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RESEARCH ARTICLE

A Generalized Two-Component Camassa-Holm System with Complex Nonlinear Terms and Waltzing Peakons

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Abstract
In this paper, we deal with the Cauchy problem for a generalized two-component Camassa-Holm system with waltzing peakons and complex higher-order nonlinear terms. By the classical Friedrichs regularization method and the transport equation theory, the local well-posedness of solutions for the generalized coupled Camassa-Holm system in nonhomogeneous Besov spaces and the critical Besov space $B^{5/2}_{2,1} \times B^{5/2}_{2,1}$ was obtained. Besides the propagation behaviors of compactly supported solutions, the global existence and precise blow-up mechanism for the strong solutions of this system are determined. In addition to wave breaking, the another one of the most essential property of this equation is the existence of waltzing peakons and multi-peaked solitray was also obtained.

Keywords Two-component Camassa-Holm · Well-posedness · Besov spaces · Blow-up criteria · Waltzing peakons · Cauchy problem

Mathematics Subject Classification 35G25 · 35L05 · 35Q50 · 35Q53 · 37K10

1 Introduction

In this paper, we propose the following Cauchy problem

$$\begin{cases} m_t + v^p m_x + a v^{p-1} v_x m = 0, & t > 0,\ x \in \mathbb{R}, \\ n_t + u^q n_x + b u^{q-1} u_x n = 0, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x),\ \ v(x,0) = v_0(x), & t = 0,\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

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where $m = u - \alpha^2 u_{xx}$, $n = v - \beta^2 v_{xx}$ $(\alpha \geq 0, \beta \geq 0)$, the constants $a, b \in \mathbb{R}$ and $p, q \in \mathbb{Z}^+$. Obviously, the system (1.1) has nonlinearities of degree $\max\{p+1, q+1\}$. If choosing $m = u, n = v$, then Equ. (1.1) is a generalized two-component Burgers type system; if ordering $m = u_{xx}, n = v_{xx}$, then Equ. (1.1) becomes a generalized two-component Hunter-Saxton type system; and if selecting $\alpha = \beta = 1$, namely, $m = u - u_{xx}, n = v - v_{xx}$, then Equ. (1.1) reads as a generalized two-component Camassa-Holm type system. In this paper, to keep our paper concise, we only focus on the coupled Camassa-Holm type system, and $m \doteq u - u_{xx}, n \doteq v - v_{xx}$.

During last few decades, due to various mathematical problems and nonlinear physics phenomena interfered, the water wave and fluid dynamics have been attracting much attention [2, 4, 36, 44]. Since the raw water wave governing equations have proven to be nearly intractable, the request for suitably simplified model equations was initiated at the early stage of hydrodynamics development. Until the early twentieth century, the study of water waves was restricted almost exclusively to the linear theory. Due to the linearization approach losing some important properties, such us the rare wave breaking, then people usually propose some nonlinear models to explain practical behaviors liking breaking waves and solitary waves [7]. The most marked example is the following dispersive nonlinear PDEs

$$u_t - \gamma u_{xxx} - \alpha^2 u_{xxt} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x, \tag{1.2}$$

where the constants $\gamma, \alpha, c_1, c_2 c_3 \in \mathbb{R}$. The Painlevé analysis method (cf.[14, 16, 28]) shown that there are only three asymptotically integrable members in this family, i.e., the famous KdV equation, the Camassa-Holm equation and the Degasperis-Procesi equation. Recently, such integrable peakon equations with cubic nonlinearity and wave breaking have been initialed: one is the Novikov equation, and the other one is the FORQ equation.

Integrable equations with soliton has been studied extensively since they usually have very delicate properties including infinite higher-order symmetries, infinitely many conservation laws, Lax pair, bi-Hamiltonian structure, which can be solved by the inverse scattering method, and so on. Discovering a new integrable equation may be accomplished via different methods. One of ways is the approach proposed by Fokas and Fuchssteiner [20] where the Korteweg-de Vries equation, the Camassa-Holm equation, and the Hunter-Saxton equation are derived in a unified way. The approach is based on the following fact: If θ_1, θ_2 are two Hamiltonian operators and for arbitrary number k their combination $\theta_1 + k\theta_2$ is also Hamiltonian, then

$$q_t = -(\theta_2 \theta_1^{-1}) q_x. \tag{1.3}$$

is an integrable equation. Now, letting $\theta_1 = \partial_x$ and $\theta_2 = \partial_x + \gamma \partial_x^3 + \frac{\alpha}{3}(q\partial_x + \partial_x q)$, where α and γ are constants, then Equ. (1.3) reads as the celebrated KdV equation (see [30]). If choosing $\theta_1 = \partial_x + \nu \partial_x^3$ and θ_2 as above with $q = u + \nu u_{xx}$, then Equ. (1.3) yields the following equation

$$u_t + u_x + \nu u_{xxt} + \nu u_{xxx} + \alpha u u_x + \frac{\alpha\nu}{3}(u u_{xxx} + 2u_x u_{xx}) = 0, \tag{1.4}$$

which could be reduced to the well-known CH equation through selecting the parameters appropriately and making a change of variables with some scaling (see [5, 6, 9]). If setting $\theta_1 = \nu\partial_x^3$ and $\theta_2 = \partial_x + (q\partial_x + \partial_x q)$ with $q = \nu u_{xx}$, then Equ. (1.3) leads to the HS equation (see [1, 27]):

$$u_{xxt} + \frac{1}{\nu}u_x + u u_{xxx} + 2u_x u_{xx} = 0. \tag{1.5}$$

Actually, in light of the Fokas-Fuchssteiner framework [20], one may generate generalized KdV-type or CH-type equations possessing bi-Hamiltonian structure(without regard to the integrability) and infinitely many conserved quantities. For instance, letting $\theta_1 = \partial_x$ and $\theta_2 = \beta\partial_x + \gamma\partial_x^3 + \frac{\alpha}{k+2}(q^k\partial_x + \partial_x q^k)$, where α, γ are constants and $k \in \mathbb{Z}^+$ in Equ. (1.3) produces the following generalized Korteweg-de Vries equation (see [10, 29]):

$$q_t + \beta\partial_x q + \gamma\partial_x^3 q + \alpha q^k q_x = 0. \tag{1.6}$$

And choosing $\theta_1 = \partial_x + \nu\partial_x^3$ and $\theta_2 = \beta\partial_x + \gamma\partial_x^3 + \alpha[(b-1)q\partial_x + \partial_x q]$ with $q = u + \nu u_{xx}$ in Equ. (1.3) generates the following CH-b family equation [25, 32]:

$$u_t + \beta u_x + \nu u_{xxt} + \gamma u_{xxx} + \alpha(b+1)u u_x + \alpha\nu(u u_{xxx} + b u_x u_{xx}) = 0. \tag{1.7}$$

Taking $b = 3, \beta = \gamma = 0, \nu = -1$ in Equ. (1.7) yields the remarkable DP equation (see [14, 15]).

Furthermore, let us take $\theta_1 = \partial_x(1 + \nu\partial_x^2)^k$ and $\theta_2 = \beta k\partial_x q^{k-1} + \gamma k\partial_x^3 q^{k-1} + \alpha[(b-1)q\partial_x q^{k-1} + \partial_x q^k]$ with $q = u + \nu u_{xx}$, then Equ. (1.3) yields the following generalized b-equation with nonlinearity of degree $k+1$ [22]:

$$u_t + \beta\partial_x u^k + \nu u_{xxt} + \gamma\partial_x^3 u^k + \alpha(b+1)u^k u_x + \alpha\nu(u^k u_{xxx} + b u^{k-1} u_x u_{xx}) = 0. \tag{1.8}$$

Taking $k = 2$ in Equ. (1.8) gives the Novikov equation through choosing the parameters appropriately and making a change of variables with some scaling [26, 33]. If choosing $\theta_1 = \partial_x - \partial_x^3$ and $\theta_2 = q^2\partial_x + q_x\partial_x^{-1}q\partial_x$ with $q = u - u_{xx}$, then Equ. (1.3) reads as

$$q_t + 2q^2 u_x + q_x(u^2 - u_x^2) = 0, \tag{1.9}$$

which is actually the FORQ equation [19, 21, 34, 35].

The other attractive feature of the CH types equation (1.8) with $\beta = \gamma = 0$ and $\nu = -1$ is: it admits the following peakon solutions [22]:

on the line: $u_c(x,t) = \sqrt[k]{c} \cdot e^{-|x-ct|}$; on the circle: $u_c(x,t) = \sqrt[k]{c} \cdot \cosh([x-ct]_\pi - \pi)$,

with

$$[x-ct]_\pi \doteq x-ct-2\pi\left[\frac{x-ct}{2\pi}\right]. \tag{1.10}$$

Equations (1.8) also has the multi-peakon solutions by the following unified form (cf.[22]):

$$\text{on the line: } u(t,x)=\sum_{i=1}^{N}p_i(t)\cdot e^{-|x-q_i(t)|}; \text{ on the circle: } u(t,x) = \sum_{i=1}^{N}p_i(t)\cdot\cosh([x-q_i(t)]_\pi-\pi),$$

here the peak positions $q_i(t)$ and amplitudes $p_i(t)$ satisfy

$$p_j' = \left(\sum_{i=1}^{N}p_i e^{-|q_j-q_i(t)|}\right)^k,$$
$$q_j' = (b-k)p_j\left(\sum_{i=1}^{N}p_i e^{-|q_j-q_i|}\right)^{k-1}\left(\sum_{i=1}^{N}p_i sgn(q_j-q_i)e^{-|q_j-q_i|}\right).$$

In recent years, the famous CH equation has been generalized to integrable two component Camassa-Holm models. One of them is the following form

$$\begin{cases} m_t = um_x + k_1u_xm + \sigma\rho\rho_x, & t>0, x\in\mathbb{R},\\ \rho_t = k_2\partial_x(u\rho), & t>0, x\in\mathbb{R},\\ u(x,0)=u_0(x), \rho(x,0)=\rho_0(x), & t=0, x\in\mathbb{R}. \end{cases} \tag{1.11}$$

Obviously, the 2CH and the 2DP are contained in Equ. (1.11) as two special cases with $k_1=2, k_2=\sigma=\pm1$ and with $k_1=3$, respectively. Constantin and Ivanov [8] derived the 2CH in the condition of shallow water theory. where the variable $\rho(x,t)$ is in connection with the horizontal deviation of the surface from equilibrium and the variable $u(x, t)$ describes the horizontal velocity of the fluid, and all are measured in dimensionless units [8]. The 2DP was shown to have solitons, kink, and antikink solutions [41]. Escher, Kohlmann and Lenells studied the geometric properties of the 2DP and local well-posedness in various function spaces [18]. However, peakon and superposition of multi-peakons were not investigated yet.

Motivated by the work of Cotter, Holm, Ivanov and Percival for the Cross-Coupled Camassa-Holm in [11] (called the CCCH equation, i.e., Equ. (1.1) with the choice of $a=b=2$ and $p=q=1$), we first deduce Equ. (1.1), and then study its wave-breaking criteria and peakon dynamical system. The CCCH can be derived from a variational principle by an Euler-Lagrange system with the following Lagrangian [11]

$$L(u,v)=\int_{\mathbb{R}}(uv+u_xv_x)dx.$$

And the Euler-Poincaré system in one dimension defined as follows,

$$\begin{aligned} \partial_t m &= -ad^*_{\delta h/\delta m} m = -(vm)_x - mv_x \quad \text{and} \quad v \doteq \frac{\delta h}{\delta m} = K * n, \\ \partial_t n &= -ad^*_{\delta h/\delta n} n = -(un)_x - nu_x \quad \text{and} \quad u \doteq \frac{\delta h}{\delta n} = K * m, \end{aligned} \tag{1.12}$$

with $K(x,y) = \frac{1}{2}e^{-|x-y|}$ and the Hamiltonian $h(n,m) = \int_{\mathbb{R}} nK * mdx = \int_{\mathbb{R}} mK * ndx$, this Hamiltonian system own a two-component singular momentum map [11]

$$m(x,t) = \sum_{a=1}^{M} m_a(t)\delta(x - q_a(t)), n(x,t) = \sum_{a=1}^{N} n_b(t)\delta(x - r_b(t)). \tag{1.13}$$

Such a formal waltzing peakons, multi-peakon and compactons of the CCCH are given in [11]. In [17], the authors given a geometrical interpretation for the CCCH system along with a large class of peakon equations. Recently, the Cauchy problem of Equ. (1.1) has been studied extensively. The local well-posedness, the condition lead to global existence or wave-breaking, continuity and analyticity of the data-to-solution map, and persistence properties for the CCCH system were discussed in [24, 31, 37, 38].

Inspired by the argument on the approximate solutions for the CH-type equations in [12] and [39, 40], we want to obtain the local well-posedness for Equ. (1.1) by the transport equations theory and classical Friedrichs regularization method. However, comparing with the one appearing in [12, 39, 40], the nonlinear terms of Equ. (1.1) is very complicated. Unlike the regular procedure, we will use the original Equ. (1.1) rather than the nonlocal form (see (3.18) below) since the fact: $\|u\|_{B^s_{l,r}} \approxeq \|m\|_{B^{s-2}_{l,r}}$. The key to show the local well-posedness through the Littlewood-Paley decomposition and nonhomogeneous Besov spaces is to prove the following inequality

$$\|u_k(t)\|_{B^s_{l,r}} + \|v_k(t)\|_{B^s_{l,r}} \le \frac{\|u_0\|_{B^s_{l,r}} + \|v_0\|_{B^s_{l,r}}}{\left(1 - 2\kappa Ct(\|u_0\|_{B^s_{l,r}} + \|v_0\|_{B^s_{l,r}})^\kappa\right)^{1/\kappa}} \text{ with } \kappa = \max\{p, q\},$$

and we obtain this inequality by mathematical induction, which involved the degree of the nonlinearities. This result specifically reads as follows.

Theorem 1.1 *Assume that the Besov indexes* $1 \le l, r \le +\infty$ *and* $s > \max\{2 + \frac{1}{l}, \frac{5}{2}, 3 - \frac{1}{l}\}$. *Let* $(u_0, v_0) \in B^s_{l,r} \times B^s_{l,r}$. *Then there exists a lifespan* $T > 0$ *such that the Equ.* (1.1) *has a unique solution* $(u, v) \in E^s_{l,r}(T) \times E^s_{l,r}(T)$, *moveover, the map* $(u_0, v_0) \mapsto (u, v)$ *is continuous from a neighborhood of the initial data* (u_0, v_0) *in* $B^s_{p,r} \times B^s_{p,r}$ *into*

$$\mathcal{C}([0,T];B^{s'}_{l,r}) \cap \mathcal{C}^1([0,T];B^{s'-1}_{l,r}) \times \mathcal{C}([0,T];B^{s'}_{l,r}) \cap \mathcal{C}^1([0,T];B^{s'-1}_{l,r})$$

for every $s' < s$ *when* $r = +\infty$, *and* $s' = s$ *whereas* $r < +\infty$.

Remark 1.1 We known that $B^s_{2,2}(\mathbb{R}) = H^s$. Thus, under the condition $m_0, n_0 \in H^s$ with $s > \frac{1}{2}$, i.e., $(u_0, v_0) \in H^s \times H^s$ with $s > \frac{5}{2}$, the above theorem implies that there exists a lifespan $T > 0$ such that the Cauchy problem (1.1) has a unique solution $m, n \in \mathcal{C}([0,T];H^s) \cap \mathcal{C}^1([0,T];H^{s-1})$, and the map $(m_0, n_0) \mapsto (m, n)$ is continuous from a neighborhood of the initial data (m_0, n_0) in $H^s \times H^s$ into $\mathcal{C}([0,T];H^s) \cap \mathcal{C}^1([0,T];H^{s-1}) \times \mathcal{C}([0,T];H^s) \cap \mathcal{C}^1([0,T];H^{s-1})$.

For any $s' < 5/2 < s$, we have the following imbed relationship:

$$H^s \hookrightarrow B^{\frac{5}{2}}_{2,1} \hookrightarrow H^{\frac{5}{2}} \hookrightarrow B^{\frac{5}{2}}_{2,\infty} \hookrightarrow H^{s'},$$

which implies that H^s and $B^s_{2,1}$ are very close, so, we next establish the local well-posedness solution for Equ. (1.1) in the critical Besov space $B^{5/2}_{2,1} \times B^{5/2}_{2,1}$.

