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ORIGINAL PAPER

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Stabilization estimates for the Brinkman–Forchheimer–Kelvin–Voigt equation backward in time

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Abstract The final value value problem for the Brinkman–Forchheimer–Kelvin–Voigt equations is analysed for quadratic and cubic types of Forchheimer nonlinearity. The main term in the Forchheimer equations is allowed to be fully anisotropic. It is shown that the solution depends continuously on the final data provided the solution satisfies an a priori bound in L^3. The technique employed avoids the use of a specialist method for an improperly posed problem such as logarithmic convexity.

1 Introduction

Improperly posed problems occur in many branches of real life and have attracted the attention of many writers over a period of years and interest in this area is highly active, see e.g. Agmon [1], Agmon and Nirenberg [2], Ames and Epperson [3], Ames et al. [5], Ames and Straughan [4], Carasso [9], Carasso [10], Carasso [11], Chirita [12], Chirita et al. [14], Chirita and Zampoli [13], Franchi and Straughan [20], John [25], Lattès and Lions [30], Mophou and Warma [33], Payne and Straughan [37], Payne et al. [38]. One technique which has proved extremely useful in finding information to an improperly posed problem is that of John [25] who demonstrated how one may recover a class of stabilization estimates by requiring an a priori bound at a particular place or set of times. This paper has been an inspiration to many subsequent works and has led to the establishment of continuous dependence estimates which, in particular, are of much practical use in extrapolating from the past where one may employ the derived estimates in conjunction with numerical methods to obtain accurate results; this is explained in detail by Carasso [9], Carasso [10], Carasso [11]. Specific studies of deriving estimates for improperly posed backward in time problems are contained in Chirita [12], Ciarletta [15], Crispo et al. [16], Franchi and Straughan [20], Galdi and Straughan [21], Lattès and Lions [30], Mophou and Warma [33], Passarella et al. [36], Straughan [39].

The theory of Kelvin–Voigt fluids has been extensively analysed in the Russian literature, see e.g. Oskolkov [34], Oskolkov and Shadiev [35]. This theory yields a class of viscoelastic fluids and properties of solutions have been studied in detail by e.g. Kalantarov et al. [28], Kalantarov and Zelik [27], Damazio et al. [17]. Our aim is to analyse equations for a Kelvin–Voigt fluid in a porous medium which is of Brinkman–Forchheimer type. Many studies of fluid flow in a Brinkman–Forchheimer porous medium have appeared where the saturating fluid is viscous or even viscoelastic. However, studies of qualitative properties of solutions to the equations for a Brinkman–Forchheimer–Kelvin–Voigt theory are relatively recent, see Anh and Trang [6], Su and Qin [41],

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Mohan [32], Thuy [42]. All of these articles deal with the forward in time problem. In this article we focus on the backward in time problem where we consider initial data but we wish to calculate the solution backward in time to determine the previous behaviour. Such work for the Navier–Stokes equations is well known and uniqueness results and continuous dependence estimates, see e.g. Crispo et al. [16], Galdi and Straughan [21], Franchi and Straughan [20], Ames and Straughan [4], Straughan [39], have led to useful bounds which may be employed in conjunction with a suitable carefully defined numerical technique to even compute solutions backward in time, see the important work of Carasso [10,11].

