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Cranks for partitions with bounded largest part

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Abstract

We study cranks for the function $p(n, m)$, enumerating partitions of n with parts of size at most m by considering “multiplicity-based statistics.” For a known infinite family of partition congruences modulo each prime ℓ , we give conditions under which a multiplicity-based statistic is a crank witnessing those congruences. Surprisingly, we find there are always several cranks witnessing the congruences in this infinite family. In addition, we show that Dyson’s rank of a partition actually witnesses a closely related infinite family of partition congruences modulo every prime ℓ .

1 Introduction

Freeman Dyson, in 1944 [4], requested a *direct* proof of Ramanujan's [14] celebrated congruences for the partition function

$$p(5n + 4) \equiv 0 \pmod{5} \tag{1}$$

$$p(7n + 5) \equiv 0 \pmod{7} \tag{2}$$

$$p(11n + 6) \equiv 0 \pmod{11} \tag{3}$$

[that] *will demonstrate by cross-examination of the partitions themselves the existence of five exhaustive, and equally numerous subclasses* [4].

Dyson observed empirically that the largest part of the partition minus the smallest part, which he called the *rank*, when considered modulo 5, seemed to divide the partitions of $5n + 4$ into five equally populated subclasses, thereby witnessing (1). Similarly, the rank also appeared to divide the partitions of $7n + 5$ into seven equally populated subclasses, thereby witnessing (2). Among the results in this paper, we show that Dyson's rank actually witnesses infinitely many congruences for $P(n, m)$, the number of partitions of n into parts of size at most m , with at least one part of size m (which are equinumerous with partitions of n into exactly m parts).

Noticing that the rank does not classify the partitions of $11n + 6$ into equally populated subclasses, Dyson conjectured the existence of a similar statistic witnessing (3), which he named the *crank* of a partition, in case such a statistic would eventually be discovered. To this day, any such statistic on partitions that witnesses a divisibility property of a partition function that is not the rank is referred to as a crank.

Dyson's conjecture on the rank was proved via q -series in 1954 by Atkin and Swinnerton-Dyer [2]. It was not until 1988 that Andrews and Garvan [1] found a crank (often referred to as *the* crank) for (1), (2) and (3). Two years later Garvan, Kim, and Stanton [5] finally gave direct combinatorial proofs for (1), (2), (3), and $p(25n + 24) \equiv 0 \pmod{25}$ by providing new cranks for each of these congruences, along with explicit bijections for the equally populated crank classes. However, Dyson's seemingly simple request for a direct proof that the rank witnesses Ramanujan's first two congruences for the partition function has not yet been resolved.

Our focus here is on the restricted partition function $p(n, m)$, which enumerates the number of partitions of n into parts from the set $[m] = \{1, 2, \dots, m\}$. It is well known that the sequence $\{p(n, m) \pmod{M}\}_{n \geq 0}$ is periodic [6, 11, 13]. Moreover, there are also infinite collections of divisibility patterns in arithmetic progressions for $p(n, m)$ [8, 9, 10]. In this paper, we investigate partition statistics that we call *multiplicity-based statistics*, which we shorten to *MB statistics* throughout this paper. An MB statistic is simply a linear combination of the multiplicities of the parts of a partition. Working with one key infinite family of congruences in arithmetic progressions, we provide conditions for an MB statistic to be a crank witnessing these congruences. For infinitely many values of m

and n and every prime ℓ , we show that these cranks classify partitions of n into parts from the set $[m]$ into ℓ equally populated subclasses.

In Section 2, we begin by giving the necessary definitions surrounding multiplicity-based statistics, and we state The Interval Theorem, the family of congruences we find are witnessed by multiplicity-based cranks. In Section 2.1, we give a condition for an MB statistic to be a crank for The Interval Theorem, and we give two cranks that witness these congruences for every prime ℓ . In Section 2.2, we show that one of these two cranks, when recast in the context of partitions with largest part m , is actually equivalent to Dyson's rank, so that Dyson's rank actually witnesses partition congruences modulo every prime ℓ . In Section 2.3, we discuss the notion of crank equivalence, where we may actually have different MB statistics that classify partitions into subclasses in the same way up to a permutation of the subclasses. In addition, we find many inequivalent cranks that witness The Interval Theorem, and we count them.

2 Definitions, Congruences, and MB statistics

Here we set forth the notation for the main objects of study in this paper, partitions into parts of size at most m .

Definition 2.1. Let \mathfrak{p} be the set of all integer partitions, $\mathfrak{p}(n)$ be the set of all integer partitions of n , and $\mathfrak{p}(n, m)$ be the set of all partitions of n into parts from the set $[m]$. To indicate $\lambda \in \mathfrak{p}(n)$, we write $\lambda \vdash n$. The function $p(n)$ denotes the number of partitions of n and $p(n, m)$ denotes the number of partitions into parts from the set $[m]$.

The sequence $\{p(n, m) \pmod{\ell}\}_{n \geq 0}$ is periodic [6, 11, 13]. In this paper, we treat the following key infinite family of congruences for $p(n, m)$ modulo a prime ℓ which stem from the periodicity of $\{p(n, m) \pmod{\ell}\}_{n \geq 0}$. These congruences fall in intervals of consecutive arithmetic progressions.

Theorem 2.2 (The Interval Theorem). [8, 9] *For any prime ℓ , any non-negative integer k , and any $2 \leq m \leq \ell + 1$, we have*

$$p(\ell \text{lcm}(m)k - v, m) \equiv 0 \pmod{\ell} \tag{4}$$

for $0 < v < \binom{m+1}{2}$.