Theorem 1.2 *Suppose that* $z_0 \doteq (u_0, v_0) \in B^{\frac{5}{2}}_{2,1} \times B^{\frac{5}{2}}_{2,1}$. *Then there exists a lifespan* $T = T(z_0) > 0$ *and a unique solution* $z = (u, v)$ *verify that the Cauchy problem* (1.1)

$$z = z(\cdot, z_0) \in \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1}) \cap \mathcal{C}^1([0,T];B^{\frac{3}{2}}_{2,1}) \times \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1}) \cap \mathcal{C}^1([0,T];B^{\frac{3}{2}}_{2,1}).$$

Furthermore, *the solutions continuously depend on the initial data*, *i.e.*, *the mapping*

$$\begin{aligned} z_0 \mapsto z(\cdot, z_0) :\, & B^{\frac{5}{2}}_{2,1} \times B^{\frac{5}{2}}_{2,1} \mapsto \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1}) \cap \mathcal{C}^1([0,T];B^{\frac{3}{2}}_{2,1}) \\ & \times \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1}) \cap \mathcal{C}^1([0,T];B^{\frac{3}{2}}_{2,1}) \end{aligned}$$

is continuous.

In order to get the precise blow-up scenario, we need the following equivalent theorem:

Theorem 1.3 *Suppose that the initial data* $(m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ *with* $s > \frac{1}{2}$, *and* (m, n) *be the corresponding solution to the Cauchy problem* (1.1), *and* $T^*_{m_0,n_0} > 0$ *is the maximum time of existence for the solution of the Equ.* (1.1). *Then*

$$\int_0^{T^*_{m_0,n_0}} \left(\|n\|^p_{L^\infty} + \|m\|_{L^\infty}\|n\|^{p-1}_{L^\infty} + \|m\|^q_{L^\infty} + \|n\|_{L^\infty}\|m\|^{q-1}_{L^\infty} \right) d\tau < \infty,$$

provided that $T^*_{m_0,n_0} < \infty$.

It is will known that the solution of Camassa-Holm type equations occurs blowup only in the form of breaking waves, namely, the solution remains bounded but its slope about the space becomes unbounded in finite time [40]. Next, we establish the accurate blowup scenarios for sufficiently regular solutions to the Equ. (1.1).

Theorem 1.4 *Let* $z_0 = (u_0, v_0) \in L^1 \cap H^s$ *with* $s > 5/2$, *and* T *be the lifespan of the solution* $z(x,t) = (u(x,t), v(x,t))$ *to Equ.* (1.1) *with the initial data* z_0. *If* $p = 2a, q = 2b$, *then every solution* $z(x, t)$ *to Equ.* (1.1) *remains globally regular in all time. If* $p > 2a$ (*or* $q > 2b$), *then the corresponding solutions* $z(x, t)$ *blow up in a finite time iff* $(v^p)_x$ (*or* $(u^q)_x$) *are unbounded at* $-\infty$ *in a finite time. If* $p < 2a$ (*or* $q < 2b$), *then the corresponding solutions* $z(x, t)$ *blow up in a finite time iff* $(v^p)_x$ (*or* $(u^q)_x$) *tends to* $+\infty$ *in a finite time.*

Let us now give a sufficient condition for the global existence of the solutions to Equ. (1.1).

Theorem 1.5 *Assume that* $u_0 \in H^s \cap W^{2,\frac{p}{a}}$ *and* $v_0 \in H^s \cap W^{2,\frac{q}{b}}$ *with* $s > 5/2$ *and* $0 \le a \le p, 0 \le b \le q$. *Then the solution to Equ.* (1.1) *remains smooth for all time.*

As per [11, 43], the CCCH system might not be completely integrable. However, it does have peakon and multi-peakon solutions which display interesting dynamics property with both oscillation and propagation. Furthermore, if the two initial values u_0 or v_0 of in Equ. (1.1) have a compact support, then the compact property will be succeed to u and v at all times $t \in [0, T)$.

Theorem 1.6 *Supposed that the initial data* $(u_0, v_0) \in H^s \times H^s$ *with* $s > 5/2$, *and* $m_0 = (1 - \partial_x^2)u_0$ (*or* $n_0 = (1 - \partial_x^2)v_0$) *have a compact support, and* $T = T(u_0, v_0) > 0$ *be the maximal existence time to the corresponding initial data. Then the* $\mathcal{C}^1$ *functions* $x \mapsto m(x,t)$ (*or* $x \mapsto n(x,t)$) *also have a compact support, for any* $t \in [0, T)$.

Finally, we will exhibit that Equ. (1.1) not only admits peaked solitary wave but also possesses multi-peaked solitray wave solutions.

Theorem 1.7 *Let the constant* $c > 0$, *Equ.* (1.1) *has the single peaked solitary wave in the form*

$$\text{on the line: } u(t,x) = \alpha e^{-|x-ct-x_0|}, \;\; v(t,x) = \beta e^{-|x-ct-x_0|}; \tag{1.14}$$

$$\text{on the circle: } u(t,x) = \frac{\alpha}{\cosh(\pi)}\cosh([x-ct]_\pi - \pi), \;\; v(t,x) = \frac{\beta}{\cosh(\pi)}\cosh([x-ct]_\pi - \pi), \tag{1.15}$$

which are global weak solutions iff $\alpha = c^{1/q}, \beta = c^{1/p}$. *Moreover, the multi-peaked solitary wave solutions for Equ.* (1.1) *takes on the form of*

$$\text{in the non-periodic case: } u(t,x) = \sum_{i=1}^{M} f_i(t)e^{-|x-g_i(t)|}, \;\; v(t,x) = \sum_{j=1}^{N} h_j(t)e^{-|x-k_j(t)|}; \tag{1.16}$$

$$\text{in the periodic case: } u(t,x) = \sum_{i=1}^{M} f_i(t)\cosh([x-g_i(t)]_\pi - \pi),\ \ v(t,x) = \sum_{j=1}^{N} h_j(t)\cosh([x-k_i(t)]_\pi - \pi), \tag{1.17}$$

whose peaked positions $g_i(t), k_j(t)$ *and amplitudes* $f_i(t), h_j(t)$ *satisfy the following dynamical system*

$$\begin{aligned} &\dot{g}_i = v^p(g_i),\ \ \dot{f}_i = (p-a)v^{p-1}(g_i)\langle v_x(g_i)\rangle f_i,\\ &\dot{h}_j = u^q(k_j),\ \ \dot{k}_j = (q-b)u^{q-1}(k_j)\langle u_x(k_j)\rangle h_j, \end{aligned} \tag{1.18}$$

where $\langle f(x)\rangle = \frac{1}{2}(f(x^-)+f(x^+))$, *and the notation* $[x-ct]_\pi$ *defined by Equ.* (1.10).

The entire paper is organized as follows. In next section, we obtain the local well-posedness solution in Besov spaces of Equ. (1.1) through proving Theorems 1.1−1.2. In section 3, our goal is twofold, one is to get the condition leads to global existence and blow up phenomena, and the another is to analyze the propagation behaviors start from compactly supported solutions to the problem (1.1), see Theorems 1.3−1.6 for the details. In section 4, the peakon and multi-peakons are derived through proving Theorem 1.7.

2 Local Well-Posedness to Equ. (1.1) in the Besov Spaces

In present section, we will establish the local well-posedness for the initial-value problem Equ. (1.1) in the Besov spaces, i.e., prove Theorem 1.1 and 1.2. The properties of the Besov spaces and the Littlewood-Paley theory can be found in [39, 40].

2.1 Local Well-Posedness to Equ. (1.1) in the Besov Spaces $B^s_{l,r}$

At the beginning we introduce the following definition.

Definition 2.1 For $T>0, s\in\mathbb{R}$ and $1\le l\le +\infty$ and $s\neq 2+\frac{1}{l}$, we define

$$\begin{aligned} E^s_{l,r}(T) &\doteq \mathcal{C}([0,T];B^s_{l,r})\cap \mathcal{C}^1([0,T];B^{s-1}_{l,r}) \quad \text{if } r<+\infty,\\ E^s_{l,\infty}(T) &\doteq L^\infty([0,T];B^s_{l,\infty})\cap Lip([0,T];B^{s-1}_{l,\infty}), \end{aligned}$$

and $E^s_{l,r} \doteq \cap_{T>0}E^s_{l,r}(T)$.

First, we get the uniqueness and continuity for the solution to the Equ. (1.1) with respect to the initial data, and we denote the generic constant $C>0$ is only depending on l, r, s, p, q, $|a|$, $|b|$.

Lemma 2.1 *Assume that* $1 \leq l, r \leq +\infty$ *and the index* $s > \max\{2+\frac{1}{l}, \frac{5}{2}, 3-\frac{1}{l}\}$, *and* $(u_i, v_i) \in \{L^\infty([0,T];B^s_{l,r}) \cap \mathcal{C}([0,T];\mathcal{S}')\}^2$ $(i=1,2)$ *be two given solutions of the Cauchy problem* (1.1) *with respect to the initial data* $(u_i(0), v_i(0)) \in B^s_{l,r} \times B^s_{l,r}$ $(i=1,2)$, *and denote* $u_{12} = u_1 - u_2, v_{12} = v_1 - v_2$. *Therefore,*

(*i*) *If* $s \neq 4 + 1/l$ *and* $s > \max\{1+\frac{1}{l}, \frac{3}{2}\}$, *then*

$$\|u_{12}\|_{B^{s-1}_{l,r}} + \|v_{12}\|_{B^{s-1}_{l,r}} \leq \left(\|u_{12}(0)\|_{B^{s-1}_{l,r}} + \|u_{12}(0)\|_{B^{s-1}_{l,r}}\right) \exp\left(C\int_0^t \Gamma_s(t,\cdot)d\tau\right), \tag{2.1}$$

for every $t \in [0,T]$, *where*

$$\Gamma_s(t,\cdot) = \Bigg(\|v_1\|^p_{B^s_{l,r}} + \|u_1\|^q_{B^s_{l,r}} + \|u_2\|_{B^s_{l,r}} \sum_{i=0}^{p-1} \|v_1\|^{p-1-i}_{B^s_{l,r}} \|v_2\|^j_{B^s_{l,r}} + \|v_2\|_{B^s_{l,r}} \sum_{j=0}^{q-1} \|u_1\|^{q-1-i}_{B^s_{l,r}} \|u_2\|^j_{B^s_{l,r}}\Bigg).$$

(*ii*) *If* $s = 4 + 1/l$, *then*

$$\|u_{12}\|_{B^{s-1}_{l,r}} + \|v_{12}\|_{B^{s-1}_{l,r}} \leq C\left(\|u_{12}(0)\|_{B^{s-1}_{l,r}} + \|u_{12}(0)\|_{B^{s-1}_{l,r}}\right)^\theta \Gamma_s^{1-\theta}(t,\cdot) \exp\left(C\theta\int_0^t \Gamma_s(t,\cdot)d\tau\right),$$

for every $t \in [0,T]$, *where* $\theta \in (0,1)$(*i.e.*, $\theta = \frac{1}{2}(1-\frac{1}{2l})$) *and* $\Gamma_s(t,\cdot)$ *as in case* (*i*).

Proof The hypothesis of this theorem implies that $m_{12} = m_1 - m_2, n_{12} = n_1 - n_2$, and $u_{12}, v_{12} \in L^\infty([0,T];B^s_{l,r}) \cap \mathcal{C}([0,T];\mathcal{S}')$, this gets that $u_{12}, v_{12} \in \mathcal{C}([0,T];B^{s-1}_{l,r})$, and (u_{12}, v_{12}) and m_{12}, n_{12} solves the transport equations

$$\begin{cases} \partial_t m_{12} + v_1^p \partial_x m_{12} = -[v_1^p - v_2^p]\partial_x m_2 - \frac{a}{p}\partial_x v_1^p m_{12} - \frac{a}{p}[\partial_x v_1^p - \partial_x v_2^p] m_2, \\ \partial_t n_{12} + u_1^q \partial_x n_{12} = -[u_1^q - u_2^q]\partial_x n_2 - \frac{b}{q}\partial_x u_1^q n_{12} - \frac{b}{q}[\partial_x u_1^q - \partial_x u_2^q] n_2, \\ m_{12}|_{t=0} = m_{12}(0) \doteq m_1(0) - m_2(0), \quad n_{12}|_{t=0} = n_{12}(0) \doteq n_1(0) - n_2(0). \end{cases}$$

According to Lemma 2.2 (i) in [40], we have

$$\begin{aligned} \|m_{12}\|_{B^{s-3}_{l,r}} \leq & \|m_{12}(0)\|_{B^{s-3}_{l,r}} + C\int_0^t \left(\|\partial_x v_1^p\|_{B^{s-4}_{l,r}} + \|\partial_x v_1^p\|_{B^{\frac{1}{l}}_{p,r}\cap L^\infty}\right)\|m_{12}\|_{B^{s-3}_{l,r}} d\tau \\ & + C\int_0^t \|[v_1^p - v_2^p]\partial_x m_2 - \frac{a}{p}\partial_x v_1^p m_{12} - \frac{a}{p}[\partial_x v_1^p - \partial_x v_2^p] m_2\|_{B^{s-3}_{l,r}} d\tau. \end{aligned} \tag{2.2}$$

Since $s > \max\{2+\frac{1}{l}, \frac{5}{2}, 3-\frac{1}{l}\} \geq 2 + \frac{1}{l}$, we obtain

$$\|\partial_x v_1^p\|_{B^{s-4}_{l,r}} + \|\partial_x v_1^p\|_{B^{\frac{1}{l}}_{l,r}\cap L^\infty} \le 2\|\partial_x v_1^p\|_{B^{s-2}_{l,r}} \le C\|v_1\|^p_{B^s_{l,r}}.$$

Since the property $(1-\partial_x^2) \in OP(S^2)$, by Proposition 2.2 (7) in [40], for all $s \in \mathbb{R}$, we obtain that

$$\|u_i\|_{B^s_{l,r}} \approxeq \|m_i\|_{B^{s-2}_{l,r}} \text{ and } \|v_i\|_{B^s_{l,r}} \approxeq \|n_i\|_{B^{s-2}_{l,r}}.$$

If $\max\{2+\frac{1}{l}, \frac{5}{2}\} < s \le 3 + \frac{1}{l}$, by Proposition 2.5 (2) in [40] and $B^{s-2}_{l,r}$ being an algebra, we arrive at

$$\begin{aligned} &\|[v_1^p - v_2^p]\partial_x m_2 - \frac{a}{p}\partial_x v_1^p m_{12} - \frac{a}{p}[\partial_x v_1^p - \partial_x v_2^p]m_2\|_{B^{s-3}_{l,r}} + \|\partial_x v_1^p - \partial_x v_2^p\|_{B^{s-3}_{l,r}}\|m_2\|_{B^{s-2}_{l,r}} \\ &\le C\left(\|v_1\|^p_{B^{s-1}_{l,r}}\|u_{12}\|_{B^{s-1}_{l,r}} + \|u_2\|_{B^s_{l,r}}\|v_{12}\|_{B^{s-1}_{l,r}}\sum_{i=0}^{p-1}\|v_1\|^{p-1-i}_{B^s_{l,r}}\|v_2\|^i_{B^s_{l,r}}\right). \end{aligned} \tag{2.3}$$

For the case $s > 3 + \frac{1}{l}$, the inequality (2.3) also holds true since that $B^{s-3}_{l,r}$ is an algebra. Thus,

$$\begin{aligned} \|u_{12}\|_{B^{s-1}_{l,r}} \le \|u_{12}(0)\|_{B^{s-1}_{l,r}} + C\int_0^t \Bigg(\|v_1\|^p_{B^s_{l,r}}\|u_{12}\|_{B^{s-1}_{l,r}} \\ +\|u_2\|_{B^s_{l,r}}\|v_{12}\|_{B^{s-1}_{l,r}}\sum_{i=0}^{p-1}\|v_1\|^{p-1-i}_{B^s_{l,r}}\|v_2\|^i_{B^s_{l,r}}\Bigg)(\tau)d\tau. \end{aligned}$$

The second component v can be treat by the similar way, and get the following inequality

$$\begin{aligned} \|v_{12}\|_{B^{s-1}_{l,r}} \le \|v_{12}(0)\|_{B^{s-1}_{l,r}} + C\int_0^t \Bigg(\|u_1\|^p_{B^s_{l,r}}\|v_{12}\|_{B^{s-1}_{l,r}} \\ +\|v_2\|_{B^s_{l,r}}\|u_{12}\|_{B^{s-1}_{l,r}}\sum_{j=0}^{q-1}\|u_1\|^{q-1-i}_{B^s_{l,r}}\|u_2\|^j_{B^s_{l,r}}\Bigg)(\tau)d\tau. \end{aligned}$$

Therefore

$$\begin{aligned} \|u_{12}\|_{B^{s-1}_{l,r}} + \|v_{12}\|_{B^{s-1}_{l,r}} \le \|u_{12}(0)\|_{B^{s-1}_{l,r}} + \|v_{12}(0)\|_{B^{s-1}_{l,r}} \\ + C\int_0^t \left(\|u_{12}\|_{B^{s-1}_{l,r}} + \|v_{12}\|_{B^{s-1}_{l,r}}\right)\Gamma_s(\tau,\cdot)d\tau. \end{aligned}$$

Using Gronwall's lemma, we obtain (i).