Applications of Kelvin–Voigt fluids and Brinkman–Forchheimer porous materials are very important in everyday engineering and industrial workings. For example, Wu et al. [45] employ Brinkman–Forchheimer theory in the treatment by acid of carbonate reservoirs. They argue that acid which is injected dissolves material in the vicinity of the wellbore which leads to the creation of channels which improve flow between the reservoir and the well. By employing Brinkman–Forchheimer theory they simulate the wormhole like procedure found in real life. Gidde and Pawar [22] employ Kelvin–Voigt fluid theory to describe polydimethylsiloxane in a micropump, Jayabal et al. [24] employ the same class of fluids to model skin in the context of the cosmetics industry. Further use of of a Kelvin–Voigt fluid is by Jozwiak et al. [26] who model the dynamic behaviour of biopolymer materials. The complex shear moduli of a Kelvin–Voigt fluid model are used by Erdel et al. [19] to calculate time-dependent coefficients for anomalous diffusion in a living cell nucleus. Askarian et al. [8] employ Kelvin–Voigt fluid models in their analysis for the foundation for pipes conveying industrial fluids. One very important use of Kelvin - Voigt fluids is in the field of viscous dampers which are employed to reduce the effects of vibrations in large civil engineering structures, see e.g. Greco and Marano [23], Lewandowski and Chorazyczewski [31], Xu et al. [46]. Very high building structures like the Burj Khalifa in Dubai require viscous dampers to control oscillations. Another example is in the tower Taipei 101 in the city of Taipei. This tower is 1667 feet high and is very close to a fault line in the Earth's crust. It was thus very important in its construction to be able to withstand typhoons and earthquakes. To achieve this the building Taipei 101 employs a 730 ton mass damper which is connected to eight viscous fluid dampers which act like shock absorbers when the mass damper moves.

We believe this is the first analysis of a model for the flow in a Brinkman–Forchheimer–Kelvin–Voigt porous material for the backward in time problem.We believe our continuous dependence estimates will be useful for both analytical and numerical studies. We stress that we employ an energy type method even in the backward in time situation. The continuous dependence achieved is of regular type and does not need to be of Hölder type as it is for example in the analiogous Navier–Stokes problem. The reason for this is the presence of the Kelvin–Voigt term which acts to regularize the solution even in the backward in time problem. For the forward in time problem the regularization effect of the Kelvin–Voigt term is already known as is very well explained by Damazio et al. [17]. This non-Hólder continuous dependence result should lead to useful numerical schemes when employed in a manner similar to the work of Carasso [10,11].

2 Brinkman–Forchheimer–Kelvin–Voigt theory

The basic equations for Brinkman–Forchheimer–Kelvin–Voigt theory for flow of a viscoelastic fluid in a porous medium have form

$$\begin{aligned} v_{i,t} + v_j v_{i,j} - \lambda \Delta v_{i,t} &= -p_{,i} + \nu \Delta v_i - \xi_{ij} v_j - b|\mathbf{v}|v_i, \\ v_{i,i} &= 0. \end{aligned} \tag{1}$$

This represents flow of an incompressible fluid and $v_i,\ p$ are the velocity and pressure, respectively, at time t and position $\mathbf{x}$. We employ standard indicial notation in conjunction with the Einstein summation convention throughout, so for example,

$$v_j v_{i,j} \equiv \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} \equiv u\frac{\partial v_i}{\partial x} + v\frac{\partial v_i}{\partial y} + w\frac{\partial v_i}{\partial z},$$

for $i = 1, 2, 3$, and $\mathbf{v} = (u, v, w) \equiv (v_1, v_2, v_3)$. In equation (1) Δ is the Laplacian in $\mathbb{R}^3$, λ is the Kelvin–Voigt coefficient, ν is the kinematic viscosity, ξ_{ij} is the anisotropic Darcy tensor, and b is the Forchheimer coefficient. Equations (1) hold on $\Omega \times \{t > 0\}$, where Ω is a bounded domain in $\mathbb{R}^3$ with boundary Γ. Equations (1) are to be solved subject to boundary data

$$v_i(\mathbf{x}, t) = h_i(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma, \tag{2}$$

and for initial data

$$v_i(\mathbf{x}, 0) = q_i(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{3}$$

We wish to establish continuous dependence of a solution to the boundary-initial value problem (1-3) for the backward in time case where we consider t being negative. As we show later this yields an improperly posed problem and our goal is to derive continuous dependence estimates on compact intervals of time.

