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RESEARCH ARTICLE

Multi-Symplectic Simulation on Soliton-Collision for Nonlinear Perturbed Schrödinger Equation

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Abstract

Seeking solitary wave solutions and revealing their interactional characteristics for nonlinear evolution equations help us lot to comprehend the motion laws of the microparticles. As a local nonlinear dynamic behavior, the soliton-collision is difficult to be reproduced numerically. In this paper, the soliton-collision process in the nonlinear perturbed Schrödinger equation is simulated employing the multi-symplectic method. The multi-symplectic formulations are derived including the multi-symplectic form and three local conservation laws of the nonlinear perturbed Schrödinger equation. Employing the implicit midpoint rule, we construct a multi-symplectic scheme, which is equivalent to the Preissmann box scheme, for the nonlinear perturbed Schrödinger equation. The elegant structure-preserving properties of the multi-symplectic scheme are illustrated by the tiny maximum absolute residual of the discrete multi-symplectic structure at each time step in the numerical simulations. The effects of the perturbation strength on the soliton-collision in the nonlinear perturbed Schrödinger equation are reported in the numerical results in detail.

Keywords Nonlinear perturbed Schrödinger equation · Multi-symplectic method · Soliton collision · Hamiltonian function

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1 Introduction

To formulate the temporal evolution of the state of the microparticles, Schrödinger introduced the wave function and proposed the well-known Schrödinger equation in 1926 [1], which initiated the research on the quantum mechanics and aroused considerable interests in the last century because of the wide applications [2–5] in the atomic physics field, the nuclear physics field and so on.

Many applications of the nonlinear Schrödinger equation (NLSE) are developed based on its soliton existence and properties [6, 7], particularly for the soliton interactions. Based on the perturbation theory for nonlinear systems, Malomed [8–10] investigated the soliton-collision for the nonlinear perturbed Schrödinger equation and found some interesting nonlinear phenomena during the collision process. Kanna and Lakshmanan [11] presented the exact bright one-/two-soliton solutions of the integrable N-coupled NLSE employing the Hirota method and presented some amplitude profiles of the soliton collisions. Dmitriev et al. [12] investigated the 2-soliton solution of the NLSE without any perturbation and anticipated that inelastic soliton collisions may occur when the perturbation is considered. For the discrete NLSE, Papacharalampous et al. [13] revealed some novel characteristics of the soliton collisions. Soljacic et al. [14] presented the reduction approach of the soliton collisions in an arbitrary number of coupled NLSE. Wang et al. [15] found that the both the inelastic soliton solution and the elastic soliton solution would appear in a general coupled NLSE. Liu et al. [16] and Xie et al. [17] reported the existence of the elastic collision between the dark N-solitons for the fourth-order NLSE with the weak dispersion. Dai et al. [18] investigated the periodical collision characteristics of two hollow solitons in $(2+1)$-dimensional nonlocal NLSE theoretically. Ilati and Dehghan [19] studied interaction phenomena between three solitons of coupled NLSE with a weak damping. Yu et al. [20] analysed the collision characteristics between two anti-dark soliton solutions in the $(3+1)$-dimensional NLSE with a dissipation rate. Rao et al. [21] obtained the multi-soliton solutions and reproduced the elastic soliton collision in the nonlocal M-component NLSE. Recently, studies on N-soliton solutions were reported for novel nonlocal integrable nonlinear Schrödinger type equations by combing two nonlocal reductions [22], which enriched the theoretical analysis on the nonlinear Schrödinger type equation. In addition, Prinari [23] summarized the developments of the inverse scattering transform method applying on the NLSE and offered some perspectives about future directions on this interesting subject.

In fact, both the soliton solution and the soliton collision are local nonlinear behaviors of the evolution equations. To investigate the local geometric characteristics of the Hamiltonian systems described by the conservative partial differential equation (PDE), Bridges [24, 25] proposed the multi-symplectic method deriving from the symplectic method [26–28] for the Hamiltonian systems described by the conservative ordinary differential equation (ODE). The most prominent contribution of the multi-symplectic theory is the preservation on the three local conservation laws (including the conservation of the multi-symplectic structure, the local energy and the local momentum) in the numerical simulation excellently [29]. Inspired by the structure-preserving properties of the (multi-) symplectic method, we developed the generalized multi-symplectic method [30, 31] for non-conservative infinite-dimensional systems and applied it in the solving of several complex dynamic problems [32–37]. Thus, the multi-symplectic approach will be proposed to investigate the soliton collision characteristics in the nonlinear perturbed Schrödinger equation.

Simulating the evolution of the soliton solutions of the Schrödinger equation by using the multi-symplectic method isn't new at all. Based on the multi-symplectic theory, Wang et al. [38] introduced a differential scheme for a strongly coupled NLSE and found that the proposed scheme could preserve not only the multi-symplectic structure but also the conservation quantity on the mass. Wang et al. [39] derived two semi-explicit schemes for the NLSE and proved that these schemes are equivalent to the multi-symplectic Euler box scheme. Aydin and Karasoezen [40] developed a six-point multi-symplectic scheme for the coupled NLSE by using the Preissmann discrete method. Chen et al. [41] proposed a splitting form of the multi-symplectic scheme to solve the coupled NLSE and found that the splitting scheme can preserve all local conservative quantities listed in the multi-symplectic framework. Hong and Kong [42] employed two multi-symplectic schemes derived from the Runge–Kutta approach and the Fourier spectral approach to solve the fourth-order Schrödinger equations respectively. Qian et al. [43] proposed a semi-explicit splitting scheme by using the multi-symplectic splitting method to solve a 3-coupled NLSE. Bai et al. [44] studied the numerical behaviors of the multi-symplectic integrator for the NLSE with inclusions of finite delta potentials. In our previous job [45], we separated the local energy/momentum dissipations from the simulation results of the linear-damped NLSE by using the generalized multi-symplectic method.