Next, we apply the interpolation method to deal with the critical case: $s = 4 + 1/l$. Indeed, $s-1 = 3+\frac{1}{l} = \theta\left(2+\frac{1}{2l}\right) + (1-\theta)\left(4+\frac{1}{2l}\right)$ provide $\theta = \frac{1}{2}(1-\frac{1}{2l}) \in (0,1)$. According to Proposition 2.2(5) in [40] and the above inequality, we have

$$
\begin{aligned}
&\|u_{12}\|_{B^{3+1/l}_{l,r}} + \|v_{12}\|_{B^{3+1/l}_{l,r}} \\
&\quad \le \|u_{12}\|^{\theta}_{B^{2+1/2l}_{l,r}} \|u_{12}\|^{1-\theta}_{B^{4+1/2l}_{l,r}} + \|v_{12}\|^{\theta}_{B^{2+1/2l}_{l,r}} \|v_{12}\|^{1-\theta}_{B^{4+1/2l}_{l,r}} \\
&\quad \le \left(\|u_{12}\|_{B^{2+1/2l}_{l,r}} + \|v_{12}\|_{B^{2+1/2l}_{l,r}} \right)^{\theta} \left(\|u_{12}\|^{1-\theta}_{B^{4+1/2l}_{l,r}} + \|v_{12}\|^{1-\theta}_{B^{4+1/2l}_{l,r}} \right) \\
&\quad \le C\Big(\|u_{12}(0)\|_{B^{s-1}_{l,r}} + \|u_{12}(0)\|_{B^{s-1}_{l,r}} \Big)^{\theta} \Gamma_s^{1-\theta}(t,\cdot) \exp\left(C\theta \int_0^T \Gamma_s(t,\cdot) d\tau \right),
\end{aligned}
$$

which yields (ii). □

Next, in order to prove the local existence theorem 1.1, we start establish the approximate solutions to Equ. (1.1) by the famous Friedrichs regularization approach.

Lemma 2.2 *Let* $u(0) = v(0) := 0$. *Thus, there exists a sequence* $(u_k, v_k) \in \mathcal{C}(\mathbb{R}^+; B^{\infty}_{p,r})^2$ *verifying*

$$
(T_k) \quad \begin{cases} \partial_t m_{k+1} + v_k^p \partial_x m_{k+1} + \frac{a}{p} \partial_x v_k^p m_k = 0, \\ \partial_t n_{k+1} + u_k^q \partial_x n_{k+1} + \frac{b}{q} \partial_x u_k^q n_k = 0, \\ m_{k+1}(0) = S_{k+1} m(0), n_{k+1}(0) = S_{k+1} n(0), \end{cases}
$$

and there is a lifespan $T > 0$ *such that the sequence of smooth functions enjoying the following properties*:

(i) *The sequence* $(u_k, v_k)_{k\in\mathbb{N}}$ *is uniformly bounded in the spaces* $E^s_{p,r}(T) \times E^s_{p,r}(T)$.
(ii) *The sequence* $(u_k, v_k)_{k\in\mathbb{N}}$ *is a Cauchy sequence in* $\mathcal{C}([0,T]; B^{s-1}_{p,r}) \times \mathcal{C}([0,T]; B^{s-1}_{p,r})$.

Proof The fact $S_{k+1} u_0 \in B^{\infty}_{l,r}$ and Lemma 2.3 in [40] enables us to show that the equation (T_k) exists a global solution by induction, moveover, which belongs to $\mathcal{C}(\mathbb{R}^+; B^{\infty}_{l,r})^2$ for all $k \in \mathbb{N}$.

For $s \neq 3 + \frac{1}{l}$ and $\max\left\{2 + \frac{1}{2}, \frac{5}{2}, 3 - \frac{1}{l}\right\}$, Lemma 2.1 (i) implies

$$
\begin{aligned}
\|m_{k+1}\|_{B^{s-2}_{l,r}} \le{}& \exp\left(C \int_0^t \left(\|\partial_x v_k^p(\tau)\|_{B^{s-4}_{l,r}} + \|\partial_x v_k^p(\tau)\|_{B^{\frac{1}{l}}_{p,r} \cap L^\infty} \right) d\tau \right) \|m(0)\|_{B^{s-2}_{l,r}} \\
&+ C \int_0^t \exp\left(C \int_\tau^t \left(\|\partial_x v_k^p(\tau')\|_{B^{s-4}_{l,r}} + \|\partial_x v_k^p(\tau')\|_{B^{\frac{1}{l}}_{p,r} \cap L^\infty} \right) d\tau' \right) \\
&\|\partial_x v_k^p m_k\|_{B^{s-2}_{l,r}} d\tau.
\end{aligned}
$$

is true for all $k \in \mathbb{N}$. Due to $s > 2 + \frac{1}{l}$, we know that $B^{s-2}_{l,r}$ is an algebra and $B^{s-2}_{l,r} \hookrightarrow L^\infty$. Thus, we have

$$\|m_{k+1}\|_{B^{s-2}_{l,r}} \le \exp\left(C\int_0^t \|v_k(\tau)\|^p_{B^s_{l,r}}\,d\tau\right)\|m(0)\|_{B^{s-2}_{l,r}} + C\int_0^t \exp\left(C\int_\tau^t \|v_k(\tau')\|^p_{B^s_{l,r}}\,d\tau'\right)\|v_k\|^p_{B^s_{l,r}}\|m_k\|_{B^{s-2}_{l,r}}d\tau,$$

and also for $\|n_{k+1}\|_{B^{s-2}_{l,r}}$. Thus, adding the two resulted inequalities yields

$$\begin{aligned}
&\|u_{k+1}\|_{B^s_{l,r}} + \|v_{k+1}\|_{B^s_{l,r}}\\
&\le \exp\left(C\int_0^t \left(\|v_k(\tau)\|^p_{B^s_{l,r}} + \|u_k(\tau)\|^q_{B^s_{l,r}}\right)d\tau\right)\left(\|u(0)\|_{B^s_{l,r}} + \|v(0)\|_{B^s_{l,r}}\right)\\
&+ C\int_0^t \exp\left(C\int_\tau^t \left(\|v_k(\tau')\|^p_{B^s_{l,r}} + \|u_k(\tau')\|^q_{B^s_{l,r}}\right)d\tau'\right)\left(\|v_k\|^p_{B^s_{l,r}}\|u_k\|_{B^s_{l,r}} + \|u_k\|^q_{B^s_{l,r}}\|v_k\|_{B^s_{l,r}}\right)d\tau\\
&\le e^{CU_k(t)}\left(\|u(0)\|_{B^s_{l,r}} + \|v(0)\|_{B^s_{l,r}} + C\int_0^t e^{-CU_k(\tau)}\left(\|v_k\|_{B^s_{l,r}} + \|u_k\|_{B^s_{l,r}}\right)^{\kappa+1}d\tau\right),
\end{aligned} \tag{2.4}$$

where $U_k(t) := \int_0^t \left(\|v_k(\tau)\|_{B^s_{l,r}} + \|u_k(\tau)\|_{B^s_{l,r}}\right)^\kappa d\tau \ge \int_0^t \left(\|v_k(\tau)\|^\kappa_{B^s_{l,r}} + \|u_k(\tau)\|^\kappa_{B^s_{l,r}}\right)d\tau$ and $\kappa = \max\{p, q\}$. Choosing $0 < T < \frac{1}{2\kappa C\left(\|u_0\|_{B^s_{l,r}} + \|v_0\|_{B^s_{l,r}}\right)^\kappa}$, and suppose by induction that

$$\|u_k(t)\|_{B^s_{l,r}} + \|v_k(t)\|_{B^s_{l,r}} \le \frac{\|u_0\|_{B^s_{l,r}} + \|v_0\|_{B^s_{l,r}}}{\left(1 - 2\kappa Ct(\|u_0\|_{B^s_{l,r}} + \|v_0\|_{B^s_{l,r}})^\kappa\right)^{1/\kappa}} \doteq \frac{Z_0}{\left(1 - 2\kappa CtZ_0^\kappa\right)^{1/\kappa}} \tag{2.5}$$

for all $t \in [0, T)$.

Noticing

$$\exp\left(C\int_\tau^t \left(\|v_k(\tau')\|_{B^s_{l,r}} + \|u_k(\tau')\|_{B^s_{l,r}}\right)^\kappa d\tau'\right) \le \exp\left(\int_\tau^t \left(\frac{CZ_0^\kappa}{\left(1 - 2\kappa CZ_0^\kappa\tau'\right)}\right)d\tau'\right) = \left(\frac{1 - 2\kappa CZ_0^\kappa\tau}{1 - 2\kappa CZ_0^\kappa t}\right)^{\frac{1}{2\kappa}},$$

and substituting (2.5) and the above inequality into (2.4), one obtain

$$
\begin{aligned}
&\|u_{k+1}\|_{B^s_{l,r}} + \|v_{k+1}\|_{B^s_{l,r}} \\
&\quad \le \frac{Z_0}{\left(1-2\kappa C Z_0^\kappa t\right)^{\frac{1}{2\kappa}}} + \frac{C}{\left(1-2\kappa C Z_0^\kappa t\right)^{\frac{1}{2\kappa}}} \int_0^t \left(1-2\kappa C Z_0^\kappa \tau\right)^{\frac{1}{2\kappa}} \frac{Z_0^{\kappa+1}}{\left(1-2\kappa C Z_0^\kappa \tau\right)^{\frac{\kappa+1}{\kappa}}} d\tau \\
&\quad \le \frac{Z_0}{\left(1-2\kappa C Z_0^\kappa t\right)^{\frac{1}{2\kappa}}} + \frac{Z_0}{-2w\left(1-2\kappa C Z_0^\kappa t\right)^{\frac{1}{2\kappa}}} \int_0^t \frac{d(1-2\kappa C Z_0^\kappa \tau)}{(1-2\kappa C Z_0^\kappa \tau)^{\frac{2\kappa+1}{2\kappa}}} \\
&\quad \le \frac{Z_0}{\left(1-2\kappa C Z_0^\kappa t\right)^{\frac{1}{2\kappa}}} + \frac{Z_0}{\left(1-2\kappa C Z_0^\kappa t\right)^{\frac{1}{2\kappa}}} \left(\frac{1}{\left(1-2\kappa C Z_0^\kappa t\right)^{\frac{1}{2\kappa}}} - 1 \right) \\
&\quad = \frac{Z_0}{\left(1-2\kappa C t Z_0^\kappa\right)^{1/\kappa}},
\end{aligned}
$$

which implies that the sequence $(u_k, v_k)_{k\in\mathbb{R}}$ is uniformly bounded in the spaces $\mathcal{C}([0,T];B^s_{l,r}) \times \mathcal{C}([0,T];B^s_{l,r})$. The linear equation (T_k) and the proofs of Lemma 2.1 implies that the sequence $(\partial_t u_k, \partial_t v_k)_{k\in\mathbb{R}}$ is uniformly bounded in the spaces $\mathcal{C}([0,T];B^{s-1}_{l,r}) \times \mathcal{C}([0,T];B^{s-1}_{l,r})$. Hence, the sequence $(u_k, v_k)_{k\in\mathbb{R}}$ is uniformly bounded in the spaces $E^s_{l,r}(T) \times E^s_{l,r}(T)$.

Let us now show that the sequence $(u_k, v_k)_{n\in\mathbb{R}}$ is a Cauchy sequence in $\mathcal{C}([0,T];B^{s-1}_{l,r}) \times \mathcal{C}([0,T];B^{s-1}_{l,r})$. In fact, from the equation (T_k), for all $k, j \in \mathbb{N}$, we have,

$$
\begin{cases}
(\partial_t + v^p_{k+j}\partial_x)(m_{k+j+1} - m_{k+1}) = F(t,x), \\
(\partial_t + u^q_{k+j}\partial_x)(n_{k+j+1} - n_{k+1}) = G(t,x),
\end{cases}
$$

where

$$
\begin{aligned}
F(t,x) &= (v^p_{k+j} - v^p_k)\partial_x m_{k+1} + \frac{a}{p}(m_k - m_{k+j})\partial_x v^p_{k+j} + \frac{a}{p} m_k \partial_x (v^p_k - v^p_{k+j}), \\
G(t,x) &= (u^q_{k+j} - u^q_k)\partial_x n_{k+1} + \frac{b}{q}(n_k - n_{k+j})\partial_x u^q_{k+j} + \frac{b}{q} n_k \partial_x (u^q_k - u^q_{k+j}).
\end{aligned}
$$

Apparently, we have

$$
\|F(t,\cdot)\|_{B^{s-3}_{l,r}} + \|G(t,\cdot)\|_{B^{s-3}_{l,r}} \le C\Big(\|u_{k+j} - u_k\|_{B^{s-1}_{l,r}} + \|v_{k+j} - v_k\|_{B^{s-1}_{l,r}} \Big) H(t),
$$

with

$$
\begin{aligned}
H(t) = \Bigg[&\Big(\|u_k\|_{B^s_{l,r}} + \|u_{k+1}\|_{B^s_{l,r}} \Big) \sum_{i=0}^{p-1} \|v_{k+j}\|^{p-1-i}_{B^s_{l,r}} \|v_k\|^i_{B^s_{l,r}} + \|v_{k+j}\|^p_{B^s_{l,r}} \\
&+ \Big(\|v_k\|_{B^s_{l,r}} + \|v_{k+1}\|_{B^s_{l,r}} \Big) \sum_{i=0}^{q-1} \|u_{k+j}\|^{q-1-i}_{B^s_{l,r}} \|u_k\|^i_{B^s_{l,r}} + \|u_{k+j}\|^q_{B^s_{l,r}} \Bigg].
\end{aligned}
$$

For $s > \max\{2 + \frac{1}{l}, 3 - \frac{1}{l}, \frac{5}{2}\}$ and $s \ne 3 + \frac{1}{l}, 4 + \frac{1}{l}$, using a similar argument in the proof of Lemma 2.1, on can arrive

$$
\begin{aligned}
V_{k+1}^{j}(t) &\doteq \|u_{k+j+1}-u_{k+1}\|_{B_{l,r}^{s-1}}+\|v_{k+j+1}-v_{k+1}\|_{B_{l,r}^{s-1}}\\
&\le \exp\left(CU_{k+j}(t)\right)\Big(\|u_{k+j+1}(0)-u_{k+1}(0)\|_{B_{l,r}^{s-1}}+\|v_{k+j+1}(0)-v_{k+1}(0)\|_{B_{l,r}^{s-1}}\\
&\quad +C\int_0^t \exp\left(-CU_{k+j}(\tau)\right)\Big(\|F(\tau)\|_{B_{l,r}^{s-3}}+\|G(\tau)\|_{B_{l,r}^{s-3}}\Big)d\tau\Big)\\
&\le \exp\left(CU_{k+j}(t)\right)\Big(V_{k+1}^{j}(0)+C\int_0^t \exp\left(-CU_{k+j}(\tau)\right)V_k^j(\tau)H(\tau)d\tau\Big).
\end{aligned}
$$

Proposition 2.1 in [40] gives

$$
\begin{aligned}
\|u_{k+j+1}(0)-u_{k+1}(0)\|_{B_{l,r}^{s-1}} &= \|S_{k+j+1}u(0)-S_{k+1}u(0)\|_{B_{l,r}^{s-1}} = \Big\|\sum_{d=k+1}^{k+j}\Delta_d u_0\Big\|_{B_{l,r}^{s-1}}\\
&\le C\Big(\sum_{d=k}^{k+j+1}2^{-dr}2^{drs}\|\Delta_d u_0\|_{L^p}^r\Big)^{\frac{1}{r}} \le C2^{-k}||u_0||_{B_{l,r}^{s}}.
\end{aligned}
$$

By the same way, we can get

$$
\|v_{k+j+1}(0)-v_{k+1}(0)\|_{B_{l,r}^{s-1}} \le C2^{-k}||v_0||_{B_{l,r}^{s}}.
$$

Due to the sequence $\{u_k, v_k\}_{k\in\mathbb{N}}$ being uniformly bounded in the spaces $E_{l,r}^{s}(T)$, we can get a constant $C_T > 0$ independent of k, i and verifying

$$
V_{k+1}^{j}(t) \le C_T\Big(2^{-k}+\int_0^t V_k^j(\tau)d\tau\Big),\ \forall t\in[0,T].
$$

Arguing by the induction procedure, we have

$$
\begin{aligned}
V_{k+1}^{j}(t) &\le C_T\Big(2^{-k}\sum_{i=0}^{k}\frac{(2TC_T)^i}{i!}+C_T^{k+1}\int\frac{(t-\tau)^k}{k!}d\tau\Big)\\
&\le 2^{-k}\Big(C_T\sum_{i=0}^{k}\frac{(2TC_T)^i}{i!}\Big)+C_T\frac{(TC_T)^{k+1}}{(k+1)!},
\end{aligned}
$$

when $k\to\infty$, we get the desired result. The interpolation method leads to the critical case $s = 4+\frac{1}{l}$, which yields the desired result. □

Therefore, we can finish the proof of the existence and uniqueness for the solution of Equ. (1.1) in the nonhomogeneous Besov space.