3 Continuous dependence backward in time

To investigate continuous dependence backward in time we suppose Γ is sufficiently regular to allow application of the divergence theorem and we let (u_i, p_1) and (v_i, p_2) be two solutions to (1)-(3) for $t < 0$, such that these solutions satisfy the boundary data (2) for the same functions h_i, but for different initial data functions $q_i^1(\mathbf{x})$ and $q_i^2(\mathbf{x})$. Define the difference variables w_i, π, and q_i by

$$w_i = u_i - v_i, \quad \pi = p_1 - p_2, \quad q_i = q_i^1 - q_i^2. \tag{4}$$

Next replace t by $-t$ and then the difference solution satisfies the boundary-initial value problem

$$\begin{aligned} w_{i,t} - \lambda \Delta w_{i,t} &= w_j u_{i,j} + v_j w_{i,j} + \pi_{,i} - \nu \Delta w_i + \xi_{ij} w_j + b(|\mathbf{u}|u_i - |\mathbf{v}|v_i), \\ w_{i,i} &= 0, \end{aligned} \tag{5}$$

on $\Omega \times \{t > 0\}$, together with the boundary and initial data

$$\begin{aligned} w_i(\mathbf{x}, t) &= 0, \quad \mathbf{x} \in \Gamma, t > 0, \\ w_i(\mathbf{x}, 0) &= q_i(\mathbf{x}), \quad \mathbf{x} \in \Omega. \end{aligned} \tag{6}$$

Let the Darcy tensor $\xi_{ij}(\mathbf{x})$ be symmetric and satisfy the bound

$$|\xi_{ij}| \leq \hat{\xi}, \quad \forall \mathbf{x} \in \Omega^-, \tag{7}$$

where Ω^- is the closure of Ω. Define further the class of solutions $\mathcal{M}$ by those which satisfy (1-3) for $t < 0$ and are such that

$$\sup_{[0,T]} \| \mathbf{u} \|_3 \leq M, \quad \sup_{[0,T]} \| \mathbf{v} \|_3 \leq M, \tag{8}$$

where $\| \cdot \|_p$ denotes the norm on $L^p(\Omega)$ and $T < \infty$ is given. We then have

Theorem 1 *Let (u_i, p) be a solution to (1)–(3) which belongs to class $\mathcal{M}$ for $t < 0$. Then this solution depends continuously upon the initial data.*

Proof To demonstrate continuous dependence we let (u_i, p_1) and (v_i, p_2) be two solutions to (1)-(3) as defined above. The difference solution (w_i, π) then satisfies the boundary-initial value problem (5),(6) above.
Let now $(\cdot, \cdot)$ and $\| \cdot \|$ denote the inner product and norm on $L^2(\Omega)$.
Multiply (5) by w_i and integrate over Ω. After the use of boundary conditions we may obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2} \| \mathbf{w} \|^2 + \frac{\lambda}{2} \| \nabla \mathbf{w} \|^2\right) = & \ (w_i, w_j u_{i,j}) + (w_i, v_j w_{i,j}) \\ & + (\pi_{,i}, w_i) + \nu \| \nabla \mathbf{w} \|^2 + (\xi_{ij} w_j, w_i) \\ & + b(|\mathbf{u}|u_i - |\mathbf{v}|v_i, w_i). \end{aligned} \tag{9}$$

The second and third terms on the right of (9) may be shown to be zero after integration by parts and use of the boundary and incompressibility conditions. Integrate by parts on the first term on the right of (9) and rearrange the last term to obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2} \| \mathbf{w} \|^2 + \frac{\lambda}{2} \| \nabla \mathbf{w} \|^2\right) = & - (w_{i,j}, w_j u_i) + \nu \| \nabla \mathbf{w} \|^2 \\ & + (\xi_{ij} w_j, w_i) + b(|\mathbf{u}|w_i, w_i) \\ & + b([|\mathbf{u}| - |\mathbf{v}|]v_i, w_i). \end{aligned} \tag{10}$$