The above contributions on the multi-symplectic analysis for the NLSE are mainly on the evolution of the soliton solutions. However, reproducing the soliton-collision phenomenon in the NLSE is more challenging because that the complex interchange of energy between the solitons is contained in the soliton-collision phenomenon. Thus, we will investigate the soliton-collision phenomenon in the nonlinear perturbed Schrödinger equation based on the multi-symplectic method in this paper. The multi-symplectic formulations with several local conservation laws is presented for the nonlinear perturbed Schrödinger equation firstly. Then, a Preissmann box scheme is constructed for the multi-symplectic PDEs to simulate the soliton-collision phenomenon in the nonlinear perturbed Schrödinger equation. In the numerical results reported in this paper, the effects of the perturbation strength on the soliton-collision are illustrated in detail.

2 Multi-Symplectic Formulations of Nonlinear Perturbed Schrödinger Equation

Considering the following nonlinear perturbed Schrödinger equation [9] in this work,

$$i\partial_t u + \partial_{xx} u + 2|u|^2 u - \varepsilon u \partial_x\left(|u|^2\right) = 0, \tag{1}$$

where u is a complex wave function, $i = \sqrt{-1}$, ε is a small positive coefficient. It is needed to clarify that the nonlinear perturbed Schrödinger equation given by Eq. (1) is non-integrable and the nonlinear perturbation term $\varepsilon u \partial_x\left(|u|^2\right)$ is a nonlinear damping term [9] resulting in the dynamic symmetry breaking [30, 31] of the Eq. (1). In fact, it has been proved that some coupled nonlinear Schrödinger equations without any perturbation are completely integrable [46, 47], which motivate us to consider the non-integrable case in this paper. In our previous jobs [30, 31], the damping term was taken into account in the coefficient matrices and the effects of which on the local conservation laws were considered. In this paper, for a small positive coefficient ε, the following approximate multi-symplectic structure of the NLSE will be presented when a tiny nonlinear term in the Hamiltonian function is neglected.

To transform the nonlinear perturbed Schrödinger Eq. (1) from a complex PDE to real coupled PDEs, let $u = p + iq$ referring to Refs. [38–40, 45], then,

$$\begin{cases} -\partial_t q + \partial_{xx} p + 2(p^2 + q^2)p - \varepsilon p \partial_x (p^2 + q^2) = 0 \\ \partial_t p + \partial_{xx} q + 2(p^2 + q^2)q - \varepsilon q \partial_x (p^2 + q^2) = 0 \end{cases}. \tag{2}$$

Introducing the canonical momenta as $\partial_x p = v,\ \ \partial_x q = w$ and the state vector as $\mathbf{z} = [p, q, v, w]^{\mathrm{T}}$, Eqs. (2) can be rewritten as a multi-symplectic PDE [24, 25, 48] approximately,

$$\mathbf{M}\partial_t \mathbf{z} + \mathbf{K}\partial_x \mathbf{z} = \nabla_{\mathbf{z}} S(\mathbf{z}), \tag{3}$$

where $S(\mathbf{z}) = (p^2 + q^2)^2 + \frac{1}{2}(v^2 + w^2) - \varepsilon(pv + qw)(p^2 + q^2)$ is the approximate Hamiltonian function (a tiny nonlinear term in which is neglected to make the system integrable approximately); $\mathbf{M}, \mathbf{K}$ are skew-symmetric matrices given by,

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \mathbf{K} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

It is needed to clarify that, $S(\mathbf{z})$ is called as the approximate Hamiltonian function because that there is a small nonlinear term is neglected. With this approximation, Eq. (3) is completely integrable system that isn't equivalent to Eq. (1) strictly. But the neglected term is tiny. Thus, we can approximate system (3) contains the local nonlinear characteristics of the NLSE given by Eq. (1).

Equation (3) is defined as the multi-symplectic PDE for the existence of the conservation law on the multi-symplectic structure [24] formulated as,

$$\partial_t(\mathrm{d}p \wedge \mathrm{d}q) + \partial_x(\mathrm{d}p \wedge \mathrm{d}v + \mathrm{d}q \wedge \mathrm{d}w) = 0, \tag{4}$$

where $\wedge$ denotes the wedge product operator.

The above approximate multi-symplectic structure for Eq. (1) implies that the nonlinear term $\varepsilon u \partial_x\left(|u|^2\right)$ will not break the symmetry of the NLSE if a small nonlinear term in the Hamiltonian function is neglected. Thus, different from [9], we named the nonlinear term $\varepsilon u \partial_x\left(|u|^2\right)$ as perturbation instead of damping term in the framework of multi-symplectic theory because that we always define the approximate symmetric form of the infinite-dimensional system as generalized multi-symplectic form [30, 31] if a damping term is contained in the system.

The important appendant of the symmetry of Eq. (3) is two local conservation quantities, i.e., the local energy and the local momentum.

Performing the inner product of Eq. (3) with $\partial_t \mathbf{z}$, we can get,

$$\langle \partial_t \mathbf{z}, \mathbf{K}\partial_x \mathbf{z}\rangle = \langle \partial_t \mathbf{z}, \nabla_{\mathbf{z}} S(\mathbf{z})\rangle, \tag{5}$$

with $\langle \partial_t \mathbf{z}, \mathbf{M}\partial_t \mathbf{z}\rangle = 0$ because of $\mathbf{M} = -\mathbf{M}^{\mathrm{T}}$.

