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Evaluation of Black Holes in an Evolving Universe

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EVALUATION OF BLACK HOLES IN AN EVOLVING UNIVERSE

A Thesis

by

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ABSTRACT

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There are various solutions to the Einstein field equations that represent different physical assumptions, but how to represent multiple black holes within an expanding universe remains an area of open interest. The first step to resolving this question involves evaluating spacetime models that contain a single black hole in an expanding universe. Here, we are primarily interested in understanding the energy distribution of black hole models by solving Einstein's equations using the associated spacetime metric and comparing the propagation of waves within the model against other known spacetime models. Specifically, we will evaluate the combined Schwarschild-de Sitter solution under a coordinate transformation into time-evolving coordinates and develop a new metric for an evolving black hole in static coordinates.

DEDICATION

This thesis is dedicated to Rachel Scherer, my partner in life. Rachel, my life has changed with you in it, and without your love and support none of this would have ever been possible.

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I would like to express my sincerest thanks to Dr. Karen Yagdjian for serving as my instructor, mentor, and committee adviser. I would also like to express my gratitude to my committee for all there attention and support.

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I would also like to thank my father John F. Naan and sister Abby Reilly. Their support throughout my graduate studies has only been overshadowed by their support through the long and somewhat tenuous journey that came before.

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CHAPTER I

INTRODUCTION

The primary objective of this thesis is to evaluate the properties and behavior of expanding spacetime models containing non-stationary black holes. Specifically, we explore the result of the known Schwarszchild-de Sitter static-spacetime under a coordinate transformation into a non-static reference frame. Once in this new reference frame we will calculate the d'Alembert covariant wave operator to understand how waves propagate within this background spacetime and compare the result to the limit cases of both Schwarzschild and de Sitter spacetimes separately. Additionally, we will explore a new spacetime model that accounts for the gravitational effects from a static, though not stationary, black hole evolving within an expanding universe. We will be interested in comparing this new spacetime model against a known black-hole toy model and the standard Friedmann-Lemaître-Robertson-Walker ($FLRW$) spacetime. We will find that this new metric can be used to evaluate wave propagation in both the toy model and $FLWR$ space despite having a unique stress-energy distribution.

Although our primary objective, as stated above, relates to understanding black hole models, the understanding of black holes by using Einstein's theory of general relativity is a significant departure from the more familiar notion of classical Newtonian mechanics and involves a much more complex mathematical framework. Thus, this paper can be considered in three parts. The first part (Chapters 2-8) will focus on explaining the major mathematical objects and operations involved describing general relativity. In this part we will be introducing the mathematical object of a manifolds and discussing how vectors, coordinates, vector spaces, fields, dual spaces, and tensors operate in regards this new object. The next part (Chapters 9-12) focus on the geometry of manifolds, and how this geometry can be related to matter through the usage of the Einstein field equations. The final part (Chapters 13-15) provides the main results of this thesis where we evaluate a black hole within expanding spacetimes.

The discussion of manifolds will follow closely with John Lee's *Introduction to Smooth Manifolds* [5]. Barrett O'Neill's *Semi-Riemannian Geometry* [7] will provide the structure for both the geometric and relativistic sections. Lee's and ONeill's work will provide the details for the included definitions, theorems and propositions and the original sources should be referenced by an interested reader. Supplemental information regarding manifolds or geometry will be pulled in from Sean Carroll's *Spacetime and Geometry* [1], John Lee's *Riemannian Manifolds* [4], Robert Wald's *General Relativity* [9], and Stavrov's *Curvature of Space and Time, with an Introduction to Geometric Analysis* as relevant.

The background on the familiar spacetimes borrows heavily from Moller's *The Theory of Relativity* and Carroll's *Spacetime and Geometry* [1]. The comparison for the propagation of waves in the Limit cases and the evaluation of the new metric leverages Yagdjian's *Integral Transform Approach to Time-Dependent Partial Differential Equations* [10] and Hawking's *The Large Scale Structure of Space-time* [2]. Additional reference to Landau's *The Classical Theory of Fields* is used sparingly.

For all parts, we will begin each chapter with a conceptual overview that will describe the importance of the chapter followed by a section reviewing any prerequisite material if needed but not introduced previously. We will then proceed into the more technical aspects of the topic that may include formal definitions, theorems, and/ or derivations. Then, as relevant, we will review examples or derivations for background chapters, or we will walk through analysis for the primary chapters. Each chapter will end with a summary that will pull out the major take-aways that are needed for future sections.

CHAPTER II

MANIFOLDS

Conceptual Introduction

It is common to first learn about objects in space. That is we learn about physical objects represented within a background space. However, it is often convenient to think of the object without deference to some larger space. To do that we need some object is both the space itself and provides the information necessary to understand the object. This object is a manifold.

In this chapter we introduce the formal definition for a manifold. For our purposes, we will quickly become interested in a specific type of 4-dimensional manifold that will be the background object upon which we will measure and observe physical phenomena. However, the main objective of this chapter is just to formally introduce a topological manifold with a smooth structure.

The mathematical theory of manifolds rests on top of the subject of topology. As such, we will begin with some topological definitions. After this review of topology we will be able to move into our main discussion of manifolds. These definitions will allow us to formally define a n-dimensional manifold as a topological object that locally resembles $\mathbb{R}^n$ space (i.e., it is locally flat). However, as a topological structure only, a manifold does not admit constructs for geometric properties such as volume or curvature. To account for these properties we must add some additional structure for "smoothness" that will allow for some form of calculus to be performed.

Prerequisite Concepts

The definition for a manifold is an extension of topics in topology. As such, we introduce some preliminary definitions here.

Definition 2.1 (Topology) *Let X be a set. A topology on X is a collection, τ, of open subsets of X satisfying:*

(i) X and $\emptyset$ are open

(ii) The union of any family of open subsets is open

(iii) The intersection of any finite family of open subsets is open

For the below, let X and Y be topological spaces.

Definition 2.2 (Continuous Map) *A map $F : X \to Y$ is said to be continuous if for each open subset $U \subseteq Y$, the preimage $F^{-1}(U)$ is open in X.*

Definition 2.3 (Homeomorphism) *A continuous bijective map $F : X \to Y$ with a continuous inverse is called a homeomorphism. If there is a homeomorphism between X and Y, then X and Y are said to be homomorphic.*

Definition 2.4 (Hausdorff Space) *If for every pair of distinct points $p, q \in X$ there exists disjoint open subsets $U, V \in X$ such that $p \in U$ and $q \in V$, then X is said to be a Hausdorff Space.*

Definition 2.5 (Topological Basis) *A collection of $\mathscr{B}$ of open subsets of X is said to be a basis for the topology of X if every open subset of X is the union of some collection of $B \in \mathscr{B}$. If $p \in X$, a neighborhood basis at p is a collection of $\mathscr{B}_p$ of neighborhoods of p such that every neighborhood of p contains at least one $B \in \mathscr{B}_p$*

Definition 2.6 (First & Second Countable) *A topological space X is said to be*

(i) first-countable if there is a countable neighborhood basis at each point

(ii) second-countable if there is a countable basis for the topology

Definition 2.7 (Smooth Maps) *Let $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$. A Map $F: U \to V$ is said to be smooth, C^∞, (or infinitely differentiable) if each of its component functions has continuous partial derivatives of all orders.*

Definition 2.8 (Diffeomorphism) *If $F: U \to V$ is bijective with a smooth inverse map, then F is called a diffeomorphism. In particular, a diffeomorphism is a homeomorphism.*

Smooth Manifolds

Topological Manifolds

The first structure we are looking for is that of a topological manifold. This is a mathematical structure with its own topology, and it does not rely upon begin embedded in some other space.

Definition 2.9 (n-Dimensional Topological Manifold) *Let M be a topological space. We say that M is a topological n-manifold (with n dimensions) if the following properties hold:*

(i) M is a Hausdorff Space

(ii) M is second-countable

(iii) M is locally Euclidean of dimension n

Property (iii) is essentially saying that for each point $p \in M$ we can find

a) an open subset $U \subseteq M$ containing p,

b) an open subset $\widehat{U} \subseteq \mathbb{R}^n$, and

c) a homeomorphism $\varphi: U \to \widehat{U}$.

Here the pair (U, φ) is called a chart (or coordinate chart). We refer to φ as the local coordinate map, U as the coordinate domain of ϕ (or coordinate neighborhood of p). The the component functions $(x^1, \cdots, x^n)$ of $\varphi(p) = \big(x^1(p), \cdots, x^n(p)\big)$ are called local coordinates on U. At times we may be interested in emphasizing the fact that we are working with the local coordinates and will represent the chart as $(U, (x^i))$. We will discuss coordinates in detail in Chapter IV. We will be continuing our discussion on manifolds for the next few chapters, but a some additional things about topological manifolds to note here include:

- Each manifold is defined with a distinct dimension and homeomorphisms do not exist between manifolds of different dimensions.
- Properties (i) and (ii) are inherited by a subspace or finite product.
- Every open subset of a topological n-manifold is itself a topological n-manifold.

Smoothness on a Topological Manifold

The topological manifold allows for studying properties such as compactness, connectivity, and classification up to homeomorphisms, but we don't yet have a way to perform the familiar operations from calculus. In order to fix this we will need to add 'smoothness' to our manifold. To accomplish this goal, we will take a look at a the charts on the manifold and determine what it means for the charts to be smoothly compatible.

Suppose that (U,φ) and (V,ψ) are two charts on a topological manifold, M. If the domains of these charts overlap, $U\cap V\neq\emptyset$, then the composite map $\psi\circ\varphi^{-1}:\ \varphi(U\cap V)\to\psi(U\cap V)$ is called the transition map from φ to ψ. Note that since the transition map is a composition of two homeomorphisms, it is also homeomorphism. Two charts are said to be smoothly compatible if either:

(i) $U\cap V=\emptyset$, or

(ii) the transition map, $\psi\circ\varphi^{-1}$, is a diffeomorphism

An *atlas*, $\mathscr{A}$, on a manifold M, is a collection of charts whose domains cover the manifold M. If any two charts from $\mathscr{A}$ are smooth compatible, we say that the atlas $\mathscr{A}$ is a *smooth atlas*.

We could define a smooth structure on a manifold, M, by giving M a smooth atlas, $\mathscr{A}$, and defining a function $f:\ M\to\mathbb{R}$ to be smooth if and only if the transition map, $f\circ\varphi_i^{-1}$, is smooth (in the sense of ordinary calculus) for each coordinate chart, $(U_i,\varphi_i)\in\mathscr{A}$. However, this structure, although smooth, would not necessarily be unique as there may be multiple atlas choices. Therefore, to identify a unique structure we define a *maximal atlas* as a smooth atlas, $\mathscr{A}$, that cannot be properly contained in any larger smooth atlas. And finally we have the conditions

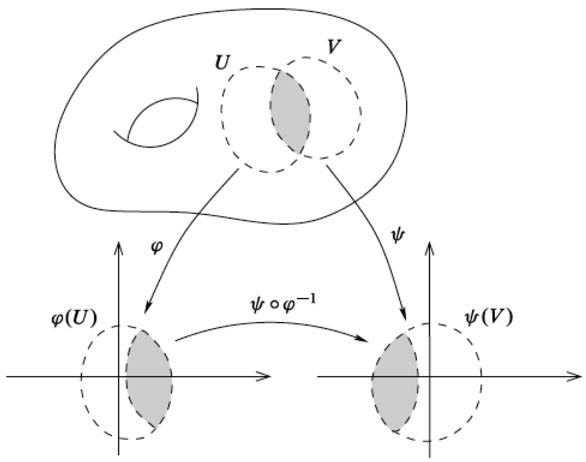

Figure 2.1: Transition Map

necessary to define a smooth structure on a topological manifold.

Definition 2.10 (C^∞-Manifold) *A smooth (or C^∞) manifold is a pair $(M, \mathscr{A})$, where M is a topological manifold, and $\mathscr{A}$ is a maximal smooth atlas on M.*

Smooth Maps

With a smooth structure as introduced above, we are able to discuss smooth maps between manifolds.

Definition 2.11 (Smooth Maps Between Manifolds) *Let M, N be smooth manifolds, and let $F : M \to N$ be any map. We say that F is a smooth map if for every $p \in M$, there exists smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.*

To prove that a map $F : M \to N$ is smooth directly from the definition requires, in part, that for each $p \in M$ we prove the existence of the coordinate domains of U containing p and V containing $F(p)$ such that $F(U) \subseteq V$. We can use this requirement to see that smoothness actually implies continuity.

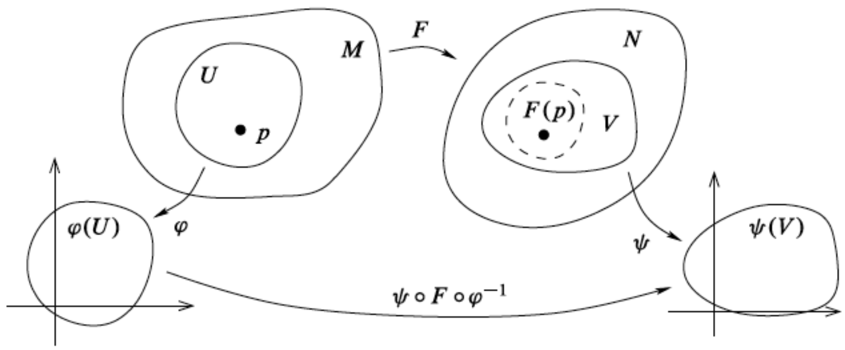

Figure 2.2: Smooth Maps between Manifolds

Proposition 2.12 *Every smooth map is continuous*

Proof: Suppose M and N are smooth manifolds, and that $F:\ M \to N$ is smooth. Given $p \in M$, smoothness of F means there are smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$, such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1}:\ \varphi(U) \to \psi(V)$ is smooth, hence continuous. Since $\varphi:\ U \to \varphi(U)$ and $\psi:\ V \to \psi(V)$ are homeomorphisms, this implies

$$F|_U = \psi^{-1} \circ \left(\psi \circ F \circ \varphi^{-1}\right) \circ \varphi : U \to V$$

which is a composition of continuous maps. Since F is continuous in a neighborhood of each point, $p \in M$, it is continuous on all of M. A few other important ways of characterizing smoothness of maps are below.

Proposition 2.13 (Equivalent Characterizations of Smoothness) *Suppose M and N are smooth manifolds and that $F:\ M \to N$ is a map. Then F is smooth iff either of the following are satisfied:*

(a) For every $p \in M$, there exists smooth charts (U, φ) containing p and (V, ψ) containing $F(p) \in N$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi\left(U \cap F^{-1}(V)\right)$ to $\psi(V)$.

(b) F is continuous and there exists smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \phi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha\left(U_\alpha \cap F^{-1}(V_\beta)\right)$ to $\psi_\beta(V_\beta)$.

Proposition 2.14 (Smoothness is Local) *Let M and N be smooth manifolds, and let $F: M \to N$ be a map.*

(a) If every $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Corollary 2.15 (Gluing Lemma for Smooth Maps) *Let M and N be smooth manifolds, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Suppose that for each $\alpha \in A$, we are given a smooth map $F_\alpha: U_\alpha \to N$ such that the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}, \forall \alpha, \beta$. Then, there exists a unique map $F: M \to N$ such that $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.*

This now gives us a definition for the coordinate representation of a map between manifolds.

Definition 2.16 (Coordinate Representation of F) *Let M and N be smooth manifolds, and $F: M \to N$ be a smooth map. If (U, φ) and (V, ψ) are coordinate charts for M and N, respectively, then $\widehat{F}: \psi \circ F \circ \varphi^{-1}$ is called the coordinate representation of F with respect to the given coordinates, and it maps the set $\varphi\left(U \cap F^{-1}(V)\right)$ to $\psi(V)$*

It is worth noting a few common ways to prove that a particular map is smooth:

(i) Write the map in smooth local coordinates and recognize its component functions as compositions of smooth elementary functions.

(ii) Exhibit the map as a composition of maps that are known to be smooth.

(iii) Use some special-purpose theorem that applies to the particular case under consideration.

An important case of smooth maps between manifolds is that of a diffeomorphisms.

Definition 2.17 (Diffeomorphism Between Manifolds) *If M and N are smooth manifolds, a diffeomorphism from M to N is a smooth bijective map $F: M \to N$ that has a smooth inverse. We say that M and N are diffeomorphic if there exists a diffeomorphism between them (sometimes represented by $M \approx N$).*

Being diffeomophic is an equivalence relation on the class of all smooth manifolds. That is, just as two topological spaces are considered "the same" if there is a homeomorphism between them, two smooth manifolds are essentially indistinguishable if they are diffeomorphic.

Theorem 2.18 (Inverse Function Theorem for Manifolds) *Suppose M and N are smooth manifolds, and $F : M \to N$ is a smooth map. If $p \in M$ is a point such that dF_p is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Proposition 2.19 (Properties of Local Diffeomorphisms) *The following are elementary properties of local diffeomorphisms:*

(a) Every composition of local diffeomorphisms is a local diffeomorphism.

(b) Every finite product of lcal diffeomorphisms between smooth manifolds is a local diffeomorphism.

(c) Every local diffeomorphism is a local homeomorphism and an open map.

(d) The restriction of a local diffeomprhism to an open submanifold is a local diffeomorphism.

(e) Every diffeomorphism is a local diffeomorphism.

(f) Every bijective local diffeomorphis is a diffeomorphism.

(g) A map between smooth manifolds is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a coordinate representation that is a local diffeomorphism.

Submanifolds

We take a moment here to introduce a particular type of manifold that will be discussed in greater detail in later sections.

Definition 2.20 (Open Submanifolds) *Let M be a smooth n-manifold and let $U \subseteq M$ be any open subset. Define an atlas on U by*

$$\mathscr{A}_U = \{smoothcharts\,(V,\phi) \text{ for } M \text{ such that } V \subseteq U\}$$

. Endowed with this smooth structure, any open subset of M is an open submanifold of M.

Manifolds With Boundary

In order to introduce the manifold with a boundary we begin by defining the closed n-dimensional upper-half space.

Definition 2.21 (n-Dimensional Upper-Half Space) *The n dimensional upper-half space is the positive definite subspace of $\mathbb{R}^n$,*

$$\mathbb{H}^n = \{\left(x^1,\ldots,x^n\right) \in \mathbb{R}^n : \; x^n \geq 0\}.$$

The definition for a n-dimensional topological manifold with boundary follows from the original definition for a manifold with the exception that each point is homeomorphic either to an open subset of $\mathbb{R}^n$ or an open subset of $\mathbb{H}^n$.

A chart for a manifold with boundary is exactly the same as our earlier definition for a chart except that now we may have need to distinguish between the interior points that are homeomorphic to the open subset of $\mathbb{R}^n$ and the boundary points that are homeomorphic to the open subset of $\mathbb{H}^n$. Points that lie on the boundary of the manifold or half-upper space will be denoted as ∂M or $\partial\mathbb{H}^n$ respectively.

Smoothness on a manifold with boundary is defined just as it was earlier for manifolds without boundary.

Lie Groups

Lie groups turn out to be very useful in the study of manifolds. This section serves as an introduction to Lie groups that will be built upon in future sections. We will begin with the definition of the Lie group as a particular kind of smooth manifold. We will then show how the Lie group has some useful properties that allow us to relate one point to another within the manifold.

Definitions

Recall that a set G with an operation, $(G,\cdot)$ is a group if we have:

(i) G is associative under $(\cdot)$: $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$, for all $g_1, g_2, g_3 \in G$,

(ii) The identity, e, is in G: $e \cdot g = g \cdot e = g$, $\forall g \in G$,

(iii) The inverse is in G: $\forall g \in G$, $\exists g^{-1} \in G$ such that $g \cdot g^{-1} = e$.

Now, if the set G also has a topology defined on it where the multiplication and inversion maps are continuous, it is called a *topological group*.

Further, there is also a classification for groups with a smooth manifold structure.

Definition 2.22 (Lie Group) *If G is a smooth manifold and is also a group in the algebraic sense, with the property that the multiplication map, $m: G \times G \to G$, and inversion map, $i: G \to G$, given by:*

$$m(g,h) = gh, \qquad i(g) = g^{-1},$$

are both smooth, then G is a Lie group.

An alternative definition for a Lie group can be given by the following proposition.

Proposition 2.23 (Alternate Definition for Lie Group) *If G is a smooth manifold with a group structure such that the map $G \times G \to G$ given by $(g,h) \to gh^{-1}$ is smooth, then G is a Lie group.*

One of the most important properties of a Lie group is that there exists a global diffeomorphism that systematically maps any point in G to another. Such a map is called a translation map.

Definition 2.24 (Left/Right Translation) *Let G be a Lie group, and let $g \in G$ define maps $L_g, R_g : G \to G$ such that*

$$L_g(h) = gh, \qquad R_g(h) = hg,$$

then, these maps are called left translation and right translation respectively.

Notice that the translation can be written as a composition of smooth functions

$$L_g : \ G \xrightarrow{i_g(h)} G \times G \xrightarrow{m} G \tag{2.1}$$

$$R_g : \ G \xrightarrow{i_g(h)} G \times G \xrightarrow{m} G \tag{2.2}$$

Thus, the translation is smooth. Further, since a smooth inverse, (i.e., $L_{g^{-1}}$), exists the translation is a diffeomorphism.

Chapter Summary

We began this section with the definition of a topological manifold, M, that provided us with the ability to analyze such properties like connectivity, compactness, and equivalence up to homeomorphism. However, we are interested in analyzing geometric properties such as curvature and volume, and this structure alone does not allow for this type of investigation. Thus, on top of the topological structure we have imposed a smooth structure. This smoothness is defined based upon a maximal set of smooth compatible coordinate charts that cover the manifold called a maximal atlas, $\mathscr{A}$. We thus achieved a complete definition of a smooth, or $C^\infty-$ manifold as a pair $(M, \mathscr{A})$. With this structure defined we were able to find a number of smooth maps with particular importance.

(i) **Smooth Function, f, and its Coordinate Representation** We found that for any smooth function $f : \ M \to \mathbb{R}^k$, we can work inside of the coordinate representation of the function f as the function $\hat{f} : f \circ \varphi^{-1} \to \mathbb{R}^k$. This means that although we are dealing with a coordinate free manifold, we can represent any smooth function f on the manifold using local coordinates with the chart $(U, \varphi(x^i))$.

(ii) **Smooth Maps Between Manifolds** Similar to the smooth function, given a smooth map $F : M \to N$ between smooth manifolds M and N, we can represent the function F using local coordinates as $\hat{F} = \psi \circ F \circ \varphi^{-1}$ and $F|_U = \psi^{-1} \circ \hat{F} \circ \varphi : \ U \to V$. Where (U, φ) and (V, ψ) are charts on M and N respectively.

We concluded this section by identifying the equivalence relations between two smooth manifolds as a diffeomorphism.

So at this point, we are able to work with smooth functions in a coordinate representation with a domain on the manifold and codomain of either $\mathbb{R}$, $\mathbb{R}^k$, or another manifold. Where $f : \ M \to \mathbb{R}$ called out separately as it is an important subset of $f : \ M \to \mathbb{R}^k$ that creates a vector space over $\mathbb{R}$, and is typically denoted as $C^\infty(M)$.

Also within this section we defined a Lie group and found that it is both a smooth manifold and a group. We went on to see that we could form additional lie groups through:

(a) Lie group homomorphisms,

(b) Embedded subgroups,

(c) Group actions,

(d) Equivariant maps, and

(e) Semidirect products

We introduced a number of important Lie groups, and showed how they can be represented as contained in either $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$. We found when the group has an isomorphism to either of these groups, there exists a faithful representation of the group G, where the representation $\rho : \ G \to GL(V)$ is injective for some finite-dimensional vector space, V.

Overall, this section provide a way for us to think about our smooth manifolds as groups, and this concept will serve us well as we move into further sections.

CHAPTER III

THE TANGENT SPACE

Conceptual Introduction

When we look at an object in space, we are able to identify how the object changes from one point to another near by point by considering the directional rate of change from the one point to the other, that is we can calculate the directional derivative. When working with an object embedded within the familiar Euclidean space, this directional derivative lives in the tangent space that is also embedded within the larger Euclidean space. However, in the case of manifolds this is not necessarily the case. We will see that, though not necessarily part of the manifold itself, there exists particular vector space associated with each point on a manifold that we call the tangent space for the point. Further, as we are interested in the manifold object itself, and not just a point on the manifold, we will find that there is a particular way we can associate the tangent space of one point to the tangent space of another.

In this chapter we will begin by reviewing the requisite concepts of geometric tangent vectors and vector spaces in order to establish some motivation for what these this could look like on a manifold. However, the main concept of this chapter is the tangent space to a point on the manifold. Beginning with tangent vectors in $\mathbb{R}^n$, we will construct 1:1 correspondence between geometric tangent vectors and linear maps (called derivations) from $C^\infty(\mathbb{R}^n)$ to $\mathbb{R}$ that satisfy the product rule. With this we will define a tangent vector on a smooth manifold as a derivation of $C^\infty(M)$ at a point. We then show how any smooth coordinate chart, (U, φ), gives a natural isomorphism from the space of tangent vectors to M at a point p to the space of tangent vectors to $\mathbb{R}^n$ at $\varphi(p)$. Thus, we find that a smooth coordinate chart forms a basis for each tangent space. The next section of this chapter will cover comparisons of one tangent space to another and provide us with a tool for establishing linear approximations to the manifold at a point. The last section will look at tangent vectors on a manifold considered as 'velocity' vectors of a curve on the manifold and provide a different perspective for our consideration of these objects.

Prerequisites

Geometric Tangent Vectors

Let $\mathbb{R}^n_a$ denote the geometric tangent space to $\mathbb{R}^n$ at a point a. Then a geometric tangent vector in $\mathbb{R}^n$ is an element $v_a \in \mathbb{R}^n_a$ for some point $a \in \mathbb{R}^n$. Note here that although $\mathbb{R}^n_a$ is essentially $\mathbb{R}^n$, $\mathbb{R}^n_a$ and $\mathbb{R}^n_b$ form disjoint sets.

We can take the directional derivative of the tangent vector v_a as a map $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$

$$\begin{aligned} D_v|_a f &= D_v f(a) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(a+tv) \end{aligned} \tag{3.1}$$

This definition of directional derivative is linear and satisfies the product rule

$$D_v|a(fg) = f(a)D_v|a(g) + g(a)D_v|a(f). \tag{3.2}$$

And if we let our vector be written in component form $v = v^i e_i$ where e_i is the standard basis, we can write the directional derivative more compactly

$$D_v|_a f = v^i \frac{\partial f}{\partial x^i}(a)\,. \tag{3.3}$$

Geometric Tangent Vector Space

With this review of the geometric tangent vector in mind we can define a derivation at a point.

Definition 3.1 (Derivation) *Let $a \in \mathbb{R}^n$. A map $\omega : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a derivation at a if*

(i) ω is linear over $\mathbb{R}$

(ii) ω follows the product rule: $\omega(fg) = f(a)\omega g + g(a)\omega f$

Note that a derivation is a linear functional. Now, Letting $T_a\mathbb{R}^n$ be the set of all derivations of $C^\infty(\mathbb{R}^n)$ at the point a, we find that $T_a\mathbb{R}^n$ forms a vector space we call the tangent space.

Proposition 3.2 *Let $a \in \mathbb{R}^n$.*

(i) For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation at point a.

(ii) The map $v_a \to D_v|a$ is an isomorphism from $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$

Now this brings us to

Corollary 3.3 *For any $a \in \mathbb{R}^n$, the n derivations*

$$\left.\frac{\partial}{\partial x^1}\right|_a, \cdots, \left.\frac{\partial}{\partial x^n}\right|_a \qquad \text{defined by} \qquad \left.\frac{\partial}{\partial x^i}\right|_a f = \frac{\partial f}{\partial x^i}(a)$$

form a n-dimensional basis for $T_a\mathbb{R}^n$.

Tangent Space to Manifolds

We begin by looking at a specific case of smooth maps that we will refer to as a smooth function.

Definition 3.4 (Smooth Function) *Let M be a smooth n-manifold, and $k \in \mathbb{N}$. A function $f : M \to \mathbb{R}^k$ is called a smooth function if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$.*

The most important of these functions is $f : M \to \mathbb{R}$ with the set of all such functions denoted as $C^\infty(M)$. Since sums and constant multiples of smooth functions are smooth, $C^\infty(M)$ is a vector space over $\mathbb{R}$.

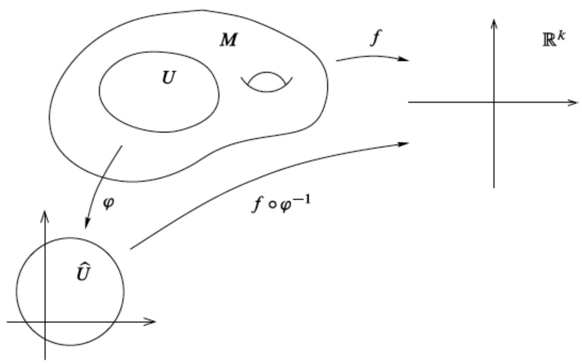

Figure 3.1: Smooth Functions

A function $f:\ M \to \mathbb{R}^k$ combined with a chart (U, φ) can provide a coordinate representation.

Definition 3.5 (Coordinate Representation of f) *Given a function $f:\ M \to \mathbb{R}^k$ and a chart (U, φ) for M, the function $\widehat{f}:\ \varphi(U) \to \mathbb{R}^k$ defined by $\widehat{f}(x) = f \circ \varphi^{-1}(x)$ is called the coordinate representation of f where $x \in \widehat{U}$.*

And so, by definition f is smooth if and only if its coordinate representation is smooth in some smooth chart around each point.

Now are able to carry over our definitions for derivations and tangent space from above directly to our manifold.

Definition 3.6 (Derivation on a Manifold) *Let M be a smooth manifold and $p \in M$ be a point in M. A linear map $v:\ C^\infty(M) \to \mathbb{R}$ is called a derivation at point p if:*

$$v(fg) = f(p)v(g) + g(p)v(f), \qquad \forall f, g \in C^\infty(M). \tag{3.4}$$

So the set of all derivations of $C^\infty(M)$ at p, T_pM, is a vector space we call the tangent space of M at p. Any element of T_pM is a tangent vector at p.

Linear Approximations of a Smooth Map on a Manifold

We first defined geometric tangent vectors from $T_a\mathbb{R}^n$, and then introduced the more abstract tangent spaces on our smooth manifold, T_pM. In order to discuss how these relate we will have to explore the way smooth maps affect tangent vectors. In the case of a smooth map between Euclidean spaces, the total derivative of the map at a point (represented by the Jacobian matrix) is a linear map that represents a linear approximation to the map near the given point. However, the concept of linear maps do not directly correlate to manifolds as a manifold need not be linear. Instead we look at linear maps between tangent spaces.

Definition 3.7 (Differential of F at p) *Let M and N be smooth manifolds with $F : M \to N$ as a smooth map between them. For each $p \in M$ we define the differential of F at p as the map*

$$dF_p :\ T_pM \to T_{F(p)}N \tag{3.5}$$

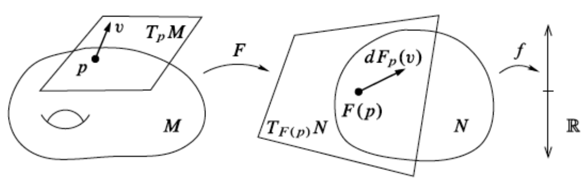

Figure 3.2: The Differential

With this definition, given a tangent vector $v \in T_pM$, we let $dF_p(v)$ be the derivation in the tangent space of N at $F(p)$ that acts on $f \in C^\infty(N)$

$$dF_p(v)(f) = v(f \circ F) \tag{3.6}$$

Recall that $C^\infty(N)$ is the set of all smooth functions $f :\ N \to \mathbb{R}$. So, if $F :\ M \to N$ is a smooth map, then $f \circ F :\ M \to \mathbb{R}$ is a smooth map. Further, the operator $dF_p(f) :\ C^\infty(N) \to \mathbb{R}$ is both (i) linear, and (ii) satisfies the product rule. Thus, the operator $dF_p(v)$ is a derivation at $F(p)$.

One Tangent Space to the Other

We can use the differential to associate the tangent space of a point on the manifold to Euclidean tangent space. To do so we will need to see that tangent vectors act locally.

Proposition 3.8 (Tangent Spaces Act Locally) *Let M be a smooth manifold, $p \in M$, and $v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p than $vf = vg$.*

Proposition 3.9 (The Tangent Space to an Open Submanifold) *Let M be a smooth manifold, $U \subseteq M$ be and open subset, and let $\iota : U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism.*

The first proposition tells us that even though the tangent space is defined in terms of smooth functions on the whole manifold and the coordinate charts act only on a subset of the manifold, the tangent vectors act locally. This means we can connect a tangent space in the submanifold defined by a coordinate chart to tangent spaces to the whole manifold. The second proposition shows how the differential of the inclusion map from T_pU to T_pM works as an isomorphism. This gives us a way to treat $d\iota_p(v)$ as though it is the same derivation as v. In particular, any tangent vector $v \in T_pM$ can be unambiguously applied to functions defined only in a neighborhood of p.

The Natural Identification for Tangent Spaces: Treating Vector Spaces, and the Tangent Space of Both Manifolds and Vector Spaces as the Same

We will see that if we are working with a vector space, since there is a natural smooth manifold associated with the vector space, we are able to associate the vector space, the tangent to the vector space at a point, p, and the tangent to the manifold at the point p

Proposition 3.10 (Dimension of the Tangent Space) *If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional vector space.*

Proposition 3.11 (The Tangent Space to a Vector Space) *Suppose V is a finite dimensional vector space with its standard smooth manifold structure. For each point $a \in V$, the map $v \to D_v|_a$ defined by*

$$D_v|_a = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f\,(a+tv)$$

is a canonical isomorphism from V to T_aV, such that for any linear map $L:\ V \to W$ the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\cong"] \arrow[d, "L"'] & T_aV \arrow[d, "dL_a"] \\ W \arrow[r, "\cong"'] & T_{L_a}W \end{tikzcd}$$

Figure 3.3: Isomorphism between Tangent Space and Vector Space

So if M is an open submanifold of a vector space V, then we can combine our identifications $T_pM \leftrightarrow T_pV \leftrightarrow V$ to obtain a canonical identification of each tangent space to M with V. There is also a natural identification of a tangent space to the product of manifolds.

Proposition 3.12 (The Tangent Space to a Product Manifold) *Let $M_1, \cdots, M_k$ be smooth manifolds, and for each j, let $\pi_j :\ M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \cdots, p_k) \in M_1 \times \cdots \times M_k$, the map*

$$\alpha :\ T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k \tag{3.7}$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \cdots, d(\pi_k)_p(x)) \tag{3.8}$$

is an isomorphism.

Thus we will identify $T_{(p,q)}(M \times N)$ with $T_pM \oplus T_qN$ and treat T_pM and T_qN as subspaces of $T_{(p,q)}(M \times N)$.

Tangent Vector as the Velocity of Curves

Our discussion of tangent vectors leads to a concept of velocity on a curve of the manifold.

Definition 3.13 (Curve in M) *If M is a manifold, a curve in M is a continuous map $\gamma : J \to M$, where $J \subseteq \mathbb{R}^n$ is an interval.*

So, if $t_0 \in J$, the velocity of curve γ at t_0 is simply the tangent space to the point $\gamma(t_0) \in M$.

$$\gamma'(t_0) = d\gamma\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) \in T_{\gamma(t_0)}M \tag{3.9}$$

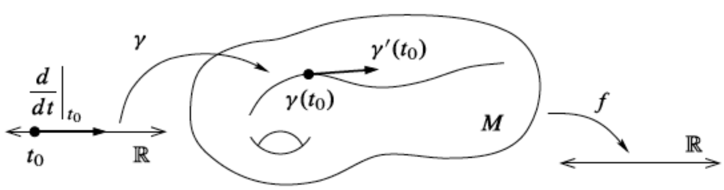

Figure 3.4: Velocity of a Curve

Notice here we have $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}$ instead of the partial derivative because it is the standard basis vector in $T_{t_0}\mathbb{R}$.

The velocity vector (as a tangent vector) acts on functions by

$$\begin{aligned} \gamma'(t_0)f &= d\gamma\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) f && (3.10)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} (f \circ \gamma) && (3.11)\\ &= (f \circ \gamma)'(t_0). && (3.12) \end{aligned}$$

So $\gamma'(t_0)$ is the derivation at $\gamma(t_0)$ obtained by taking the derivative of a function along γ.

If we have a smooth chart $(U,(x^i))$, we have the coordinate representation of γ as $\gamma(t) = (\gamma^1(t),\cdots,\gamma^n(t))$ for t sufficiently close to t_0, and so the coordinate formula for the differential (or velocity) is given by

$$\gamma'(t_0) = \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}(t_0)\,\frac{\partial}{\partial x^i}\bigg|_{\gamma(t_0)}. \tag{3.13}$$

Which tells us that the velocity of γ at t_0 is the tangent vector whose components in a coordinate basis are the derivatives of the component functions of γ.

It turns out that every tangent vector on M is the velocity of some curve.

Proposition 3.14 *Suppose M is a smooth manifold, and $p \in M$. Every $v \in T_pM$ is the velocity of some smooth curve in M.*

It turns out that this concept of velocity provides a useful way to determine the differential of a smooth function.

Proposition 3.15 (Computing the Differential Using a Velocity Vector) *Suppose $F: M \to N$ is a smooth map, $p \in M$ and $v \in T_pM$. Then,*

$$dF_p(v) = (F \circ \gamma)'(0) \tag{3.14}$$

for any smooth curve $\gamma\colon J \to M$ such that $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$.

Chapter Summary

Within this section we introduce the method to measure linear approximations on manifolds to obtain information about curvature. Although the section is structured to build the abstract foundations, the practical takeaways are best summarized by going in reverse.

We concluded this section by seeing that the velocity of a curve provides a convenient way to obtain the differential of a smooth map between smooth manifolds $F: M \to N$. Further the set of all curve velocities on a smooth n-manifold form a tangent bundle TM that is itself a smooth $2n$-manifold. The elements of TM are the ordered pairs containing the tangent vectors of M, $v \in T_pM$, and the point, $p \in M$. We found that the global differential is the smooth map $dF: TM \to TN$. We also found that there is a natural projection that comes with the structure of the tangent bundle $\pi: TM \to N$ that brings us from the tangent bundle to a particular tangent space at a given point.

We also saw that given a curve γ on a manifold parameterized by t in an interval J, with $t_0 \in J$, we can obtain the velocity, $\dot{\gamma}$ at a point, $\gamma(t_0)$, that provides us with the differential of a smooth map between manifolds. We can then use the differential in coordinates to obtain

(i) the coordinate free Jacobian matrix providing us with the total differential of the function F providing the best linear approximation of the manifold at a given point.

(ii) The change of coordinates between two smooth charts on the manifold that contain the same point.

CHAPTER IV

COORDINATE REPRESENTATIONS

Conceptual Introduction

We have seen that a manifold is an object that is locally flat containing possibly many different coordinate charts, that each point on a manifold comes with a tangent space, and that we can discuss the relationship between tangent space through the notion of a differential. However, all of these concepts do not rely on any particular choice of coordinates. In this chapter we revisit these topics, but focus on how the coordinate representation works.

Tangent Vectors in Coordinates

We first construct the coordinate vectors using the knowledge we have obtained from the previous section.

a We start with a smooth manifold, M, and a smooth chart (U,φ).

b Since (U,φ) is smooth, it is a member of a maximal atlas, and thus by definition for any $f:\ M\to\mathbb{R}$ ($f\in C^\infty(M)$), $f\circ\varphi^{-1}$ is smooth. And in particular φ is a diffeomorphism from U to an open subset $\hat{U}\subseteq\mathbb{R}^n$.

c Since φ is a diffeomorphism, we know that $d\varphi_p:\ T_pU\to T_{\varphi(p)}\mathbb{R}^n$.

d Further, we know that the differential of the inclusion map $d\iota_p:\ T_pU\to T_pM$ is an isomorphism. So $d\varphi_p:\ T_pM\to T_{\varphi(p)}\mathbb{R}^n$.

e We also found that the derivations $\left.\frac{\partial}{\partial x^1}\right|_{\varphi(p)},\cdots,\left.\frac{\partial}{\partial x^n}\right|_{\varphi(p)}$ form a basis of $T_{\varphi(p)}\mathbb{R}^n$. So, the preimage of these vectors form a basis of T_pM.

f We will denote these preimage vectors as one of the following:

$$\left.\frac{\partial}{\partial x^i}\right|_p = (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = (d\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right). \tag{4.1}$$

g Unwinding this definition we find that the preimage vector acts on a function $f \in C^\infty(U)$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} (f \circ \varphi^{-1}) = \frac{\partial \hat{f}}{\partial x^i}(\hat{p}), \tag{4.2}$$

where $\hat{f} = f \circ \varphi^{-1}$ is the coordinate representation of f, and $\hat{p} = (p^1, \cdots, p^n) = \varphi(p)$ is the coordinate representation of p.

h So, the preimage vectors $\left.\frac{\partial}{\partial x^i}\right|_p$ are just the derivation that takes the ith partial derivative of the coordinate representation of f at the coordinate representation of p. We call these vectors the coordinate vectors at p associated with the given coordinate system.

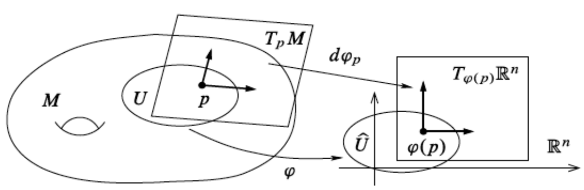

Figure 4.1: Tangent Vectors in Coordinates

This is summarized in the following proposition.

Proposition 4.1 *Let M be a smooth n-manifold, and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart $(U, (x^i))$ containing p, the coordinate vectors $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ form a basis for T_pM.*

Thus we see that a tangent vector $v \in T_pM$, can be represented as a linear combination

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p. \tag{4.3}$$

We call the ordered basis $(\partial/\partial x^i|_p)$ the coordinate basis for T_pM, and the numbers (v^i) the components of v (with respect to the coordinate basis).

If v is known, its components can be computed from its action on the coordinate functions. That is, for each j we have,

$$v(x^j) = \left(v^i \left. \frac{\partial}{\partial x^i} \right|_p \right) (x^j) = v^i \frac{\partial x^j}{\partial x^i} (p) = v^j. \tag{4.4}$$

Where the last equality follows from the fact that $\frac{\partial x^j}{\partial x^i}$ is zero for all $i \neq j$, and 1 when $i = j$.

Differential in Coordinates

To see how differentals look in coordinates we start by exploring mappings from tangents in Euclidean space. Let $F : U \to V$ be a smooth map where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open subsets of Euclidean space. For any $p \in U$, we can determine the matrix $dF_p : \; T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ in terms of the standard coordinate basis. Letting f be an arbitrary function, and with (x^i) as domain coordinates and (y^j) codomain coordinates, we can compute the action of dF_p on a typical basis vector as

$$dF_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) f = \left. \frac{\partial}{\partial x^i} \right|_p (f \circ F) \tag{4.5}$$

$$= \frac{\partial f}{\partial y^j} (F(p)) \frac{\partial F^j}{\partial x^i} (p) \tag{4.6}$$

$$= \left(\frac{\partial F^j}{\partial x^i} (p) \left. \frac{\partial}{\partial y^j} \right|_{F(p)} \right) f. \tag{4.7}$$

Thus, we have

$$dF_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \frac{\partial F^j}{\partial x^i} (p) \left. \frac{\partial}{\partial y^j} \right|_{F(p)}. \tag{4.8}$$

Or as in matrix form we have

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^i}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^i}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}, \tag{4.9}$$

which is just the Jacobian matrix of F at p. That is, it is the matrix representation of the total derivative $DF(p): \mathbb{R}^n \to \mathbb{R}^m$.

Now following the above procedure for the more general case of $F: M \to N$ where M, N are smooth manifolds. Choosing smooth coordinate charts (U, φ) containing $p \in M$ and (V, ψ) containing $F(p)$, we have coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1}: \varphi(U \cap F^{-1}(V)) \to \psi(V)$ and $\hat{p} = \varphi(p)$. So we get

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = dF_p\left(d(\varphi^{-1})_{\hat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}}\right)\right) \tag{4.10}$$

$$= d(\psi^{-1})_{\hat{F}(\hat{p})}\left(d\hat{F}_{\hat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}}\right)\right) \tag{4.11}$$

$$= d\left(\psi^{-1}\right)_{\hat{F}(\hat{p})}\left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \left.\frac{\partial}{\partial y^j}\right|_{\hat{F}(\hat{p})}\right) \tag{4.12}$$

$$= \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \left.\frac{\partial}{\partial y^j}\right|_{\hat{F}(\hat{p})} \tag{4.13}$$

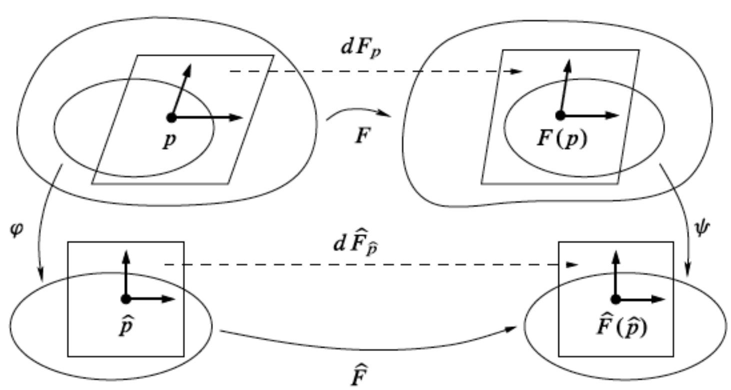

Figure 4.2: Differential in Coordinates

This result tells us that dF_p is represented in coordinate bases by the Jacobian matrix of the coordinate representation of F. That is we now have coordinate-free representation of the Jacobian matrix.

Change of Coordinates

We now look at how the coordinate change from one smooth chart to another works. If we have our smooth manifold M that has two smooth charts (U,φ) and (V,ψ) that each contain point $p \in M$, we can express these charts in terms of their coordinate representations as $(U,(x^i))$ and $(V,(\tilde{x}^i))$. Since we have two choices of coordinate representation at point p, any tangent vector at point p can be represented by either the basis $(\partial/\partial x^i|_p)$ or the basis $(\partial/\partial \tilde{x}^i|_p)$.

Now we have our transition map from φ to ψ defined earlier as $\psi \circ \varphi^{-1} : \ \varphi(U \cap V) \to \psi(U \cap V)$. Our differential $d(\psi \circ \varphi^{-1})|_{\varphi(p)}$ becomes

$$d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\left. \frac{\partial}{\partial x^i} \right|_{\varphi(p)} \right) = \frac{\partial \tilde{x}^j}{\partial x^i} (\varphi(p)) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_{\psi(p)}. \tag{4.14}$$

With this differential in hand, from the definition for coordinate vectors we get

$$\begin{aligned}
\left. \frac{\partial}{\partial x^i} \right|_p &= d(\varphi^{-1})|_{\varphi(p)} \left(\left. \frac{\partial}{\partial x^i} \right|_{\varphi(p)} \right) && (4.15)\\
&= d(\psi^{-1})|_{\psi(p)} \circ d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\left. \frac{\partial}{\partial x^i} \right|_{\varphi(p)} \right) && (4.16)\\
&= d(\psi^{-1})|_{\psi(p)} \left(\frac{\partial \tilde{x}^j}{\partial x^i} (\varphi(p)) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_{\psi(p)} \right) && (4.17)\\
&= \frac{\partial \tilde{x}^j}{\partial x^i} (\hat{p}) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_p. && (4.18)
\end{aligned}$$

Applying this to $v = v^i \left.\frac{\partial}{\partial x^i}\right|_p = \tilde{v}^j \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p$ we get

$$v^i \left.\frac{\partial}{\partial x^i}\right|_p = v^i \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p \tag{4.19}$$

$$\Rightarrow v^i \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p = \tilde{v}^j \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p \tag{4.20}$$

$$\Rightarrow \tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p})\, v^i, \tag{4.21}$$

which provides a rule for coordinates to transform.

Chapter Summary

In this chapter we saw how to represent tangent vectors and differentials in coordinates as well as how to perform a change in coordinates. Specifically, we found that

a) Each tangent vector $v \in T_pM$, can be represented by the linear combination $v = v^i \left.\frac{\partial}{\partial x^i}\right|_p$, where $\left.\left(\partial/\partial x^i\right)\right|_p$ is the coordinate basis for T_pM.

b) The differential is represented in coordinates as

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial \widehat{F}^j}{\partial x^i} \left.\frac{\partial}{\partial y^j}\right|_{\widehat{F}(\widehat{p})}, \tag{4.22}$$

which tells us that the differential is the manifold version of the Jacobian matrix or total derivative that provides a linear approximation.

c) A vector on the manifold is the same object regardless of the chosen coordinate chart, and so given two coordinates x^i and $\widetilde{x}^j$ we have

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p = v^j \left.\frac{\partial}{\partial x^j}\right|_p, \tag{4.23}$$

which in turn tells us that we can transform from one coordinate to another by

$$v^j = \frac{\partial j \widetilde{x}^j}{\partial x^i}(\widehat{p})\, v^i. \tag{4.24}$$

CHAPTER V

MANIFOLDS FROM MANIFOLDS

Conceptual Introduction

In this chapter we will take a look at the tangent bundle manifold, relationships between manifolds, submanifolds, and Lie groups. The main objective for this chapter is to introduce a number of ways wayst that manifolds relate to each other and specific cases where we can even generate new manifolds from existing ones.