Next, use the triangle inequality $|\mathbf{w}| \geq |\mathbf{u}| - |\mathbf{v}|$, and the bound for ξ_{ij} to deduce that

$$\frac{d}{dt}\left(\frac{1}{2} \parallel \mathbf{w} \parallel^2 + \frac{\lambda}{2} \parallel \nabla \mathbf{w} \parallel^2\right) \leq -(w_{i,j}, w_j u_i) + \nu \parallel \nabla \mathbf{w} \parallel^2 + \hat{\xi} \parallel \mathbf{w} \parallel^2 + b \int_\Omega [|\mathbf{u}| + |\mathbf{v}|] \, |\mathbf{w}|^2 dx. \tag{11}$$

Now, employ the Cauchy-Schwarz and Hölder inequalities and the Sobolev inequality for the embedding of $H_0^1(\Omega) \subset L^6(\Omega)$ to obtain

$$\begin{aligned} -(w_{i,j}, w_j u_i) &\leq \parallel \nabla \mathbf{w} \parallel \left(\int_\Omega |\mathbf{w}|^2 |\mathbf{u}|^2 dx\right)^{1/2} \\ &\leq \parallel \nabla \mathbf{w} \parallel \parallel \mathbf{w} \parallel_6 \parallel \mathbf{u} \parallel_3 \\ &\leq c_1 \parallel \nabla \mathbf{w} \parallel^2 \parallel \mathbf{u} \parallel_3, \end{aligned} \tag{12}$$

where c_1 is the Sobolev constant. One further requires the estimates

$$\int_\Omega |\mathbf{u}| \, |\mathbf{w}|^2 \, dx \leq \parallel \mathbf{u} \parallel_{3/2} \parallel \mathbf{w} \parallel_6^2 \leq c_1^2 \parallel \mathbf{u} \parallel_{3/2} \parallel \nabla \mathbf{w} \parallel^2 \tag{13}$$

where we have employed Hölder's inequality and the Sobolev inequality. Next, from the Cauchy-Schwarz inequality

$$\int_\Omega |\mathbf{u}|^{3/2} dx \leq m^{1/2} \left(\int_\Omega |\mathbf{u}|^3 dx\right)^{1/2}, \tag{14}$$

where m is the Lebesgue measure of Ω. Employing (14) in (13) we then obtain

$$\int_\Omega |\mathbf{u}| \, |\mathbf{w}|^2 dx \leq c_1^2 m^{1/3} \parallel \mathbf{u} \parallel_3 \parallel \nabla \mathbf{w} \parallel^2 . \tag{15}$$

An analogous estimate holds with $\mathbf{v}$ replacing $\mathbf{u}$. Hence, we may now employ (12) and (15) in (11) to find

$$\frac{d}{dt}\left(\frac{1}{2} \parallel \mathbf{w} \parallel^2 + \frac{\lambda}{2} \parallel \nabla \mathbf{w} \parallel^2\right) \leq c_2 \parallel \nabla \mathbf{w} \parallel^2 + \hat{\xi} \parallel \mathbf{w} \parallel^2, \tag{16}$$

where $c_2 = (2bc_1^2 m^{1/3} + c_1)M + \nu$. Let $\alpha = \max\{2\hat{\xi}, 2c_2/\lambda\}$ and then from (16) one obtains

$$\frac{dE}{dt} \leq \alpha E, \quad t \in [0, T], \tag{17}$$

where

$$E(t) = \frac{1}{2} \parallel \mathbf{w} \parallel^2 + \frac{\lambda}{2} \parallel \nabla \mathbf{w} \parallel^2 . \tag{18}$$

Upon integration (17) yields

$$\frac{1}{2} \parallel \mathbf{w}(t) \parallel^2 + \frac{\lambda}{2} \parallel \nabla \mathbf{w}(t) \parallel^2 \leq e^{\alpha t} E(0), \tag{19}$$

$\forall t \in [0, T]$. The theorem is thus proved.

