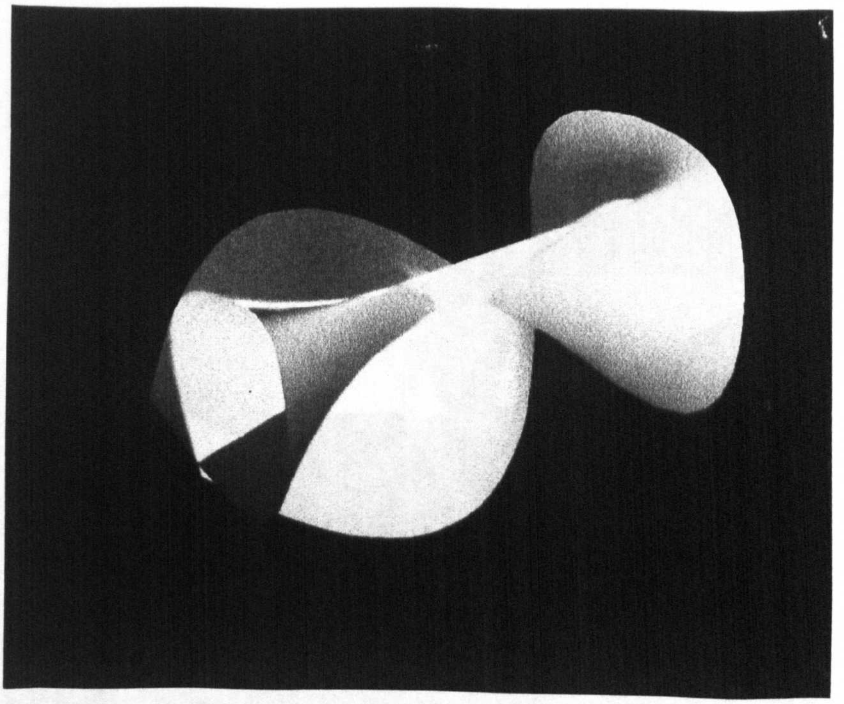


Fig. 7 $B_2^-$

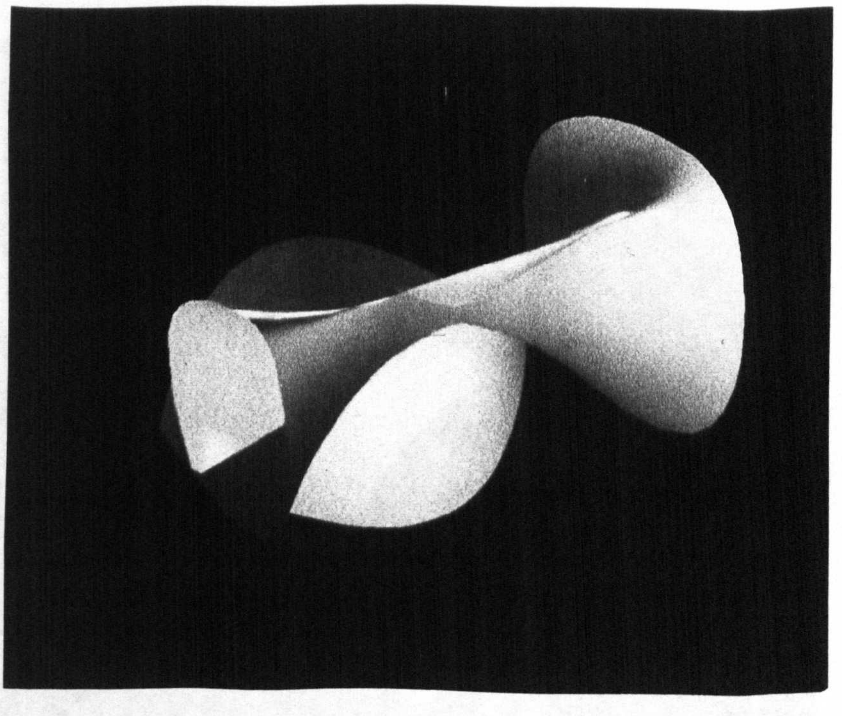


Fig. 8 $B_2^-$

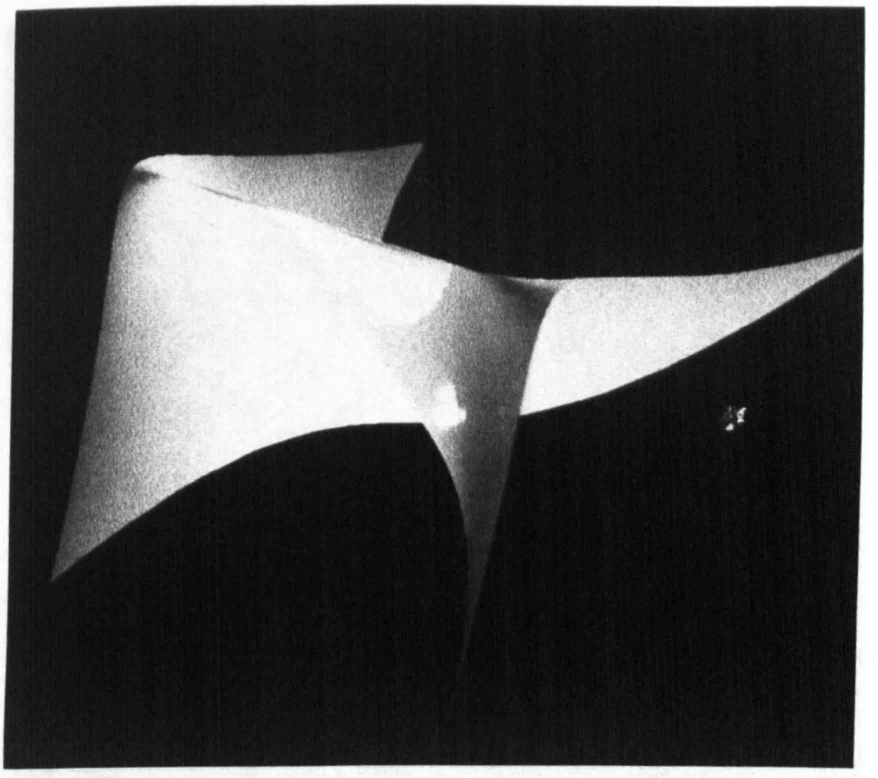


Fig. 9 $C_3^+$

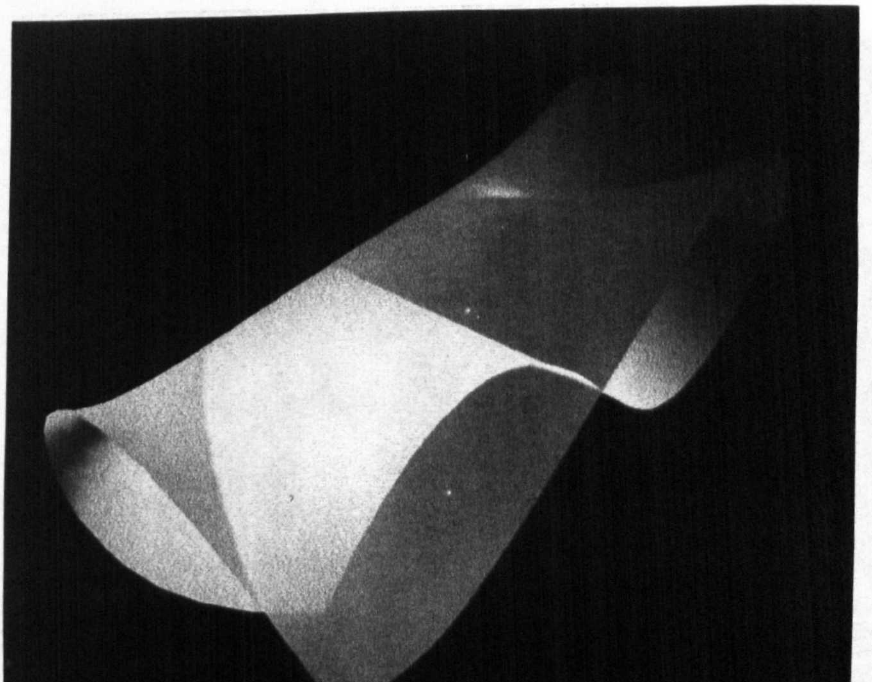


Fig. 10 $C_3^-$

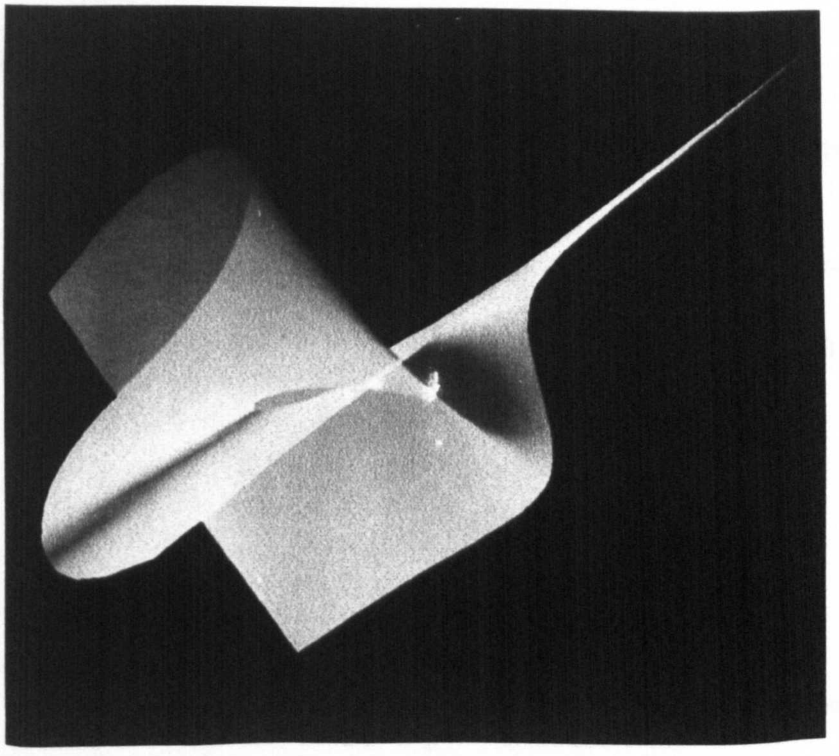


Fig. 11 $H_2$

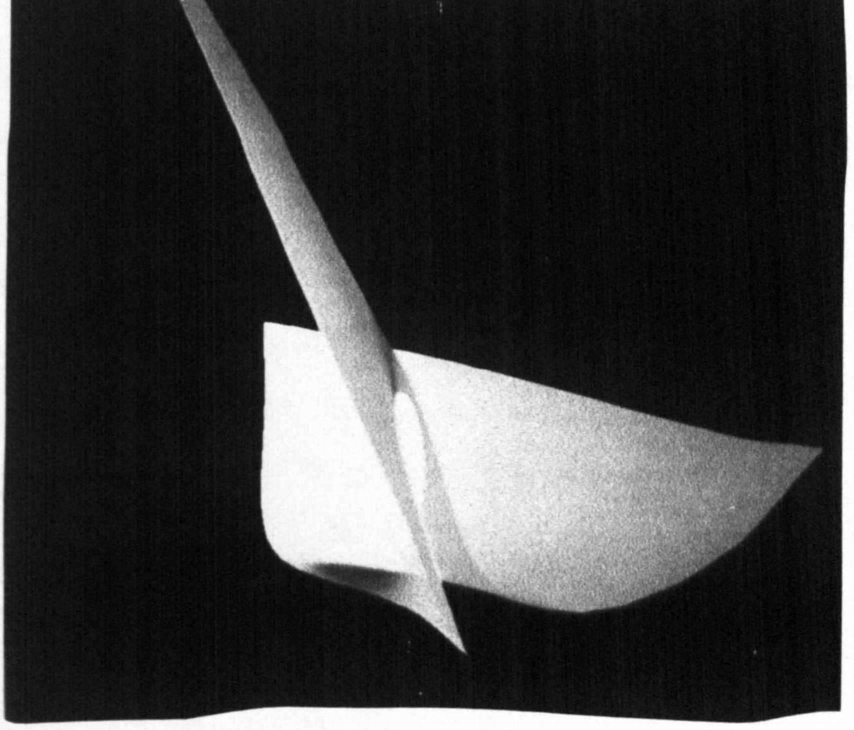


Fig. 12 $H_2$

# Appendix B – Codimension Calculations

## B.1 Calculations from Chapter 3

We give the calculations of $\mathcal{A}_e$-codimensions of the simple singularities of space curves, as listed in Theorems 3.1.1 and 3.1.2.

(i) $f(t) = (t^2, t^{2m+1}, 0)$. This is $2m+1$ determined and the $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(2t, (2m+1)t^{2m}, 0)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

So the vectors in the tangent space are

$$\begin{array}{ll} (t^i, 0, 0) & 0 \le i \le 2m+1 \\ (0, t^{2i}, 0) & 0 \le i \le m \\ (0, t^{2m+1}, 0) & \\ (0, 0, t^{2i}) & 0 \le i \le m \\ (0, 0, t^{2m+1}) & \end{array}$$