Example 2.3. We display the collection of congruences in arithmetic progressions from The Interval Theorem for the case $\ell = 5$ with $2 \leq m \leq 6$.

| | | |
|-----------------------------------|-------------------------------------|-------------------------------------|
| $\underline{m = 2}$ | $\underline{m = 5}$ | $\underline{m = 6}$ |
| $p(10k - 1, 2) \equiv 0 \pmod{5}$ | $p(300k - 1, 5) \equiv 0 \pmod{5}$ | $p(300k - 1, 6) \equiv 0 \pmod{5}$ |
| $p(10k - 2, 2) \equiv 0 \pmod{5}$ | $p(300k - 2, 5) \equiv 0 \pmod{5}$ | $p(300k - 2, 6) \equiv 0 \pmod{5}$ |
| | $p(300k - 3, 5) \equiv 0 \pmod{5}$ | $p(300k - 3, 6) \equiv 0 \pmod{5}$ |
| $\underline{m = 3}$ | $p(300k - 4, 5) \equiv 0 \pmod{5}$ | $p(300k - 4, 6) \equiv 0 \pmod{5}$ |
| $p(30k - 1, 3) \equiv 0 \pmod{5}$ | $p(300k - 5, 5) \equiv 0 \pmod{5}$ | $p(300k - 5, 6) \equiv 0 \pmod{5}$ |
| $p(30k - 2, 3) \equiv 0 \pmod{5}$ | $p(300k - 6, 5) \equiv 0 \pmod{5}$ | $p(300k - 6, 6) \equiv 0 \pmod{5}$ |
| $p(30k - 3, 3) \equiv 0 \pmod{5}$ | $p(300k - 7, 5) \equiv 0 \pmod{5}$ | $p(300k - 7, 6) \equiv 0 \pmod{5}$ |
| $p(30k - 4, 3) \equiv 0 \pmod{5}$ | $p(300k - 8, 5) \equiv 0 \pmod{5}$ | $p(300k - 8, 6) \equiv 0 \pmod{5}$ |
| | $p(300k - 9, 5) \equiv 0 \pmod{5}$ | $p(300k - 9, 6) \equiv 0 \pmod{5}$ |
| $\underline{m = 4}$ | $p(300k - 1, 5) \equiv 0 \pmod{5}$ | $p(300k - 10, 6) \equiv 0 \pmod{5}$ |
| $p(60k - 1, 4) \equiv 0 \pmod{5}$ | $p(300k - 10, 5) \equiv 0 \pmod{5}$ | $p(300k - 11, 6) \equiv 0 \pmod{5}$ |
| $p(60k - 2, 4) \equiv 0 \pmod{5}$ | $p(300k - 11, 5) \equiv 0 \pmod{5}$ | $p(300k - 12, 6) \equiv 0 \pmod{5}$ |
| $p(60k - 3, 4) \equiv 0 \pmod{5}$ | $p(300k - 12, 5) \equiv 0 \pmod{5}$ | $p(300k - 13, 6) \equiv 0 \pmod{5}$ |
| $p(60k - 4, 4) \equiv 0 \pmod{5}$ | $p(300k - 13, 5) \equiv 0 \pmod{5}$ | $p(300k - 14, 6) \equiv 0 \pmod{5}$ |
| $p(60k - 5, 4) \equiv 0 \pmod{5}$ | $p(300k - 14, 5) \equiv 0 \pmod{5}$ | $p(300k - 15, 6) \equiv 0 \pmod{5}$ |
| $p(60k - 6, 4) \equiv 0 \pmod{5}$ | | $p(300k - 16, 6) \equiv 0 \pmod{5}$ |
| $p(60k - 7, 4) \equiv 0 \pmod{5}$ | | $p(300k - 17, 6) \equiv 0 \pmod{5}$ |
| $p(60k - 8, 4) \equiv 0 \pmod{5}$ | | $p(300k - 18, 6) \equiv 0 \pmod{5}$ |
| $p(60k - 9, 4) \equiv 0 \pmod{5}$ | | $p(300k - 19, 6) \equiv 0 \pmod{5}$ |
| | | $p(300k - 20, 6) \equiv 0 \pmod{5}$ |

Given this large family of congruences, one might hope that there is some crank statistic that witnesses these congruences by classifying the partitions being counted into ℓ equally populated subclasses. Throughout this paper, we consider a special class of partition statistics which we call *multiplicity-based statistics* or *MB statistics* defined below.

Definition 2.4. Let λ be a partition of n into parts from the set $[m]$. We write λ in “multiplicity notation,” so that $\lambda = (1^{e_1}, 2^{e_2}, \dots, m^{e_m})$ is the partition with exactly e_i parts of size i for each $i \in [m]$. We define a *multiplicity-based statistic* or *MB statistic* $\tau = (\tau_1, \tau_2, \dots, \tau_m) \in \mathbb{Z}^m$ to be a function $\tau : \mathfrak{p}(n, m) \rightarrow \mathbb{Z}$ such that

$$\tau(\lambda) = \sum_{i=1}^m \tau_i e_i. \quad (5)$$

The function $\tau(\lambda)$ is simply a linear combination of the multiplicities of the parts of λ .

Below we establish notation for treating the way in which MB statistics classify partitions into subclasses.

Definition 2.5.

- For a given partition statistic τ , define $\mathcal{M}_\tau(r, n, m)$ to be the set of partitions λ of n into parts from $[m]$ such that $\tau(\lambda) = r$, and define $M_\tau(r, n, m) = |\mathcal{M}_\tau(r, n, m)|$.

- Given an MB-statistic τ , we can produce a generating function for $M_\tau(r, n, m)$.

$$f_\tau(z, q) = \sum_{n=0}^{\infty} \sum_{r=-\infty}^{\infty} M_\tau(r, n, m) z^r q^n = \prod_{i=1}^m \frac{1}{1 - z^{\tau_i} q^i}. \quad (6)$$

- For a given partition statistic τ and a positive integer ℓ , we allow τ to classify the partitions of n into ℓ subclasses by letting $\mathcal{M}_\tau(r, \ell, n, m)$ be the set of partitions λ of n into parts from $[m]$ such that $\tau(\lambda) \equiv r \pmod{\ell}$. Also, define $M_\tau(r, \ell, n, m) = |\mathcal{M}_\tau(r, \ell, n, m)|$.