We start out as usual with a number of preliminary concepts related to topology, linear algebra, and group theory that will be leveraged in the main sections.

Prerequisites

Recall that the rank of a linear transformation is the dimension of the image under the transformation. Further, if we are free to choose bases independently for the domain and codomain, then rank is the only property that distinguishes linear maps.

Definition 5.1 (Rank of Maps Between Manifolds) *Let M and N be smooth manifolds with $p \in M$. If $F : M \to N$ is a smooth map, we define the rank of F at p to be the rank of the linear map*

$$dF_p : T_pM \to T_{F(p)}N.$$

Because the rank of a linear map is never higher than the dimension of either its domain or codomain, it is bounded from above. If the rank of dF_p is equal to this upper bound, we say that F has full rank at p, and if it has full rank everywhere, we say F has full rank. If F has the same rank, r, at every point we say that it has constant rank.

Definition 5.2 (Level Set) *Let $\Phi: M \to N$ be any map and $c \in N$, we call the set $\Phi^{-1}(c)$ a level set of Φ.*

Recall that a topological space X is *disconnected* if it has two disjoint nonempty open subsets whose union is X, otherwise it is considered *connected*. A *connected subset* of X is a subset that is a connected space when endowed with the subspace topology. A *connected component* of X is a connected subset of X that is not properly contained in any larger connected subset.

Two more important topological notions are that of embeddings and proper maps.

Definition 5.3 (Topological Embedding) *Let X and Y be topological spaces with a continuous injective map $F: X \to Y$ between them. F is called a topological embedding if it is a homeomorphism onto its image $F(X) \subseteq Y$ in the subspace topology.*

Definition 5.4 (Conditions for Proper Maps) *Let X and Y be topological spaces with $F: X \to Y$ as a continuous map. Than the following are sufficient conditions for F to be considered proper:*

(i) if X is compact and Y is Hausdorff,

(ii) if F is a closed map with compact fibers (the preimage is compact for every $y \in Y$), and

(iii) if F is a topological embedding with a closed image.

An action by a Lie group on another manifold makes up an important application of Lie groups to smooth manifolds, and so it is important to review the notions of group actions.

Definition 5.5 (Group Action on Manifold) *Let G be a group and M a set. A left action of G on M is a map $G \times M \to M$, often written as $(g, p) \to g \cdot p$, that satisfies*

(i) $g_1 \cdot (g_2 p) = (g_1 g_2) \cdot p$ for all $g_1, g_2 \in G$ and all $p \in M$;

(ii) $e \cdot p = p$ for all $p \in M$.

And a right action of G on M is a map $M \times G \to M$ that satisfies

(i) $(p \cdot g_1) \cdot g_2 = p \cdot (g_1 g_2)$ for all $g_1, g_2 \in G$ and for all $p \in M$;

(ii) $p \cdot e = p$ for all $p \in M$

If the group and set have additional structures, such as topological or manifold structures, we find:

(a) If G happens to be a group with a topology (that is a topological group) and M is a topological space, then the action is continuous if the map $F: G \times M \to M$ (or $F: M \times G \to M$ for right actions) is continuous.

(b) Further if G is a Lie group and M is a smooth manifold, then the action $\theta : G \times M \to M$ is said to be a smooth action.

If instead of looking at the global action, we want to focus on the action of a group element $g \in G$ on a point $p \in M$ we denote it $\theta_g(p)$.

A few noteworthy actions include:

- for each $p \in M$ the orbit of p, denoted by $G \cdot p$ is the set of all images of p under the action by elements of G:

$$G \cdot p = \{g \cdot p : g \in G\}$$

- for each $p \in M$, the isotropy group (or stabilizer of p), denoted by G_p is the set of elements of G that fix p:

$$G_p = \{g \in G : g \cdot p = p\}$$

The action is said to be free if every isotropy group is trivial

Note that for the general definition of group action, the additional properties of continuity or smoothness are not implied. Further, when thinking about smooth actions, we are really considering smooth maps between smooth manifolds. So, in this case for a group action, we need a group that is also a smooth manifold, thus we are looking at a Lie group.

Tangent Bundle

Recall that the set of all tangent vectors at a point on the manifold is called the tangent space. It turns out that the set of all tangent spaces on a manifold is itself a manifold itself, called the tangent bundle. Although important for many reasons, for now it is important to recognize that the tangent bundle counts as a manifold in our discussion maps between manifolds above.

Definition 5.6 (Tangent Bundle) *Let M be a smooth manifold. The tangent bundle of M, denoted TM, is the disjoint union of the tangent spaces at all points of M,*

$$TM = \bigsqcup_{p \in M} T_pM.$$

An element of the tangent bundle is often represented as the ordered pair (p, v) where the first element $p \in M$ is the point on the manifold, and the second $v \in T_pM$ is the particular vector from the tangent space of the point.

There is a natural projection map on the tangent bundle, $\pi : \; TM \to M$, which sends each vector to the point at which it is tangent: $\pi(p, v) = p$. In the case of a smooth manifold, this turns out to be more than just a collection of vector spaces.

Proposition 5.7 (Tangent Bundle on a Smooth Manifold is a Manifold Itself) *For an smooth n-manifold, M, the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection $\pi : \; TM \to M$ is smooth.*

Just as the differential of a smooth function $dF \; : \; T_pM \to T_qN$, we can take the global differential of the tangent bundle, $dF : \; TM \to TN$ which makes a smooth map between tangent bundles.

Proposition 5.8 *If $F : \; M \to N$ is a smooth map, then its global differential $dF : \; TM \to TN$ is a smooth map.*

Further, if F is a diffeomorphism, then the global differential is a diffeomorphism too.

Types of Relations between Manifolds

The differential provides a local approximation. The manifolds that provide the best local approximations have a particular relationship to each other. Submersion, Immersion and Embedding. The characteristic feature to look for in these cases is a constant rank (i.e., dimension of the image).

Submersions and Immersions

Definition 5.9 (Smooth Submersion) *A smooth map $F : M \to N$ is called a smooth submersion if its differential is surjective at each point (or equivalently if rank F = dim N*

Definition 5.10 (Smooth Immersion) *A smooth map $F : M \to N$ is called a smooth immersion if its differential is injective at each point (or equivalently if rank F = dim M*

Proposition 5.11 (Submersions and Immersions) *Let M and N be smooth manifolds with $p \in M$ and $F : M \to N$ be smooth. If dF_p is surjective, then p has a neighborhood U such that $F|_U$ is a submersion (that is, $rank(F) = dim(N)$). If dF_p is injective, then p has a neighborhood U shuch that $F|_U$ is an immersion (that is, $rank(F) = dim(M)$).*

Theorem 5.12 (Global Rank Theorem) *Let M and N be smooth manifolds, and suppose $F : M \to N$ is a smooth map of constant rank. Then,*

(a) If F is surjective, then it is a smooth submersion.

(b) If F is injective, then it is a smooth immersion.

(c) If F is bijective, then it is a diffeomorphism.

Theorem 5.13 (Local Immersion Theorem for Manifolds with Boundary) *Suppose that M is a smooth m-manifold with boundary, N is a smooth n-manifold, and $F : M \to N$ is a smooth immersion. For any $p \in \partial M$, there exists a smooth boundary chart (U, ϕ) for M centered at p and a smooth coordinate chart (V, ψ) for N centered at $F(p)$ with $F(U) \subseteq V$ in which F has the coordinate representation:*

$$\widehat{F}\left(x^1, \ldots, x^m\right) = \left(x^1, \ldots, x^m, 0, \ldots, 0\right).$$

Embeddings

We find the particular case of an immersion where there is also a topological embedding of particular interest.

Definition 5.14 (Smooth Embedding) *Let M and N be smooth manifolds. A smooth embedding is a smooth immersion, $F : M \to N$, that is also a topological embedding.*

Proposition 5.15 (Criteria for Smooth Embedding) *Suppose M and N are smooth manifolds and $F : M \to N$ is an injective smooth immersion. If any of the following holds, then F is a smooth embedding.*

(i) F *is an open or closed map.*

(ii) F *is a proper map.*

(iii) M *is compact.*

(iv) M *has empty boundary and* $dim(M) = dim(N)$.

Theorem 5.16 (Local Embedding Theorem) *Suppose M and N are smooth manifolds, and $F : M \to N$ is a smooth map. Then F is a smooth immersion if and only if every point in M has a neighborhood $U \subseteq M$ such that $F|_U : U \to N$ is a smooth embedding.*

Theorem 5.17 (Rank Theorem) *Let M and N be smooth manifolds with dimensions m and n respectively, and let $F : M \to N$ be a smooth map with constant rank r. Then, for each $p \in M$ there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form*

$$\hat{F}(x^1, \cdots, x^r, x^{r+1}, \cdots, x^m) = (x^1, \cdots, x^r, 0, \cdots, 0). \tag{5.1}$$

In particular, if F is a smooth submersion, this becomes

$$\hat{F}(x^1, \cdots, x^r, x^{r+1}, \cdots, x^m) = (x^1, \cdots, x^n). \tag{5.2}$$

and if F is a smooth immersion, it is

$$\hat{F}(x^1,\cdots,x^m) = (x^1,\cdots,x^m,0,\cdots,0). \tag{5.3}$$

Submanifolds

We begin with embedded submanifolds.

Definition 5.18 (Embedded Submanifold) *Let M be a smooth manifold with or without boundary. An embedded submanifold, or regular submanifold, of M is a subset $S \subseteq M$ that is a manifold, without boundary, in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \overset{M}{\hookrightarrow}$ is a smooth embedding.*

In the case where $S \subseteq M$ is a manifold with boundary, we have an *embedded smooth manifold with boundary*.

If the inclusion map is a proper map, then we say that the embedding is a *proper embedding*.

The general case of an embedded submanifold is the immersed submanifold.

Definition 5.19 (Immersed Submanifold) *Let M be a smooth manifold with or without boundary. An immersed submanifold of M is a subset $S \subseteq M$ endowed with a topology that makes it a topological manifold, and a smooth structure with respect to which the inclusion map $S \overset{M}{\hookrightarrow}$ is a smooth immersion.*

Just as in the case of the embedded submanifold, if $S \subseteq M$ is a topological manifold with boundary, then we say it is an *immersed submanifold with boundary*.

We define the codimension of S in M as $dim\, M - dim\, S$.

Theorem 5.20 (Constant-Rank Level Set Theorem) *Let M and N be smooth manifolds, and let $\Phi :\ M \to N$ be a smooth map with constant rank r. Each level set of Φ is a properly embedded submanifold of codimension r in M.*

Proposition 5.21 (Images of Immersions as Submanifolds) *Suppose M and N are smooth manifolds with $F: N \to M$ as an injective smooth immersion. Let $S = F(N)$. Then S has a unique topology and smooth structure such that it is a smooth submanifold of M and such that $F: N \to S$ is a diffeomorphism onto its image.*

We next take a look at a particularly important example of a properly embedded submanifold.

Definition 5.22 (Regular Domain) *Let M be a smooth manifold with or without boundary. A properly embedded codimension-0 submanifold with boundary is called a regular domain in M.*

Proposition 5.23 *Suppose M is a smooth manifold without boundary and $D \subseteq M$ is a regular domain. The topological interior and boundary o D are equal to its manifold interior and boundary, respectively.*

We conclude this section with some properties for submanifolds with boundaries.

Proposition 5.24 *Let M be a smooth manifold with or without boundary.*

(i) *Every open subset of M is an embedded codimension-0 submanifold with (possibly empty) boundary.*

(ii) *If N is a smooth manifold with boundary and $F: N \to M$ is a smooth embedding, then with the subspace topology $F(N)$ is a topological manifold with boundary, and it has a smooth structure making it into an embedded submanifold with boundary in M.*

(iii) *An Embedded submanifold with boundary in M is properly embedded if and only if it is closed.*

(iv) *If $S \subseteq M$ is an immersed submanifold with boundary, then for each $p \in S$ there exists a neighborhood of U of $p \in S$ that is embedded in M.*

Lie Group Homomorphisms

Recall that a group homomorphism is a map, F, between two groups $(G,\cdot)$ and $(H,*)$ such that $F(g_1\cdot g_2) = F(g_1) * F(g_2)$, for $\forall g_1, g_2 \in G$, and if F is bijective it is called an isomorphsim.

Similarly, if G and H are Lie groups, F is a Lie group homomorphism if F is smooth and G and H are isomorphic Lie groups if F^{-1} is smooth (that is, if G and H are diffeomorphic).

Theorem 5.25 *Every Lie group homomorphism has constant rank*

And a related corollary that follows from this and the global rank theorem is

Corollary 5.26 *A Lie group homomorphism is a Lie group isomorphism if and only if it is bijective.*

An interesting Lie group homomorphism is the conjugation map.

Definition 5.27 (Conjugation by $g\in G$) *If G is a Lie group with $g\in G$, the conjugation by g is the map $C_g:\ G\to G$ given by $C_g(h) = ghg^{-1}$.*

Since group multiplication and inversion are smooth, C_g is smooth. Further if $h_1, h_2 \in G$,

$$C_g(h_1h_2) = g\,(h_1h_2)\,g^{-1} \tag{5.4}$$

$$= gh_1\left(g^{-1}g\right)h_2g^{-1} \tag{5.5}$$

$$= \left(gh_1g^{-1}\right)\left(gh_2g^{-1}\right) \tag{5.6}$$

$$= C_g(h_1)C_g(h_2) \tag{5.7}$$

So, C_g is a homomorphism. Further, since $C_{g^{-1}}$ is the inverse, C_g is an isomorphism. Worth noting, is that a subgroup $H\subseteq G$ is said to be normal if $C_g(H) = H$ for every $g\in G$.

Lie Subgroups

From our earlier discussion we see that a Lie subgroup of G is a subgroup of G endowed with a topology and smooth structure making it into a Lie group and an immersed submanifold of G.

Proposition 5.28 (Embedded Subgroups are Lie Subgroups) *Let G be a Lie group, and suppose $H \subseteq G$ is a subgroup that is also an embedded submanifold, then H is a Lie subgroup.*

Recall that if G is a group and $S \subseteq G$, then the subgroup generated by S is the smallest subgroup containing S (i.e., the intersection of all subgroups containing S).

Proposition 5.29 *Suppose G is a Lie group, and $W \subseteq G$ is any neighborhood of the identity.*

(i) W generates an open subgroup of G.

(ii) If W is connected, it generates a connected open subgroup of G.

(iii) If G is connected, then W generates G.

Example 5.30 (Given G is connected) *With $det(G) > 0$ let*

$$G = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{5.8}$$

It is clear that $G \in GL(2,\mathbb{R})$, so G is a connected Lie group. If we take an interval W of the identity we have:

$$\begin{aligned} W &= \left(\begin{pmatrix} a-\varepsilon & b \\ c & d-\varepsilon \end{pmatrix}, \begin{pmatrix} a+\varepsilon & b \\ c & d+\varepsilon \end{pmatrix} \right), && (5.9) \\ &= (G-\varepsilon, G+\varepsilon) && (5.10) \end{aligned}$$

If we look at a translation $\varphi : G \to G$ defined as $\varphi(G) = G + \varepsilon$ we find

$$W = \left(\varphi(G), \varphi^{-1}(G)\right) \tag{5.11}$$

$$= G \tag{5.12}$$

Thus W generates G

$\square$

Example 5.31 (Given G is not connected) *Consider*

$$G = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{5.13}$$

with $det(G) \neq 0$ and W as an interval around the identity. Then G is not connected since it is the union of two connected components corresponding to the positive and negative values of the determinant. So we cannot use (iii) to show that W generates G.

$\square$

If G is a Lie group, the connected component of G containing the identity is called the identity component of G.

Proposition 5.32 *Let G be a Lie group and let G_0 be its identity component. Then G_0 is a normal subgroup of G, and is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0.*

The below two propositions show how to produce Lie groups from more than just subgroups.

Proposition 5.33 *Let $F: G \to H$ be a Lie group homomorphism. The kernel of F is a properly embedded Lie subgroup of G, whose codimension is equal to the rank of F.*

Proposition 5.34 *If $F : G \to H$ is an injective Lie group homomorphism, the image of F has a unique smooth manifold structure such that $F(G)$ is a Lie subgroup of H and $F : G \to F(G)$ is a Lie group isomorphism.*

For Lie subgroups the property of closedness and embeddedness are not independent. That is, every embedded Lie subgroup is properly embedded.

Theorem 5.35 *Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. Then, H is closed in G if and only if it is embedded.*

Equivariant Maps

Definition 5.36 (Equivariant Map) *Let G be a Lie group, and let M,N be smooth manifolds endowed with smooth left or right G-actions. $F : M \to N$ is said to be equivariant with respect to the given G-action, if for each $g \in G$,*

$$F(g \cdot p) = g \cdot F(p) \qquad \textit{(for left actions)} \tag{5.14}$$

$$F(p \cdot g) = F(p) \cdot g \qquad \textit{(for right actions)} \tag{5.15}$$

This class of maps proves useful in determining if certain maps have a constant rank.

Theorem 5.37 (Equivariant Rank Theorem) *Let M and N be smooth manifolds and let G be a Lie group. Suppose $F : M \to N$ is a smooth map that is equivariant with respect to a transitive smooth G-action on M and any smooth G-action on N. Then F has constant rank.*

Thus, if F is surjective, it is a smooth submersion; if it is injective, it is a smooth immersion; and if it is bijective it is a diffeomorphism.

The orbit map is an important example of the equivariant rank theorem.

Definition 5.38 (Orbit Map) *Suppose G is a Lie group, M is a smooth manifold, and $\theta : G \times M \to M$ is a smooth left action. The orbit map $\theta^{(p)} : G \to M$is defined by*

$$\theta^{(p)}(g) = g \cdot p, \qquad \forall p \in M$$

Proposition 5.39 (Properties of the Orbit Map) *Suppose θ is a smooth left action of a Lie group G on a smooth manifold M. For each $p \in M$, the orbit map $\theta^{(p)} : G \to M$ is smooth and has constant rank, so the isotropy group $G_p = \left(\theta^{(p)}\right)^{-1}$ is a properly embedded Lie subgroup of G. If $G_p = \{e\}$, then $\theta^{(p)}$ is an injective smooth immersion, so the orbit $G \cdot p$ is an immersed submanifold of M.*

Semidirect Products

With group actions, we now have a new tool for constructing Lie groups. First lets define the semidirect product.

Definition 5.40 (Semidirect Product) *Let H, N be Lie groups, and $\theta : H \times N \to N$ be a smooth left action of H on N. If for each $h \in H$, the map $\theta_h : N \to N$ is a group automorphism of N we say θ is an action by automorphism.*

Given such an action by automorphism, we define a new Lie group $N \rtimes_\theta H$, called a semidirect product of H and N as a smooth manifold where $N \rtimes_\theta H$ is just the Cartesian product $N \times H$; but the group multiplication is defined by

$$(n,h)(n',h') = (n\theta_h(n'), hh'). \tag{5.16}$$

We will see that semidirect products contain properties that define subsets as Lie groups.

Proposition 5.41 (Properties of Semidirect Products) *Suppose N and H are Lie groups, and θ is a smooth action of H on N by automorphisms. Let $G = N \rtimes_\theta H$.*

- *(i) The subsets $\widetilde{N} = N \times \{e\}$ and $\{e\} \times \widetilde{H}$ are closed Lie subgroups of G isomorphic to N and H respectively.*
- *(ii) $\widetilde{N}$ is a normal subgroup of $\widetilde{G}$*
- *(iii) $\widetilde{N} \cap \widetilde{H} = \{(e,e)\}$ and $\widetilde{N}\widetilde{H} = G$*

And it turns out that many Lie groups can be generated by semidirect products of subgroups.

Proposition 5.42 (Characterization of Semidirect Products) *Suppose G is a Lie group, $N, H \subseteq G$ are closed Lie subgroups such that N is normal, $N \cap H = \{e\}$, and $NH = G$. Then, the map $(n,h) \rightarrow nh$ is a Lie group isomorphism between $N \rtimes_\theta H$ and G, where $\theta : H \times N \rightarrow N$ is the action by conjugation: $\theta_h(n) = hnh^{-1}$.*

Lie Group Examples

We have thus far examined a number of ways to generate Lie groups from other groups. In this section we demonstrate a hierarchy of very important groups using some of the methods discussed above.

Example 5.43 (General Linear Group, $GL(n)$) *We begin by the largest and most general Lie group called the general linear group, $GL(n)$. The general linear group is the set of all $n \times n$ invertible matrices. In essence it is any linear transformation in n-dimensional space. Within the general linear group, the only thing preserved or invariant are the points in space themselves.*

The following are examples of Lie groups obtained from embedded sub groups of the general linear group.

Example 5.44 (Complex General Linear Group, $GL(n,\mathbb{C})$) *Our next example is formed by considering the most general field of numbers we can take from $GL(n)$. It turns out that this is the complex numbers, $\mathbb{C}$. So the subset of $GL(n)$ such that the elements are invertible $n \times n$ matrixies with complex entries.*

We see that $GL(n,\mathbb{C})$ is an open submanifold of the vector space $M(n,\mathbb{C})$ and thus it is a $(2n)^n$-dimensional smooth manifold. It is a Lie group because the matrix products and inverses are smooth functions of the real and imaginary parts of the matrix enteries.

Example 5.45 (Real General Linear Group) *We find with similar reasoning that the set of all invertible $n \times n$ matrices with real entries, $GL(n,\mathbb{R})$, is a Lie subgroup of $GL(n,\mathbb{C})$.*

We can also find Lie groups from homomorphisms.

Example 5.46 (Special Linear, $SL(n,\mathbb{R})$) *The set of all $n \times n$ real-valued matrices with determinant of 1 is called the special linear group of degree n, $SL(n,\mathbb{R})$. If we look at the determinant homomorphism, det : $GL(n,\mathbb{R}) \to \mathbb{R}^*$, where R^* is the set of all non-zero reals. We find the kernal of the homomorphism is a properly embedded Lie subgroup.* $(prop\, 1.3.5)$

Example 5.47 (Special Linear, $SL(n,\mathbb{C})$) *Similarlly to the above we see that the set of all $n \times n$ complex-valued matrices with determinant of 1, called the special linear group of degree n, $SL(n,\mathbb{C})$, is a Lie subgroup of $GL(n,\mathbb{C})$.*

Finally we see that there are a number of Lie groups obtained as the result of group actions.

Example 5.48 (The Orthogonal Group, $O(n)$) *A real $n \times n$ matrix is considered orthogonal if as a linear map $A: \mathbb{R}^n \to \mathbb{R}^n$ it preserves the inner product:*

$$(Ax) \cdot (Ay) = x \cdot y, \qquad \text{for all } x, y \in \mathbb{R}^n \tag{5.17}$$

If we define a smooth map $\Phi: GL(n,\mathbb{R}) \to M(n,\mathbb{R})$ by $\Phi(A) = A^T A$. Then the orthogonal group, $O(n)$ is equal to the level set $\Phi^{-1}(I_n)$. From the equivariant rank theorem it can be shown that $O(n)$ is an embedded Lie group

Example 5.49 (Special Orthogonal Group, $SO(n)$) *The special orthogonal group, $SO(n)$, is defined as*

$$SO(n) = O(n) \cap SL(n,\mathbb{R}) \subseteq GL(n,\mathbb{R}) \tag{5.18}$$

So, $SO(n)$ is an open subgroup of $O(n)$ consisting of matrices of positve determinant, and thus is an embedded Lie group.

Example 5.50 (Unitary Group, $U(n)$) *For any complex matrix A, tha adjoint of A is the matrix A^* formed by conjiugating the enteries of A and taking the transpose.*

$$A^* = \overline{A}^T \tag{5.19}$$

Observe that $(AB)* = (\overline{AB})^T = \overline{B}^T\overline{A}^T = B^*A^*$. So, for any positive integer n, the unitary group of degree n is the subgroup $U(n) \subseteq GL(n,\mathbb{C})$ consisting of complex $n \times n$ matrices whhose columns form an orthonormal basis for $\mathbb{C}^n$ with respect to the Hermitian dot product $z \cdot w = \sum_i z^i \overline{w}^i$. That is, $U(n)$ consists of matrices A such that $A^*A = I_n$.

Example 5.51 (Special Unitary Group, $SU(n)$) *The special unitary group, $SU(n)$ is defined by $SU(n) = U(n) \cap SL(n,\mathbb{C})$ we can see that $SU(n)$ is clearly embedded within $U(n)$ and $SL(n,\mathbb{C})$. Further, since inclusion maps show that the composition of $SU(n) \hookrightarrow U(n) \hookrightarrow GL(n,\mathbb{C})$ we find that $SU(n)$ is also embedded in $GL(n,\mathbb{C}$*

These above examples can be represented by the following group hierarchy.

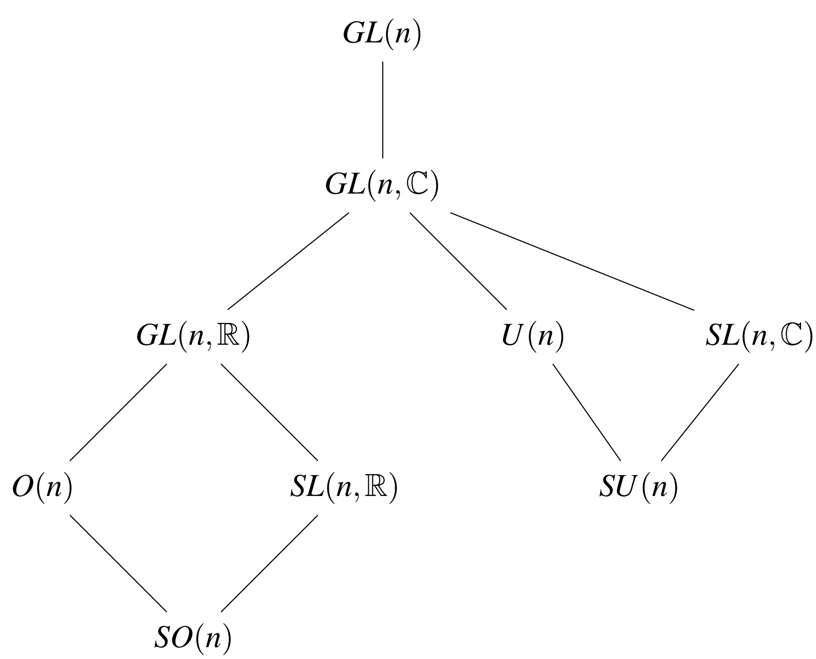

Figure 5.1: Group Hierarchy

Representations

We see from our examples that many groups can be realized as some subgroup of $GL(n)$ which can lead to the natural question of if all Lie groups are of this form. In order to understand this question we must introduce the notion of group representations.

Recall, if V be a finite dimensional vector space, then $GL(V)$ denotes the group of invertible linear transformations on V. This is a Lie group isomorphic to $GL(n)$ where $dim(V) = n$.

Definition 5.52 (Representation of Lie Group) *If G is a Lie group, a (finite dimensional) representation of G is a Lie group homomorphism from G to $GL(V)$ for some finite-dimensional vector space, V.*

If a representation $\rho:\ G \to GL(V)$ is injective, the representation is said to be faithful.

It turns out that a representation is faithful if and only if it is isomorphic to a Lie subgroup of $GL(n,\mathbb{C})$ or $GL(n,\mathbb{R})$. Although, not every Lie group admits a faithful representation.

Lastly, there is a relationship between actions an representations.

Proposition 5.53 *Let G be a Lie group and V be a finite-dimensional vector space. A smooth left action of G on V is linear if and only if it is of the form $g \cdot x = \rho(g)x$ for some representation ρ of G.*

Chapter Summary

Here we saw a number of ways to relate different manifolds and even create new manifolds from existing ones. In our first main section we built upon the earlier concepts of the tangent space, and we found a very important related manifold called the tangent bundle. This particular manifold becomes very important in our quest to identify the relation between two different points on a manifold.

The next two sections deal with particular types of ways that manifolds can relate to each other, and then we see how these relations provide us a way to think about submanifolds.

The last section goes deeper into the concept of a Lie group. Specifically, we see here how it becomes possible to generate Lie groups from an existing group, and a hierarchical example of a number of specific Lie groups.

CHAPTER VI

VECTOR FIELDS: INTRODUCTION

Conceptual Introduction

In multivariable calculus we found that vector fields were very useful in describing differences in behavior over an entire region. Here we extend both the notion and usefulness of vector fields to manifolds. Due to the breath of material, this concept will be split across the next two chapters. Within this chapter we introduce the definition of a vector field on a manifold. We then proceed to describe how vector fields provide us with a particular frame of reference on a manifold. We will also discuss how we can associate tangent vectors with vector fields, and how vector fields work with smooth maps. We conclude this section with a few ways to find new vector fields from known ones.

Vector Fields on Manifolds

We will setup the discussion of vector fields by first introducing the definition of vector fields on manifolds. We will then proceed to define local and global frames on the manifolds. Finally we will revisit derivations and show how we can now talk about derivations of $C^{\infty}(M)$ rather than just derivations at a point. This section will provide the foundation for the remainder of the section.

Definition 6.1 (Vector Fields on Manifolds) *Let M be a smooth manifold. A vector field on M is a section of the map $\pi : TM \to M$. Specifically, a vector field is a continuous map $X : M \to TM$, usually written $p \mapsto X_p$, with the property that:*

$$\pi \circ X = Id_M, \tag{6.1}$$

or equivalently, $X_p \in T_pM$ for each $p \in M$ (Where here X_p represents the value of X at p). If the map $X : M \to TM$ is smooth, then it is a smooth vector field.

Also of interest is the case where the map X is not continuous.

Definition 6.2 (Rough Vector Field on M**)** *If a map $X : M \to TM$ such that $\pi \circ X = Id_M$, but is not necessarily continuous, then it is called a rough vector field.*

If M is a smooth n-manifold, X is a rough vector field, and $\left(U, (x^i)\right)$ is any smooth coordinate chart for M, then we can write the value of X at any point $p \in U$ in terms of the coordinate basis vectors:

$$X_p = X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p. \tag{6.2}$$

And this defines n functions $X^i : U \to \mathbb{R}$, called the component functions of X in the given chart.

Proposition 6.3 (Smoothness Criterion for Vector Field) *Let M be a smooth manifold with or without boundary, and let $X : M \to TM$ be a rough vector field. If $\left(U, (x^i)\right)$ is any smooth coordinate chart on M, then the restriction of X to U is smooth if and only if its component functions with respect to this chart are smooth.*

An important example of vector fields is the vector field under a given chart.

Example 6.4 (Coordinate Vector Field) *If $\left(U, (x^i)\right)$ is any smooth chart on a manifold M, then the assignment*

$$p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p$$

determines a vector field on U, called the i^{th} coordinate vector field and is denoted by $\frac{\partial}{\partial x^i}$ or ∂_i. It is smooth because its components are constants.

Next we see that for every point p on our smooth manifold M, there exists a global vector field.

Proposition 6.5 (Existence of X on M) *Let M be a smooth manifold. Given $p \in M$ and $v \in TpM$, there is a smooth global vector field X on M such that $X_p = v$.*

The set of all smooth vector fields on M is denoted, $\mathscr{X}(M)$, and it forms a vector space under pointwise addition and scalar multiplication. The zero of $\mathscr{X}(M)$ is the zero vector field whose value at each $p \in M$ is $0 \in T_pM$.

If $f \in C^\infty(M)$ is a real-valued function and $X \in \mathscr{X}(M)$ is a smooth vector field we define multiplication of $fX : M \to TM$ by:

$$(fX)_p = f(p)X_p \tag{6.3}$$

It turns out that both the addition and multiplication operations form smooth vector fields.

Proposition 6.6 *Let M be a smooth manifold with or without boundary.*

(i) If X and Y are smooth vector fields on M and $f, g \in C^\infty(M)$, then $fX + gY$ is a smooth vector field.

(ii) The set of all smooth vector fields $\mathscr{X}(M)$ is a module over the ring $C^\infty(M)$.

Revisiting the vector field X in terms of a smooth coordinate chart, we can now write it in terms of coordinates using equations between vector fields instead of an equation between vectors at a point:

$$X = X^i \frac{\partial}{\partial x^i}, \tag{6.4}$$

where X^i is the i^{th} component of the vector field X in the given coordinates.

Local and Global Frames

We now begin a discussion of local and global frames. Although smooth local frames are common, smooth global frames are not. However, we will use these concepts later on when we show that Lie groups admit of smooth global frames. For now we will just introduce the definitions.

Definition 6.7 (Linearly Independent Vector Fields) *Let M be a smooth n-manifold. An ordered k-tuple, $(X_1, \cdots, X_k)$, of vector fields defined on some subset $A \subseteq M$ is said to be linearly independent if $(X_1|_p, \cdots, X_k|_p)$ is a linearly independent k-tuple in T_pM for each $p \in A$.*

Definition 6.8 (Span of the Tangent Bundle) *Let M be a smooth n-manifold. If the ordered k-tuple, $(X_1|_p, \cdots, X_k|_p)$, spans T_pM at each $p \in A$, then the k-tuple is said to span the tangent bundle.*

Definition 6.9 (Frames for M) *A local frame for a smooth n-manifold, M, is an ordered n-tuple of vector fields, $(E_1, \cdots, E_n)$, defined on an open subset $U \subseteq M$ that is linearly independent and spans the tangent bundle. We will sometimes refer to the n-tuple as (E_i) instead of writing the full $(E_1, \cdots, E_n)$. Further:*

a) If $U = M$, then we have a global frame.

b) If each E_i is smooth, we have a smooth frame.

Note that from the definition for local frame, we can see that $(E_1|_p, \cdots, E_n|_p)$ creates a basis of T_pM for each $p \in U$.

The last concept for this section is that of parallelizable manifolds.

Definition 6.10 (Parallelizable manifolds) *If a smooth manifold admits a smooth global frame it is said to be parallelizable.*

Vector Fields as Derivations of $C^\infty(M)$

Now if $X \in \mathscr{X}(M)$ is a smooth vector field and f is a real-valued function defined on $U \subseteq M$, then we can obtain a new function $Xf: U \to \mathbb{R}$ defined by

$$(Xf)(p) = X_p f. \tag{6.5}$$

Notice here the order of multiplication. Since X is a smooth vector field on M and $f: M \to \mathbb{R}$:

(a) Xf is real-valued function obtained by applying vector field X to f.

(b) fX is the vector field on U that is the scalar multiple of X.

Because the action of a tangent vector on a function is determined locally, we get the following criteria for smoothness of vector fields.

Proposition 6.11 (Smoothness Criteria for Vector Fields) *Let M be a smooth manifold, and let $X: M \to TM$ be a rough vector field. The following are equivalent:*

(i) X is smooth.

(ii) For every $f \in C^\infty(M)$, the function Xf is smooth on M.

(iii) For every open subset $U \subseteq M$ and every $f \in C^\infty(M)$, the function Xf is smooth on U.

With these equivalent criteria for smoothness, we are able to define a derivation (as apposed to a derivation at a point as we have done earlier).

Definition 6.12 (Derivation) *Let $X \in \mathscr{X}(M)$ be a smooth vector field on the smooth manifold M. Then we see that X defines a linear (over $\mathbb{R}$) map $D: C^\infty(M) \to C^\infty(M)$ that takes $f \mapsto Xf$. If D satisfies the product rule*

$$X(fg) = fX(g) + gX(f), \qquad \forall f,g \in C^\infty(M); \tag{6.6}$$

then D is a derivation of $C^\infty(M)$.

It turns out that derivations can be associated with smooth vector fields.

Proposition 6.13 *Let M be a smooth manifold. A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some smooth vector field $X \in \mathscr{X}(M)$.*

Vector Fields and Smooth Maps

Definition 6.14 (F-related) *Let $F : M \to N$ be a map between smooth manifolds M and N, and $X \in \mathscr{X}(M)$ is a smooth vector field on M. If there exists a smooth vector field on N, $Y \in \mathscr{X}(N)$ such that $dF_p(X_p) = Y_{F(p)}$, then we say that vector fields X and Y are F-related.*

It is important to note that not every smooth map F and vector field $X \in \mathscr{X}(M)$ has an F-related vector field $Y \in \mathscr{X}(N)$. However, if there is, we see that there is a specific way that F-related vector fields act on smooth functions.

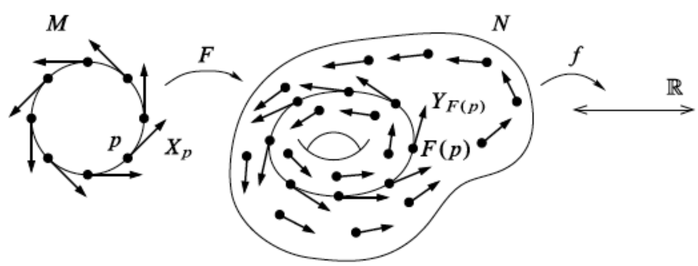

Figure 6.1: F-related Vector Fields

Proposition 6.15 (F-related Vector Fields and Smooth Functions) *Let $F : M \to N$ be a smooth map between manifolds, $X \in \mathscr{X}(M)$, and $Y \in \mathscr{X}(N)$. Then X and Y are F-related if and only if for every smooth real-valued function f defined on an open subset of N,*

$$X(f \circ F) = (Yf) \circ F. \tag{6.7}$$

Proposition 6.16 (Condition for Unique F-related Vector Fields) *Suppose M and N are smooth manifolds, and $F : M \to N$ is a diffeomorphism. For every $X \in \mathscr{X}(M)$, there is a unique smooth vector field on N that is F-related to X.*

Definition 6.17 (Pushforward of X by F) *Suppose M and N are smooth manifolds, and $F: M \to N$ is a diffeomorphism. The unique smooth vector field on N that is F-related to X is called the pushforward of X by F, and is denoted by F_*X.*

The pushforward, F_*X, can be explicitly defined by

$$(F_*X)_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right). \tag{6.8}$$

Corollary 6.18 *Suppose $F: M \to N$ is a diffeomorphism and $X \in \mathscr{X}(M)$. For any $f \in C^\infty(M)$,*

$$((F_*X)f) \circ F = X(f \circ F). \tag{6.9}$$

Finding New Vector Fields

Vector Fields and Submanifolds

Consider a region $S \subseteq M$. For $X \in \mathscr{X}(M)$, it does not mean that $X|_S$ is a vector field on a an immersed or embedded submanifold $S \subseteq M$. This is because for a point $p \in S$, the vector X_p may not lie in the tangent subspace of $T_pS \subseteq T_pM$.

Definition 6.19 (Tangent to Submanifold S) *Given a point $p \in S$, a vector field $X \in \mathscr{X}(M)$ is said to be tangent to S at p if $X_p \in T_pS \subseteq T_pM$. If X_p is tangent to S for every point $p \in S$, then we say that X is tangent to S.*

Proposition 6.20 *Let M be a smooth manifold, $S \subseteq M$ be an embedded submanifold, and let X be a smooth vector field on M ($X \in \mathscr{X}(M)$). Then X is tangent to S if and only if $(Xf)|_S = 0$ for every $f \in C^\infty(M)$ such that $f|_S \equiv 0$*

To prove this, notice that if $S \subseteq M$ is an immersed submanifold with $Y \in \mathscr{X}(S)$, then if $X \in \mathscr{X}(S)$ is ι-related to Y, with $\iota: S \hookrightarrow M$ as the inclusion map, we have $Y_p = d\iota_p(X_p)$ is in the image of $d\iota_p$ for each $p \in S$. That is Y is tangent to S. We find that the converse is true as well.

Proposition 6.21 (Restricting Vector Fields to Submanifolds) *Let M be a smooth manifold, let $S \subseteq M$ be an immersed submanifold, and let $\iota : S \hookrightarrow M$ denote the inclusion map. If $Y \in \mathscr{X}(M)$ is tangent to S, then there is a unique smooth vector field on S, denoted by $Y|_S$, that is ι-related to Y.*

Lie Brackets

Thus far we have defined a vector field on a smooth manifold, discussed how to represent vector fields in coordinate frames, how to apply vector fields to smooth functions, and when this application results in derivations of $C^\infty(M)$. We then looked when a smooth map relates two vector fields, and finally looked at when a vector field is tangent to a submanifold. We now take a look at ways of combining two smooth vector fields to obtain another vector field.

Let $X, Y \in \mathscr{X}(M)$ be smooth vector fields on a smooth manifold M, and let $f : M \to \mathbb{R}$ be a smooth function.

We can use these two vector fields of X and Y to define a new operator on f

(i) Apply X to f with the operation $f \mapsto Xf$. Since $f \in C^\infty(M)$, the function Xf is another smooth function on M.

(ii) Now apply Y to the new smooth function Xf. we find the result, $YXf = Y(Xf)$, is another smooth function. However, this operation $f \mapsto YXf$ does not necessarily satisfy the product rule.

(iii) Next apply the same vector fields but in the opposite order to obtain XYf. Similarly, this is a smooth function that does not satisfy the product rule.

(iv) Now subtract these two results to obtain a new operation:

$$XYf - YXf = (XY)f - (YX)f \tag{6.10}$$

$$= (XY - YX)f \tag{6.11}$$

$$= [X,Y]f \tag{6.12}$$

We call $[X,Y] = (XY - YX)$ the Lie bracket of X and Y. Now if we let $f,g \in C^\infty(M)$ we find that

$$[X,Y](fg) = X(Y(fg)) - Y(X(fg)) \tag{6.13}$$
$$= X(fY(g) + gY(f)) - Y(fX(g) + gX(f)) \tag{6.14}$$
$$= (X(f)Y(g) + fXY(g) + X(g)Y(f) + gXY(f))$$
$$- (Y(f)X(g) + fYX(g) + Y(g)X(f) + gYX(f)) \tag{6.15}$$
$$= X(f)Y(g) + fXY(g) + X(g)Y(f) + gXY(f)$$
$$- Y(f)X(g) - fYX(g) - Y(g)X(f) - gYX(f) \tag{6.16}$$
$$= fXY(g) + gXY(f) - fYX(g) - gYX(f) \tag{6.17}$$
$$= (fXY(g) - fYX(g)) + (gXY(f) - gYX(f)) \tag{6.18}$$
$$= f(XY(g) - YX(g)) + g(XY(f) - YX(f)) \tag{6.19}$$
$$= f[X,Y]g + g[X,Y]f. \tag{6.20}$$

So, even though the individual smooth functions XYf and YXf did not generally satisfy the product rule, we found that $[X,Y]$ does.

Recall

- $f \in C^\infty(M)$ means that $f:\ M \to \mathbb{R}$ is a smooth function
- A derivation at a point $p \in M$ on $C^\infty(M)$ is a linear map $v:\ C^\infty \to \mathbb{R}$ that satisfies the product rule, and the set of all derivations on $C^\infty(M)$ is a vector space called the tangent space to M at p and is represented by T_pM.
- The tangent bundle is the disjoint union of all tangent spaces to M over all $p \in M$ and is represented by TM. That is

$$TM = \bigsqcup_{p \in M} T_pM$$

- A smooth vector field is a smooth continuous map $X : M \to TM$

Thus, the Lie bracket of any pair of smooth vector fields is a smooth vector field.

There is a simple formula for determining the value of the Lie bracket at a point $p \in M$: $[X,Y]_p f = X_p(Yf) - Y_p(Xf)$. However, this formula involves computing terms with second derivatives of f that will cancel out, and thus provide us with limited utility. We will thus consider a different method to calculate Lie brackets that accounts for these cancellations using coordinates.

Proposition 6.22 (Coordinate Formula for the Lie Bracket) *Let $X,Y \in \mathscr{X}(M)$ be vector fields on a smooth manifold M, and let $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ be coordinate representations for X and Y in terms of arbitrary local coordinates (x^i) for M. Then we have the following expression.*

$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \tag{6.21}$$

$$= \left(XY^j - YX^j \right) \frac{\partial}{\partial x^j}. \tag{6.22}$$

The following are properties of Lie brackets.

Proposition 6.23 (Properties of Lie Brackets) *The Lie bracket satisfies the following identities for all $X,Y,Z \in \mathscr{X}(M)$:*

(i) BILINEARITY*: For $a,b \in \mathbb{R}$,*

$$\begin{aligned} [aX+bY,Z] &= a[X,Z] + b[Y,Z], \\ [Z,aX+bY] &= a[Z,X] + b[Z,Y]. \end{aligned} \tag{6.23}$$

(ii) ANTISYMMETRY

$$[X,Y] = -[Y,X]. \tag{6.24}$$

(iii) JACOBI IDENTITY

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{6.25}$$

(iv) For $f, g \in C^\infty(M)$,

$$[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X. \tag{6.26}$$

The Lie Algebra of a Lie Group

The Lie bracket has an important application within Lie groups. Recall that a Lie group G acts smoothly and transitively on itself by left translation $L_g(h) = gh$. A vector field X on G is said to be *left-invariant* if it is invariant under all left translations, in the sense that it is L_g-related to itself for every $g \in G$. This means

$$d(L_g)_{g'}(X_{g'}) = X_{gg'}, \qquad \forall g, g' \in G. \tag{6.27}$$

Since L_g is a diffeomorphism we can abbreviate this as

$$(L_g)_* X = X, \qquad \forall g \in G. \tag{6.28}$$

It turns out that the set of all left-invariant vector fields on G is a linear subspace of $\mathscr{X}(G)$. Further, this set is closed under Lie brackets.

Proposition 6.24 *Let G be a Lie group, and suppose X and Y are smooth left invariant vector fields on G. Then $[X,Y]$ is also left-invariant.*

This brings us to Lie algebras.

Definition 6.25 (Lie Algebra) *A Lie algebra (over $\mathbb{R}$) is a real vector space $\mathfrak{g}$ endowed with a map called the bracket from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$, denoted by $(X,Y) \mapsto [X,Y]$, that satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:*

(i) BILINEARITY*: For* $a, b \in \mathbb{R}$*,*

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y]. \tag{6.29}$$

(ii) ANTISYMMETRY

$$[X, Y] = -[Y, X]. \tag{6.30}$$

(iii) JACOBI IDENTITY

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \tag{6.31}$$

We have a number of intuitive relations with Lie algebras. Letting $\mathfrak{g}$ be a Lie algebra we see:

(a) A linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a Lie subalgebra of $\mathfrak{g}$ if it is closed under brackets.

(b) If $\mathfrak{h}$ is a Lie algebra, a linear map $A : \mathfrak{g} \to \mathfrak{h}$ is called a Lie algebra homomorphism if it preserves brackets: $A[X, Y] = [AX, AY]$, and an invertible Lie algebra homomorphism is called a Lie algebra isomorphism.

Now, if G is a Lie group, the set of all smooth left-invariant vector fields on G is a Lie subalgebra of $\mathscr{X}(G)$, and is thus a Lie algebra. This Lie algebra is called the *Lie algebra of G*, and is denoted $Lie(G)$.

Theorem 6.26 (Dimensionality of $Lie(G)$) *Let G be a Lie group. The evaluation map $\varepsilon : Lie(G) \to T_\varepsilon G$, given by $\varepsilon(X) = X_\varepsilon$, is a vector space isomorphism. Thus, $Lie(G)$ is finite-dimensional, with dimension equal to $dim(G)$.*

This lead us to the following two corollaries.

Corollary 6.27 *Every left-invariant rough vector field on a Lie group is smooth.*

Corollary 6.28 *Every Lie group admits a left-invariant smooth global frame, and is therefore parallelizable.*

So, just as we can view the tangent space of a smooth manifold at a given point as a linear model (or approximation) of the manifold at that point, the Lie algebra provides us with a linear model of the Lie group that reflects many of the properties of the group. Additionally, since a finite-dimensional Lie algebra is a purely linear-algebraic object, we can use it to evaluate properties that we might not be able to get from the group itself. We can now use the above information to introduce the most important non-abelian Lie group's associated algebra.

Proposition 6.29 (Lie Algebra of the General Linear Group) *The composition of the natural maps*

$$Lie\left(GL\left(n,\mathbb{R}\right)\right) \to T_{I_n}GL(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R}) \tag{6.32}$$

gives a Lie algebra isomorphism between $Lie\left(GL\left(n,\mathbb{R}\right)\right)$ and the matrix algebra $\mathfrak{gl}(n,\mathbb{R})$.

And we get the following corollary for abstract vector spaces.

