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Constrained quantization for a uniform distribution with respect to a family of constraints

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CONSTRAINED QUANTIZATION FOR A UNIFORM DISTRIBUTION WITH RESPECT TO A FAMILY OF CONSTRAINTS

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Abstract. In this paper, with respect to a family of constraints for a uniform probability distribution we determine the optimal sets of n-points and the nth constrained quantization errors for all positive integers n. We also calculate the constrained quantization dimension and the constrained quantization coefficient. The work in this paper shows that the constrained quantization dimension of an absolutely continuous probability measure depends on the family of constraints and is not always equal to the Euclidean dimension of the underlying space where the support of the probability measure is defined.

1. Introduction

Let P be a Borel probability measure on $\mathbb{R}^k$ equipped with a metric d induced by a norm $\|\cdot\|$ on $\mathbb{R}^k$, and $r \in (0, \infty)$. Then, for $n \in \mathbb{N}$, the *nth constrained quantization error* for P, of order r with respect to a family of constraints $\{S_j : j \in \mathbb{N}\}$ with S_1 nonempty, is defined as

$$V_{n,r} := V_{n,r}(P) = \inf\Big\{\int \min_{a\in\alpha} d(x,a)^r dP(x) : \alpha \subseteq \bigcup_{j=1}^{n} S_j,\ 1 \leq \text{card}(\alpha) \leq n\Big\}, \tag{1}$$

where $\text{card}(A)$ represents the cardinality of the set A. The number

$$V_r(P;\alpha) := \int \min_{a\in\alpha} d(x,a)^r dP(x)$$

is called the distortion error for P, of order r, with respect to a set $\alpha \subseteq \mathbb{R}^k$. Write $V_{\infty,r}(P) := \lim_{n\to\infty} V_{n,r}(P)$. Then, the number $D_r(P)$ defined by

$$D_r(P) := \lim_{n\to\infty} \frac{r \log n}{-\log(V_{n,r}(P) - V_{\infty,r}(P))},$$

if it exists, is called the *constrained quantization dimension* of P of order r and is denoted by $D_r(P)$. For any $\kappa > 0$, the number

$$\lim_{n\to\infty} n^{\frac{r}{\kappa}}(V_{n,r}(P) - V_{\infty,r}(P)), \tag{2}$$

if it exists, is called the *κ-dimensional constrained quantization coefficient* for P of order r. Constrained quantization has recently been introduced, see [PR1, PR2]. Unconstrained quantization, which traditionally in the literature is known as quantization, is a special case of constrained quantization. For unconstrained quantization, one can see [DFG, DR, GG, GL, GL1, GL2, GL3, GN, KNZ, P, P1, R1, R2, R3, Z1, Z2]. If $\int d(x,0)^r dP(x) < \infty$ is satisfied, then the infimum in (1) exists (see [PR1]). A set $\alpha \subseteq \bigcup_{j=1}^{n} S_j$ for which the infimum in (1) exists and does not contain more than n elements is called an *optimal set of n-points* for P. In unconstrained quantization, the optimal sets of n-points are referred to as *optimal sets of n-means*. This paper deals with the Euclidean metric induced by the Euclidean norm $\|\cdot\|$. Thus, instead of writing $V_r(P;\alpha)$ and $V_{n,r} := V_{n,r}(P)$ we will write them as $V(P;\alpha)$ and $V_n := V_n(P)$. Let us take the family $\{S_j : j \in \mathbb{N}\}$ of constraints, that occurs in (1) as follows:

$$S_j = \{(x,y) : -\frac{1}{j} \leq x \leq 1 \text{ and } y = x + \frac{1}{j}\} \tag{3}$$

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for all $j \in \mathbb{N}$. Let P be a Borel probability measure on $\mathbb{R}^2$ which has support the closed interval $\{(x, y) : 0 \leq x \leq 1,\, y = 0\}$. Moreover, P is uniform on its support. In this paper we determine the optimal sets α_n of n-points for P such that

$$\alpha_n \subseteq \bigcup_{j=1}^{n} S_j$$

for all $n \in \mathbb{N}$. We also calculate the constrained quantization dimension and the constrained quantization coefficient for P. As mentioned in Remark 4.3, unlike the unconstrained quantization dimension of an absolutely continuous probability measure, the constrained quantization dimension of an absolutely continuous probability measure depends on the family of constraints and is not always equal to the Euclidean dimension of the space where the support of the probability measure is defined (for some more details, one can see [PR1]).