We are interested in MB statistics that witness congruences by dividing $\mathfrak{p}(n, m)$ into subclasses that are equally populated.

Definition 2.6. If the MB statistic $\tau : \mathfrak{p}(n, m) \rightarrow \mathbb{Z}$ is equally distributed over every residue class modulo ℓ , we say that τ is a *crank* modulo ℓ , witnessing the ℓ -divisibility of $p(n, m)$. That is, if $M_\tau(i, \ell, n, m) = p(n, m)/\ell$ for each $0 \leq i \leq \ell - 1$, then τ is a *crank* modulo ℓ .

2.1 Cranks Witnessing The Interval Theorem

Surprisingly, every congruence given in The Interval Theorem is witnessed by a crank. In Corollary 2.11, we show that there are in essence two universal cranks witnessing each and every one of these congruences for all m and ℓ . In addition, we produce a collection of MB statistics, each of which is a crank for The Interval Theorem. The collection grows ever larger with m and ℓ , and we quantify this in Theorem 2.18.

When some ℓ is prescribed, we write $\widehat{(x_i)_{i=1}^m}$ to denote the tuple $(x_i)_{i=1}^m$ with the component x_ℓ omitted whenever $m \geq \ell$.

Theorem 2.7. *An MB statistic τ is a crank for the congruences of The Interval Theorem if the components of the tuple $\widehat{\left(\frac{\tau_i}{i}\right)_{i=1}^m}$ are distinct modulo ℓ , and $\tau_\ell \not\equiv 0 \pmod{\ell}$.*

Example 2.8. The components of $\widehat{\left(\frac{2}{1}, \frac{0}{2}, \frac{-1}{3}\right)}$ for the MB statistic $\tau = (2, 0, -1)$ are distinct modulo 5, hence τ is a crank for The Interval Theorem. This crank reads “twice the number of 1s minus the number of 3s”. For example, the crank value of $\lambda = (1^2, 2^3, 3^6)$ is computed as $(2 \times 2) + (0 \times 3) - (1 \times 6) = -2 \equiv 3 \pmod{5}$. Table 1 displays the 70 partitions of $\mathfrak{p}(26, 3)$ classified by the crank $\tau = (2, 0, -1)$ modulo 5 into five equally populated subclasses.

We require the following two lemmas to prove Theorem 2.7.

Lemma 2.9. [12, 15] *Let ℓ a prime, $\zeta = e^{2\pi i/\ell}$, and $a_i \in \mathbb{Z}$ for $1 \leq i \leq \ell$. If $\sum_{i=1}^{\ell} a_i \zeta^i = 0$ then $a_i = a_j$ for all i, j .*

Table 1: The set $\mathfrak{p}(26, 3)$ classified into five equally populated subclasses under the crank $\tau = (2, 0, -1)$ modulo 5.

| $\tau(\lambda) \equiv 0 \pmod{5}$ | $\tau(\lambda) \equiv 1 \pmod{5}$ | $\tau(\lambda) \equiv 2 \pmod{5}$ | $\tau(\lambda) \equiv 3 \pmod{5}$ | $\tau(\lambda) \equiv 4 \pmod{5}$ |
|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| $(1^1, 2^2, 3^7)$ | $(1^2, 2^0, 3^8)$ | $(1^0, 2^1, 3^8)$ | $(1^5, 2^0, 3^7)$ | $(1^3, 2^1, 3^7)$ |
| $(1^8, 2^0, 3^6)$ | $(1^6, 2^1, 3^6)$ | $(1^4, 2^2, 3^6)$ | $(1^2, 2^3, 3^6)$ | $(1^0, 2^4, 3^6)$ |
| $(1^5, 2^3, 3^5)$ | $(1^3, 2^4, 3^5)$ | $(1^1, 2^5, 3^5)$ | $(1^9, 2^1, 3^5)$ | $(1^7, 2^2, 3^5)$ |
| $(1^2, 2^6, 3^4)$ | $(1^0, 2^7, 3^4)$ | $(1^{11}, 2^0, 3^5)$ | $(1^6, 2^4, 3^4)$ | $(1^4, 2^5, 3^4)$ |
| $(1^{12}, 2^1, 3^4)$ | $(1^{10}, 2^2, 3^4)$ | $(1^8, 2^3, 3^4)$ | $(1^3, 2^7, 3^3)$ | $(1^{14}, 2^0, 3^4)$ |
| $(1^9, 2^4, 3^3)$ | $(1^7, 2^5, 3^3)$ | $(1^5, 2^6, 3^3)$ | $(1^{13}, 2^2, 3^3)$ | $(1^1, 2^8, 3^3)$ |
| $(1^6, 2^7, 3^2)$ | $(1^{17}, 2^0, 3^3)$ | $(1^{15}, 2^1, 3^3)$ | $(1^0, 2^{10}, 3^2)$ | $(1^{11}, 2^3, 3^3)$ |
| $(1^{16}, 2^2, 3^2)$ | $(1^4, 2^8, 3^2)$ | $(1^2, 2^9, 3^2)$ | $(1^{10}, 2^5, 3^2)$ | $(1^8, 2^6, 3^2)$ |
| $(1^3, 2^{10}, 3^1)$ | $(1^{14}, 2^3, 3^2)$ | $(1^{12}, 2^4, 3^2)$ | $(1^{20}, 2^0, 3^2)$ | $(1^{18}, 2^1, 3^2)$ |
| $(1^{13}, 2^5, 3^1)$ | $(1^1, 2^{11}, 3^1)$ | $(1^9, 2^7, 3^1)$ | $(1^7, 2^8, 3^1)$ | $(1^5, 2^9, 3^1)$ |
| $(1^{23}, 2^0, 3^1)$ | $(1^{11}, 2^6, 3^1)$ | $(1^{19}, 2^2, 3^1)$ | $(1^{17}, 2^3, 3^1)$ | $(1^{15}, 2^4, 3^1)$ |
| $(1^0, 2^{13}, 3^0)$ | $(1^{21}, 2^1, 3^1)$ | $(1^6, 2^{10}, 3^0)$ | $(1^4, 2^{11}, 3^0)$ | $(1^2, 2^{12}, 3^0)$ |
| $(1^{10}, 2^8, 3^0)$ | $(1^8, 2^9, 3^0)$ | $(1^{16}, 2^5, 3^0)$ | $(1^{14}, 2^6, 3^0)$ | $(1^{12}, 2^7, 3^0)$ |
| $(1^{20}, 2^3, 3^0)$ | $(1^{18}, 2^4, 3^0)$ | $(1^{26}, 2^0, 3^0)$ | $(1^{24}, 2^1, 3^0)$ | $(1^{22}, 2^2, 3^0)$ |