Corollary 6.30 *If V is any finite-dimensional real vector space, the composition of the canonical isomorphism in*

$$Lie\left(GL\left(V\right)\right)) \to T_{I_n}GL(V) \to \mathfrak{gl}(V) \tag{6.33}$$

yields a Lie algebra isomorphism between $Lie\left(GL\left(V\right)\right)$ and $\mathfrak{gl}(V)$.

Chapter Summary

In this section we introduced the notion of a vector field on a manifold, and provided a way to think about the coordinate charts in terms of the vector field of the manifold rather than just at a point on the manifold.

We then used the notion of vector fields to define frames for the manifold which can be thought of as a corollary to a basis of a vector space in linear algebra.

We then related derivations of $C^\infty(M)$ to vector fields, and provided a way for us to think of how vector fields act on a function $f \in C^\infty(M)$. We extended this concept by showing how (and when) we can "pushforward" a vector field from one manifold, M to another manifold N under a map $F: M \to N$.

We then defined the Lie bracket operator and showed that it too is a smooth vector field. We then combined the notion of left-invariant vector fields with the Lie bracket to show that the Lie bracket of two left-invariant vector fields is itself invariant. Thus the set of all left-invariant vector fields, $Lie(M)$ it satisfies the definition of a Lie algebra.

We then saw that every left-invariant vector field on a lie group is smooth and that every Lie group admits a left-invariant smooth global frame. Although we will have more to say about it in the future, this means that every Lie group is parallelizable.

Finally, we showed how there is an isomorphism between the Lie group of the General Linear group, the evaluation of the general linear group around the identity and the Lie algebra of the general linear group. Further, this result generalizes to abstract vector spaces.

CHAPTER VII

VECTOR FIELDS: PATHS AND MOTION ON A MANIFOLD

Conceptual Introduction

In the last section we introduced the notion of a vector field on a manifold. Here we build upon that material to specifically talk about how vector fields help to describe paths and motion on a manifold. For paths, or integral curves, we talk about how they are another way to represent vectors on manifold (as velocity), how they are scale, and how they translate. Lastly, we talk about how integral curves "pushforward" with F-related vector fields.

For motion, or flows, we see that we can take a collection of integral curves to generate a flow on our manifold. After reviewing a number of properties and types of flow, we consider how motion along a curve in a flow can be expressed through pushing back from each tangent space along the curve to the tangent space at the point of reference. This particular procedure leverages the push-back of the differential of the flow, and is called the Lie Derivative.

We conclude this section with a few brief notes on commuting and time dependent vector fields.

Integral Curves

Having now introduced vector fields we can now talk about integral curves.

Definition 7.1 (Integral Curve) *If V is a vector field on M, an integral curve of V is a differentiable curve $\gamma: J \to M$ whose velocity at each point is equal to the value of V at that point:*

$$\gamma'(t) = V_{\gamma(t)}, \qquad \forall t \in J.$$

Suppose that V is a smooth vector field on M, and let $\gamma\colon\; J \to M$ be a smooth curve. On a smooth coordinate domain $U \subseteq M$, we can write γ in local coordinates as $\gamma(t) = (\gamma^1(t), \cdots \gamma^n(t))$. Then we have the condition for $\gamma'(t) = V_{\gamma(t)}$ to be an integral curve as

$$\dot{\gamma}^i(t)\, \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} = V^i\left(\gamma(t)\right) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)},$$

which is no more than the following system of autonomous ordinary differential equations (ODEs)

$$\begin{aligned} \dot{\gamma}^1 &= V^1\left(\gamma^1, \cdots, \gamma^n(t)\right) \\ &\vdots \\ \dot{\gamma}^n &= V^n\left(\gamma^1, \cdots, \gamma^n(t)\right). \end{aligned} \tag{7.1}$$

We begin by seeing that there exists, at least locally, an integral curve that begins at each point.

Proposition 7.2 *Let V be a smooth vector field on a smooth manifold M. For each $p \in M$, there exist $\varepsilon > 0$ and a smooth curve $\gamma\colon\; (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at p.*

We also see that affine reparameterizations of rescaling and translation affect an integral curve. That is first, an integral curve scales with the vector field when multiplied by a real number.

Lemma 7.3 (Rescaling Lemma) *Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma\colon\; J \to M$ be an integral curve of V. For any $a \in \mathbb{R}$, the curve $\widetilde{\gamma}(t) = \gamma(at)$ is an integral field of the vector field aV, where $\widetilde{J} = \{t :\; at \in J\}$.*

And second, a translation of the integral curves parameter is still an integral curve.

Lemma 7.4 (Translation Lemma) *Let V be a smooth vector field on a smooth manifold M, $J \subseteq \mathbb{R}$ be an interval, and let $\gamma\colon\; J \to M$ be an integral curve of V. For any $b \in \mathbb{R}$, the curve $\hat{\gamma}\colon\; \hat{J} \to M$ defined by $\hat{\gamma(t)} = \gamma(t+b)$ is also an integral curve of V, where $\hat{J} = \{t :\; t+b \in J\}$.*

For the last item of this section we take a look at how an integral curve transforms under F-related vector fields.

Proposition 7.5 (Naturality of Integral Curves) *Suppose M and N are two smooth manifolds with $F: M \to N$ as a smooth map between them. Then $X \in \mathscr{X}(M)$ and $Y \in \mathscr{X}(N)$ are F-related if and only if F takes integral curves of X to integral curves of Y. That is, for each integral curve γ of X, $F \circ \gamma$ is an integral curve of Y.*

Flows

We begin with the definition for a Global Flow.

Definition 7.6 (Global Flow) *A global flow on a smooth manifold M is a continuous left $\mathbb{R}$-action on M. That is, a global flow is a continuous map $\theta: \mathbb{R} \times M \to M$ satisfying the following for all $s,t \in \mathbb{R}$ and $p \in M$.*

(i) $\theta(t, \theta(s, p)) = \theta(t+s, p)$, *and*

(ii) $\theta(0, p) = p$

Proposition 7.7 *Let $\theta: \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. The assignment $p \mapsto V_p = \theta^{(p)\prime}(0) \in T_pM$, called the infinitesimal generator V of θ is a smooth vector field on M, and each curve $\theta^{(p)}$ is an integral curve of V.*

The Fundamental Theorem on Flows

We have seen that a smooth global flow generates a smooth vector field where the integral curves of the vector field are the same as the curves defined by the flow. We will now look at when a smooth vector field generates a smooth global flow. We will begin with defining a few relevant concepts.

Definition 7.8 (Flow Domain of M) *If M is a manifold, a flow domain for M is an open subset $\mathscr{D} \subseteq \mathbb{R} \times M$ such that for each $p \in M$, the set $\mathscr{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathscr{D}\}$ is an open interval containing* 0.

Definition 7.9 (Flow on M) *If M is a manifold, a flow on M is a continuous map $\theta : \mathscr{D} \to M$ that satisfies the additive group laws:*

(i) $\theta(t, \theta(s,p)) = \theta(t+s, p)$, *$\forall s \in \mathscr{D}^{(p)}$ and $\forall t \in \mathscr{D}^{\theta(s,p)}$, such that $s+t \in \mathscr{D}^{(p)}$*

(ii) $\theta(0,p) = p$, *$\forall p \in M$.*

A flow on M is sometimes referred to a local flow in order to differentiate it from the global flow we defined above.

Now just as we have $\mathscr{D}^{(p)}$ to represent the set of t's in the flow domain, we also define the set of p's in the flow domain:

$$M_t = \{p \in M : (t,p) \in \mathscr{D}\}.$$

So, with these definitions we see that

$$p \in M_t \iff t \in \mathscr{D}^{(p)} \iff (t,p) \in \mathscr{D} \tag{7.2}$$

So, we now have a way to determine a smooth vector field with its integral curves from a smooth flow restricted to $\mathscr{D}$ rather than all $\mathbb{R} \times M$ as we did above.

Proposition 7.10 *If $\theta : \mathscr{D} \to M$ is a smooth flow, then the infinitesimal generator V of θ is a smooth vector field, and each curve $\theta^{(p)}$ is an integral curve of V.*

This restriction to $\mathscr{D}$ gives rise to questions about the size of interval contained within $\mathscr{D}$ and the size of integral curves. For this reason we define the concepts of a maximal integral curve and a maximal flow.

Definition 7.11 (Maximal Integral Curve) *A maximal integral curve is one that cannot be extended to an integral curve on any larger open interval.*

Definition 7.12 (Maximal Flow) *A maximal flow is a flow that admits no extension to a flow on a larger flow domain.*

This all brings us to the main result within this section.

Theorem 7.13 (Fundamental Theorem on Flows) *Let V be a smooth vector field on a smooth manifold M. There is a unique smooth maximal flow $\theta : \mathscr{D} \to M$ whose infinitesimal generator is V. This flow has the following properties:*

(i) For each $p \in M$, the curve $\theta^{(p)} : \mathscr{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p.

(ii) If $s \in \mathscr{D}^{(p)}$, then $\mathscr{D}^{(\theta(s,p))}$ is the interval $\mathscr{D}^{(p)} - s = \{t - s : t \in \mathscr{D}^{(p)}\}$.

(iii) For each $t \in \mathbb{R}$, the set M_t is open in M, and $\theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

So first we saw that given a flow we get a vector field. Now we see that given a vector field we get a maximal flow that gives us a maximal integral curve starting at each point in the manifold. Further we see that we can move from an integral curve at one point to any other point in the curve with a simple translation, and that we can reverse each flow with a parameter reversal.

Just as we saw how the integral curves behave between two F-related vector fields, we now see the same for flows.

Proposition 7.14 (Naturality of Flows) *Suppose M and N are smooth manifolds, $F : M \to N$ is a smooth map, $X \in \mathscr{X}(M)$, and $Y \in \mathscr{X}(N)$. Let θ be the flow of X and η be the flow of Y. If X and Y are F-related, then for each $t \in \mathbb{R}$, $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:*

$$\begin{tikzcd} M_t \arrow[r, "F"] \arrow[d, "\theta_t"'] & N_t \arrow[d, "\eta_t"] \\ M_{-t} \arrow[r, "F"'] & N_{-t} \end{tikzcd}$$

Figure 7.1: Naturality of Flows

Thus it follow that we can push forward the flow of X under F if F is a diffeomorphism.

Corollary 7.15 *Let $F : M \to N$ be a diffeomorphism. If $X \in \mathscr{X}(M)$ and θ is the flow of X, then the flow of F_*X is $\eta_t = F \circ \theta_t \circ F^{-1}$ with the domain of $N_t = F(M_t)$ for each $t \in \mathbb{R}$.*

Complete Vector Fields

The vector fields that emit smooth global flows are important enough to deserve their own name.

Definition 7.16 (Complete Vector Fields) *We say a smooth vector field is complete if each of its maximal integral curves is defined for all $t \in \mathbb{R}$.*

Since it is not always easy to determine if a vector field is complete we provide here a few ways to determine if it is complete or not.

Lemma 7.17 (Uniform Time Lemma) *Let V be a smooth vector field on a smooth manifold M, and let θ be its flow. Suppose there is a positive number ε such that for every $p \in M$, the domain of $\theta^{(p)}$ contains $(-\varepsilon, \varepsilon)$. Then V is complete.*

Theorem 7.18 *Every compactly supported smooth vector field on a smooth manifold is complete.*

Corollary 7.19 *On a compact smooth manifold, every smooth vector field is complete.*

Theorem 7.20 *Every left-invariant vector field on a Lie group is complete.*

We also find a condition on the interval that excludes the ability for a curve to be contained within a submanifold.

Lemma 7.21 (Escape Lemma) *Suppose M is a smooth manifold and $V \in \mathscr{X}(M)$. If $\gamma : J \to M$ is a maximal integral curve of V whose domain J has a finite least upperbound b, then for any $t_0 \in J$, $\gamma([t_0, b))$ is not contained in any compact subset of M.*

Flowouts

As flows are useful tools in describing the geometry of manifolds, it is useful to see how flows behave around submanifolds.

Theorem 7.22 (Flowout Theorem) *Suppose M is a smooth manifold, $S \subseteq M$ is an embedded k-dimensional submanifold, and $V \in \mathscr{X}(M)$ is a smooth vector field that is nowhere tangent to S. Let $\theta : \mathscr{D} \to M$ be the flow of V, let $\Omega = (\mathbb{R} \times S) \cap \mathscr{D}$, and let $\Phi = \theta|_\Omega$. Then we have*

(i) $\Phi : \Omega \to M$ *is an immersion.*

(ii) $\frac{\partial}{\partial t} \in \mathscr{X}(\Omega)$ *is Φ-related to V.*

(iii) There exists a smooth positive function $\delta : S \to \mathbb{R}$ such that he restriction of Φ to Ω_δ is injective, where $\Omega_\delta \subseteq \Omega$ is the flow domain:

$$\Omega_\delta = \{(t, p) \in \Omega : |t| < \delta(p)\} \tag{7.3}$$

Thus, $\Phi(\Omega_\delta)$ is an immersed submanifold of M containing S, and V is tangent to this submanifold.

(iv) If S has codimension of 1, then $\Phi|_{\Omega_\delta}$ is a diffeomorphism onto an open submanifold of M.

The submanifold $\Phi(\Omega_\delta) \subseteq M$ is called the flowout from S along V.

Regular Points and Singular Points

We find that the behavior of integral curves around singular points is different from that of regular points.

Definition 7.23 (Regular and Singular Points) *Let V be a vector field on M. A point $p \in M$ is a singular point of V if $V_p = 0$, and a regular point otherwise.*

Proposition 7.24 *Let V be a smooth vector field on a smooth manifold M, and let $\theta : \mathscr{D} \to M$ be the flow generated by V. If $p \in M$ is a singular point of V, then $\mathscr{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)}$ is the constant curve $\theta^{(p)} \equiv p$. If p is a regular point, then $\theta^{(p)} : \mathscr{D}^{(p)} \to M$ is a smooth immersion.*

There is a special name for a point in a curve that is remains unchanged under the parameter $t \in \mathbb{R}$.

Definition 7.25 (Equilibrium Point of θ) *If $\theta : \mathscr{D} \to M$ is a flow, then an equilibrium point of θ is a point $p \in M$ if $\theta(t,p) = p$ for all $t \in \mathscr{D}^{(p)}$.*

So we see that the equilibrium points of a smooth flow are precisely the singular points of its infinitesimal generator.

The behavior around a regular point is described below.

Theorem 7.26 (Canonical Form Near a Regular Point) *Let V be a smooth vector field on a smooth manifold M, and let $p \in M$ be a regular point of V. There exist smooth coordinates (s^i) on some neighborhood of p in which V has the coordinate representation $\left(\frac{\partial}{\partial s^1}\right)$. If $S \subseteq M$ is any embedded hypersurface with $p \in S$ and $V_p \notin T_pS$, then the coordinates can also be chosen so that s^1 is a local defining function for S.*

Example 7.27 (Canonical Form) *Let $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$. The flow of W is the map $\theta : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $\theta_t(x,y) = (x\cos t - y\sin t, x\sin t + y\cos t)$. So for each $t \in \mathbb{R}$, θ_t rotates the plane through an angle t about the origin.*

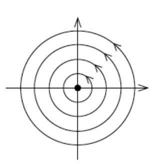

Figure 7.2: Flow Example

Notice that the point $(1,0)$ is a regular point of W ($\partial/\partial y|_{1,0} \neq 0$). Because W has a nonzero y-coordinate there, we can take S to be the x-axis parameterized by $X(s) = (s,0)$. We *define* $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $\Psi(t,s) = \theta_t(s,0) = (s\cos t, s\sin t)$*. Solving locally for* (s,t) *in terms of* (x,y) *gives us the coordinate map*

$$(t,s) = \Psi^{-1}(x,y) = \left(\tan^{-1}\left(\frac{y}{x}\right), \sqrt{x^2+y^2}\right),$$

and we get W *with a different coordinate representation.*

Thus, up to a diffeomorphism, local shifts in integral curves around a regular point behave the same as translations along parallel coordinate lines in $\mathbb{R}^n$.

Lie Derivatives

We know that $v \in T_pM$ is a tangent vector of M at a point p, and by definition, v acts on a smooth function $f \in C^\infty(M)$ such that vf is interpreted as the directional derivative of f in the direction of v at point p. Further, we have seen how we can consider v as a velocity function, and interpret vf as the ordinary derivative of f along the curve where the initial velocity is v.

However, in the case of vector fields we have to consider a curve $\gamma(t)$ that passes through the multiple different tangent spaces from $T_{\gamma(0)}M$ to $T_{\gamma(t)}$ for t in the interval $J = [0,T)$. So, in order to handle this we essentially push back from the point $\gamma(t)$ to the point $\gamma(0)$ by looking at their tangent spaces of $T_{\gamma(t)}M$ and $T_{\gamma(0)}M$ and consider them as vector fields V and W. We can then use differential of the inverse flow. We call this result the Lie derivative.

Definition 7.28 (Lie Derivative of W with respect to V) *Let M be a smooth manifold. Let γ be an integral curve on M starting at point p so that $\gamma(0) = p$. There exists a flow θ such that $\gamma(t) = \theta_t(p)$. We replace the tangent spaces of $T_{\gamma(0)}M$ and $T_{\gamma(t)}M$ with vector fields V_p and $V_{\theta_t(p)}$ respectively. Then we define the Lie derivative of W with respect to V as:*

$$\begin{aligned}(\mathscr{L}_V W)_p &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) \\ &= \lim_{t\to 0} \frac{d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t}.\end{aligned} \tag{7.4}$$

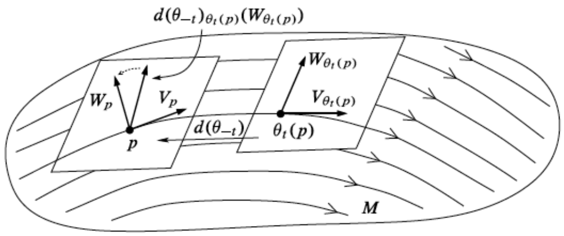

Figure 7.3: Lie Derivative

Since the flow is often difficult to explicitly determine, we often use a rather simple formula for determining the Lie derivative.

Theorem 7.29 *If M is a smooth manifold and $V,W \in \mathscr{X}(M)$, then $\mathscr{L}_V W = [V,W]$.*

The geometric interpretation of this is that the Lie bracket is the directional derivative of the second vector field along the flow of the first. A number of additional properties follow.

Corollary 7.30 *Suppose M is a smooth manifold, and $V,W,X \in \mathscr{X}(M)$.*

(i) $\mathscr{L}_V W = -\mathscr{L}_W V$.

(ii) $\mathscr{L}_V [W,X] = [\mathscr{L}_V W, X] + [W, \mathscr{L}_V X]$.

(iii) $\mathscr{L}_{[V,W]} X = \mathscr{L}_V \mathscr{L}_W X - \mathscr{L}_W \mathscr{L}_V X$.

(iv) If $g \in C^\infty(M)$, then $\mathscr{L}_V (gW) = (Vg)W + g\mathscr{L}_V W$.

(v) If $F : M \to N$ is a diffeomorphism, then $F_*(\mathscr{L}_V X) = \mathscr{L}_{F_* V} F_* X$.

Commuting Vector Fields

It turns out that commutability and invariance are related.

We say that V and W commute if $VWf = WVf$ for every smooth function f. Or equivalently we see that

$$
\begin{aligned}
VWf &= WVf && (7.5)\\
&\Rightarrow VWf - WVf = 0 && (7.6)\\
&\Rightarrow (VW - WV)f = 0 && (7.7)\\
&\Rightarrow [V,W]f = 0 && (7.8)\\
&\Rightarrow [V,W] \equiv 0. && (7.9)
\end{aligned}
$$

If θ is a smooth flow, a vector field W is said to be invariant under θ if W is θ_t-related to itself for each t. Recall that $M_t = \{p \in M : (t,p) \in \mathscr{D}\}$. We see that W is invariant under θ means that $W|_{M_t}$ is θ_t-related to $W|_{M_{-t}}$ for each t, or equivalently $d(\theta_t)_p(W_p) = W_{\theta_t(p)}$ for all $(t,p) \in \mathscr{D}$.

With these concepts of commutability and invariance in mind we can find a relation between the two.

Theorem 7.31 *For smooth vector fields V and W on a smooth manifold M, the following are equivalent:*

(i) V and W commute.

(ii) W is invariant under the flow of V.

(iii) V is invariant under the flow of W.

Taking $V = W$ we get the following corollary.

Corollary 7.32 *Every smooth vector field is invariant under its own flow.*

Commuting Flows

We now turn our attention to commuting flows.

Definition 7.33 (Commuting Flows) *Let θ and Ψ be flows on the manifold M. We say that θ and Ψ commute if for every $p \in M$, whenever J and K are open intervals containing 0 such that one of the expressions*

$$\theta_t \circ \Psi_s(p) \quad -or- \quad \Psi_s \circ \theta_t(p)$$

is defined for all $(s,t) \in J \times K$, both are defined and they are equal.

Theorem 7.34 *Smooth vector fields commute if and only if their flows commute*

Commuting Frames

Recall that a local frame for a smooth n-manifold, M, is an n-tuple (E_i) of vector fields defined on an open subset $U \subseteq M$ such that $(E_i|_p)$ forms a basis for T_pM at each $p \in U$. We now define a commuting frame.

Definition 7.35 (Commuting Frame) *A smooth local frame (E_i) for a manifold M is called a commuting frame if $[E_i, E_j] = 0$ for all i and j. (Commuting frames are also called holonimic frames.)*

It is easy to see that every coordinate frame is a commuting frame. Also, from above, we know that Lie brackets are invariantly defined. It turns out that commuting vector fields are a necessary and sufficient condition for a smooth frame to be locally expressible as a coordinate frame.

Theorem 7.36 (Canonical Form for Commuting Vector Fields) *Let M be a smooth n-manifold, ad let $(V_1, \cdots, V_k)$ be a linearly independent k-tuple of smooth commuting vector fields on an open subset $W \subseteq M$. For each $p \in W$, there exists a smooth coordinate chart $(U, (s^i))$ centered at p such that $V_i = \frac{\partial}{\partial s^i}$ for $i = 1, \cdots, k$. If $S \subseteq W$ is an embedded codimension-k submanifold and p*

is a point of S such that T_pS is complementary to the span of $(V_1|_p, \cdots, V_k|_p)$, then the coordinates can also be chosen such that $S \cap U$ is the slice defined by $s^1 = \cdots = s^k = 0$.

Thus we can establish a process for finding explicit coordinates that put a set of commuting vector fields into canonical form, as long as their flows can be found explicitly.

(a) Begin with a $(n-k)$-dimensional submanifold S whose tangent space at p is complementary to the span of $(V_1|_p, \cdots, V_k|_p)$.

(b) Define Φ by starting at an arbitrary point in S and following the k flows successively for k arbitrary times. (Because the flows commute, the order does not matter.)

Example 7.37 (Canonical Form for Commuting Vectors) *Consider the following two vector fields on $\mathbb{R}^2$:*

$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad \text{and} \quad W = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

We can clearly see that:

$$\begin{aligned}[V,W] &= \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \quad (7.10)\\ &= 0. \quad (7.11)\end{aligned}$$

Our previous example shows that the flow of V is

$$\theta_t(x,y) = (x\cos t - y\sin t, x\sin t + y\cos t). \quad (7.12)$$

We can find the flow for W of

$$\eta_t(x,y) = (e^t x, e^t y). \quad (7.13)$$

At $p=(1,0)$, V_p and W_p are linearly independent. Because $k=n=2$, we can take the subset S to be the single point $\{(1,0)\}$, and define $\Phi:\ \mathbb{R}^2 \to \mathbb{R}^2$ by:

$$\Phi(s,t) = \eta_t \circ \theta_s(1,0) = (e^t \cos s, e^t \sin s). \tag{7.14}$$

We can then solve for $(s,t) = \Phi^{-1}(x,y)$ explicitly in a neighborhood of p to obtain the coordinate map:

$$(s,t) = \left(\tan^{-1}\frac{y}{x}, \log\sqrt{x^2+y^2}\right). \tag{7.15}$$

Time Dependent Vector Fields

We have thus far been exploring differential equations that do not depend explicitly on the independent variable t. For those that do depend on t we introduce time-dependent vector fields.

Definition 7.38 (Time-Dependent Vector Field) *Let M be a smooth manifold. A time-dependent vector field on M is a continuous map $V:\ J \times M \to TM$, where $J \subseteq \mathbb{R}$ is an interval, such that $V(t,p) \in T_pM$ for each $(t,p) \in J \times M$.*

With the vector field defined, we will define the integral curve for a time dependent vector field.

Definition 7.39 (Integral Curve of a Time-Dependent Vector Field) *If V is a time-dependent vector field on M, then and integral curve of V is a differentiable curve $\gamma:\ J_0 \to M$, where J_0 is an interval contained in J, such that*

$$\gamma'(t) = V(t,\gamma(t)), \quad \forall t \in J_0$$

So for every ordinary vector field $X \in \mathscr{X}(M)$ we can find the time dependent vector field defined on $\mathbb{R} \times M$ by setting $V(t,p) = X_p$.

As a substitute for the more general theorem of flows above, we now have the following.

Theorem 7.40 (Fundamental Theorem on Time-Dependent Flows) *Let M be a smooth manifold, $J \subseteq \mathbb{R}$ be an open interval, and let $V: J \times M \to TM$ be a smooth time-dependent vector field on M.*

There exists an open subset $\mathscr{E} \subseteq J \times J \times M$ and a smooth map $\Psi: \mathscr{E} \to M$ called the time-dependent flow of V, with the following properties:

(i) *$\forall t_0 \in J$ and $\forall p \in M$, the set $\mathscr{E}^{(t_0,p)} = \{t \in J : (t, t_0, p) \in \mathscr{E}\}$ is an open interval containing t_0, and the smooth curve $\Psi^{(t_0,p)}: \mathscr{E}^{(t_0,p)} \to M$ given by $\Psi^{(t_0,p)}(t) = \Psi(t,t_0,p)$ is the unique maximal integral curve of V with initial condition $\Psi^{(t_0,p)}(t_0) = p$.*

(ii) *If $t_1 \in \mathscr{E}^{(t_0,p)}$ and $q = \Psi^{(t_0,p)}(t_1)$, then $\mathscr{E}^{(t_1,q)} = \mathscr{E}^{(t_0,p)}$ and $\Psi^{(t_1,q)} = \Psi^{(t_0,p)}$.*

(iii) *For each $(t_1,t_0) \in J \times J$, the set $M_{t_1,t_0} = \{p \in M : (t_1, t_0, p) \in \mathscr{E}\}$ is open in M, and the map $\Psi_{t_1,t_0}: M_{t_1,t_0} \to M$ defined by $\Psi_{t_1,t_0} = \Psi(t_1,t_0,p)$ is a diffeomorphism from M_{t_1,t_0} to M_{t_0,t_1} with inverse Ψ_{t_0,t_1}.*

(iv) *If $p \in M_{t_1,t_0}$ and $\Psi_{t_1,t_0} \in M_{t_2,t_1}$, then $p \in M_{t_2,t_0}$ and $\Psi_{t_2,t_1} \circ \Psi_{t_1,t_0}(p) = \Psi_{t_2,t_0}(p)$.*

Thus we find the geometric conditions for first order partial differential equations.

Chapter Summary

In this section we introduced the notion of integral curves as a differential curve on a manifold whose value at each point equals that of the same vector field on the manifold. We then saw that we can use this notion of integral curves to identify F-related manifolds thereby allowing us to push-forward from one manifold to another. We next saw that the vector field can be thought of as a flow of motion. We then saw a procedure, called the Lie Derivative, where we can reverse the flow of an integral curve, and by treating tangent spaces as vector fields, we can compare all the points along the curve as if they were in the original starting reference point's tangent space. We then found that commuting vector fields provide us a necessary and sufficient condition for invariance. Finally, we saw that the notion of a time dependent vector field can provide us the geometric conditions for first order partial differential equations.

CHAPTER VIII

COVECTORS, COTANGENT, COFRAME, AND COFIELDS

Conceptual Introduction

From Linear algebra, we know that every vector space has a dual space such that if $v \in V$, then all smooth maps ω such that $\omega : V \to \mathbb{R}$ make up a dual space, V^*. This dual space is made up of covectors that can be thought of as functions that act on a vector taking it into the set of real numbers. It turns out that this concept of duality plays a very important role in manifold theory. Not only do the notions of vectors, tangents, frames and fields have a corollary in dual space, but this dual space will eventually provide us the space for the operations of gradient and line integrals. Thus, it is this dual space that will be providing the foundation for geometry, and particularly for out interests, for curvature.

Abstraction of Tangent Bundle

We find that the tangent bundle, TM, that we learned about earlier is actually an abstraction of a broader concept itself. The tangent bundle is considered a specific example of vector bundle which is aptly named as it is really a bundle of vectors that make up a space with a map that brings a slice of the bundle to the manifold.

Vector Bundles

Definition 8.1 (Vector Bundle) *Let M be a topological space. A real vector bundle of rank k over M is a topological space E together with a surjective continuous map $\pi : E \to M$, called its projection, satisfying the following:*

(i) For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional real vector space.

(ii) For each $p \in M$, there esist a neighborhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called a local trivialization of E over U), satisfying

- *$\pi_U \circ \Phi = \pi$ (where $\pi_U : U \times \mathbb{R}^k \to U$ is the projection).*
- *For each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.*

If both M and E are smooth manifolds, then π is a smooth map. Further, the local trivializations can be chosen to be diffeomorphisms making E a smooth vector bundle. Any local trivialization that is a diffeomorphism onto its image is a smooth local trivialization. The space E is called the total space of the bundle, and M is called its base.

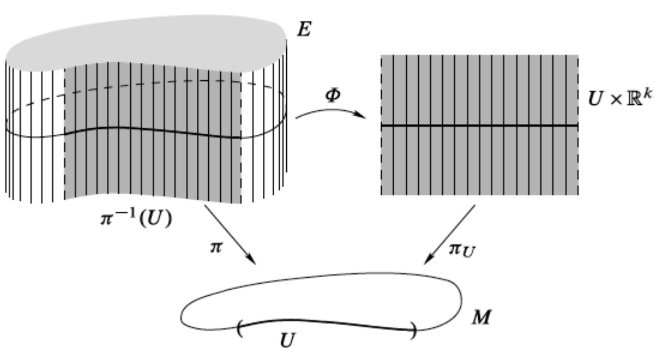

Figure 8.1: Local Trivialization of a Vector Bundle

Definition 8.2 (Trivial Bundle) *If there exists a local trivialization of E over all of M (a global trivialization), then E is homeomorphic to the product space $M \times \mathbb{R}^k$, and we say that E is a trivial bundle. If $E \to M$ is a smooth bundle that admits a smooth global trivialization, then E is diffeomorphic to $M \times \mathbb{R}^k$.*

Example 8.3 (Product Bundle) *A simple example of a rank k vector bundle is $E = M \times \mathbb{R}^k$ with $\pi = \pi_1 : M \times \mathbb{R}^k \to M$ as its projection. This type of bundle is called a product bundle. If M is a smooth manifold, then E is smoothly trivial.*

As mentioned, the tangent bundle is a vector bundle. Let M be a smooth manifold, and (U, φ) be a smooth chart with coordinate functions (x^i). We can define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by:

$$\Phi\left(v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = (p, (v^1, \cdots, v^n)). \tag{8.1}$$

This is linear on fibers and satisfies $\pi_1 \circ \Phi = \pi$. Now, since the composite map

$$\pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^n \xrightarrow{\varphi \times \mathtt{ID}_{\mathbb{R}^n}} \varphi(U) \times \mathbb{R}^n,$$

is a diffeomorphism and $\varphi \times \mathtt{ID}_{\mathbb{R}^n}$ is a diffeomorphism, we find that Φ is a diffeomorphism. Thus, Φ satisfies all the conditions for a smooth local trivialization. Thus as stated, we see that the tangent bundle is a specific instance of a vector bundle.

Proposition 8.4 (The Tangent Bundle as a Vector Bundle) *Let M be a smooth n-manifold, and let TM be its tangent bundle. With its standard projection map, natural vector space structure on each fiber, topology, and smooth structure, we find that the tangent bundle, TM, is a smooth vector bundle of rank n over M.*

Next we see a simple from in the overlap of two smooth local trivializations.

Lemma 8.5 *Let $\pi : E \to M$ be a smooth vector bundle of rank k over M. Suppose $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two smooth local trivializations of E with $U \cap V \neq \emptyset$. Then, there exists a smooth map $\tau : U \cap V \to GL(k,\mathbb{R})$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the form:*

$$\Phi \circ \Psi^{-1}(p,v) = (p, \tau(p)v),$$

where $\tau(p)v$ denotes the usual action of the $k \times k$ matrix $\tau(p)$ on the vector $v \in \mathbb{R}^k$

So we see that the overlap of the composition for trivialization is just the point of the vector bundle represented by the pair for the manifold point and the action at that point for a given vector.

Next we provide a way to construct vector bundles from trivializations.

Lemma 8.6 (Vector Bundle Chart Lemma) *Let M be a smooth manifold, and suppose that for each $p \in M$ we are given a real vector space E_p of some fixed dimension k. Let $E = \bigsqcup_{p \in M} E_p$, and let $\pi : E \to M$ be the map that takes each element of E_p to the point p. Suppose further we are given the following data:*

(i) an open cover $\{U_\alpha\}_{\alpha \in A}$ of M;

(ii) for each $\alpha \in A$, a bijective map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a vector space isomorphism from E_p to $p \times \mathbb{R}^k \cong \mathbb{R}^k$;

(iii) for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, a smoth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k,\mathbb{R})$ such that the map $\Phi_\alpha \circ \Phi_b eta^{-1}$ from $(U_\alpha \cap U_\beta) \times \mathbb{R}^k$ to itself has the form:

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p,v) = (p, \tau_{\alpha\beta}(p)v).$$

Then E has a unique topology and smooth structure making it into a smooth manifold and a smooth rank-k vector bundle over M, with π as projection and $\{(U_\alpha, \Phi_\alpha)\}$ as smooth local trivializations.

Local and Global Vector Bundles

We again begin with a definition.

Definition 8.7 (Sections of Vector Bundles) *Let $\pi : E \to M$ be a vector bundle. A local section of E is a continuous map $\sigma : U \to E$ for some open subset $U \subseteq M$ satisfying $\pi \circ \sigma = \mathtt{ID}_U$. If $U = E$ we say that σ is a global section of E (sometimes referred to as a cross section or just section). If σ is not necessarily continuous, we call it a rough section of E.*

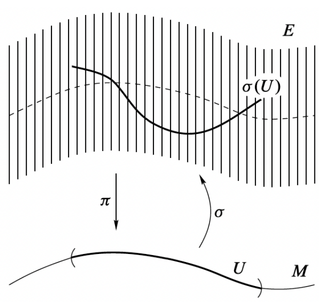

Figure 8.2: Vector Bundle Section

Notice $\sigma(p)$ is an element of the fiber E_p for each $p \in M$. Also, it is worth noting that the local section of E over $U \subseteq M$ is the same as the global section of the restricted bundle $E|_U$.

The zero section of E is the global section $\zeta : M \to E$ defined by

$$\zeta(p) = 0 \in E_p,\ \forall p \in M.$$

And the support of a section σ is the closure of the set $\{p \in M : \sigma(p) \neq 0\}$

The set of all smooth global sections of the total space E of a vector bundle is a vector space often denoted by $\Gamma(E)$. However, specific examples often get their own symbols just like $\mathscr{X}(M)$ represented the space of smooth sections of the tangent bundle TM.

As a vector field, we find that we can multiply by smooth real valued functions:

$$(f\sigma)(p) = f(p)\sigma(p).$$

Local and Global Frames

The earlier concepts of frames extends readily to vector bundles. For a vector bundle $E \to M$ and open subset $U \subseteq M$, a k-tuple of local sections $(\sigma_1, \cdots, \sigma_k)$ of E over U is *linearly independent* if their values $(\sigma_1(p), \cdots, \sigma_k(p))$ form a linearly independent k-tuple in E_p for each $p \in U$. The k-tuple is said to *span* E if their values span E_p for each $p \in M$. With this we can define local and global frames.

Definition 8.8 (Frame for E over U) *If $\pi : E \to M$ is a vector bundle. A local frame of E over U is an ordered k-tuple $(\sigma_1, \cdots, \sigma_k)$ of linearly independent local sections over U that span E. If $U = E$, then the k-tuple is called a global frame.*

Notice that for a local frame, their values $(\sigma_1(p), \cdots, \sigma_k(p))$ is a basis for the fiber E_p for each $p \in U$. The frame is often abbreviated with (σ_i).

So we have now extended our earlier notions of frames for a manifold with frames for vector bundles, and will often use the term frame interchangeably. To round this out we have the below proposition to extend the notion of complete local frames to vector bundles.

Proposition 8.9 (Completion of Local Frames for Vector Bundles) *Let $\pi : E \to M$ be a vector bundle of rank k.*

(i) If $(\sigma_1, \cdots, \sigma_m)$ is a linearly independent m-tuple of smooth local sections of E over an open subset $U \subseteq M$, with $1 \leq m \leq k$, then for each $p \in U$ there exists smooth sections $\sigma_{m+1}, \cdots, \sigma_k$ defined on some neighborhood V of p such that $(\sigma_1, \cdots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.

(ii) If $(v_1, \cdots, v_m)$ is a linearly independent m-tuple of elements of E_p for some $p \in M$, with $1 \leq m \leq k$, then there exists a smooth local frame (σ_i) for E over some neighborhood of p such that $\sigma_i(p) = v_i$ for $i = 1, \cdots, m$.

(iii) *If $A \subseteq M$ is a closed subset and $(\tau_1, \cdots, \tau_k)$ is a linearly independent k-tuple of sections of $E|_A$ that are smooth in the sense that σ extends to a smooth local section of E in a neighborhood of each point, then there exists a smooth local frame $(\sigma_1, \cdots, \sigma_k)$ for E over some neighborhood of A such that $\sigma_i|_A = \tau_i$, for $i = 1, \cdots, k$.*

We find a close connection between triviality and frames.

Proposition 8.10 *Every smooth local frame for a smooth vector bundle is associated with a smooth local trivialization.*

Corollary 8.11 *A smooth vector bundle is smoothly trivial if and only if it admits a smooth global frame.*

Corollary 8.12 *Let $\pi : E \to M$ be a smooth vector bundle of rank k. Let (V, φ) be a smooth chart on M with coordinate functions (x^i), and suppose there exists a smooth local frame (σ_i) for E over V. Define $\widetilde{\varphi} : \pi^{-1}(V) \to \varphi(V) \times \mathbb{R}^k$ by*

$$\widetilde{\varphi}(v^i \sigma_i(p)) = (x^1(p), \cdots, x^n(p), v^1, \cdots, v^k).$$

Then $(\pi^{-1}(V), \widetilde{\varphi})$ is a smooth coordinate chart for E.

We have seen how smoothness of vector fields can be related to their component functions in any smooth charts. Further, smoothness of sections of vector bundles can be characterized in terms of local frames.

Proposition 8.13 (Local Frame Criterion for Smoothness) *Let $\pi : E \to M$ be a smooth vector bundle, and let $\tau : M \to E$ be a rough section. If (σ_i) is a smooth local frame for E over an open subset $U \subseteq M$, then τ is smooth on U if and only if its component functions with respect to (σ_i) are smooth.*

These propositions related to a correspondence between local frames and local trivializations leads us to a uniqueness result on the tangent bundle TM.

Proposition 8.14 (Uniqueness of the Smooth Structure on TM) *Let M be a n-manifold. The natural topology and smooth structure on TM are the unique ones with respect to which $\pi : TM \to M$ is a smooth vector bundle with the given vector space structure on the fibers, and such that all coordinate vector fields are smooth local sections.*

Covectors

We will start in general finite-dimensional vector spaces, and then move the discussion to manifolds.

Definition 8.15 (Covector on Vector Space, V) *Let V be a finite dimensional real vector space. A covector on V is a real valued linear functional $\omega : V \to \mathbb{R}$.*

The set of all covectors forms a new real valued vector space and is called the dual space of V, denoted by V^*. The next few propositions introduce a few important facts pertaining to the dual space.

Proposition 8.16 *Let V be a finite-dimensional vector space. Given any basis $(E_1, \dots, E_n)$ for V, let $\varepsilon^1, \dots, \varepsilon^n \in V^*$ be the covetors defined by*

$$\varepsilon^i(E_j) = \delta^i_j$$

where δ^i_j is the Kronecker delta symbol. Then $(\varepsilon_1, \dots, \varepsilon_n)$ is a basis for V^*, called the dual basis to (E_j). Thus, $dim\, V^* = dim\, V$.

From this we see that for a finite dimensional vector space V its dual space V^* has the same dimension.

Further, ε^i is the linear functional that picks out the ith component of a vector with respect to its basis. That is, if (E_j) is a basis for V and (ε^i) is its dual basis, then for any $v \in V$ we have

$$\varepsilon^i(v) = \varepsilon^i(v^j E_j) \tag{8.2}$$

$$= v^j \varepsilon^i(E_j) \tag{8.3}$$

$$= v^j \delta^i_j \tag{8.4}$$

$$= v^i \tag{8.5}$$

So just as we can represent any vector $v \in V$ with basis (E_i) by $v = v^i E_i$, we can represent any covector $\omega \in V^*$ in terms of its dual basis by

$$\omega = \omega_i \varepsilon^i, \tag{8.6}$$

where the components are determined by $\omega_i = \omega(E_i)$.

The action of the covector of ω on a vector $v = v^j E_j \in V$ is

$$\omega(v) = \omega_i v^i. \tag{8.7}$$

Having now introduced the covector, its basis representation and its action on a vector we take a look at the dual map.

Definition 8.17 (Dual Map) *Let V and W be vector spaces, and let $A:\ V \to W$ be a linear map. We define the dual map (or transpose of A) as the linear map $A^*:\ W^* \to V^*$ where*

$$(A^*\omega)(v) = \omega(Av) \quad \forall \omega \in W^*,\ v \in V.$$

Proposition 8.18 (Properties of the Dual Map) *The dual map satisfies the following properties:*

(i) $(A \circ B)^* = B^* \circ A^*$.

(ii) $(\mathtt{ID}_V)^*:\ V^* \to V^*$ *is the identity map of* V^*.

Since a dual vector space is a vector space we see that there is a second dual space $V^{**} = (V^*)^*$. And so for each vector space V there is a natural basis independent map $\xi : V \to V^{**}$ such that for each vector $v \in V$ we have a linear functional $\xi(v) : V^* \to \mathbb{R}$ where $\xi(v)(\omega) = \omega(v), \quad \forall \omega \in V^*$.

Proposition 8.19 *For any finite-dimensional vector space V, the map $\xi : V \to V^{**}$ is an isomorphism.*

So, although there is no canonical isomorphism $V \cong V^*$, we see there is a canonical isomorphism $V \cong V^{**}$ without reference to any basis. The real number $\omega(v)$ is often denoted by $\langle v, \omega \rangle$ or $\langle \omega, v \rangle$ respectively representing the action of a vector on a covector or the action of a covector on a vector. In particular, if $v \in V$ with the basis E_j, and $\omega \in V^*$ with the basis ε^i then $\langle E_j, \varepsilon^i \rangle = \langle \varepsilon^i, E_j \rangle = \delta^i_j$. It is important to note that notation is representing the action of vectors on covectors (or vice versa), and it should not be confused with the inner product which shares the same notation. The particular usage is be clear from context.

Having introduced the more general concept of covectors, we can now take a look at covectors on a manifold.

Cotangent Space

Recall that the tangent space of a manifold M at a point p is a vector space. So naturally there exists a vector space dual to T_pM.

Definition 8.20 (Cotangent Space at p) *Let T_pM be the tangent space on a manifold M at point p. Then the dual space to T_pM is the space $T^*_PM = (T_pM)^*$. Elements of T^*_PM are called tangent covectors at p, but they may also called dual vectors, covectors or covairant vectors at p.*

Let us take a look at how covectors transform, and in particular compare this to how the vectors transform.

Take an arbitrary smooth chart $(U, (x^i))$ on a smooth manifold M. For each $p \in U$ the coordinate basis $\left.\frac{\partial}{\partial x^i}\right|_p$ gives rise to a dual space T^*_PM, denoted for the moment as $\lambda|_p$. Any covector

$\omega \in T_p^*M$ can be written uniquely as $\omega = \omega_i \lambda^i|_p$, where the component ω_i is defined by

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right). \tag{8.8}$$

Now suppose there is another chart $(V, (\widetilde{x}^j))$ that also contains the point p (that is $p \in V$ too). We can let $\left(\widetilde{\lambda}_j|_p\right)$ be the dual basis to $\left(\left.\frac{\partial}{\partial \widetilde{x}^j}\right|_p\right)$. Recall that we have seen that the coordinate vector fields transform as

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\left.\frac{\partial}{\partial \widetilde{x}^j}\right|_p. \tag{8.9}$$

Well, we can now express the same covector ω with respect to both coordinates by

$$\omega = \widetilde{\omega}_j \widetilde{\lambda}^j|_p = \omega_j \lambda^j|_p. \tag{8.10}$$

Computing the components ω_i in terms of $\widetilde{\omega}_j$ we find

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) \tag{8.11}$$

$$= \omega\left(\frac{\partial \widetilde{x}^j}{\partial x^i}(p)\left.\frac{\partial}{\partial \widetilde{x}^j}\right|_p\right) \tag{8.12}$$

$$= \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\,\omega\left(\left.\frac{\partial}{\partial \widetilde{x}^j}\right|_p\right) \tag{8.13}$$

$$= \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\,\widetilde{\omega}_j. \tag{8.14}$$

Thus we have obtained the transformation rules for tangent vectors:

$$\widetilde{v}^j = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\,v^i. \tag{8.15}$$

And the transformation rules for tangent covectors:

$$\omega_i = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\,\widetilde{\omega}_j. \tag{8.16}$$

Having discussed tangent covectors we will relate this back to our earlier chapters and take a look at the cotangent bundle as a vector bundle.

Covector Fields

We had defined the tangent bundle as the disjoint union tangent spaces. It follows naturally that the cotangent bundle would be the disjoint union of the cotangent, or dual, spaces. It also follows naturally that the cotangent bundle would then also be a vector bundle.

Definition 8.21 (Cotangent Bundle of M) *Let M be a smooth manifold. The cotangent bundle of M is the disjoint union of the dual space over all $p \in M$:*

$$T^*M = \bigsqcup_{p \in M} T_p^*M.$$

The cotangent bundle of M has a natural projection map $\pi : T^*M \to M$ sending $\omega \in T_p^*M$ to $p \in M$. Similar to the above we can define the coordinate covector fields:

(a) Begin with a smooth coordinate chart $(U, (x^i))$.

(b) Then for each $p \in U$ denote the basis for the dual space T_p^*M dual to $\left(\frac{\partial}{\partial x^i} \bigg|_p \right)$ by $\lambda^i|_p$.

(c) We now have n maps $\lambda^1, \ldots, \lambda^n : U \to T^*M$ called the coordinate covector field.

Proposition 8.22 (The Cotangent Bundle as a Vector Bundle) *Let M be a smooth n-manifold. With its standard projection map and the natural vectors space structure on each fiber, the cotangent bundle T^*M has a unique topology and smooth structure making it into a smooth rank-n vector bundle over M for which all coordinate covector fields are smooth local sections.*

Just like the tangent bundle, smooth local coordinates of M result in smooth local coordinates for its cotangent bundle. If $(U,(x^i))$ is a smooth chart on M. Then, the map from $\pi^{-1}(U)$ to $\mathbb{R}^{2n}$ given by

$$\xi_i\lambda^i|_p \mapsto (x^1(p),\ldots,x^n(p),\xi_1,\ldots,\xi_n), \tag{8.17}$$

is a smooth coordinate chart for T^*M. We call (x^i,ξ_i) the natural coordinates for T^*M associated with (x^i).

In any smooth local coordinates an open subset $U\subseteq M$, a rough covector field ω can be written in terms of the coordinate covector fields (λ^i) as $\omega=\omega_i\lambda^i$ for n-functions $\omega_i:\ U\to\mathbb{R}$ called the component functions of ω where

$$\omega_i(p)=\omega_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right). \tag{8.18}$$

If ω is a rough covector field and X is a vector field on M, then we can form a function $\omega(x):$ $M\to\mathbb{R}$ by

$$\omega(X)(p)=\omega_p(X_p),\quad \forall p\in M. \tag{8.19}$$

A section of T^*M is called a covector field or a differential 1-form. In general, covector fields are assumed continuous, but they can also be assumed to be rough or smooth. If we write $\omega=\omega_i\lambda^i$ and $X=X^j\frac{\partial}{\partial x^j}$ in terms of local coordinates, then $\omega(X)$ has the local coordinate representation $\omega(X)=\omega_iX^i$. We have a number of ways to check for the smoothness of the covector field.

Proposition 8.23 (Smoothness Criteria for Covector Fields) *Let M be a smooth manifold, and let $\omega:\ M\to T^*M$ be a rough covector field. The following statements are equivalent:*

(i) ω is smooth.

(ii) In every smooth coordinate chart, the component functions of ω are smooth.

