Chapter

Introductory Chapter: The Generalizations of the Fourier Transform

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1. Introduction

In the world of physical science, important physical quantities such as sound, pressure, electric current, voltage, and electromagnetic fields vary with time t. Such quantities are labeled as signals/waveforms. Exemplified by signals with examples such as oral signals, optical signals, acoustic signals, biomedical signals, radar, and sonar. Indeed, signals are very common in the real world. Time-frequency analysis is a vital aid in signal analysis, which is concerned with how the frequency of a function (or signal) behaves in time, and it has evolved into a widely recognized applied discipline of signal processing. The signals can be classified under various categories. It could be done in terms of continuity (continuous v/s discrete), periodicity(periodic v/s aperiodic), stationarity(stationary v/s non-stationary), and so on. Most of the signals in nature are non-stationary (i.e., whose spectral components change with time) and apt presentation of such non-stationary signals need frequency analysis, which is local in time, resulting in the time-frequency analysis of signals. Although time frequency analysis of signals had its origin almost 70 years ago, there has been major development of the time-frequency distribution approach in the last three decades. The basic idea of these methods is to develop a joint function of time and frequency, known as a time-frequency distribution, that can describe the energy density of a signal simultaneously in both time and frequency domains. In signal processing, time-frequency analysis comprises those techniques that study signal in both the time and frequency domains simultaneously, using various time-frequency representations/tools known as integral transformations. An integral transform maps a function/signal from one function space into another function space *via* integration, where some of the properties of the original function might be more easily characterized and manipulated than in the original function space. The integral transforms are essentially considered from the functional analysis viewpoint and as a useful technique of mathematical physics.

The classical Fourier transform (FT) is an integral transform introduced by Joseph Fourier in 1807 [1], is one of the most valuable and widely-used integral transforms that converts a signal from time versus amplitude to frequency versus amplitude. Thus FT can be considered as the time-frequency representation tool in signal processing and analysis. A fundamental limitation of the Fourier transform is that the all properties of a signal are global in scope. Information about local features of the signal, such as changes in frequency, becomes a global property of the signal in the frequency domain. In order to circumvent these drawbacks of FT, authors in Ref. [2] introduced the generalizations of FT that includes short-time Fourier transform (STFT) by performing the FT on a block-by-block basis rather than to process the entire signal at once. In spite of the fact that STFT did much to ameliorate the limitations of FT, still in some cases the STFT cannot track the signal dynamics properly for a signal with both very high frequencies of short duration and very low frequencies of long duration. To overcome these drawbacks of FT and STFT different novel generalizations of the classical Fourier transform came into existence *viz.*: the fractional Fourier transform (FRFT), the Fresnal transform, the linear canonical transform (LCT), the quadratic-phase Fourier transform (QPFT), and so on. As a generalization of classical Fourier transform, the FRFT, the LCT, the QPFT gained its ground intermittently and profoundly influenced several branches of science and engineering including signal and image processing, quantum mechanics, neural networks, differential equations, optics, pattern recognition, radar, sonar, and communication systems.

2. Fourier transform and its generalizations

2.1 Fourier transform

Joseph Fourier [1] in 1822 published first work about Fourier transform, which is an integral transform that converts a mathematical function from the time domain to the frequency domain. Fourier transform measures the frequency component of a given function. The Fourier transform has evolved into a widely recognized discipline of harmonic analysis and has been successfully applied in diverse scientific and engineering pursuits [3–6].

Let us begin with definition of the classical Fourier transform.

Definition 1. *The FT of any signal* $x(t) \in L^2(\mathbb{R})$ *is defined and denoted as*

$$\mathscr{F}[x(t)](\xi) = \hat{x}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi t} x(t) dt, \tag{1}$$

and corresponding inversion formula is given by

$$\mathscr{F}^{-1}(\mathscr{F}[x(t)](\xi))(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi t} \mathscr{F}[x(t)](\xi) d\xi. \tag{2}$$