Lemma 2.10. *Given a prime ℓ , set $\zeta = \exp(2\pi i/\ell)$. For any MB statistic, if $f_\tau(\zeta, q) - q^D f_\tau(\zeta, q)$ reduces to a polynomial in q of degree $d < D$, then for $0 < v < D - d$ and $k \geq 1$,*

(i) $p(Dk - v, m) \equiv 0 \pmod{\ell}$, and

(ii) $\tau(\lambda)$ is a crank witnessing the congruence above.

That is, $M_\tau(a, \ell, Dk - v, m) = M_\tau(b, \ell, Dk - v, m)$ for all a, b .

Proof. Given τ , suppose $f_\tau(\zeta, q) - q^D f_\tau(\zeta, q)$ reduces to a polynomial in q of degree $d < D$. Then, for $n > d$,

$$\sum_{r=0}^{\ell-1} (M_\tau(r, \ell, n, m) - M_\tau(r, \ell, n - D, m)) \zeta^r = 0. \quad (7)$$

Thus the coefficient $M_\tau(r, \ell, n, m) - M_\tau(r, \ell, n - D, m)$ is equal to some constant c_n for every r by Lemma 2.9.

We now prove (ii) by induction on k . For $d < n < D$, we have for all r ,

$$M_\tau(r, \ell, n, m) - M_\tau(r, \ell, n - D, m) = M_\tau(r, \ell, n, m) - 0 = c_n.$$

Now suppose $M_\tau(a, \ell, Dk - v, m) = M_\tau(b, \ell, Dk - v, m)$ for all a, b . Since by

(7)

$$\sum_{r=0}^{\ell-1} (M_{\tau}(r, \ell, D(k+1) - v, m) - M_{\tau}(r, \ell, Dk - v, m)) \zeta^r = 0,$$

and by our induction hypothesis

$$\sum_{r=0}^{\ell-1} (M_{\tau}(r, \ell, Dk - v, m)) \zeta^r = 0,$$

we have

$$\sum_{r=0}^{\ell-1} (M_{\tau}(r, \ell, D(k+1) - v, m)) \zeta^r = 0.$$

Thus by Lemma 2.9, $M_{\tau}(a, \ell, D(k+1) - v, m) = M_{\tau}(b, \ell, D(k+1) - v, m)$ for all a, b . Hence (ii) holds by induction, and (i) follows. \blacksquare

We now prove Theorem 2.7.

Proof. Set $\zeta = \exp(2\pi i/\ell)$. So that we may invoke Lemma 2.10, we describe conditions on the components τ_i of τ such that $f_{\tau}(\zeta, q) - q^D f_{\tau}(\zeta, q)$ reduces to a polynomial in q . We begin by setting $D = \ell \text{lcm}(m)$ where $2 \leq m \leq \ell + 1$. Consider

$$f_{\tau}(\zeta, q) - q^{\ell \text{lcm}(m)} f_{\tau}(\zeta, q) = \frac{1 - q^{\ell \text{lcm}(m)}}{\prod_{j=1}^m (1 - \zeta^{\tau_j} q^j)}. \quad (8)$$

The rational function in (8) reduces to a polynomial in q if the multiset of roots of the denominator is contained in the multiset of roots of the numerator. We now examine these multisets of roots.

The numerator $1 - q^{\ell \text{lcm}(m)}$ in (8) has a set of $\ell \text{lcm}(m)$ distinct roots described by

$$\left\{ \exp\left(\frac{s2\pi i}{\ell \text{lcm}(m)}\right) \mid 0 \leq s < \ell \text{lcm}(m) \right\}. \quad (9)$$

The set of roots of the denominator in (8) is

$$\bigcup_{j=1}^m \{q \mid \zeta^{\tau_j} q^j = 1\} = \bigcup_{j=1}^m \{q \mid q^j = \zeta^{-\tau_j}\} = \bigcup_{j=1}^m \bigcup_{c=1}^j \left\{ \exp\left(\frac{-\tau_j 2\pi i}{j\ell} + \frac{c2\pi i}{j}\right) \right\}. \quad (10)$$

The roots of the denominator (10) are each an $\ell \text{lcm}(m)^{\text{th}}$ root of unity and hence are members the set of roots of the numerator (9). It follows then, that the difference (8) is a polynomial if the roots of the denominator are distinct.

We now prove that (8) reduces to a polynomial if and only if τ satisfies the conditions in Theorem 2.7.