(iii) *Each point of M is contained in some coordinate chart in which ω has smooth component functions.*

(iv) *For every smooth vector field $X \in \mathscr{X}(M)$, the function $\omega(X)$ is smooth on M.*

(v) *For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X): U \to \mathbb{R}$ is smooth on U.*

Coframes

Just as we defined frames we can define coframes.

Definition 8.24 (Coframe) *Let M be a smooth manifold, and $U \subseteq M$ be an open subset. A local coframe for M over U is an ordered n-tuple of covector fields $(\varepsilon^1, \ldots, \varepsilon^n)$ defined on U such that $(\varepsilon^i|_p)$ forms a basis for T_p^*M at each point $p \in U$. If $E = U$, it is called a global coframe.*

Given a local frame $(E_1, \ldots, E_n)$ for TM over an open subset U, a coframe dual to (E_j) is the uniquely determined rough local coframe $(\varepsilon^1, \ldots, \varepsilon^n)$ over U such that $(\varepsilon^i|_p)$ is the dual basis to $(E_i|_p)$ for each $p \in U$. This condition is equivalent to $\varepsilon^i(E_j) = \delta^i_j$. As an example, in a smooth chart, the coordinate frame $\left(\partial/\partial x^i\right)$ and the coordinate coframe (λ^i) are dual to each other.

Lemma 8.25 *Let M be a smooth manifold. If (E_i) is a rough local frame over an open subset $U \subseteq M$ and (ε^i) is its dual coframe, then (E_i) is smooth if and only if (ε^i) is smooth.*

Given a local coframe (ε^i) over an open subset $U \subseteq M$, every rough covector field ω on U can be expressed in terms of the coframe as $\omega = \omega_i \varepsilon^i$ for some functions $\omega_1, \ldots, \omega_n : U \to \mathbb{R}$, called the component functions of ω with respect to the given coframe. The component functions are determined by $\omega_i = \omega(E_i)$, where (E_i) is the frame dual to (ε^i). This leads to another way of characterizing smoothness of covector fields.

Proposition 8.26 (Coframe Criterion for Smoothness of Covector Fields) *Let M be a smooth manifold, and let ω be a rough covector field on M. If (ε^i) is a smooth coframe on an open subset $U \subseteq M$, then ω is smooth on U if and only if its component functions with respect to (ε^i) are smooth.*

The real vector space of all smooth covector fields on M is denoted by $\mathscr{X}^*(M)$. Elements of $\mathscr{X}^*(M)$ are smooth sections of a vector bundle and thus can be multiplied by smooth real-valued functions. With $f \in C^\infty(M)$ and $\omega \in \mathscr{X}^*(M)$, the covector field $f\omega$ is defined by

$$(f\omega)_p = f(p)\omega_p. \tag{8.20}$$

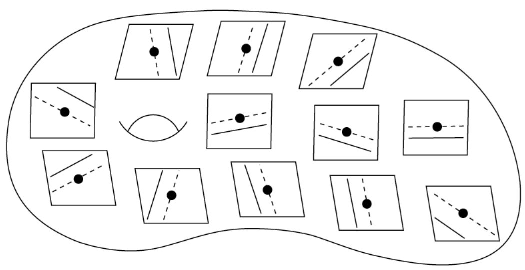

Figure 8.3: Covector Field

Geometrically we think of a vector field on M as an arrow attached to each point of M. A covector field can be visualized as a pair of hyperplanes (a codimension-1 linear subspace) in each tangent space, one through the origin and another parallel to it and varying continuously from point to point.

The Gradient as the Differential of a Function

The familiar notion of the gradient from calculus defined by

$$\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}, \tag{8.21}$$

does not make sense independently of coordinates. That means that we cannot interpret the partial derivatives of a smooth function, f, as the as the coordinate independent gradient of a vector field. So, we need to reconsider this for our purposes. It turns out that we can interpret the partial derivatives of the smooth function, f, as the components of a covector field.

We begin by introducing our definition of the differential as a covector, and then take a look about how this definition can be interpreted in a familiar way.

Definition 8.27 (The Differential of f as a Covector Field) *Let f be a smooth real-valued function on a smooth manifold M. We define a covector field df, called the differential of f, by*

$$df_p(v) = vf \quad \forall v \in T_pM. \tag{8.22}$$

Proposition 8.28 *The differential of a smooth function is a smooth covector field.*

Let us look at the coordinate representation for df. Let (x^i) be smooth coordinates on an open subset $U \subseteq M$, and let (λ^i) be the coframe on U. We can write df in coordinates as $df_p = A_i(p)\lambda^i|_p$ for some functions $A_i : U \to \mathbb{R}$ so from our definitions we get:

$$\begin{aligned} A_i(p) &= df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) && (8.23)\\ &= \left.\frac{\partial}{\partial x^i}\right|_p f && (8.24)\\ &= \frac{\partial f}{\partial x^i}(p). && (8.25) \end{aligned}$$

So we have

$$\begin{aligned} df_p &= A_i(p)\lambda^i|_p && (8.26)\\ &= \frac{\partial f}{\partial x^i}(p)\,\lambda^i|_p. && (8.27) \end{aligned}$$

Next, lets look as the special case where f is one of the coordinate functions $x^j : U \to \mathbb{R}$

$$\begin{aligned} df_p &= dx^j|_p && (8.28)\\ &= \frac{\partial x^j}{\partial x^i}(p)\,\lambda^i|_p && (8.29)\\ &= \delta_i^j\lambda^i|_p && (8.30)\\ &= \lambda^j|_p. && (8.31) \end{aligned}$$

So we see that the covector field λ^j is just the differential of x^j. So, we can rewrite our formula for the differential of f as

$$df_p = \frac{\partial f}{\partial x^i}(p)\, dx^i|_p. \tag{8.32}$$

Or written in terms of covector fields instead of covectors as

$$df = \frac{\partial f}{\partial x^i} dx^i. \tag{8.33}$$

We also have a number of familiar properties associate with the differential.

Proposition 8.29 (Properties of the Differential) *Let M be a smooth manifold, and let $f, g \in C^\infty(M)$.*

(i) If a and b are constants, then $d(af + bg) = ad(f) + bdg$.

(ii) $d(fg) = fdg + gdf$.

(iii) $d(f/g) = (gdf - fdg)/g^2$ on the set where $g \neq 0$.

(iv) If $J \subseteq \mathbb{R}$ is an interval containing the image of f, and $h :\ J \to \mathbb{R}$ is a smooth function, then $(h \circ f) = (h' \circ f)df$.

(v) If f is constant, then $df = 0$.

Where we can take property (v) above even further.

Proposition 8.30 (Functions with Vanishing Differentials) *If f is a smooth real valued function on a smooth manifold M, then $df = 0$ if and only if f is constant on each component of M.*

We also find that the differential can be used to defined the derivative of a function along a curve.

Proposition 8.31 (Derivative of a Function Along a Curve) *Let M be a smooth manifold, $\gamma : J \to M$ be a smooth curve, and $f : M \to \mathbb{R}$ be a smooth function. Then the derivative of the real-valued function $f \circ \gamma : J \to \mathbb{R}$ is given by*

$$(f \circ \gamma)'(t) = df_{\gamma(t)}\left(\gamma'(t)\right). \tag{8.34}$$

We have now defined a differential of a smooth function, f, at a point $p \in M$ in two ways:

a) df_p as a linear map from T_pM to $T_{f(p)}\mathbb{R}$, and

b) df_p as a covector at p

Since , by definition, df_p as a covector at p is a linear map from T_pM to $\mathbb{R}$. With the canonical isomorphism between $\mathbb{R}$ and $T_{f(p)}\mathbb{R}$, we can consider df_p as in a and b above as the same object.

Similarly, if γ is a smooth curve in M, we can interpret $(f \circ \gamma)'(t)$ as:

(a) the smooth curve $f \circ \gamma$, in $\mathbb{R}$, where $(f \circ \gamma)'(t)$ is the velocity at the point $f \circ \gamma(t) \in T_{f\circ\gamma(t)}\mathbb{R}$, or

(b) $f \circ \gamma$ can be considered simply as a real-valued function of one real variable, and thus $(f \circ \gamma)'(t)$ is just its ordinary derivative equal to $df_{\gamma(t)}(\gamma'(t))$

Pullbacks of Covector Fields

We have seen that a smooth map yields a linear map on tangent vectors called the differential. Considering the dual of this map leads this to a new linear map on covectors going in the opposite direction.

Definition 8.32 (Cotangent Map of F) *Let $F : M \to N$ be a smooth map between smooth manifolds, and let $p \in M$ be arbitrary. The differential $dF_p : T_pM \to T_{F(p)}N$ yields a dual linear map*

$$dF_p^* : T_{F(p)}^*N \to T_p^*M, \tag{8.35}$$

called the cotangent map of F (or pullback by F of p).

From the definitions we see that dF_p^* is characterized by

$$dF_p^*(\omega)(v) = \omega(dF_p(v)), \quad \forall \omega \in T^*_{F(p)}N,\ v \in T_pM. \tag{8.36}$$

Recall that a pointed smooth manifold is an ordered pair (M, p) where M is a smooth manifold and $p \in M$. The assignments $(M, p) \mapsto T_p^*M$ and $F \mapsto dF_p^*$ yield a contravariant functor from the category of pointed smooth manifolds to the category of vector spaces.

Recall also, that pushforwards of vector fields under smooth maps are defined only in the case of diffeomorphisms or Lie group homomorphisms. However, covector fields always pull back to covector fields.

Definition 8.33 (Pullback of ω by F) *Let $F : M \to N$, and let ω be a covector field on N. A pullback of ω by F as the rough covector field $F^*\omega$ on M defined by*

$$(F^*\omega)_p = dF_p^*(\omega_{F(p)}). \tag{8.37}$$

The pullback acts on a vector $v \in T_pM$ by

$$(F^*\omega)_p(v) = \omega_{F(p)}\left(F_p(v)\right). \tag{8.38}$$

Some additional properties of the pullback are listed below.

Proposition 8.34 *Let $F : M \to N$ be a smooth map between smooth manifolds. Suppose u is a continuous real-valued functions on N, and ω is a covector field on N. Then*

$$F^*(u\,\omega) = (u \circ F)F^*\omega. \tag{8.39}$$

If in addition u is smooth, then

$$F^*du = d(u \circ F). \tag{8.40}$$

Proposition 8.35 *Suppose $F : M \to N$ is a smooth map between smooth manifolds, and let ω be a covector field on N. Then $F^*\omega$ is a continuous covector field on M. If ω is smooth, then so is $F^*\omega$.*

In coordinate form, we have two equal ways of representing the pullback of ω. With $\omega = \omega_j dy^j$, $F : M \to N$, and $(V,(y^j))$ as a smooth chart on N. And letting $p \in M$ with $U = F^{-1}(V)$ as a neighborhood of P we find

$$F^*\omega = F^*(\omega_j dy^j) \tag{8.41}$$

$$= (\omega_j \circ F)F^* dy^j \tag{8.42}$$

$$= (\omega_j \circ F)d(y^j \circ F) \tag{8.43}$$

and

$$F^*\omega = (\omega_j \circ F)d(y^j \circ F) \tag{8.44}$$

$$= (\omega_j \circ F)dF^j, \tag{8.45}$$

where F^j is the jth component function of F in these coordinates. We find the computation of the pullback in coordinates is very simple in the following example.

Example 8.36 (Pullback in Coordinates) *Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be the map given by*

$$(u,v) = F(x,y,z) = (x^2 y, y \sin z),$$

and let $\omega \in \mathscr{X}^*(\mathbb{R}^2)$ be the covector field

$$\omega = udv + vdu.$$

Then we can compute the pullback of $F * \omega$ as

$$F^*\omega = (u \circ F)d(v \circ F) + (v \circ F)d(u \circ F) \tag{8.46}$$

$$= (x^2y)d(y\sin z) + (y\sin z)d(x^2y) \tag{8.47}$$

$$= x^2y(\sin z dy + y\cos z dz) + y\sin z(y2xdx + x^2dy) \tag{8.48}$$

$$= 2xy^2\sin z dx + 2x^2y\sin z dy + x^2y^2\cos z dz. \tag{8.49}$$

Line Integrals

Covectors are also used in establishing a coordinate free sense of line integrals.

Definition 8.37 (Line Integral of ω) *Let $[a,b] \subseteq \mathbb{R}$ be a compact interval, and ω be a smooth covector field on $[a,b]$. Letting t be the standard coordinate on $\mathbb{R}$, then ω can be written $\omega_t = f(t)dt$ for some smooth function $f: [a,b] \to \mathbb{R}$. We define the integral of ω over $[a,b]$ to be*

$$\int_{[a,b]} \omega = \int_a^b f(t)dt. \tag{8.50}$$

We find a relation between the pullback and the line integral.

Proposition 8.38 (Diffeomorphism Invariance of the Integral) *Let ω be a smooth covector field on the compact interval $[a,b] \subseteq \mathbb{R}$. If $\varphi: [c,d] \to [a,b]$ is an increasing diffeomorphism (meaning that $t_1 < t_2 \Rightarrow \varphi(t_1) < \varphi(t_2)$), then*

$$\int_{[c,d]} \varphi^*\omega = \int_{[a,b]} \omega \tag{8.51}$$

Now, with a smooth manifold M we define a curve segment below.

Definition 8.39 (Curve Segment) *Let M be a smooth manifold. We define a curve segment in M as a continuous curve $\gamma: [a,b] \to M$ whose domain is a compact interval.*

If γ has an extension to a smooth curve defined in a neighborhood of each endpoint, then we say γ is a smooth curve segment. If there exists a finite partition $a = a_0 < a_1 < \cdots < a_k = b$ of $[a,b]$ such that $\gamma|_{[a_{i-1},a_i]}$ is smooth for each i we say that γ is a piecewise smooth curve segment.

With piecewise smooth line segment defined we see that we can find one on any connected smooth manifold.

Proposition 8.40 *If M is a connected smooth manifold, any two points of M can be joined by a piecewise smooth curve segment.*

We can now look at our line integral over a curve gamma.

Definition 8.41 (Line integral of ω over γ) *Let $\gamma\colon [a,b] \to M$ be a smooth curve segment and ω be a covector field on M. We define the line integral of ω over γ to be the real number*

$$\int_\gamma \omega = \int_{[a,b]} \gamma^* \omega \tag{8.52}$$

We find a number of familiar properties of the line integral.

Proposition 8.42 (Properties of Line Integrals) *Let M be a smooth manifold. Suppose γ : $[a,b] \to M$ is a piecewise smooth curve segment on M, and $\omega, \omega_1, \omega_2 \in \mathscr{X}^*(M)$.*

(i) *For any $c_1, c_2 \in \mathbb{R}$ we have*

$$\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2.$$

(ii) *If γ is a constant map, then $\int_\gamma \omega = 0$.*

(iii) *$\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$ where $a < c < b$, then*

$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

(iv) If $F: M \to N$ is any smooth map and $\eta \in \mathscr{X}^*(N)$, then

$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$$

The most significant feature of line integrals it that they are invariant of the reparametrization.

We will first formally introduce the definition for rearametrization, and then we will discuss the invariance.

Definition 8.43 (Reparametrization) *Let M be a smooth manifold. If $\gamma: [a,b] \to M$ and $\widetilde{\gamma}: [c,d] \to M$ are piecewise smooth curve segments, we say that $\widetilde{\gamma}$ is a reparametrization of γ if $\widetilde{\gamma} = \gamma \circ \varphi$ for some diffeomorphism $\varphi: [c,d] \to [a,b]$.*

If φ is an increasing function, we say that $\widetilde{\gamma}$ is a forward reparametrization. If φ is a decreasing function, we say that $\widetilde{\gamma}$ is a backwards reparametrization.

Proposition 8.44 (Parameter Independence of Line Integrals) *Suppose M is a smooth manifold, $\omega \in \mathscr{X}^*(M)$, and γ is a piecewise smooth curve segment in M. For any reparametrization $\widetilde{\gamma}$ of γ, we have:*

$$\int_{\widetilde{\gamma}} \omega = \begin{cases} \int_\gamma \omega & \text{if } \widetilde{\gamma} \text{ is a forward reparametrization,} \\ \int_{\widetilde{\gamma}} \omega & \text{if } \widetilde{\gamma} \text{ is a backward reparametrization.} \end{cases} \tag{8.53}$$

We can also express the line integral as a regular integral.

Proposition 8.45 *If $\gamma: [a,b] \to M$ is a piecewise smooth curve segment, the line integral of ω over γ can be expressed as*

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}\left(\gamma'(t)\right) dt. \tag{8.54}$$

Notice that for $f \in C^\infty(M)$ and $\gamma\colon\ [a,b] \to M$, we can express the line integral of the differential as

$$\int_\gamma df = \int_a^b df_{\gamma(t)}\left(\gamma'(t)\right)dt \tag{8.55}$$
$$= \int_a^b (f\circ\gamma)'(t)dt \tag{8.56}$$
$$= (f\circ\gamma)(b) - (f\circ\gamma)(a) \tag{8.57}$$
$$= f(\gamma(b)) - f(\gamma(a)) \tag{8.58}$$

So evaluating the line integral of a differential is trivial.

Theorem 8.46 (Fundamental Theorem for Line Integrals) *Let M be a smooth manifold. Suppose f is a smooth real-valued function on M and γ is a piecewise smooth curve segment in M. Then*

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

Conservative Covector Fields

We just saw how it is easy to calculate the line integral in a covector field when it can be represented as the differential of a smooth function. This type of field is given its own name.

Definition 8.47 (Exact Covector Field) *A smooth covector field ω is said to be exact (or an exact differential) on M if there is a function $f \in C^\infty(M)$ such that $\omega = df$*

The function f of an exact differential is called the potential for the covector field ω. Although, the function f need not be unique, any two potential functions must only differ by a constant at each point.

From the fundamental theorem for line integrals, we can see that in an exact covector field, the line integral of a curve γ depends only on the endponts. In the case where the value of the endpoints are the same we have

$$\gamma(b) = \gamma(a) \tag{8.59}$$

$$\Rightarrow \gamma(b) - \gamma(a) = 0 \tag{8.60}$$

$$\Rightarrow \int_{\gamma} df = 0, \tag{8.61}$$

and we call γ a closed curve segment. We say that a smooth covector field ω is conservative if the line integral of ω over every piecewise smooth closed curve segment is zero. Conservative covector fields can also be characterized by path independence.

Proposition 8.48 *A smooth covector field ω is conservative if and only if its line integrals are path-independend, in the sense that $\int_{\gamma} \omega = \int_{\widetilde{\gamma}} \omega$ whenever γ and $\widetilde{\gamma}$ are piecewise smooth curve segments with the same starting and ending points.*

Theorem 8.49 *Let M be a smooth manifold. A smooth covector field on M is conservative if and only if it is exact.*

We find that there is a property of exact covector fields that will help us determine if a covector field is exact. Suppose that $\omega \in \mathscr{X} * (M)$ is exact. Let f be the potential for ω, and let $(U, (x^i))$ be any smooth chart on M. Because f is smooth, we know that the order of partial derivatives on U do not matter. That is

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}. \tag{8.62}$$

We can write ω in coordinates as $\omega_i dx^i$. Then $\omega = df$ implies that the component functions are $\omega_i = \frac{\partial f}{\partial x^i}$. Likewise, $\omega_j = \frac{\partial f}{\partial x^j}$. We can now use this to expand upon the above.

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} \tag{8.63}$$

$$\Rightarrow \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right) = \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) \tag{8.64}$$

$$\Rightarrow \frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}. \tag{8.65}$$

Definition 8.50 (Closed Covector Field) *We say that a smooth covector field is closed if its components in every smooth chart satisfy* $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ *for each pair of indices* i *and* j.

As demonstrated, getting from exact to closed is straight forward.

Proposition 8.51 *Every exact covector field is closed.*

Getting from closed to exact takes a little more work. First lets consider some statements equivalent to being closed.

Proposition 8.52 *Let* ω *be a smooth covector field on a smooth manifold* M. *The following are equivalent:*

(i) ω *is closed.*

(ii) ω *satisfies* $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ *in some smooth chart around every point.*

(iii) For any open subset $U \subseteq M$ *and smooth vector fields* $X, Y \in \mathscr{X}(U)$,

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X,Y]).$$

So we can quickly show many covector fields are not exact using (ii) to establish a counter example.

Another way to check exactness is with pullpacks.

Corollary 8.53 *Suppose $F : M \to N$ is a local diffeomorphism. Then the pullback $F^* : \mathscr{X}^*(N) \to \mathscr{X}^*(M)$ takes closed covector fields to closed covector fields, and exact ones to exact ones.*

This gives us a way to see exactness if we are pulling back from an exact covector field. This is particularly helpful if we introduce the star-shaped subset.

Definition 8.54 (Star-shaped Subset) *If V is a finite dimensional vector space, a subset $U \subseteq V$ is said to be starshaped if there is a point $c \in U$ such that for every $x \in U$ the line segment from c to x is entirely contained in U.*

Theorem 8.55 (Poincaré Lemma for Covector Fields) *If U is a star-shaped open subset of $\mathbb{R}^n$ or the upper-space $\mathbb{H}^n$, then every closed covector field on U is exact.*

Thus we can find local exactness directly from closed covector fields.

Corollary 8.56 (Local Exactness of Closed Covector Fields) *Let ω be a closed covector field on a smooth manifold M. Then every point of M has a neighborhood on which ω is exact.*

Above we see a number of ways to find if the covector field is exact. If you know that your covector field, ω, is exact, finding a potential function, f, for ω can be done by taking $\frac{\partial f}{\partial x^i}$ and integrating by $\frac{\partial f}{\partial x^j}$ according to the typical calculus process.

Chapter Summary

We began this chapter with the concept of the vector bundle. We saw that the tangent bundle is in fact a specific vendor bundle. Further we saw how we can relate overlapping trivializations to the general linear group. We concluded with a short cut for making a set into a vector bundle by constructing local trivializations with a smooth overlap.

We then introduce the concepts of frames on a vector bundle, and found that they are really a generalization of the frames constructed earlier on Manifolds. We then related frames with trivializations, and found that our the topology and smooth structure of our tangent bundle are unique.

Next we defined the dual space on a manifold and formally introduced all the related objects. Of primary interest, is that the dual space provides the domain to perform the gradient operation and measure the line integral.

CHAPTER IX

COORDINATE-FREE REPRESENTATIONS ON A MANIFOLD

Conceptual Introduction

Earlier we defined how we can represent objects in coordinates, and we have seen how they can be useful in performing calculations. However, since coordinates are arbitrary, it is often useful to avoid moving into any particular coordinate reference frame, and work independently of coordinates. The tool for this type of operation is the tensor. We also find that the tensor, just like vectors, can be looked at on a vector space and have covariant and contravariant types. However, as tensors are a more general object they can also be of mixed type and used for various purposes as an object that takes in vectors and covectors to transform them into scalars. Tensors can be thought of as multilinear machines that take in any number of vectors or covectors and spit out a scalar. Although the notion of a tensor may be new, we have seen it before. Vectors, dual vectors and even scalars are particular examples of tensors.

Tensor Product Space

Multilinear Algebra

Tensors are essentially just multilinear functions of one or more variables.

Definition 9.1 (Multilinear) *Suppose $V_1, \ldots, V_k$ and W are vector spaces. A map $F : V_1 \times \ldots \times V_k \to W$ is said to be multilinear if it is a linear function of each variable separately when the others are held fixed. That is for each i,*

$$F(v_1, \ldots, av_i + a'v_i', \ldots, v_k) = aF(v_1, \ldots, v_i, \ldots, v_k) + a'F(v_1, \ldots, v_i', \ldots, v_k).$$

We denote the set of all multilinear maps $V_1,\ldots,V_k$ to W as $L(V_1,\ldots,V_k;W)$, and this set is a vector space under pointwise addition and scalar multiplication.

Example 9.2 (Tensor Product of Covectors) *Suppose V is a vector space, and $\omega,\eta\in V^*$. Define a function*

$$\omega\otimes\eta(v_1,v_2)=\omega(v_1)\eta(v_2),$$

where the product on the right is just ordinary multiplication of real numbers. The linearity of ω and η guarantees that $\omega\otimes\eta$ is a bilinear function of v_1 and v_2, so $\omega\otimes\eta\in L(V,V;\mathbb{R})$. Specifically, if (e^1,e^2) denotes the standard basis for $(\mathbb{R}^2)^*$, then $e^1\otimes e^2((w,x),(y,z))=wz$.

The generalization of this example provides us with the definition for a tensor product.

Definition 9.3 (Tensor Product) *Let $V_1,\ldots,V_k,W_1,\ldots,W_l$ be real vector spaces and suppose $F\in L(V_1,\ldots,V_k;\mathbb{R})$, and $G\in L(W_1,\ldots W_l;\mathbb{R})$. The tensor product is the function*

$$F\otimes G:\ V_1\times\ldots\times V_k\times W_1\times\ldots\times W_l\to\mathbb{R},$$

defined by

$$F\otimes G(v_1,\ldots,v_k,w_1,\ldots,w_k)=F(v_1,\ldots,v_k)G(w_1,\ldots,w_l).$$

Since F and G are multilinear it follows the $F\otimes G\in L(V_1,\ldots,V_k,W_1,\ldots,W_l;\mathbb{R})$.

The tensor product operation is associative. As a result we see that we can unambiguously write the tensor product of any number of multilinear functions without parenthesis. Further, the tensor product is useful in that a basis for any space of multilinear functions can be formed by taking all possible tensor products of basis covectors.

Proposition 9.4 (A Basis for the Space of Multilinear Functions) *Let $V_1,\ldots,V_k$ be real vector spaces of dimensions $n_1,\ldots,n_k$, respectively. For each $j\in\{1,\ldots,k\}$, let $(E_1^{(j)},\ldots,E_{n_j}^{(j)})$ be a basis for V_j, and let $(\varepsilon_{(j)}^1,\ldots,\varepsilon_{(j)}^{n_j})$ be the corresponding dual basis for V_j^*. Then the set*

$$\mathscr{B}=\left\{\varepsilon_{(1)}^{i_1},\ldots,\varepsilon_{(k)}^{i_k}:\ (1\le i_1\le n_1),\ldots,(1\le i_k\le n_k)\right\},$$

is a basis for $L(V_1,\ldots,V_k;\mathbb{R})$, which has dimension equal to $n_1\cdots n_k$.

Suppose $F\in L(V_1,\ldots,V_k;\mathbb{R})$. for each ordered k-tuple $(i_1,\ldots,i_k)$ of integers with $1\le i_j\le n_j$, define a number

$$F_{i_1,\ldots,i_k}=F\left(E_{i_1}^{(1)},\ldots E_{i_k}^{(k)}\right). \tag{9.1}$$

We find that

$$F=F_{i_1,\ldots,i_k}\varepsilon_{(1)}^{i_1}\otimes\cdots\otimes\varepsilon_{(k)}^{i_k}. \tag{9.2}$$

This means that the multilinear function F is completely determined by its action on all possible sequences of basis vectors.

Abstract Tensor Products of Vector Spaces

We begin the construction of the abstract tensor products by defining the formal linear combination and free vector space.

Definition 9.5 (Formal Linear Combination of a Set) *Let S be a set. A formal linear combination of elements of S is a function $f:\ S\to\mathbb{R}$ such that $f(s)=0$ for all but a finite number of elements $s\in S$.*

Definition 9.6 (Free Vector Space on a Set) *Let S be a set. The free (real) vector space on S, denoted by $\mathscr{F}(S)$, is the set of all formal linear combinations of the elements of S. The set $\mathscr{F}(S)$ forms a vector space under pointwise addition and scalar multiplication.*

Since $\mathscr{F}(S)$ is the set of all formal linear combinations of S, for each $x \in S$, we have a function $\delta_x \in \mathscr{F}(S)$ such that

$$\delta_x(s) = \begin{cases} 1: & s = x \\ & \\ 0: & s \neq x \end{cases} . \tag{9.3}$$

We can instead identify δ_x with x itself as

$$\delta_x(s) = \begin{cases} x: & s = x \\ & \\ 0: & s \neq x \end{cases} . \tag{9.4}$$

In this case $S \subseteq \mathscr{F}(S)$, and each element $f \in \mathscr{F}(S)$ can be written uniquely in the familiar form for linear combinations as

$$f = \sum_{i=1}^{m} a_i x_i, \tag{9.5}$$

where $x_1, \ldots, x_m \in S$ such that $f(x_i) \neq 0$, and $a_i = f(x_i)$.

We thus find that S is a basis for $\mathscr{F}(S)$ and is finite dimensional if and only if S is a finite set.

Proposition 9.7 (Characteristic Property of the Free Vector Space) *For any set S and any vector space W, every map $A: S \to W$ has a unique extension to a linear map $\overline{A}: \mathscr{F}(S) \to W$.*

With this we can now construct the abstract tensor product of vector spaces.

a) Let $V_1, \ldots, V_k$ be real vector spaces. We can form the free vector space $\mathscr{F}(V_1 \times \cdots \times V_k)$, which is the set of all finite formal linear combinations of k-tuples $(v_1, \ldots, v_k)$ with $v_i \in V_i$ for $i = 1, \ldots, k$.

b) Next let $\mathscr{R}$ be the subspace of $\mathscr{F}(V_1 \times \cdots \times V_k)$ spanned by all elements of the following forms:

$$(v_1, \ldots, av_1, \ldots, v_k) - a(v_1, \ldots, v_i, \ldots, v_k), \tag{9.6}$$

$$(v_1, \ldots, v_i + v_i', \ldots, v_k) - (v_1, \ldots, v_i, \ldots, v_k) - (v_1, \ldots, v_i', \ldots, v_k), \tag{9.7}$$

where $v_i,\ v_i' \in V_i$, $i \in \{1, \ldots, k\}$, and $a \in \mathbb{R}$.

c) We can now define the tensor product of the spaces $V_1, \ldots, V_k$, denoted $V_1 \otimes, \cdots, \otimes, V_k$, to be the following vector space:

$$V_1 \otimes \cdots \otimes V_k = \mathscr{F}(V_1 \times \cdots \times V_k)/\mathscr{R}. \tag{9.8}$$

d) Then by letting $\Pi :\ \mathscr{F}(V_1 \times \cdots \times V_k) \to V_1 \otimes \cdots \otimes V_k$ be the natural projection from the free vector space onto the tensor product of the vector spaces, we find the (abstract) tensor product of $v_1, \ldots, v_k$ is the equivalence class of an element $(v_1, \ldots, v_k) \in V_1 \otimes \cdots \otimes V_k$, denoted by

$$v_1 \otimes \cdots \otimes v_k = \Pi(v_1, \ldots, v_k). \tag{9.9}$$

We have thus constructed our abstract tensor product. Notice that multilinearity follows directly from our definition. Further, the definition implies that every element of $V_1 \otimes \cdots \otimes V_k$ can be expressed as a linear combination of the form $v_1 \otimes \cdots \otimes v_k$ for $v_i \in V_i$. However, it is not necessarily the case that every element of the tensor product space is of the form $v_1 \otimes \cdots \otimes v_k$.

Proposition 9.8 (Characteristic Property of the Tensor Product Space) *Let $V_1, \ldots, V_k$ be finite-dimensional real vector spaces. If $A :\ V_1 \times \cdots \times V_k \to X$ is any multilinear map into vector space X and π is the map $\Pi :\ \mathscr{F}(V_1 \times \cdots \times V_k) \to V_1 \otimes \cdots \otimes V_k$ defined, as above, by $\pi(v_1, \ldots, v_k) = v_1 \otimes \cdots \otimes v_k$, then there is a unique linear map $\widetilde{A} :\ V_1 \otimes \cdots \otimes V_k \to X$ such that the following diagram commutes:*

$$
\begin{tikzcd}
V_1 \times \cdots \times V_k \arrow[r, "A"] \arrow[d, "\pi"'] & X \\
V_1 \otimes \cdots \otimes V_k \arrow[ur, dashed, "\widetilde{A}"'] &
\end{tikzcd}
$$

Figure 9.1: Characteristic Property of the Tensor Product Space

So, we see that the tensor product is uniquely determined, up to isomorphism, by the characteristic property.

We saw earlier how we can construct a basis of multilinear functions. We now see a corollary for tensor product spaces.

Proposition 9.9 (A Basis for the Tensor Product Space) *Suppose $V_1, \ldots, V_k$ are real vector spaces of $n_1, \ldots, n_k$ dimensions, respectively. For each $j = 1, \ldots, k$, suppose $\left(E_1^{(j)}, \ldots, E_{n_k}^{(j)}\right)$ is a basis for V_j. Then the set*

$$
\mathscr{C} = \left\{ E_{i_1}^{(1)} \otimes \cdots \otimes E_{i_k}^{(k)} : \ (1 \leq i_1 \leq n_1), \ldots, (1 \leq i_k \leq n_k) \right\},
$$

is a basis for $V_1 \otimes \cdots \otimes V_k$, which therefore has dimension equal to $n_1 \cdots n_k$.

There is a manner in which we can think of tensor product spaces as associative.

Proposition 9.10 (Associativity of Tensor Product Spaces) *Let V_1, V_2, V_3 be finite-dimensional vector spaces. There are unique isomorphims*

$$
V_1 \otimes (V_2 \otimes V_3) \cong V_1 \otimes V_2 \otimes V_3 \cong (V_1 \otimes V_2) \otimes V_3,
$$

under which elements of the forms $v_1 \otimes (v_2 \otimes v_3), v_1 \otimes v_2 \otimes v_3, (v_1 \otimes v_2) \otimes v_3$ all correspond.

We can relate the abstract and the more concrete versions of the tensor product through an isomorphism.

Proposition 9.11 (Abstract Vs. Concrete Tensor Products) *If $V_1,\ldots,V_k$ are finite-dimensional vector spaces, there is a canonical isomorphism*

$$V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1,\ldots,V_k;\mathbb{R}),$$

under which the abstract tensor product $v_1 \otimes \cdots \otimes v_k = \Pi(v_1,\ldots,v_k)$ corresponds to the concrete definition for the tensor product defined as $\omega^1 \otimes \cdots \otimes \omega^k(v_1,\ldots,v_k) = \omega^1(v_1)\cdots\omega^k(v_k)$.

Further, when dealing with finite-dimensional vector spaces we obtain another canonical identification

$$V_1 \otimes \cdots \otimes V_k \cong L(V_1^*,\ldots,V_k^*;\mathbb{R}). \tag{9.10}$$

Covariant and Contravariant Tensors on a Vector Space

Within this section we will simply introduce the definition for covariant tensors and contravariant tensors on a vectors space and identify their basis.

Definition 9.12 (Covariant k-Tensor on a Vector Space) *Let V be a finite-dimensional real vector space. If k is a positive integer, a covariant k-tensor on V is an element of the k-fold tensor product $V^* \otimes \cdots \otimes V^*$, which we typically think of as a real-valued multilinear function of k elements of V:*

$$\alpha : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

The number k is called the rank of α.

We denote the set of all covariant k-tensors on V by

$$T^k(V^*) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ copies}}. \tag{9.11}$$

We can denote the space of contravariant tensors on V of rank k to be the vector space

$$T^k(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}}. \tag{9.12}$$

Because we are assuming that V is finite dimensional, we can identify this space with the set of multilinear functionals of k covectors:

$$T^k(V) \cong \left\{\text{multilinear functions } \alpha : \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \to \mathbb{R}\right\}. \tag{9.13}$$

More generally, we can define a mixed type tensor.

Definition 9.13 (Mixed Tensors on a Vector Space) *Let V be a vector space. We define the space of mixed tensors on V of type (k,l) as*

$$T^{(k,l)}(V) = \underbrace{V \otimes \cdots \otimes V}_{k \textit{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \textit{ copies}}.$$

Note that $T^{(k,l)}$ can sometimes be denoted by T^k_l or even T^l_k so it is important to understand the context behind the notation.

Some mixed type spaces are identical. For instance:

a) $T^{(0,0)}(V) = T^0(V^*) = T^0(V) = \mathbb{R}$,

b) $T^{(0,1)} = T^1(V^*) = V^*$,

c) $T^{(1,0)} = T^1(V) = V$,

d) $T^{(0,k)} = T^k(V^*)$,

e) $T^{(k,0)} = T^k(V)$.

With V as a finite dimensional vector space, the choice of basis for V automatically yields bases for all of the tensor spaces over V.

Corollary 9.14 *Let V be an n-dimensional real vector space. Suppose (E_i) is any basis for V and (ε^j) is the dual basis for V^*. Then the following sets constitute bases for the tensor spaces over V:*

(i) For $T^k(V^*)$:

$$\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} : 1 \leq i_1, \ldots, i_k \leq n\}.$$

(ii) For $T^k(V)$:

$$\{E_{i_1} \otimes \cdots \otimes E_{i_k} : 1 \leq i_1, \ldots, i_k \leq n\}.$$

(iii) For $T^{(k,l)}(V)$.:

$$\{E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l} : 1 \leq i_1, \ldots, i_k, j_1, \ldots, j_l \leq n\}.$$

Therefore, $dim\, T^k(V^*) = dim\, T^k(V) = n^k$ and $dim\, T^{(k,l)}(V) = n^{k+l}$.

In particular, once a basis is chosen for V, every covariant k-tensor $\alpha \in T^k(V^*)$ can be written uniquely in the form

$$\alpha = \alpha_{i_1,\ldots,i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}, \tag{9.14}$$

where the n^k coefficients $\alpha_{i_1,\ldots,i_k}$ are determined by

$$\alpha_{i_1,\ldots,i_k} = \alpha(E_{i_1}, \ldots, E_{i_k}). \tag{9.15}$$

For example, $T^2(V^*)$ is the space of bilinear forms on V, and every bilinear form can be written as $\beta = \beta_{ij}\varepsilon^i \otimes \varepsilon^j$ for some uniquely determined $n \times n$ matrix (β_{ij}).

Symmetric and Alternating Tensors

In general, changing the order of the arguments within a covariant tensor do not have a predictable result. However, there are two special tensors that do. In particular,

(a) The result of *symmetric* covariant tensors is unchanged when rearranging the arguments.

(b) The result of *alternating* covariant tensors is a sign change when rearranging the arguments.

Symmetric Tensors

Definition 9.15 (Symmetric covariant k-tensor) *Let V be a finite-dimensional vector space. A covariant k-tensor α on V is said to be symmetric if its value is unchanged by interchanging any pair of arguments. That is, whenever $1 \leq i \leq j \leq k$,*

$$\alpha(v_1,\dots,v_i,\dots,v_j\dots v_k) = \alpha(v_1,\dots,v_j,\dots,v_i\dots v_k).$$

Given the space of all covariant k-tensors on V, $T^k(V^*)$, we find that the set of symmetric covariant k-tensors is a linear subspace of $T^k(V^*)$. We denote this subspace by $\Sigma^k(V^*)$. There is a natural projection from $T^k(V^*)$ to $\Sigma^k(V^*)$ defined as follows:

(a) Let S_k denote the symmetric group on k elements (The group of permutations of the set $\{1,\dots,k\}$.

(b) Given a k-tensor α and a permutation $\sigma \in S_k$ define a new k-tensor, ${}^{\sigma}\alpha$, by

$$^{\sigma}\alpha(v_1,\dots,v_k) = \alpha(v_{\sigma(1)},\dots,v_{\sigma(k)}). \tag{9.16}$$

(c) Next we can define the projection $Sym : \ T^k(V^*) \to \Sigma^k(V^*)$ called symmetrization by

$$Sym\,\alpha = \frac{1}{k!}\sum_{\sigma\in S_k} {}^{\sigma}\alpha. \tag{9.17}$$

So in terms of action we have

$$(Sym\ \alpha)(v_1,\ldots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k}\alpha(v_{\sigma(1)},\ldots,v_{\sigma(k)}) \tag{9.18}$$

Proposition 9.16 (Properties of Symmetrization) *Let α be a covariant tensor on a finite-dimensional vector space.*

(i) Sym α is symmetric.

(ii) Sym $\alpha = \alpha$ if and only if α is symmetric.

If α and β are symmetric tensors, then the tensor product $\alpha \otimes \beta$ is not in general symmetric. So to insure a symmetric result we define a new product called the symmetric product defined by

$$\alpha\beta = Sym\ (\alpha \otimes \beta) \tag{9.19}$$

So this action is

$$\alpha\beta(v_1,\ldots,v_{k+l}) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}\alpha(v_{\sigma(1)},\ldots,v_{\sigma(k)})\beta(v_{\sigma(k+1)},\ldots,v_{\sigma(k+l)})$$

Proposition 9.17 (Properties of the Symmetric Product) *Let α,β,γ be symmetric tensors and $a,b\in\mathbb{R}$.*

(i) The symmetric product is symmetric and bilinear:

$$\alpha\beta = \beta\alpha \tag{9.20}$$

$$(a\alpha + b\beta)\gamma = a\alpha\gamma + b\beta\gamma \tag{9.21}$$

$$= \gamma(a\alpha + b\beta). \tag{9.22}$$

(ii) If α and β are covectors, then

$$\alpha\beta = \frac{1}{2}\big(\alpha \otimes \beta + \beta \otimes \alpha\big).$$

Alternating Tensors

Definition 9.18 (Alternating Covariant Tensor) *Let V be a finite dimensional vector space. A covariant k-tensor α on V is said to be alternating (or skew-symmetric) if it changes sign whenever two of its arguments are interchanged. Thus, for all vectors $v_1, \ldots, v_k \in V$ and every pair of distinct indices i, j it satisfies*

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots v_k) = -\alpha(v_1, \ldots, v_j, \ldots v_i, \ldots, v_k)$$

Alternating covariant k-tensors are also known as exterior forms, multilinear covectors, or k-covectors.

The subspace of all alternating covariant k-tensors on a vector space V is known as $\Lambda^k(V^*) \subseteq T^k(V^*)$.

Tensors and Tensor Fields on Manifolds

Having formally introduced the definition of tensors, we can now start talking about tensors on manifolds. To do so we start with the definitions for the tensor bundle.

Definition 9.19 (Bundle of Covariant k-tensors on M)

$$T^kT^*M = \bigsqcup_{p \in M} T^k\big(T_p^*M\big).$$

Definition 9.20 (Bundle of Contravariant k-tensors on M)

$$T^kTM = \bigsqcup_{p \in M} T^k\left(T_pM\right).$$

Definition 9.21 (Bundle of Mixed Tensors of type (k,l))

$$T^{(k,l)}TM = \bigsqcup_{p \in M} T^{(k,l)}\left(T_pM\right).$$

Just like we had for the tensors on vector spaces we have some natural identifications:

(a) $T^{(0,0)}TM = T^0T^*M = T^0TM = M \times \mathbb{R}$,

(b) $T^{(0,1)}TM = T^1T^*M = T^*M$,

(c) $T^{(1,0)}TM = T^1TM = TM$,

(d) $T^{(0,k)}TM = T^kT^*M$,

(e) $T^{(k,o)}TM = T^KTM$.

Any one of these bundles is called a tensor bundle over M, and any section of a (covariant, contravairant, or mixed) bundle is a tensor field on M.

Smooth sections of tensor bundles, $\Gamma(T^kT^*M)$, $\Gamma(T^kTM)$, or $\Gamma(T^{(k,l)}TM)$, are infinite-dimensional vector spaces over $\mathbb{R}$ and modules over $C^\infty(M)$.

We can express these bundles in coordinates in any smooth coordinate representation (x^i) as

$$A = \begin{cases} A_{i_1\ldots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}, & A \in \Gamma(T^kT^*M); \\ A^{i_1,\ldots,i_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \cdots \frac{\partial}{\partial x^{i_k}}, & A \in \Gamma(T^kTM); \\ A^{i_1,\ldots,i_k}_{j_1,\ldots,j_l} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}, & A \in \Gamma(T^{(k,l)}TM). \end{cases} \tag{9.23}$$

Where just like in vectors the $A_{i_1\ldots i_k}$, $A^{i_1,\ldots,i_k}$, or $A^{i_1,\ldots,i_k}_{j_1,\ldots,j_l} \frac{\partial}{\partial x^{i_1}}$ are called the component functions of A.

We find that it is possible for us to identify smooth tensor fields based upon the properties of the vectors in vector spaces of its domain.

Proposition 9.22 (Smoothness Criteria for Tensor Fields) *Let M be a smooth manifold, and let $A: M \to T^kT^*M$ be a rough section. The following statements are equivalent.*

(i) A is smooth.

(ii) In every smooth coordinate chart, the component functions of A are smooth.

(iii) Each point of M is contained in some coordinate chart in which A has smooth component functions.

(iv) If $X_1 \dots X_k \in \mathscr{X}(M)$, then the function $A(X_1, \dots X_k): M \to \mathbb{R}$, defined by

$$A(X_1,\dots,X_k)(p) = A_p\big(X_1|_p,\dots,X_k|_p\big),$$

is smooth.

(v) Whenever $X_1,\dots,X_k$ are smooth vector fields defined on some open subset $U \subseteq M$, the function $A(X_1,\dots,X_k)$ is smooth on U.

From (iv) above, we see that if A is a smooth covariant k-tensor field on M and $X_1,\dots,X_k$ are smooth vector field, then $A(X_1,\dots,X_k)$ is a smooth real-valued function on M. Thus A induces a map

$$\underbrace{\mathscr{X}(M)\times\cdots\times\mathscr{X}(M)}_{k\text{ copies}} \to C^\infty(M).$$

Not only is this induced map multilinear over $\mathbb{R}$, but it is also multilinear over $C^\infty(M)$ which brings us the the characteristic of smooth tensor fields.

Lemma 9.23 (Tensor Characterization Lemma) *A map*

$$\mathscr{A}: \underbrace{\mathscr{X}(M)\times\cdots\times\mathscr{X}(M)}_{k\text{ copies}} \to C^\infty(M), \tag{9.24}$$

is induced by a smooth covariant k-tensor field as above if and only if it is multilinear over $C^\infty(M)$.

Lastly, a covariant tensor field whose value at each point is a symmetric tensor is called a symmetric tensor field. Similarly, a covariant tensor field with a alternating tensor at each point is called an alternating tensor field. We will have more to say about these as we proceed.

The Contraction (or Trace) of Tensors

There exists an operation that shrinks a (k,l)-tensor to a $(k-1,l-1)$-tensor.

Lemma 9.24 *There exists a unique $C^\infty(M)$ function $\mathrm{C}\colon \mathscr{T}_1^1(M) \to C^\infty(M)$ called $(1,1)$-contraction (or trace), such that $\mathrm{C}(X \otimes \omega) = \omega X$ for all $X \in \mathscr{X}(M)$ and $\omega \in \mathscr{X}^*(M)$.*

We will sometimes denote the contraction operation by Tr rather than C.

The notion of contraction can be extended to any rank tensor. For instance suppose $A \in T_k^l(M)$ and $1 \leq i \leq k$ and $1 \leq j \leq l$. Fix the covectors $\omega^1, \ldots, \omega^{k-1}$ and vector fields $V_1, \ldots, V_{l-1}$. Then the function

$$(\omega, V) \to A(\omega^1, \ldots, \omega_i, \ldots, \omega^k, V_1, \ldots, V_j, \ldots, V_l)$$

is a $(1,1)$ tensor that can be written as

$$A(\omega^1, \ldots, ;, \ldots, \omega^{k-1}, V_1, \ldots, ;, \ldots, V_{l-1}).$$

Applying the contraction of A over i, j,is the application of the $(1,1)$-contraction to this tensor produces a real-valued function denoted by

$$(Tr^i_j A)(\omega^1, \ldots, \omega^{k-1}, V_1, \ldots, V_{l-1}).$$

In coordinates, suppose A is a $(2,3)$-tensor field, then the contraction $Tr_3^1(A)$ is the $(1,2)$-tensor field given by

$$
\begin{aligned}
(Tr_3^1 A)_{ij}{}^{k} &= (Tr_3^1)\left(dx^k, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &(9.25)\\
&= \sum_m A\left(dx^m, dx^k, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^m}\right) &(9.26)\\
&= A_{ijm}^{mk}. &(9.27)
\end{aligned}
$$

Which leads us to the following corollary.

Corollary 9.25 *Let $1 \leq i \leq k$ and $1 \leq j \leq l$. Relative to a coordinate system, if $A \in T_l^k(M)$ has components $A_{j_1,\ldots,j_l}^{i_1,\ldots,i_k}$, then $Tr_j^i A$ has components*

$$\sum_m A_{j_1,\ldots,m_j,\ldots,j_l}^{i_1,\ldots,m_i,\ldots,i_k}. \tag{9.28}$$

Pullbacks of Tensor Fields

Just like covector fields, covariant tensor fields can be pulled back by a smooth map to yield tensor fields on the domain.

Definition 9.26 (Pointwise Pullback of α by F at p) *Suppose $F: M \to N$ is a smooth map. For any point $p \in M$ and any k-tensor $\alpha \in T^k\left(T_{F(p)}^*N\right)$, the pointwise pullback of α by F at p is a tensor $dF_p^*(\alpha) \in T^k(T_p^*M)$, defined by*

$$dF_p^*(\alpha)(v_1,\ldots,v_k) = \alpha\big(dF_p(v_1),\ldots,dF_p(v_k)\big),$$

where $v_1,\ldots,v_k \in T_pM$.