Example 1. *Consider a function* $x(t) = \begin{cases} e^{-\alpha t} & \text{for} \quad t \geq 0, \alpha > 0, \\ 0 & \text{otherwise}; \end{cases}$, *then the Fourier transform of* $x(t)$ *is obtained as*

$$\begin{aligned} \mathscr{F}[x(t)](\xi) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\xi t} e^{-\alpha t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty (\cos \xi t - i \sin \xi t) e^{-\alpha t} dt \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \int_0^\infty \cos \xi t e^{-\alpha t} dt - i \int_0^\infty \sin \xi t e^{-\alpha t} dt \right\} \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \frac{\alpha}{\alpha^2 + \xi^2} - \frac{i\xi}{\alpha^2 + \xi^2} \right\}. \end{aligned}$$

Example 2. *Consider the function*

$$x(t) = \begin{cases} \sin 3t & \text{for} \quad -\pi \le t \le \pi, \\ 0 & \text{otherwise}. \end{cases}$$

Then the Fourier transform of $x(t)$ is obtained as

$$\begin{aligned} \mathscr{F}[x(t)](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\cos \xi t - i \sin \xi t) \sin 3t dt \\ &= \frac{-i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin \xi t \sin 3t dt \\ &= \frac{i3\sqrt{2} \sin \xi\pi}{\sqrt{\pi}(\xi^2 - 9)}. \end{aligned}$$

Next, we shall study some properties of FT.

Theorem 1 (Translation)**.** *The Fourier transform of any function $x(t-k)$ is given by*

$$\mathscr{F}[x(t-k)](\xi) = e^{-i\xi k} \mathscr{F}[x(t)](\xi). \tag{3}$$

Proof. From Definition 1, we have

$$\begin{aligned} \mathscr{F}[x(t-k)](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi t} x(t-k) dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi(u+k)} x(u) du \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi k} e^{-i\xi u)} x(u) du \\ &= \frac{1}{\sqrt{2\pi}} e^{-i\xi k} \int_{\mathbb{R}} e^{-i\xi u)} x(u) du \\ &= e^{-i\xi k} \mathscr{F}[x(t)](\xi). \end{aligned}$$

This completes the proof. □

Theorem 2 (Modulation)**.** *The Fourier transform of any function $e^{i\xi_0 t} x(t)$ is given by*

$$\mathscr{F}\left[e^{i\xi_0 t} x(t)\right](\xi) = \mathscr{F}[x(t)](\xi - \xi_0). \tag{4}$$

Proof. From Definition 1, we have

$$\begin{aligned} \mathscr{F}\left[e^{i\xi_0 t} x(t)\right](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi t} e^{i\xi_0 t} x(t) dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i(\xi - \xi_0)t} x(t) dt \\ &= \mathscr{F}[x(t)](\xi - \xi_0). \end{aligned}$$

This completes the proof. □

Theorem 3 (Orthogonality relation)**.** *The Fourier transform of the functions $x(t)$ and $y(t)$ in $L^2(\mathbb{R})$ satisfies the following orthogonality relation*

$$\langle \mathcal{F}[x(t)], \mathcal{F}[y(u)] \rangle = \langle x(t), y(u) \rangle. \tag{5}$$

Proof. We have

$$\begin{aligned}
\langle \mathcal{F}[x(t)], \mathcal{F}[y(u)] \rangle &= \int_{\mathbb{R}} \mathcal{F}[x(t)](\xi)\overline{\mathcal{F}[y(u)](\xi)}d\xi \\
&= \int_{\mathbb{R}} \mathcal{F}[x(t)](\xi)\overline{\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi u} y(u)du\right)}d\xi \\
&= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi t} x(t)dt\right)\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\xi u}\overline{y(u)}du\right)d\xi \\
&= \int_{\mathbb{R}^2} x(t)\overline{y(u)}\left(\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi(u-t)}d\xi\right)dtdu \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} x(t)\overline{y(u)}\delta(u-t)dtdu \\
&= \int_{\mathbb{R}} x(t)\overline{y(u)}dt \\
&= \langle x(t), y(u) \rangle.
\end{aligned}$$

This completes the proof. □

Note: If we take $x(t) = y(t)$, the orthogonality relation yields Plancherel' s Theorem for the Fourier transforms that states the energy of a signal ln the time domain, is the same as the energy in the frequency domain given as

$$\|\mathcal{F}(x(t))\| = \|x(t)\|. \tag{6}$$

Next, we show that the inverse Fourier operator is the adjoint of the Fourier operator.