In one direction, suppose τ does not satisfy the conditions in Theorem 2.7, so that either $\tau_\ell \equiv 0 \pmod{\ell}$ or for some $w < y$ with $w, y \neq \ell$, we have $\frac{\tau_w}{w} \equiv \frac{\tau_y}{y} \pmod{\ell}$.

The first case, $\tau_\ell \equiv 0 \pmod{\ell}$, is only relevant if $m = \ell$ or $\ell + 1$. Then, in the far right side of (10), when $j = \ell$, as c ranges from 1 to ℓ , we get the set of all ℓ^{th} roots of unity. When $j = 1$ and $c = 1$ we get $\exp(-\tau_1 2\pi i/\ell)$, which is also an ℓ^{th} root of unity. Hence we have a repeated root in (10), so (8) does not reduce to a polynomial.

In the second case, where $\frac{\tau_w}{w} \equiv \frac{\tau_y}{y} \pmod{\ell}$, we have $\tau_w \equiv \frac{w\tau_y}{y} \pmod{\ell}$. For $j = w$ and $c = w$, in the far right side of (10) we get

$$\exp\left(\frac{-\tau_w 2\pi i}{w\ell} + \frac{w 2\pi i}{w}\right) = \exp\left(\frac{-w\tau_y 2\pi i}{yw\ell} + 2\pi i\right) = \exp\left(\frac{-\tau_y 2\pi i}{y\ell}\right).$$

When $j = y$ and $c = y$ in the far right side of (10), we get $\exp(-\tau_y 2\pi i/y\ell)$ again. Hence we have a repeated root in (10), so (8) does not reduce to a polynomial.

In the other direction, supposing τ does satisfy the conditions in Theorem 2.7, we now show that (8) reduces to a polynomial in q of degree $\ell \text{lcm}(m) - \binom{m+1}{2}$. As before, the roots of the numerator $1 - q^{\ell \text{lcm}(m)}$ in (8) are the set of all $\ell \text{lcm}(m)$ roots of unity, the roots of the denominator (10) are each an $\ell \text{lcm}(m)^{\text{th}}$ root of unity, and thus the difference (8) is a polynomial if the roots of the denominator (10) are distinct.

Suppose to the contrary that the roots of the denominator are not distinct. Then either

$$\exp\left(\frac{-\tau_j 2\pi i}{j\ell} + \frac{c 2\pi i}{j}\right) = \exp\left(\frac{-\tau_\ell 2\pi i}{\ell^2} + \frac{c 2\pi i}{\ell}\right)$$

for some $j \neq \ell$, or

$$\exp\left(\frac{-\tau_j 2\pi i}{j\ell} + \frac{c 2\pi i}{j}\right) = \exp\left(\frac{-\tau_k 2\pi i}{k\ell} + \frac{c 2\pi i}{k}\right)$$

for some $j \neq k$, where $j, k \neq \ell$.

In the first case, which is only relevant for $m = \ell$ or $\ell + 1$, we have

$$\exp\left(\frac{-\tau_j 2\pi i}{j\ell} + \frac{c 2\pi i}{j}\right) = \exp\left(\frac{-\tau_\ell 2\pi i}{\ell^2} + \frac{c 2\pi i}{\ell}\right),$$

which implies

$$2\pi i \left(\frac{-\tau_j}{j\ell} + \frac{c}{j}\right) = 2\pi i \left(\frac{-\tau_\ell}{\ell^2} + \frac{c}{\ell} + x\right)$$

for some integer x . Thus

$$-\tau_j \ell + c\ell^2 = -\tau_\ell j + c j \ell + x j \ell^2,$$

which implies $\tau_\ell \equiv 0 \pmod{\ell}$, contradicting the conditions on τ in Theorem 2.7.

In the second case, we have

$$\exp\left(\frac{-\tau_j 2\pi i}{j\ell} + \frac{c 2\pi i}{j}\right) = \exp\left(\frac{-\tau_k 2\pi i}{k\ell} + \frac{c 2\pi i}{k}\right),$$

which implies

$$2\pi i \left(\frac{-\tau_j}{j\ell} + \frac{c}{j}\right) = 2\pi i \left(\frac{-\tau_k}{k\ell} + \frac{c}{k} + x\right)$$

for some integer x . Thus

$$-\tau_j k + c k \ell = -\tau_k j + c j \ell + x j k \ell,$$

which implies $\frac{\tau_j}{j} \equiv \frac{\tau_k}{k} \pmod{\ell}$, contradicting the conditions on τ in Theorem 2.7.

Thus $f_\tau(\zeta, q) - q^{\ell \text{lcm}(m)} f_\tau(\zeta, q)$ reduces to a polynomial in q of degree $\ell \text{lcm}(m) - \binom{m+1}{2}$ if and only if $\frac{\tau_j}{j} \not\equiv \frac{\tau_k}{k} \pmod{\ell}$ for $j \neq k$, $j, k \neq \ell$, and $\tau_\ell \not\equiv 0 \pmod{\ell}$. Since these conditions on τ match those from Theorem 2.7, we have that such MB statistics are cranks for The Interval Theorem. ■

Surprisingly, there are always two very simple MB statistics that are cranks witnessing The Interval Theorem.

Corollary 2.11. *The MB-statistics α , the number of parts excluding those of size $\ell + 1$, and β , the number of parts excluding parts of size 1, are cranks witnessing The Interval Theorem.*

Proof. For $2 \leq m \leq \ell + 1$,

$$\alpha = \begin{cases} (\alpha_1, \alpha_2, \dots, \alpha_m) = (1, 1, \dots, 1, 1) & \text{if } 2 \leq m \leq \ell \\ (\alpha_1, \alpha_2, \dots, \alpha_{\ell+1}) = (1, 1, \dots, 1, 0) & \text{if } m = \ell + 1 \end{cases}$$

$$\beta = (\beta_1, \beta_2, \dots, \beta_m) = (0, 1, \dots, 1, 1)$$

In each case, both of the functions α and β satisfy Theorem 2.7, and the proof is complete. ■

2.2 Dyson's Rank and $P(n, m)$.

We now show that Dyson's original rank statistic witnesses an infinite family of partition congruences.