Definition 9.27 (Pullback of A by F) *Suppose $F:\ M \to N$ is a smooth map and A is a covariant k-tensor on N. For any point $p \in M$ the pullback of A by F is a rough vector field F^*A on M defined by*

$$(F^*A)_p = dF_p^*(A_{F(p)}).$$

The pullback of A by F is thus a tensor field that acts on vectors $v_1,\ldots,v_k \in T_pM$ by

$$(F^*A)_p(v_1,\ldots,v_k) = A_{F(p)}(dF_p(v_1),\ldots,dF_p(v_k)).$$

Proposition 9.28 (Properties of Tensor Pullbacks) *Suppose $F:\ M \to N$ and $G:\ N \to P$ are smooth maps, A and B are covariant tensor fields on N, and f is a real valued function on N.*

(i) $F^*(fB) = (f \circ F)F^*B$,

(ii) $F^*(A \otimes B) = F^*A \otimes F^*B$,

(iii) $F^*(A+B) = F^*A + F^*B$,

(iv) $(G \circ F)^*B = F^*(G^*B)$,

(v) $(Id_N)^*B = B$,

(vi) *F^*B is a (continuous) tensor field, and is smooth if B is smooth.*

These properties tell us that the pullback of a covariant k-tensor field is computed in the same manner as the pullback of a covector field.

Corollary 9.29 *Let $F:\ M \to N$ be a smooth map, and let B be a covariant k-tensor field on N. If $p \in M$ and (y^i) are smooth coordinates for N on a neighborhood of $F(p)$, then F^*B has the following expression in the neighborhood of p:*

$$F^*B = F^*(B_{i_1,\ldots,i_k} dy^{i_1} \otimes \cdots \otimes dy^{i_k}) \tag{9.29}$$

$$= (B_{i_1,\ldots,i_k} \circ F) d(y^{i_1} \circ F) \otimes \cdots \otimes d(y^{i_k} \circ F). \tag{9.30}$$

In the special case of diffeomorphisms, tensor fields of any type (including mixed type) can be pushed forward and pulled back.

Differential Forms

Definition 9.30 (Differential l-Form) *A differential l-form is a skew-symmetric covariant tensor field of type* $(0,l)$.

The theory of differential forms is a generalization of covector fields, and allows us to extend a number of multivariable calculus concepts to the tensor fields. We will introduce these concepts as they become necessary to our purposes.

However, we begin with a particular lemma that begins our discussion.

Lemma 9.31 *Let μ be a n-form. If $V_i = \sum_{j=1}^{n} A_{ij} W_j$ for $1 \leq i \leq n$, then*

$$\mu(V_1, \ldots, V_n) = (detA)\mu(W_1, \ldots, W_n). \tag{9.31}$$

Lie Derivatives of Tensor Fields

The Lie derivative operation can be extended to tensor fields.

Definition 9.32 (Lie Derivative of a Covariant Tensor Field) *Let A be a smooth covariant tensor field on M, V be a smooth vector field on M, and the flow θ_t be a diffeomorphism from a neighborhood of $p \in M$ to $\theta_t(p)$. The Lie derivative of A with respect to V, denoted by $\mathscr{L}_V A$, is defined by*

$$(\mathscr{L}_V A)_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\theta_t^* A)_p \tag{9.32}$$

$$= \lim_{t \to 0} \frac{d(\theta_t)_p^* \left(A_{\theta_t(p)}\right) - A_p}{t}, \tag{9.33}$$

provided the derivative exists.

Lemma 9.33 *With A as a smooth covariant tensor field on M and V a smooth vector field on M, the derivative $(\mathscr{L}_V A)_p$ exists for every $p \in M$, and defines $\mathscr{L}_V A$ as a smooth tensor field on M.*

Proposition 9.34 *Let M be a smooth manifold and let $V \in \mathscr{X}(M)$. Suppose f is a smooth real-valued function (regarded as the 0-tensor field) on M, and A, B are smooth covariant tensor fields on M.*

(i) $\mathscr{L}_V f = V f$.

(ii) $\mathscr{L}_V(fA) = (\mathscr{L}_V f)A + f(\mathscr{L}_V A)$.

(iii) $\mathscr{L}_{A\otimes B} = (\mathscr{L}_V A)\otimes B + A \otimes (\mathscr{L}_V B)$.

(iv) If $X_1, \ldots, X_k$ are smooth vector fields and A is a smooth k-tensor field,

$$\begin{aligned}\mathscr{L}_V(A(X_1,\ldots,X_k) &= (\mathscr{L}_V A)(X_1,\ldots,X_k) + A(\mathscr{L}_V X_1,\ldots,X_k)\\ &\quad + \cdots + A(X_1,\ldots,\mathscr{L}_V X_k).\end{aligned}$$

This shows us that we are able to determine the Lie bracket without having to first determining the flow.

Corollary 9.35 *If V is a smooth vector field and A is a smooth covariant k-tensor field, then for any smooth vector fields $X_1, \ldots, X_k$,*

$$\begin{aligned}\mathscr{L}_V A(X_1,\ldots,X_k) &= V(A(X_1,\ldots,X_k)) - A([V,X_1],\ldots,X_k)\\ &\quad + \cdots + A(X_1,\ldots,[V,X_k]).\end{aligned} \tag{9.34}$$

Corollary 9.36 *If $f \in C^\infty(M)$, then $\mathscr{L}_V(df) = d(\mathscr{L}_V f)$.*

The coordinate free formulas require the calculation of the local behavior of the Lie derivative near a point $p \in M$, but there is a simple way to perform these calculations in local coordinates.

Example 9.37 *Suppose A is an arbitrary smooth covariant 2-tensor field, and V is a smooth vector field. We compute the Lie derivative $\mathscr{L}_V A$ in smooth local coordinates (x^i). Notice first* that $\mathscr{X}_V dx^i = d(\mathscr{L}_V x^i) = d(Vx^i) = dV^i$. Thus

$$\mathscr{L}_V A = \mathscr{L}_V (A_{ij} dx^i \otimes dx^j) \tag{9.35}$$

$$= (\mathscr{L}_V A_{ij}) dx^i \otimes dx^j + A_{ij} (\mathscr{L}_V dx^i) \otimes dx^j + A_{ij} dx^i \otimes (\mathscr{L}_V dx^j) \tag{9.36}$$

$$= VA_{ij} dx^i \otimes dx^j + A_{ij} dV^i \otimes dx^j + A_{ij} dx^i \otimes dV^j \tag{9.37}$$

$$= \left(VA_{ij} + A_{kj} \frac{\partial V^k}{\partial x^i} + A_{ik} \frac{\partial V^k}{\partial x^j} \right) dx^i \otimes dx^j. \tag{9.38}$$

Similar to the Lie derivative on the vector field, we can use the Lie derivative of a tensor field to determine invariance.

Definition 9.38 (Tensor Field Invariance) *Let A be a smooth tensor field on M and θ be a flow on M. We say that A is invariant under θ if for each t, the map θ_t pulls A back to itself wherever it is defined:*

$$d(\theta_t)_p^* \left(A_{\theta_t(p)} \right) = A_p,$$

for all (t, p) the flow domain of θ. If θ is a global flow, this is equivalent to $\theta_t^* A = A$ for all $t \in \mathbb{R}$.

The following proposition shows that we can use Lie derivatives to compute t-derivatives for $t \neq 0$.

Proposition 9.39 *Suppose M is a smooth manifold and $V \in \mathscr{X}(M)$. If $\partial(M) \neq \emptyset$, with V tangent to $\partial(M)$, and θ as the flow of M. For any smooth covariant tensor field A and any (t_0, p) in the domain of θ,*

$$\left. \frac{\partial}{\partial t} \right|_{t=t_0} (\theta_t^*)_p = (\theta_{t_0})^* (\mathscr{L}_V A))_p$$

Thus we see that a tensor field is invariant under the flow of a smooth vector field if its Lie derivative is zero.

Theorem 9.40 *Let M be a smooth manifold and let $V \in \mathscr{X}(M)$. A smooth covariant tensor field A is invariant under the flow of V if and only if $\mathscr{L}_V A = 0$.*

Chapter Summary

Our major take away from this chapter is that the tensor is a multi-linear object that operates on any number of vectors and covectors to produce a scalar. Since this object works on general vectors and covectors, it operates without the need for any specified coordinate reference frame, and is thus considered a coordinate-free representation.

Also introduced within this chapter is the definition of a tensor product, and that this operation can be used to form a basis for the space of multi-linear functions by taking all the possible tensor products of basis covectors.

We saw that the characteristic property of the tensor product space is that for any multi-linear map into a vector space there is a unique linear map from the tensor product space into the same vector space.

We also introduced the notion of a tensor tensor bundle, and saw how they are represented in smooth coordinates. For instance, a covariant 2-tensor A can be represented in coordinates as

$$A = A_{ij} dx^i \otimes dx^j, \qquad A \in \Gamma\left(T^2 T^* M\right). \tag{9.39}$$

We will find this form of particular interest as we move to Geometry in the next section.

CHAPTER X

GEOMETRY ON A MANIFOLD

Conceptual Introduction

In this chapter we begin to look at geometry on a manifold. We find that we are able to make geometric statements about a manifold if there exists a particular tensor that allows us to determine the distance between any two points. In general, if this tensor exists, we say that the manifold has a scalar product. However, there are two specific types of manifold that of great importance. The first is a Riemannian manifold where the scalar product is positive definite. The other is called a pseduo-Riemannian (or semi-Riemannian) manifold, and it allows for the indefiniteness of the scalar product. We find that the latter manifold is more important in the study of relativity.

Prerequisites

Symmetric Bilinear Forms

Geometry on a manifold involves a specific kind of covariant $(0,2)$-tensor on tangent spaces. Recall from our earlier section that a symmetric bilinear form is a covariant 2-tensor $b: V \times V \to \mathbb{R}$ such that $b(v,w) = b(w,v)$ for all $v, w \in T_pM$.

Definition 10.1 (Definiteness) *A symmetric bilinear form b on a vector space V can take on a few different forms:*

(i) If $v \neq 0$ implies $b(v,v) > 0$, then we say that b is positive definite. If $b(v,v) \geq 0$, $\forall v \in V$, then we say that b is positive semidefinite.

(ii) If $v \neq 0$ implies $b(v,v) < 0$, then we say that b is negative definite. If $b(v,v) \leq 0$, $\forall v \in V$, then we say that b is negative semidefinite.

(iii) If $b(v,w) = 0$ implies $v = 0$, $\forall w \in V$, then we say that b is nondegenerate.

If b is a symmetric bilinear form on V than for any subspace W of V the restriction $b|_{W\times W}$, denoted simply as $b|_W$, is symmetric and bilinear. If b is (semi-) definite, then so is $b|_W$. However, in order to deal with negative definiteness we need a way to keep track of the negative values for the components within the symmetric bilinear form.

Definition 10.2 (Index of Symmetric Bilinear Form) *The index, k, of a symmetric bilinear form b on V is the largest integer that is the dimension of a subspace $W \subseteq V$ on which $b|_W$ is negative definite.*

So $0 \leq k \leq dim\, V$, and $k = 0$ if and only if b is positive semidefinite.

With this we also can define the signature of the bilinear form.

Definition 10.3 (Signature of a Bilinear Form) *For a bilinear form $b : V \times V \to \mathbb{R}$ with index k, the pair $(k, (dim\, V - k))$ is called the signature of b.*

For each bilinear form, there also exits a function that often makes things easier.

Definition 10.4 (Quadratic Form of a Bilinear Form) *Let $b : V \times V \to \mathbb{R}$ be a symmetric bilinear form on the finite dimensional vector space V. The function $q : V \to \mathbb{R}$ defined by $q(v) = b(v,v)$ is called the associated quadratic form of b.*

Notice that the information from the bilinear form is completely encoded within the quadratic form since we can recreate the bilinear form through the polarization identity:

$$b(v,w) = \frac{1}{2}\left(q(v+w) - q(v) - q(w)\right).$$

The quadratic form of basis vectors gives us a specific result.

Definition 10.5 (Matrix of a Bilinear Form Relative to a Basis) *Let b be a symmetric bilinear form on a finite dimensional vector space V. If $e_1, \ldots, e_n$ is a basis for V, then the $n \times n$ matrix $b_{ij} = b(e_i, e_j)$ is called the matrix of b relative to $e_1, \ldots, e_n$.*

Notice that for $v, w \in V$,

$$
\begin{aligned}
b(v,w) &= b\left(\sum v^i e_i, \sum w^j e_j\right) && (10.1)\\
&= \sum v^i b\left(e_i, \sum w^j e_j\right) && (10.2)\\
&= \sum w^j \sum v^i b\left(e_i, e_j\right) && (10.3)\\
&= \sum w^j \sum v^i b_{ij} && (10.4)\\
&= b_{ij} w^j v^i. && (10.5)
\end{aligned}
$$

So, since b is symmetric, the matrix b_{ij} is symmetric and determines b. This leads us to the following lemma.

Lemma 10.6 *A symmetric bilinear form is nodegenerate if and only if its matrix relative to any basis is invertible.*

Scalar Products

With these features of bilinear forms in mind, we come to our first main geometric feature: The scalar product.

Definition 10.7 (Scalar Product) *A scalar product, g, on a vector space V is a nondegenerate symmetric bilinear form on V.*

A finite dimensional vector space equipped with a scalar product, g is called a scalar product space. A non-degenerate scalar product that can be equal to zero $g = 0$ is called indefinite.

Definition 10.8 (Null Vector) *A vector $v \in V$ where the quadratic form $q(v) = 0$, but $v \neq 0$ is called null.*

It follows from this definition that null vectors exist in a vector scalar product space (V, g) if and only if g is indefinite.

Definition 10.9 (Orthogonal Vectors) *Two vectors $v, w \in V$ are considered orthogonal, written $v \perp w$ if $g(v, w) = 0$*

So, if g is indefinite, we cannot picture orthogonal vectors as being right angles anymore because, a null vector is orthogonal with itself.

If V is a scalar product space, we can generate a subspace based upon a condition of orthogonality.

Definition 10.10 (Orthogonal Subspace of V) *If W is a subspace of the scalar product space V, let*

$$W^{\perp} = \{v \in V : \ v \perp W\}.$$

Then $W^{\perp}$ (called W $perp$) is a subspace of V.

Note that in general $W + W^{\perp} \neq V$, and so we can't think of $W^{\perp}$ as the orthogonal complement of V. However, we do have the following properties:

Lemma 10.11 (Properties of $W^{\perp}$) *Let W be a subspace of the scalar product space V. Then,*

a $dim\, W + dim\, W^{\perp} = dim\, V = n$

b $\left(W^{\perp}\right)^{\perp} = W.$

Note here that the nondegeneracy of g on all of V is equivalent to $V^{\perp} = 0$

Lemma 10.12 *A subspace W of V is nondegenerate if and only if V is the direct sum of W and $W^{\perp}$.*

Just as we had some minor generalizations for the notions of orthogonality, we have a few for the definition of norm and unit vector too. Since the quadratic from $g(v)$ can be negative a vector v in a scalar product space V we define:

(a) The *norm* of v: $|v| = |g(v,v)|^{1/2}$, and

(b) A *unit* vector, u, is a vector with norm equal to 1, $g(u,u) = \pm 1$.

Definition 10.13 (Orthonormal Basis) *A set of mutually orthogonal unit vectors is said to be orthonormal, and for a scalar product space V, with dim $V = n$, any set of n orthonormal vectors in V forms a orthonormal basis for V.*

Lemma 10.14 *A nontrivial scalar product space, $V \neq 0$, has an orthonormal basis.*

So, for a scalar product, g, relative to an orthonormal basis $e_1, \ldots, e_n$ for V is the diagonal matrix defined by

$$g(e_i, e_j) = \delta_{ij} g(e_j, e_j), \tag{10.6}$$

where $g(e_j, e_j) = \pm 1$ depending upon the signature.

Lemma 10.15 *Let $e_1, \ldots, e_n$ be an orthonormal basis for V, with $\varepsilon_i = g(e_i, e_i)$. Then each $v \in V$ can be uniquely written by*

$$v = \sum \varepsilon_i g(v, e_i) e_i. \tag{10.7}$$

We can now express the index k of the scalar product g as the index of V, denoted by $k = ind\, V$.

Lemma 10.16 *For any orthonormal basis $e_1, \ldots, e_n$ of V the number of negative signs in the signature $e_1, \ldots, e_n)$ is the index k of V.*

So, for a nondegenerate subspace W of the scalar product space V, we find that

$$ind\, V = ind\, W + ind\, W^{\perp} \tag{10.8}$$

Now we can look at transformations that preserve the scalar product.

Definition 10.17 (Preservation of Scalar Product Under Mapping) *Let V and $\widehat{V}$ have scalar products g and $\widehat{g}$ respectively. If a linear transformation $T : V \to \widehat{V}$ has the property that*

$$\widehat{g}(Tv, Tw) = g(v, w),$$

then we say that the transformation T preserves the scalar products.

Notice that transformation T preserves scalar products if and only if it preservers their associated quadratic forms.

Definition 10.18 (Linear Isometry) *Let W be a nondegenerate subspace of the scalar product space V. A linear isomorphism $T :\ V \to W$ that preserves scalar products is called a linear isometry.*

Note that this implies that a linear transformation $T :\ V \to W$ is an isomorphism if and only if $dim\, V = dim\, W$ and T preserves their scalar products.

Thus, we conclude with the following lemma.

Lemma 10.19 *Scalar product spaces V and W have the same dimension and index if and only if there exists a linear isometry from V to W.*

The Riemannian Manifold

We begin with a definition.

Definition 10.20 (Riemannian Metrics) *A Riemannian metric on a smooth manifold M is a covariant 2-tensor field g, ($g \in \mathscr{T}^2(M)$), that is both*

(i) symmetric: $g(X,Y) = g(Y,X)$ for vectors $X,Y \in T_pM$, and

(ii) positive definite: $g(X,X) \geq 0$ and $g(X,X) = 0$ iff $X = 0$, $\forall X \in T_pM$

So, a Riemannian metric determines an inner product on each tangent space T_pM, denoted by $\langle X,Y \rangle := g(X,Y)$, $\forall X,Y \in T_pM$.

Definition 10.21 (Riemannian Manifold) *A smooth manifold, M, equipped with a Riemannian metric, g, is called a Riemannian manifold, and is denoted by (M,g).*

Just as in Euclidean geometry, if $p \in M$, we can define the length or norm of any tangent vector $X \in T_pM$ with the metric $|X| := \langle X,X \rangle^{1/2} = \sqrt{g(X,X)}$. Similarly, for any two vectors $X,Y \in T_pM$ we can define the angle between them as the unique $\theta \in [0,\pi]$ such that

$$\cos\theta = \frac{\langle X,Y \rangle}{|X||Y|}.$$

However, we will see later that we will be able to generalize this formula.

We now look to find how to tell if two Riemannian manifolds are the same.

Definition 10.22 (Equivalence between Riemannian manifolds) *Let (M,g) and $(\widetilde{M},\widetilde{g})$ be Riemannian manifolds. A diffeomorphism $\varphi: (M,g) \to (\widetilde{M},\widetilde{g})$ is an isometry if $\varphi^*\widetilde{g} = g$.*

Being isometric is an equivalence class on the class of all Reimannian manifolds. In fact, the study of Riemannian manifolds is primarily concerned with invariant properties, that is the properties that are preserved under isometries.

Lets take a moment to consider the case where $M = \widetilde{M}$ in our definition above.

Definition 10.23 (Isometry of M) *An isometry $\varphi : (M, g) \to (M, g)$ is called an isometry of M.*

It is easy to see that both the inverse of an isometry and the composition of isometries are themselves isometries.

Definition 10.24 (The Isometry Group) *The set of all isometries of M forms a group called the isometry group of M, and it is denoted by $\mathscr{I}(M)$.*

At this point we will now turn to local representations of the metric.

(a) Notice if $(E_1, \ldots, E_n)$ is any local frame for the tangent bundle TM, and $(\varphi^1, \ldots, \varphi^n)$ is its dual coframe, a Riemannian metric can be written locally as

$$g = g_{ij}\varphi^i \otimes \varphi^j.$$

(b) By definition, the coefficient matrix defined by $g_{ij} = \langle E_i, E_j \rangle$ is symmetric in i and j and depends smoothly on $p \in M$. So in a coordinate frame g has the form

$$g = g_{ij}dx^i \otimes dx^j. \tag{10.9}$$

(c) Further, recall that the symmetric product of two 1-forms ω and η, denoted by juxtaposition with no product symbols, is

$$\omega\eta := \frac{1}{2}(\omega \otimes \eta + \eta \otimes \omega) \tag{10.10}$$

(d) So, from the symmetric product and properties of tensors, for $dx^i dx^j$ we find

$$\begin{aligned} dx^i dx^j &= \frac{1}{2}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right) && (10.11)\\ &= \frac{1}{2}\left(dx^i \otimes dx^j + dx^i \otimes dx^j\right) && (10.12)\\ &= dx^i \otimes dx^j. && (10.13) \end{aligned}$$

(e) Thus, because of the symmetry in $g_{ij} = \langle E_i, E_j\rangle = \langle E_j, E_i\rangle$, we find that in a coordinate frame g has the form

$$g = g_{ij} dx^i dx^j \tag{10.14}$$

Pseudo-Riemannian Manifolds

As hinted above, it is possible to loosen up the positive definite requirement for the Riemannian manifold. It turns out that in doing so we obtain another manifold structure that is uniquely suited for modeling General Relativity.

Definition 10.25 (Metric Tensor) *A metric tensor (or just metric), g, on a smooth manfold M is a symmetric nondegenerate* $(0,2)-$*tensor field on M of constant index.*

A smooth manifold M paired with a metric tensor g, denoted (M,g), is called a pseudo-Riemannian manifold (or semi-Riemannian Manifold). When dealing with metric tensors it is important to keep note of the signature. When the signature is $(+,\dots,+)$, it is a Riemannian metric. When it is $(-,+,\dots,+)$ it is a special kind of pseudo-Riemannian metric called the Lorentizan metric. Additionally, any two metric tensors g_1, g_2 on a smooth manifold M actually define two different pseudo-Riemannian manifolds (M,g_1) and (M,g_2), where the metrics g_1 and g_2 have opposite signatures but are otherwise the same.

Just as above, we often use $\langle\cdot,\cdot\rangle$ to denote g. That is

(a) $g(v,w) = \langle v,w\rangle \in \mathbb{R}$ for $v,w \in T_pM$, and

(b) $g(v,w) = \langle v,w\rangle \in C^\infty(M)$ for $v,w \in \mathscr{X}(M)$

We are able to express the metric tensor locally just as we did above for the Riemannian metric.

(a) $g_{ij} = \langle x_i, x_j\rangle = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$ where $1 \le i, j \le n$.

(b) The matrix g_{ij} is symmetric: $g_{ij} = g_{ji}$.

(c) For vector fields $V,W \in \mathscr{X}(M)$, we have

$$g(V,W) = \langle V,W\rangle = g_{ij} V^i W^j. \tag{10.15}$$

(d) The metric tensor on M can be written

$$\begin{aligned} g &= g_{ij} dx^i \otimes dx^j \\ &= g_{ij} dx^i dx^j \end{aligned} \tag{10.16}$$

Recall that we identified the canonical isomorphism $\mathbb{R}^n \cong T_p\mathbb{R}^n$, and so we find that the familiar dot product on $\mathbb{R}^n$ gives rise to a metric

$$\langle v_p, w_p \rangle = v \cdot w = \sum v^i w^i \tag{10.17}$$

So we can refer to any geometric context in $\mathbb{R}^n$ to a Riemannian manifold through Euclidean n-space. However, when lessening the positive definite restriction for the Riemannian metric we have to keep note of the index k of our bilinear form g. So, for an integer k where $0 \leq k \leq n$, changing the first k plus signs to minus signs gives us the associated metric tensor

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{k} v^i w^i + \sum_{j=k+1}^{n} v^i w^i \tag{10.18}$$

The resulting semi-Euclidean space $\mathbb{R}^n_k$ reduces to $\mathbb{R}^n$ in the case of $k = 0$.

We can simplify our notation further if we fix the notation

$$\kappa_i = \begin{cases} -1, & \text{for } 1 \leq i \leq k \\ +1, & \text{for } k+1 \leq i \leq n. \end{cases} \tag{10.19}$$

In this case we end up with a general expression for the metric tensor for the semi-Euclidean space $\mathbb{R}^n_k$ as

$$g = \kappa_i du^i \otimes du^i. \tag{10.20}$$

Line Element

Suppose $q(v) = \langle v, v \rangle$ for each $v \in T_pM$. Then for each point of M, q gives the associated quadratic form of the scalar product at $p \in M$. Thus q determines the metric tensor. We call this metric tensor the *line element*, and denote it by ds^2,

$$ds^2 = g_{ij}dx^i dx^j. \tag{10.21}$$

Note that for $X \in \mathscr{X}(M)$ and $f \in C^\infty(M)$, we have $ds^2 = q(fV) = f^2 q(V) \in C^\infty(M)$. So, ds^2 is not a tensor field.

Just as we did for the Riemannian manifold and metric, we can determine the geometric properties for semi-Riemannian manifolds from the line element:

(a) The *norm* of a tangent vector $X \in T_pM$ is defined by

$$|X| := \sqrt{ds^2} = \langle X, X \rangle^{1/2} = \sqrt{g(X,X)}. \tag{10.22}$$

(b) The *angle between two tangent vectors*, where $\theta \in [0, \pi]$ is the angle and $X, Y \in T_pM$ is defined by

$$\cos\theta = \frac{\langle X, Y \rangle}{|X||Y|}. \tag{10.23}$$

(c) Two tangent vectors $X, Y \in T_pM$, separated by the angle θ, are orthogonal if $\cos\theta = 0$, or equivalently

$$ds^2 = 0 \Rightarrow x \perp y, \quad \forall x \in X \text{ and } \forall y \in Y. \tag{10.24}$$

Given a submanifold P of M, we have a way to find the metric associated with P.

Definition 10.26 (Metric of a Submanifold) *Let P be a submanifold of a semi-Riemannian manifold M. If the pullback $\iota^*(g)$, where ι is the inclusion map, is a metric tensor on P, then P is semi-Riemannian submanifold of M.*

Finally we consider the case of semi-Riemannian product manifolds.

Lemma 10.27 *Let M and N be semi-Riemannian manifolds with metrics g_M and g_N respectivly. If π and σ are the projections of $M \times N$ onto M and N respectively, then*

$$g = \pi^*(g_M) + \sigma^*(g_N),$$

is a metric tensor of the semi-Riemannian product manifold $M \times N$.

This definition extends to the case of a product of any finite number of semi-Riemannian manifolds. For instance

$$\underbrace{\mathbb{R}^1_1 \times \cdots \times \mathbb{R}^1_1}_{v \text{ factors}} \times \underbrace{\mathbb{R}^1 \times \cdots \times \mathbb{R}^1}_{n-v \text{ factors}} = \mathbb{R}^v_v \times \mathbb{R}^{n-v}. \tag{10.25}$$

Or in the specific case of $n = 4$ with index 1,

$$\mathbb{R}^1_1 \times \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^1_1 \times \mathbb{R}^3. \tag{10.26}$$

Isomotries

The definition for an isometry is the same for the semi-Riemannian manifold as it is for the Riemannian manifold.

Definition 10.28 (Isometry Between Semi-Riemannian Manifolds) *Let M and N be semi-Riemannian manifolds with metrics g_M and g_N. An isometry from M to N is a diffeomorphism $\phi : M \to N$ that preserves metric tensors: $\phi^*(g_N) = g_M$.*

Just as with Riemannian manifolds we see that for semi-Riemannian manifolds:

(a) The identity map of a semi-Riemannian manifold is an isometry.

(b) A composition of isometries is and isometry.

(c) The inverse map of an isometry is an isometry.

Semi-Riemannian geometry is primarily concerned with the study of invariants or of the objects that are preserved under isometries.

Lemma 10.29 *If $\psi : V \to W$ is a linear isometry of scalar product spaces, then with V and W as semi-Riemannian manifolds defined by the metric $\langle v_p, w_p \rangle = \langle v, w \rangle$, we find that $\psi : V \to W$ is an isometry of semi-Riemannian manifolds.*

So, if V is a scalar product space with $dim\, V = n$ and index k, then as a semi-Riemannian manifold, V is isometric to $\mathbb{R}^n_k$. Further if M is an arbitrary semi-Riemannian manifold, its metric tensor makes each of its tangent spaces a semi-Euclidean space of the same dimension and index as M itself.

Chapter Summary

In this chapter we were able to see that the existence of a particular metric that allows us to determine the distance between two points on the manifold is sufficiant for us to begin talking about geometry. In particular we saw that there are two common types of manifolds defined by such a metric:

a) the Riemannian metric, and

b) the semi-Riemannian metric.

This latter metric, and associated manifold, provides us with the basis for our future discussion of spacetimes.

CHAPTER XI

CURVATURE

Conceptual Introduction

We find that the key to identifying curvature on a manifold lies in being able to connect one tangent space to another. In fact, if there exists a covariant directional derivative that can link tangent spaces, we call it a connection. We find that there exists a particular connection on all semi-Riemannian manifolds called the Levi-Civita Connection. This particular connection is used in obtaining the Riemannian curvature tensor that obtains the curvature of one vector field in relation to 3 covector fields. That is, we can think of the Riemannian curvature tensor as a multilinear function on individual tangent vectors. The Riemannian tensor can be used to obtain two other measures of curvature. The Ricci tensor is obtained through contracting the Riemannian tensor to obtain a covariant 2-tensor. Another contraction brings us to the Ricci scalar. We will find that the combination of the Ricci tensor and Ricci scalar play a significant role in relating geometry and gravitation.

We also saw that there is a useful result to obtaining the covariant directional derivative between the various basis vectors called the Christoffel Symbol. This notion of variation between basis vectors provides us the ability to come up with the notions of parallel, and thus geodesic as the path with zero acceleration.

The Levi-Civita Connection

We previously defined the notion of a directional derivative of a vector field with respect to another, but this was done only through use of a flow, and not locally. Here we instead look for a connection that will serve this purpose without the need for understanding the flow.

We will begin with a definition for the covariant derivative on a semi-Euclidean space in natural coordinates and then look to generalize our findings.

Definition 11.1 (Natural Coordinate Covariant Derivative) *Let $u^1,\ldots,u^n$ be the natural coordinates on $\mathbb{R}^n_k$. If V and $W = W^i \frac{\partial}{\partial u^i}$ are vector fields on $\mathbb{R}^n_k$, then we define the natural covariant derivative of W with respect to V as*

$$D_V W = V(W^i)\frac{\partial}{\partial u^i}.$$

We are interested in extending this notion onto the semi-Riemannian manifold, and to do so will define a connection using its key properties.

Definition 11.2 (Connection) *A connection D on a smooth manifold M is a function $D : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ such that:*

(i) $D_V W$ is $C^\infty(M)$-linear in V:

$$D_{(f+g)V} W \ = D_{fV} W + D_{gV} W \qquad (f,g \in C^\infty(M)).$$

(ii) $D_V W$ is $\mathbb{R}$-linear in W:

$$D_V(a+b)W \ = D_V(aW) + D_V(bW) \qquad (a,b \in \mathbb{R}).$$

(iii) $D_V(W)$ has a product rule in W:

$$D_V(fW) = (Vf)W + fD_V W \text{ for } f \in C^\infty(M).$$

The operation $D_V W$ reads as the covariant derivative of W in the direction of V with respect to to the connection D. We have the following implications from this definition:

a) $D_V W$ is a tensor in W.

b) $D_V W$ is not a tensor in V

We will next show the existence (not uniqueness) of a natural connection on $\mathbb{R}^n_k$.

Proposition 11.3 *Let M be a semi-Riemannian manifold. If $V \in \mathscr{X}(M)$, let V^* be the covector field (one-form) on M such that*

$$V^*(X) = \langle V, X \rangle, \qquad \forall X \in \mathscr{X}(M).$$

Then the function $V \mapsto V^*$ is a $C^\infty(M)$-linear isomorphism from $\mathscr{X}(M)$ to $\mathscr{X}^*(M)$.

Thus we find that we can always transform a vector field into a corresponding covector field and covector fields into vector fields. It turns out that the related fields $V \leftrightarrow \omega$ contain exactly the same information, and are thus called *metrically equivalent*. In general we can think of metrically equivalent as maintaining length and inner product under the mapping.

Definition 11.4 (Levi-Civita Connection) *Let M be a semi-Riemannian manifold. A Levi-Civita connection, D, is a connection such that*

(i) $[V, W] = D_V W - D_W V$, *and*

(ii) $X\langle V, W\rangle = \langle D_X V, W\rangle + \langle V, D_X W\rangle$,

for all $X, V, W \in \mathscr{X}(M)$.

We see that the Levi-Civita connection exists on any semi-Riemannian manifold.

Theorem 11.5 *On a semi-Riemannian manifold M there exists a unique Levi-Civita connection*

As we have seen in the past, there exists a characteristic form of the Levi-Civita connection that will allow us to check to make sure that we are in fact dealing with this connection. This characteristic property is defined by the Koszul formula.

Definition 11.6 (Koszul formula)

$$\langle D_V W, X\rangle = \frac{1}{2}\left(V\langle W, X\rangle + W\langle X, V\rangle - X\langle V, W\rangle - \langle V, [W, X]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle\right). \quad (11.1)$$

Christoffel Symbols

Considering the case where we are taking the directional derivative of the basis vectors with respect to the basis vectors we define what is called as the Christoffel symbols.

We pause here for a moment to recall the interpretation of the Lie bracket $[V,W]$ as a measure of the rate of change of W in the direction of V. So if we think in the case where V and W are orthoganal coordinate fields our first condition above becomes

$$[V,W]=0 \Longrightarrow D_V W = D_W V.$$

Definition 11.7 (Christoffel Symbols) *Let $\left(U,(x^i)\right)$ be a smooth chart on the semi-Riemannian manifold M. We define the Christoffel symbols for the coordinate system as the real-valued functions Γ^k_{ij} on U such that*

$$D_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right)=\Gamma^k_{ij}\frac{\partial}{\partial x^k}.$$

Notice that since $\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right]=0$, we have $D_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right)=D_{\frac{\partial}{\partial x^j}}\left(\frac{\partial}{\partial x^i}\right)$. Thus,

$$\Gamma^k_{ij}=\Gamma^k_{ji}. \tag{11.2}$$

So, in the case of for orthonormal coordinates the Christoffel symbol is symmetric in its lower indices. However, it is important to remember that, despite the notation, the Christoffel symbol is not in fact a tensor. This is the reason we use the term symbol instead of tensor.

Proposition 11.8 *For a smooth coordinate chart $\left(U,(x^i)\right)$ on a semi-Riemannian manifold M,*

$$D_{\frac{\partial}{\partial x^i}}\left(W^j\frac{\partial}{\partial x^j}\right)=\sum_k\left(\frac{\partial W^k}{\partial x^i}+\sum_j\Gamma^k_{ij}W^j\right)\frac{\partial}{\partial x^k}, \tag{11.3}$$

where the Christoffel symbols are given by

$$\Gamma^k_{ij}=\frac{1}{2}\sum_m g^{km}\left(\frac{\partial g_{jm}}{\partial x^i}+\frac{\partial g_{im}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^m}\right). \tag{11.4}$$

We can take a look at the specific instance of the semi-Euclidean space $\mathbb{R}^n_k$ to obtain some insight into the geometric interpretation of the Christoffel symbol.

a) The matrix of the metric tensor with regard to the normal basis is

$$g_{ij}=\varepsilon_j\delta_{ij},\qquad \varepsilon_j=\begin{cases}-1, & 1\leq j\leq k\\ +1, & k+1\leq j\leq n\end{cases}. \tag{11.5}$$

b) The Christoffel symbol are zero:

$$\begin{aligned}\Gamma^k_{ij}&=\frac{1}{2}\sum_m g^{km}\left(\frac{\partial g_{jm}}{\partial x^i}+\frac{\partial g_{im}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^m}\right) &(11.6)\\ &=0. &(11.7)\end{aligned}$$

So, in semi-Euclidean space, a vector field V is considered parallel to $\mathbb{R}^n_k$ if $D_XV=0$ for each $X\in\mathscr{X}(M)$. Further, this tells us that if $\Gamma^k_{ij}=0$, then the coordinate grid lines are parallel.

Covariant Derivative on Tensor Fields

The definition for a covariant derivative can easily be extended from vector fields to tensor fields using the $\mathbb{R}$-linear and product rule properties of the connection.

Definition 11.9 (Covariant Derivative on Tensor Fields) *Let V be a vector field on a semi-Riemannian manifold M. The Levi-Civita covariant derivative D_V is the unique tensor derivation* *on M such that*

$$D_V f = V f, \qquad \forall f \in C^\infty(M),$$

and $D_V W$ is the Levi-Civita covariant derivative for all $W \in \mathscr{X}(M)$.

Definition 11.10 (Covariant Differential) *The covariant differential of a (k,l)-tensor A on M is the $(k,l+1)$-tensor DA such that*

$$(DA)(\omega^1,\ldots,\omega^k,X_1,\ldots,X_l,V) = (D_V A)(\omega^1,\ldots,\omega_k,X_1,\ldots,X_l)$$

for all $V, X_i \in \mathscr{X}(M)$ and $\omega^j \in \mathscr{X}^*(M)$.

Notice that for a $(0,0)$-tensor, then for all $V \in \mathscr{X}(M)$ we have

$$(Df)(V) = D_V f \tag{11.8}$$

$$= V f \tag{11.9}$$

$$= df(V) \tag{11.10}$$

That is as we would expect, the covariant differential of $(0,0)$-tensor f is just your ordinary differential.

Just as for a vector field, a tensor field is considered parallel provided its covariant derivative is zero, $D_V A = 0$, $\forall V \in \mathscr{X}(M)$. Further, since DA is a tensor, we can of course represent it in coordinate form. However, this process is often very tedious, and as we will find out, often unnecessary.

Parallel Translation

In general we are not able to linearly associate one tangent space on a manifold to another at a different point. However, by defining what we mean as parallel tangent spaces, we find we can we can consider cases where a connection allows us to think of one tangent space being transported to another in a parallel manner.

Consider the simple case of a vector field, Z, on a semi-Riemannian manifold, M, where we assign to each point on a curve $\gamma(t)$ its tangent vector at that point, $p \mapsto \gamma'(t)|_p$, then the set of all smooth vector fields of such curves, γ is module over $\mathscr{X}(J)$, where $t \in J$ is the domain of the curve. There is a natural way to define the vector rate of change of this vector field $\mathscr{X}(\gamma)$.

Proposition 11.11 (Induced Covariant Derivative) *Let $\gamma\colon J \to M$ be a curve in a semi-Riemannian manifold M. Then there is a unique function $Z \mapsto Z' = \frac{DZ}{dt}$ from $\mathscr{X}(\gamma)$ to $\mathscr{X}(\gamma)$, called the induced covariant derivative, such that*

(i) $(aZ_1 + bZ_2)' = aZ_1 + bZ_2 \qquad \forall (a, b \in \mathbb{R})$.

(ii) $(hZ)' = \frac{\mathrm{d}h}{\mathrm{d}t} Z + hZ' \qquad \forall h \in C^\infty(J)$.

(iii) $(V_\gamma)'(t) = D_{\gamma'(t)}(V) \qquad \forall t \in J,\ V \in \mathscr{X}(M)$.

Further,

(iv) $\frac{\mathrm{d}}{\mathrm{d}t}\langle Z_1, Z_2\rangle = \langle Z_1', Z_2\rangle + \langle Z_1, Z_2'\rangle$.

Looking at this in coordinates with $x = x^i \frac{\partial}{\partial x^i} = x^i \partial_i \in \mathscr{X}(M)$, where we assume that the curve γ lies within a single coordinate frame (x^i), then for $Z \in \mathscr{X}(\gamma)$ we get the following.

$$
\begin{aligned}
Z' &= \frac{\mathrm{d}}{\mathrm{d}t} Z && (11.11)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(Z x^i \partial_i\right)|_\gamma && (11.12)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(Z^i \partial_i\right)|_\gamma && (11.13)\\
&= \frac{\mathrm{d}Z^i}{\mathrm{d}t}\partial_i\bigg|_\gamma + Z^i \frac{\mathrm{d}\partial_i}{\mathrm{d}t}\bigg|_\gamma && (11.14)\\
&= \frac{\mathrm{d}Z^i}{\mathrm{d}t}\partial_i\bigg|_\gamma + Z^i \left(\partial_i|_\gamma\right)' && (11.15)\\
&= \frac{\mathrm{d}Z^i}{\mathrm{d}t}\partial_i + Z^i D_{\gamma'}(\partial_i) && (11.16)\\
&= \left(\frac{\mathrm{d}Z^k}{\mathrm{d}t} + \Gamma^k_{ij}\frac{\mathrm{d}d(x^i \circ \gamma)}{\mathrm{d}t} Z^j\right)\partial_k \qquad . && (11.17)
\end{aligned}
$$

And as noted above $Z' = 0$ means that Z is parallel. This in consideration of the last formula

$$\left(\frac{\mathrm{d}Z^k}{\mathrm{d}t} + \Gamma^k_{ij}\frac{\mathrm{d}d(x^i \circ \gamma)}{\mathrm{d}t}Z^j\right)\partial_k = 0 \tag{11.18}$$

gives us a system of k linear differential equations. This system is well understood, and from the uniqueness and existence theorems we find the following proposition.

Proposition 11.12 *For a curve $\gamma\colon J \to M$, let $t \in J$ and $z \in T_{\gamma(t)}M$. Then, there is a unique parallel vector field Z on γ such that $Z(t) = z$.*

Definition 11.13 (Parallel Translation) *Let $\gamma\colon\ J \to M$, be a curve on a semi-Riemannian manifold M. Let $a, b \in J$. The function $P = P_a^b(\gamma)\colon\ T_pM \to T_qM$ such that $z \mapsto Z(b)$ is called parallel translation along γ from $p = \gamma(a)$ to $q = \gamma(b)$.*

It turns out that parallel translation is a linear isometry. However, it is important to note that, in general, parallel translation depends upon the path taken. That is the two curves joining points p and q may have different functions connecting their tangent spaces.

Geodesics

Notice that in the special case where we have a curve $\gamma : J \to M$, and the smooth vector field on $M|_\gamma$: $Z \in \mathscr{X}(\gamma)$ where $Z = \gamma'$, then $Z' = \gamma''$, and is considered the acceleration. This case gives us a way to think about the geometry of curvature as acceleration, and we can use it to define a generalized concept of a straight line on a manifold.

Definition 11.14 (Geodesic) *Let (x^i) be a coordinate system on $U \subseteq M$. A curve γ in U is called a geodesic of M if and only if its coordinate functions $x^k \circ \gamma$ satisfy*

$$\frac{\mathrm{d}^2(x^k \circ \gamma)}{\mathrm{d}t^2} + \Gamma^k_{ij}\frac{\mathrm{d}(x^i \circ \gamma)}{\mathrm{d}t}\frac{\mathrm{d}(x^j \circ \gamma)}{\mathrm{d}t} = 0, \qquad (1 \le k \le n).$$

It is common for us to think of x^i as the coordinate representation of $x^i \circ \gamma$ in order to simplify the notation. Since it is generally clear from the context that we are dealing with the curve γ

this shouldn't cause any confusion, and provides us with a cleaner formula for a geodesic:

$$\frac{\mathrm{d}^2(x^k)}{\mathrm{d}t^2} + \Gamma^k_{ij} \frac{\mathrm{d}(x^i)}{\mathrm{d}t} \frac{\mathrm{d}(x^j)}{\mathrm{d}t} = 0, \qquad (1 \leq k \leq n). \tag{11.19}$$

Again, this is a well understood system of second-order differential equations, and so we have the correlated existence and uniqueness lemmas.

Lemma 11.15 (Existence) *If $v \in T_pM$, then there exists an interval J about* 0 *and a unique geodesic $\gamma\colon J \to M$ such that $\gamma'(0) = v$.*

In general, we think of γ as the curve begining at $p \in M$ with initial velocity v, and this curve is uniquely determined by v at p.

Lemma 11.16 (Uniqueness) *Let $\alpha, \beta : J \to M$ be geodesics. If there is a number $a \in J$ such that $\alpha'(a) = \beta'(a)$ then $\alpha = \beta$.*

Proposition 11.17 *Given any tangent vector $v \in T_pM$, there is a uniqe geodesic γ_v in M such that:*

(i) The initial velocity of γ_v is v, that is $\gamma_v'(0) = v$.

(ii) The domain I_v is the largest possible. Thus, if $\alpha :\ J \to M$ is a geodesic with initial velocity v, then $J \subseteq I$ and $\alpha = \gamma_v|_J$

We refer to the geodesic γ_v as the *maximal geodesic* or *geodesically inextendible*. In the case where every maximal geodesic is defined on the entire real line, we say that the manifold is *geodesically complete*.

Lemma 11.18 *Let $\gamma\colon I \to M$ be a nonconstant geodesic. A reparamertization $\gamma \circ h\colon J \to M$ is a geodesic if and only if h has the form $h(t) = at + b$.*

If a nongeodesic curve has a reparameterization that translates into a geodesic curve we call it a pregeodesic.

This has a significance for the geodesic equation. If a system of second-order equations is given by smooth functions, then its solutions are smooth not just in the parameter, but simultaneously in the parameter, initial values, and initial first derivatives. This gives us the following lemma.

Lemma 11.19 *Let v be a tangent vector to M, then by definition $v \in TM$. Tus there exists a neighborhood N of v in TM and an interval I around 0 such that $(w,s) \mapsto \gamma_w(s)$ is well-defined smooth function from $N \times I$ into M.*

Just as we can take γ' as a new variable, we can consider the second order ODE as a pair of first order ODEs, we can represent a geodesic in M by integral curves in the tangent bundle TM.

Proposition 11.20 *There is a vector field G on TM such that the projection $\pi: TM \to M$ establishes a one-to-one correspondence between (maximal) integral curves of G and (maximal) geodesics of M.*

The Exponential Map

Consider a semi-Riemannian manifold M. We can collect all the geodesics starting at a point $0 \in M$ into a single mapping.

Definition 11.21 (The Exponential Map) *Let $0 \in M$, and let $\mathscr{D}_0$ be the set of vectors $v \in T_0M$ such that the maximal geodesic γ_v is defined at least on $[0,1]$. We define the exponential map of M at 0 is the function*

$$exp_0 : \mathscr{D}_0 \to M,$$

such that $exp_0(v) = \gamma_v(1)$ for all $v \in \mathscr{D}_0$.

Proposition 11.22 *For each $0 \in M$ there exists a neighborhood $\widetilde{U}$ of 0 in T_0M on which the exponential map exp_0 is a diffeomorphism onto a neighborhood U of 0 in M.*

Recall that we call a subset S of a vector space starshaped if $v \in S$ implies $tv \in S$ for any $t \in [0,1]$. We use the starshaped concept to define a normal neighborhood on M.

Definition 11.23 (Normal Neighborhood) *If U is a neighborhood of 0 on M and $\widetilde{U}$ is a starshaped neighborhood of 0 on T_0M, we call U a normal neighborhood of 0.*

Proposition 11.24 *If U is a normal neighborhood of 0 in M, then for each point $p \in U$ there is a unique geodesic $\sigma:\ [0,1] \to U$ from 0 to p in U. Further $\sigma'(0) = exp_0^{-1}(p) \in \widetilde{U}$.*

Definition 11.25 (Broken Geodesic) *A broken geodesic is a piecewise smooth curve segment whose smooth subsegments are geodesics.*

Lemma 11.26 *A semi-Riemannian manifold M is connected if and only if any two points of M can be joined by a broken geodesic.*

There is another important definition associated with the relation between the point 0 on M and the tangent space T_0M.

Definition 11.27 (Normal Coordinate System) *Let U be a normal neighborhood of 0 on M. With the orthonormal coordinate system $e_1,\ldots,e_n$ as the basis of T_0M (that is $\langle e_i, e_j\rangle = \delta_{ij}\varepsilon_j$), the normal coordinate system $\xi = (x^1,\ldots,x^n)$ determined by $e_1,\ldots,e_n$ assigns to each point $p \in U$ the vector coordinates relative to $e_1,\ldots,e_n$ of the cooresponding point in T_0M*

$$exp_0^{-1}(p) = x^i(p)e_i \in \widetilde{U} \subset T_0M, \qquad (p \in U).$$

Proposition 11.28 *If $x^1,\ldots,x^n$ is a normal coordinate system at 0 in M, then for all i,j,k*

(i) $g_{ij} = \delta_{ij}\varepsilon_j$.

(ii) $\Gamma^k_{ij}(0)=0$.

So, at the point 0, though not everywhere, the metric and the Christoffel symbols of a normal coordinate system are semi-Euclidean. Since tensors are defined pointwise, this provides us a powerful tool for calculating the covariant differential of a tensor, as promised earlier. In general we can think of the covariant differential DA as the partial derivative with respect coordinates of the tensor A's component functions (just as we would in the semi-Euclidean space with natural coordinates).