Proof of Theorem 1.1 Let us first show that the limit $(u, v)\in E_{l,r}^{s}(T)\times E_{l,r}^{s}(T)$ and satisfies system (1.1). Proposition 2.2(6)and Lemma 2.2 in [40] means that

$$
(u, v)\in L^{\infty}([0,T];B_{l,r}^{s})\times L^{\infty}([0,T];B_{l,r}^{s}).
$$

Combining an interpolation argument with Lemma 2.2 gets

$$(u_k, v_k) \to (u, v) \text{ in } \mathcal{C}([0,T]; B^{s'}_{l,r}) \times \mathcal{C}([0,T]; B^{s'}_{l,r}), \quad \text{as } k \to \infty, \text{ for all } s' < s.$$

Taking limit in the equation T_k reveals that $(u, v) \in \mathcal{C}([0,T]; B^{s'-1}_{l,r}) \times \mathcal{C}([0,T]; B^{s'-1}_{l,r})$ and satisfy the Cauchy problem (1.1) for all $s' < s$. Note the fact $B^s_{l,r}$ is an algebra as $s > 2 + \frac{1}{l}$, and applying the Lemma 2.2 and Lemma 2.3 in [40] produces $(u, v) \in E^s_{l,r}(T) \times E^s_{l,r}(T)$.

At the end, the continuity on the initial data in the spaces

$$\mathcal{C}([0,T]; B^{s'}_{l,r}) \cap \mathcal{C}^1([0,T]; B^{s'-1}_{l,r}) \times \mathcal{C}([0,T]; B^{s'}_{l,r}) \cap \mathcal{C}^1([0,T]; B^{s'-1}_{l,r}) \quad \text{for all } s' < s,$$

can be proved through the use of Lemma 2.1 and a interpolation argument. While the continuity in the spaces $\mathcal{C}([0,T]; B^s_{l,r}) \cap \mathcal{C}^1([0,T]; B^{s-1}_{l,r}) \times \mathcal{C}([0,T]; B^s_{l,r}) \cap \mathcal{C}^1([0,T]; B^{s-1}_{l,r}$ when $r < \infty$ can be obtained by a sequence of viscosity approximation solutions $(u_\epsilon, v_\epsilon)_{\epsilon > 0}$ for the initial-value problem (1.1) which converges uniformly in these spaces. The proof of Theorem 1.1 is completed.

2.2 Local Well-Posedness for Equ. (1.1) in Critical Besov Space

In present section, local well-posedness of the solution for Equ. (1.1) in critical Besov spaces was established. Inspired by the argument of local existence about CH type equations [13], by the famous Friedrichs regularization method, one can construct the approximate solutions of Equ. (1.1).

Lemma 2.3 *Given* $(u^0, v^0) = 0$ *and the initial data* $(u_0, v_0) \in B^{\frac{5}{2}}_{2,1} \times B^{\frac{5}{2}}_{2,1}$. *Then there exists a sequence* $\{(u^k, v^k)\}_{k \in \mathbb{N}} \in \mathcal{C}(\mathbb{R}^+; B^\infty_{2,1})$ *satisfy the linear Cauchy problem* (T_k) (*see, Lemma* 2.1). *Furthermore, the solutions* (u^k, v^k) *enjoying the following two properties*:

(i) *The sequence* $(u^k, v^k)_{k \in \mathbb{N}}$ *is uniformly bounded in the spaces* $E^{\frac{5}{2}}_{2,1}(T) \times E^{\frac{5}{2}}_{2,1}(T)$.

(ii) *The sequence* $(u^k, v^k)_{k \in \mathbb{N}}$ *is a Cauchy sequence in* $\mathcal{C}([0,T]; B^{\frac{3}{2}}_{2,\infty}) \times \mathcal{C}([0,T]; B^{\frac{3}{2}}_{2,\infty})$

.

Proof Firstly, we claim that the sequence $(u^k, v^k)_{k \in \mathbb{N}}$ (defined by (T_k)) is a Cauchy sequence in $\mathcal{C}([0,T); B^{\frac{3}{2}}_{2,\infty}) \times \mathcal{C}([0,T); B^{\frac{3}{2}}_{2,\infty})$, then by $||u^k||_{B^{\frac{5}{2}}_{2,1}} + ||v^k||_{B^{\frac{5}{2}}_{2,1}} \leq M$ and the interpolation inequality imply that $(u^k, v^k)_{k \in \mathbb{N}}$ tends to (u^k, v^k) in $\mathcal{C}([0,T); B^s_{2,1})$ for all $s < \frac{5}{2}$. This argument is very similar to the proof of Lemma 2.2, we omit the details here for concise. ☐

The stability of the solution to Equ. (1.1) was obtain by the following lemma:

Lemma 2.4 *Set* $\bar{\mathbb{N}} \doteq \mathbb{N} \cup \infty$, *for any initial data* $z_0 \doteq (u_0, v_0) \in B^{\frac{5}{2}}_{2,1} \times B^{\frac{5}{2}}_{2,1}$, *then there exists a neighborhood V correspond to* z_0 *in* $B^{\frac{5}{2}}_{2,1} \times B^{\frac{5}{2}}_{2,1}$ *and a positive time T satisfy that every solution z of the initial-value problem* (1.1) *is continuous.*

Proof At the beginning, we will claim that the map Φ in $\mathcal{C}([0,T];B^{\frac{3}{2}}_{2,1}) \times \mathcal{C}([0,T];B^{\frac{3}{2}}_{2,1})$ is continuous. Fix $\delta > 0$ and $z_0 \in B^{\frac{5}{2}}_{2,1} \times B^{\frac{5}{2}}_{2,1}$, we prove that there exists two positive constants T and M verify that the solution $z = \Phi(z) \in \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1})^2$ and satisfies $\|z\|_{L^\infty([0,T];B^{\frac{5}{2}}_{2,1})^2} \le M$ for any $z_0' \in B^{\frac{5}{2}}_{2,1} \times B^{\frac{5}{2}}_{2,1}$ and $\|z_0' - z_0\|_{B^{\frac{5}{2}}_{2,1} \times B^{\frac{5}{2}}_{2,1}} \le \delta$. Indeed,we already get that if we fix a time $T > 0$ satisfy that $T < \frac{1}{2\kappa C \|z_0\|^{\kappa}_{B^s_{l,r}}}$, form the proof of the local well-posedness, then

$$||z'(t)||_{B^{\frac{5}{2}}_{2,1}} \le \frac{C||z_0'||_{B^{\frac{5}{2}}_{2,1}}}{\left(1 - 2\kappa Ct||z_0'||^{\kappa}_{B^{\frac{5}{2}}_{2,1}}\right)^{1/\kappa}} \quad \text{for all } t \in [0,T]. \tag{2.6}$$

Due to $\|z_0' - z_0\|_{B^{\frac{5}{2}}_{2,1}} \le \delta$, it follows that $\|z_0'\|_{B^{\frac{5}{2}}_{2,1}} \le \|z_0\|_{B^{\frac{5}{2}}_{2,1}} + \delta$. Here, one can actually choose some suitable constant C verify that

$$T = \frac{1}{4\kappa C(\|z_0\|_{B^{\frac{5}{2}}_{2,1}} + \delta)^{\kappa}} < \min\left\{ \frac{1}{2\kappa C \|z_0'\|^{\kappa}_{B^s_{l,r}}}, \frac{1}{2C} \right\}$$

and $M = 2^{1/\kappa}(\|z_0\|_{B^{\frac{5}{2}}_{2,1}} + \delta)$. Substitute this uniform bounds into Lemma 2.4 yields

$$\|\Phi(v_0) - \Phi(u_0)\|_{L^\infty(0,T;B^{\frac{5}{2}}_{2,1})} \le \delta e^{2\kappa CM^{\kappa}T}.$$

Hence Φ is Hölder continuous from $B^{\frac{5}{2}}_{2,1} \times B^{\frac{5}{2}}_{2,1}$ into $\mathcal{C}([0,T];B^{\frac{3}{2}}_{2,1}) \times \mathcal{C}([0,T];B^{\frac{3}{2}}_{2,1})$.

On the other hand, we claim that the map Φ in $\mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1}) \times \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1})$ is continuous. Let $z^\infty(0) \doteq (u^\infty(0), v^\infty(0)) \in B^{\frac{5}{2}}_{2,1} \times B^{\frac{5}{2}}_{2,1}$ and $(z_k(0))_{k\in\mathbb{N}} \doteq (u_k(0), v_k(0))_{k\in\mathbb{N}}$ tend to z_0^∞ in $B^{\frac{5}{2}}_{2,1} \times B^{\frac{5}{2}}_{2,1}$ as $k \to \infty$. Let $z_k \doteq (u_k, v_k)$ be the solution of the initial-value problem (1.1) correspond to the initial data $z_k(0)$. Through the above procedure we may obtain

$$\sup_{k\in\mathbb{N}} \|z_k\|_{L^\infty_T(B^{\frac{5}{2}}_{2,1})} \leq M, \text{ for any } n \in \mathbb{N}, t \in T. \tag{2.7}$$

Apparently, proving $z_k \to z_\infty$ in the spaces $\mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1}) \times \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1})$ is equivalent to proving that $m_k = u_k - \partial_x^2 u_k, n_k = v_k - \partial_x^2 v_k$ tends to $p^{(\infty)} = u^{(\infty)} - u^{(\infty)}_{xx}, q^{(\infty)} = v^{(\infty)} - v^{(\infty)}_{xx}$ in the spaces $\mathcal{C}([0,T];B^{\frac{1}{2}}_{2,1})$ as $k \to \infty$.

Let us recall that (u_k, v_k) solves the linear transport equation:

$$\begin{cases} \partial_t m_k + v_k^p \partial_x m_k + \frac{a}{p}\partial_x v_k^p m_k = 0, \\ \partial_t n_k + u_k^q \partial_x n_k + \frac{b}{q}\partial_x u_k^q n_k = 0, \\ u_k|_{t=0} = u_k(0), v_k|_{t=0} = v_k(0). \end{cases} \tag{2.8}$$

Applying the Kato theory [13], we decompose the solution (m_k, n_k) into $m_k = \alpha_k + \beta_k, n_k = \phi_k + \varphi_k$ with

$$\begin{cases} \left[\partial_t + v_k^p \partial_x\right] \alpha_{(n)} = -\frac{a}{p}\partial_x v_k^p m_k + \frac{a}{p}\partial_x v_\infty^p m_\infty, \\ \left[\partial_t + u_k^q \partial_x\right] \phi_k = -\frac{b}{q}\partial_x u_k^q n_k + \frac{b}{q}\partial_x u_\infty^q n_\infty, \\ u_k|_{t=0} = u_k(0) - u_\infty(0), v_k|_{t=0} = v_k(0) - v_\infty(0), \end{cases} \tag{2.9}$$

and

$$\begin{cases} \left[\partial_t + v_k^p \partial_x\right] \alpha_k = -\frac{a}{p}\partial_x v_\infty^p m_\infty, \\ \left[\partial_t + u_k^q \partial_x\right] \phi_k = -\frac{b}{q}\partial_x u_\infty^q n_\infty, \\ u_k|_{t=0} = u_\infty(0), v_k|_{t=0} = v_\infty(0). \end{cases} \tag{2.10}$$

Using properties of Besov spaces(cf. [13]), it is easily see that the sequence $(m_k, n_k)_{k\in\overline{\mathbb{N}}}$ are uniformly bounded in the spaces $\mathcal{C}([0,T];B^{\frac{1}{2}}_{2,1})$. Moreover,

$$\begin{aligned} \|\frac{a}{p}\partial_x v_k^p m_k - \frac{a}{p}\partial_x v_\infty^p m_\infty\|_{B^{\frac{1}{2}}_{2,1}} &\leq C\|\partial_x v_k^p\|_{B^{\frac{1}{2}}_{2,1}} \|m_k - m_\infty\|_{B^{\frac{1}{2}}_{2,1}} \\ &+ C\left(\|\partial_x v_k^p - \partial_x v_k^\infty\|_{B^{\frac{3}{2}}_{2,1}} \right) \|m_\infty\|_{B^{\frac{1}{2}}_{2,1}}. \end{aligned}$$

In light of the product law in the Besov spaces and Lemma 4.3 in [13] to equation (2.10) yields

$$\begin{aligned} \|\alpha_{(n)}\|_{B^{\frac{1}{2}}_{2,1}} \leq \exp\left\{ C\int_0^t \|v_k^p(\tau)\|_{B^{\frac{3}{2}}_{2,1}}\, d\tau \right\} \cdot \\ \left(\|m_k(0) - m_\infty(0)\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \left\| \frac{a}{p}\partial_x v_k^p m_k - \frac{a}{p}\partial_x v_\infty^p m_\infty \right\|_{B^{\frac{1}{2}}_{2,1}} d\tau \right). \end{aligned} \tag{2.11}$$

By the argument in the first step, we can get that $(u_k, v_k)_{n\in\overline{\mathbb{N}}\doteq\mathbb{N}\cup\{\infty\}}$ is uniformly bounded in the spaces $\mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1})\times\mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1})$, and which tends to the limit (u^∞, v^∞) in the spaces $\mathcal{C}([0,T];B^{\frac{3}{2}}_{2,1})\times\mathcal{C}([0,T];B^{\frac{3}{2}}_{2,1})$ as $k\to\infty$. Therefore, adopting Proposition 3 in [13] reveals that (α_k, ϕ_k) tends to (m_∞, n_∞) in the spaces $\mathcal{C}([0,T];B^{\frac{1}{2}}_{2,1})\times\mathcal{C}([0,T];B^{\frac{1}{2}}_{2,1})$. Therefore, adding this convergence result into the estimates (2.7) and (2.11), for large enough $n\in\mathbb{Z}^+$ leads to

$$\begin{aligned}
&\|m_k-m_\infty\|_{B^{\frac{1}{2}}_{2,1}}+\|n_k-n_\infty\|_{B^{\frac{1}{2}}_{2,1}}\\
&\quad\le\varepsilon+CM^{p+q}e^{CM^{p+q}T}\Bigg[\|m_k(0)-m_\infty(0)\|_{B^{\frac{1}{2}}_{2,1}}+\|n_k(0)-n_\infty(0)\|_{B^{\frac{1}{2}}_{2,1}}\\
&\quad+\int_0^t\left(\|m_k-m_\infty\|_{B^{\frac{1}{2}}_{2,1}}+\|n_k-n_\infty\|_{B^{\frac{1}{2}}_{2,1}}\right)d\tau+\int_0^t\|u_k-u_\infty\|_{B^{\frac{1}{2}}_{2,1}}\\
&\quad+\|v_k-v_\infty\|_{B^{\frac{1}{2}}_{2,1}}\,d\tau\Bigg].
\end{aligned}$$