Let $P(n, m)$ enumerate the partitions of n into parts from the set $[m]$ with at least one part of size m . This function is also equal to the number of partitions of n into exactly m parts. It is well known that $p(n, m) = P(n + m, m)$, and we may recast The Interval Theorem in terms of $P(n, m)$.

Proposition 2.12. *For any prime ℓ , any non-negative integer k , and any $2 \leq m \leq \ell + 1$, we have*

$$P(\ell \text{lcm}(m)k + m - v, m) \equiv 0 \pmod{\ell} \quad (11)$$

for $0 < v < \binom{m+1}{2}$.

In this new context, Dyson's rank reprises its original role.

Theorem 2.13. *For any prime ℓ , Dyson's rank modulo ℓ witnesses the partition congruences in Proposition 2.12 for $2 \leq m \leq \ell$.*

Proof. Let $\mathfrak{P}(n, m)$ be the set of partitions of n with largest part m . Let $\mu : \mathfrak{p}(n, m) \rightarrow \mathfrak{P}(n + m, m)$ be the bijection defined by adding a part of size m . By Corollary 2.11, we have that α classifies $\mathfrak{p}(\ell \text{lcm}(m)k - v, m)$ into ℓ equally populated subclasses. For $\mathfrak{p}(\ell \text{lcm}(m)k - v, m)$, multiplying α by -1 and adding the constant $m - 1$ merely relabels these subclasses. Thus the new statistic $m - \alpha - 1$ also classifies $\mathfrak{p}(\ell \text{lcm}(m)k - v, m)$ into ℓ equally populated subclasses. Now notice that for any $\lambda \in \mathfrak{p}(\ell \text{lcm}(m)k - v, m)$ such that $m - \alpha(\lambda) - 1 = x$, $\mu(\lambda)$ will have Dyson rank x , because m is the largest part of $\mu(\lambda)$, and $\mu(\lambda)$ has exactly $\alpha(\lambda) + 1$ parts. Thus Dyson's rank modulo ℓ is a witness for the partition congruences in Proposition 2.12. \blacksquare

Example 2.14. In Table 2, we list all twenty-one partitions of $\mathfrak{P}(16, 3)$ classified by Dyson's rank modulo 3 into three equally populated subclasses.

Table 2: $\mathfrak{P}(16, 3)$

| 0 (mod 3) | 1 (mod 3) | 2 (mod 3) |
|----------------------|----------------------|----------------------|
| $(1^1, 2^0, 3^5)$ | $(1^4, 2^0, 3^4)$ | $(1^2, 2^1, 3^4)$ |
| $(1^0, 2^2, 3^4)$ | $(1^3, 2^2, 3^3)$ | $(1^1, 2^3, 3^3)$ |
| $(1^5, 2^1, 3^3)$ | $(1^2, 2^4, 3^2)$ | $(1^7, 2^0, 3^3)$ |
| $(1^4, 2^3, 3^2)$ | $(1^8, 2^1, 3^2)$ | $(1^0, 2^5, 3^2)$ |
| $(1^{10}, 2^0, 3^2)$ | $(1^1, 2^6, 3^1)$ | $(1^6, 2^2, 3^2)$ |
| $(1^3, 2^5, 3^1)$ | $(1^7, 2^3, 3^1)$ | $(1^5, 2^4, 3^1)$ |
| $(1^9, 2^2, 3^1)$ | $(1^{13}, 2^0, 3^1)$ | $(1^{11}, 2^1, 3^1)$ |

2.3 Crank Equivalence

Theorem 2.7 shows that there are many cranks witnessing the congruences described in The Interval Theorem. One might ask whether or not each of these cranks is genuinely different, or if several different cranks might classify partitions into subclasses in the same way, with the labels on the subclasses merely rearranged.

Definition 2.15. We call two MB statistics *equivalent modulo ℓ* if their classifications of partitions into ℓ subclasses are the same, with the labels on the subclasses possibly rearranged. That is, two MB statistics τ and σ are equivalent if there exists a permutation ϕ of $\{0, 1, \dots, \ell-1\}$ such that $\mathcal{M}_\tau(r, \ell, n, m) = \mathcal{M}_\sigma(\phi(r), \ell, n, m)$ for all $0 \leq r < \ell$.

In Example 2.3, with $k = 1$, we have $p(291, 6) \equiv 0 \pmod{5}$. In this case, the cranks α and β described in Corollary 2.11 are *not equivalent*. Table 3 shows that when each are taken modulo 5, α classifies the partitions of 291 differently than β . In particular, we see two partitions separated by α into different subclasses, that are placed into the same subclass by β .

Table 3: Distinct subclasses generated by different crank functions.

| $\lambda \vdash 291$ | $\alpha(\lambda) \pmod{5}$ | $\beta(\lambda) \pmod{5}$ |
|--------------------------------------|----------------------------|---------------------------|
| $(1^{288}, 2^0, 3^1, 4^0, 5^0, 6^0)$ | 4 | 1 |
| $(1^{289}, 2^1, 3^0, 4^0, 5^0, 6^0)$ | 0 | 1 |

In Theorem 2.16, we show that the cranks α and β witnessing The Interval Theorem are, in general, not equivalent.