This gives a total of $4m + 6$ linearly independent vectors and so the $\mathcal{A}_e$-codimension is given by

$$\begin{aligned} \mathcal{A}_e\text{-codimension}(f) &= \dim(J^{2m+1}(1,3)) - (4m+6) \\ &= 2m \end{aligned}$$

(ii) $f(t) = (t^3, t^{3m+1}, 0)$. This is $6m-1$ determined and the $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(3t^2, (3m+1)t^{3m}, 0)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

So the vectors in the tangent space are

$$\begin{array}{ll} (t^i, 0, 0) & i = 0,\ 2 \le i \le 6m-1 \\ (0, t^{3i}, 0) & 0 \le i \le 2m-1 \\ (0, t^{3m+1+3i}, 0) & 0 \le i \le m-1 \\ (0, t^{6m-1}, 0) & \\ (0, 0, t^{3i}) & 0 \le i \le 2m-1 \\ (0, 0, t^{3m+1+3i}) & 0 \le i \le m-1 \end{array}$$

This gives $12m$ vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{-codimension}(f) = 18m - 12m = 6m$$

(iii) $f(t) = (t^3, t^{3m+2}, 0)$. This is $6m+1$ determined and the $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(3t^2, (3m+2)t^{3m+1}, 0)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

So the vectors in the tangent space are

$$\begin{array}{ll}
(t^i, 0, 0) & i = 0,\ 2 \le i \le 6m+1 \\
(0, t^{3i}, 0) & 0 \le i \le 2m \\
(0, t^{3m+2+3i}, 0) & 0 \le i \le m-1 \\
(0, t^{6m+1}, 0) & \\
(0, 0, t^{3i}) & 0 \le i \le 2m \\
(0, 0, t^{3m+2+3i}) & 0 \le i \le m-1
\end{array}$$

There are $12m+4$ vectors and so

$$\mathcal{A}_e\text{−codimension}(f) = 18m + 6 - (12m+4) = 6m+2$$

(iv) $f(t) = (t^3, t^{3m+1} + t^{3n+2}, 0)$ where $1 \le m \le n < 2m$. This is $6m-1$ determined and the $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(3t^2, (3m+1)t^{3m} + (3n+2)t^{3n+1}, 0)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

So the vectors in the tangent space are

$$\begin{array}{ll}
(t^i, 0, 0) & i = 0,\ 2 \le i \le 6m-1 \\
(0, t^{3i}, 0) & 0 \le i \le 2m-1 \\
(0, t^{3m+1+3i}, 0) & 0 \le i \le m-1 \\
(0, t^{3n+2+3i}, 0) & 0 \le i \le 2m-n-1 \\
(0, 0, t^{3i}) & 0 \le i \le 2m-1 \\
(0, 0, t^{3m+1+3i}) & 0 \le i \le m-1
\end{array}$$