The right-hand side of Eq. (5) can be reduced as $\langle \partial_t \mathbf{z}, \nabla_{\mathbf{z}} S(\mathbf{z})\rangle = \partial_t S(\mathbf{z})$ and the left-hand side of Eq. (5) can be split into two terms as $\langle \partial_t \mathbf{z}, \mathbf{K}\partial_x \mathbf{z}\rangle = \frac{1}{2}\partial_t\langle \mathbf{z}, \mathbf{K}\partial_x \mathbf{z}\rangle + \frac{1}{2}\partial_x\langle \partial_t \mathbf{z}, \mathbf{K}\mathbf{z}\rangle$. Then, the local energy conservation law can be derived from Eq. (5),

$$\partial_t[S(\mathbf{z}) - \frac{1}{2}\langle \mathbf{z}, \mathbf{K}\partial_x \mathbf{z}\rangle] + \frac{1}{2}\partial_x\langle \mathbf{z}, \mathbf{K}\partial_t \mathbf{z}\rangle = 0 \tag{6}$$

Following the outline of the deducing process for the local energy conservation law presented above, the local momentum conservation law can be obtained by taking the inner product of Eq. (3) with $\partial_x \mathbf{z}$,

$$\partial_x[S(\mathbf{z}) - \frac{1}{2}\langle \mathbf{z}, \mathbf{M}\partial_t \mathbf{z}\rangle] + \frac{1}{2}\partial_t\langle \mathbf{z}, \mathbf{M}\partial_x \mathbf{z}\rangle = 0. \tag{7}$$

3 A Multi-Symplectic Scheme for Nonlinear Perturbed Schrödinger Equation

When Bridges proposed the multi-symplectic method [24], several classic differential discretization approaches were evaluated to check whether they can be used to preserve the multi-symplectic structure of the infinite-dimensional Hamiltonian systems or not. Among which, the Preissmann box discretization method [49] for the Hamiltonian PDEs derived from the implicit midpoint rule for ODE is a robust approach that can preserve the three conservation laws [24] perfectly. In the following discretization process, the numerical approximation $\mathbf{z}(j\Delta t, k\Delta x)$ is denoted by $\mathbf{z}_j^k$ with the time step length Δt and the spatial step length Δx.

When the implicit midpoint method is used to discretize the PDEs (3) in the time and space separately, the Preissmann box scheme can be formulated as,

$$\mathbf{M}\delta_t^+ \mathbf{z}_j^{k+1/2} + \mathbf{K}\delta_x^+ \mathbf{z}_{j+1/2}^k = \nabla_{\mathbf{z}} S\left(\mathbf{z}_{j+1/2}^{k+1/2}\right), \tag{8}$$

where δ_t^+ and δ_x^+ are the forward differences for ∂_t and ∂_x respectively, the midpoints are defined as

$\mathbf{z}_j^{k+1/2} = \frac{1}{2}\left(\mathbf{z}_j^{k+1} + \mathbf{z}_j^k\right), \mathbf{z}_{j+1/2}^k = \frac{1}{2}\left(\mathbf{z}_{j+1}^k + \mathbf{z}_j^k\right), \mathbf{z}_{j+1/2}^{k+1/2} = \frac{1}{2}\left(\mathbf{z}_{j+1}^{k+1/2} + \mathbf{z}_j^{k+1/2}\right)$ and so on.

Substituting $\mathbf{M}, \mathbf{K}, \mathbf{z}, S(\mathbf{z})$ into Eq. (8), we can get,

$$\begin{aligned} &\frac{q_{j+1}^{k+1/2} - q_j^{k+1/2}}{\Delta t} - \frac{v_{j+1/2}^{k+1} - v_{j+1/2}^k}{\Delta x} \\ &= 2\left[\left(p_{j+1/2}^{k+1/2}\right)^2 + \left(q_{j+1/2}^{k+1/2}\right)^2\right] p_{j+1/2}^{k+1/2} - \varepsilon p_{j+1/2}^{k+1/2}\left[p_{j+1/2}^{k+1/2} v_{j+1/2}^{k+1/2} + q_{j+1/2}^{k+1/2} w_{j+1/2}^{k+1/2}\right], \end{aligned} \tag{9}$$

$$\begin{aligned} &-\frac{p_{j+1}^{k+1/2} - p_j^{k+1/2}}{\Delta t} - \frac{w_{j+1/2}^{k+1} - w_{j+1/2}^k}{\Delta x} \\ &= 2\left[\left(p_{j+1/2}^{k+1/2}\right)^2 + \left(q_{j+1/2}^{k+1/2}\right)^2\right] q_{j+1/2}^{k+1/2} - \varepsilon q_{j+1/2}^{k+1/2}\left[p_{j+1/2}^{k+1/2} v_{j+1/2}^{k+1/2} + q_{j+1/2}^{k+1/2} w_{j+1/2}^{k+1/2}\right], \end{aligned} \tag{10}$$

$$\left(p_{j+1/2}^{k+1} - p_{j+1/2}^k\right) \Big/ \Delta x = v_{j+1/2}^{k+1/2}, \tag{11}$$

$$\left(q_{j+1/2}^{k+1} - q_{j+1/2}^k\right) \Big/ \Delta x = w_{j+1/2}^{k+1/2}. \tag{12}$$

Following the outline of our previous job [45], the intermediate variables v and w can be eliminated, and a new scheme equivalent to the Preissmann box scheme can be obtained,

$$\begin{aligned} &\frac{q_{j+1}^{k+3/2} - q_j^{k+3/2} + q_{j+1}^{k+1/2} - q_j^{k+1/2}}{\Delta t} - 2\frac{p_{j+1/2}^{k+2} - 2p_{j+1/2}^{k+1} + p_{j+1/2}^k}{(\Delta x)^2} \\ &= 2\left[\left(p_{j+1/2}^{k+3/2}\right)^2 + \left(q_{j+1/2}^{k+3/2}\right)^2\right] p_{j+1/2}^{k+3/2} - \varepsilon p_{j+1/2}^{k+3/2}\left[p_{j+1/2}^{k+3/2} v_{j+1/2}^{k+3/2} + q_{j+1/2}^{k+3/2} w_{j+1/2}^{k+3/2}\right], \end{aligned} \tag{13}$$

$$\begin{aligned} &-\frac{p_{j+1}^{k+3/2} - p_j^{k+3/2} + p_{j+1}^{k+1/2} - p_j^{k+1/2}}{\Delta t} - 2\frac{q_{j+1/2}^{k+2} - 2q_{j+1/2}^{k+1} + q_{j+1/2}^k}{(\Delta x)^2} \\ &= 2\left[\left(p_{j+1/2}^{k+3/2}\right)^2 + \left(q_{j+1/2}^{k+3/2}\right)^2\right] q_{j+1/2}^{k+3/2} - \varepsilon q_{j+1/2}^{k+3/2}[p_{j+1/2}^{k+3/2} v_{j+1/2}^{k+3/2} + q_{j+1/2}^{k+3/2} w_{j+1/2}^{k+3/2}]. \end{aligned} \tag{14}$$