By the Gronwall's inequality, we have

$$\begin{aligned}
&\|m_k-m_\infty\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1})}+\|n_k-n_\infty\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1})}\\
&\quad\le C\left(\|m_k(0)-m_\infty(0)\|_{B^{\frac{1}{2}}_{2,1}}+\|n_k(0)-n_\infty(0)\|_{B^{\frac{1}{2}}_{2,1}}+\varepsilon\right)
\end{aligned}$$

where the constant C depends only on the constants M and T. Continuity of the map Φ in the spaces $\mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1})\times\mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1})$ is now completed. Using the operator ∂_t to original equations (1.1), then repeating the above procedure to the obtaining system in views of $(\partial_t u, \partial_t v)$, we obtain that the map Φ in the spaces $\mathcal{C}^1([0,T];B^{\frac{3}{2}}_{2,1})\times\mathcal{C}^1([0,T];B^{\frac{3}{2}}_{2,1})$ is continuous. □

Proof of Theorem 1.2 From Lemma 2.3, one can get that the sequence $\{u^n, v^n\}_{n\in\mathbb{N}}$ is uniformly bounded in the Besov spaces $E^{\frac{5}{2}}_{2,1}\times E^{\frac{5}{2}}_{2,1}$ with $E^{\frac{5}{2}}_{2,1}=\mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1})\cap\mathcal{C}^1([0,T];B^{\frac{3}{2}}_{2,1})$, Lemma 2.5 further show that the sequence $\{u^n, v^n\}_{n\in\mathbb{N}}$ tends to the limit $(u,v)=(u^\infty, v^\infty)$ in the spaces $\mathcal{C}([0,T];B^{\frac{3}{2}}_{2,1})\times\mathcal{C}([0,T];B^{\frac{3}{2}}_{2,1})$ as $k\to\infty$. In other words $\{u^n, v^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}^\infty\left([0,T];B^{\frac{3}{2}}_{2,\infty}\right)\times\mathcal{L}^\infty\left([0,T];B^{\frac{3}{2}}_{2,\infty}\right)$ and converges to some limit function $(u,v)\in\mathcal{L}^\infty\left([0,T];B^{\frac{3}{2}}_{2,\infty}\right)\times\mathcal{L}^\infty\left([0,T];B^{\frac{3}{2}}_{2,\infty}\right)$, and $(u,v)\in E^{\frac{5}{2}}_{2,1}\times E^{\frac{5}{2}}_{2,1}$ is indeed a solution of (1.1). Furthermore, which is convergence in the spaces $\mathcal{C}([0,T], B^{s_1}_{2,1})$, $s_1<5/2$ through interpolation theorem.

On the other hands, Lemma 2.4 also obtain that the map $\Phi : z_0 \to z = (u, v)$ in the spaces $\mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1}) \times \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1}) \cap \mathcal{C}^1([0,T];B^{\frac{3}{2}}_{2,1}) \times \mathcal{C}^1([0,T];B^{\frac{3}{2}}_{2,1})$ is continuous. Let us pass the limit in the system (T_k) (see, Lemma 2.1), one can easy get that the pair (u, v) is a solution to Equ. (1.1) and verifies

$$(u,v) \in \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1}) \cap \mathcal{C}^1([0,T];B^{\frac{3}{2}}_{2,1}) \times \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1}) \cap \mathcal{C}^1([0,T];B^{\frac{3}{2}}_{2,1}).$$

The Lemma 2.3 implies that the continuity with the initial data $(u_0, v_0) \in \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1}) \times \mathcal{C}([0,T];B^{\frac{5}{2}}_{2,1})$. Now, we only need to prove the uniqueness and stability of strong solutions to (1.1). Assume that $(u_i, v_i) \in E^{\frac{5}{2}}_{2,1} \times E^{\frac{5}{2}}_{2,1}$ are two solutions of (1.1) with $m_i = (1-\partial_x^2)u_i, n_i = (1-\partial_x^2)u_i, i = 1,2$. Then $m_{12} := m_1 - m_2, n_{12} := n_1 - n_2$ solves the transport equations

$$\begin{cases} \partial_t m_{12} + v_1^p \partial_x m_{12} = -[v_1^p - v_2^p]\partial_x m_2 - \frac{a}{p}\partial_x v_1^p m_{12} - \frac{a}{p}[\partial_x v_1^p - \partial_x v_2^p] m_2, \\ \partial_t n_{12} + u_1^q \partial_x n_{12} = -[u_1^q - u_2^q]\partial_x n_2 - \frac{b}{q}\partial_x u_1^q n_{12} - \frac{b}{q}[\partial_x u_1^q - \partial_x u_2^q] n_2, \\ m_{12}|_{t=0} = m_{12}(0) \doteq m_1(0) - m_2(0), \quad n_{12}|_{t=0} = n_{12}(0) \doteq n_1(0) - n_2(0). \end{cases}$$

By a similar argument in the proof of Lemma 2.1, we can easy get

$$\begin{aligned} &e^{-CA(t)}(\|m_{12}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} + \|n_{12}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}) \\ &\quad \le \|m_{12}(0)\|_{B^{-\frac{1}{2}}_{2,\infty}} + \|n_{12}(0)\|_{B^{-\frac{1}{2}}_{2,\infty}} + C\mathbf{M}\int_0^t e^{-CU(\tau)}(\|m_{12}(\tau)\|_{B^{-\frac{1}{2}}_{2,\infty}} \\ &\quad + \|n_{12}(\tau)\|_{B^{-\frac{1}{2}}_{2,\infty}})d\tau \end{aligned}$$

where $U(t) = \int_0^t (\|m_{12}(\tau)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|n_{12}(\tau)\|_{B^{\frac{1}{2}}_{2,\infty}})d\tau$, and used (2.5) that $\|m_i(\tau)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|n_i(\tau)\|_{B^{\frac{1}{2}}_{2,\infty}} \le \left(\frac{1-2\kappa C Z_0^\kappa \tau}{1-2\kappa C Z_0^\kappa T}\right)^{\frac{1}{2\kappa}} =: \mathbf{M}$. If we define $W(t) = e^{-CA(t)}$ $(\|m_{12}(t)\|_{B^{-\frac{1}{2}}_{2,1}} + \|n_{12}(t)\|_{B^{-\frac{1}{2}}_{2,1}})$, then get

$$\begin{aligned} W(t) &\le c\left(W(0) + \int_0^t W(\tau)\ln\left(e + \frac{C}{W(\tau)}\right)d\tau\right) \\ &\le c\left(W(0) + \int_0^t W(\tau)\left(1 - \ln\frac{W(\tau)}{C}\right)d\tau\right) \end{aligned}$$

If we set $\mu(r) = r(1-\ln r)$ which satisfies the condition $\int_0^1 \frac{dr}{\mu(r)}$. A simple calculation shows that $\mathcal{M}(x) = \ln(1-\ln x)$, we deduce that $\rho(t) \le eC^{\exp\int_{t_0}^t} - \gamma(\tau)d\tau$, if $c > 0$. By virtue of Osgood lemma (cf. Lemma 3.4. in [3]) with $\rho(t) = \frac{W(t)}{C}$, we verify that

$$W(t) \le CW(0)^{\exp\{-Ct\}} \le CW(0)^{\exp\{-CT\}}$$

which leads to

$$\|W(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \le C\|W(0)\|^{\exp\{-CT\}}_{B^{-\frac{1}{2}}_{2,\infty}} \le C_T\|W(0)\|_{B^{-\frac{1}{2}}_{2,1}}$$

Next, we apply the interpolation argument ensures that

$$\sup_{t\in[0,T]}\|W(t)\|_{B^{s'}_{2,1}} \le \|W(t)\|^{\theta}_{B^{-\frac{1}{2}}_{2,1}}\|W(t)\|^{1-\theta}_{B^{\frac{1}{2}}_{2,1}} \le \|W(t)\|^{\theta}_{B^{-\frac{1}{2}}_{2,\infty}}\|W(t)\|^{1-\theta}_{B^{\frac{1}{2}}_{2,1}} \le C\|W(0)\|^{\theta\exp\{-CT\}}_{B^{-\frac{1}{2}}_{2,1}},$$

where $\theta = \frac{1}{2} - s' \in (0,1]$. The above inequality implies the uniqueness. Consequently, we prove the theorem 1.2. □

3 Blow-Up Criterion

In present section, we shall build up a blow-up criteria for Equ. (1.1). We first recall two useful lemmas as follows.

Lemma 3.1 (See [40]) *If the Sobolev index* $r > 0$, *then* $H^r \cap L^\infty$ *is an algebra, and*

$$||fg||_{H^r} \le c(||f||_{L^\infty}||g||_{H^r} + ||g||_{L^\infty}||h||_{H^r}),$$

where the constant c depend only on r.

Lemma 3.2 (*See* [40]) *If Sobolev index* $r > 0$, *then*

$$||[D^r, f]g||_{L^2} \le c(||\partial_x f||_{L^\infty}||D^{r-1}g||_{L^2} + ||D^r f||_{L^2}||g||_{L^\infty}),$$

where the constant c depend only on r.

Proof of Theorem 1.3 This theorem can be proved by an inductive method with respect to the Sobolev index s. The proof consist by the following three steps.

Step 1. For the cases $s \in \left(\frac{1}{2}, 1\right)$, Using the Theorem 3.2 in [23] to the equations (1.1), one gets

$$\|m(t)\|_{H^s} \le \|m_0\|_{H^s} + C\int_0^t \left(\|m\partial_x v^p(\tau)\|_{H^s} + \|m(\tau)\|_{H^s}\|\partial_x v^p(\tau)\|_{L^\infty}\right)d\tau \quad (3.1)$$

for all $t \in (0, T^*_{m_0,n_0})$. Let $u = G * m = (1-\partial_x^2)^{-1}m, v = G * n = (1-\partial_x^2)^{-1}n$, where $G = \frac{1}{2}e^{-|x|}$ and $*$ stands for the convolution on $\mathbb{R}$. Then $u_x = \partial_x G * m$ where $\partial_x G(x) = -\frac{1}{2}sign(x)e^{-|x|}$. As per the Young inequality, we have

$$\begin{aligned} \|u\|_{L^\infty} &\le \|G\|_{L^1}\|m\|_{L^\infty} \le C\|m\|_{L^\infty}, \|v\|_{L^\infty} \le C\|n\|_{L^\infty}, \\ \|u_x\|_{L^\infty} &\le \|\partial_x G\|_{L^1}\|m\|_{L^\infty} \le C\|m\|_{L^\infty}, \|v_x\|_{L^\infty} \le C\|n\|_{L^\infty}, \\ ||u_{xx}||_{L^\infty} &\le ||u-u_{xx}||_{L^\infty} + ||u||_{L^\infty} \le C\|m\|_{L^\infty}. \end{aligned} \tag{3.2}$$

Utilizing Eq. (3.2), $\|u_x\|_{H^s} \le C\|m\|_{H^s}, \|v_x\|_{H^s} \le C\|n\|_{H^s}$ and the Moser-type estimates leads to

$$\begin{aligned} \|m\partial_x v^p(\tau)\|_{H^s} &\le C\big(\|\partial_x v^p\|_{L^\infty}\|m\|_{H^s} + \|m\|_{L^\infty}\|\partial_x v^p\|_{H^s}\big) \\ &\le C\Big(\|n\|^p_{L^\infty}\|m\|_{H^s} + \|m\|_{L^\infty}\|n\|^{p-1}_{L^\infty}\|n\|_{H^s}\Big), \end{aligned} \tag{3.3}$$

and

$$\|m(\tau)\|_{H^s}\|\partial_x v^p(\tau)\|_{L^\infty} \le C\|n\|^p_{L^\infty}\|m\|_{H^s}. \tag{3.4}$$

Plugging Eqs. (3.3) and (3.4) into (3.1) generates

$$\|m(t)\|_{H^s} \le \|m_0\|_{H^s} + C\int_0^t \Big((\|n\|^p_{L^\infty} + \|m\|_{L^\infty}\|n\|^{p-1}_{L^\infty})(\|m\|_{H^s} + \|n\|_{H^s})\Big)\,d\tau. \tag{3.5}$$

Similarly, For the second equation of the system (1.1) leads to

$$\|n(t)\|_{H^s} \le \|n_0\|_{H^s} + C\int_0^t \Big((\|m\|^q_{L^\infty} + \|n\|_{L^\infty}\|m\|^{q-1}_{L^\infty})(\|m\|_{H^s} + \|n\|_{H^s})\Big)\,d\tau.$$

Therefore

$$\begin{aligned} \|m(t)\|_{H^s} + \|n(t)\|_{H^s} &\le \|m_0\|_{H^s} + \|n_0\|_{H^s} \\ &+ C\int_0^t \Big(\|n\|^p_{L^\infty} + \|m\|_{L^\infty}\|n\|^{p-1}_{L^\infty} + \|m\|^q_{L^\infty} + \|n\|_{L^\infty}\|m\|^{q-1}_{L^\infty}\Big)(\|n\|_{H^s} + \|m\|_{H^s})\,d\tau. \end{aligned}$$

Then, Using the Gronwall's inequality yields

$$\begin{aligned} &\|m(t)\|_{H^s} + \|n(t)\|_{H^s} \le (\|m_0\|_{H^s} + \|n_0\|_{H^s}) \\ &\exp\left\{C\int_0^t \Big(\|n\|^p_{L^\infty} + \|m\|_{L^\infty}\|n\|^{p-1}_{L^\infty} + \|m\|^q_{L^\infty} + \|n\|_{L^\infty}\|m\|^{q-1}_{L^\infty}\Big)d\tau\right\}. \end{aligned} \tag{3.6}$$

Moreover, if there exists a maximal time $T^*_{m_0,n_0} < \infty$ verify

$$\int_0^{T^*_{m_0,n_0}} \Big(\|n\|^p_{L^\infty} + \|m\|_{L^\infty}\|n\|^{p-1}_{L^\infty} + \|m\|^q_{L^\infty} + \|n\|_{L^\infty}\|m\|^{q-1}_{L^\infty}\Big)d\tau < \infty,$$

then Equ. (3.6) implies that the following inequality holds

$$\limsup_{t\to T^*_{m_0,n_0}} (\|m(t)\|_{H^s} + \|n(t)\|_{H^s}) < \infty. \tag{3.7}$$

which is contradicted to the assumption $T^*_{m_0,n_0} < \infty$.