Theorem 2.16. *For any prime $\ell \geq 3$ and $3 \leq m \leq \ell + 1$, the cranks $\alpha \pmod{\ell}$ and $\beta \pmod{\ell}$ are not equivalent. That is, they classify the partitions described in The Interval Theorem in two genuinely different ways.*

Proof. To show that α and β divide the partitions of n into different subclasses, we construct partitions $\lambda \vdash n$ and $\mu \vdash n$ such that $\beta(\lambda) = \beta(\mu)$, but $\alpha(\lambda) \neq \alpha(\mu)$.

Let $\lambda = (1^{n-3}, 2^0, 3^1, 4^0, 5^0, \dots, m-1^0, m^0)$, and $\mu = (1^{n-2}, 2^1, 3^0, 4^0, 5^0, \dots, m-1^0, m^0)$. Then $\alpha(\lambda) = n-2$ and $\beta(\lambda) = 1$, while $\alpha(\mu) = n-1$ and $\beta(\mu) = 1$.

Hence, $\beta(\lambda) = \beta(\mu)$ while $\alpha(\lambda) = \alpha(\mu) - 1$. Thus, the cranks α and β produce distinct subclasses of partitions as they witness The Interval Theorem. \blacksquare

At the same time, every MB statistic has many equivalents. In Proposition 2.17, we give two transformations that generate equivalent MB statistics.

Proposition 2.17. *Let $\tau = (\tau_1, \tau_2, \dots, \tau_m)$ be an MB statistic. Given a prime ℓ and a constant $a \not\equiv 0 \pmod{\ell}$, define $a\tau = (a\tau_1, a\tau_2, \dots, a\tau_m)$. Then τ and $a\tau$ are equivalent modulo ℓ .*

Let b be any integer and define $\tau + b(1, 2, \dots, m) = (\tau_1 + b, \tau_2 + 2b, \dots, \tau_m + bm)$. Then τ and $\tau + b(1, 2, \dots, m)$ are equivalent modulo ℓ .

Proof. Define a permutation ϕ^\times of $\{0, 1, \dots, \ell-1\}$ by $\phi^\times(r) = ar \pmod{\ell}$. For each partition $\lambda = (1^{e_1}, 2^{e_2}, \dots, m^{e_m})$ of n , there is some $r \in \{0, 1, \dots, \ell-1\}$ such that $\lambda \in \mathcal{M}_\tau(r, \ell, n, m)$. Then $a\tau(\lambda) \equiv ar \pmod{\ell}$, and thus $\lambda \in \mathcal{M}_{a\tau}(\phi^\times(r), \ell, n, m)$. Thus τ and $a\tau$ are equivalent.

Further define a permutation ϕ^+ of $\{0, 1, \dots, \ell - 1\}$ by $\phi^+(r) = r + bn \pmod{\ell}$. Then, since $(1, 2, \dots, m)(\lambda) = n$, we have $(\tau + b(1, 2, \dots, m))(\lambda) \equiv r + bn \pmod{\ell}$. Thus $\lambda \in \mathcal{M}_{\tau+b(1,2,\dots,m)}(\phi^+(r), \ell, n, m)$, hence τ and $\tau + b(1, 2, \dots, m)$ are equivalent. \blacksquare

By indicating a distinct representative of each equivalence class of cranks, we are able to count the number of inequivalent MB statistics that are cranks for The Interval Theorem.

Theorem 2.18. *For any prime $\ell \geq 3$ and $3 \leq m \leq \ell + 1$, the number of inequivalent MB statistics generated by Theorem 2.7 that witness The Interval Theorem is exactly*

- $\frac{(\ell-2)!}{(\ell-m)!}$ for $2 \leq m < \ell$, and
- $(\ell - 1)!$ for $m = \ell, \ell + 1$.

Proof. We show that $\{\tau \in \{0, 1, \dots, \ell - 1\}^m : \tau = (0, 1, \tau_3, \tau_4, \dots, \tau_m)\}$ is a set of inequivalent cranks for The Interval Theorem if the components of $\left(\frac{\tau_i}{i}\right)_{i=1}^m$ are distinct modulo ℓ , and $\tau_\ell \neq 0$.

Consider two different MB statistics $\tau = (0, 1, \tau_3, \tau_4, \dots, \tau_m)$ and $\sigma = (0, 1, \sigma_3, \sigma_4, \dots, \sigma_m)$ in $\{0, 1, \dots, \ell - 1\}^m$, and let j be the smallest such that $\tau_j \neq \sigma_j$. Without loss of generality, assume $\tau_j \neq 0$ (if $\tau_j = 0$, switch the roles of τ and σ).

Let n be some integer of the form $\ell \text{lcm}(m)k - v$ meeting the hypotheses The Interval Theorem. To show that τ and σ divide the partitions of n into different subclasses, we construct partitions $\gamma \vdash n$ and $\kappa \vdash n$ such that $\tau(\gamma) = \tau(\kappa)$, but $\sigma(\gamma) \neq \sigma(\kappa)$.

Let

$$\gamma = (1^{n-j\ell}, 2^0, 3^0, \dots, j^\ell, (j+1)^0, \dots, m-1^0, m^0).$$

Choose a to be the least non-negative residue of $(\tau_j)^{-1}$ modulo ℓ , and let

$$\kappa = (1^{n-j(\ell-a)-2}, 2^1, 3^0, 4^0, \dots, j^{\ell-a}, (j+1)^0, \dots, m-1^0, m^0).$$

Then $\tau(\gamma) = \ell\tau_j \equiv 0 \pmod{\ell}$ and $\tau(\kappa) = 1 + (\ell - a)\tau_j = 1 + \ell\tau_j - 1 \equiv 0 \pmod{\ell}$. However, $\sigma(\gamma) = \ell\sigma_j \equiv 0 \pmod{\ell}$, while $\sigma(\kappa) = 1 + (\ell - a)\sigma_j = 1 + \ell\sigma_j - a\sigma_j \not\equiv 0 \pmod{\ell}$, since $\sigma_j \neq \tau_j \equiv a^{-1} \pmod{\ell}$.