Riemannian Curvature

We will begin this section by defining the Riemannian curvature tensor.

Definition 11.29 (Riemannian Curvature Tensor) *Let M be a semi-Riemannian manifold with Levi-Civita connection, D. The function $R : \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ given by*

$$R_{XY}Z = D_{[X,Y]}Z - [D_X, D_Y]Z$$

is a $(1,3)$-tensor field on M called the Riemannian curvature tensor of M.

Due to the Characteristic property of tensors, we can think of R_{XY} as a multilinear function on individual tangent vectors.

Definition 11.30 (Curvature Operator) *Let $x, y \in T_pM$. The linear operator*

$$R_{xy} : T_pM \to T_pM,$$

That sends each z to $R_{xy}z$ is called the curvature operator.

The curvature operator can be written either as R_{xy} or as the typical operator notation of $R(x,y)$ depending upon the desired emphasis, but the two are equivalent.

Proposition 11.31 *With $x, y, z, y, w \in T_pM$, the following are considered the symmetries of curvature:*

(i) Skew symmetric: $R_{xy} = -R_{yx}$.

(ii) Skew adjoint: $\langle R_{xy}v, w\rangle = -\langle R_{xy}w, v\rangle$.

(iii) First Bianchi identity: $R_{xy}z + R_{yz}x + R_{zx}y = 0$.

(iv) Symmetry by pairs: $\langle R_{xy}v, w \rangle = \langle R_{vw}x, y \rangle$.

Now since R is a tensor, we can take its differential DR that results in a $(1,4)$-tensor. We can interpret this as assigning to four vector fields the (vector field) value $(D_Z R)_{XY} V$. This leads us to the second Binachi identity.

Proposition 11.32 (Second Bianchi Identity) *If $x, y, z \in T_pM$, then*

$$(D_z R)(x,y) + (D_x R)(y,z) + (D_y R)(z,x) = 0.$$

Lemma 11.33 *On the coordinate neighborhood of a coordinate system $x^1, \ldots, x^n$ we have*

$$R_{\partial_k \partial_l}(\partial_j) = R^i_{jkl} \partial_i,$$

where the components of R are given by

$$R^i_{jkl} = \partial_l \Gamma^i_{kj} - \partial_k \Gamma^i_{lj} + \Gamma^i_{lm} \Gamma^m_{kj} - \Gamma^i_{km} \Gamma^m_{lj}.$$

Since Γ is explicitly determined by the metric, we have an explicit formula for curvature in terms of the metric tensor. However, it is worth noting that due to the complexities of these calculations it is often advantageous to leverage theoretical results and exploit the distinctive features of the manifold M to obtain curvature.

Sectional Curvature

General Sectional Curvature

Since the Riemannian curvature tensor is rather complicated we will explore a simpler method to compute R. We begin by considering a tangent plane.

Definition 11.34 (Tangent Plane, Π) *A two-dimensional subspace, Π, of the tangent space T_pM is called a tangent plane to M at p. We define the metric in this space for tangent vectors v and w as*

$$Q(v,w) = \langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2.$$

The implications are as follows:

(a) Π is nondegenerate if and only if $Q(v,w) \neq 0$ for any basis v,w of Π.

(b) The absolute value $|Q(v,w)|$ represents the area of the parallelogram with sides v and w.

(c) $Q(v,w)$ is positive if $g|_\Pi$ is definite, and $Q(v,w)$ is negative if it is indefinite.

Definition 11.35 (Sectional Curvature, K) *The sectional curvature, K, of M is the real valued function on the set of all nondegenerate tangent planes defined by*

$$K(v,w) = \frac{\langle R_{vw}v, w\rangle}{Q(v,w)}.$$

Lemma 11.36 *Let Π be a nondegenerate tangent plane to M at the point p. The sectional curvature $K(v,w)$ is independent of the choice of basis v,w for Π.*

So, we see by definition that the Riemannian curvature, R, determines the sectional curvature, K. However, we also find the converse to be true, and we find that K also determines R. To prove this latter point a way to associate scalar product spaces with a nondegenerate plane.

Lemma 11.37 *Given vectors v,w in a scalar product space, there exists $\overline{v},\overline{w}$, arbitrarily close to v and w, respectively, that span a nondegenerate plane.*

Thus we can consider any pair of vectors in a scalar product space as the limit of two vectors that span a nondegenerate plane. Now for the key feature that allows us to determine R from K.

Proposition 11.38 *If $K = 0$ for every nondegenerate plane in T_pM, then $R_{xy}z = 0$ for all $x, y, z \in T_pM$.*

Although this proposition may appear simple, there is much that we can determined from it. We consider a semi-Riemannian manifold M to be flat if $R = 0$ at every point. But we see that M is flat if and only if $K = 0$. Lets use this to define a property of *curvaturelike* for a function.

Definition 11.39 (Curvaturelike) *We call a multilinear function $F : T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R}$ curvaturelike provided that F satisfies the curvature symmetry properties for $(v, w, z, y) \mapsto \langle R_{vw}x, y\rangle$.*

Now from our simple proposition we find that the curvaturelike function $F(v, w, v, w) = 0$ for all $v, w \in T_pM$ spans a nondegenerate plane.

Corollary 11.40 *Let F be a curvature like function on T_pM such that*

$$K(v, w) = \frac{F(v, w, v, w)}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2},$$

whenever v and w span a nondegenerate plane. Then,

$$\langle R_{vw}x, y\rangle = F(v, w, x, y), \tag{11.20}$$

for all $v, w, x, y \in T_pM$.

Thus, we can use K to determine R.

Also worth noting is that if K is constant, then the manifold is said to have *constant curvature*. In this case our formula for R is simplified.

Corollary 11.41 *If M has constant curvature C, then*

$$R_{xy}z = C\left(\langle z, x\rangle y - \langle z, y\rangle x\right).$$

Gaussian Curvature

In two-dimensions, the sectional curvature becomes a particular real-valued function on M known as Gaussian curvature. We will now explore this specific instance in a little more detail. Suppose M is a 2-dimensional semi-Riemannian manifold (that is a semi-Riemannian surface). For a coordinate system u, v in M the metric tensor is traditionally notated as

$$E = g_{11} = \langle \partial_u, \partial_u \rangle, \quad F = g_{12} = g_{21} = \langle \partial_u, \partial_v \rangle, \quad G = g_{22} = \langle \partial_v, \partial_v \rangle.$$

Then, we get the line element of

$$ds^2 = E du^2 + 2F du dv + G dv^2. \tag{11.21}$$

and two-dimensional subspace metric of

$$Q(\partial_u, \partial_v) = EG - F^2. \tag{11.22}$$

The non-zero Christoffel symbols are found to be

$$Q\Gamma^1_{11} = \begin{vmatrix} E_u/2 & F \\ F_u - (E_v/2) & G \end{vmatrix}, \qquad Q\Gamma^2_{11} = \begin{vmatrix} E & E_v/2 \\ F & F_u - (E_v/2) \end{vmatrix},$$

$$Q\Gamma^1_{12} = \begin{vmatrix} E_v/2 & F \\ G_u/2 & G \end{vmatrix}, \qquad Q\Gamma^2_{12} = \begin{vmatrix} E & E_v/2 \\ F & G_u/2 \end{vmatrix}, \tag{11.23}$$

$$Q\Gamma^1_{22} = \begin{vmatrix} F_v - (G_u/2) & F \\ G_v/2 & G \end{vmatrix}, \qquad Q\Gamma^2_{22} = \begin{vmatrix} E & F_v - (G_u/2) \\ F & G_v/2 \end{vmatrix}.$$

We then find the geodesic equations by substitution in

$$\begin{aligned} u'' + \Gamma_{11}^{1} u'^2 + 2\Gamma_{12}^{2} u'v' + \Gamma_{22}^{1} v'^2 = 0, \\ v'' + \Gamma_{11}^{2} u'^2 + 2\Gamma_{12}^{2} u'v' + \Gamma_{22}^{2} v'^2 = 0. \end{aligned} \tag{11.24}$$

However, since M is 2-dimensional, T_pM is the only tangent plane at p. So the sectional curvature, K, is a real-valued function on M that we call the Gaussian curvature of M.

Since this is second order non-linear differential equation, it becomes rather challenging to solve for K. However, we can consider the following case.

Proposition 11.42 *Let u, v be an orthogonal coordinate system in a semi-Riemannian surface, so $F = \langle \partial_u, \partial_v \rangle = 0$. Then,*

(i) We have the following covariant derivatives

$$D_{\partial_u}\partial_u = \frac{E_u}{2E}\partial_u - \frac{E_v}{2G}\partial_v,$$

$$D_{\partial_v}\partial_v = -\frac{G_u}{2E}\partial_u + \frac{G_v}{2G}\partial_v,$$

$$D_{\partial_u}\partial_v = D_{\partial_v}\partial_u = \frac{E_v}{2E}\partial_u + \frac{G_u}{2G}\partial_v.$$

(ii) With $e = |E|^{1/2}$ and $g = |G|^{1/2}$, and let $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = Sgn(G)$. Then

$$K = \frac{-1}{eg}\left[\varepsilon_1 \left(\frac{g_u}{e}\right)_u + \varepsilon_2 \left(\frac{e_v}{g}\right)_v\right].$$

So in the special case of a 2-dimensional semi-Riemannian surface we find an explicit formula for the sectional curvature called the Gaussian curvature in the case where the coordinate system is orthogonal.

Type-Changing Metric Contraction

We discussed the operations of raising/ lowering indexes and tensor contraction in the Tensors chapter, but we find out that these operations preserve the metric tensor (that is the argument and result are metrically equivalent under the operation). As such we can regard any tensor obtained by these operations as different representations of the same object. Further, these operations commute under the covariant and differential operators.

Lemma 11.43 *Covariant derivatives, D_V, and the covariant differential D commute with both type-changing and contraction.*

For instance, with a tensor A and metric g we have

$$D_V\left(\downarrow_1^a A\right) = D_V\left(C_1^a\left(g \otimes A\right)\right) \tag{11.25}$$

$$= C_1^a\left(g \otimes D_V A\right) \tag{11.26}$$

$$= \downarrow_1^a \left(D_V A\right). \tag{11.27}$$

Frame Fields on a Curve

Recall from the Vector chapter that an orthonormal normal basis for a tangent space T_pM is called a frame on M.

Definition 11.44 (Frame Field) *Let M be a n-dimensional semi-Riemannian manifold. The set $E_1, \ldots, E_n$ of n mutually orthogonal unit vector fields is called a frame field since it assigns a frame to each point $p \in M$.*

Recall from the scalar product section that we can represent any $v \in T_pM$ by orthonormal expansion as

$$v = \varepsilon_i g(v, e_j) e_i, \tag{11.28}$$

where $\varepsilon_i = g(e_i, e_i)$. So, in terms of a frame field we have

$$V = \varepsilon_i \langle V, E_i \rangle E_i, \qquad \text{where } \varepsilon_i = \langle E_i, E_i \rangle.$$

And for two vector fields $V,W \in \mathscr{X}(M)$ we find

$$\langle V,W\rangle = \langle \varepsilon_i \langle V,E_i\rangle E_i, \varepsilon_j \langle W,E_j\rangle E_j\rangle \tag{11.29}$$

$$= \varepsilon_i \varepsilon_j \langle\langle V,E_i\rangle, \langle W,E_j\rangle\rangle E_i E_j \tag{11.30}$$

$$= \varepsilon_i \langle V,E_i\rangle\langle W,E_i\rangle \tag{11.31}$$

For tensor algebra this provides a simpler means for point-wise calculations since coordinate vectors are orthonormal at the origin of a normal coordinate system. However, for tensor calculus, we must keep in mind that unlike $[\partial_i, \partial_j]$, the brackets $[E_i, E_j]$ are, in general, not zero.

We can also consider a frame field on a curve.

Definition 11.45 (Frame Field on a Curve) *Lat $\alpha : I \to M$ be a curve on M. A frame field on a curve α is a set of mutually orthogonal unit vector fields $E_1, \ldots, E_n$ on α.*

It turns out that we are able to define $E_i \in \mathscr{X}(\alpha)$ to be parallel.

Corollary 11.46 *If $\alpha : I \to M$ is a curve and $e_1, \ldots, e_n$ is a frame at $\alpha(0)$, then there is a unique parallel frame field $E_1, \ldots, E_n$ on α such that $E_i(0) = e_i$.*

So now we can leverage the advantages of orthonormality, parallel transport, and global definition. It follows that on M frame fields are defined locally:

a) Given any frame $e_1, \ldots, e_n$ in a tangent space T_0M, choose a normal neighborhood U of 0.
b) Then extend this frame to a frame field $E_1, \ldots, E_n$ on U by parallel transport along radial geodesics.
c) From the theory of differential equations, the vector fields E_i are smooth.

Some Differential Operators

Here we define a number of familiar vector calculus concepts generalized for our application on manifolds explicitly defined by the metric tensor g.

The Gradient

Definition 11.47 (Gradient, *grad***)** *The gradient of a function $f \in C^\infty(M)$, ∇f (or grad f), is the vector field metrically equivalent to the dffeerential $df \in \mathscr{X}^*(M)$. Thus, for all $X \in \mathscr{X}(M)$,*

$$\langle \nabla f, X\rangle = df(X) \tag{11.32}$$

$$= Xf \tag{11.33}$$

So, in terms of a coordinate system $df = \frac{\partial f}{\partial x^i} dx^i$, and thus

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \partial_j \tag{11.34}$$

Divergence

Definition 11.48 (Divergence, *div***)** *The divergence of a tensor field A is the contraction of the new covariant slot in its covariant differential DA with one of its original slots.*

Two common instances where the divergence is unique are stated below.

(a) If V is a vector field, then $div\, V = C(DV) \in C^\infty(M)$. Thus for a frame field

$$div\, V = \varepsilon_i \langle D_{E_i} V, E_i\rangle,$$

and for a coordinate system

$$div\, V = V^i_{;i} = \frac{\partial V^i}{\partial x^i} + \Gamma^i_{ij} V^j.$$

(b) If A is a symmetric $(0,2)$-tensor, then

$$div\, A = C_{13}(DA) = C_{23}(DA) \in \mathscr{X}^*(M).$$

For a frame field,

$$(div\, A)(X) = \varepsilon_i (D_{E_i} A)(E_i, X),$$

while for coordinates,

$$(div\, A)_i = g^{rs} A_{ri;s} = A^s_{i;s}.$$

The Hessian

Definition 11.49 (Hessian) *The Hessian of a function $f \in C^\infty(M)$ is its second covariant differential, $H^f = D(Df)$.*

The Laplacian

Definition 11.50 (Laplacian, $\triangle$) *The Laplacian of a function $f \in C^\infty(M)$ is the divergence of its gradient: $\triangle f = div(grad f) \in C^\infty(M)$.*

Since covariant differentials commute with type-changing, the Laplacian can be viewed as the contraction of its Hessian

$$\begin{aligned}
\triangle f &= div(grad\ f) && (11.35)\\
&= CD(grad\ f) && (11.36)\\
&= CD(\uparrow^1_1 df) && (11.37)\\
&= C \uparrow^1_1 D(df) && (11.38)\\
&= (C \uparrow^1_1) H^f && (11.39)\\
&= C_{12}(H^f). && (11.40)
\end{aligned}$$

The Laplacian has the following coordinate expression.

$$\begin{aligned}
\triangle f &= g^{ij} H_{ij} && (11.41)\\
&= g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right) && (11.42)
\end{aligned}$$

Ricci and Scalar Curvature

Though dealing with Riemannian curvature is rather complicated, we find the contraction of Riemannian curvature gives us simpler invariants.

Definition 11.51 (Ricci Curvature Tensor) *Let R be the Riemannian curvature tensor of M. The Ricci curvature tensor, denoted by Ric, of M is the contraction $C_3^1(R) \in \mathscr{T}_2^0(M)$, whose components relative to a coordinate system are $R_{ij} = R^m_{ijm}$.*

Because of the symmetry of the Riemann tensor, R, the only non-zero contractions are $\pm Ric$.

Lemma 11.52 *The Ricci curvature tensor, Ric, is symmetric, and it is given relative to a frame field by*

$$Ric(X,Y) = \varepsilon_m \langle R_{XE_m}Y, E_m \rangle,$$

where $\varepsilon_m = \langle E_m, E_m \rangle$.

Since the Riemann tensor is determined from the sectional curvature, K, it follows that the Ricci curvature , Ric, is also determined from K.

Notice that we can reconstruct $Ric(u,u)$, with unit vectors u, at each point p. Further, if $e_1, \ldots, e_n$ is a frame at p such that $u = e_1$ we get

$$Ric(u,u) = \varepsilon_m \langle R_{u,e_m}(u), e_m \rangle \tag{11.43}$$

$$= \langle u,u \rangle \sum K(u, e_m) \tag{11.44}$$

From the Ricci curvature tensor, there is another curvature metric that is of significant importance.

Definition 11.53 (Scalar Curvature) *The scalar curvature S of M is the contraction $C(Ric) \in C^\infty$ of its Ricci tensor.*

In coordinates we have

$$S = g^{ij} R_{ij} \tag{11.45}$$

$$= g_{ij} R^k_{ijk}. \tag{11.46}$$

Contracting relative to a frame gives us

$$S = \sum_{i \neq j} K(E_i, E_j) \tag{11.47}$$

$$= 2 \sum_{i<j} K(E_i, E_j) \tag{11.48}$$

And from the second Bianchhi identity we find

$$dS = 2div\, Ric. \tag{11.49}$$

We will see in future sections that the Einstein gravitational tensor is related to the Ricci and scalar curvature tensors.

Chapter Summary

This chapter provides us with the key values for determining the curvature:

a) The Levi-Civita connection D, where

   i. $[V,W] = D_V W - D_W V$

   ii. $X\langle V,W\rangle = \langle D_X V, W\rangle + \langle V, D_X W\rangle$,

   for all smooth vector fields X, V, and W on M.

b) The Riemannian curvature tensor R_{XY} as the function from $\mathscr{X}^3 \to \mathscr{X}$ such that

$$R_{XY}Z = D_{[X,Y]}Z - [D_X, D_Y]Z,$$

where R_{XY} is a $(1,3)$-tensor field on M since it acts on the vector field Z and the covector fields $D_{[X,Y]}$, D_X, and D_Y.

c) The Ricci curvature tensor is the contraction of the Riemannian curvature tensor resulting in the $(0,2)$ tensor field.

d) The Ricci scalar is the contraction of the Ricci curvature tensor resulting in the $(0,0)$-tensor field, or rather a scalar field.

We also saw a convenient value obtained from the covariant derivative between two covariant basis vectors called the Christoffel symbol, and how this symbol helped us identify parallel transport between different tangent spaces as well as the notion of a geodesic or path of zero acceleration.

CHAPTER XII

ADDITIONAL GEOMETRIC CONCEPTS

Conceptual Introduction

We have already discussed the key aspects of manifold curvature, but there are a few more geometric concepts that will come in handy when evaluating spacetime manifolds. The first is local isomotries and the Lorentz geometry which provide us a way to simplify certain models through the usage of symmetries. The other is the notions of product manifolds, or in particular the warped product manifold that will come in handy when we are evaluating expanding spacetime.

Local Isometries

Here we introduce the notion of local equivalence, and provide a number of examples of features that are preserved under local isometries. We then loosen the criteria of isometries to define a homothety, and find that this relation preservers Levi-Civita connections.

Definition 12.1 (Local Isometry) *A smooth map $\phi : M \to N$ of semi-Riemannian manifolds is a local isometry provided each differential map $d\phi : T_pM \to T_{\phi(p)}N$ is a linear isometry.*

The following are examples of objects preserved with isometries that due to thier pointwise nature are also preserved by local isometries:

(a) The induced covariant derivative on a curve

(b) Parallel translation

(c) Geodesics

(d) Exponential maps

(e) Riemannian curvature tensors

(f) Sectional curvature

We find that local isometries on a connected manifold have an additional uniqueness property.

Proposition 12.2 *Let $\phi, \psi : M \to N$ be local isometries of a connected semi-Riemannian manifold M. If there is a point $p \in M$ such that $d\phi_p = d\psi_p$ then $\phi = \psi$.*

The next two definitions are used to understand the conditions under which the Levi-Civita connection is preserved.

Definition 12.3 (Conformal Mapping) *A smooth mapping $\psi : M \to N$ of semi-Riemannian manifolds is conformal provided $\psi^*(g_N) = hg_M$ for some function C^∞ function $h \neq 0$.*

A specific type of conformal mapping occurs in the case of a diffeomorphism.

Definition 12.4 (Homothety) *A diffeomorphism $\psi : M \to N$ of semi-Riemannian manifolds such that $\psi^*(g_N) = cg_M$ where $c \neq 0$ is a constant, is called a homethety.*

Notice that an isometry is just the case where the constant under the homothety is just 1.

Lemma 12.5 *Hometheties preserve Levi-Civita connections.*

Lorentz Geometry

Here we begin to outline how a Lorentz manifold serves as a useful object for the modeling of physical phenomena. We will find that each tangent space of a Lorentz manifold is linearly isomorphic to the semi-euclidean space $\mathbb{R}^n_k$. We will then consider the causal character of vectors in this space.

The Gauss Lemma

We have seen in our previous section that the exponential map takes vectors $t \mapsto tx$ in T_0M to radial geodesics γ_x. We find that the orthogonality to radial directions is also maintained.

Lemma 12.6 (Gauss Lemma) *Let* $0 \in M$ *and* $0 \neq x \in T_0M$. *If,* $v_x, w_x \in T_x(T_0M)$ *with* v_x *radial, then*

$$\langle d\,exp_0(v_x), d\,exp_0(w_x) \rangle = \langle v_x, w_x \rangle$$

In particular, since we had v_x as a radial vector we find that the radial length is preserved. Further, we can view this lemma as a partial isometry with the principle distortions in the the directions orthogonal to the radial directions in T_0M.

We then use this lemma to setup a comparison between the neighborhood on the tangent space T_0M, $\widetilde{U}$ and the corresponding normal neighborhood U on M. For simplicity we will denote $\left(exp_0|_{\widetilde{U}}\right)^{-1}$ simply as exp^{-1}.

We thus have the following:

(a) if $\widetilde{q}$ is the function $v \mapsto \langle v, v \rangle$ on T_oM, the corresponding function $\widetilde{q} \circ exp^{-1}$ is denoted q.

(b) Hyperquadrics appear in $T_oM \cong \mathbb{R}^n_k$, as level hypersurfaces $\widetilde{Q} = \widetilde{q}^{-1}(x), c \neq 0$. The diffeomorphic image of $\widetilde{Q} \cap \widetilde{U}$ under exp_0 is the hypersurface $Q = q^{-1}(c)$ in $U \subseteq M$, called a local hyperquadric at o.

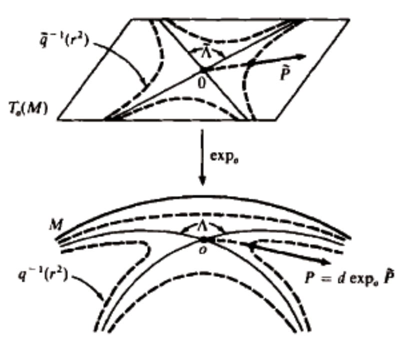

Figure 12.1: Exp Hyperquadric

(c) For indefinite metrics, if $\widetilde{\Lambda}$ is the nullcone $\widetilde{q}^{-1}(0) - 0$ in T_oM, then the diffeomorphic image of $\widetilde{\Lambda} \cap \widetilde{U}$ under exp_o is the local nullcone $\Lambda(0) = q^{-1}(0) - 0$. Thus $\Lambda(0)$ considts of initial segments of all null geodesics starting at 0, and in U as in T_oM the two families of hyperquadrics ($c < 0, c > 0$) are separated by the nullcone.

(d) If $\widetilde{P}$ is the position vector field $v \mapsto v_v$ on $\widetilde{U} \subseteq T_oM$ then the transferred vector field $P = dexp_o(\widetilde{P})$ is called the local position vector field at o. Like $\widetilde{P}$ it is radial, that is tangent to all radial geodesics emanating from o.

Which leads us to the following corollary.

Corollary 12.7 *The local position vector field P at o is orthogonal to every local hyperquadric of M at o. Furthermore P is both orthogonal and tangent to the local nullcone $\Lambda(0)$.*

Lorentz Causal Character

We begin with a scalar product space with index 1 and a dimension $n \geq 2$ (called a *Lorentz vector space*) and a definition for causal character.

Definition 12.8 (Causal Character) *Let $v \in T_pM$. The tangent vector v is said to be one of the three following causal characters:*

(i) spacelike if $\langle v, v \rangle > 0$ or $v = 0$;

(ii) timelike if $\langle v, v \rangle < 0$;

(iii) null or lightlike if $\langle v, v \rangle = 0$.

Definition 12.9 (Nullcone) *The set of all null vectors in T_pM forms what is called as the nullcone.*

It turns out that the causal character of a vector v is the same as the subspace $\mathbb{R}v$ that it generates. Thus, if V is our Lorentz vector space, with the scalar product g. If $W \subseteq V$ is a subspace it must have one of the below causal characters:

(a) W is spacelike if $g|_W$ is positive definite.

(b) W is timelike if $g|_W$ is nondegenerate of index 1.

(c) W is lightlike if $g|_W$ is degenerate.

It turns out that timelike and spacelike subspaces have an orthogonal relation.

Lemma 12.10 *If z is a timelike vector in a Lorentz vector space, V, then the subspace $z^{\perp}$ is spacelike and V is the direct sum $\mathbb{R}z + z^{\perp}$.*

This means that a subspace W is timelike if and only if $W^{\perp}$ is spacelike. Similarlly, this also means that W is lightlike if and only if $W^{\perp}$ is lightlike.

The easiest causal character to deal with is that of spacelike since within these spaces the Schwarz inequality always holds:

$$|\langle v, w \rangle| \leq |v||w|, \qquad \text{with equality only when v and w are collinear.} \tag{12.1}$$

In consideration of timelike subspaces ($n \geq 2$), we find the following.

Lemma 12.11 *Let W be a subspace of dimension ≥ 2 in a Lorentz vector space. Then the following are equivalent:*

(i) W is timelike, hense is itself a Lorentz vector space.

(ii) W contains two linearly independent null vectors.

(iii) W contains a timelike vector

And lastly, we consider the lightlike subspaces.

Lemma 12.12 *For a subspace W of a Lorentz vector space, the following are equivalent.*

(i) W is lightlike, that is degenerate.

(ii) W *contains a null vector, but not a timelike vector.*

(iii) $W \cup \Lambda = L - 0$*, where* L *is a one-dimensional subspace and* Λ *is the nullcone of* V.

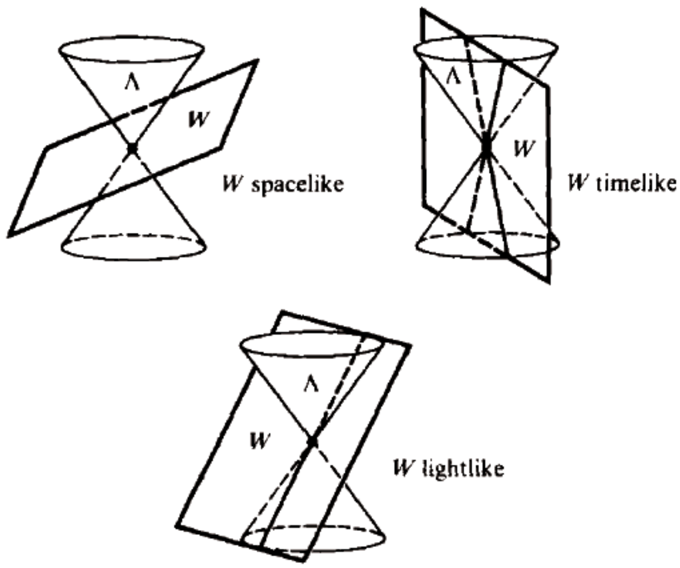

Figure 12.2: Causal Character of Subspace W

If P is a submanifold of a Lorentz manifold, then for each $p \in P$ if we have the subspace T_pP with the same causal character in T_pM, then we say the subspace P possesses that causal character.

Timecones

Let $\mathscr{T}$ be the set of all timelike vectors in a Lorentz vector space V. For $u \in \mathscr{T}$

$$C(u) = \{v \in \mathscr{T} : \langle u, v \rangle < 0\}$$

is the timecone of V containing u. There exists an opposite timecone

$$C(-u) = -C(u) = \{v \in \mathscr{T} : \langle u, v \rangle > 0\}.$$

Further, since $u^{\perp}$ is spacelike, it follows that $\mathscr{T}$ is the disjoint union of these two timecones.

This leads us to a way to identify the comparative location between two timelike vectors.

Lemma 12.13 *Timelike vectors v and w are in the same timecone if and only if $\langle v, w\rangle < 0$.*

There are a number of analogs to the Lorentz space from an inner product space.

Proposition 12.14 *Let v and w be timelike vectors in a Lorentz vector space. Then*

(i) The Schwarz inequality is reversed: $|\langle v, w\rangle| \geq |v||w|$, with equality if and only if v and w are collinear.

(ii) If v and w are in the same timecone of V, then there is a unique number $\varphi \geq 0$, called the hyperbolic angle between v and w such that

$$\langle v, w\rangle = -|v||w|\cosh\varphi.$$

(iii) If v and w are in the same timecone of V, then $|v| + |w| \leq |v + w|$ with equality if and only if v and w are collinear.

Notice in particular that (iii) above implies that a straight line is no longer the shortest path between two points. In fact cutting a corner would make a trip longer. This particular feature turns out to be a fundamental reason these objects are used in the study of relativity.

Now, relating the above specifically to our Lorentz manifold M. Since the tangent space T_pM is a vector space all the above applies. Specifically, for each tangent space T_pM there are two timecones that are really indistinguishable from one another. We can of course pick one, a process known as to *time orient T_pM*.

Lemma 12.15 *A Lorentz manifold M, is time-orientable if and only if there exists a timelike vector field $X \in \mathscr{X}(M)$.*

Semi-Riemannian Product Manifolds

In this section we will see how the geometry of the semi-Riemannian product $M \times N$ relies upon the geometries of the semi-Riemannian manifolds M and N.

We begin with the notion of lift.

Definition 12.16 (Lift of a Function) *Let $f \in C^\infty(M)$. The lift of f to the product manifold $M \times N$ is $\widetilde{f} = f \circ \pi \in C^\infty(M \times N)$, where π is the natural projection.*

Definition 12.17 (Lift of a Tangent Vector) *Let $M \times N$ be a semi-Riemannian product manifold, $x \in T_pM$ and $q \in N$. The lift $\widetilde{x}$ of x to (p,q) is the unique vector in $T_{(p,q)}M$ such that $d\pi(\widetilde{x}) = x$.*

Definition 12.18 (Lift of a Vector Field) *Let $M \times N$ be a semi-Riemannian product manifold, $X \in \mathscr{X}(M)$, $\pi :\ M \times N \to M$, and $\sigma :\ M \times N \to N$. The lift of X to $M \times N$ is the vector field $\widetilde{X}$ whose value at each (p,q) is the lift of X_p to (p,q). Thus, the lift of $X \in \mathscr{X}(M)$ to $M \times N$ is the unique element $\widetilde{X}$ of $\mathscr{X}(M \times N)$ that is both π-related to X and σ-related to the zero vector field on N.*

These three above definitions that describe a lift from a semi-Riemannian manifold M to the semi-Riemannian product manifold $M \times N$ are considered *horizontal lifts*, and the set of all such lifts is denoted as $L(M)$. Similar definitions exist for lifts from the semi-Riemannian manifold N to the product manifold $M \times N$ and are called *vertical lifts*, denoted by $L(N)$

Proposition 12.19 *Let $M \times N$ be a semi-Riemannian product manifold. If $X, Y \in L(M)$ and $V, W \in L(N)$, then*

(i) D_XY *is the lift of* ${}^MD_XY \in \mathscr{X}(M)$.

(ii) D_VW *is the lift of* ${}^ND_VW \in \mathscr{X}(N)$.

(iii) $D_VX = D_XV = 0$.

Corollary 12.20 *Let $M \times N$ be a semi-Riemannian product manifold.*

(i) A curve $\gamma(s) = (\alpha(s), \beta(s))$ on $M \times N$ is a geodesic if and only if its projections α in M and β in N are both geodesics.

(ii) The semi-Riemannian product manifold $M \times N$ is complete if and only if both M and N are complete.

Relating this to the definition for curvature yields the following result.

Corollary 12.21 *Let $M \times N$ be a semi-Riemannian product manifold. If $X, Y, Z \in L(M)$ and $U, V, W \in L(N)$, then:*

(i) $R_{XY}Z$ is the lift of ${}^{M}R_{XY}Z$ on M.

(ii) $R_{VW}U$ is the lift of ${}^{N}R_{VW}U$ on N.

(iii) R is zero for any other coices from $X, \ldots, W$.

These results tell us that the sectional curvature of any horizontal non-degenerate plane to the product manifold $M \times N$ has the same curvature as the projection onto M. Similarly, the sectional curvature for vertical non-degenerate plane is the same as the projection onto N. And finally, there is always some flatness on a product manifold since any non-degenerate plane spanned by a vertical and horizontal vector has $K = R = 0$.

Warped Products

Introduction to Warped Product

Definition 12.22 (Warped Product) *Suppose B and F are semi-Riemannian manifolds, and let $f > 0$ be a smooth function on B. The warped product $M = B \times_f F$ is the product manifold $B \times F$ furnished with the metric tensor*

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F),$$

where π and σ are the projections from $B \times F$ onto B and F respectively.

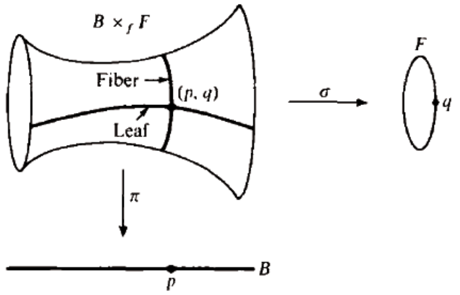

Figure 12.3: Warped Product: Fiber and Leaf

Moving forward we will continue to use the standard notation $\langle , \rangle$ for the metric on the base, B, but will use $(,)$ for the metric on the fiber, F, with its Levi-Civita connection denoted by ∇.

Using the above definition, in explicit form, we find that if x is tangent to $B \times F$ at (p,q) then,

$$\langle x,x \rangle = \langle d\pi(x), d\pi(x) \rangle + f^2(p)\,(d\sigma(x), d\sigma(x))\,.$$

For a manifold $M = B \times_f F$, we call f the *warping function*, and for $(p,q) \in M$ we call $\pi^{-1}(p) = p \times F$ the *fibers* and $\sigma^{-1}(q) = B \times q$ the *leaves*. It is clear that each fiber and leaf are semi-Riemannian submanifolds of M.

Further the warped metric is characterized by:

(a) For each $q \in F$, the map $\pi|(B \times q)$ is an isometry onto B.

(b) For each $p \in B$, the map $\sigma|(p \times F)$ is a positive homethety onto F, with scale factor $\frac{1}{f(p)}$.

(c) For each $(p,q) \in M$, the leaf $B \times q$ and the fiber $p \times F$ are orthogonal at (p,q).

We consider vector tangents to leaves to be *horizontal* and those tangent to fibers to be *vertical*. We will denote the orthogonal projection of $T_{(p,q)}M$ onto its horizontal subspace $T_{(p,q)}(B \times q)$ by $\mathscr{H}$, and similarly for the orthogonal projection of $T_{(p,q)}M$ onto its vertical subspace of $T_{(p,q)}(p \times F)$ by $\mathscr{V}$.

Notice here that in our particular case of general relativity, the subspaces $\mathscr{V}$ are giving us the tangent of the space geometry for each time p. As such, we will primarily be interested in the shape tensor of the fibers. Or rather,

(a) We can use the tangent, *tan*, for the projection $\mathscr{V}$ onto $T_{(p,q)}(p \times F)$

(b) We can use the normal, *nor*, for the projection $\mathscr{H}$ onto $T_{(p,q)}(B \times q)$.

(c) $T_{(p,q)}(B \times q) = \left(T_{(p,q)}(p \times F)\right)^{\perp}$.

Now consider the relationships from B and F to $M = B \times_f F$. Although the lifting function from the base appears straight forward, the lift from the fiber, in general, involves the warping function f. So, let us first consider a lift from the gradient on the base, and then consider the Levi-Civita connections of M.

Lemma 12.23 *If $h \in C^{\infty}(B)$, then the gradient of the lift $h \circ \pi$ of h to $M = B \times_f F$ is the lift to M of the gradient of h on B.*

We can now relate the Levi-Civita connection of M with those of B and F. For simplicity, we will denote $h \circ \pi$ simply as h and similarly $grad(h \circ \pi)$ as $grad\, h$.

Proposition 12.24 *Let $M = B \times_f F$ be a warped product. If $X, Y \in \mathscr{L}(B)$ and $V, W \in \mathscr{L}(F)$, then*

(i) $D_X Y \in \mathscr{L}(B)$ *is the lift of* $D_X Y$ *on* B.

(ii) $D_X V = D_V X = \left(\frac{Xf}{f}\right) V$.

(iii) $nor\, D_V W = -\left(\frac{\langle V,W\rangle}{f}\right) grad\, f$.

(iv) $\tan D_V W \in \mathscr{L}(F)$ *is the lift of* $\nabla_V W$ *on* F.

This provides us some details of the shape of the leaves and fibers.

Corollary 12.25 *The leaves $B \times q$ of a warped product are totally geodesic; and the fibers $p \times F$ are totally umbilic (that is F bends towards from the normal curvature vector in its spacelike directions at each point).*

Warped Product Geodesics

Just as in any manifold, we can parameterize a curve γ on a warped product manifold $M = B \times_f F$ as $\gamma(s) = (\alpha(s), \beta(s))$ where α and β are the projections of γ onto B and F, respectively.

Proposition 12.26 *A curve $\gamma = (\alpha, \beta)$ in $M = B \times_f F$ is a geodesic if and only if*

(i) $\alpha'' = (\beta', \beta')(f \circ \alpha)\, grad\, f$ *in B.*

(ii) $\beta'' = \frac{-2}{f \circ \alpha} \frac{d(f \circ \alpha)}{ds} \beta'$ *in F.*

Curvature of Warped Products

It is also possible to relate the curvature of the warped product to the curvatures of its base and fiber.

Proposition 12.27 *Let $M = B \times_f F$ be a warped product with Riemannian curvature tensor R. If $X, Y, Z \in \mathscr{L}(B)$ and $U, V, W \in \mathscr{L}(F)$, then*

1. $R_{XY}X \in \mathscr{L}(B)$ *is the lift of* ${}^B R_{XY}Z$ *on B.*
2. $R_{VX}Y = \left(\frac{H^f(X,Y)}{f}\right)V$*, where H^f is the Hessian of f.*
3. $R_{XY}V = R_{VW}X = 0.$
4. $R_{XV}W = \left(\frac{\langle V,W\rangle}{f}\right) D_X(grad\, f).$
5. $R_{VW}U = {}^F R_{VW}U - \left(\frac{\langle grad\, f, grad\, f\rangle}{f^2}\right)(\langle V,U\rangle W - \langle W,U\rangle V).$

Similarly, we can now determine the relation of the Ricci curvature tensors.

Corollary 12.28 *On a warped product $M = B \times_f F$ with $d = dimF > 1$, let X, Y be horizontal and V, W be vertical. Then*

(i) $Ric(X,Y) = {}^B Ric(X,Y) - \frac{d\left(H^f(X,Y)\right)}{f}.$

(ii) $Ric(X,V) = 0.$

(iii) $Ric(V,W) = {}^F Ric(V,W) - \langle V,W\rangle f^\sharp$*, where*

$$f^\sharp = \frac{\triangle f}{f} + (d-1)\frac{\langle grad\, f, grad\, f\rangle}{f^2}$$

Chapter Summary

This brief chapter provided some background on how within the Lorentz geometry we are able to leverage the invariance properties of symmetries. We also looked a the product manifold, and in particular the warped product manifold which will help us in understanding the spacetime metric for expanding space.

CHAPTER XIII

EINSTEIN'S THEORY OF RELATIVITY

Conceptual Introduction

Einstein's theory of relativity is a theory of gravity. However, as opposed to gravity being an attractive force between two massive objects, Einstein posited that gravity is actually what we experience as a result of massive objects curving spacetime. Thus as gravity is a result of curvature, the theory relativity is developed using the mathematical objects defined in the earlier chapters. Specifically, spacetime is a semi-Riemannian manifold, free fall is a geodesic path, and motion is relative to the frame of reference. The discussion of relativity is typically divided into two specific cases. The simpler case is that of Special relativity where we work in the special instance where gravitational forces are absent and the space is thus flat. General relativity involves systems where gravitational forces are present and the space is therefore curved. The curvature of space in General relativity makes this case signifigantly more difficult to work with. In this chapter we will begin with a preliminary discussion of Newtonian space-time, and use this as the foundation upon which to define flat spacetime. We will then proceed to explain the specific features of special relativity before concluding with the full case of general relativity explained by Einstein's field equations.

Prerequisites

Newtonian Space

Prior to Einstein's work physics was modeled as a Euclidean space. However, we would think of it today more formally as a particular manifold since there is no preferred coordinate system in space.

Definition 13.1 (Newtonian Space) *Newtonian space is a Euclidean* 3*-space,* $\mathbb{E}$*, or in other words, a Riemannian manifold isometric to* $\mathbb{R}^3$ *(with dot product).*

When we think of motion in a Newtonian space, it makes sense to take a look at Newton's three laws of motion:

a) Force: $F = ma$, where F is a force, m is the mass of an object, and a is the acceleration of the object.

b) Inertia: A body at rest remains at rest, and an non-accelerated object moves at a constant velocity unless acted upon by an outside force.

c) Action: Every action has an equal and opposite reaction.

Now, we are able to manipulate the force equation into an acceleration equation as $a = \frac{F}{m}$. Further, looking at the inertia law it is clear that some object at rest has the same position, r, for all time. We can call the position vector $r(t) = C$. If the object is in a constant non-accelerated motion, its position at any time can be denoted by position vector $r(t) = C + vt$, where v is the constant velocity and $r(0) = C$. Notice that we can find the velocity v by the ordinary derivative of the position vector with respect to time, $\frac{\mathrm{d}r}{\mathrm{d}t} = v$, regardless of the starting position of C. However, if the object were not moving at a constant velocity but was in fact actually accelerating, then the position vector $r(t)$ would point to a position on the curve $\alpha(t)$ and would no longer be linear. We could still find the velocity as the instant rate of change for the position as $\alpha'(t)$, but we would also be able to find the acceleration as the instantaneous rate of change of the velocity $\alpha''(t)$. Further combining the force and acceleration laws we find that

$$F = ma \tag{13.1}$$

$$= m\alpha''(t) \tag{13.2}$$

Thus we find position, velocity, acceleration, and force in the Newtonian system to be functions of time. That is motion reflects the 3-dimensional path in space and position is fixed by time. If we think of motion as the path traversed by a particle, we can define a Newtonian particle as the collection of positions a relatively small object takes over a time interval.

Definition 13.2 (Newtonian particle) *Let $I \subset \mathbb{R}$ be an interval. A Newtonian particle is a curve $\alpha : I \to \mathbb{E}$ in Newtonian space.*

Though dynamic problems may involve a change in mass, we will think of mass, m, as a positive constant in Newtonian mechanics. We can also add in Newton's laws for energy and momentum.

Definition 13.3 (Newtonian Equations) *If $\alpha : I \to E$ is a Newtonian particle of mass m, then*

(i) The momentum of α is the vector field of $m\alpha' = mv$ on α.

(ii) The scalar momentum is the function $m|\alpha'|$ on I.

(iii) The kinetic energy of α is the function $\frac{mv^2}{2}$ on I, where $v = |\alpha'|$.

These definitions lead us to Newtonian space having preferred coordinate systems.

Definition 13.4 (Euclidean Coordinate System) *A Euclidean coordinate system for $\mathbb{E}$ is an isometry $\xi : \mathbb{E} \to \mathbb{R}^3$.*

In particular, since an isometry is a diffeomorphism, ξ is a coordinate system for the manifold $\mathbb{E}$. Further for any coordinate system (x^i), $d\xi\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial u^i}$. Hence, ξ is Euclidean if and only if $g_{ij} = \delta_{ij}$ for $1 \le i, j \le 3$.

The following three examples demonstrate the convenience of Euclidean coordinate system in dealing with straight line problems:

(a) geodesics have affine coordinates $x^i(\gamma(t)) = a^i t + b^i$,

(b) distantly parallel tangent vectors have the same components, and

(c) the distance from p to q is given by the usual Pythagorean formula.

Newtonian Space-time

With time considered as an in interval of $\mathbb{R}$, we can formalize concept of a space-time.

Definition 13.5 (Newtonian space-time) *Newtonian space-time is the Riemannian product manifold $\mathbb{R}^1 \times \mathbb{E}$ of Newtonian time and Newtonian space.*

This definition can be regarded as slightly contrived since the metric of this product space does not align to any particular physical significance, however it does serve well as an adequate starting point for our future discussion of relativity. It is in this sense that the product manifold definition of space-time is valuable.

One notion that is present in both Newton's system and that of relativity is the idea of an event being a point on the manifold.

Definition 13.6 (Event) *A point of the space-time (or spacetime) manifold is called an event.*

Thus in the case of the Newtonian space-time manifold, an event is the point (t,x) on the space-time manifold $\mathbb{R}^1 \times \mathbb{E}$. Further, in the Newtonian system an event can be thought of as an instantaneous happening at a particular time $t \in \mathbb{R}^1$ and position $x \in \mathbb{E}$. We denote the natural projections of $\mathbb{R}^1 \times \mathbb{E}$ onto $\mathbb{R}^1$ and $\mathbb{E}$ as T and S respectively. T then is the universal clock that lets us measure the time interval between any two event points.

We find that the collection of events makes up another object.

Definition 13.7 (Worldline) *A worldline in space-time (or spacetime) is a one-dimensional submanifold, $W \subset M$, such that for any regular curve α on M the projection onto W is the image $\alpha(I)$ where I is an interval of proper time.*

Although we will formally introduce proper time later, for Newtonian space-time the worldline, W, is a one-dimensional submanifold W such that $T|_W$ is a diffeomorphism onto an interval $I \subset \mathbb{R}^1$. For a particle traveling along a path $\alpha : \ I \to E$, the graph $\{(t, \alpha(t)) : \ t \in \mathbb{R}^1\}$ is its worldline. Conversely, given the worldline, we have $\alpha(t) = S \circ (T|_W)^{-1}$. We can see that

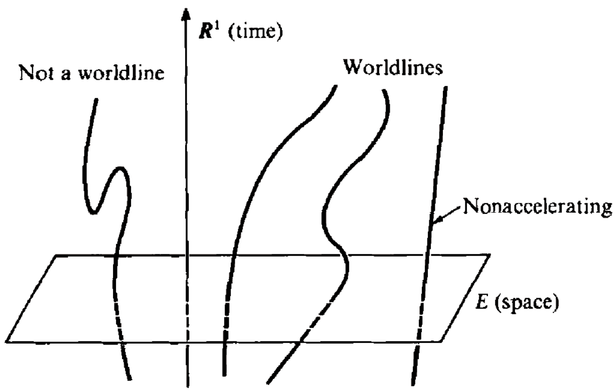

Figure 13.1: Newtonian Worldline

the worldline can provide us information about inertia. For instance a non-curved straight line indicates that we have a constant non-accelerating path through space-time.

These concepts of the space-time, event, and worldline will be expanded upon to develop the foundation for relativity.

A Spacetime Manifold

Although Newtonian mechanics does not pose great inaccuracies on the comparatively small human scale, we find that as we look at large scale systems some difficulties arise:

(a) The first difficult arises from the fact that no material object has ever been found to travel faster than light, but the speed of light plays no specific role in Newtonian motion. For instance, if a particle is traveling towards an observer at a speed of $c/10$ and emits a light beam traveling at the speed of c, the Newtonian addition of velocities says the beam reaches the observer with a velocity of $c + c/10$ greater that of c.

(b) The second difficult arises from the expectation that the speed of light is a constant. However, in the Newtonian system example from above, if the particle were traveling away from the observer a velocity of $c - c/10$ would be observed. Thus, the speed of light in the Newtonian system is relative to motion, but we would expect it to be absolute and everywhere the same.

(c) Our last difficulty arises from the definition of inertial. In Newtonian theory, a particle is either at rest (inertial) or not (non-inertial). However, although the Newtonian system treats motion absolutely, non-accelerating particles at great distance can not be distinguished from being at rest or traveling at a constant velocity.

The first item tells us that we want the speed of light to be an upper bound velocity for interaction between particles. The second item says that we need the speed of light to be constant and the same everywhere. And the third tells us that we need a system that treats motion as relative to a given frame of reference and not as an absolute.