Step 2. For the cases $s \in [1,2)$, we differentiate the system (1.1) with respect to x yields

$$\begin{aligned} &\partial_t(m_x) + v^p \partial_x(m_x) = -\frac{a+p}{p}(v^p)_x m_x - \frac{a}{p}(v^p)_{xx} m, \\ &\partial_t(n_x) + u^q \partial_x(n_x) = -\frac{b+q}{q}(u^q)_x n_x - \frac{b}{q}(u^q)_{xx} n. \end{aligned} \tag{3.8}$$

By Theorem 3.2 in [23], we have

$$\|\partial_x m(t)\|_{H^{s-1}} \le \|\partial_x m_0\|_{H^{s-1}} + C\int_0^t \|(v^p)_x m_x + (v^p)_{xx} m\|_{H^{s-1}}\, d\tau + C\int_0^t \|m_x\|_{H^{s-1}} \|(v^p)_x\|_{L^\infty} d\tau \tag{3.9}$$

According to the Moser-type estimates in [40] and (3.2), we obtain

$$\begin{aligned} \|(v^p)_{xx} m\|_{H^{s-1}} &\le C\big(\|(v^p)_{xx}\|_{L^\infty}\|m\|_{H^{s-1}} + \|m\|_{L^\infty}\|(v^p)_{xx}\|_{H^{s-1}}\big) \\ &\le C\Big(\|n\|^p_{L^\infty}\|m\|_{H^s} + \|m\|_{L^\infty}\|n\|^{p-1}_{L^\infty}\|n\|_{H^s}\Big), \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \|m_x \partial_x v^p\|_{H^{s-1}} &= \|\partial_x[m(v^p)_x] - (v^p)_{xx} m\|_{H^{s-1}} \le C\big(\|m_x v^p\|_{H^s} + \|(v^p)_{xx} m\|_{H^{s-1}}\big) \\ &\le C\Big(\|n\|^p_{L^\infty}\|m\|_{H^s} + \|m\|_{L^\infty}\|n\|^{p-1}_{L^\infty}\|n\|_{H^s}\Big). \end{aligned} \tag{3.11}$$

Plugging Eqs. (3.10) and (3.11) into Equ. (3.9) gives

$$\|\partial_x m(t)\|_{H^{s-1}} \le \|\partial_x m_0\|_{H^{s-1}} + C\int_0^t \Big(\|n\|^p_{L^\infty}\|m\|_{H^s} + \|m\|_{L^\infty}\|n\|^{p-1}_{L^\infty}\|n\|_{H^s}\Big)\, d\tau.$$

By a similar argument to the second equation in (3.8) produces

$$\begin{aligned} &\|\partial_x m(t)\|_{H^{s-1}} + \|\partial_x n(t)\|_{H^{s-1}} \le \|\partial_x m_0\|_{H^{s-1}} + \|\partial_x n_0\|_{H^{s-1}} \\ &\quad + C\int_0^t \Big(\|n\|^p_{L^\infty} + \|m\|_{L^\infty}\|n\|^{p-1}_{L^\infty} + \|m\|^q_{L^\infty} + \|n\|_{L^\infty}\|m\|^{q-1}_{L^\infty}\Big)(\|n\|_{H^s} + \|m\|_{H^s})\, d\tau \end{aligned}$$

Considering the estimate for (3.6) and the fact

$$\|\partial_x m(t,\cdot)\|_{H^{s-1}} \le C\|m(t)\|_{H^s}, \quad \|\partial_x n(t,\cdot)\|_{H^{s-1}} \le C\|n(t)\|_{H^s},$$

one may see that (3.6) holds for all $s \in [1,2)$. Repeating the above procedure as shown in Step 1, thus, Theorem 1.3 holds for all $s \in [1,2)$.

Step 3. Let us assume that Theorem 1.3 holds for the cases $k-1 \le s < k$ and $2 \le k \in \mathbb{N}$. By the mathematical induction, we shall claim that it is true for $k \le s < k+1$ as well. Differentiating the system (1.1) k times with respect to the space variant x leads to

$$\partial_t(\partial_x^k m_x) + v^p \partial_x^k(m_x) = -\frac{a}{p}\partial_x^k[(v^p)_x m] - \sum_{l=0}^{k-1} C_k^l \partial_x^{k-l}(v^p)\partial_x^l(m_x),$$
$$\partial_t(\partial_x^k n_x) + u^q \partial_x^k(n_x) = -\frac{b}{q}\partial_x^k[(u^q)_x n] - \sum_{l=0}^{k-1} C_k^l \partial_x^{k-l}(v^p)\partial_x^l(n_x). \tag{3.12}$$

According to Lemma 2.2 in [40], we get

$$\begin{aligned}\|\partial_x^k m(t)\|_{H^{s-k}} \leq & \|\partial_x^k m_0\|_{H^{s-k}} + C\int_0^t \|(v^p)_x(\tau)\|_{L^\infty}\|\partial_x^k m(\tau)\|_{H^{s-n}}\,d\tau \\ & + C\int_0^t \left(\left\| \sum_{l=0}^{k-1} C_k^l \sum_{l=0}^{k-1} C_k^l \partial_x^{k-l}(v^p)\partial_x^l(m_x) \right\|_{H^{s-k}} + \|\partial_x^k[(v^p)_x m]\|_{H^{s-k}} \right) d\tau.\end{aligned} \tag{3.13}$$

By Sobolev embedding inequality and Moser-type estimate, we derive

$$\|\partial_x^k[(v^p)_x m]\|_{H^{s-k}} \leq C\|(v^p)_x m\|_{H^s} \leq C\Big(\|n\|_{L^\infty}^p \|m\|_{H^s} + \|m\|_{L^\infty}\|n\|_{L^\infty}^{p-1}\|n\|_{H^s} \Big), \tag{3.14}$$

where we applied $H^{s-\frac{1}{2}+\epsilon_0}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ $(s \geq 2)$ and

$$\begin{aligned}&\left\| \sum_{l=0}^{k-1} C_n^l \partial_x^{k-l} v^p \partial_x^{l+1} m \right\|_{H^{s-k}} \\ &\leq C\sum_{l=0}^{k-1}\left(\left\|\partial_x^{k-l}v^p\right\|_{L^\infty}\left\|\partial_x^{l+1}m\right\|_{H^{s-k}} + \left\|\partial_x^{k-l}v^p\right\|_{H^{s-k}}\left\|\partial_x^{l+1}m\right\|_{L^\infty}\right) \\ &\leq C\sum_{l=0}^{n-1}\left(\|v^p\|_{H^{k-l+\frac{1}{2}+\epsilon_0}}\|m\|_{H^{s-k+l+1}} + \|v^p\|_{H^{s-l}}\|m\|_{H^{l+1+\frac{1}{2}+\epsilon_0}}\right) \\ &\leq C\|n\|^p_{H^{s-\frac{1}{2}+\epsilon_0}}\|m\|_{H^s},\end{aligned} \tag{3.15}$$

with $\epsilon_0 \in \left(0, \frac{1}{4}\right)$ and $H^{\frac{1}{2}+\epsilon_0}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Plugging Eqs. (3.14) and (3.15) into Eq. (3.13) leads to

$$\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} + C\int_0^t \left(\|n\|^p_{H^{s-\frac{1}{2}+\epsilon_0}} + \|m\|_{H^{s-\frac{1}{2}+\epsilon_0}}\|n\|^{p-1}_{H^{s-\frac{1}{2}+\epsilon_0}} \right)(\|m(\tau)\|_{H^s} + \|n(\tau)\|_{H^s})\,d\tau.$$

and

$$\begin{aligned}&\|m(t)\|_{H^s} + \|n(t)\|_{H^s} \leq \|m_0\|_{H^s} + \|n_0\|_{H^s} + C\int_0^t (\|m(\tau)\|_{H^s} + \|n(\tau)\|_{H^s}) \\ &\times\left(\|n\|^p_{H^{s-\frac{1}{2}+\epsilon_0}} + \|m\|_{H^{s-\frac{1}{2}+\epsilon_0}}\|n\|^{p-1}_{H^{s-\frac{1}{2}+\epsilon_0}} + \|m\|^q_{H^{s-\frac{1}{2}+\epsilon_0}} + \|n\|_{H^{s-\frac{1}{2}+\epsilon_0}}\|m\|^{q-1}_{H^{s-\frac{1}{2}+\epsilon_0}} \right) d\tau\end{aligned} \tag{3.16}$$

Then, by Gronwall's inequality, we obtain

$$\|m(t)\|_{H^s} + \|n(t)\|_{H^s} \leq (\|m_0\|_{H^s} + \|n_0\|_{H^s})$$
$$\exp\left\{ C\int_0^t \left(\|n\|^p_{H^{s-\frac{1}{2}+\epsilon_0}} + \|m\|_{H^{s-\frac{1}{2}+\epsilon_0}}\|n\|^{p-1}_{H^{s-\frac{1}{2}+\epsilon_0}} + \|m\|^q_{H^{s-\frac{1}{2}+\epsilon_0}} + \|n\|_{H^{s-\frac{1}{2}+\epsilon_0}}\|m\|^{q-1}_{H^{s-\frac{1}{2}+\epsilon_0}} \right) \right\}. \tag{3.17}$$

If there exist a maximal existence time $T^*_{m_0,n_0} < \infty$ varify that

$$\int_0^{T^*_{m_0,n_0}} \left(\|n\|^p_{L^\infty} + \|m\|_{L^\infty}\|n\|^{p-1}_{L^\infty} + \|m\|^q_{L^\infty} + \|n\|_{L^\infty}\|m\|^{q-1}_{L^\infty} \right) d\tau < \infty,$$

then by the solution uniqueness in Theorem 1.1, we know that $\|n\|^p_{H^{s-\frac{1}{2}+\epsilon_0}} + \|m\|_{H^{s-\frac{1}{2}+\epsilon_0}}\|n\|^{p-1}_{H^{s-\frac{1}{2}+\epsilon_0}} + \|m\|^q_{H^{s-\frac{1}{2}+\epsilon_0}} + \|n\|_{H^{s-\frac{1}{2}+\epsilon_0}}\|m\|^{q-1}_{H^{s-\frac{1}{2}+\epsilon_0}}$ is uniformly bounded in $t \in (0, T^*_{m_0,n_0})$. As per the mathematical induction assumption, we obtained a contradiction that

$$\limsup_{t\to T^*_{m_0,n_0}} (\|m(t)\|_{H^s} + \|n(t)\|_{H^s}) < \infty.$$

Therefore, we complete the proof of Theorem 1.3. □

To prove Theorem 1.4, let us rewrite the initial-value problem of the transport equation (1.1) as follows

$$\begin{cases} u_t + v^p u_x + I_1(u,v) = 0, \\ v_t + u^q v_x + I_2(u,v) = 0, \end{cases} \tag{3.18}$$

with the functions

$$\begin{cases} I_1(u,v) = (1-\partial_x^2)^{-1}[av^{p-1}v_x u + (p-a)v^{p-1}v_x u_{xx}] + p(1-\partial_x^2)^{-1}\partial_x(v^{p-1}v_x u_x), \\ I_2(u,v) = (1-\partial_x^2)^{-1}[bu^{q-1}u_x v + (q-b)u^{q-1}u_x v_{xx}] + q(1-\partial_x^2)^{-1}\partial_x(u^{q-1}u_x v_x). \end{cases}$$

Let us first provide the sufficient conditions lead to global existence for the solutions to Equ. (1.1).

Theorem 3.1 *Assume that T be the maximal time of the solution $z = (u,v)$ to the Cauchy problem (1.1) with the initial data z_0 $z_0 = (u_0, v_0) \in H^s \times H^s$ $(s > 5/2)$. Morover, if there exists a positive constant M satisfies that*

$$\Gamma \doteq \left(||u||^{q-1}_{L^\infty} + ||v||^{p-1}_{L^\infty} \right)\left(||u_x||_{L^\infty} + ||v_x||_{L^\infty} \right) \leq M, \quad t \in [0, T),$$

then the solution $z(t,\cdot)$ with the $H^s \times H^s$-norm does not blow up on $[0, T)$.

Proof The local well-posedness was guaranteed by Theorem 1.1.

Using the operator D^s to the system (3.18), multiplying the result system by $D^s u$ and $D^s v$, respectively. Then integrating the obtained system over $\mathbb{R}$, we may arrive at

$$\frac{1}{2}\frac{d}{dt}||u||_{H^s}^2 + (v^p u_x, u)_s + (u, I_1(u,v))_s = 0, \tag{3.19}$$

$$\frac{1}{2}\frac{d}{dt}||v||_{H^s}^2 + (u^q v_x, v)_s + (v, I_2(u,v))_s = 0, \tag{3.20}$$

with

$$I_1(u,v) = (1-\partial_x^2)^{-1}\left[\frac{a}{p}(v^p)_x u + \frac{p-a}{p}(v^p)_x u_{xx}\right] + (1-\partial_x^2)^{-1}\partial_x[(v^p)_x u_x],$$
$$I_2(u,v) = (1-\partial_x^2)^{-1}\left[\frac{b}{q}(u^q)_x v + \frac{q-b}{q}(u^q)_x v_{xx}\right] + (1-\partial_x^2)^{-1}\partial_x[(u^q)_x v_x].$$

Now, we estimate the right-hand side of (3.19),

$$\begin{aligned}|(v^p u_x, u)_s| &= |(D^s v^p u_x, D^s u)_0| = |([D^s, v^p]u_x, D^s u)_0 + (v^p D^s u_x, D^s u)_0| \\ &\le ||[D^s, v^p]u_x||_{L^2}||D^s u||_{L^2} + \frac{1}{2}|((v^p)_x D^s u, D^s u)_0| \\ &\le c(||(v^p)_x||_{L^\infty}||u||_{H^s}^2 + ||u_x||_{L^\infty}||v^p||_{H^s}||u||_{H^s}).\end{aligned}$$

In the above inequality, we applied Lemma 3.2 with $r = s$ is used.

By Lemma 3.1 and the mathematical induction, we have $||v^p||_{H^s} \le p||v||_{L^\infty}^{p-1}||v||_{H^s}$ and

$$|(v^p u_x, u)_s| \le c||v||_{L^\infty}^{p-1}(||v_x||_{L^\infty} + ||u_x||_{L^\infty})(||v||_{H^s} + ||u||_{H^s})^2.$$

Therefore, we obtain

$$\begin{aligned}|(I_1(z), u)_s| &\le ||I_1(z)||_{H^s}||u||_{H^s} \le c(||(v^p)_x u||_{H^{s-2}} + ||(v^p)_x u_{xx}||_{H^{s-2}} + ||(v^p)_x u_x||_{H^{s-1}})||u||_{H^s} \\ &\le c||(v^p)_x||_{L^\infty}||u||_{H^s}^2,\end{aligned}$$

which reveals

$$\frac{1}{2}\frac{d}{dt}||u||_{H^s}^2 \le c||v||_{L^\infty}^{p-1}(||v_x||_{L^\infty} + ||u_x||_{L^\infty})(||u||_{H^s} + ||v||_{H^s})^2.$$

In a similar way, from (3.20) we can get the estimate for $||v||_{H^s}^2$. So, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(||u||_{H^s} + ||v||_{H^s}\big)^2 &\le \frac{d}{dt}\big(||u||_{H^s}^2 + ||v||_{H^s}^2\big) \\ &\le c\Gamma\big(||u_x||_{L^\infty}, ||v_x||_{L^\infty}\big)(||u||_{H^s} + ||v||_{H^s})^2.\end{aligned}$$

Adopting the assumption of the theorem and the Gronwall's inequality imply

$$||u||_{H^s} + ||v||_{H^s} \le \exp(cMt)(||u_0||_{H^s} + ||v_0||_{H^s}),$$

which completes the proof of Theorem 3.1. ☐

Now, we use Theorem 3.1 to show the blow-up scenario for Equ. (1.1).

Proof of Theorem 1.4 Let $z = (u, v)$ and T according to the assumption of the theorem. Multiplying both sides of Equ. (1.1) by m, and integrating the result equation by parts, one can get

$$\frac{d}{dt}\int_{\mathbb{R}} m^2 dx = 2\frac{d}{dt}\int_{\mathbb{R}} mm_t dx = -2\int_{\mathbb{R}} m(v^p m_x + \frac{a}{p}(v^p)_x m)dx = \frac{p-2a}{p}\int_{\mathbb{R}} m^2 (v^p)_x dx.$$
$$\frac{d}{dt}\int_{\mathbb{R}} n^2 dx = \frac{q-2b}{q}\int_{\mathbb{R}} n^2 (u^q)_x dx. \tag{3.21}$$

We also notice

$$\|u(t,\cdot)\|_{H^2}^2 \le \|m(t,\cdot)\|_{L^2}^2 \le 2\|u(t,\cdot)\|_{H^2}^2, \quad \|v(t,\cdot)\|_{H^2}^2 \le \|n(t,\cdot)\|_{L^2}^2 \le 2\|n(t,\cdot)\|_{H^2}^2.$$

Casting $p = 2a, q = 2b$ in Eq. (3.21) yields

$$\|u_x(t,\cdot)\|_{L^\infty}^2 \le \|u(t,\cdot)\|_{H^2}^2 \le \|m(t,\cdot)\|_{L^2}^2 = \|m(0,\cdot)\|_{L^2}^2 < \infty, \quad \|v_x(t,\cdot)\|_{L^\infty}^2 \le \|n(0,\cdot)\|_{L^2}^2 < \infty.$$

In view of Theorem 3.1 and Sobolev inequality $\|u(t,\cdot)\|_{L^\infty}^2 \le \|u(t,\cdot)\|_{H^1}^2$, one may see that every solution to the Cauchy problem (1.1) remains globally regular in time.