The most fascinating characteristics of the multi-symplectic scheme is the preservation of the three discretization conservation laws. As a typical multi-symplectic scheme, the three discretization conservation laws of the Preissmann box scheme (8) will be given in detail.

First discretization conservation law is the discretization multi-symplectic conservation law. The variation equation of the Preissmann box scheme (8) is,

$$\mathbf{M}\delta_t^+ \mathrm{d}\mathbf{z}_j^{k+1/2} + \mathbf{K}\delta_x^+ \mathrm{d}\mathbf{z}_{j+1/2}^k = S_{\mathbf{zz}}(\mathbf{z}_{j+1/2}^{k+1/2})\mathrm{d}\mathbf{z}_{j+1/2}^{k+1/2}. \tag{15}$$

Taking the wedge product of $\mathrm{d}\mathbf{z}_{j+1/2}^{k+1/2}$, we can get,

$$\mathrm{d}\mathbf{z}_{j+1/2}^{k+1/2} \wedge \mathbf{M}\delta_t^+ \mathrm{d}\mathbf{z}_j^{k+1/2} + \mathrm{d}\mathbf{z}_{j+1/2}^{k+1/2} \wedge \mathbf{K}\delta_x^+ \mathrm{d}\mathbf{z}_{j+1/2}^{k} = \mathrm{d}\mathbf{z}_{j+1/2}^{k+1/2} \wedge S_{\mathbf{zz}}(\mathbf{z}_{j+1/2}^{k+1/2})\mathrm{d}\mathbf{z}_{j+1/2}^{k+1/2}. \tag{16}$$

For the Hessian matrix contained in the right end item of Eq. (16) is symmetric, i.e., $S_{\mathbf{zz}}(\mathbf{z}_{j+1/2}^{k+1/2}) = [S_{\mathbf{zz}}(\mathbf{z}_{j+1/2}^{k+1/2})]^{\mathrm{T}}$, the right end item of Eq. (16) is zero, so,

$$\mathrm{d}\mathbf{z}_{j+1/2}^{k+1/2} \wedge \mathbf{M}\delta_t^+ \mathrm{d}\mathbf{z}_j^{k+1/2} + \mathrm{d}\mathbf{z}_{j+1/2}^{k+1/2} \wedge \mathbf{K}\delta_x^+ \mathrm{d}\mathbf{z}_{j+1/2}^{k} = 0, \tag{17}$$

in which,

$$\begin{aligned}
\mathrm{d}\mathbf{z}_{j+1/2}^{k+1/2} \wedge \mathbf{M}\delta_t^+ \mathrm{d}\mathbf{z}_j^{k+1/2} &= \frac{1}{2\Delta t}\left(\mathrm{d}\mathbf{z}_{j+1}^{k+1/2} + \mathrm{d}\mathbf{z}_j^{k+1/2}\right) \wedge \mathbf{M}\left(\mathrm{d}\mathbf{z}_{j+1}^{k+1/2} - \mathrm{d}\mathbf{z}_j^{k+1/2}\right)\\
&= \frac{1}{2\Delta t}\left(\mathrm{d}\mathbf{z}_{j+1}^{k+1/2} \wedge \mathbf{M}\mathrm{d}\mathbf{z}_{j+1}^{k+1/2} - \mathrm{d}\mathbf{z}_j^{k+1/2} \wedge \mathbf{M}\mathrm{d}\mathbf{z}_j^{k+1/2}\right)\\
&= \frac{1}{2}\delta_t^+\left(\mathrm{d}\mathbf{z}_j^{k+1/2} \wedge \mathbf{M}\mathrm{d}\mathbf{z}_j^{k+1/2}\right)\\
&= \frac{1}{2}\delta_t^+(\mathrm{d}p_j^{k+1/2} \wedge \mathrm{d}q_j^{k+1/2}),
\end{aligned} \tag{18}$$

$$\begin{aligned}
\mathrm{d}\mathbf{z}_{j+1/2}^{k+1/2} \wedge \mathbf{K}\delta_x^+ \mathrm{d}\mathbf{z}_{j+1/2}^{k} &= \frac{1}{2\Delta x}\left(\mathrm{d}\mathbf{z}_{j+1/2}^{k+1} + \mathrm{d}\mathbf{z}_{j+1/2}^{k}\right) \wedge \mathbf{K}\left(\mathrm{d}\mathbf{z}_{j+1/2}^{k+1} - \mathrm{d}\mathbf{z}_{j+1/2}^{k}\right)\\
&= \frac{1}{2\Delta x}\left(\mathrm{d}\mathbf{z}_{j+1/2}^{k+1} \wedge \mathbf{K}\mathrm{d}\mathbf{z}_{j+1/2}^{k+1} - \mathrm{d}\mathbf{z}_{j+1/2}^{k} \wedge \mathbf{K}\mathrm{d}\mathbf{z}_{j+1/2}^{k}\right)\\
&= \frac{1}{2}\delta_x^+\left(\mathrm{d}\mathbf{z}_{j+1/2}^{k} \wedge \mathbf{K}\mathrm{d}\mathbf{z}_{j+1/2}^{k}\right)\\
&= \frac{1}{2}\delta_x^+\left(\mathrm{d}p_{j+1/2}^{k} \wedge \mathrm{d}v_{j+1/2}^{k} + \mathrm{d}q_{j+1/2}^{k} \wedge \mathrm{d}w_{j+1/2}^{k}\right).
\end{aligned} \tag{19}$$