2. Preliminaries

In this section, we give some basic notations and definitions which we have used throughout this paper. Notice that for any elements $p, q \in \mathbb{R}^2$, if e an element on the boundary of their Voronoi regions, then

$$\rho(p, e) - \rho(q, e) = 0,$$

where for any two elements (a, b) and (c, d) in $\mathbb{R}^2$, $\rho((a, b), (c, d))$ represents the squared Euclidean distance between the two elements. Such an equation is known as a *canonical equation*. Let P be a Borel probability measure on $\mathbb{R}^2$ which is uniform on its support the line segment given by

$$J := \{(x, y) : 0 \leq x \leq 1 \text{ and } y = 0\}.$$

Let P_1, P_2 be the marginal distributions of P, i.e., $P_1(A) = P(A \times \mathbb{R})$ for all $A \in \mathfrak{B}$, and $P_2(B) = P(\mathbb{R} \times B)$ for all $B \in \mathfrak{B}$, where $\mathfrak{B}$ is the Borel σ-algebra on $\mathbb{R}$. Then, $P = P_1 \times P_2$. Thus, for any $A \in \mathfrak{B}$ and $B \in \mathfrak{B}$, $P(A \times B) = P_1(A)P_2(B)$. Since P has support $[0, 1] \times \{0\}$, we have

$$1 = P([0, 1] \times \{0\}) = P_1([0, 1])P_2(\{0\}),$$

i.e., $P_1([0, 1]) = 1$ and $P_2(\{0\}) = 1$, i.e., P_1 and P_2 have supports $[0, 1]$ and $\{0\}$, respectively. The probability density function $f(x, y)$ for P is given by

$$f(x, y) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \text{ and } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$dP(x, 0) = P(dx \times \{0\}) = P_1(dx) = dP_1(x) = f(x, 0)dx.$$

Thus, we see that the probability distribution P is uniform on its support J is equivalent to say that P_1 is a uniform distribution on the closed interval $[0, 1]$. Let $\mathbf{X} = (X, Y)$ be a random vector with probability distribution P. Then, notice that for any $0 \leq a < b \leq 1$, we have

$$E(\mathbf{X} : \mathbf{X} \in [a, b] \times \{0\}) = \int_{[a,b]\times\{0\}} (x, 0)dP(x, 0) = \Big(\int_a^b x\, dP_1(x), 0\Big) = \Big(\frac{a + b}{2}, 0\Big). \tag{4}$$

With respect to a finite set $\alpha \subset \mathbb{R}^2$, by the *Voronoi region* of an element $a \in \alpha$, it is meant the set of all elements in $\mathbb{R}^2$ which are nearest to a among all the elements in α, and is denoted by $M(a|\alpha)$.

In this paper, we investigate the constrained quantization for the probability measure P with respect to the family of constraints given by

$$S_j = \{(x, y) : -\frac{1}{j} \leq x \leq 1 \text{ and } y = x + \frac{1}{j}\} \text{ for all } j \in \mathbb{N}, \tag{5}$$

i.e., the constraints S_j are the line segments joining the points $(-\frac{1}{j}, 0)$ and $(1, 1+\frac{1}{j})$ which are parallel to the line $y = x$. The perpendicular on a constraint S_j passing through a point $(x, x + \frac{1}{j}) \in S_j$ intersects the real line at the point $(2x+\frac{1}{j}, 0)$ where $-\frac{1}{j}\leq x\leq 1$; and it intersects J if $0\leq 2x+\frac{1}{j}\leq 1$, i.e., if

$$-\frac{1}{2j}\leq x\leq \frac{1}{2}-\frac{1}{2j}. \tag{6}$$

Thus, for all $j\in\mathbb{N}$, there exists a one-one correspondence between the elements $(x, x+\frac{1}{j})$ on S_j and the elements $(2x+\frac{1}{j}, 0)$ on the real line if $-\frac{1}{j}\leq x\leq 1$. Thus, for all $j\in\mathbb{N}$, there exist bijective functions U_j such that

$$U_j(x, x+\frac{1}{j})=(2x+\frac{1}{j}, 0) \text{ and } U_j^{-1}(x,0)=\Big(\frac{1}{2}(x-\frac{1}{j}), \frac{1}{2}(x-\frac{1}{j})+\frac{1}{j}\Big), \tag{7}$$

where $-\frac{1}{j}\leq x\leq 1$.