Consider the distance light travels over a distance x in meters for some fixed time t in seconds. This is calculated as $cx(t)$ seconds or approximately 3×10^8 meters. Now anything traveling slower than the speed of light will cover less ground in the same amount of time. Similarly, anything traveling faster than the speed of light will travel farther in the same amount of time.

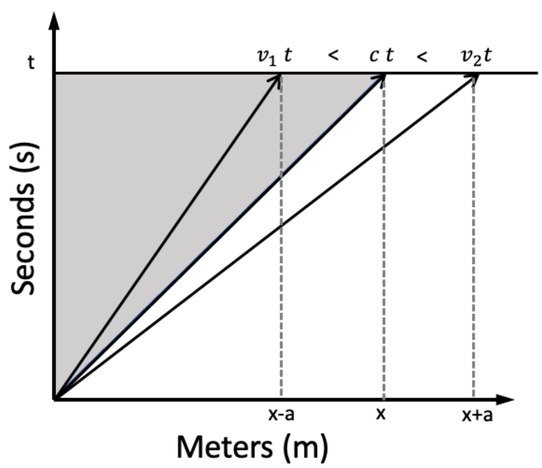

Figure 13.2: Traveling Slower and Faster than Light

Now, for us it is important to consider all spacial directions and not just a single x direction. So in considering all the possible paths that a particle can travel from a given event without traveling faster then the speed of light we would essentially need sweep out a cone with the vertex at the event opening upward where the wall of the cone represents the path of light from the event.

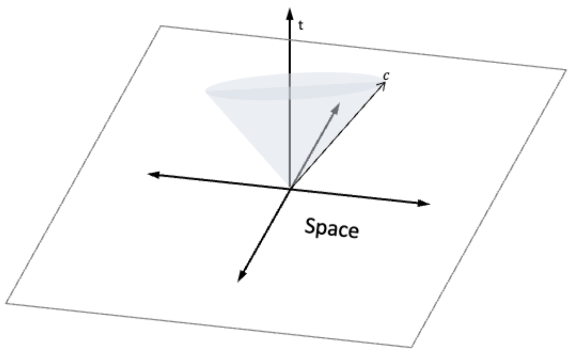

Figure 13.3: Reachable Regions in Space from a Point

Next, notice that a cone embedded within a 3-dimensional Euclidean space can be generated with the formula $\sqrt{x^2+y^2}+t<h$, where h represents the height of the vertex.

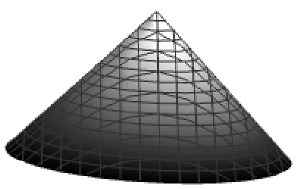

Figure 13.4: Cone: $\sqrt{x^2+y^2}+t<h$

And we can invert the cone by reversing the sign of t (and adjusting the origin to the vertex).

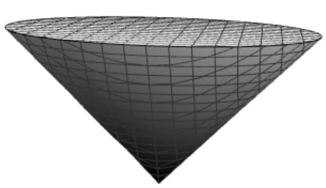

Figure 13.5: Cone: $\sqrt{x^2+y^2}-t<h$

Finally, we have been talking about what regions an event can influence in the time limits imposed by the speed of light. If we are also influenced in the all the events that can influence a given event we find that we obtain both an upward opening and downward opening cone connected at the vertex. This is known as the causal cone, light cone, or time cone.

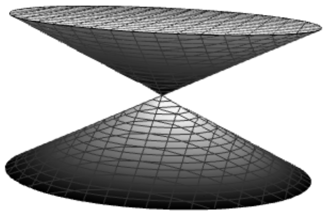

Figure 13.6: Causal Cone

Thus, due to the constant and limiting feature of the speed of light, any event in space should only be influenced from within the lower cone (or past), and should only be able to influence within the upper cone (or future).

We can adopt a similar procedure in order to create this behavior on semi-Riemannian manifold, by changing change the sign of the time-coordinate of the Newtonian line element of $\mathbb{R}^1 \times \mathbb{E}$. The result is the metric for $\mathbb{R}^1_1 \times \mathbb{E}$, or equivalently the flat manifold $\mathbb{R}^4_1$, given by the line element

$$ds^2 = -c^2\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \tag{13.3}$$

$$= -c^2dt^2 + dx^2 + dy^2 + dz^2 \qquad \text{(Cartisean Representation)} \tag{13.4}$$

$$= -c^2dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad \text{(Spherical Representation)}\,. \tag{13.5}$$

So here, if ds^2 is negative we are looking at the distance from an event to a point in spacetime separated by time. If ds^2 is positive, we are looking at the distance from an event to a point in spacetime separated by space. And finally, if ds^2 is zero we are on the edge of the cone where we are in the path with light. We will see specific features related to this cone soon, but for now we are led to a new notion of spacetime where space and time are combined within the same object. Whereas Newtonian space-time was a product of 1-dimensional time and 3-dimensional space, we introduce a new definition for spacetime.

Definition 13.8 (Spacetime) *A spacetime is a connected time-oriented four-dimensional Lorentz manifold.*

Minkowski Spacetime and Special Relativity

Flat $\mathbb{R}^4_1$ spacetime is of particular of interest as it serves as the foundation for Special Relativity, and is formally called the Minkowski spacetime.

Definition 13.9 *A Minkowski spacetime,* **M***, is a spacetime that is isometric to Minkowski* 4-*space,* $\mathbb{R}^4_1$.

Just as in Newtonian space-time, points in **M** are called events, and paths of particles are called worldlines. A few other important features of Minkowski spacetime are:

a) The time orientation of **M** is called the future, and its negative the past.

b) A tangent vector in a future causal cone is called future-pointing.

c) A causal curve is future-pointing if all its velocity vectors are future-pointing.

d) There exists no canonical time function on **M**.

This last point is another important feature of this spacetime. In Newtonian mechanics there was a notion of simultaneity where two events could be said to occur at different points at the same time. However, we no longer have have just one time function but many. As such, it is important for us to distinguish how to measure differences in time. To do so we consider a path of a particle parameterized by a particular time. We will start by defining relativistic particles.

Definition 13.10 (Material Particle) *A material particle in* **M** *is a timelike future-pointing curve* $\alpha : I \to M$ *such that* $|\alpha'(\tau)| = 1$ *for all* $\tau \in I$*. The parameter* τ *is called the proper time of the particle.*

Only intervals of proper time are significant, and we are not interested in isolated values of τ alone. For instance, if α is a particle moving from $\alpha(a) = p$ to $\alpha(b) = q$, then the arclength of the segment $b - a$ is what we are concerned with instead of the individual values of a or b. The segment is called the elapsed proper time between points p and q.

Definition 13.11 (Lightlike Particle) *A lightlike particle is a future-pointing null geodesic* $\gamma : I \to M$.

As in the Newtonian system, material particles have mass. However, lightlike particles are massless. To date, physicists has identified three types of lightlight particles: photons, neutrinos, and gravitons. Having a notion of massive and lightlight particles we can now consider their motion in a relativistic system. We begin with an updated notion of inertial.

Definition 13.12 (Freely Falling) *A particle moving under only the influence of gravity is said to be freely falling.*

In relation to our manifold $\mathbf{M}$, a particle whose world line is geodesic is freely falling. In the case of relativity, we consider this concept the replacement for the Newtonian notion of inertial. Notice that in flat space, where there are no gravitational forces, the geodesic becomes linear and inertial relates to a constant velocity.

Definition 13.13 (Lorentz Coordinate System) *A Lorentz (or inertial) coordinate system in* $\mathbf{M}$ *is a time-oriented-preserving isometry* $\xi : \mathbf{M} \to \mathbb{R}^4_1$.

Lemma 13.14 *Given a frame* e_0, e_1, e_2, e_3 *in* T_pM *such that* e_0 *is future-pointing, there is a unique Lorentz coordinate system* ξ *such that* $\left.\frac{\partial}{\partial x^i}\right|_p = \partial_i|_p = e_i$ *for* $0 \leq i \leq 3$.

Einstein posited that every physical law had to have an expression independent of coordinates. Tensors allowed us to work independent of coordinates, and as we have seen, the introduction of coordinates has the added complexity of distinguishing between intrinsic properties and that of coordinate properties. This problem is alleviated by choosing coordinates that are well suited to the particular intrinsic data at hand.

We are now able to combine the features of the time cone, motion, and reference frames in the Minkowski spacetime to see some features of Special Relativity. Since Minkowski spacetime is isometric to $\mathbb{R}^4_1$, we know the following:

a) For any points $p,q \in M$ there exists a unique geodesic σ with $\sigma(0) = p$ and $\sigma(1) = q$.

b) There is a natural linear isometry $T_pM \cong T_qM$, called distant parallelism.

c) Each exponential map $exp_p : T_pM \to M$ is an isometry.

Thus, M viewed from p looks geometrically the same as T_pM viewed from 0.

In terms of Lorentz coordinate system ξ, distantly parallel tangent vectors have the same components, and $\overrightarrow{pq} = \left(x^i(q) - x^i(p)\right)\partial_i$. Further, we are able to move the notion of causality from the tangent space to the manifold M itself, and for an event $p \in M$:

a) The *future timecone* of p is the set

$$\{q \in M : \overrightarrow{pq} \text{ is timelike and future pointing}\}.$$

b) The negative of the future timecone is called the *past timecone*.

c) The *lightcone*, $\Lambda(p) \subset M$ is the union of the past and future lightcones.

d) A point q that is in neither causal cone of p is *spacelike relative to the event p* ($\overrightarrow{pq}$ is spacelike).

Thus, the usage of the term causal references the direction of influence related to a given event $p \in M$.

a) An event $p \in M$ can only influence events that lie within the future causal cone.

b) Only events in the past causal cone, can influence a given event $p \in M$.

It is however important to notice that given an event $p \in M$, most events (i.e., those that are spacelike) can neither influence nor be influenced by the point p.

Notice the distinction between this definition of causality and that implied by Newtonian mechanics. In Newtonian mechanics, there is a notion of simultaneity where multiple events can occur at a specific fixed time. However, in relativistic causality only events within the light cone have causal effects.

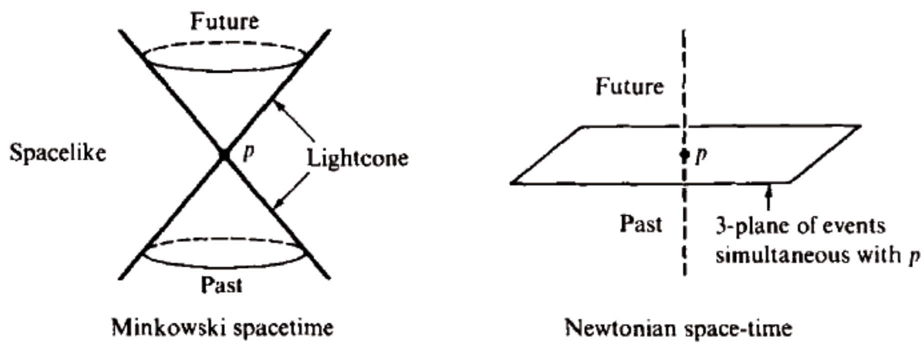

Figure 13.7: Past and Future of an Event

As hinted earlier, there is a specific notion of separation that accounts for the different causal relation between events.

Definition 13.15 (Separation) *For $p, q \in M$ the number $pq = |\overrightarrow{pq}| \geq 0$ is called the separation between p and q.*

We find some features that we would expect from our section on Lorentz geometry.

Lemma 13.16 *If $\overrightarrow{op}$ is spacelike and $\overrightarrow{oq}$ is timelike, then any two of the following imply the third:*

(i) $\overrightarrow{pq}$ *is lightlike,*

(ii) $\overrightarrow{op} \perp \overrightarrow{oq}$*,*

(iii) $op = oq$*.*

Proposition 13.17 *Let p and q be two events within the same timecone of the event o such that $op \perp pq$, and $\varphi = \angle poq$. Then*

(i) $oq^2 = op^2 - pq^2$

(ii) $op = oq \cosh \varphi$*,*

(iii) $pq = oq \sinh \varphi$

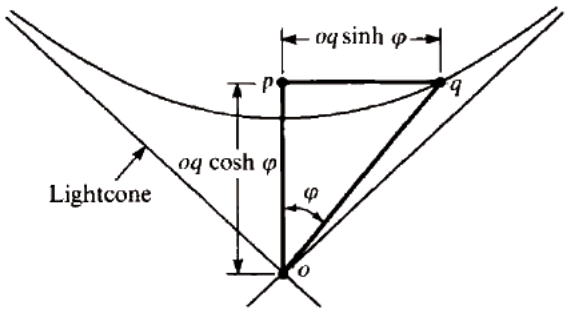

Figure 13.8: Timecone Trigonometry

The first item in this proposition, (i), is actually a replacement for the typical Pythagorean formula. The other two items tell us that projections are hyperbolic rather than circular. Further, upon reflection we can find a few more important implications:

a) A timelike projection op is always greater than the timelike projection oq.

b) If $\overrightarrow{oq}$ is timelike future-pointing, then oq is the measure of elapsed proper time $L(\sigma)$ of the unique freely falling material particle from o to q.

c) If $\overrightarrow{oq}$ is lightlike, then $oq = 0$ and there is a lightlike particle through o and q.

d) If $\overrightarrow{pq}$ is spacelike, then $pq \geq 0$ is the distance from p to q as measured by any freely falling observer perpendicular to $\overrightarrow{pq}$.

This provides us with the foundation for relative motion, but it is worth some time to specifically review an observer frame.

Observed Particles

Here we spend a little more time on the notion of an observer.

Definition 13.18 (Observer) *An observer in M is a material particle.*

The special name is given as a new way to consider coordinates. If ξ is a Lorentz coordinate system, the x^0 axis of ξ can be considered to trace out a world like for the observer ω. The natural parametrization of ω has $x^0\omega(t) = t$, and so t is the proper time for the observer ω. We saw in the last section that any such freely falling observer has many such associated Lorentz coordinate systems.

Definition 13.19 (ξ-time & ξ-position) *Let ξ be a Lorentz coordinate system in M. For each event $p \in M$:*

(i) The number $x^0(p)$ is called the ξ-time of p.

(ii) The point $\overrightarrow{p} = \left(x^1(p), x^2(p), x^3(p)\right) \in \mathbb{R}^3$ is called the ξ-position of p.

We can now consider the ξ-time and ξ-position as measurements taken by the freely falling observer. Further we can build out the relationship between this observer, or material particle, and that of a Newtonian particle.

a) Let $\alpha: I \to M$ be a particle.

b) For each parameter value $s \in I$, the ξ-time of the event, $\alpha(s)$, is given by $t = x^0(\alpha(s))$; and its ξ-position is $\left(x^1(\alpha(s)), x^2(\alpha(s)), x^3(\alpha(s))\right)$.

c) Since α is causal (i.e., within a time cone) and future pointing,

$$\frac{\mathrm{d}(x^0 \circ \alpha)}{\mathrm{d}s} = -\langle \alpha', \partial_0 \rangle > 0.$$

Thus $x^0 \circ \alpha$ is a diffeomorphism of I onto some interval $J \in \mathbb{R}^1$.

(d) Let $u: J \to I$ be the inverse of $x^0 \circ \alpha$.

(e) At ξ-time $t \in J$, the ξ-position of α is

$$\overrightarrow{\alpha}(t) = \Big(x^1\left(\alpha\left(u(t)\right)\right), x^2\left(\alpha\left(u(t)\right)\right), x^3\left(\alpha\left(u(t)\right)\right)\Big).$$

So from the measurements of α in M, we now have a curve $\overrightarrow{\alpha}: J \to \mathbb{R}^3$, where $\alpha \cong E_0$ and E_0 is the coordinate slice where $x^0 = 0$. We call $\overrightarrow{\alpha}$ the *ξ-associated Newtonian particle of α*.

It is possible to look at the relationship between the causal particle and the Newtonian particle to consider how relativistic concepts modify Newtonian theory.

Lemma 13.20 *Let γ be a lightlike particle in M. For Lorentz coordinate system ξ, the associated Newtonian particle $\overrightarrow{\gamma}$ of γ is a straight line in $\mathbb{R}^3$ with speed* 1.

Proposition 13.21 *Let ξ be a Lorentz coordinate system in M. If $\overrightarrow{\alpha} : J \to \mathbb{R}^3$ is the associated Newtonian particle of a material particle $\alpha : I \to M$, then*

(i) The speed $|\frac{\mathrm{d}\overrightarrow{\alpha}}{\mathrm{d}t}|$ of $\overrightarrow{\alpha}$ is $v = \tanh\varphi$ where φ is the Lorentz angle between $\alpha' = \frac{\mathrm{d}\alpha}{\mathrm{d}\tau}$ and the coordinate vector partial$_0 = \frac{\partial}{\partial x^0}$ of ξ. In particular $0 \leq v < 1$.

(ii) The time τ of α and its ξ-time t are related by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}(x^0 \circ \alpha)}{\mathrm{d}\tau} \tag{13.6}$$

$$= -\langle \alpha', \partial_0 \rangle \tag{13.7}$$

$$= \cosh\varphi \tag{13.8}$$

$$= \frac{1}{\sqrt{1-v^2}} \tag{13.9}$$

$$\geq 1. \tag{13.10}$$

So we can now consider interpretations that arise if we move away from coordinates, and focus on the observations of a freely falling observer ω.

(a) [*Time*] For any Lorentz coordinate system ξ, let E_t be the coordinate 3-plane through $\omega(t)$ and perpendicular to ω given by $x^0 = t$. Then, the coordinate x^0 imposes ω's proper time on t for all of M, with E_t consisting of those events that ω "considers" to be simultaneous with $\omega(t)$.

(b) [*Space*] For an observer ω, the ξ-associated Newtonian particle $\overrightarrow{\omega}$ is constant. Thus, E_0 is called the *restspace* of ω.

(c) [*Space*] For any s,t, orthogonal projection $E_s \to E_t$ sends each $p \in E_s$ to the unique point $q \in E_t$ such that $\overrightarrow{pq}$ is parallel to ω. In terms of Lorentz coordinates, this map just changes the x^0 coordinates.

(d) [*Speed*] With ω as x^0 coordinate curve, ω' is always distantly parallel to ∂_0. Thus $\varphi(t)$ is the hyperbolic angle between $\alpha'(t)$ and ω'.

- The function $v = \left|\frac{\mathrm{d}\vec{\alpha}}{\mathrm{d}t}\right|$ measures the speed of α relative to the observer ω.
- The function $\varphi = \tanh^{-1} v$ is the 4-velocity (parameter) of α with respect to the observer ω.

(e) [*Time Dialiation*] For a particle with proper time τ, from (2) in the preceding proposition, the faster the particle is moving relative to the observer the slowerr rthe particle's clock (τ) runs relative to the observer's clock (t).

(f) [*Distance*] For $\overrightarrow{pq}$ to be orthogonal to an observer ω, $x^0(p) = x^0(q)$. This means that p and q are in the same hyperplane E_t, and their separation is normal Euclidean distance.

General Relativity

In general relativity, the flat Minkowski spacetime is replaced with spacetime of arbitrary curvature. It is in this sense that we regard gravity as curvature of spacetime. This section provides with both the cosmological theory and defining equations for general relativity.

Cosmology

The primary mathematical object of general relativity is time-oriented connected four-dimensional Lorentz manifolds. There are a number of primitive principles of general relativity.

Special Relativity is a Special Case of General Relativity

In special relativity the manifold of interest is the flat Minkowski manifold, and so we think of special relativity as the case where the effects of gravity are negligible.

General Relativity is Locally Approximated by Special Relativity

Here if we have a spacetime M and an event p in M, we find that the tangent space is $T_pM \cong \mathbb{R}^4_1$, and the exponential map exp_p provides a local comparison.

Gravity Dominates in the Large

Physics has identified four forces:

a) Electromagnetism

b) Strong Nuclear

c) Weak Nuclear

d) Gravity

Electromagnetism is stronger than gravity, but there are both positive and negative forces essentially canceling each other out. The nuclear forces are much stronger than gravity, but have a very small range. As such when we think about the universe, with its vast distances, we find that gravity is the dominant force.

Free Fall is Geodesic; Matter Curves Spacetime

If we think of the Minkowski metric of special relativity as the local representation of general relativity, then locally we have particles moving in straight lines or:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} = 0 \tag{13.11}$$

Now this is not a tensorial equation, but we can use the chain rule to write:

$$\begin{aligned} \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} &= \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\partial}{\partial x^\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} && (13.12) \\ &= \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\partial_\nu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} && (13.13) \end{aligned}$$

Now we simply replace the partial derivative with the covariant one in order to generalize this into curved space:

$$\begin{aligned} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\partial_\nu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} &\Rightarrow \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\nabla_\nu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} && (13.14) \\ &= \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} && (13.15) \end{aligned}$$

Then the general relativistic version of the Newtonian relation in flat spacetime is simply the geodesic equation:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0 \tag{13.16}$$

Thus we consider freely falling particles to move along geodesics.

Gravity as Curvature

A freely falling observer follows a geodesic. As the geodesic is a particular curve, the observer is able to measure gravitational variations as tidal force. Consider first the following definition for a variation.

Definition 13.22 (Variation) *A variation of a curve segment $\alpha :\ [a,b] \to M$ is a two parameter mapping*

$$x :\ [a,b] \times (-\delta, \delta) \to M,$$

such that $\alpha(u) = x(u,0)$ for all $a \leq u \leq b$.

We can consider the variation vector field of x as the vector field V on a curve segment α given by $V(u) = x_v(u,v) = x_v(u,0)$. Each $V(u)$ is the initial velocity of the traverse curve $v \to x(u,v)$. Thus for $\delta > 0$ sufficiently small, the vector field V is an infinitesimal model of the variation of x. We can consider the geodesic variation or one-parameter family of geodesics where every longitudinal curve of x is geodesic.

Definition 13.23 (Jacobi Vector Field) *If γ is a geodesic, a vector field Y on γ that satisfies the Jacobi differential equation $Y'' = R_{Y\gamma'}(\gamma')$ is called the Jacobi vector field.*

From this definition we have the following lemma.

Lemma 13.24 *The variational vector field of a geodesic variation is a Jacobi field.*

Because of this result, we can call the Jacobi equation the equation of geodesic deviation. Since a freely falling particle follows a geodesic, we can think of a geodesic variation x of γ as a one-parameter family of freely falling particles. This would mean that the variation vector field V

gives the position, relative to the geodesic curve γ of arbitrarily nearby particles. Thus the derivative V' gives the relative velocity and V'' gives the relative acceleration. If we consider assigning these particles unit mass we can read the Jacobi equation $V'' = R_{V\gamma'}\gamma'$ as Newtons second law ($acceleration = 1 \cdot force$) with curvature vector $R_{V\gamma'}\gamma'$ in the role of force. This force is called the *tidal force*.

Next let us consider a vector field Y on a curve $\alpha : I \to M$. We say Y is tangent to α, Y^T, if $Y = f\alpha'$ for some smooth function $f \in C^\infty(I)$, and we say Y is perpendicular to α, $Y^\perp$, if $\langle Y, \alpha' \rangle = 0$. If $|\alpha'| > 0$, then each tangent space $T_{\alpha(s)}M$ has a direct sum decomposition $\mathbb{R}\alpha' + \alpha'^T$. Hence, each vector field Y on α has a unique expression $Y = Y^T + Y^\perp$. Further, if γ is a geodesic then if:

a) $Y \perp \gamma \Rightarrow Y' \perp \gamma$.

b) Y is tangent to $\gamma \Rightarrow Y'$ is tangent to γ.

c) γ is not null, then $(Y')^T = (Y^T)'$, and $(Y')^\perp = (Y^\perp)'$.

We can now define the tidal force operator.

Definition 13.25 (Tidal Force Operator) *For a vector $0 \neq v \in T_p(M)$, the tidal force operator $F_v : v^\perp \to v^\perp$ is given by $F_v(y) = R_{yv}v$.*

In summary:

a) Freely falling particles follow geodesics.

b) The relative position of neighbor particles are given by Jacobi fields Y on γ_0.

c) Changes in relative position result from relative acceleration Y'', which by the Jacobi equation is $R_{Y\gamma'}\gamma'$.

So, curvature, in its role as tidal force, replaces the Newtonian notion of gravitation. In general, an instantaneous observer $u \in T_pM$ measures gravity by the tidal force operator $F_v : u^\perp \to u^\perp$.

Sources of Gravity

The term “matter" is an ambiguous term that intuitively refers to the “stuff" in the universe. In Newtonian mechanics, it is the mass of matter that determines gravity, but in general relativity it is the energy-momentum of matter that is the important feature. The energy-momentum content of matter in a given spacetime is expressed infinitesimally by the stress-energy tensor, T_{ij}. The usage of this infinitesimal quantity by an instantaneous observer replaces the global action within the Newtonian system with a direct-contact differential action. If two spacetimes have an isometry that preserves the matter models, then the two spacetimes are considered *physically equivalent*.

The Einstein Field Equations

Matter is gravitationally significant only as a carrier of energy-momentum, so for this effect as a source of gravitation we must look to the stress-energy tensor T_{ij}. Thus, the task is to find how the stress-energy tensor, T_{ij}, relates to the curvature tensor.

The Einstein Tensor

Definition 13.26 (Einstein Gravitational Tensor) *The Einstein gravitational tensor of a spacetime M is*

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij},$$

where R_{ij} represents the Ricci tensor, R is the scalar curvature, and g_{ij} is the spacetime metric.

Lemma 13.27 *Two points related to the gravitational tensor are:*

(i) G_{ij} *is a symmetric* $(0,2)$*-tensor field with divergence zero.*

(ii) $R_{ij} = G - \frac{1}{2}(G_i^j)g_{ij}$, *where* (G_i^j) *represents the typical contraction operation on the gravitational tensor.*

Definition 13.28 (The Einstein Equation) *If M is a spacetime containing matter with stress-energy tensor T_{ij}, then*

$$G_{ij} = \kappa T_{ij},$$

where $G_i j$ is the Einstein gravitational tensor and kappa is the Einstein gravitational constant.

Note that we are able to relate the Einstein gravitational constant, κ, to the Newtonian gravitational constant G by

$$\kappa = \frac{8\pi G}{c^4}. \tag{13.17}$$

However, we will typically either refer to the constant simply as κ or rescale the system so that $\kappa = 1$. Context will make each approach apparent.

The Einstein equation implies that the stress-energy tensor is a $(0,2)$-tensor with divergence zero. We can make the following two generalizations,

a) $G_{ij} = \kappa T_{ij}$, tells how matter determines Ricci curvature, and

b) $div\, T_{ij} = 0$, tells how Ricci curvature moves this matter.

In the specific case where $T_{ij} = 0$, that is, if M is Ricci flat, then M is said to be a *vacuum* (or *empty*).

The Stress-Energy Tensor

We have thus far seen how one side of the Einstein equation is governed by the curvature of the spacetime manifold. We now take a look at the other side and how curvature is proportional to the stress-energy tensor. But we must first consider a more formal account of matter.

From Einstein's famous equation $E = mc^2$, or more precisely $E^2 = c^2p^2 + \left(mc^2\right)^2$, we know that there is an equivalence between matter (and momentum) and energy. We can also consider a collection of particles (for our purposes here we will focus on massive time-like particles) as being contained within a fluid, and their movements thereby governed by familiar conservation laws of fluid dynamics. In particular we look at a perfect fluid.

The flow of a fluid can be described by the path of particles in a spacetime M. Although, this could be defined in a discrete manner, it is easier to deal with a smooth case. Consider a 4-velocity of a flow given by a timelike unit vector field U on M. We can think of the integral curves of U as the average worldlines of the “molecules" of the fluid.

Definition 13.29 (Perfect Fluid) *A perfect fluid on a spacetime M is a triple (U, ρ, p) where:*

(i) U *is a timelike future-pointing unit vector field on M called the flow vector field.*

(ii) $\rho \in C^\infty(M)$ *is the energy density function.*

(iii) $p \in C^\infty(M)$ *is the pressure function.*

(iv) The stress-energy tensor is:

$$T = (\rho + p)U^* \otimes U^* + pg,$$

where U^* is the one-form metrically equivalent to U.

Definition 13.30 *The stress-energy tensor, or energy-momentum tensor, is a symmetric $(0,2)$-tensor that describes the dynamics of the energy distribution within a given spacetime.*

In its matrix representation, we have:

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix} \tag{13.18}$$

As a perfect fluid, the physical representation of the components can be interpreted as follows:

a) T^{00} represents the energy density, ρ, in the rest frame.

b) $T^{0i} = T^{i0}$ represents the momentum density in the i^{th} direction.

c) T^{ii} represents the unit force in the i^{th} direction, or the pressure p_i (specifically $p_i = p$ in the case of our perfect fluid).

In more mathematical terms, if $X,Y \perp U$, then the above formula for $T^{\mu\nu}$ is equivalent to the following three equations.

$$\begin{aligned} T(U,U) &= \rho \\ T(X,U) &= T(U,X) = 0 \\ T(X,Y) &= p\langle X,Y\rangle \end{aligned} \tag{13.19}$$

Lastly, for a perfect fluid the divergence of the stress-energy tensor is zero, and results in the following consequences.

Proposition 13.31 *If (U,ρ,p) is a perfect fluid,*

(i) $U\rho = -(\rho+p)\,div\,U$ *(energy equation)*

(ii) $(\rho+p)D_U U = -grad_{\perp}\,p$ *(force equation)*

where the spatial pressure gradient $grad_{\perp}\,p$ is the component of grad p orthogonal to U.

This first formula gives us the time rate of change of energy density as measured from U. The second formula is an analog of Newton's $F = ma$ where mass m is replaced by $(\rho+p)$ and the acceleration a is replaced by the spacial acceleration $D_U U$; and the resulting force F is replaced by the spacial pressure gradient $-grad_{\perp}\,p$.

Though we have defined a perfect fluid, we still don't exactly know how to create a spacetime model of one. In both Newtonian mechanics and special relativity we have fixed geometries, but with the arbitrary curvature allowed in general relativity, models only make sense if the stress-energy tensor is physically realistic. Thus building spacetimes in general relativity is a matching game between spacetime geometry and physical matter.

Chapter Summary

Here we introduced the concept of spacetime as a time oriented 4-dimensional Lorentz manifold. We took a look at Minkowski geometry as the geometry of special relativity. We then moved into a discussion of General Relativity by first discussing the cosmological theory, and then the Einstein field equations that relate the geometry of the spacetime to the matter distribution.

CHAPTER XIV

SPACETIME GEOMETRIES

Conceptual Introduction

There are many different spacetime geometries that provide solutions to Einstein's equations. However, solutions are generally not of interest unless the spacetime geometry has a corresponding stress-energy tensor that implies a matter distribution with a significant physical interpretation. As a result, a solution of the Einstein field equations can be approached by starting either with the matter distribution, represented by the stress-energy tensor, or by the spacetime geometry, defined by the metric tensor. In this chapter we will look at both of these approaches. After a brief review of some relevant prerequisites, we will consider how some basic assumptions of physical phenomena can simplify our equations. Then, in the third section we take a look at a solution named after the physicist Willem de Sitter that represents a homogeneous universe with a cosmological horizon. Next, in the fourth section will consider the solution named after Karl Schwarzschild that represents the gravitational forces in an empty space around a particle point mass with an event horizon, such as a black hole. In the fifth, section we will derive a spacetime metric that contains both the event and cosmological horizons, called the Schwarzschild-de Sitter metric. Also within this section, we will obtain the corresponding stress-energy tensor, discuss its physical implications, and identify any restrictions that may be present within this combined spacetime. The last spacetime to be introduced is the Friedmann-Lemaître-Robertson-Walker ($FLRW$) due to the contributions of each of these people. As usual, we will conclude this chapter with a summary that will highlight the important take-aways for the remaining chapters.

This chapter will lean heavily upon the concepts of both the metric tensor and the stress-energy tensor that were discussed in previous chapters. In addition, it is important to have an understanding of what cosmological and event horizons are as well as what is meant by a vacuum dominated universe. We will proceed to introduce the notions with only as much rigor as is necessary to follow the proceeding sections.

Prerequisites

Static Spacetimes

Roughly speaking a static spacetime is a spacetime where the spacial components do not depend upon time. The formal definition of a static spacetime is based on Lie derivatives and Killing fields. Recall that a Lie derivative $\mathscr{L}_X$ applied to a vector field Y results in the Lie bracket $[X,Y]$ and is interpreted as the rate of change of Y along the flow of X. We can also apply the Lie derivative to a covariant tensor field.

Proposition 14.1 *If $X \in \mathscr{X}(M)$ and $A \in \mathscr{T}_s^0(M)$, then*

$$\mathscr{L}_X A = \lim_{t \to 0} \frac{1}{t}\left(\psi_t^*(A) - A\right),$$

where $\{\psi_t\}$ is the flow of X.

With the recognition that the metric tensor g is a covariant 2-tensor, we can now define a Killing field.

Definition 14.2 (Killing Vector Field) *A Killing vector field on a semi-Riemannian manifold is a vector field X for which the Lie derivative of the metric tensor vanishes:*

$$\mathscr{L}_X g = 0.$$

It is also worth noting some implications of a Killing vector field.

Proposition 14.3 *The following conditions on a vector field X are equivalent:*

(i) *X is Killing.*

(ii) $X\langle V,W\rangle = \langle [X,V],W\rangle + \langle V,[X,W]\rangle$ *for all* $V,W \in \mathscr{X}(M)$.

(iii) *DX is skew-adjoint relative to g. That is,* $\langle D_V X,W\rangle + \langle D_W X,V\rangle = 0$, $\forall\, V,W \in \mathscr{X}(M)$.

We can now define a static spacetime.

Definition 14.4 (Static Spacetime) *A spacetime M is static relative to an observer field U provided that U is irrotational and there is a smooth function $f > 0$ on M such that fU is a Killing vector field.*

Note that any local flow $\{\psi_t\}$ of fU consists of isometries. Further each ψ_t preservers U-observers although there may be a change to the proper time parameterization. With U irrotational, we have the restspace of the spacial manifold S, and again $\psi_t(S)$ is also a restspace. All this tells us that, at least locally, the spacial universe always looks the same to a U-observer.

Definition 14.5 (Standard Static Spacetime) *Let S be a three-dimensional Riemannian manifold, I an open interval, and $f > 0$ be a smooth function on S. Let t and σ be the projections of $I \times S$ to I and S, respectively. The standard static spacetime $I\ {}_f\!\times S$ is the manifold $I \times S$ with the line element*

$$-f(\sigma)^2 dt^2 + ds^2,$$

where ds^2 is the lift of the line element of S.

Now, this is just the warped product of $S \times_f I$ with the time coordinate written first.

Lemma 14.6 *For $I\ {}_f\!\times S$,*

(i) *∂_t is a Killing vector field with global flow isometries given by* $\psi_t(s,p) = (s+t,p)$.

(ii) *The observer field $U = \partial_t/g$ is synchronizable, with $U = -f\,\mathrm{grad}\,t$, hence U is irrotational.*

(iii) *The restspaces $t \times S$ of U are isometric under the flow isometries ψ_t, and all are isometric under σ to S.*

Each static spacetime acts locally like a standard static spacetime.

Proposition 14.7 *A spacetime M is static relative to an observer field U if and only if for each $p \in M$ there is a U-preserving isometry of a standard static spacetime onto a neighborhood of p.*

Vacuum Energy and the Cosmological Constant

Vacuum energy is the characteristic energy density for empty space (Carroll, 172). If we consider a nonzero energy density with a Lorentz invariant stress energy tensor, then it would be proportional to the metric,

$$T_{ij}^{(Vac)} = -\rho_{Vac}\eta_{ij}, \tag{14.1}$$

which generalizes to

$$T_{ij}^{(Vac)} = -\rho_{Vac}g_{ij}. \tag{14.2}$$

We then find that the vacuum solution looks just like that of a perfect fluid, with

$$p_{vac} = -\rho_{vac}. \tag{14.3}$$

If we consider the Einstein equations,

$$R_{ij} - \frac{1}{2}R = \kappa T_{ij} \tag{14.4}$$

We can decouple the vacuum energy from the stress energy tensor as

$$R_{ij} - \frac{1}{2}R = \kappa \left(T_{ij}^{(Matter)} + T_{ij}^{(Vac)} \right) \tag{14.5}$$

$$= \kappa \left(T_{ij}^{(Matter)} - \rho_{Vac} g_{ij} \right). \tag{14.6}$$

Reorganizing we get

$$R_{ij} - \frac{1}{2}R + \kappa \rho_{Vac} g_{ij} = \kappa T_{ij}^{(Matter)} \tag{14.7}$$

$$\Rightarrow R_{ij} - \frac{1}{2}R + \Lambda g_{ij} = \kappa T_{ij}^{(Matter)}, \tag{14.8}$$

where this second equation can be considered the alternate Einstein equation and Λ is the cosmological constant that represents the vacuum energy.

This derivation of the cosmological constant differs from the usage originally intended by Einstein. Originally, the cosmological constant was added in order to maintain a static model in an expanding universe, and it turns out that this linear combination of the Einstein tensor with the metric, such as we have here, is the most general modification we can make in order to not significantly alter the basic properties of the original Einstein equation (Wald, 99). Today the constant is used in various models to explain differences in observed phenomena and predicted outcomes, including this instance of vacuum energy. With the expansion of the universe energy density will decrease. Thus, in the limits of expansion, vacuum energy will dominate the universe. With this we can see that our conditions are modeling the entire universe with a limit of expansion, that is a vacuum-dominated universe (Carroll, 335).

Cosmological and Event Horizons

A cosmological horizon is the set of points that act as the outward limit of where an observer can be located. Interestingly for an isotropic universe, if an observer is located near a section of the horizon the space between them and the near horizon looks the same as the space between them and the distant horizon.

The event horizon describes the set of points where a particle traveling towards a mass is unable to move away from it again. That is, if a particle is moving towards a point mass, it's radius from the center of mass is decreasing. Once the particle travels beyond the event horizon, it becomes impossible for the radius from the center of mass to increase.

Physical and Coordinate Singularities

A singularity is a point where the geometry of spacetime breaks down. In some cases, it is actually a feature of the spacetime manifold itself. However, in other cases it is just a feature of the particular coordinate system used. It is important to keep in mind that coordinates can be just local representations of the spacetime manifold. That is, coordinates are just a short hand reference to a particular coordinate chart inside the maximal atlas of the smooth manifold. These coordinates need not be part of a global chart and often are not. This means that not only may there be different coordinate representations for the same spacetime, there may necessarily need to be multiple coordinate representations in order to account for all of the spacetime manifold. As a consequence of local coordinate representations, there may be singularities that are artificial. That is, some singularities are a result of the particular coordinate system in use. In other cases the singularity may be physical. If a singularity is physical, then it must be present after any coordinate transformation. However, it may be represented differently. For example, a physical singularity occurring in space may, under the appropriate coordinate transformations, be represented as a singularity occurring in time although the underlining spacetime manifold remains unchanged. Similarly, artificial singularities, or local coordinate singularities, may potentially be moved or even removed by an appropriate transformation to a different coordinate chart within the spacetime manifold's chart atlas. Singularities are just one example where we see that there may be benefits to one coordinate representation over another depending upon the questions that are being asked.

Common Model Assumptions

If we consider the universe on very large scales we find the stars, nebula, and even galaxies to have an approximately even distribution. Thus, it is reasonable for us to consider the all matter in the universe to be evenly dispersed, that is it is reasonable for us to assume a homogeneous universe. Additionally, we have come to believe that there is nothing particularly special about our placement within the universe. We may therefore also assume that the universe should generally look the same if we happened to be located in a different region of the universe, that is we assume an isotropic universe. This leads us to the two most basic assumptions, that the universe is both homogeneous and isotropic.

Spherically Symmetric

These two assumptions mean that no matter where we move in space, each direction we look appears the same. Mathematically this means we are talking about spherical symmetry in space (although not necessarily in time). This symmetry simplifies the model by allowing us to leverage a familiar geometry. Lets look then at what this symmetry looks like.

Recall that, in general the semi-Riemannian geometry is a study of objects that are invariant or remain unchanged under coordinate transformations. In the case of a 3 dimensional sphere in Cartesian coordinates ($x^2+y^2+z^2=r^2$), we are looking at the following features that remain invariant:

(a) the radius: $r=\sqrt{x^2+y^2+z^2}$,

(b) the differential radius: dr,

(c) the metric, $\langle g,g\rangle$: $dx^2+dy^2+dz^2=dr^2+r^2d\theta^2+r^2sin^2\theta d\phi^2$,

(d) the total differential: $xdx+ydy+zdz=rdr$,

(e) time: both t and dt.

These particular invariants allow us to construct a general form of the line element. Since the metric is quadratic, with variables r and t, in its simplest form we get:

$$ds^2 = A(r,t)dt^2 + B(r,t)dr^2 + C(r,t)\left(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) + 2D(r,t)drdt. \tag{14.9}$$

Consider the coordinate variable change, $r'^2 = r^2 C(r,t)$. Then, dropping the prime, our line element is:

$$ds^2 = O(r,t)dt^2 + M(r,t)dr^2 + r^2\left(d\theta^2 + \sin^\theta d\phi^2\right) + 2N(r,t)drdt. \tag{14.10}$$

Now, we can make the cross terms go to zero through a time coordinate change that makes the vector potentials vanish. Since $N(r,t)$ only depends on r and t we can use the transformation: $t' = f(r',t)$. Then, again dropping the primes we get the line element

$$ds^2 = -a(r,t)c^2dt^2 + b(r,t)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{14.11}$$

Static and Spherically Symmetric

Another possible assumption is that the model is not dependent upon time, or that it is static. With the continued model assumption of spherical symmetry, this is simply updating our line element from above so that our coefficients a and b no longer depend upon time. That is, we are looking at

$$ds^2 = -a(r)c^2dt^2 + b(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{14.12}$$

Lets try to relate this to our more general definitions within the previous chapter. Recall our definition for static is a line element of the form

$$-f(\sigma)^2 dt^2 + d\Sigma^2. \tag{14.13}$$

If we consider the function f from the definition equal to our function a from above and $d\Sigma$ the lift of the line element $b(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$, we find that our definition for a static metric is satisfied. Further, since $b(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ is the line element of the sphere, we have spherical symmetry of space.

Coefficients for the Static and Spherical Symmetric System

We now look to the Christoffel symbols to help us determine the curvature tensor. From the line element, we see that the metric tensor, g_{ij} is

$$g_{ij} = \begin{bmatrix} -a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}. \tag{14.14}$$

And thus the inverse metric, g^{ij} is

$$g^{ij} = \begin{bmatrix} -\frac{1}{a} & 0 & 0 & 0 \\ 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}. \tag{14.15}$$

Recalling that the Christoffel symbols are defined by

$$\Gamma^k_{ij} = g^{km}\frac{1}{2}\left(\frac{\partial}{\partial x^i}g_{jm} + \frac{\partial}{\partial x^j}g_{im} - \frac{\partial}{\partial x^m}g_{ij}\right), \tag{14.16}$$

we have the following non-zero connection coefficients

$$\begin{array}{llll} & \Gamma^0_{10} = \frac{a'}{2a}, & & \\ \Gamma^1_{00} = \frac{a'}{2b}, & \Gamma^1_{11} = \frac{b'}{2b}, & \Gamma^1_{22} = -\frac{r}{b}, & \Gamma^1_{33} = -\frac{r}{b}\sin^2\theta, \\ & \Gamma^2_{12} = \frac{1}{r}, & \Gamma^2_{33} = -\sin\theta\cos\theta, & \\ & \Gamma^3_{13} = \frac{1}{r}, & \Gamma^3_{23} = \cot\theta. & \end{array} \tag{14.17}$$

Curvature

From the Christoffel symbols we are able to determine the curvature. Recall that the Ricci curvature tensor is defined using the metric tensor and Christoffel symbols by:

$$R_{ij} = \frac{\partial}{\partial x^k}\Gamma^k_{ij} - \frac{\partial}{\partial x^j}\Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^m_{km} - \Gamma^k_{im}\Gamma^m_{kj}. \tag{14.18}$$

So, we get the non-zero components of the Ricci tensor, R_{ij}, as

$$\begin{aligned} R_{00} &= \frac{a''}{2b} - \frac{(a'))^2}{3ab} + \frac{a'}{rb} - \frac{a'b'}{4b^2} \\ R_{11} &= -\frac{a''}{2a} + \frac{(a')^2}{4a} + \frac{a'b'}{4ab} + \frac{b'}{rb} \\ R_{22} &= -\frac{ra'}{2ab} - \frac{rb'}{2b^2} - \frac{1}{b} - \cot^2\theta + \csc^2\theta \\ R_{33} &= \frac{rb'\sin^2\theta}{2b^2} - \frac{ra'\sin^2\theta}{2ab} - \frac{\sin^2\theta}{b} + \sin^2\theta \end{aligned} \quad . \tag{14.19}$$

With this we contract the Ricci tensor, R_{ij}, to obtain the Ricci scalar, R, of:

$$R = -\frac{a''}{ab} + \frac{(a')^2}{2a^2b} - \frac{2a'}{rab} + \frac{a'b'}{2ab^2} + \frac{2b'}{rb^2} - \frac{2}{r^2b} + \frac{2}{r^2}. \tag{14.20}$$

The Einstein and de Sitter Universe

With these components in hand we can look to solve Einstein's original equation:

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} = \kappa T_{ij} \tag{14.21}$$

Einstein's Solution

We will start as Einstein did and keep our assumptions of a homogeneous, isotropic, static, and spherically symmetric universe. Thus we continue with the line element:

$$ds^2 = -ac^2 dt^2 + b dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right). \tag{14.22}$$

However, we need some way to deal with the presence of matter represented by the stress energy tensor. If we were to consider each element of mass independently, the model would quickly become too complicated for any exact solutions. Instead, we will consider the case of a perfect fluid which is exactly what Einstein originally did. In this way, we will be able to use the familiar conservation laws from fluid dynamics that were introduced in the previous section.

Lets begin by looking at stress energy tensor for a perfect fluid,

$$T_{ij} = \begin{bmatrix} -c^2\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}. \tag{14.23}$$

This, combined with the constant κ, gives us the right hand side of the Einstein equation.

Now, turning our attention to the left-hand side of the Einstein equation, we can plug the curvature values into the Einstein tensor, G_{ij}. For ease of computation, we will consider the contracted form of the Einstein tensor, G_i^j, as

$$\begin{cases} G_0^0 = -\frac{b'}{rb^2} + \frac{1}{r^2 b} - \frac{1}{r^2} = \kappa\rho c^2. \\ G_1^1 = \frac{a'}{rab} + \frac{1}{r^2 b} - \frac{1}{r^2} = -\kappa p, \end{cases} \tag{14.24}$$

The off-diagonal components are all zero, and as the other diagonal components are just different representations of the pressure, they add no new information.

Our third and last governing equation comes from the laws of conservation, $\partial_k T_i^k = 0$. We have:

$$\frac{\mathrm{d}p}{\mathrm{d}r} + \left(\rho c^2 + p\right)\frac{a'}{2a} = 0. \tag{14.25}$$

Since $\frac{\mathrm{d}p}{\mathrm{d}r} = 0$ for a perfect fluid, this equation simplifies to

$$\left(\rho c^2 + p\right)\frac{a'}{2a} = 0 \tag{14.26}$$

Or rather we have either

$$a' = 0, \tag{14.27}$$

or

$$\left(\rho c^2 + p\right) = 0. \tag{14.28}$$

The former choice leads us to the Einstein solution,whereas the latter is considered the de Sitter solution. We will focus our attention on the de Sitter case.

The de Sitter Universe

For the de Sitter solution, the leading assumption is

$$\rho c^2 + p = 0. \tag{14.29}$$

However, this is not a straight forward choice. In fact, it implies that with density, $\rho \neq 0$, and pressure, $p \geq 0$, when appropriately scaled (i.e., set c equal to unity),

$$\rho + p = 0 \tag{14.30}$$

$$\Rightarrow p = -\rho. \tag{14.31}$$

This tells us that we have a negative, or vacuum density. So, in the case of the de Sitter universe we are dealing with the vacuum energy density ρ_{vac} as discussed above. We then are interested in the adjusted Einstein equations

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = \kappa T_{ij}, \tag{14.32}$$

where the vacuum density is represented inside the cosmological constant Λ.