Substituting Eqs. (18) and (19) into Eq. (17), the discrete multi-symplectic conservation law is obtained,

$$\delta_t^+\left(\mathrm{d}p_j^{k+1/2} \wedge \mathrm{d}q_j^{k+1/2}\right) + \delta_x^+\left(\mathrm{d}p_{j+1/2}^{k} \wedge \mathrm{d}v_{j+1/2}^{k} + \mathrm{d}q_{j+1/2}^{k} \wedge \mathrm{d}w_{j+1/2}^{k}\right) = 0. \tag{20}$$

For the existence of the numerical truncation error of the multi-symplectic scheme given by Eqs. (13–14), the residual of the discrete multi-symplectic structure isn't zero on each grid point. We will record the maximum absolute residual of the discrete multi-symplectic structure at each time step in the numerical simulation as,

$$\Delta_j = \max_k \left|\delta_t^+(\mathrm{d}p_j^{k+1/2} \wedge \mathrm{d}q_j^{k+1/2}) + \delta_x^+(\mathrm{d}p_{j+1/2}^{k} \wedge \mathrm{d}v_{j+1/2}^{k} + \mathrm{d}q_{j+1/2}^{k} \wedge \mathrm{d}w_{j+1/2}^{k})\right|. \tag{21}$$

Δ_j can be used to assess the structure-preserving characteristic of the multi-symplectic scheme given by Eqs. (13–14) directly. To avoid the tedious mathematical deduction, although the discrete form of the local energy conservation law and the discrete form of the local momentum conservation law can also be used to illustrate the structure-preserving characteristics of the multi-symplectic scheme given by Eqs. (13–14), they will not be presented in this section.

4 Numerical Experiments

In the following numerical experiments, the structure-preserving properties of the multi-symplectic scheme given by Eqs. (13–14) will be assessed by using the maximum absolute residual of the discrete multi-symplectic structure at each time step firstly. Then, the soliton-collision phenomena in the nonlinear perturbed Schrödinger Eq. (1) will be reproduced with different values of ε. In the following numerical simulations, we let the time step size $\Delta t = 0.05$ and the spatial step size $\Delta x = 0.1$ in the domain $x \times t = [-100, 100] \times [0, 120]$.

Referring to the results of Refs. [8, 9], the acceleration of the soliton solution of the nonlinear perturbed Schrödinger Eq. (1) can be formulated as $\dot{V} = \frac{128}{15}\varepsilon\eta^4$, where V is the moving speed of the soliton solution, η is a constant. In this paper, we let $\eta_1 = 0.5, \eta_2 = 2$ (the subscript denotes the two soliton solutions) and $V(0) = 1$. The initial condition of the soliton solutions of the nonlinear perturbed Schrödinger Eq. (1) considered in the following simulation is,

$$u_1(0, x) = i\mathrm{sech}(x)\mathrm{e}^{\frac{i}{2}x}, \quad u_2(0, x) = 4i\mathrm{sech}(4x)\mathrm{e}^{\frac{i}{2}x}. \tag{22}$$

The collision processes of the two soliton solutions $(u_1(t, x),\ u_2(t, x))$ are simulated by using the multi-symplectic scheme given by Eqs. (13–14) with different perturbation strengths (in the simulations, we consider $\varepsilon = 0,\ 0.01, 0.05, 0.1$ respectively). The maximum absolute residual of the discrete multi-symplectic structure at each time step is recorded, see Fig. 1.

From Fig. 1, it can be found that, the maximum absolute residuals of the discrete multi-symplectic structure in each time step are less than 5×10^{-8} in each case, which implies that the multi-symplectic conservation law is preserved perfectly in the simulations and the following numerical results are credible.

The collision process of the two soliton solutions of the nonlinear perturbed Schrödinger Eq. (1) for each case (including $\varepsilon = 0$; $\varepsilon = 0.01$; $\varepsilon = 0.05$; $\varepsilon = 0.1$) is simulated. The evolutions of the amplitude profiles for the wave form in the soliton-collision process denoted by $|u|$ for each case are shown in Figs. 2, 4, 6 and 8 respectively, where $|u| = \left[\frac{1}{4}(u_1 + u_2 + \overline{u}_1 + \overline{u}_2)^2 + \frac{1}{4}(u_1 + u_2 - \overline{u}_1 - \overline{u}_2)^2\right]^{1/2}$, $\overline{u}_1$ and $\overline{u}_2$ denote the conjugate of u_1 and u_2 respectively. To further illustrate the soliton-collision process, the evolutions of the central positions for the solitons are shown in Fig. 3, 5, 7 and 9 .