In the following sections we give the main results of the paper.

3. Optimal sets of n-points and the nth constrained quantization errors

In this section, we calculate the optimal sets of n-points and the nth constrained quantization errors for all $n\in\mathbb{N}$. Let us first give the following lemma.

Lemma 3.1. *Let $\alpha_n\subseteq\bigcup\limits_{j=1}^n S_j$ be an optimal set of n-points for P such that*

$$\alpha_n:=\{(a_j, b_j): 1\leq j\leq n\},$$

where $a_1<a_2<a_3<\cdots<a_n$. Then, $\alpha_n\subseteq S_n$ and $(a_j, b_j)=U_n^{-1}(E(\boldsymbol{X}:\boldsymbol{X}\in M((a_j, b_j)|\alpha_n)))$, where $M((a_j, b_j)|\alpha_n)$ are the Voronoi regions of the elements (a_j, b_j) with respect to the set α_n for $1\leq j\leq n$.

Proof. Let $\alpha_n:=\{(a_j, b_j): 1\leq j\leq n\}$, as given in the statement of the lemma, be an optimal set of n-points. Take any $(a_q, b_q)\in\alpha_n$. Since $\alpha_n\subseteq\bigcup\limits_{j=1}^n S_j$, we can assume that $(a_q, b_q)\in S_t$, i.e., $b_q=a_q+\frac{1}{t}$ for some $1\leq t\leq n$. Since the Voronoi region of (a_q, b_q), i.e., $M((a_q, b_q)|\alpha_n)$ has positive probability, we can assume that $M((a_q, b_q)|\alpha_n)$ intersects the support of P at the points $(a,0)$ and $(b,0)$, where $0\leq a<b\leq 1$. Hence, the distortion error contributed by (a_q, b_q) in its Voronoi region $M((a_q, b_q)|\alpha_n)$ is given by

$$\begin{aligned}\int_{M((a_q,b_q)|\alpha_n)}\rho((x,0),(a_q,b_q))\,dP &= \int_a^b \|(x,0)-(a_q,b_q)\|^2\,dx\\ &= \frac{1}{3}(b-a)\Big(a^2-3(a+b)a_q+ab+3a_q^2+b^2+3b_q^2\Big)\\ &= \frac{1}{3}(b-a)\Big(a^2-3(a+b)a_q+ab+3\Big(a_q+\frac{1}{t}\Big)^2+3a_q^2+b^2\Big).\end{aligned}$$

The above expression is minimum if $a_q=\frac{at+bt-2}{4t}$. Now, putting $a_q=\frac{at+bt-2}{4t}$, we have the above distortion error as

$$\frac{(b-a)\left(t^2\left(5a^2+2ab+5b^2\right)+12t(a+b)+12\right)}{24t^2}.$$

Since $1\leq t\leq n$, the above distortion error is minimum if $t=n$. Thus, for $t=n$, we see that $(a_q, b_q)\in S_n$, and

$$a_q=\frac{1}{2}(\frac{a+b}{2}-\frac{1}{n}), \text{ and } b_q=a_q+\frac{1}{n}=\frac{1}{2}(\frac{a+b}{2}-\frac{1}{n})+\frac{1}{n},$$

which implies

$$(a_q, b_q)=U_n^{-1}\Big(\Big(\frac{a+b}{2}, 0\Big)\Big),$$

which by (4) yields that

$$(a_q, b_q)=U_n^{-1}(E(\mathbf{X}:\mathbf{X}\in M((a_j, b_j)|\alpha_n))).$$

Since $(a_q, b_q)\in\alpha_n$ is chosen arbitrarily, the proof of the lemma is complete. □

Remark 3.2. By (6) and (7), and Lemma 3.1, we can conclude that all the elements in an optimal set of n-points must lie on S_n between the two elements $U_n^{-1}(0,0)$ and $U_n^{-1}(1,0)$, i.e., between the two elements $(-\frac{1}{2n}, \frac{1}{2n})$ and $(\frac{n-1}{2n}, \frac{n+1}{2n})$. If this fact is not true, then the constrained quantization error can be strictly reduced by moving the elements in the optimal set between the elements $(-\frac{1}{2n}, \frac{1}{2n})$ and $(\frac{n-1}{2n}, \frac{n+1}{2n})$ on S_n, in other words, the x-coordinates of all the elements in an optimal set of n-points must lie between the two numbers $-\frac{1}{2n}$ and $\frac{n-1}{2n}$ (see Figure 1).