There are $14m-n-1$ vectors and so

$$\mathcal{A}_e\text{−codimension}(f) = 18m - (14m-n-1) = 4m+n+1$$

(v) $f(t) = (t^3, t^{3m+1} + t^{3n+2}, 0)$ where $1 \le n < m \le 2n$. This is $6n+1$ determined and the $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(3t^2, (3m+1)t^{3m} + (3n+2)t^{3n+1}, 0)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

So the vectors in the tangent space are

$$\begin{array}{ll}(t^i,0,0) & i=0,\ 2\le i\le 6n+1\\ (0,t^{3i},0) & 0\le i\le 2n\\ (0,t^{3m+1+3i},0) & 0\le i\le 2n-m\\ (0,t^{3n+2+3i},0) & 0\le i\le n-1\\ (0,0,t^{3i}) & 0\le i\le 2n\\ (0,0,t^{3m+1+3i}) & 0\le i\le n-1\end{array}$$

There are $14n-m+4$ vectors and so

$$\mathcal{A}_e\text{−codimension}(f)=18m+6-(14n-m+4)=4n+m+2$$

(vi) $f(t)=(t^4,t^5,0)$. This is 11 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f=\mathcal{E}_1\langle(4t^3,5t^4,0)\rangle+f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll}(t^i,0,0) & i=0,\ 3\le i\le 11\\ (0,t^i,0) & i=0,4,5,6,\ 8\le i\le 11\\ (0,0,t^i) & i=0,4,5,8,9,10\end{array}$$

This gives 24 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{−codimension}(f)=36-24=12$$

(vii) $f(t)=(t^4,t^5+t^7,0)$. This is 11 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f=\mathcal{E}_1\langle(4t^3,5t^4+6t^7,0)\rangle+f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll}(t^i,0,0) & i=0,\ 3\le i\le 11\\ (0,t^i,0) & i=0,\ 4\le i\le 11\\ (0,0,t^i) & i=0,4,5,8,9,10\end{array}$$

This gives 25 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{−codimension}(f)=36-25=11$$

(viii) $f(t)=(t^4,t^6+t^{2m+1},0)$, which is $2m+9$ determined. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f=\mathcal{E}_1\langle(4t^3,6t^5+(2m+1)t^{2m},0)\rangle+f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

The vectors in this are

$$\begin{array}{ll}
(t^i,0,0) & i=0,\ 3\le i\le 2m+9 \\
(0,t^{2i},0) & i=0,\ 2\le i\le m+4 \\
(0,t^{2i+1},0) & m\le i\le m+4 \\
(0,0,t^{2i}) & i=0,\ 2\le i\le m+4 \\
(0,0,t^{2m+3}) &
\end{array}$$

Thus we have $4m+22$ vectors and the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{-codimension}(f) = 6m+30-(4m+22) = 2m+8.$$

(ix) $f(t) = (t^4,t^7,0)$. This is 17 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3,7t^6,0)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll}
(t^i,0,0) & i=0,\ 3\le i\le 17 \\
(0,t^i,0) & i=0,4,7,8,10,11,12,14,15,16,17 \\
(0,0,t^i) & i=0,4,7,8,11,12,14,15,16
\end{array}$$

This gives 36 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{-codimension}(f) = 54-36 = 18$$

(x) $f(t) = (t^4,t^7+t^9,0)$. This is 17 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3,7t^6+9t^8,0)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll}
(t^i,0,0) & i=0,\ 3\le i\le 17 \\
(0,t^i,0) & i=0,4,\ 7\le i\le 17 \\
(0,0,t^i) & i=0,4,7,8,11,12,14,15,16
\end{array}$$

This gives 38 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{-codimension}(f) = 54-38 = 16$$

(xi) $f(t) = (t^4,t^7+t^{13},0)$. This is 17 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3,7t^6=13t^{12},0)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3\le i\le 17 \\ (0,t^i,0) & i=0,4,7,8,\ 10\le i\le 17 \\ (0,0,t^i) & i=0,4,7,8,11,12,14,15,16 \end{array}$$

This gives 37 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{-codimension}(f) = 54 - 37 = 17$$

(xii) $f(t) = (t^3, t^{3m+1}, t^{3n+2})$ where $1 \le m \le n < 2m$. This is $3n+2$ determined and the $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(3t^2,(3m+1)t^{3m},(3n+2)t^{3n+1})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

So the vectors in the tangent space are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 2\le i\le 3n+2 \\ (0,t^{3i},0) & 0\le i\le n \\ (0,t^{3m+1+3i},0) & 0\le i\le n-m \\ (0,t^{3n+2},0) & \\ (0,0,t^{3i}) & 0\le i\le n \\ (0,0,t^{3m+1+3i}) & 0\le i\le n-m \\ (0,0,t^{3n+2}) & \end{array}$$

There are $7n-2m+8$ vectors and so

$$\mathcal{A}_e\text{-codimension}(f) = 9n+9-(7n-2m+8) = 2n+2m+1$$

(xiii) $f(t) = (t^3, t^{3m+1}+t^{3n+2}, t^{3l+2})$ where $1 \le m \le n < l < 2m$. This is $3l+2$ determined and the $\mathcal{A}_e$-tangent space is

$$\begin{aligned} T\mathcal{A}_e.f =& \mathcal{E}_1\langle(3t^2,(3m+1)t^{3m}+(3n+2)t^{3n+1},(3l+2)t^{3l+1})\rangle \\ &+ f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \end{aligned}$$