Theorem 4. *Let $x(t)$ and $y(t)$ in $L^2(\mathbb{R})$, then*

$$\langle \mathcal{F}[x(t)](\xi), y(\xi) \rangle = \left\langle x(t), \mathcal{F}^{-1}[y](t) \right\rangle. \tag{7}$$

Proof. We have

$$\begin{aligned}
\langle \mathcal{F}[x(t)], y(t) \rangle &= \int_{\mathbb{R}} \mathcal{F}[x(t)](\xi)\overline{y(\xi)}d\xi \\
&= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi t} x(t)dt\right)\overline{y(\xi)}d\xi \\
&= \int_{\mathbb{R}} x(t)\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi t}\overline{y(\xi)}d\xi\right)dt \\
&= \int_{\mathbb{R}} x(t)\overline{\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\xi t} y(\xi)d\xi\right)}dt \\
&= \int_{\mathbb{R}} x(t)\overline{\mathcal{F}^{-1}[y](t)}dt \\
&= \left\langle x(t), \mathcal{F}^{-1}[y](t) \right\rangle.
\end{aligned}$$

This completes the proof. □

Theorem 5. *Let $x(t)$ and $y(t)$ in $L^2(\mathbb{R})$, then*

$$\mathcal{F}[(x*y)](\xi) = \sqrt{2\pi}\,\mathcal{F}[x(t)](\xi)\mathcal{F}[y(t)](\xi), \tag{8}$$

where $x*y$ denotes the convolution of the functions $x(t)$ and $y(t)$ and is given by

$$(x*y)(t) = \int_{\mathbb{R}} x(t)y(u-t)dt.$$

Proof. By applying definition of Fourier transform to the convolution of the functions $x(t)$ and $y(t)$, we obtain

$$\begin{aligned}
\mathcal{F}[(x*y)](\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(x*y)(u)e^{-i\xi u}du \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} x(t)y(u-t)dt\right)e^{-i\xi u}du \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} x(t)y(v)e^{-i\xi(t+v)}dvdt \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\xi t}x(t)y(v)e^{-i\xi v}dvdt \\
&= \sqrt{2\pi}\left\{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi t}x(t)dt\right\}\left\{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi v}y(v)dv\right\} \\
&= \sqrt{2\pi}\mathcal{F}[x(t)](\xi)\mathcal{F}[y(t)](\xi).
\end{aligned}$$

This completes the proof. □

2.2 Windowed Fourier transform

Definition 2. *Let Ψ be a given window function in $L^2(\mathbb{R})$, then the window Fourier transform (WFT) of any function $x(t) \in L^2(\mathbb{R})$ is defined and denoted as*

$$\mathcal{V}_\Psi[x(t)](b,\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi t}x(t)\overline{\Psi(t-b)}dt, \quad b,\xi \in \mathbb{R}. \tag{9}$$

Further, the WFT (9) can be rewritten as

$$\mathcal{V}_\Psi[x(t)](b,\xi) = \mathcal{F}\left[x(t)\overline{\Psi(t-b)}\right]. \tag{10}$$

Applying inverse FT (2), (10) yields

$$\begin{aligned}
x(t)\overline{\Psi(t-b)} &= \mathcal{F}^{-1}[\mathcal{V}_\Psi[x(t)](b,\xi)] \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\xi t}\mathcal{V}_\Psi[x(t)](b,\xi)d\xi
\end{aligned} \tag{11}$$

Multiplying (11) both sides by $\Psi(t-b)$ and then integrating with respect to db, we get

$$x(t)\|\Psi\|^2 = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\xi t}\mathcal{V}_\Psi[x(t)](b,\xi)\Psi(t-b)d\xi db.$$

Equivalently, we have

$$x(t) = \frac{1}{\sqrt{2\pi}\|\Psi\|^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\xi t}\mathcal{V}_\Psi[x(t)](b,\xi)\Psi(t-b)d\xi db. \tag{12}$$

Eq. (12) gives the inversion formula corresponding to WFT (9).