Remark 1 It is noteworthy that we have proved continuous dependence in the final value problem by means of an energy-like technique. For the Navier–Stokes version of the Brinkman–Forchheimer equation where the Kelvin–Voigt term is not present such an energy technique fails and one requires a specialist method like logarithmic convexity. In addition the bounds imposed upon u_i and v_i are weaker than those required in the analogous Navier–Stokes theory, cf. eg. Franchi and Straughan [20]. Furthermore, the continuous dependence, equation (19), is not of Hölder type as in the Navier–Stokes case. When the dependence is of Hölder type the exponent of the initial data measure depends on the Hölder coefficient and the estimate holds on $[0, T)$ and is weaker as t approaches T^-. The result (19) hinges on the presence of the Kelvin–Voigt term $\lambda \|\nabla w\|^2$.

Remark 2 If we were to enclose the fluid and porous medium for equations (1) in an isolated cannister, shake the system, and then hold it steady at time $t = 0$ it is of interest to observe the behaviour for negative time. Such a motion is known as isolated cannister flow, see Dunn and Fosdick [18]. For this case we take equations (1) for $t < 0$ and select the boundary data $h_i \equiv 0$ in (2). Then we multiply equation (1) by v_i for $t < 0$ to obtain (reversing time by switching $t \to -t$)

$$\frac{d}{dt}\left(\frac{1}{2} \parallel \mathbf{v} \parallel^2 + \frac{\lambda}{2} \parallel \nabla \mathbf{v} \parallel^2\right) = \nu \parallel \nabla \mathbf{v} \parallel^2 + (\xi_{ij} v_j, v_i) + b \parallel \mathbf{v} \parallel_3^3 . \tag{20}$$

Suppose now

$$\xi_{ij}\eta_i\eta_j \geq 0, \quad \forall \eta_i \neq 0, \tag{21}$$

then write the ν term as $\nu = \nu\gamma + \nu(1-\gamma)$, some $0 < \gamma < 1$, and one finds

$$\frac{d}{dt}\left(\frac{1}{2} \parallel \mathbf{v} \parallel^2 + \frac{\lambda}{2} \parallel \nabla \mathbf{v} \parallel^2\right) \geq \gamma\nu \parallel \nabla \mathbf{v} \parallel^2 + (1-\gamma)\nu\lambda_1 \parallel \mathbf{v} \parallel^2, \tag{22}$$

where $\lambda_1(\Omega)$ is the first eigenvalue in the membrane problem for Ω. Pick

$$\gamma = \frac{2\lambda\lambda_1}{1 + 2\lambda\lambda_1}$$

and then for

$$k = \frac{2\lambda_1\nu}{1 + 2\lambda\lambda_1}$$

one may derive from (22)

$$\frac{dF}{dt} \geq kF, \tag{23}$$

where

$$F = \frac{1}{2} \parallel \mathbf{v} \parallel^2 + \frac{\lambda}{2} \parallel \nabla \mathbf{v} \parallel^2 .$$

Hence,

$$F(t) \geq F(0)\exp(kt). \tag{24}$$

This shows that in general the boundary-initial value problem (1–2) for $t < 0$ is non-well posed for all time.

Remark 3 If we consider isolated cannister flow without the Kelvin–Voigt term, i.e. set $\lambda = 0$ in (1), then instead of (20) we obtain

$$\frac{d}{dt}\frac{1}{2} \parallel \mathbf{v} \parallel^2 = \nu \parallel \nabla \mathbf{v} \parallel^2 + (\xi_{ij} v_j, v_i) + b \parallel \mathbf{v} \parallel_3^3 . \tag{25}$$

Adopting (21) and using Hölder's inequality with now $G(t) = \|\mathbf{v}\|^2/2$, we find from (25),

$$G' \geq 2\nu\lambda_1 G + \frac{2^{3/2}}{m^{2/3}} G^{3/2}. \tag{26}$$

Under suitable conditions on $G(0)$, inequality (26) leads to blow-up in finite time and nonexistence of the solution for all time.