From Figs. 2, 3, 4, 5, 6, 7, 8, 9, we can find that the effects of the perturbation strength on the soliton-collision in the nonlinear perturbed Schrödinger Eq. (1) are remarkable. When $\varepsilon = 0$, the essence of the soliton-collision in the nonlinear perturbed Schrödinger Eq. (1) is the superposition of these two soliton solutions. In this process, the two soliton solutions move in the same direction and

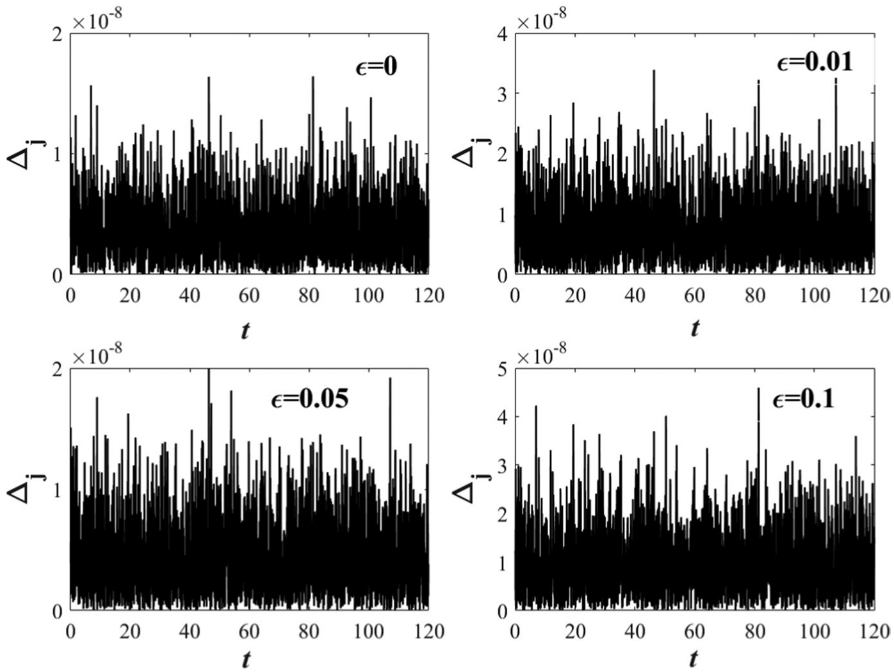

Fig. 1 The maximum absolute residual of the discrete multi-symplectic structure

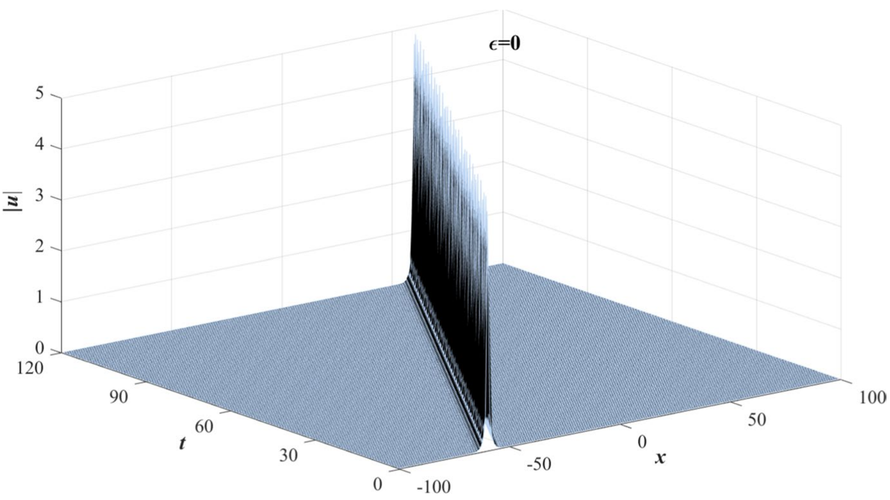

Fig. 2 Amplitude profile of the soliton-collision with $\varepsilon = 0$

this moving direction will not be changed with the time elapse, which is agreed with the results reported in Refs. [14, 50]. When $\varepsilon > 0$, the soliton-collision phenomena are more remarkable with the increase of ε. In this process, the moving