Theorem 6 (Orthogonality relation). *For any two functions* $x(t), y(t)$ *in* $L^2(\mathbb{R})$, *we have following relation*

$$\langle \mathcal{V}_\Psi[x(t)](b,\xi), \mathcal{V}_\Psi[y(t)](b,\xi)\rangle = \|\Psi\|^2\langle x(t), y(t)\rangle. \tag{13}$$

Proof. By Definition (2), we have

$$\begin{aligned}
&\langle \mathcal{V}_\Psi[x(t)](b,\xi), \mathcal{V}_\Psi[y(t)](b,\xi)\rangle \\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{V}_\Psi[x(t)](b,\xi)\overline{\mathcal{V}_\Psi[y(t)](b,\xi)}d\xi db \\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{V}_\Psi[x(t)](b,\xi)\overline{\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi t}y(t)\overline{\Psi(t-b)}dt\right)}d\xi db \\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\xi t}\mathcal{V}_\Psi[x(t)](b,\xi)d\xi\right)\overline{y(t)}\Psi(t-b)dtdb.
\end{aligned} \tag{14}$$

By virtue of Eq. (11), (14) yields

$$\begin{aligned}
&\langle \mathcal{V}_\Psi[x(t)](b,\xi), \mathcal{V}_\Psi[y(t)](b,\xi)\rangle \\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} x(t)\overline{\Psi(t-b)}\Psi(t-b)\overline{y(t)}dtdb \\
&\quad= \int_{\mathbb{R}} x(t)\overline{y(t)}dt\int_{\mathbb{R}}\overline{\Psi(t-b)}\Psi(t-b)db \\
&\quad= \|\Psi\|^2\langle x(t), y(t)\rangle.
\end{aligned} \tag{15}$$

This completes the proof. □

Next, we introduce the fractional Fourier transform as a generalization of the classical Fourier transform.

2.3 Fractional Fourier transform

It is well known that when one performs the FT two times, the time-reverse operation is obtained. When one performs the FT three times, the inverse FT is obtained. Furthermore, performing the FT four times is equivalent to performing an identity operation. Now, one may think what will be obtained when the FT is performed a non-integer number of times The fractional Fourier transform (FRFT) can be viewed as performing the FT $\{2\alpha/\pi\}$ times, where $\{2\alpha/\pi\}$ can be a non-integer value. The fractional Fourier transform (FRFT) has played an important role in signal processing [7] optics [8, 9], image processing [10], and quantum mechanics [11]. As a generalization of the conventional Fourier transform (FT), the FRFT implements an order parameter which acts on the conventional Fourier transform operator and can process time-varying signals and non-stationary signals. With variation of the fractional parameter, the FRFT transforms the signal into the fractional Fourier domain representation, which is oriented by corresponding rotation angle with respect to the time axis in the counter-clockwise direction. Using a global kernel, the FRFT shows the overall fractional Fourier domain contents. Hence, the time-frequency representation should be extended to the time-fractional Fourier frequency domain. Let us define fractional Fourier transform.

Definition 3. *Let $x(t)$ be a signal in $L^2(\mathbb{R})$, then the fractional Fourier transform of $x(t)$ is defined as*

$$\mathcal{F}_\alpha[x(t)](\xi) = \int_{\mathbb{R}} K_\alpha(t,\xi)x(t)dt, \tag{16}$$

where α is a angular parameter and $K_\alpha(t,\xi)$ is the kernel of the FRFT and is given by

$$K_\alpha(t,\xi) = \begin{cases} \sqrt{\dfrac{1-i\cot\alpha}{2\pi}}e^{\frac{i}{2}(t^2+\xi^2)\cot\alpha - i\xi t\csc\alpha} & for \quad \alpha \neq n\pi, \\ \delta(t-\xi) & for \quad \alpha = 2n\pi, \\ \delta(t+\xi) & for \quad \alpha = (2n\pm 1)\pi, \quad n\in\mathbb{Z}. \end{cases} \tag{17}$$

and the corresponding inversion formula is also a FRFT with angle $-\alpha$ and is given by

$$x(t) = \mathcal{F}_{-\alpha}\{\mathcal{F}_\alpha[x(t)](\xi)\}(t) = \int_{\mathbb{R}} \mathcal{F}_\alpha[x(t)](\xi)K_{-\alpha}(t,\xi)d\xi. \tag{18}$$

It is easy to see that, when $\alpha = 0, \pi/2, \pi$ and $3\pi/2$, the FRFT is reduced to the identity operation, the FT, time-reverse operation, and the IFT, respectively.