Hence $\tau(\gamma) = \tau(\kappa)$ while $\sigma(\gamma) \neq \sigma(\kappa)$, and thus the cranks τ and σ produce genuinely different divisions into subclasses as they witness the The Interval Theorem.

On the other hand, for $m < \ell$, Since each $i \in [\ell - 1]$ is invertible modulo ℓ , the total number of (possibly equivalent) cranks generated in Theorem 2.7 is equal to the total number of tuples $\left(\frac{\tau_i}{i}\right)_{i=1}^m = \left(\frac{\tau_i}{i}\right)_{i=1}^m$ that have distinct components modulo ℓ , which is $\ell!/(\ell - m)!$. Let $\tau = (\tau_1, \tau_2, \dots, \tau_m)$ be any such crank. By Proposition 2.17, we see that τ is equivalent to several other cranks, which we now count. Since there ℓ constants b modulo ℓ and $\ell - 1$ non-zero

constants a modulo ℓ , there are $\ell(\ell - 1)$ potential equivalents to τ , which we now show are all different.

Suppose for some a, b, c, d , we have that $a\tau + b(1, 2, \dots, m) = c\tau + d(1, 2, \dots, m)$. Then

$$a\tau_1 + b = c\tau_1 + d \quad (12)$$

and

$$a\tau_2 + 2b = c\tau_2 + 2d. \quad (13)$$

Subtracting twice (12) from (13), we get $a(\tau_2 - 2\tau_1) = c(\tau_2 - 2\tau_1)$. Notice that since $\tau_1/1$ and $\tau_2/2$ are distinct, we have $a = c$. If $\tau_1 = 0$, we also have $b = d$ and we are done. Otherwise, subtracting τ_2/τ_1 times (12) from (13) we get $b(2 - \tau_2/\tau_1) = d(2 - \tau_2/\tau_1)$. Notice again that since $\tau_1/1$ and $\tau_2/2$ are distinct, we have $b = d$. Thus we have $\ell(\ell - 1)$ different cranks that are equivalent to τ . Hence, for $2 \leq m < \ell$, there are at most $(\ell!/(\ell - m)!)/\ell(\ell - 1) = (\ell - 2)!/(\ell - m)!$ inequivalent cranks witnessing The Interval Theorem.

The case $m = \ell$ follows identically to the case $m = \ell - 1$, but with an additional factor of $\ell - 1$ total (possibly equivalent) cranks generated in Theorem 2.7 from the choice of some $\tau_\ell \not\equiv 0 \pmod{\ell}$. Hence, for $m = \ell$, there are at most $((\ell - 1)\ell!/(\ell - \ell)!)/\ell(\ell - 1) = (\ell - 1)!$ inequivalent cranks witnessing The Interval Theorem.

The case $m = \ell + 1$ follows identically to the case $m = \ell$, but with an additional trivial factor of 1 total cranks generated in Theorem 2.7, since once τ_1 through τ_ℓ are generated, $\tau_{\ell+1}$ is uniquely determined. Hence, for $m = \ell + 1$, there are again at most $(\ell - 1)!$ inequivalent cranks witnessing The Interval Theorem. \blacksquare

3 Future Work

The Interval Theorem describes partition congruences that occur in arithmetic progressions with period $\ell \text{lcm}(m)$. There exists a related infinite family of partition congruences having a much smaller period.

Theorem 3.1. [7]

For any prime ℓ , any non-negative integer k , and $0 < u < \frac{\ell-1}{2}$, we have

$$p(\text{lcm}(\ell)k - u\ell, \ell) \equiv 0 \pmod{\ell}. \quad (14)$$

Theorem 2.7 shows that the MB statistic $\boldsymbol{\tau} = (1, 1, \dots, 1, 0, -1)$, otherwise described as *the number of parts of size 1 through $\ell - 2$, minus the number of parts of size ℓ* , is a crank for The Interval Theorem. Theorem 2.7 does not show that $\boldsymbol{\tau}$ is a crank for Theorem 3.1. However, by producing quasipolynomial formulas for $M_{\boldsymbol{\tau}}(r, \ell, n, m)$ for each r , since we find that the relevant constituents are the same as r varies, we discover that $\boldsymbol{\tau}$ is also a crank witnessing the congruences of Theorem 3.1 for $\ell = 3, 5$, and 7.

For example, for k a non-negative integer, we find after performing nine individual computations that

$$\begin{aligned} M_{\tau}(0, 3, 18k + 3, 3) &= M_{\tau}(1, 3, 18k + 3, 3) = M_{\tau}(2, 3, 18k + 3, 3) \\ &= 9k^2 + 6k + 1, \\ M_{\tau}(0, 3, 18k + 9, 3) &= M_{\tau}(1, 3, 18k + 9, 3) = M_{\tau}(2, 3, 18k + 9, 3) \\ &= 9k^2 + 12k + 4, \text{ and} \\ M_{\tau}(0, 3, 18k + 15, 3) &= M_{\tau}(1, 3, 18k + 15, 3) = M_{\tau}(2, 3, 18k + 15, 3) \\ &= 9k^2 + 18k + 9. \end{aligned}$$

Thus the truth of

$$M_{\tau}(0, 3, 6k + 3, 3) = M_{\tau}(1, 3, 6k + 3, 3) = M_{\tau}(2, 3, 6k + 3, 3) = \frac{p(6k + 3, 3)}{3}$$

follows. For more information about computing the constituents of quasipolynomials for $p(n, m)$, see [3].

Despite only a small amount of evidence, we boldly offer the following.

Conjecture 3.2. *The MB statistic $\tau = (1, 1, \dots, 1, 0, -1)$ is a crank witnessing the congruences of Theorem 3.1 for all primes ℓ .*

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