Lemma 3.3. *Let α_n be an optimal set of n-points for P. Then, $U_n(\alpha_n)$ is an optimal set of n-means for P.*

Proof. By Lemma 3.1, $\alpha_n \subseteq S_n$ for all $n \in \mathbb{N}$. Let $\alpha_n := \{(a_j, b_j) : 1 \leq j \leq n\}$ be an optimal set of n-points for P such that $a_1 < a_2 < \cdots < a_n$. Then, by Remark 3.2, we have $-\frac{1}{2n} \leq a_1 < a_2 < \cdots < a_n \leq \frac{n-1}{2n}$. Moreover, as $(a_j, b_j) \in S_n$, we have $b_j = a_j + \frac{1}{n}$ for all $1 \leq j \leq n$.

Notice that the boundary of the Voronoi region of the element (a_1, b_1) intersects the support of P at the elements $(0,0)$ and $(a_1 + a_2 + \frac{1}{n}, 0)$, the boundaries of the Voronoi regions of (a_j, b_j) for $2 \leq j \leq n-1$ intersect the support of P at the elements $(a_{j-1} + a_j + \frac{1}{n}, 0)$ and $(a_j + a_{j+1} + \frac{1}{n}, 0)$, and the boundary of the Voronoi region of (a_n, b_n) intersects the support of P at the elements $(a_{n-1} + a_n + \frac{1}{n}, 0)$ and $(1, 0)$. Thus, the distortion error due to the set α_n is given by

$$\begin{aligned}
V(P; \alpha_n) &= \int_{\mathbb{R}} \min_{a \in \alpha_n} \|(x,0) - a\|^2 dP(x) \\
&= \int_0^{a_1+a_2+\frac{1}{n}} \rho((x,0), (a_1, a_1 + \frac{1}{n}))\, dx + \sum_{i=2}^{n-1} \int_{a_{i-1}+a_i+\frac{1}{n}}^{a_i+a_{i+1}+\frac{1}{n}} \rho((x,0), (a_i, a_i + \frac{1}{n}))\, dx \\
&\quad + \int_{a_{n-1}+a_n+\frac{1}{n}}^{1} \rho((x,0), (a_n, a_n + \frac{1}{n}))\, dx.
\end{aligned}$$

Since $V(P; \alpha_n)$ gives the optimal error and is differentiable with respect to a_i for all $1 \leq i \leq n$, we have $\frac{\partial}{\partial a_i} V(P; \alpha_n) = 0$ implying

$$\frac{1}{n} + 2a_1 = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} = 1 - \frac{1}{n} - 2a_n.$$

Then, we can assume that there is a constant d depending on n, such that

$$\frac{1}{n} + 2a_1 = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} = 1 - \frac{1}{n} - 2a_n = d \tag{8}$$

yielding

$$a_2 = d + a_1,\ a_3 = 2d + a_1,\ a_4 = 3d + a_1,\ \cdots,\ a_n = (n-1)d + a_1$$

i.e.,

$$a_j = (j-1)d + a_1 \text{ for } 2 \leq j \leq n. \tag{9}$$

Again, by (8), we have

$$a_1 = \frac{1}{2}(d - \frac{1}{n}) \text{ and } a_n = \frac{1}{2}(1 - \frac{1}{n} - d). \tag{10}$$

Putting the above values of a_1 and a_n in the expression $a_n = (n-1)d + a_1$, and then upon simplification, we have $d = \frac{1}{2n}$. Putting the values of d by (9) and (10), we have

$$a_j = \frac{2j-3}{4n} \text{ for } 1 \leq j \leq n. \tag{11}$$

Then, we see that the boundary of the Voronoi region of the element (a_1, b_1) intersects the support of P at the elements $(0,0)$ and $(a_1 + a_2 + \frac{1}{n}, 0)$, i.e., at the elements