The continuous dependence estimate (19) relies on the bounds (8). The theorem only holds if this is true. In general, we do not know if a solution to (1) exists for all t. Consider the simple model

$$\begin{aligned} u_t - \lambda \Delta u_t &= -\nu \Delta u + \xi u + b u^{1+\alpha}, \qquad \alpha > 0, \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \\ u(\mathbf{x}, t) &= 0, \qquad \text{on } \Gamma, \end{aligned} \tag{27}$$

where the domain is $\Omega \times \{t > 0\}$ and u_0 is positive. Use the eigenfunction method, cf. Straughan [40, pp. 10,11], for which one introduces

$$\begin{aligned} \Delta\phi + \lambda_1\phi = 0, \qquad & \mathbf{x} \in \Omega, \\ \phi = 0, \qquad & \mathbf{x} \in \Gamma, \end{aligned}$$

where λ_1 is the first eigenvalue and $\phi > 0$ is the first eigenfunction of the membrane problem for Ω. Set

$$H(t) = \int_\Omega \phi u \, dx,$$

then for (27) one finds

$$\frac{dH}{dt} \geq aH + cH^{1+\alpha}, \tag{28}$$

where

$$a = \frac{\nu\lambda_1 + \xi}{1 + \lambda\lambda_1}, \qquad c = \frac{b}{\Phi^{1+\alpha}(1 + \lambda\lambda_1)},$$

$\Phi = \int_\Omega \phi \, dx$. Inequality (28) is integrated to find

$$H^\alpha(t) \geq \frac{H^\alpha(0)e^{a\alpha t}}{1 - h(e^{a\alpha t} - 1)}, \tag{29}$$

where

$$h = \frac{bH^\alpha(0)}{(\nu\lambda_1 + \xi)\Phi^{1+\alpha}}.$$

For initial data such that $H(0) > 0$, inequality (29) leads to global nonexistence, with for suitable initial data, blow-up in finite time. The blow-up time for the right hand side of (29) is

$$T = \frac{1}{a\alpha} \log\Big(1 + \frac{1}{h}\Big). \tag{30}$$

Note that for λ large, T is large, whereas for b large, T is small, so λ does act to regularize.

The fact that the simple model (27) suggests blow-up in finite time does occur does not prove anything for the backward in time problem for (1). There are three immediate differences with the model. Equations (1) are a system, v_i satisfies the constraint $v_{i,i} = 0$, and the presence of the convective term $-v_i v_{i,j}$. For some partial differential equation systems the convective term can prevent blow-up with forcing terms up to u^2, see Straughan [40, pp. 37–39], and the references therein.

4 Continuous dependence with a second Forchheimer term

Some writers employ a stronger Forchheimer term than that in (1), cf. Antontsev and Khompysh [7], Ugurlu [43], Wang and Lin [44]. And then on may wish to consider the boundary-initial value problem consisting of (2) and (3) but with (1) replaced by

$$\begin{aligned} v_{i,t} + v_j v_{i,j} - \lambda\Delta v_{i,t} &= -p_{,i} + \nu\Delta v_i - \xi_{ij} v_j - b|\mathbf{v}|v_i - c|\mathbf{v}|^2 v_i, \\ v_{i,i} &= 0. \end{aligned} \tag{31}$$

One may establish continuous dependence on the initial data for a solution to Eq. (31) backward in time by proceeding as in Sect. 3. In addition to the analysis of Sect. 3 one encounters upon multiplying the difference equation by w_i a term of form

$$\begin{aligned} c(|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i)w_i &= c\left(|\mathbf{u}|^2 w_i w_i + [|\mathbf{u}|^2 - |\mathbf{v}|^2] v_i w_i\right) \\ &= c\left(|\mathbf{u}|^2 w_i w_i + [u_j + v_j] w_j v_i w_i\right) \\ &\leq \frac{3c}{2} w_i w_i \left(|\mathbf{u}|^2 + |\mathbf{v}|^2\right). \end{aligned} \tag{32}$$