If $p > 2a$ (or $q > 2b$) and the slope of the function v^p (or u^q) is lower bounded or if $p < 2a$ (or $q < 2b$) and the slope of the function v^p (or u^q) is upper bounded on $[0, T) \times \mathbb{R}$, then there exists a positive constant $M > 0$ verify that

$$\frac{d}{dt}\int_{\mathbb{R}} m^2 dx \le M\int_{\mathbb{R}} m^2 dx, \quad \frac{d}{dt}\int_{\mathbb{R}} n^2 dx \le M\int_{\mathbb{R}} n^2 dx.$$

In view of the Gronwall's inequality, we obtain

$$||m(t,\cdot)||_{L^2} \le ||m(0,\cdot)||_{L^2}\exp\{Mt\}, \quad ||n(t,\cdot)||_{L^2} \le ||n(0,\cdot)||_{L^2}\exp\{Mt\} \quad \forall t \in [0, T),$$

This inequality and Theorem 3.1 imply that the solution of Equ. (1.1) does not blow up in a finite time.

On the other hand, combing Theorem 3.1 and Sobolev's imbedding theorem give that if the slope of the functions v^p, u^q becomes unbounded either lower or upper in a finite time, then the solution will blow up in a finite time. This complete the proof of Theorem. □

Next, let us consider the following Cauchy problem:

$$\begin{cases} \phi_t = v^p(t, \phi(t,x)), & \text{for all } (t,x) \in [0,T) \times \mathbb{R}, \\ \varphi_t = u^q(t, \varphi(t,x)), & \text{for all } (t,x) \in [0,T) \times \mathbb{R}, \\ \phi(0,x) = x, \varphi(0,x) = x, \ x \in \mathbb{R}, \end{cases} \tag{3.22}$$

where u, v denote the solution to the problem (1.1). Adopting classical ordinary differential equations theory leads to the results on ϕ, φ, which are key to the blow-up scenarios.

Lemma 3.3 *Let $T > 0$ be the lifespan of the solution to Equ.* (1.1)*with correspond to $u_0, v_0 \in H^s(s > 5/2)$. Then the system* (3.22) *exists a unique solution $\phi, \varphi \in \mathcal{C}^1([0,T),\mathbb{R})$. and the map $\phi(t,\cdot), \varphi(t,\cdot)$ is an increasing diffeomorphism over $\mathbb{R}$, where*

$$\phi_x(t,x) = e^{\int_0^t (v^p)_\phi(s,\phi(s,x))ds} > 0, \varphi_x(t,x) = e\int_0^t (u^q)_\varphi(s,\varphi(s,x))ds > 0,$$

for all $(t,x) \in [0,T) \times \mathbb{R}$.

Proof Theorem 1.1 leads to $u, v \in \mathcal{C}([0,T);H^s) \cap \mathcal{C}^1([0,T);H^{s-1})$. Thus, both functions $u(t,x)$, $v(t,x)$ and $u_x(t,x)$, $v_x(t,x)$ are bounded, Lipschitz in space and $\mathcal{C}^1$ in time. As per the classical existence and uniqueness theorem of ODEs, equation (3.22) exists a unique solution $\phi, \varphi \in \mathcal{C}^1([0,T),\mathbb{R})$.

Differentiating both sides of equation (3.22) respect to x yields

$$\begin{cases} \frac{d}{dt}\phi_x = (v^p)_\phi(t,\phi(t,x))\phi_x, & (t,x) \in [0,T) \times \mathbb{R}, \\ \frac{d}{dt}\varphi_x = (u^q)_\varphi(t,\varphi(t,x))\phi_x, & (t,x) \in [0,T) \times \mathbb{R}, \\ \phi_x(0,x) = 1, \varphi_x(0,x) = 1, & x \in \mathbb{R}, \end{cases}$$

which implies

$$\phi_x(t,x) = e^{\int_0^t (v^p)_\phi(s,\phi(s,x))ds} > 0, \varphi_x(t,x) = e^{\int_0^t (u^q)_\varphi(s,\varphi(s,x))ds} > 0.$$

Employing the Sobolev embedding theorem gives, for every $T' < T$,

$$\sup_{(\tau,x)\in[0,T)\times\mathbb{R}} |(v^p)_x(\tau,x)| < \infty, \quad \sup_{(\tau,x)\in[0,T)\times\mathbb{R}} |(u^q)_x(\tau,x)| < \infty.$$

So, there exists two constants $K_1, K_2 > 0$ such that $\phi_x \geq e^{-K_1 t}, \varphi_x \geq e^{-K_2 t}$ for $\tau \in [0,T), x \in \mathbb{R}$, which yields the Lemma 3.3. □

Lemma 3.4 *Let z and T be as in the statement of the Lemma* 3.3. *Then, we have*

$$\begin{cases} m(t,\phi(t,x))\phi_x^{\frac{a}{p}}(t,x) = m_0(x), & t \in [0,T), x \in \mathbb{R} \\ n(t,\varphi(t,x))\varphi_x^{\frac{a}{p}}(t,x) = n_0(x), & t \in [0,T), x \in \mathbb{R}. \end{cases} \tag{3.23}$$

Moreover, if there exist $M_1 > 0$ and $M_2 > 0$ such that $\frac{a}{p}(v^p)_\phi(t,\phi) \geq -M_1$ and $\frac{b}{q}(u^q)_\varphi(t,\varphi) \geq -M_2$, then

$$||m(t,\cdot)||_{L^\infty} = ||m(t,\phi(t,\cdot))||_{L^\infty} \leq \exp\{2M_1T\}||m_0(\cdot)||_{L^\infty}, t \in [0,T), x \in \mathbb{R}$$

and

$$||n(t,\cdot)||_{L^\infty} = ||n(t,\varphi(t,\cdot))||_{L^\infty} \leq \exp\{2M_2T\}||n_0(\cdot)||_{L^\infty}, t \in [0,T), x \in \mathbb{R}.$$

Furthermore, if $\int_{\mathbb{R}} |m_0(x)|^{p/a}dx$*(or* $\int_{\mathbb{R}} |n_0(x)|^{q/b}dx$ *) converge with* $a \neq 0$*(or* $b \neq 0$*), then*

$$\int_{\mathbb{R}} |m(t,x)|^{p/a}dx = \int_{\mathbb{R}} |m_0(x)|^{p/a}dx, t \in [0,T).$$

(Respectively, $\int_{\mathbb{R}} |n(t,x)|^{p/b}dx = \int_{\mathbb{R}} |n_0(x)|^{q/b}dx, t \in [0,T)$ *for all* $t \in [0,T)$*).*

Proof Noticing $\frac{d\phi_x(t,x)}{dt} = \phi_{xt} = (v^p)_\phi(t,\phi(t,x))\phi_x(t,x)$, we differentiate the left-hand side of the first equation in (3.23) with respect to the variable t, and recall the first equation in (1.1), we obtain

$$\begin{aligned}
&\frac{d}{dt}\{m(t,\phi(t,x))\phi_x^{a/p}(t,x)\}\\
&\quad= \left[m_t(t,\phi) + m_\phi(t,\phi)\phi_t(t,x)\right]\phi_x^{a/p}(t,x) + \frac{a}{p}m(t,\phi)\phi_x^{(a-p)/p}(t,x)\phi_{xt}(t,x)\\
&\quad= \left[m_t(t,\phi) + m_\phi(t,\phi)v^p(t,\phi) + \frac{a}{p}m(t,\phi)(v^p)_\phi(t,\phi)\right]\phi_x^{a/p}(t,x)\\
&\quad= 0.
\end{aligned}$$

In a similar way, we would arrive at

$$\frac{d}{dt}\{n(t,q(t,x))\varphi_x^{b/q}(t,x)\} = 0,$$

which implies that the function $m(t,\phi(t,x))\phi_x^{a/p}(t,x)$ and $n(t,\varphi(t,x))\varphi_x^{b/q}(t,x)$ are independent on the time t. By (3.22), we know $\phi_x(x,0) = 1$. So, Eq. (3.23) holds.

Combining Lemma 3.3, Eq. (3.23), and $\phi_x(0,x) = 1$, one can get

$$\begin{aligned}
||m(t,\cdot)||_{L^\infty} &= ||m(t,\phi(t,\cdot))||_{L^\infty} = ||\phi_x^{-a/p}m_0||_{L^\infty}\\
&= \left\|\exp\left\{-\frac{a}{p}\int_0^t (v^p)_\phi(s,\phi(s,x))ds\right\}m_0(\cdot)\right\|_{L^\infty}\\
&\le \exp\{2M_1T\}||m_0(\cdot)||_{L^\infty}, t \in [0,T),
\end{aligned}$$

moreover,

$$\begin{aligned}
\int_{\mathbb{R}} |m_0(x)|^{p/a}dx &= \int_{\mathbb{R}} |m(t,\phi(t,x))|^{p/a}\phi_x(t,x)dx = \int_{\mathbb{R}} |m(t,\phi(t,x))|^{p/a}d\phi(t,x)\\
&= \int_{\mathbb{R}} |m(t,x)|^{p/a}dx, t \in [0,T),
\end{aligned}$$

which concludes the proof of the lemma. □

Let us now come to prove Theorems 1.5–1.6 using Lemma 3.4.

Proof of Theorem 1.5 Since $u_0 \in H^s \cap W^{2,\frac{p}{a}}$ for $s > 5/2$, Lemma 3.4 tells us that

$$\int_{\mathbb{R}} |m(t,x)|^{\frac{p}{a}} dx \leq \int_{\mathbb{R}} |m_0(x)|^{\frac{p}{a}} dx \leq ||u_0||_{W^{2,\frac{p}{a}}} \quad \text{if } 0 < a \leq p,$$

and

$$||m(t,x)||_{L^\infty} \leq ||m(0,x)||_{L^\infty} \quad \text{if } a = 0,$$

which imply $m = (1-\partial_x^2)u \in L^{\frac{p}{a}}$ and therefore $u \in W^{2,\frac{p}{a}}$. Sobolev imbedding theorem implies that $W^{2,\frac{p}{a}} \subset \mathcal{C}^1$ for $0 \leq a \leq p$. Therefore, $||u||_{L^\infty}$ and $||u_x||_{L^\infty}$ are uniformly bounded for all $t \in [0,T)$. Theorem 1.4 guarantees the Theorem 1.5 is ture, i.e., the solution of the problem (1.1) is global existence. □

Proof of Theorem 1.6 Since the initial date $(u_0, v_0) \in H^s \times H^s$ $(s > 5/2)$, $m_0 = (1-\partial_x^2)u_0$ has a compact support. Without loss of generality, suppose that m_0 is supported in the compact interval $[a, b]$. Lemma 3.3 ensure that $\phi_x(x,t) > 0$ on the interval $\mathbb{R} \times [0,T)$. Lemma 3.4 conclude that, for any $t \in [0,T)$, the $\mathcal{C}^1$ function $m(x, t)$ exists compact support in $[\phi(a,t), \phi(b,t)]$. □

4 The Peaked Traveling Wave Solutions of Equs.(1.1)

In present section, in order to prove Theorem 1.9, we construct some appropriate sequences of peakon solutions by the method of undetermined coefficients. First, let us show that the peakon formulas (1.14–1.15) and multi-peakon formulas (1.16–1.17) define some weak solutions to Equ. (1.1) both on a circle and on a line, respectively.

Proof of Theorem 1.7 **The non-periodic peakon solution in the form of** (1.14). Without loss of generality, we set $x_0 = 0$. First, Rewriting the model (1.1) as

$$\begin{cases} u_t + v^p u_x + I_1(u,v) = 0, \\ v_t + u^q v_x + I_2(u,v) = 0, \end{cases} \tag{4.1}$$

where

$$\begin{cases} I_1(u,v) = (1-\partial_x^2)^{-1}[av^{p-1}v_x u + (p-a)v^{p-1}v_x u_{xx}] + p(1-\partial_x^2)^{-1}\partial_x(v^{p-1}v_x u_x), \\ I_2(u,v) = (1-\partial_x^2)^{-1}[bu^{q-1}u_x v + (q-b)u^{q-1}u_x v_{xx}] + q(1-\partial_x^2)^{-1}\partial_x(u^{q-1}u_x v_x). \end{cases}$$

Noticing that

$$u_t = \text{sgn}(x-ct)cu, u_x = -\text{sgn}(x-ct)u, v_t = \text{sgn}(x-ct)cu, v_x = -\text{sgn}(x-ct)v,$$

then we have

$$u_t + v^p u_x = -(-cu + v^p u)\text{sgn}(x-ct), v_t + u^q v_x = -(-cv + u^q v)\text{sgn}(x-ct). \tag{4.2}$$

A simple computation reveals

$$
\begin{aligned}
I_1(u,v) &= \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}[av^{p-1}v_y u + (p-a)v^{p-1}v_y u_{yy}](t,y)dy \\
&\quad + \frac{p}{2}\partial_x \int_{\mathbb{R}} e^{-|x-y|}(v^{p-1}v_y u_y)(t,y)dy \\
&= -\frac{a\alpha\beta^p}{2}\int_{\mathbb{R}} \mathrm{sgn}(y-ct)e^{-|x-y|}e^{-(p+1)|y-ct|}(t,y)dy \\
&\quad + \frac{(p-a)\alpha\beta^p}{4}\int_{\mathbb{R}} \partial_y\big[\mathrm{sgn}(y-ct)e^{-|y-ct|}\big]^2 e^{-|x-y|}e^{-(p-1)|y-ct|}dy \\
&\quad - \frac{p\alpha\beta^p}{2}\int_{\mathbb{R}} \mathrm{sgn}^2(y-ct)\mathrm{sgn}(x-y)e^{-|x-y|}e^{-(p+1)|y-ct|}dy \\
&= \alpha\beta^p \int_{\mathbb{R}} \big[-\frac{a}{2}\mathrm{sgn}(y-ct) - \frac{3p-a}{4}\mathrm{sgn}^2(y-ct)\mathrm{sgn}(x-y) \\
&\quad + \frac{(p-a)(p-1)}{4}\mathrm{sgn}^3(y-ct)\big] \\
&\qquad e^{-|x-y|-(p+1)|y-ct|}(t,y)dy.
\end{aligned}
$$

For the case $x < ct$, we derive

$$
\begin{aligned}
I_1(u,v) &= \alpha\beta^p\Big(\frac{(a-p)(p+2)}{4}\int_{-\infty}^{x} e^{(p+2)y-x-(p+1)ct}dy \\
&\quad + \frac{p(a+4-p)}{4}\int_{x}^{ct} e^{-py-x+(p+1)ct}dy \\
&\quad + \frac{(p-a)(p+2)}{4}\int_{ct}^{\infty} e^{-(p+2)y+x+(p+1)ct}dy\Big) \\
&= \alpha\beta^p\big(-e^{(p+1)(x-ct)} + e^{x-ct}\big).
\end{aligned}
$$

For the case $x > ct$, we deduce

$$
\begin{aligned}
I_1(u,v) &= \alpha\beta^p\Big(\frac{(a-p)(p+2)}{4}\int_{-\infty}^{ct} e^{(p+2)y-x-(p+1)ct}dy \\
&\quad + \frac{p(p-a-4)}{4}\int_{ct}^{x} e^{-py-x+(p+1)ct}dy \\
&\quad + \frac{(p-a)(p+2)}{4}\int_{x}^{\infty} e^{-(p+2)y+x+(p+1)ct}dy\Big) \\
&= \alpha\beta^p\big(e^{(p+1)(x-ct)} - e^{x-ct}\big).
\end{aligned}
$$

Consequently, we obtain

$$
I_1(u,v) = (-\beta^p u + v^p u)\mathrm{sgn}(x-ct), \tag{4.3}
$$

and

$$
I_2(u,v) = (-\alpha^q v + u^q v)\mathrm{sgn}(x-ct). \tag{4.4}
$$

Combining Equs. (4.2–4.4) with the assumption $\alpha = c^{1/q}, \beta = c^{1/p}$, we get that the first equation of the system (4.1) holds on the line in the sense of distribution.