Assuming that $u(t) = e^{it^2\cot\alpha/2}x(t)$, then for $\alpha \neq n\pi$ the FRFT (16) can be rewritten as

$$\mathcal{F}_\alpha[x(t)](\xi) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}e^{i\xi^2\cot\alpha/2}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi t\csc\alpha}u(t)\right) \tag{19}$$

$$= \sqrt{\frac{1-i\cot\alpha}{2\pi}}e^{i\xi^2\cot\alpha/2}\mathcal{F}[u](\xi\csc\alpha). \tag{20}$$

It is clear from (20) that the FRFT can be viewed as a chirp-Fourier-chirp transformation.

Next, we highlight some properties of FRFT.

Theorem 7. *Let $x(t), y(t) \in L^2(\mathbb{R})$ and $k, \xi_0 \in \mathbb{R}$, then the FRFT satisfies following properties:*

1. *Translation:* $\mathcal{F}_\alpha[x(t-k)](\xi) = e^{\frac{1}{2}ik^2\cos\alpha\sin\alpha - ik\xi\sin\alpha}\mathcal{F}_\alpha[x(t)](\xi)(\xi - k\cos\alpha)$.

2. *Modulation:* $\mathcal{F}_\alpha\left[e^{i\xi_0 t}x(t)\right](\xi) = e^{i\xi_0\xi\cos\alpha - \frac{i}{2}\xi_0^2\sin\alpha\cos\alpha}\mathcal{F}_\alpha[x(t)](\xi - \xi_0\sin\alpha)$.

3. *Orthogonality Relation:* $\langle\mathcal{F}_\alpha[x(t)], \mathcal{F}_\alpha[y(t)]\rangle = \langle x(t), y(t)\rangle$.

Proof. For the sake of brevity, we omit proof of translation and modulation properties and prove only orthogonality relation.

We have

$$\begin{aligned}\langle \mathscr{F}_\alpha[x(t)], \mathscr{F}_\alpha[y(t)]\rangle &= \int_{\mathbb{R}} \mathscr{F}_\alpha[x(t)](\xi)\overline{\mathscr{F}_\alpha[y(t)](\xi)}d\xi \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} K_\alpha(t,\xi)x(t)\overline{K_\alpha(s,\xi)y(s)ds}dtd\xi \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} x(t)\overline{y(s)}\left(\int_{\mathbb{R}} K_\alpha(t,\xi)\overline{K_\alpha(s,\xi)}d\xi\right)dsdt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} x(t)\overline{y(s)}\delta(t-s)dsdt \\ &= \int_{\mathbb{R}} x(t)\overline{y(s)}dt \\ &= \langle x(t), y(t)\rangle.\end{aligned}$$

This completes the proof. □

Since the FRFT is a generalization of the FT, many properties, applications, and operations associated with FT can be generalized by using the FRFT. The FRFT is more flexible than the FT and performs even better in many signal processing and optical system analysis applications.

In the sequel, we introduce linear canonical transform, which is a generalized version of the classical Fourier transform with four parameters.

2.4 Linear canonical transform

The linear canonical transform (LCT) introduced by Moshinsky and Quesne [12] has a total of four parameters. It is not only a generalization of the FT, but also a generation of the FRFT, the scaling operation. As the FRFT, the LCT was first used for solving differential equations and analyzing optical systems. Recently, after the applications of FRFT were developed, the roles of the LCT for signal processing have also been examined. Due to the extra degrees of freedom and simple geometrical manifestation, the LCT is more flexible than other transforms and is as such suitable as well as powerful tool for investigating deep problems in science and engineering [13–16]. Now, we shall define linear canonical transform (LCT).

Definition 4. *Consider the second order matrix* $M_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Then the linear canonical transform of any* $x(t) \in L^2(\mathbb{R})$ *with respect to the uni-modular matrix* $M_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is defined by*

$$\mathscr{L}_M[x(t)](\xi) = \begin{cases} \int_{\mathbb{R}} \mathcal{K}_M(t,\xi)x(t)dt & b \neq 0 \\ \sqrt{d}\exp\left\{\dfrac{cd\xi^2}{2}\right\}f(d\xi) & b = 0. \end{cases} \tag{21}$$

where $\mathcal{K}_M(t,\xi)$ is the kernel of linear canonical transform and is given by

$$\mathcal{K}_M(t,\xi) = \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2b}\left(at^2 - 2t\xi + d\xi^2\right)}, \quad b \neq 0. \tag{22}$$

When $b \neq 0$, *the inverse LCT is given by*

$$f(t) = \mathscr{L}_{M^{-1}}\left\{\mathscr{L}_M[x((t)](\xi)\}(t) = \int_{\mathbb{R}} \mathscr{L}^M[x(t)](\xi)\overline{\mathcal{K}_M(t,\xi)}d\xi \tag{23}$$

where the kernel $\overline{\mathcal{K}_M(t,\xi)} = \mathcal{K}_{M^{-1}}(t,\xi)$ and M^{-1} denotes the inverse of matrix M.