So the vectors in the tangent space are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 2\le i\le 3l+2 \\ (0,t^{3i},0) & 0\le i\le l \\ (0,t^{3m+1+3i},0) & 0\le i\le l-m \\ (0,t^{3n+2+3i},0) & 0\le i\le l-n \\ (0,0,t^{3i}) & 0\le i\le l \\ (0,0,t^{3m+1+3i}) & 0\le i\le l-m \\ (0,0,t^{3l+2}) & \end{array}$$

There are $8l - 2m - n + 8$ vectors and so

$$\mathcal{A}_e\text{-codimension}(f) = 9l + 9 - (8l - 2m - n + 8) = l + 2m + n + 1$$

(xiv) $f(t) = (t^4, t^5, t^6)$. This is 7 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 5t^4, 6t^5)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3 \le i \le 7 \\ (0,t^i,0) & i = 0,4,5,6,7 \\ (0,0,t^i) & i = 0,4,5,6,7 \end{array}$$

This gives 16 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{-codimension}(f) = 24 - 16 = 8$$

(xv) $f(t) = (t^4, t^5, t^7)$. This is 7 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 5t^4, 7t^6)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3 \le i \le 7 \\ (0,t^i,0) & i = 0,4,5,6,7 \\ (0,0,t^i) & i = 0,4,5,7 \end{array}$$

This gives 15 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{-codimension}(f) = 24 - 15 = 9$$

(xvi) $f(t) = (t^4, t^5, t^{11})$. This is 11 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 5t^4, 11t^{10})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3 \le i \le 11 \\ (0,t^i,0) & i = 0,4,5,6,\ 8 \le i \le 11 \\ (0,0,t^i) & i = 0,4,5,\ 8 \le i \le 11 \end{array}$$

This gives 25 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{-codimension}(f) = 36 - 25 = 11$$

(xvii) $f(t) = (t^4, t^5 + t^7, t^{11})$. This is 11 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 5t^4 + 7t^6, 11t^{10})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3\le i\le 11 \\ (0,t^i,0) & i=0,\ 4\le i\le 11 \\ (0,0,t^i) & i=0,4,5,\ 8\le i\le 11 \end{array}$$

This gives 26 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{−codimension}(f) = 36 - 26 = 10$$

(xviii) $f(t) = (t^4, t^6, t^{2m+1})$, which is $2m+3$ determined. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 6t^5, (2m+1)t^{2m})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

The vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3\le i\le 2m+3 \\ (0,t^{2i},0) & i=0,\ 2\le i\le m+1 \\ (0,t^{2i+1},0) & i=m,m+1 \\ (0,0,t^{2i}) & i=0,\ 2\le i\le m+1 \\ (0,0,t^{2i+1}) & i=m,m+1 \end{array}$$

Thus we have $4m+8$ vectors and the $\mathcal{A}_e$-codimension is given by

$$\mathcal{A}_e\text{−codimension}(f) = 6m + 12 - (4m+8) = 2m+4.$$

(xix) $f(t) = (t^4, t^6 + t^{2m+1}, t^{2m+3})$, which is $2m+5$ determined. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 6t^5+(2m+1)t^{2m}, (2m+3)t^{2m+2})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

The vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3\le i\le 2m+5 \\ (0,t^{2i},0) & i=0,\ 2\le i\le m+2 \\ (0,t^{2i+1},0) & i=m,m+1,m+2 \\ (0,0,t^{2i}) & i=0,\ 2\le i\le m+2 \\ (0,0,t^{2i+1}) & i=m+1,m+2 \end{array}$$

Thus we have $4m+13$ vectors and the $\mathcal{A}_e$-codimension is given by

$$\mathcal{A}_e\text{−codimension}(f) = 6m + 18 - (4m+13) = 2m+5.$$

(xx) $f(t) = (t^4, t^6 + t^{2m+1}, t^{2m+5})$, which is $2m+7$ determined. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 6t^5+(2m+1)t^{2m}, (2m+5)t^{2m+4})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

The vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3\le i\le 2m+7 \\ (0,t^{2i},0) & i=0,\ 2\le i\le m+3 \\ (0,t^{2i+1},0) & m\le i\le m+3 \\ (0,0,t^{2i}) & i=0,\ 2\le i\le m+3 \\ (0,0,t^{2i+1}) & i=m+2, m+3 \end{array}$$

Thus we have $4m+18$ vectors and the $\mathcal{A}_e$-codimension is given by

$$\mathcal{A}_e\text{–codimension}(f) = 6m+24-(4m+18) = 2m+6.$$

(xxi) $f(t) = (t^4, t^6 + t^{2m+1}, t^{2m+9})$, which is $2m+9$ determined. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 6t^5+(2m+1)t^{2m}, (2m+9)t^{2m+8})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

The vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3\le i\le 2m+9 \\ (0,t^{2i},0) & i=0,\ 2\le i\le m+4 \\ (0,t^{2i+1},0) & m\le i\le m+4 \\ (0,0,t^{2i}) & i=0,\ 2\le i\le m+4 \\ (0,0,t^{2i+1}) & i=m+3, m+4 \end{array}$$