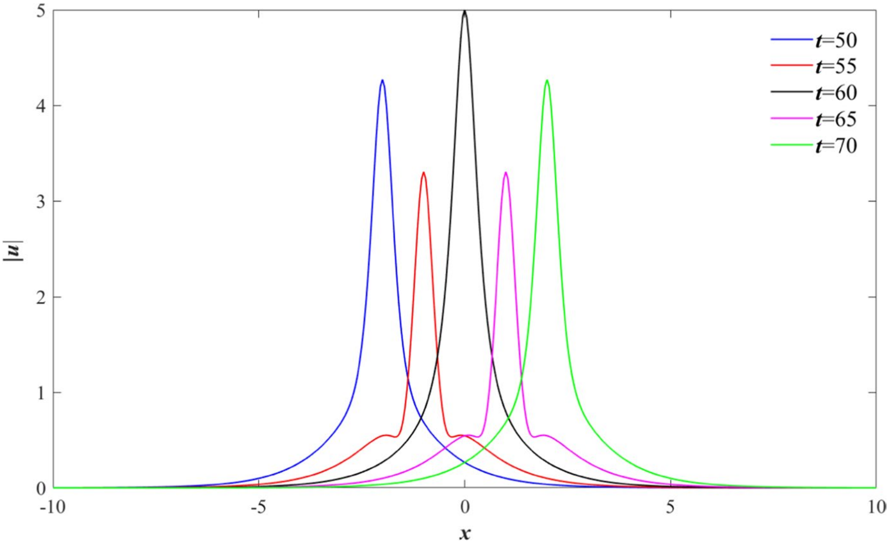

Fig. 3 Central positions of the soliton-collision with $\varepsilon = 0$

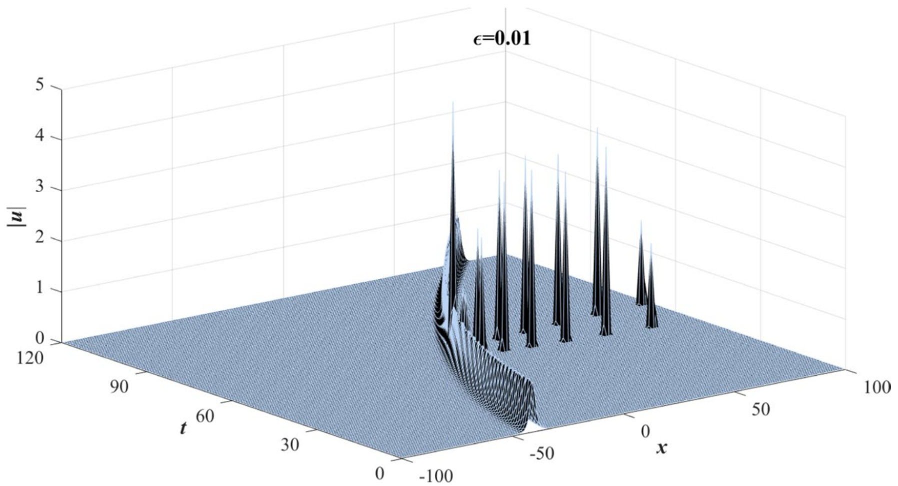

Fig. 4 Amplitude profile of the soliton-collision with $\varepsilon = 0.01$

direction of u_1 changes quicker than that of u_2. It is interesting that, when $\varepsilon = 0.1$, u_1 is almost reflected in the collision process. It is worth to mentioning that, some nonlinear phenomena, such as unphysical excitations, appear in Figs. 4, 6 and 8. Because of the existence of the perturbation denoted by ε, Eq. (1) is a non-integrable system, which has been mentioned in Sect. 2 (when a tiny nonlinear term in the Hamiltonian function is neglected, Eq. (2) can be rewritten as the

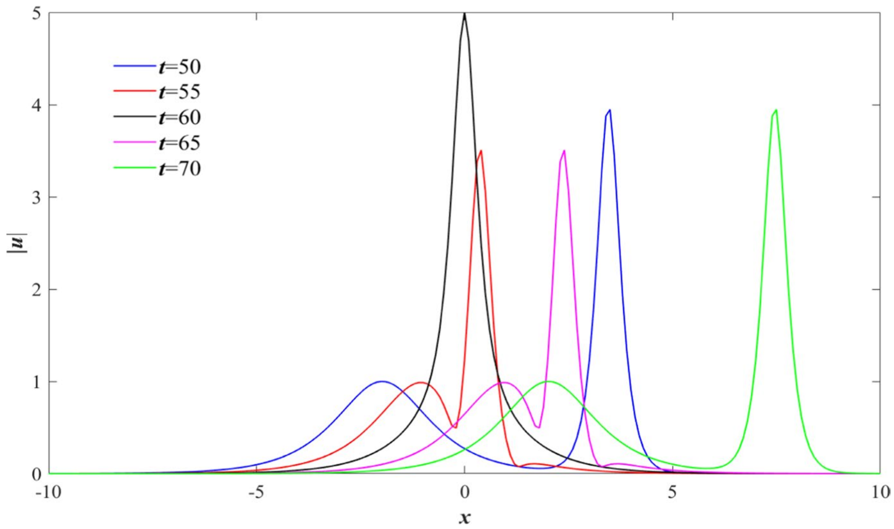

Fig. 5 Central positions of the soliton-collision with $\varepsilon = 0.01$

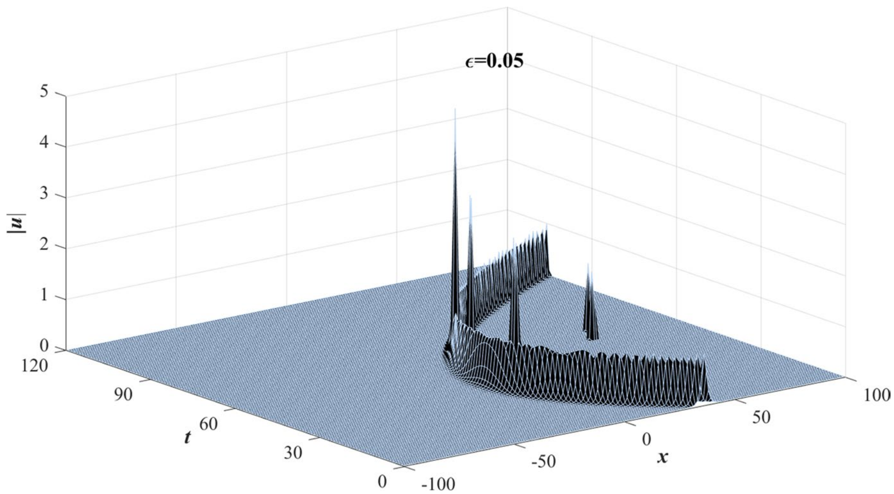

Fig. 6 Amplitude profile of the soliton-collision with $\varepsilon = 0.05$

symmetric form given by Eq. (3) approximately). Thus, the appearance of the unphysical excitations in the soliton-collision when $\varepsilon \neq 0$ is reasonable.

From the expression of the propagation speed of the soliton solution of Eq. (1) $\left(V = \int_0^t \frac{128}{15}\varepsilon\eta^4 \mathrm{d}t = \frac{128}{15}\varepsilon\eta^4 t\right)$, we can conclude that, the increase of the perturbation strength ε will increase the difference of the propagation speeds of the solitons (u_1 and u_2) at any moment. The larger difference of the propagation speeds implies the bigger difference of the momenta of the solitons, which will