For typographical convenience we write the matrix $M = (a;b;c;d)$.

By changing the matrix parameter $M = (a;b;c;d)$, the LCT boils down to various integral transforms such as:

- When $M = (0, 1, -1, 0)$, the LCT turns out to be Fourier transform(FT):

$$\mathscr{L}_M[x(t)] = \sqrt{-i}\mathscr{F}[x(t)].$$

- When $M = (0, -1,1,0)$, the LCT turns out to be inverse Fourier transform(IFT):

$$\mathscr{L}_M[x(t)] = \sqrt{i}\mathscr{F}^{-1}[x(t)].$$

- When $M = (\cos\alpha,\ \sin\alpha, -\sin\alpha,\ \cos\alpha)$, the LCT becomes the FRFT:

$$\mathscr{L}_M[x(t)] = \sqrt{e^{-i\alpha}}\mathscr{F}_\alpha[x(t)].$$

- When $M = \left(\lambda, 0, 0, \frac{1}{\lambda}\right)$, the LCT becomes a scaling operation:

$$\mathscr{L}_M[x(t)] = \sqrt{\frac{1}{\lambda}}x\left(\frac{\xi}{\lambda}\right).$$

- When $M = (1, 0, \beta, 1)$, the LCT becomes a chirp multiplication operation:

$$\mathscr{L}_M[x(t)] = e^{\frac{i}{2}\beta\xi^2}x(\xi).$$

Moreover Fresnel transform can be viewed with matrix $(1,b,0,1)$ and the Laplace transform can be obtained with $(0, i, i, 0)$.

From (21), we have for $b \neq 0$

$$\begin{aligned}
\mathscr{L}_M[x(t)](\xi) &= \int_{\mathbb{R}} \mathcal{K}_M(t,\xi)x(t)dt \\
&= \frac{1}{\sqrt{2\pi ib}} \int_{\mathbb{R}} e^{\frac{i}{2b}\left(at^2 - 2t\xi + d\xi^2\right)} x(t)dt \\
&= \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2b}d\xi^2} \int_{\mathbb{R}} e^{-\frac{i}{b}\xi t}\left(x(t)e^{\frac{i}{2b}at^2}\right)dt \\
&= \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2b}d\xi^2} \mathscr{F}[g(t)](\xi/b),
\end{aligned} \tag{24}$$

where $g(t) = x(t)e^{\frac{i}{2b}at^2}$.

Thus, it is clear from (24), that LCT can be regarded as a chirp-Fourier-chirp transformation.

Next, we investigate some basic properties associated with LCT.

Theorem 8. *Let* $x(t), y(t) \in L^2(\mathbb{R})$ *and* $k, \xi_0 \in \mathbb{R}$, *then the LCT satisfies following properties:*

1. *Translation:* $\mathscr{L}_M[x(t-k)](\xi) = e^{ikc\xi - \frac{i}{2}k^2 ac}\mathscr{L}_M[x(t)](\xi - ak)$.

2. *Modulation:* $\mathscr{L}_M\left[e^{i\xi_0 t}x(t)\right](\xi) = e^{id\xi_0\xi - \frac{i}{2}b\xi_0^2}\mathscr{L}_M[x(t)](\xi - b\xi_0)$.

3. *Parity:* $\mathscr{L}_M[x(-t)](\xi) = \mathscr{L}_M[x(t)](-\xi)$.

4. *Orthogonality Relation:* $\langle \mathscr{L}_M[x(t)], \mathscr{L}_M[y(t)]\rangle = \langle x(t), y(t)\rangle$.

Proof. To be specific, we shall only prove the translation property, the rest of the properties follows similarly.