Thus in the analogous derivation to (17) one encounters the term

$$\frac{3c}{2}\int_\Omega w_i w_i(|\mathbf{u}|^2+|\mathbf{v}|^2)dx. \tag{33}$$

This term is handled by Hölder's inequality and the Sobolev inequality as

$$\int_\Omega |\mathbf{u}|^2 w_i w_i dx \le \| \mathbf{w} \|_6^2 \| \mathbf{u} \|_3^2 \le c_1^2 \| \nabla \mathbf{w} \|^2 \| \mathbf{u} \|_3^2 . \tag{34}$$

Hence we find

$$\begin{aligned} c\int_\Omega \left(|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i\right) dx &\le \frac{3cc_1^2}{2} \| \nabla \mathbf{w} \|^2 \left(\| \mathbf{u} \|_3^2 + \| \mathbf{v} \|_3^2\right) \\ &\le 3cc_1^2 M^2 \| \nabla \mathbf{w} \|^2 . \end{aligned} \tag{35}$$

We thus arrive at an inequality like (17) but now

$$\alpha = \max\left\{2\hat{\xi}, \frac{2c_3}{\lambda}\right\} \quad \text{where } c_3 = 3cc_1 M^2 + c_2.$$

Remark 4 Similar comments to those of remark 3 apply to the model of this section. The continuous dependence estimate relies on the solution belonging to the class $\mathcal{M}$, which is in keeping with the procedure advocated by John [25].

5 Inclusion of temperature effects

Damazio et al. [17] argue that the λ term in the Kelvin–Voigt theory should be regarded as a regularizing term. Kaya [29] studies a thermal convection problem for the Navier-Stokes-Voigt equations but this article also introduces a regularization term in the energy balance equation. One may wish to introduce such terms in a non-isothermal theory for the Brinkman–Forchheimer–Kelvin–Voigt theory and then we replace (1–3) by

$$\begin{aligned} v_{i,t} + v_j v_{i,j} - \lambda \Delta v_{i,t} &= -p_{,i} + \nu \Delta v_i - \xi_{ij} v_j - b|\mathbf{v}|v_i + g_i T, \\ v_{i,i} &= 0, \\ T_{,t} + v_i T_{,i} - \chi \Delta T_{,t} &= \kappa \Delta T, \end{aligned} \tag{36}$$

in $\Omega \times \{t < 0\}$,

$$\begin{aligned} v_i(\mathbf{x},t) &= h_i(\mathbf{x},t), \quad T = H(\mathbf{x},t), \quad \mathbf{x} \in \Gamma, \\ v_i(\mathbf{x},0) &= q_i(\mathbf{x}), \quad T(\mathbf{x},0) = Q(\mathbf{x}), \quad \mathbf{x} \in \Omega. \end{aligned} \tag{37}$$

One may employ a technique similar to that of Sect. 3 to demonstrate continuous dependence on the final data for the backward in time problem for (36), (37).
The analysis employs a function E of form

$$E(t) = \frac{1}{2} \| \mathbf{w} \|^2 + \frac{\lambda}{2} \| \nabla \mathbf{w} \|^2 + \frac{1}{2} \| \theta \|^2 + \frac{\chi}{2} \| \nabla \theta \|^2, \tag{38}$$

where θ is the difference in temperature for two solutions to (36), (37) for the same boundary data, but different initial data.

6 Conclusions

We have analyzed the solutions to the Brinkman–Forchheimer–Kelvin–Voigt equations for the improperly posed backward in time problem. By employing what is essentially an energy method and a weak a priori bound on the class of solutions we have been able to produce estimates which demonstrate continuous dependence upon the final data for compact intervals of time. This should be contrasted with the analogous class of problem where Kelvin–Voigt theory is not employed and an energy method fails. The Forchheimer nonlinear term has been allowed to be quadratic in the velocity field, and then generalized to include a cubic term. We have further shown how temperature effects may be included provided a Kelvin–Voigt regularization is applied to the balance of energy equation in addition the inclusion in the momentum equation. In all cases, continuous dependence upon the final data is established under a relatively weak a priori bound.

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Declarations

Conflict of interest There are no conflicts of interest.

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