$$(0,0) \text{ and } (\frac{1}{n}, 0),$$

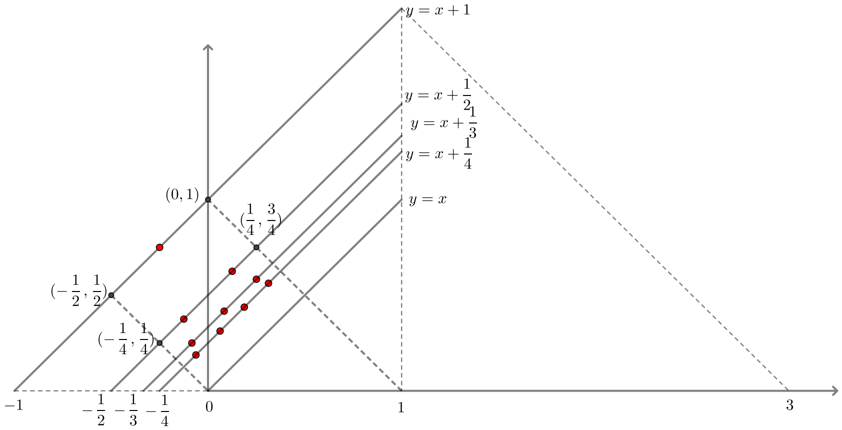

FIGURE 1. Points in the optimal sets of n-points for $1 \leq n \leq 4$.

the boundaries of the Voronoi regions of (a_j, b_j) for $2 \leq j \leq n-1$ intersect the support of P at the elements $(a_{j-1} + a_j + \frac{1}{n}, 0)$ and $(a_j + a_{j+1} + \frac{1}{n}, 0)$, i.e., at the elements

$$\Big(\frac{j-1}{n}, 0\Big) \text{ and } \Big(\frac{j}{n}, 0\Big),$$

and the boundary of the Voronoi region of (a_n, b_n) intersects the support of P at the elements $(a_{n-1} + a_n + \frac{1}{n}, 0)$ and $(1, 0)$, i.e., at the elements

$$\Big(\frac{n-1}{n}, 0\Big) \text{ and } (1, 0).$$

Thus, we deduce that for $1 \leq j \leq n$, the boundaries of the elements (a_j, b_j) in the optimal set α_n of n-points intersect the support of P at the elements $(\frac{j-1}{n}, 0)$ and $(\frac{j}{n}, 0)$. Hence, by Lemma 3.1, we have

$$(a_j, b_j) = U_n^{-1}(E(\mathbf{X} : \mathbf{X} \in [\frac{j-1}{n}, \frac{j}{n}] \times \{0\})) = U_n^{-1}((\frac{2j-1}{2n}, 0)) \text{ for } 1 \leq j \leq n.$$

We know that for the uniform distribution P, the optimal set of n-means (see [RR]) is given by

$$\Big\{\Big(\frac{2j-1}{2n}, 0\Big) : 1 \leq j \leq n\Big\}.$$

Since

$$U_n(\alpha_n) = \{U_n(a_j, b_j) : 1 \leq j \leq n\} = \Big\{\Big(\frac{2j-1}{2n}, 0\Big) : 1 \leq j \leq n\Big\},$$

the proof of the lemma is complete. □

Let us now give the following theorem, which is the main theorem of the paper.

Proposition 3.4. *A set $\alpha_n \subseteq S_n$ is an optimal set of n-points for P if and only if $U_n(\alpha_n)$ is an optimal set of n-means for P.*

Proof. Let $\alpha_n \subseteq S_n$ be an optimal set of n-points for P. Then, by Lemma 3.3, the set $U_n(\alpha_n)$ is an optimal set of n-means for P. Next assume that for a set $\beta_n \subseteq S_n$, the set $U_n(\beta_n)$ is an optimal set of n-means for P, we need to show that β_n is an optimal set of n-points for P. For the sake of contradiction, assume that there exists a set $\gamma_n \subseteq S_n$ such that γ_n is an optimal set of n-points for P and $\gamma_n \neq \beta_n$. Then, by Lemma 3.3, the set $U_n(\gamma_n)$ is an optimal set of n-means for P. Since, the optimal set of n-means for P is unique, we must have $U_n(\gamma_n) = U_n(\alpha_n)$. Since U_n is an injective function, then we have $\gamma_n = \beta_n$, which is a contradiction. Thus, the proof of the proposition is complete. □

Let us now give the following theorem, which is the main theorem of the paper.