For any real k, we have

$$\begin{aligned}
\mathscr{L}_M[x(t-k)](\xi) &= \int_{\mathbb{R}} \mathcal{K}_M(t,\xi)x(t-k)dt \\
&= \frac{1}{\sqrt{2\pi ib}}\int_{\mathbb{R}} e^{\frac{i}{2b}\left(at^2 - 2t\xi + d\xi^2\right)}x(t)dt \\
&= \frac{1}{\sqrt{2\pi ib}}\int_{\mathbb{R}} e^{\frac{i}{2b}\left(a(s+k)^2 - 2(s+k)\xi + d\xi^2\right)}x(s)ds \\
&= e^{ikc\xi - \frac{i}{2}k^2 ac}\frac{1}{\sqrt{2\pi ib}}\int_{\mathbb{R}} e^{\frac{i}{2b}\left(as^2 - 2s(\xi - ak) + d\left(\xi - ak^2\right)\right)}x(s)ds \\
&= e^{ikc\xi - \frac{i}{2}k^2 ac}\mathscr{L}_M[x(t)](\xi - ak).
\end{aligned}$$

This completes the proof. □

Finally, we will define quadratic-phase Fourier transform.

2.5 Quadratic-phase Fourier transform

The most neoteric generalization of the classical Fourier transform (FT) with five real parameters appeared *via* the theory of reproducing kernels is known as the quadratic-phase Fourier transform (QPFT) [17]. It treats both the stationary and non-stationary signals in a simple and insightful way that are involved in radar, signal processing, and other communication systems [18–25]. Here, we gave the notation and definition of the quadratic-phase Fourier transform and study some of its properties.

Definition 5. *For a real parameter set* $\Lambda = (a, b, c, d, e)$ *with* $b \neq 0$, *the quadratic-phase Fourier transform of any signal* $f \in L^2(\mathbb{R})$ *is defined as*

$$\mathcal{Q}_\Lambda[x(t)](\xi) = \int_{\mathbb{R}} K_\Lambda(t,\xi)x(t)dt, \tag{25}$$

where $k_\Lambda(t,\xi)$ is the kernel signal of the QPFT and is given by

$$K_\Lambda(t,\xi) = \frac{1}{\sqrt{2\pi}}e^{-i\left(at^2 + b\xi t + c\xi^2 + dt + e\xi\right)}, \tag{26}$$

and corresponding inversion formula is given by

$$x(t) = \mathcal{Q}_\Lambda^{-1}(\mathcal{Q}_\Lambda[x(t)](\xi))(t) = \int_{\mathbb{R}} \overline{K_\Lambda(t,\xi)}\mathcal{Q}_\Lambda[x(t)](\xi)d\xi. \tag{27}$$

The novel QPFT (5) can be considered as a cluster of several existing integral transforms ranging from the classical Fourier to the much recent special affine Fourier transform. Nevertheless, many signal processing operations, such as scaling,shifting and time reversal, can also be performed *via* the QPFT (5).

Now, we will establish some properties of the quadratic-phase Fourier transform.

Theorem 9. *Let $x(t), y(t) \in L^2(\mathbb{R})$ and $k, \xi_0 \in \mathbb{R}$, then the QPFT satisfies following properties:*

1. *Modulation:* $\mathcal{Q}_\Lambda\left[e^{i\xi_0 t}x(t)\right](\xi) = e^{i\left(c\left(b^{-2}\xi_0^2 - 2b^{-1}\xi\xi_0\right) - eb^{-1}\xi_0\right)}\mathcal{Q}_\lambda[x(t)]\left(\xi - b^{-1}\xi_0\right)$.

2. *Parity:* $\mathcal{Q}_\Lambda[x(-t)](\xi) = \mathcal{Q}_{\Lambda'}[x(t)](-\xi)$, *where* $\Lambda' = (a, b, c, -d, -e)$.

3. *conjugation:* $\mathcal{Q}_\Lambda\left[\overline{x(t)}\right](\xi) = \overline{\mathcal{Q}_{-\Lambda}[x(t)](\xi)}$, *where* $-\Lambda = (-a, -b, -c, -d, -e)$.

4. *Orthogonality Relation:* $\langle \mathcal{Q}_\Lambda[x(t)], \mathcal{Q}_\Lambda[y(t)]\rangle = \frac{1}{b}\langle x(t), y(t)\rangle$.

Proof. For the sake of brevity, we avoid proof. □

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