Thus we have $4m+23$ vectors and the $\mathcal{A}_e$-codimension is given by

$$\mathcal{A}_e\text{–codimension}(f) = 6m+30-(4m+23) = 2m+7.$$

(xxii) $f(t) = (t^4, t^7, t^9)$. This is 10 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 7t^6, 9t^8)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3\le i\le 10 \\ (0,t^i,0) & i=0,4,7,8,9,10 \\ (0,0,t^i) & i=0,4,7,8,9 \end{array}$$

This gives 20 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{–codimension}(f) = 33-20 = 13$$

(xxiii) $f(t) = (t^4, t^7, t^9 + t^{10})$. This is 10 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 7t^6, 9t^8 + 10t^9)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i = 0,\ 3 \le i \le 10 \\ (0,t^i,0) & i = 0,4,7,8,9,10 \\ (0,0,t^i) & i = 0,4,7,8,9,10 \end{array}$$

This gives 21 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{−codimension}(f) = 33 - 21 = 12$$

(xxiv) $f(t) = (t^4, t^7, t^{10})$. This is 10 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 7t^6, 10t^9)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i = 0,\ 3 \le i \le 10 \\ (0,t^i,0) & i = 0,4,7,8,10 \\ (0,0,t^i) & i = 0,4,7,8,10 \end{array}$$

This gives 19 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{−codimension}(f) = 33 - 19 = 14$$

(xxv) $f(t) = (t^4, t^7 + t^9, t^{10})$. This is 10 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 7t^6 + 9t^8, 10t^9)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i = 0,\ 3 \le i \le 10 \\ (0,t^i,0) & i = 0,4,7,8,9,10 \\ (0,0,t^i) & i = 0,4,7,8,10 \end{array}$$

This gives 20 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{−codimension}(f) = 33 - 20 = 13$$

(xxvi) $f(t) = (t^4, t^7, t^{13})$. This is 13 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 7t^6, 13t^{12})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3\le i\le 13 \\ (0,t^i,0) & i=0,4,7,8,10,11,12,13 \\ (0,0,t^i) & i=0,4,7,8,11,12,13 \end{array}$$

This gives 27 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{-codimension}(f) = 42-27=15$$

(xxvii) $f(t)=(t^4,t^7+t^9,t^{13})$. This is 13 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3,7t^6+9t^8,13t^{12})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3\le i\le 13 \\ (0,t^i,0) & i=0,4,\ 7\le i\le 13 \\ (0,0,t^i) & i=0,4,7,8,11,12,13 \end{array}$$

This gives 28 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{-codimension}(f) = 42-28=14$$

(xxviii) $f(t)=(t^4,t^7,t^{17})$. This is 17 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3,7t^6,17t^{16})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3\le i\le 17 \\ (0,t^i,0) & i=0,4,7,8,10,11,12,14,15,16,17 \\ (0,0,t^i) & i=0,4,7,8,11,12,14,15,16,17 \end{array}$$

This gives 37 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{-codimension}(f) = 54-37=17$$

(xxix) $f(t)=(t^4,t^7+t^9,t^{17})$. This is 17 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3,7t^6+9t^8,17t^{16})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i=0,\ 3\le i\le 17 \\ (0,t^i,0) & i=0,4,\ 7\le i\le 17 \\ (0,0,t^i) & i=0,4,7,8,11,12,14,15,16,17 \end{array}$$

This gives 39 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{–codimension}(f) = 54 - 39 = 15$$

(xxx) $f(t) = (t^4, t^7 + t^{13}, t^{17})$. This is 17 determined. The $\mathcal{A}_e$-tangent space is given by

$$T\mathcal{A}_e.f = \mathcal{E}_1\langle(4t^3, 7t^6 + 13t^{12}, 17t^{16})\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle$$

and the linearly independent vectors in this are

$$\begin{array}{ll} (t^i,0,0) & i = 0,\ 3 \le i \le 17 \\ (0,t^i,0) & i = 0,4,7,8,\ 10 \le i \le 17 \\ (0,0,t^i) & i = 0,4,7,8,11,12,14,15,16,17 \end{array}$$

This gives 38 vectors and so the $\mathcal{A}_e$-codimension is

$$\mathcal{A}_e\text{–codimension}(f) = 54 - 38 = 16$$

## B.2 Unfolding Calculations from Chapter 7

First from Section 7.6 we have:

$S_1^\pm$: The normal form for $S_1^+$ is $f(x,y) = (x, y^2, y^3 \pm x^2y)$. Then the $\mathcal{A}_e$ tangent space is

$$\begin{aligned} T\mathcal{A}_e.f &= \mathcal{E}_2\langle(1,0,\pm 2xy),\ (0,2y,3y^2 \pm x^2)\rangle + \mathcal{E}_3\langle(1,0,0),\ (0,1,0),\ (0,0,1)\rangle \\ &= (\mathcal{E}_2,\ \mathcal{E}_2,\ \mathcal{E}_2 - \{y\}) \end{aligned}$$

So an unfolding is given by $F_a(x,y) = (x, y^2, y^3 \pm x^2y + ay)$.