The periodic peakon solution in the forms of (1.15). We claim that Equ. (4.1) is equivalent to Equ. (1.1), let us start from the original system(1.1). Let $f \in L^1_{loc}(X)$, and the open set $X \subset \mathbb{R}$. Assume that $f' \in L^1_{loc}(X)$ and is continuous except at a single point $x_0 \in X$; then the right-handed and left-handed limits $f(x_0^{\pm})$ exist, moreover, $(T_f)' = T_{f'} + [f(x_0^+) - f(x_0^-)]\delta_{x_0}$, where T_f is the distribution associated to the function f and δ_{x_0} is the Dirac delta distribution centered at $x = x_0$. Denote $K \doteq x - ct - 2\pi\left[\frac{x-ct}{2\pi}\right] - \pi$. Noticing that

$$u_t = cu, u_x = \alpha \sinh K, u_{xx} = u - 2\alpha \sinh(\pi)\delta_{ct},$$
$$v_t = cv, v_x = \beta \sinh K, v_{xx} = v - 2\beta \sinh(\pi)\delta_{ct},$$

where δ_{ct} is the periodic Dirac delta distribution centered at $x = ct \bmod 2\pi$, we have $u - u_{xx} = 2\alpha \sinh(\pi)\delta_{ct}$ and

$$(1 - \partial_x^2)u_t = -2c\alpha \sinh \pi \delta'_{ct}.$$

Employing the hyperbolic identity $\cosh^2 x = 1 + \sinh^2 x$ yields

$$\begin{aligned}
\partial_x^2(v^p u_x) &= \alpha\beta^p \partial_x^2(\sinh K \cosh^p K) \\
&= \alpha\beta^p \partial_x(\cosh^{p+1} K + p\cosh^{p-1} K \sinh^2 K - 2\sinh(\pi)\cosh^p(\pi)\delta_{ct}) \\
&= \alpha\beta\partial_x((p+1)\cosh^{p+1} K - p\cosh^{p-1} K - 2\sinh(\pi)\cosh^p(\pi)\delta_{ct}) \\
&= \alpha\beta^p[(p+1)^2 \cosh^p K \sinh K - p(p-1)\cosh^{p-2} K \sinh K \\
&\quad - 2\sinh(\pi)\cosh^p(\pi)\delta'_{ct}].
\end{aligned}$$

Then, we find

$$\begin{aligned}
(1 - \partial_x^2)(v^p u_x) = \alpha\beta^p[&(1-(p+1)^2)\cosh^p K \sinh K + p(p-1)\cosh^{p-2} K \sinh K \\
&+ 2\sinh(\pi)\cosh^p(\pi)\delta'_{ct}].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
p\partial_x(v^{p-1}v_x u_x) &= p\alpha\beta^p\partial_x(\cosh^{p-1} K \sinh^2 K) = \alpha\beta^p\partial_x(\cosh^{p+1} K - \cosh^{p-1} K) \\
&= p\alpha\beta^p((p+1)\cosh^p K \sinh K - (p-1)\cosh^{p-2} K \sinh K),
\end{aligned}$$

$$v_x u_{xx} = \frac{\alpha\beta}{2}\partial_x \sinh^2 K = \frac{\alpha\beta}{2}\partial_x(\cosh^2 K - 1) = \alpha\beta \sinh K \cosh K,$$

and

$$av^{p-1}v_x u + (p-a)v^{p-1}v_x u_{xx} = p\alpha\beta^p \sinh K \cosh^p K.$$

Therefore, we have

$$
\begin{aligned}
& m_t + v^p m_x + a v^{p-1} v_x m \\
&\qquad = (1-\partial_x^2)u_t + (1-\partial_x^2)(v^p u_x) + a v^{p-1} v_x u + (p-a) v^{p-1} v_x u_{xx} + p\partial_x (v^{p-1} v_x u_x) \\
&\qquad = -2c\alpha \sinh \pi \delta'_{ct} + 2\alpha\beta^p \sinh(\pi)\cosh^p(\pi)\delta'_{ct}.
\end{aligned}
$$

In a similar way, for $n(t)$ we obtain

$$
n_t + v^q n_x + b v^{q-1} v_x n = -2c\beta \sinh \pi \delta'_{ct} + 2\alpha^q \beta \sinh(\pi)\cosh^p(\pi)\delta'_{ct}.
$$

So, the periodic peaked function (1.15) is a solution to the equation (1.1) iff $\alpha = \frac{c^{1/q}}{\cosh(\pi)}, \beta = \frac{c^{1/p}}{\cosh(\pi)}$.

Multi-peakon solutions (1.16) and (1.17) **for Equ.**(1.1).

Let us use an adhoc definition for $u_x(x,t), v_x(x,t)$ given by

$$
u_x(t,x) = -\sum_{i=1}^{M} \operatorname{sgn}(x-g_i(t)) f_i(t) e^{-|x-g_i(t)|}, v_x(t,x) = -\sum_{j=1}^{N} \operatorname{sgn}(x-g_j(t)) f_j(t) e^{-|x-g_j(t)|},
$$

which imply $u_x(x,t), v_x(x,t)$ are equal to $\langle u_x(x,t)\rangle = \frac{1}{2}(u_x(x^-,t)+u_x(x^+,t))$ and $\langle v_x(x,t)\rangle = \frac{1}{2}(v_x(x^-,t)+v_x(x^+,t))$, respectively. Note that

$$
u_{xx}(t,x) = u(x,t) - 2\sum_{i=1}^{M} f_i(t)\delta_{g_i(t)}, v_{xx}(t,x) = v(x,t) - 2\sum_{j=1}^{N} f_j(t)\delta_{g_j(t)},
$$

i.e., $m = u - u_{xx} = 2\sum_{i=1}^{M} f_i(t)\delta_{g_i(t)}, n = v - v_{xx} = 2\sum_{j=1}^{N} f_j(t)\delta_{g_j(t)}$, which lead to

$$
\begin{aligned}
m_t + v^p m_x + \frac{a}{p}(v^p)_x m &= 2\sum_{i=1}^{M}[-f_i\dot{g}_i\partial_x(\delta_{g_i}) + \dot{f}_i\delta_{g_i} + v^p f_i \partial_x(\delta_{g_i}) + \frac{a}{p}(v^p)_x f_i \delta_{g_i}] \\
&= 2\sum_{i=1}^{M}[(-f_i\dot{g}_i + v^p f_i)\partial_x(\delta_{g_i}) + (\dot{f}_i + \frac{a}{p}(v^p)_x f_i)\delta_{g_i}],
\end{aligned}
$$

where $\dot{f} = \partial_t f$. Casting a test function $\varphi \in C_0^\infty(\mathbb{R})$ and $(f, \delta_{g_i}) = f(g_i)$ on the equation yields

$$
\begin{aligned}
& \left(m_t + v^p m_x + \frac{a}{p}(v^p)_x m, \varphi\right) \\
&\quad = 2\sum_{i=1}^{M} -f_i\dot{g}_i\left(\partial_x\delta_{g_i}, \varphi\right) + 2\sum_{i=1}^{M} f_i\left(v^p\partial_x\delta_{g_i}, \varphi\right) + 2\sum_{i=1}^{M}\left((\dot{f}_i + \frac{a}{p}(v^p)_x f_i)\varphi, \delta_{g_i}\right) \\
&\quad = 2\sum_{i=1}^{M} f_i\dot{g}_i\left(\delta_{g_i}, \partial_x\varphi\right) - 2\sum_{i=1}^{M} f_i\left((v^p)_x\varphi + (v^p)\partial_x\varphi, \delta_{g_i}\right) + 2\sum_{i=1}^{M}\left((\dot{f}_i + \frac{a}{p}(v^p)_x f_i)\varphi, \delta_{g_i}\right) \\
&\quad = 2\sum_{i=1}^{M}\left[f_i(\dot{g}_i - v^p(g_i))\varphi_x(q_i) + (\dot{f}_i + \frac{a-p}{p}(v^p)_x(g_i) f_i)\varphi(g_i)\right].
\end{aligned}
$$

Similarly, we have

$$\left(n_t + v^q n_x + \frac{b}{q}(v^q)_x n, \varphi\right)$$
$$= 2\sum_{j=1}^{N}\left[h_j(\dot{k}_j - u^q(k_j))\varphi_x(q_j) + (\dot{h}_j + \frac{b-q}{q}(u^q)_x(k_j)h_j)\varphi(k_j)\right].$$

Accordingly, the multi-peakon is a solution to Equs.(1.1) iff $g_i(t), h_j(t)$ and amplitudes $f_i(t), k_j(t)$ verify the ODE system defined by (1.18). □

Numerical experiments of Peakons solutions to system (1.1).

Next we perform some numerical experiments to illustrate our results. Rewriting system (1.18) in a specific form:

$$\dot{g_\sigma} = \left(\sum_{j=1}^{N} h_j e^{-|g_\sigma - k_j|}\right)^p, 1 \le \sigma \le N, \tag{4.5}$$

$$\dot{k_\mu} = \left(\sum_{j=1}^{M} f_j e^{-|k_\mu - g_j|}\right)^q, 1 \le \mu \le M, \tag{4.6}$$

$$\dot{f_\sigma} = f_\sigma\left[(a-p)\sum_{j=1}^{N} sgn(g_\sigma - k_j)h_j e^{-|g_\sigma - k_j|}\left(\sum_{m=1}^{N} h_m e^{-|g_\sigma - k_m|}\right)^{p-1}, 1 \le \sigma \le N,\right. \tag{4.7}$$

$$\dot{h_\mu} = h_\mu\left[(b-q)\sum_{j=1}^{M} sgn(k_\mu - g_j)f_j e^{-|k_\mu - g_j|}\left(\sum_{m=1}^{M} f_m e^{-|k_\mu - g_m|}\right)^{q-1}, 1 \le \mu \le M.\right. \tag{4.8}$$

Now, we consider the special case with $M = N$, $f_j = h_j$, $g_j = k_j$ $p = q$ and $a = b$, then system (4.5–4.8) reduces to

$$\dot{g_\sigma} = \left(\sum_{j=1}^{M} f_j e^{-|g_\sigma - g_j|}\right)^p, \dot{f_\sigma} \tag{4.9}$$

$$=(a-p)f_\sigma \sum_{j=1}^{M} sgn(g_\sigma - g_j)f_j e^{-|g_\sigma - g_j|}\left(\sum_{j=1}^{M} f_j e^{-|g_\sigma - g_j|}\right)^{p-1}, \tag{4.10}$$

which coincide with the generalized b-family equation(cf.[42]). For $p = 1, a = 2, M = 2$ to system (4.9), we obtain the two-peakon dynamics of CH equation as follows:

$$\begin{cases} \dot{g}_1 = f_1 + f_2 e^{-|g_1 - g_2|}, \\ \dot{g}_2 = f_2 + f_1 e^{-|g_2 - g_1|}, \\ \dot{f}_1 = f_1 f_2 sgn(g_1 - g_2) e^{-|g_1 - g_2|}(f_1 + f_2 e^{-|g_1 - g_2|}), \\ \dot{f}_2 = f_1 f_2 sgn(g_2 - g_1) e^{-|g_2 - g_1|}(f_2 + f_1 e^{-|g_2 - g_1|}). \end{cases} \tag{4.11}$$

which was studied by Camassa and Holm [5]. Taking $p = q = 1, a = b = 2$, the system (1.1) yields the CCCH system(cf. Cotter et.al[11]). In case $N = M = 2$, $k_j = g_j$ and $g_1 = g_2$ with (4.5–4.8) satisfies:

$$\begin{cases} \dot{g}_1 = \dot{g}_2 = (h_1 + h_2)^p = (f_1 + f_2)^q, \\ \dot{f}_1 = 0, \ \dot{f}_2 = 0, \ \dot{h}_1 = 0, \ \dot{h}_2 = 0. \end{cases} \tag{4.12}$$

This implies $f_1 = c_1, f_2 = c_2, h_1 = c_3, h_2 = c_4, g_1 = g_2 = (c_1 + c_2)^q t + c_5 = (c_3 + c_4)^p t + c_6, c_5 = c_6$,

$$u(x,t) = (c_1 + c_2) e^{-|x - [(c_1 + c_2)^q t + c_5]|}, \ \ v(x,t) = (c_3 + c_4) e^{-|x - [(c_1 + c_2)^q t + c_5]|}. \tag{4.13}$$

It is clearly that (4.13) corresponds to our conclusion (1.14). When $p = 1, q = 2$, according to (4.12), the c_1, c_2, c_3, c_4 satisfy $(c_1 + c_2)^2 = c_3 + c_4$.

If $t = 1$, $c_5 = 3$ and $p = 1, q = 2$, and taking $c_1 + c_2 = 2, c_3 + c_4 = 4$, $c_1 + c_2 = 3, c_3 + c_4 = 9$ and $c_1 + c_2 = 4, c_3 + c_4 = 16$, respectively, the single-peakon solution given by (4.13) with correspond to red, blue, green (See Fig. 1).

Finally, if $x = 1$, $c_5 = 3$ and $p = 1, q = 2$, taking $c_1 + c_2 = 2, c_3 + c_4 = 4$, $c_1 + c_2 = 3, c_3 + c_4 = 9$ and $c_1 + c_2 = 4, c_3 + c_4 = 16$, respectively, the single-peakon solution given by Fig. 2.

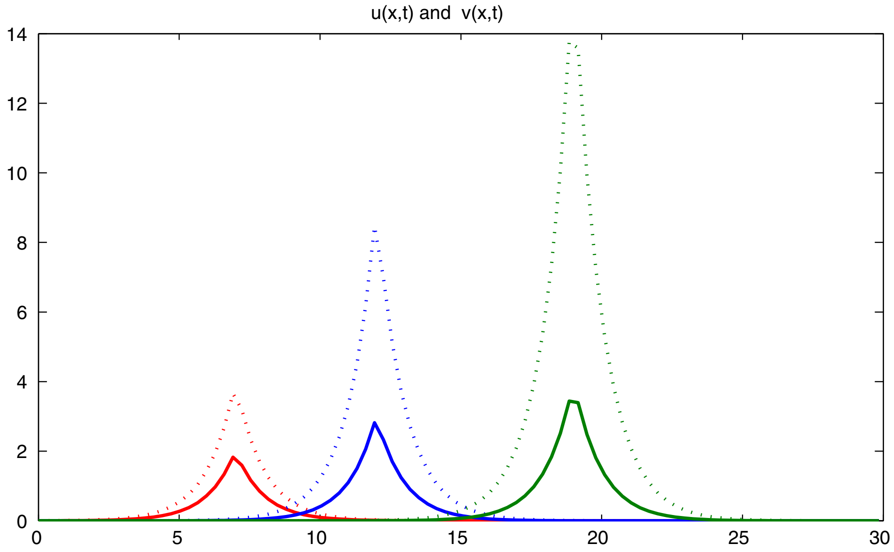

Fig. 1 Solid line: $u(x, t)$; Dashed line: $v(x, t)$

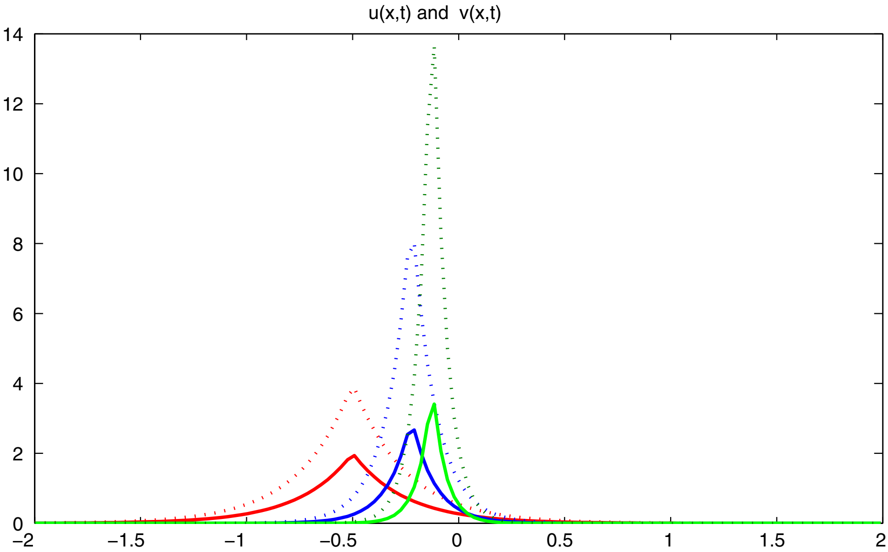

Fig. 2 Solid line: $u(x, t)$; Dashed line: $v(x, t)$

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Ethics Approval and Consent to Participate Not applicable.

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