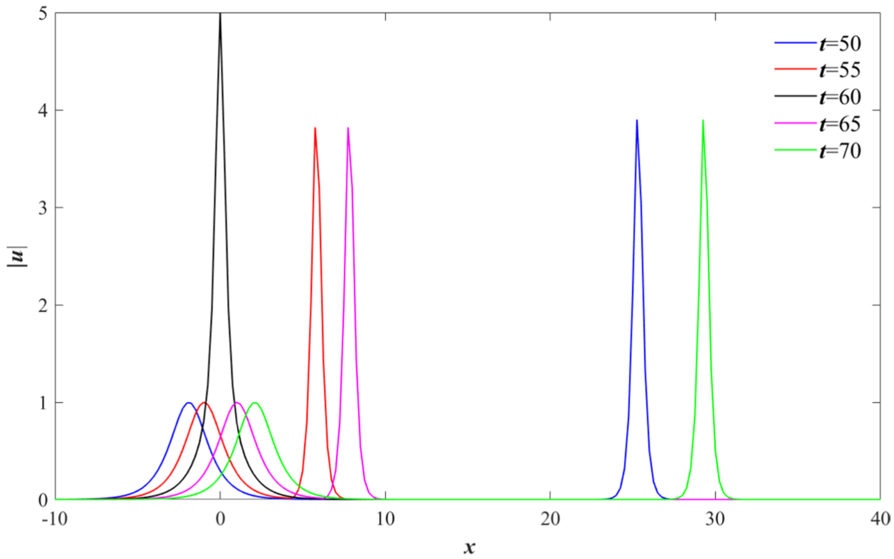

Fig. 7 Central positions of the soliton-collision with $\varepsilon = 0.05$

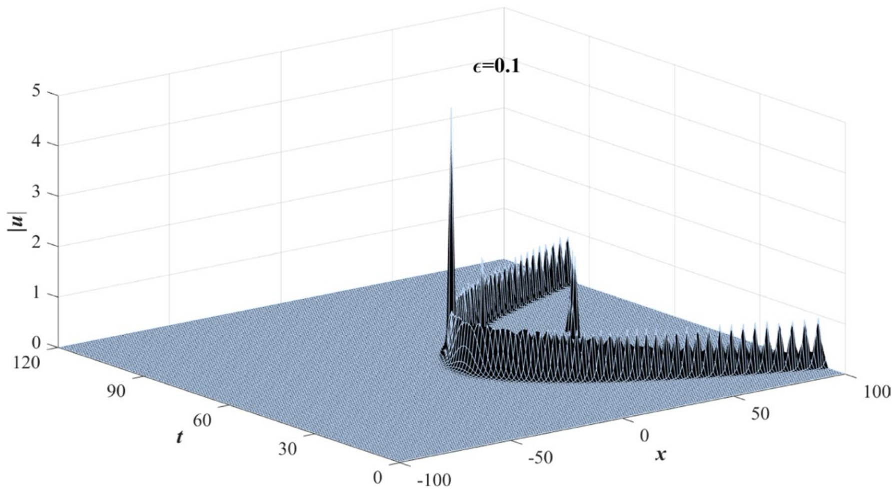

Fig. 8 Amplitude profile of the soliton-collision with $\varepsilon = 0.1$

result in the sharp change of the propagation direction of the soliton with the smaller momentum when it interacts with the soliton with the bigger momentum, see Figs. 4, 6 and 8. Thus, the perturbation strength determines the changing speed of the moving direction of the solitons during the collision process.

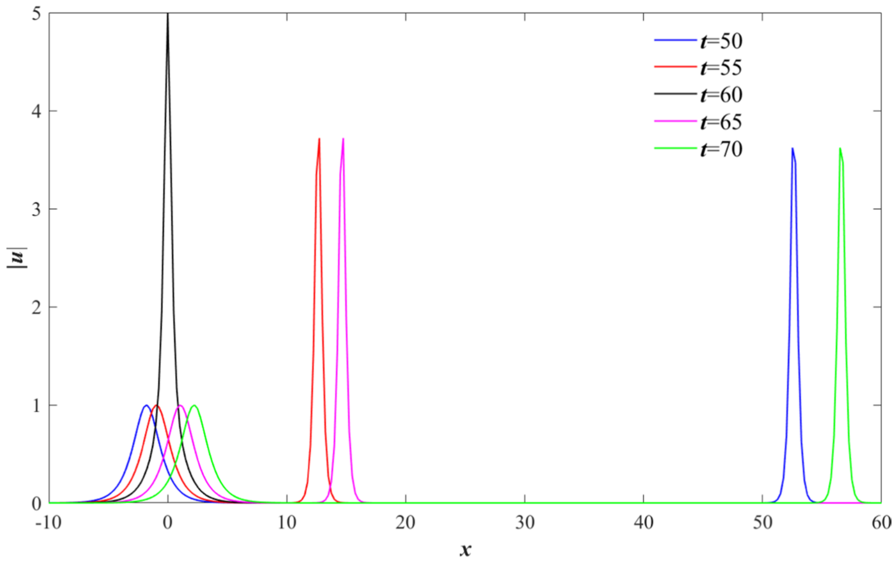

Fig. 9 Central positions of the soliton-collision with $\varepsilon = 0.1$

5 Conclusions

The NLSE describes the motion of the microparticles by the wave function, which owns broad applications in quantum physics and aroused considerable interests in the last century. Seeking the analytical solution, especially the soliton solution of the NLSE and revealing the nonlinear dynamic characteristics of which are eternal topics for physicists. In this paper, the soliton-collision phenomena in the nonlinear perturbed Schrödinger equation with different perturbation strengths are investigated employing the multi-symplectic method in detail. The standard multi-symplectic form with three local conservation laws for the nonlinear perturbed Schrödinger equation is proposed firstly. Then, a multi-symplectic scheme equivalent to the Preissmann box scheme is constructed and the discrete multi-symplectic conservation law is deduced in detail. In the numerical simulations, the tiny maximum absolute residuals of the discrete multi-symplectic structure in each time step illustrate the excellent structure-preserving properties of the constructed multi-symplectic scheme. The effects of the perturbation strength on the soliton-collision in the nonlinear perturbed Schrödinger equation are presented finally. We found that, with the increase of the perturbation strength, the effect of the perturbation strength on the soliton-collision in the nonlinear perturbed Schrödinger equation becomes more remarkable.

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Data availability Upon writing to corresponding author.

Declarations

Conflict of interest The authors declare no competing financial or non-financial interests.

Ethical approval and consent to participate Hereby we confirm that the article is not under consideration in other journals.

Consent for publication Not applicable.

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