$S_2$: The normal form for the germ is $f(x,y) = (x, y^2, y^3 + x^3y)$ and the $\mathcal{A}_e$ tangent space is

$$\begin{aligned} T\mathcal{A}_e.f &= \mathcal{E}_2\langle(1,0,3x^2y),\ (0,2y,3y^2 + x^3)\rangle + \mathcal{E}_3\langle(1,0,0),\ (0,1,0),\ (0,0,1)\rangle \\ &= (\mathcal{E}_2,\ \mathcal{E}_2,\ \mathcal{E}_2 - \{y, xy\}) \end{aligned}$$

So the unfolding is given by $f_{a,b}(x,y) = (x, y^2, y^3 + x^3y + ay + bxy)$

$S_3^\pm$: The normal form for $S_3^+$ is $f(x,y) = (x, y^2, y^3 \pm x^4y)$. The $T\mathcal{A}_e$ tangent space is

$$\begin{aligned} T\mathcal{A}_e.f &= \mathcal{E}_2\langle(1,0,\pm 4x^3y0,\ (0,2y,3y^2 \pm x^4)\rangle + \mathcal{E}_3\langle(1,0,0),\ (0,1,0),\ (0,0,1)\rangle \\ &= (\mathcal{E}_2,\ \mathcal{E}_2,\ \mathcal{E}_2 - \{y, xy, x^2y\}) \end{aligned}$$

and so the unfolding is $f_{a,b,c}(x,y) = (x, y^2, y^3 \pm x^4y + ay + bxy + cx^2y)$.

$B_2^{\pm}$**:** The normal form for $B_2^+$ is $f(x,y) = (x, y^2, x^2y \pm y^5)$. The $\mathcal{A}_e$ tangent space is

$$\begin{aligned} T\mathcal{A}_e.f &= \mathcal{E}_2\langle(1,0,2xy),\ (0,2y,x^2 \pm 5y^4)\rangle + \mathcal{E}_3\langle(1,0,0),\ (0,1,0),\ (0,0,1)\rangle \\ &= (\mathcal{E}_2,\ \mathcal{E}_2,\ \mathcal{E}_2 - \{y, y^3\}) \end{aligned}$$

and so the unfolding is $f_{a,b}(x,y) = (x, y^2, x^2y \pm y^5 + ay + by^3)$.

$B_3^{\pm}$**:** The normal form for $B_3^+$ is $f(x,y) = (x, y^2, x^2y \pm y^7)$ and the $\mathcal{A}_e$ tangent space is

$$\begin{aligned} T\mathcal{A}_e.f &= \mathcal{E}_2\langle(1,0,2xy),\ (0,2y,x^2 \pm 7y^6)\rangle + \mathcal{E}_3\langle(1,0,0),\ (0,1,0),\ (0,0,1)\rangle \\ &= (\mathcal{E}_2,\ \mathcal{E}_2,\ \mathcal{E}_2 - \{y, y^3, y^5\}) \end{aligned}$$

So the unfolding is $f_{a,b,c}(x,y) = (x, y^2, x^2y \pm y^7 + ay + by^3 + cy^5)$.

$C_3^{\pm}$**:** The normal form for $C_3^+$ is $f(x,y) = (x, y^2, xy^3 \pm x^3y)$ and the $\mathcal{A}_e$ tangent space is

$$\begin{aligned} T\mathcal{A}_e.f &= \mathcal{E}_2\langle(1,0,y^3 \pm 3x^2y),\ (0,2y,3xy^2 \pm x^3)\rangle + \mathcal{E}_3\langle(1,0,0),\ (0,1,0),\ (0,0,1)\rangle \\ &= (\mathcal{E}_2,\ \mathcal{E}_2,\ \mathcal{E}_2 - \{y, xy, x^2y, y^3\}) + (0,0,y^3 \pm 3x^2y) \end{aligned}$$

So we do have a choice of unfoldings. Choosing the one which gives a symmetrical picture gives $f_{a,b,c}(x,y) = (x, y^2, xy^3 \pm x^3y + ay + by^3 + cxy)$.

$H_2$**:** The normal form for $H_2$ is

$$f(x,y) = (x, y^3, xy + y^5)$$

and the $\mathcal{A}_e$ tangent space is given by

$$\begin{aligned} T\mathcal{A}_e.f =& \mathcal{E}_2\langle(1,0,y),\ (0,3y^2,x+5y^4)\rangle + \mathcal{E}_3\langle(1,0,0),\ (0,1,0),\ (0,0,1)\rangle \\ =& (\mathcal{E}_2 - \{y\},\ \mathcal{E}_2 - \{y\},\ \mathcal{E}_2 - \{y^2\}) + (y,0,y^2) \end{aligned}$$

We choose the unfolding $f_{a,b}(x,y) = (x, y^3 + ay, xy + y^5 + by^2)$.

$H_3$**:** The normal form for $H_3$ is

$$f(x,y) = (x, y^3, xy + y^8)$$

and the $\mathcal{A}_e$ tangent space is

$$\begin{aligned} T\mathcal{A}_e.f =& \mathcal{E}_2\langle(1,0,y),(0,3y^2,x+8y^7)\rangle + \mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \\ =& (\mathcal{E}_2 - \{y,y^3\}, \mathcal{E}_2 - \{y\}, \mathcal{E}_2 - \{y^2,y^5\}) + \{(y,0,y^2),(y^3,0,y^5)\} \end{aligned}$$

We choose the unfolding $f_{a,b,c}(x,y) = (x, y^3 + ay, xy + y^8 + by^2 + cy^5)$.