Theorem 3.5. *An optimal set of n-points for the probability distribution P is given by*
$$\Big\{\Big(\frac{1}{2}\left(\frac{2j-1}{2n}-\frac{1}{n}\right), \frac{1}{2}\left(\frac{2j-1}{2n}-\frac{1}{n}\right)+\frac{1}{n}\Big) : 1\leq j\leq n\Big\},$$
with nth constrained quantization error
$$V_n=\frac{4n^2+12n+13}{24n^2}.$$

Proof. Let $\alpha_n:=\{(a_j,b_j) : 1\leq j\leq n\}$ be an optimal set of n-points for P such that $a_1<a_2<\cdots<a_n$. By Proposition 3.4, we know that $U_n(\alpha_n)$ is an optimal set of n-means for P, i.e.,
$$U_n(\alpha_n)=\Big\{\Big(\frac{2j-1}{2n},0\Big) : 1\leq j\leq n\Big\}.$$
Since U_n is an injective function, we have
$$\alpha_n=U_n^{-1}\Big\{\Big(\frac{2j-1}{2n},0\Big) : 1\leq j\leq n\Big\}=\Big\{U_n^{-1}\Big(\frac{2j-1}{2n},0\Big) : 1\leq j\leq n\Big\}$$
i.e.,
$$\alpha_n=\Big\{\Big(\frac{1}{2}\left(\frac{2j-1}{2n}-\frac{1}{n}\right), \frac{1}{2}\left(\frac{2j-1}{2n}-\frac{1}{n}\right)+\frac{1}{n}\Big) : 1\leq j\leq n\Big\}.$$
Writing
$$(a_j,b_j)=\Big(\frac{1}{2}\left(\frac{2j-1}{2n}-\frac{1}{n}\right), \frac{1}{2}\left(\frac{2j-1}{2n}-\frac{1}{n}\right)+\frac{1}{n}\Big),$$
for $1\leq j\leq n$, we have the nth constrained quantization for n-points as
$$\begin{aligned}V_n&=\int_{\mathbb{R}}\min_{a\in\alpha_n}\|(x,0)-a\|^2dP(x)\\&=\int_0^{a_1+a_2+\frac{1}{n}}\rho((x,0),(a_1,a_1+\frac{1}{n}))\,dx+\sum_{i=2}^{n-1}\int_{a_{i-1}+a_i+\frac{1}{n}}^{a_i+a_{i+1}+\frac{1}{n}}\rho((x,0),(a_i,a_i+\frac{1}{n}))\,dx\\&\qquad+\int_{a_{n-1}+a_n+\frac{1}{n}}^{1}\rho((x,0),(a_n,a_n+\frac{1}{n}))\,dx,\end{aligned}$$
which upon simplification yields
$$V_n=\frac{4n^2+12n+13}{24n^2}.$$
Thus, the proof of the theorem is complete (see Figure 1). □

4. Constrained quantization dimension and constrained quantization coefficient

In this section, we show that the constrained quantization dimension $D(P)$ exists and equals two. We further show that the $D(P)$-dimensional constrained quantization coefficient for P exists as a finite positive number.

Theorem 4.1. *The constrained quantization dimension $D(P)$ of the probability measure P exists, and $D(P)=2$.*

Proof. By Theorem 3.5, the nth constrained quantization error is given by
$$V_n=\frac{4n^2+12n+13}{24n^2}.$$
Notice that $V_\infty=\lim\limits_{n\to\infty}V_n=\frac{1}{6}$. Hence, the constrained quantization dimension is given by
$$D(P)=\lim_{n\to\infty}\frac{2\log n}{-\log(V_n-V_\infty)}=\lim_{n\to\infty}\frac{2\log(n)}{-\log\left(\frac{4n^2+12n+13}{24n^2}-\frac{1}{6}\right)}=2,$$
which is the theorem. □

Theorem 4.2. *The $D(P)$-dimensional constrained quantization coefficient for P exists, and equals $\frac{1}{2}$.*

Proof. We have

$$V_n = \frac{4n^2+12n+13}{24n^2} \text{ and } V_\infty = \lim_{n\to\infty} V_n = \frac{1}{6},$$

and hence, using (2), we have the $D(P)$-dimensional constrained quantization coefficient as

$$\lim_{n\to\infty} n(V_n - V_\infty) = \lim_{n\to\infty} n\left(\frac{4n^2+12n+13}{24n^2} - \frac{1}{6}\right) = \frac{1}{2}.$$

Thus, the proof of the theorem is complete. □

Remark 4.3. For the absolutely continuous probability measure, considered in this paper, we have obtained that the constrained quantization dimension is two which is not equal to the Euclidean dimension of the underlying space where the support of the probability measure is defined. On the other hand, it is well-known that the unconstrained quantization dimension of an absolutely continuous probability measure always equals the Euclidean dimension of the space where the support of the probability measure is defined (see [BW]).

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