$P_3$: The normal form for $P_3$ is $f(x,y) = (x, xy + y^3, xy^2 + cy^4)$ and the $\mathcal{A}_e$-tangent space (of the stratum) is

$$\begin{aligned} T\mathcal{A}_e.f =& \mathcal{E}_2\langle(1,y,y^2),(0,x+3y^2,2xy+4cy^3)\rangle + f^*.\mathcal{E}_3\langle(1,0,0),(0,1,0),(0,0,1)\rangle \\ =& (\mathcal{E}_2, \mathcal{E}_2 - \{y\}, \mathcal{E}_2 - \{y, y^3\}) \end{aligned}$$

and so the unfolding is

$$F_{a,b,d}(x,y) = (x, xy + y^3 + ay, xy^2 + cy^4 + by + dy^3).$$

We now give the calculations of unfoldings of bi-germs from Section 7.7.

(I) $F(x,y;X,Y) = (x,y,0;X,Y,X^2 \pm Y^2)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2 - \{1\})$$

and so the unfolding is $F_a(x,y;X,Y) = (x,y,0;X,Y,X^2 \pm Y^2 + a)$.

(II) $F(x,y;X,Y) = (x,y,0;X,Y,X^2 + Y^3)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2 - \{1,Y\})$$

and so the unfolding is $F_{a,b,c}(x,y;X,Y) = (x,y,0;X,Y,X^2 + Y^3 + a + bY)$.

(III) $F(x,y;X,Y) = (x,y,0;X,Y,X^2 \pm Y^4)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2 - \{1,Y,Y^2\})$$

and so the unfolding is

$$F_{a,b,c}(x,y;X,Y) = (x,y,0;X,Y,X^2 \pm Y^4 + a + bY + cY^2).$$

(IV) $F(x,y;X,Y) = (x,y,0;Y^2,XY + Y^3,X)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2 - \{Y\},\mathcal{E}_2)$$

and so the unfolding is $F_a(x,y;X,Y) = (x,y,0;Y^2,XY + Y^3 + aY,X)$.

(V) $F(x,y;X,Y) = (x,y,0;Y^2,XY + Y^5,X)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2 - \{Y,Y^3\},\mathcal{E}_2)$$

and so the unfolding is $F_{a,b}(x,y;X,Y) = (x,y,0;Y^2,XY + Y^5 + aY + bY^3,X)$.

(VI) $F(x,y;X,Y) = (x,y,0;Y^2,XY + Y^7,X)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2 - \{Y,Y^3,Y^5\},\mathcal{E}_2)$$

and so the unfolding is

$$F_{a,b,c}(x,y;X,Y) = (x,y,0;Y^2, XY + Y^7 + aY + bY^3 + cY^5, X).$$

(VII) $F(x,y;X,Y) = (x,y,0;Y^3 \pm X^2Y, Y^2, X)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2 - \{1\}; \mathcal{E}_2 - \{Y\}, \mathcal{E}_2, \mathcal{E}_2)$$

and so the unfolding is $F_{a,b}(x,y;X,Y) = (x,y,a;Y^3 \pm X^2Y + bY, Y^2, X)$.

(VIII) $F(x,y;X,Y) = (x,y,0;Y^3 + X^3Y, Y^2, X)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2 - \{1\}; \mathcal{E}_2 - \{Y, XY\}, \mathcal{E}_2, \mathcal{E}_2)$$

and so the unfolding is $F_{a,b,c}(x,y;X,Y) = (x,y,a;Y^3 + X^3Y + bY + cXY, Y^2, X)$.

(IX) $F_{a,b,c}(x,y;X,Y) = (x,y,0;X, XY, Y^2 + X^3)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2 - \{1, X, X^2\})$$

and so the unfolding is $F_{a,b,c}(x,y;X,Y) = (x,y,0;X, XY, Y^2 + X^3 + a + bX + cX^2)$.

(X) $F_{a,b,c}(x,y;X,Y) = (x,y,0;X, Y^2, XY + Y^4)$ and the $\mathcal{A}_e$ tangent space is given by

$$T\mathcal{A}_e.F = (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2 - \{1, X, Y^2\})$$

so the unfolding is $F_{a,b,c}(x,y;X,Y) = (x,y,0;X, Y^2, XY + Y^4 + a + bX + cY^2)$

Finally, we give the calculations of unfoldings of tri-germs from Section 7.8.

(I) $F(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y,Y + X^2)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2 - \{1\})$$

and so the unfolding is $F_a(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y,Y + X^2 + a)$.

(II) $F(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y,Y + X^3)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2 - \{1, X\})$$

and so the unfolding is

$$F_{a,b}(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y,Y + X^3 + a + bX).$$

(III) $F(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y,X^2 \pm Y^2)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2; \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_2 - \{1, X\})$$

and so the unfolding is

$$F_{a,b}(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y,X^2 \pm Y^2).$$

(IV) $F(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y,Y+X^4)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2 - \{1,X,X^2\})$$

and so the unfolding is

$$F_{a,b,c}(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y,Y+X^4+a+bX+cX^2).$$

(V) $F(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y,XY+X^3)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2 - \{1,X\},\mathcal{E}_2 - \{1\})$$

and so the unfolding is

$$F_{a,b,c}(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y+a+bX,XY+X^3+c).$$

(VI) $F(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y,X^2+Y^3)$. The $\mathcal{A}_e$-tangent space is

$$T\mathcal{A}_e.F = (\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2;\mathcal{E}_2,\mathcal{E}_2,\mathcal{E}_2 - \{1,X\})$$

and so the unfolding is

$$F_{a,b,c}(x,y;\bar{x},\bar{y};X,Y) = (x,y,0;\bar{x},0,\bar{y};X,Y,X^2+Y^3).$$

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