Updating our guiding equations for the cosmological constant we have,

$$\begin{cases} G_0^0 + \Lambda = -\frac{b'}{rb^2} + \frac{1}{r^2 b} - \frac{1}{r^2} + \Lambda = \kappa\rho c^2 \\ G_1^1 + \Lambda = \frac{a'}{rab} + \frac{1}{r^2 b} - \frac{1}{r^2} + \Lambda = -\kappa p \end{cases} . \tag{14.33}$$

Next, by subtracting the latter from the former we obtain,

$$\left(-\frac{b'}{rb^2}+\frac{1}{r^2b}-\frac{1}{r^2}\right)-\left(\frac{a'}{rab}+\frac{1}{r^2b}-\frac{1}{r^2}\right)=\kappa\rho c^2+\kappa p \tag{14.34}$$

$$\Rightarrow -\frac{ab'+a'b}{rab^2}=\kappa\left(\rho c^2+p\right) \tag{14.35}$$

$$\Rightarrow -(ab)'=\kappa\left(\rho c^2+p\right)rab^2. \tag{14.36}$$

From our starting assumption of $\rho c^2+p=0$ we find,

$$\begin{aligned}(ab)'&=0\\ \Rightarrow ab&=constant.\end{aligned} \tag{14.37}$$

Thus $ab=constant$, and since a is our time coefficient we can simplify things by rescaling the time coordinate to get,

$$a=b^{-1} \tag{14.38}$$

Notice that this implies that $a'=\frac{b'}{b^2}$. Then, from our $G_0^0+\Lambda$ equation, we get

$$-\frac{b'}{rb^2}+\frac{1}{r^2b}-\frac{1}{r^2}+\Lambda=\left(\kappa\rho c^2\right) \tag{14.39}$$

$$\Rightarrow -\frac{rb'}{b^2}+\frac{1}{b}-1+\Lambda r^2=\left(\kappa\rho c^2\right)r^2 \tag{14.40}$$

$$\Rightarrow -\frac{rb'}{b^2}+\frac{1}{b}=1-\Lambda r^2+\left(\kappa\rho c^2\right)r^2 \tag{14.41}$$

$$\Rightarrow a'r+a=1-\left(\Lambda-\kappa\rho c^2\right)r^2 \tag{14.42}$$

$$\Rightarrow (ar)'=1-\left(\Lambda-\kappa\rho c^2\right)r^2. \tag{14.43}$$

And from integration

$$yr = r - \frac{\Lambda - \kappa\rho c^2}{3} r^3 + constant. \tag{14.44}$$

Next, since b is smooth, we can take the constant as zero, and we have

$$y = 1 - \frac{\Lambda - \kappa\rho c^2}{3} r^2. \tag{14.45}$$

And finally, by setting $R^{-2} = \frac{\Lambda-\kappa\rho c^2}{3}$ we have the coefficients

$$b = \left(1 - \frac{r^2}{R^2}\right)^{-1}, \quad and \quad a = \left(1 - \frac{r^2}{R^2}\right). \tag{14.46}$$

Thus our line element is

$$ds^2 = -c^2\left(1 - \frac{r^2}{R^2}\right)dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{14.47}$$

which is known as the *de Sitter line element*.

Notice here that for $r \ll R$, then the metric approximates the flat space of special relativity. Also, we a spacial singularity where

$$r = R. \tag{14.48}$$

The singularity is known as a cosmological horizon. The physical implications imply that R is the "radius of the universe", and so as r, considered as the radial distance, approaches the radius of the universe, we approach the singularity $r = R$.

The Schwarschild Universe

Above, we considered a spacetime with a uniform distribution of matter density. Now we consider The Schwarzschild spacetime that considers the gravitational effects in an empty space around a single body of mass such as a star or a planet. We will continue the model assumptions above regarding the geometric properties of the spacetime, but now we want space devoid of matter, or the vacuum solution where the stress energy tensor is zero, $T_{ij} = 0$.

Schwarzschild's Solution to the Einstein Equation

With our vacuum assumption, the stress energy tensor is zero, and the Einstein equation becomes:

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = 0. \tag{14.49}$$

Just as above, we contract the Einstein tensor for ease of computation to find:

$$\begin{cases} -\frac{b'}{rb^2} + \frac{1}{r^2 b} - \frac{1}{r^2} + \Lambda = 0 \\ \frac{a'}{rab} + \frac{1}{r^2 b} - \frac{1}{r^2} + \Lambda = 0 \\ \frac{a''}{2ab} - \frac{(a')^2}{4a^2 b} - \frac{a'}{2rab} - \frac{a'b'}{4ab^2} - \frac{b'}{2rb^2} + Lambda = 0 \end{cases} . \tag{14.50}$$

However, as a result of isometry, we are only interested in our first two equations. So, just as above we can subtract the first from the second to get:

$$\frac{a'}{rab} + \frac{b'}{rb^2} = 0 \tag{14.51}$$

$$\Rightarrow \frac{a'b + b'a}{rab^2} = 0 \tag{14.52}$$

$$\Rightarrow a'b + ab' = 0 \tag{14.53}$$

$$\Rightarrow (ab)' = 0 \tag{14.54}$$

$$\Rightarrow ab = constant. \tag{14.55}$$

Next, with $y = \frac{1}{b}$, we have $y' = -\frac{b'}{b^2}$. Then from G^1_1 we get:

$$-\frac{b'}{rb^2} + \frac{1}{r^2 b} - \frac{1}{r^2} + \Lambda = 0 \tag{14.56}$$

$$\Rightarrow -\frac{rb'}{b^2} + \frac{1}{b} - 1 + r^2\Lambda = 0 \tag{14.57}$$

$$\Rightarrow ry' + y - 1 + r^2\Lambda = 0 \tag{14.58}$$

$$\Rightarrow (yr)' = 1 - \Lambda r^2. \tag{14.59}$$

Then we simply integrate to solve for y to get

$$yr = r - \frac{\Lambda r^3}{3} - 2m \tag{14.60}$$

$$\Rightarrow y = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \tag{14.61}$$

where m is a constant of integration.

Since $y = \frac{1}{b}$, we have

$$b = \frac{1}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}}. \tag{14.62}$$

Now, going back to our our spherically symmetric and static line element of

$$ds^2 = -ac^2 dt^2 + bdr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{14.63}$$

we can plug b into the spacial line element $d\Sigma^2$ to get

$$d\Sigma^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{14.64}$$

Notice here that since Λ negligible with the exception of very large scale, we have the spacial line element

$$d\Sigma^2 = \frac{dr^2}{1-\frac{2m}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{14.65}$$

where we obtain the Euclidean line element in polar form as at the limit $r \to \infty$.

Finally, we look for the the coefficient a in order to build the full line element. Recall that we found that $ab = constant$. This clearly implies that $a = \frac{1}{b} \times constant$. Just as above, since $a(r)$ is the coefficient for the time differential term, we can just scale our time coordinate with $t' = \frac{t}{constant}$ so that we get $constant = 1$. This whole process then gives us $a = b^{-1}$. Thus we get the line element in the form of

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{14.66}$$

In this section we are specifically concerned with the gravitational effects around a given point mass, and thus the cosmological constant won't have much of an effect at this scale. Thus, assuming $\Lambda \to 0$, we have

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{14.67}$$

which is known as the *Schwarzschild (exterior) solution*.

Here we find an interesting singularity where $r = 2m$. This particular singularity is called an event horizon, and our external solution is only valid outside of this horizon where $r_0 > 2m$, where m is of course the mass of our object. This means that although the line element is perfectly valid, our exterior solution becomes invalid for the domain interval $0 < r_0 < 2m$. If we consider the case of a typical star, then the surface of the mass is outside the critical point of $2m$. However, if the surface of the mass is within the critical point, $r_0 < 2m$, then the star has disappeared and we are left with a black hole.

This raises two interesting questions:

a) Is the singularity $r = 2m$ a feature of our particular coordinate chart, or is it a physical singularity as a feature of our manifold?

b) What happens if the radius is less then twice the mass, $r_0 \leq 2m$?

To address the first question, this is in fact a physical singularity. As expected of a physical singularity such as an event horizon, despite the coordinate dependence this line element, we are unable to remove this singularity through a coordinate change. Consider the following coordinate change,

$$r = r'\left(1 + \frac{m}{2r'}\right)^2 \tag{14.68}$$

$$\Rightarrow r' = \frac{1}{2}\left(\sqrt{r^2 - 2rm} + r - m\right). \tag{14.69}$$

Again dropping the primes, we get the updated line element

$$ds^2 = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right). \tag{14.70}$$

Thus changing the coordinates removed the singularity from the projection into the spacial manifold, but we find the singularity simply moved into the projection into the time manifold at the same point. Thus the overall spacetime manifold maintained the singularity at the same point even though represented differently depending upon which coordinate chart we used.

To explore the second question. Let's consider our original time coefficient $a(r) = b(r)^{-1} = -\left(1 - \frac{2m}{r}\right)$. If $r < 2m$ then $a(r) \to -a(r)$ and $b(r) \to -b(r)$. So, our time-like vector field, ∂_t, becomes space-like, ∂_r, and our space-like vector field becomes time-like. So, our metric as constructed formally works for both regions $N := r \in (2m, \infty)$ and $B := r \in (0, 2m)$, but we do not have a connected space $N \cup B$.

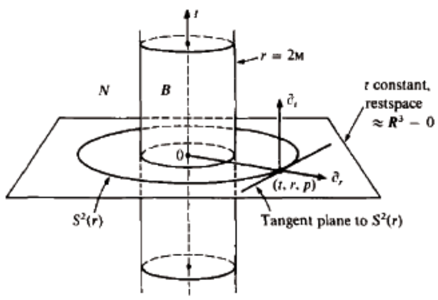

Figure 14.1: Schwarzschild Regions N and B

Notice that when $r \to 2m$ from within N, then $a(r) \to 0$. So, an at rest observer watching a particle travel into the star would see it speed up boundlessly. So, the observer would not actually see the particle cross the event horizon at $r = 2m$, however effectually the particle would be red-shifted to the point where it would not be visible.

Lets now consider what is going on in the region B where $r < 2m$. To do this we will take a look at the light cone, or specifically where θ and ϕ are constant and $ds^2 = 0$. We have

$$
\begin{aligned}
ds^2 &= 0 && (14.71)\\
\Rightarrow -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 &= 0 && (14.72)\\
\Rightarrow \left(1-\frac{2m}{r}\right)^{-1} dr^2 &= \left(1-\frac{2m}{r}\right)dt^2 && (14.73)\\
\Rightarrow \frac{dt^2}{dr^2} &= \left(1-\frac{2m}{r}\right)^{-2} && (14.74)\\
\Rightarrow \frac{dt}{dr} &= \pm\left(1-\frac{2m}{r}\right)^{-1} && (14.75)
\end{aligned}
$$

So, as $r \to 2m$, the slope approaches zero and the light cone closes up.

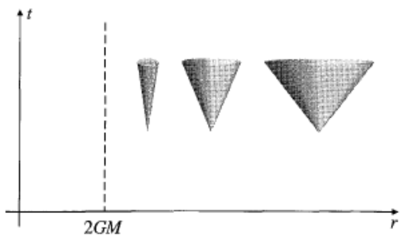

Figure 14.2: Light Cone As we Approach the Event Horizon

Our problem here is in fact due to our coordinate choice. If we were to instead introduce the coordinate transform

$$\begin{cases} r^* &= r + \frac{2m}{r} ln\left(\frac{r}{2m} - 1\right) \\ t &= \pm r^* + constant, \end{cases} \tag{14.76}$$

we end up with the updated line element

$$ds^2 = \left(1 - \frac{2m}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 \triangle_2 \tag{14.77}$$

In considering a particle crossing the event horizon we will have to switch coordinate charts to one that is valid at $r = 2m$.

We next transform these coordinates to those that are aligned to the null geodesics. That is,

$$\begin{cases} v &= t + r^* \\ u &= t - r^* \end{cases} \tag{14.78}$$

Now we have the incoming null geodesics with $v = constant$ and the outgoing as $u = constant$. If we use the original radial coordinate r, but replace t with v we obtain what is known as Eddington-Finkelstein coordinates (Carroll, 221)

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dv^2 + (dvdr + drdv) + r^2 \triangle_2 . \tag{14.79}$$

The determinant of this metric is $g = -r^4 \sin^2\theta$, which has no singluarity at $r = 2m$. So, the singularity was in fact a feature of our chosen coordinate chart, and no such singularity exists with the choice of Eddington-Finkelstein coordinates. If we follow the same procedure as above to obtain $\frac{dv}{dr}$, we have

$$ds^2 = 0 \tag{14.80}$$
$$\Rightarrow -\left(1 - \frac{2m}{r}\right) dv^2 + (dvdr + drdv) = 0 \tag{14.81}$$
$$\Rightarrow \left(1 - \frac{2m}{r}\right) dv^2 = (dvdr + drdv) \tag{14.82}$$
$$\Rightarrow \left(1 - \frac{2m}{r}\right) dv = 2dr \tag{14.83}$$
$$\Rightarrow \frac{dv}{dr} = 2\left(1 - \frac{2m}{r}\right)^{-1} \tag{14.84}$$

Or rather

$$\frac{dv}{dr} = \begin{cases} 0, & (incoming) \\ 2\left(1 - \frac{2m}{r}\right)^{-1}, & (outgoing) \end{cases} \tag{14.85}$$

So, we have lightcones that do not close, but where for all $r < 2m$, all future directed paths are projected along decreasing r. So, although we can now travel past $r = 2m$, once we do, we cannot come back.

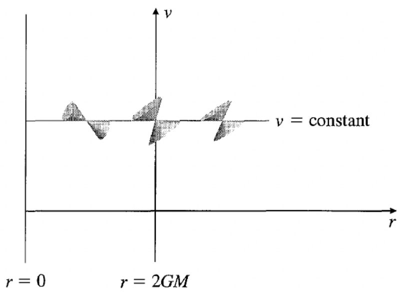

Figure 14.3: Decreasing r for all future directed paths once $r < 2m$

The Schwarzschild-de Sitter Universe

Let us consider a new spacetime that combines the event horizon of the Schwarzschild solution with the cosmological horizon of the de Sitter solution. Both of these spacetimes shared the same symmetry assumptions, but they differed in assumptions about the distribution of matter. The Schwarzschild solution assumed a perfectly empty space where the stress-energy tensor was zero. The de Sitter solution on the other hand considered a uniform distribution of a non-zero energy density. However, we found that our assumption of $(\rho c^2 + p) = 0$ in the de Sitter solution, was really providing us with the energy density of the vacuum dominated universe. By separating out the vacuum density, $T_{ij}^{(Vac)} = -\rho_{vac} g_{ij}$, from the stress-energy tensor for matter, $T_{ij}^{(Matter)}$, we were able to represent the vacuum energy in the cosmological constant. In addition, we found the cosmological constant present in both solutions. So if we consider the stress-energy of mass from the de Sitter solution to be zero, $T_{ij}^{(Matter)} = 0$, we find that

$$R^{-2} = \frac{\Lambda - \kappa\rho c^2}{3} \tag{14.86}$$

$$= \frac{\Lambda}{3}. \tag{14.87}$$

Combining this with the Schwarzschild line element containing the cosmological constant term yields

$$ds^2 = -c^2\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (14.88)$$

$$(14.89)$$

$$= -c^2\left(1 - \frac{2m}{r} - \frac{r^2}{R^2}\right)dt^2 + \left(1 - \frac{2m}{r} - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \qquad (14.90)$$

$$(14.91)$$

This result is known as the *Schwarzschild-de Sitter* line element.

A defining feature of this metric is that it converges to either of the two original metrics in the limits. That is, if we go far away from the point mass, $r \gg 2m$, or if the point mass were to become insignificant, $m \to 0$, then the solution goes to that of de Sitter solution

$$ds^2 = -c^2\left(1 - \frac{r^2}{R^2}\right)dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \qquad (14.92)$$

Similarly, if R tends toward infinity, then for any fixed value of r, we have $r \ll R$ making the last term $\left(\frac{r^2}{R^2}\right) \to 0$, leaving us with a spacetime resembling that of the Schwarzchild solution

$$ds^2 = -c^2\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \qquad (14.93)$$

Having constructed this metric geometrically from these two spacetimes, we expect the matter distribution, or the stress-energy tensor, to match up in the limits as well. To solve the Einstein equations, we begin with the metric as the determining feature of the spacetime geometry,

$$g_{ij} = \begin{pmatrix} -c^2\left(1-\frac{2m}{r}-\frac{r^2}{R^2}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2m}{r}-\frac{r^2}{R^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{14.94}$$

We then find the nonzero Christoffel symbols:

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{r^3 - mR^2}{r^2 - rR^2\left(r-2m\right)}, \tag{14.95}$$

$$\Gamma^1_{00} = -\frac{c^2\left(r^3 - R^2 m\right)\left(r^3 + R^2\left(2m-r\right)\right)}{r^3R^4}, \tag{14.96}$$

$$\Gamma^1_{11} = \frac{R^2m - r^3}{r^4 - rR^2\left(r-2m\right)}, \tag{14.97}$$

$$\Gamma^1_{22} = 2m - r + \frac{r^3}{R^2}, \tag{14.98}$$

$$\Gamma^1_{33} = \frac{\left(r^3 + 2mR^2 - rR^2\right)\sin^2\theta}{R^2}, \tag{14.99}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \tag{14.100}$$

$$\Gamma^2_{33} = -\cos\theta\sin\theta, \tag{14.101}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \tag{14.102}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta. \tag{14.103}$$

Using the Christoffel symbols, we find the Ricci curvature tensor to be

$$R_{ij} = \begin{pmatrix} \frac{3c^2r^2}{R^4} - \frac{3c^2}{R^2} + \frac{6c^2m}{rR^2} & 0 & 0 & 0 \\ 0 & -\frac{3r}{r^3+2mR^2-rR^2} & 0 & 0 \\ 0 & 0 & \frac{3r^2}{R^2} & 0 \\ 0 & 0 & 0 & \frac{3r^2\sin^2\theta}{R^2} \end{pmatrix}. \tag{14.104}$$

Which gives the Ricci scalar of

$$\mathscr{R} = \frac{12}{R^2}. \tag{14.105}$$

Solving the Einstein equation with the cosmological constant, and scaling the gravitational constant to unity, we have

$$\begin{aligned} R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} &= T_{ij} \\ &= \begin{pmatrix} -c^2\frac{\left(rR^2-r^3-2mR^2\right)\left(3-R^2\Lambda\right)}{rR^4} & 0 & 0 & 0 \\ 0 & \frac{r\left(3-R^2\Lambda\right)}{r^3+2mR^2-rR^2} & 0 & 0 \\ 0 & 0 & -\frac{r^2\left(3-R^2\Lambda\right)}{R^2} & 0 \\ 0 & 0 & 0 & -\frac{r^2\left(3-R^2\Lambda\right)}{R^2}\sin^2\theta \end{pmatrix}, \end{aligned} \tag{14.106}$$

giving us the matter distribution.

Schwarzschild-de Sitter Metric at the Limits

In this section we will confirm the above assumption that the stress energy tensor will approach that of the Schwarzschild or de Sitter in the limits. That is, in the limits, the matter distribution of the combined metric will resemble that of the individual metric.

Schwarzschild-de Sitter as $R \to \infty$

If we consider the case where $R \to \infty$, then our stress-energy tensor becomes

$$T_{ij}|_{R\to\infty} = \begin{pmatrix} \frac{c^2\Lambda(2m-r)}{r} & 0 & 0 & 0 \\ 0 & -\frac{r\Lambda}{2m-r} & 0 & 0 \\ 0 & 0 & r^2\Lambda & 0 \\ 0 & 0 & 0 & r^2\Lambda\sin^2\theta \end{pmatrix} \tag{14.107}$$

Also, since we are considering $R \to \infty$, for any fixed value of r, we have $r \ll R$. This means that we can neglect the effects of the cosmological constant, and set $\Lambda = 0$. This provides the expected result where

$$T_{ij} = 0. \tag{14.108}$$

Thus we see that if $R \to \infty$ we end up with the vacuum solution for the empty universe that we were modeling the Schwarzschild solution. Further, for any $r \ll R$, where we can neglect the cosmological constant Λ we end up with $T_{ij} = 0$. So, emptiness can be considered a physical property of the energy distribution rather than just a geometric limit.

Schwarzschild-de Sitter as $\frac{2m}{r} \to 0$

Recall for the de Sitter solution, we had $\Lambda = \frac{3}{R^2} + \kappa\rho c^2$. So, substituting this value into the stress-energy tensor gives us

$$T_{ij}|_{\Lambda \to \frac{3}{R^2}+\kappa\rho} = \begin{pmatrix} \frac{c^4\kappa\rho\left(r^3+2mR^2-rR^2\right)}{rR^2} & 0 & 0 & 0 \\ 0 & -\frac{c^2\kappa\rho rR^2}{r^3+2mR^2-rR^2} & 0 & 0 \\ 0 & 0 & c^2\kappa\rho r^2 & 0 \\ 0 & 0 & 0 & c^2\kappa\rho r^2\sin^2\theta \end{pmatrix} \tag{14.109}$$

Then, since we can split T_{ij} with

$$T_{ij} = T_{ij}^{(matter)} - \rho_{vac} g_{ij} \tag{14.110}$$

$$= -\rho_{vac} g_{ij}, \tag{14.111}$$

where the matter component is zero.

We expect that $g^{ij}T_{ij} = -\rho_{vac}\delta_{ij}$. So

$$g^{ij}T_{ij} = \begin{pmatrix} -c^2\kappa\rho & 0 & 0 & 0 \\ 0 & -c^2\kappa\rho & 0 & 0 \\ 0 & 0 & -c^2\kappa\rho & 0 \\ 0 & 0 & 0 & -c^2\kappa\rho \end{pmatrix} \tag{14.112}$$

$$= \kappa \begin{pmatrix} -c^2\rho & 0 & 0 & 0 \\ 0 & -c^2\rho & 0 & 0 \\ 0 & 0 & -c^2\rho & 0 \\ 0 & 0 & 0 & -c^2\rho \end{pmatrix}. \tag{14.113}$$

Further, for the vacuum energy case where $\left(\rho c^2 + p\right) = 0$ we have $-\rho = \frac{p}{c^2}$, and

$$g^{ij}T_{ij} = \kappa \begin{pmatrix} -c^2\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{14.114}$$

Thus, as expected, the stress energy tensor in this limit becomes that of a perfect fluid which is precisely what was modeled by the de Sitter model.

Domain of Existence

We now examine the domain in which our Schwarzshild-de Sitter model is valid. To do so we will look at both the metric, g_{ij}, and the stress-energy tensor, T_{ij}, to determine our domain of existence.

The first restriction we look at is obvious, from both tensors, since we cannot have a zero in the denominator. We see that neither r nor R is zero:

$$\begin{cases} r & \neq 0 \\ R & \neq 0 \end{cases}. \tag{14.115}$$

Each of these cases, just as in the the individual metrics, result in a physical singularity on our spacetime manifold.

Next, a brief observation of the components within each of our tensors shows that we can actually determine the restrictions from just g_{00} and T_{00}. So we are considering both,

$$g_{00} = 1 - \frac{2m}{r} - \frac{r^2}{R^2} \qquad \text{, and} \tag{14.116}$$

$$T_{00} = -c^2 \frac{\left(-r^3 - 2mR^2 + rR^2\right)\left(3 - R^2\Lambda\right)}{rR^4}. \tag{14.117}$$

From the metric, it is clear that,

$$1 - \frac{2m}{r} - \frac{r^2}{R^2} \neq 0. \tag{14.118}$$

And since we are modeling a vacuum, we know that

$$-c^2 \frac{\left(-r^3 - 2mR^2 + rR^2\right)\left(3 - R^2\Lambda\right)}{rR^4} = 0 \tag{14.119}$$

$$\Rightarrow \left(-r^3 - 2mR^2 + rR^2\right)\left(3 - R^2\Lambda\right) = 0. \tag{14.120}$$

However, notice that

$$1 - \frac{2m}{r} - \frac{r^2}{R^2} \neq 0 \tag{14.121}$$

$$\Rightarrow \left(-r^3 - 2mR^2 + rR^2\right) \neq 0. \tag{14.122}$$

Thus the restriction from the metric is also contained within the stress-energy tensor, giving us the two restrictions of

$$\begin{cases} \left(-r^3 - 2mR^2 + rR^2\right) & \neq 0 \\ \left(3 - R^2\Lambda\right) & = 0. \end{cases} \tag{14.123}$$

Restriction from Metric

Continuing with this equation and noticing that it is a depressed cubic in r, we can solve for r using Vieta's substitution.

With the coefficient of the first order term of R^2, let $r = x + \frac{R^2}{3x}$. So,

$$r^3 = \left(x + \frac{R^2}{3x}\right)^3 \tag{14.124}$$

$$= x^3 + R^2x + \frac{R^4}{3x} + \frac{R^6}{27x^3} \qquad \text{, and} \tag{14.125}$$

$$rR^2 = R^2x + \frac{R^4}{3x}. \tag{14.126}$$

Plugging this in, we get

$$\left(r^3 + 2mR^2 - rR^2\right) = x^3 + R^2x + \frac{R^4}{3x} + \frac{R^6}{27x^3} - R^2x - \frac{R^4}{3x} + 2mR^2 \tag{14.127}$$

$$= x^3 + 2mR^2 + \frac{R^6}{27x^3} \tag{14.128}$$

$$\neq 0. \tag{14.129}$$

Then multiplying each side by x^3, and substituting with $u = x^3$, we end up with

$$u^2 + 2mR^2u + \frac{R^6}{27} \neq 0. \tag{14.130}$$

Which we can solve by the quadratic formula. Keeping only the positive value, we obtain

$$u = \frac{\sqrt{27m^2R^4 - R^6}}{3\sqrt{3}} - mR^2 \tag{14.131}$$

$$= \frac{R^2}{9}\left(\sqrt{81m^2 - 3R^2} - 9m\right). \tag{14.132}$$

Moving back to the x variable, and using the roots of unity, we obtain:

$$x_1 = \sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2 - 3R^2} - 9m\right)} \tag{14.133}$$

$$x_2 = \left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2 - 3R^2} - 9m\right)} \tag{14.134}$$

$$x_3 = \left(\frac{-1+\sqrt{-3}}{2}\right)^2\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2 - 3R^2} - 9m\right)} \tag{14.135}$$

Our solutions then are $r_i = x_i - \frac{R^2}{3x_i}$, for $i = 1,2,3$. Thus, for r_1,

$$r_1 = \sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2 - 3R^2} - 9m\right)} + \frac{R^2}{3\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2 - 3R^2} - 9m\right)}} \tag{14.136}$$

$$= \frac{3\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2 - 3R^2} - 9m\right)}\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2 - 3R^2} - 9m\right)} + R^2}{3\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2 - 3R^2} - 9m\right)}} \tag{14.137}$$

$$= \frac{3\left(\frac{R^2}{9}\left(\sqrt{81m^2 - 3R^2} - 9m\right)\right)^{\frac{2}{3}} + R^2}{3\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2 - 3R^2} - 9m\right)}}. \tag{14.138}$$

For r_2, we find

$$r_2 = \left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)} + \frac{R^2}{3\left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)}} \tag{14.139}$$

$$= \frac{\left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)}\left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)} + R^2}{3\left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)}} \tag{14.140}$$

$$= \frac{-(-1)^{\frac{1}{3}}\,3\left(\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)\right)^{\frac{2}{3}} + R^2}{3\left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)}}. \tag{14.141}$$

And finally, for r_3 we get

$$r_3 = \left(\frac{-1+\sqrt{-3}}{2}\right)^2\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)} + \frac{R^2}{3\left(\frac{-1+\sqrt{-3}}{2}\right)^2\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)}} \tag{14.142}$$

$$= \frac{3\left(\frac{-1+\sqrt{-3}}{2}\right)^2\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)}\left(\frac{-1+\sqrt{-3}}{2}\right)^2\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)} + R^2}{3\left(\frac{-1+\sqrt{-3}}{2}\right)^2\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)}} \tag{14.143}$$

$$= \frac{3\left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)\right)^{\frac{2}{3}} + R^2}{3\left(\frac{-1+\sqrt{-3}}{2}\right)^2\sqrt[3]{\frac{R^2}{9}\left(\sqrt{81m^2-3R^2}-9m\right)}}. \tag{14.144}$$

So, we have found three more restrictions on the value for r. But, if we look a little closer, we seem to run into a bit of a problem if $81m^2 - 3R^2 < 0$. So, in order to maintain our real root we would also want,

$$81m^2 - 3R^2 \geq 0 \tag{14.145}$$

$$\Rightarrow 3R^2 \leq 81m^2 \tag{14.146}$$

$$\Rightarrow \sqrt{3}R \leq 9m \tag{14.147}$$

$$\Rightarrow R \leq 3\sqrt{3}m. \tag{14.148}$$

Restriction from the Stress-Energy Tensor

The second equation we obtained from the stress-energy tensor tells us that

$$R^{-2} = \frac{\Lambda}{3}. \tag{14.149}$$

However, comparing this to our original definition of R^{-2}, we see that this really just telling us that our matter density is zero, $\rho = 0$.

Restrictions at Limits

Finally, we review our restrictions in the limits. With $R \to \infty$ makes the third term in the g_{00} vanish, we maintain our event horizon at $r = 2m$. Similarly, for $\left(\frac{2m}{r}\right) \to 0$ we maintain our cosmological horizon at $r = R$.

Friedmann-Lemaître-Robertson-Walker Spacetimes

The Friedmann-Lemaître-Robertson-Walker, $FLRW$, is another example of a spacetime model built upon the assumptions of a homogeneous and isotropic universe.

Derivation of the Robertson-Walker Spacetime

a) We begin with a smooth manifold:

$$M = I \times S,$$

where I is an open interval of $\mathbb{R}^1_1$, and S is a connected 3-dimensional manifold.

b) Next let $U = \frac{\partial}{\partial t} = \partial_t$ be the lift to $I \times S$ of the standard vector field $\frac{\mathrm{d}}{\mathrm{d}t}$ on I.

c) For each $p \in S$, set the world lines of the galactic flow on M by $\gamma_p(t) = (t, p)$.

Since U gives the velocity of each such “galaxy", γ_p, each U represents the integral curves. Thus the function t will give the common proper time of all galaxies.

d) Holding t constant gives the hypersurface

$$S(t) = t \times S = \{(t, p) : \ p \in S\}.$$

e) We next denote the lift for a function $h \in C^\infty(I)$ as h, and we write h' for $Uh = \frac{\mathrm{d}h}{\mathrm{d}t}$.

The geometry of this model will follow from our physical assumptions about the galactic flow.

f) Consider each γ_p as a particle with proper time t. From isotropy we find that

$$\begin{aligned}\langle U, U\rangle &= \langle \partial_t, \partial_t\rangle \\ &= -1.\end{aligned}$$

And since the relative motion of the actual galaxies is negligible on a large-scale average, we can take each slice $S(t)$ to be a common restspace for their idealizations γ_p, requiring

$$U \perp S(t), \text{ for all } t \in I.$$

Hence, each such slice becomes a Riemannian (i.e., spacelike) hypersurface.

We formalize the isotropy condition "all spatial directions the same" as a galaxy-preserving isometry.

Definition 14.8 (Galaxy-Preserving Isometry) *Each (t,p) has a neighborhood $\mathscr{N}$ such that, given unit tangent vectors x,y to $S(t)$ at the point (t,p), there is a galaxy-preserving isometry $\phi = id \times \phi_x$ of $\mathscr{N}$ such that $d\phi(x) = y$.*

Under these conditions we find:

g) Each slice $S(t)$ has constant curvature $C(t)$.

h) For any $s,t \in I$ the natural map $\mu(s,p) = (t,p)$ from $S(s)$ to $S(t)$ is a homothety.

Since the homothety $\mu = \mu_{st}$ has scale factor $h(s,t)$, $h(s,t)^2C(t) = C(s)$. Further, since μ_{st} is a diffeomorphism h is never zero. This implies that the curvature, C, maintains the same sign: $k = -1, 0,$ or, 1.

i) Now fix $a \in I$, and define $f(a) > 0$ by $C(a)f(a)^2 = k$.

j) Assign S the metric tensor such that the map $j_a(s) = (a,s)$ from S to $S(a)$ is a homothety of scale factor $f(a)$.

k) Then, define a function $f \in C^\infty(I)$ by $f(t) = \frac{h(a,t)}{f(a)}$.

It follows that:

i. S has constant curvature k, and every injection $j_t : S \to S(t)$ is a homothety of scale factor $f(t)$.

In particular, the constant curvature of $S(t)$ is $\frac{k}{f(t)^2}$, and for vectors x,y tangent to $S(t)$

$$\langle x,y \rangle = f^2(t)(d\sigma(x), d\sigma(y)).$$

Thus the conditions above express $I \times S$ geometrically as a warped product with base I and fiber S.

Definition 14.9 (Friedmann-Lemaître-Robertson-Walker Spacetime) *Let S be a connected 3-dimensional Riemannian manifold of constant curvature $k = -1, 0,$ or, 1. Let $f > 0$ be a smooth function on an open interval I in $\mathbb{R}^1_1$. Then the warped product*

$$M(k, f) = I \times_f S$$

is called a Friedmann-Lemaître-Robertson-Walker spacetime.

Explicitly, $M(k, f)$ is the manifold $I \times S$ with the line element $-dt^2 + f^2(t)d\sigma^2$ where $d\sigma^2$ is the line element of S lifted to $I \times S$.

The standard choices for S are the simply connected ones: $\mathbb{H}^3, \mathbb{R}^3, S^3$ with curvatures of $-1, 0, +1$ respectively.

Robertson-Walker Flow

Having introduced the definition for both perfect fluids and Robertson-Walker spacetime, we must now relate the two. Specifically, we will show that for any Robertson-Walker spacetime, the flow given by the vector field $U = \partial_t$ is that of a perfect fluid.

We already know the stress-energy tensor T_{ij} as a result of the Einstein equation, but still need the density function ρ and pressure function p such that we satisfy the definition for a perfect fluid.

Theorem 14.10 *If U is the flow vector field on a Robertson-Walker spacetime $M(k, f)$, then (U, ρ, p) is a perfect fluid with energy density ρ and pressure p given by*

$$\begin{cases} \frac{8\pi\rho}{3} & = \left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} \\ -8\pi p & = 2\left(\frac{f''}{f}\right) + \left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} \end{cases} \tag{14.150}$$

Notice if we subtract the pressure formula from the density formula we find:

$$\frac{8\pi\rho}{3} + 8\pi p = \left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} - 2\left(\frac{f''}{f}\right) - \left(\frac{f'}{f}\right)^2 - \frac{k}{f^2} \tag{14.151}$$

$$\Rightarrow 2\left(\frac{f''}{f}\right) = -\frac{8\pi}{3}(\rho + 3p) \tag{14.152}$$

$$\Rightarrow 3\left(\frac{f''}{f}\right) = -4\pi(\rho + 3p) \tag{14.153}$$

Thus giving us a relation between the scale function f on the left with the pressure, p, and density, ρ, on the right.

Next we consider the dynamics of the perfect fluid, that is $div\,T = 0$. Since $D_U U = 0$ and $grad\,p = 0$ the force equation is trivial, and we end up with the energy equation in the form below.

Corollary 14.11 *For a Roberson-Walker perfect fluid,*

$$\rho' = -3(\rho + p)\frac{f'}{f}$$

This gives us the time rate of change for energy density.

Robertson-Walker Cosmology

As mentioned above, the math of the Einstein equation is irrelevant if the stress-energy tensor does not have any specific interpretation of physical phenomena. So, let us now consider what physical significance can be tied to the Robertson-Walker model by beginning with two cosmological observations.

a) Hubble found that distant galaxies are moving away from us at a rate proportional to their distance. For two galaxies γ_p and γ_q, the distance between γ_p and γ_q in $S(t)$ is $f(t)d(p,q)$, where d is the Riemannian distance in the space S. By current estimates we have

$$H_0 = \frac{f'(t_0)}{f(t_0)} = \frac{1}{18 \pm 2 \times 10^9 yr}.$$

b) For all known forms of matter, energy density dominates pressure. That is $\rho > p$. At the present time t_0, the energy density ρ_0 of our universe is estimated to be between 10^{-31} and 5×10^{-29}. The pressure p_0 is positive, but very much smaller: $\rho_0 \gg p_0 > 0$.

Since, f is greater than zero by definition and our observations have $H_0 > 0$, then this first observation tells us that $f'(t_0) > 0$. Or in other words, our universe, S, is currently expanding.

Proposition 14.12 *Let $M(k, f) = I \times_f S$. If $H_0 > 0$ for some t_0, and $\rho + 3p > 0$, then I has an initial endpoint t_* with $t_0 - H_0^{-1} < t_* < t_0$, and either*

(i) $f' > 0$. Or,

(ii) After t_0 the scale factor f has a maximum point, and I is a finite interval (t_*, t^*).

The estimate of Hubble time H_0^{-1} coupled with this proposition tells us that the universe had a definite beginning estimated to be between 10 and 20 million years ago. Further it implies that exactly one of the following will hold

a) The universe will continue to expand. Or,

b) After contraction for some time, the universe will collapse upon itself.

If the energy density ρ approaches infinity as $t \to t_*$ or $t \to t^*$, then we say that the space-time $M(k, f)$ has a *physical singularity* there. The names for these possible types of singularity are well known.

Definition 14.13 (Big Bang) *An initial singularity of $M(k, f)$ at t_* is called a big bang provided $f \to 0$ and $f' \to \infty$ as $t \to t_*$.*

Definition 14.14 (Big Bang and Big Crunch) *A final singularity is called a big crunch if $f \to 0$ and $f' \to -\infty$ as $t \to t^*$.*

The theory provides us with some information that allows us to form some conclusions given the current measurements.

Theorem 14.15 *Assume that $M(k,f) = I \times_f S$ has only physical singularities and that I is maximal. If $H_0 > 0$ for some t_0, if $\rho > 0$, and for constants a and A we have $-\frac{1}{3} < a \leq \frac{p}{\rho} \leq A$, then:*

(i) The initial singularity is a big bang.

(ii) If $k = 0$ or $k = -1$, then $I = (t_*, \infty)$ and both $f \to \infty$ and $\rho \to 0$ as $t \to \infty$.

(iii) If $k = 1$ then f reaches a maximum followed by a big crunch. Hence, I is a finite interval (t_*, t^*).

Corollary 14.16 *Let $\rho_c = \frac{3(H_0)^2}{8\pi}$, called the critical energy density. If $\rho_0 \leq \rho_c$, then $k = 0, -1$. If $\rho_0 > \rho_c$ then $k = +1$.*

So, given ρ_0 we would be able to determine the long term behavior of the universe. However, measurements are ρ_0 are difficult because we can only measure a very small portion of universe within our timecone.

Chapter Summary

Here we have introduced a number of different spacetime models that will continue to have relevance as we proceed through our remaining chapters. We saw the de Sitter universe representing homogeneous universe bounded by a cosmological horizon. We saw the Schwarzschild universe that represents the empty space surrounding a black hole. We then derived the Schwarzschild-de Sitter universe that contained both our event horizon and cosmological horizon. And finally we took a look at the Friedmann-Lemaître-Robertson-Walker universe and saw how this model aligns with our current observations for an expanding universe. We also saw how these first three models were representative of a static universe, but the $FLRW$ universe was time evolving. In our next few chapters we will see a few methods to model a black hole withing an expanding universe.

CHAPTER XV

SCHWARZSCHILD-DE SITTER METRIC IN TIME EVOLVING COORDINATES

Conceptual Introduction

The objective of this chapter is to represent the Schwarzschild-de Sitter spacetime in time evolving coordinates, and discuss the physical implications of this representation. We begin this chapter, as always, with some conceptual introductions before proceeding to the main discussion. For this chapter the preliminary concepts include:

a) Static spacetimes

b) Observer fields

c) Lemaître-Robertson transformation

d) Dimenssionless Units

e) The d'Alembert covariant wave operator

f) Stationary black hole toy model

After this introduction we will show how the Lemaître-Robertson transformation applied to the de-Sitter spacetime metric results in a Friedmann-Lemaître-Robertson-Walker spacetime with scale factor of $e^{\frac{ct}{R}}$, and is thus an evolving spacetime.

In the next section we explore transformations to the Schwarzschild-de Sitter spacetime. We will see that since the transformed Schwarzschild-de Sitter spacetime approaches the transformed de Sitter spacetime, the transformed Schwarzschild-de Sitter spacetime too is in fact evolving although not represented by a warped product.

We finish off this chapter by applying the d'Alembert covariant wave operator to the Schwarzschild-de Sitter spacetime in the Lemaître-Robertson coordinates in order to understand the propagation of waves within this background space. We compare our results from this section against the results of a known stationary black hole toy model.

Prerequisites

Observer Fields

Observer fields are also known as reference frames, and they refer to local "observations".

Definition 15.1 (Observer Field) *An observer field on an arbitrary space time M is a timelike, future-pointing, unit vector field U, and each integral curve of U is called an observer.*

So, from our definition we can think of the Observer field U as a family of U-observers filling the spactime M. The task becomes finding if and how each observer can agree on common notions of space and time.

Proposition 15.2 *For an observer field U the following statements are equivalent:*

(i) There is a restspace of U through each point $p \in M$.

(ii) If vector fields X and Y are orthogonal to U, then so is $[X,Y]$.

(iii) U is irrotational; that is $curl\,U$ is zero on vector fields $X, Y \perp U$.

We find that an observer field is geodesic if each of its observers is geodesic, that is if $D_U U = 0$.

Corollary 15.3 *An observer field U is geodesic and irrotational if and only if $curl\,U = 0$.*

The above gives us information on when the observers in U can agree on space. We now turn our attention to when they can agree on time.

Definition 15.4 (Proper Time Synchronizablen Observer Field) *An observer field U on M is proper time synchronizable provided there exists a function $t \in C^\infty(M)$ such that $U = -grad\,t$. Then t is called a proper time function on M.*

Corollary 15.5 *If an observer field U on M is proper time synchronizable, then U is geodesic and irrotational. The converse is true if M is simply connected, and thus always true locally.*

We can also weaken the proper time synchronizability condition.

Definition 15.6 (Synchronizable Observer Field) *An observer field U on M is synchronizable provided there are smooth functions $h > 0$ and t on M such that $U = -h\,grad\,t$.*

A synchronizable observer field is also irroational. But the time recognized by all-observers is more like an average time so the elapsed time between two restspaces will likely be different for each observer.

The Lemaître-Robertson Transformation

The Lamaître-Robertson transformation is given by

$$r' = \frac{r}{\sqrt{1-\frac{r^2}{R^2}}}e^{-\frac{ct}{R}}, \qquad t' = t + \frac{R}{2c}ln\left(1-\frac{r^2}{R^2}\right), \qquad \theta' = \theta, \qquad \phi' = \phi. \tag{15.1}$$

We will also want the inverse transformation.

The Inverse Lemaître-Robertson Transformation

We start by finding $r(t,r')$ with

$$\begin{aligned}
r' &= \frac{r}{\sqrt{1-\frac{r^2}{R^2}}}e^{-\frac{ct}{R}} \\
\Rightarrow (r')^2 &= \frac{r^2}{1-\frac{r^2}{R^2}}e^{-\frac{2ct}{R}} \\
&= \frac{1}{\frac{1}{r^2}-\frac{1}{R^2}}e^{-\frac{2ct}{R}} \\
\Rightarrow \frac{1}{r^2}-\frac{1}{R^2} &= \frac{1}{(r')^2}e^{-\frac{2ct}{R}} \\
\Rightarrow \frac{1}{r^2} &= \frac{1}{(r')^2 e^{\frac{2ct}{R}}}+\frac{1}{R^2} \\
&= \frac{R^2+(r')^2 e^{\frac{2ct}{R}}}{R^2(r')^2 e^{\frac{2ct}{R}}} \\
\Rightarrow r^2 &= \frac{R^2(r')^2 e^{\frac{2ct}{R}}}{R^2+(r')^2 e^{\frac{2ct}{R}}} \\
\Rightarrow r &= \frac{Rr' e^{\frac{ct}{R}}}{\sqrt{R^2+(r')^2 e^{\frac{2ct}{R}}}}. \qquad (15.2)
\end{aligned}$$

Then for $t(t',r)$ we have

$$t(t',r) = t' - \frac{R}{2c}\log\left(1-\frac{r^2}{R^2}\right). \qquad (15.3)$$

So if we take $r(t,r')$ from above, and manipulate it into the form $(1-\frac{r^2}{R^2})$ it will be easy to plug into $t(t',r)$ to get $t(t',r')$. So first we find

$$
\begin{aligned}
r^2 &= \frac{R^2(r')^2 e^{\frac{2ct}{R}}}{R^2+(r')^2 e^{\frac{2ct}{R}}} \\
\Rightarrow \frac{r^2}{R^2} &= \frac{(r')^2 e^{\frac{2ct}{R}}}{R^2+(r')^2 e^{\frac{2ct}{R}}} \\
\Rightarrow 1-\frac{r^2}{R^2} &= \frac{R^2+(r')^2 e^{\frac{2ct}{R}}-(r')^2 \exp^{\frac{2ct}{R}}}{R^2+(r')^2 e^{\frac{2ct}{R}}} \\
\Rightarrow (1-\frac{r^2}{R^2}) &= \frac{R^2}{R^2+(r')^2 e^{\frac{2ct}{R}}}.
\end{aligned}
\tag{15.4}
$$

Then we plug it in to find $t(t',r')$ as:

$$
\begin{aligned}
t &= t' - \frac{R}{2c}\log\left(\frac{R^2}{R^2+(r')^2 e^{\frac{2ct}{R}}}\right) \\
&= t' - \frac{R}{2c}\log\left(\frac{1}{1+\frac{(r')^2 e^{\frac{2ct}{R}}}{R^2}}\right) \\
&= t' - \frac{R}{2c}\left[\log(1) - \log\left(1+\frac{(r')^2 e^{\frac{2ct}{R}}}{R^2}\right)\right] \\
&= t' + \frac{R}{2c}\log\left(1+\frac{(r')^2 e^{\frac{2ct}{R}}}{R^2}\right).
\end{aligned}
\tag{15.5}
$$

Finally, bringing t together on the same side gives us:

$$
\begin{aligned}
t &= t' + \frac{R}{2c}\log\left(1 + \frac{(r')^2 e^{\frac{2ct}{R}}}{R^2}\right) \\
\Rightarrow e^t &= e^{t'}\left(1 + \frac{(r')^2 e^{\frac{2ct}{R}}}{R^2}\right)^{\frac{R}{2c}} \\
\Rightarrow e^{\frac{2ct}{R}} &= e^{\frac{2ct'}{R}}\left(1 + \frac{(r')^2 e^{\frac{2ct}{R}}}{R^2}\right) \\
\Rightarrow e^{\frac{2ct}{R}} - \frac{(r')^2 e^{\frac{2ct}{R}}}{R^2} e^{\frac{2ct'}{R}} &= e^{\frac{2ct'}{R}} \\
\Rightarrow e^{\frac{2ct}{R}}\left(1 - \frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2}\right) &= e^{\frac{2ct'}{R}} \\
\Rightarrow e^{\frac{2ct}{R}} &= e^{\frac{2ct'}{R}}\left(1 - \frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2}\right)^{-1} \\
\Rightarrow \frac{2ct}{R} &= \frac{2ct'}{R} + \log\left[\left(1 - \frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2}\right)^{-1}\right] \\
\Rightarrow t &= t' + \frac{R}{2c}\log\left[\left(1 - \frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2}\right)^{-1}\right] \\
\Rightarrow t &= t' - \frac{R}{2c}\log\left(1 - \frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2}\right). \qquad (15.6)
\end{aligned}
$$

Now we can take this and simply use it in $r(t,r')$ to get $r(t',r')$. We begin with

$$
r(t,r') = \frac{Rr' e^{\frac{ct}{R}}}{\sqrt{R^2 + (r')^2 e^{\frac{2ct}{R}}}}. \qquad (15.7)
$$

Starting with the denominator we get

$$
\begin{aligned}
\sqrt{R^2+(r')^2 e^{\frac{2ct}{R}}} &= \sqrt{R^2+(r')^2 e^{\frac{2c}{R}\left[t'-\frac{R}{2c}\log\left(1-\frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2}\right)\right]}} \\
&= \sqrt{R^2+(r')^2 e^{\frac{2ct'}{R}}\left(1-\frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2}\right)^{-1}} \\
&= R\sqrt{1+\frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2-(r')^2 e^{\frac{2ct'}{R}}}} \\
&= R\sqrt{\frac{R^2-(r')^2 e^{\frac{2ct'}{R}}+(r')^2 e^{\frac{2ct'}{R}}}{R^2-(r')^2 e^{\frac{2ct'}{R}}}} \\
&= \frac{R^2}{\sqrt{R^2-(r')^2 e^{\frac{2ct'}{R}}}}. \qquad (15.8)
\end{aligned}
$$

And the numerator yields

$$
\begin{aligned}
Rr'e^{\frac{ct}{R}} &= Rr'e^{\frac{c}{R}\left[t'-\frac{R}{2c}\log\left(1-\frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2}\right)\right]} \\
&= \frac{Rr'e^{\frac{ct'}{R}}}{\sqrt{1-\frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2}}} \\
&= \frac{R^2 r' e^{\frac{ct'}{R}}}{\sqrt{R^2-(r')^2 e^{\frac{2ct'}{R}}}}. \qquad (15.9)
\end{aligned}
$$

When we bring these together it simplifies to

$$
\begin{aligned}
r &= \frac{\frac{R^2 r' e^{\frac{ct'}{R}}}{\sqrt{R^2-(r')^2 e^{\frac{2ct'}{R}}}}}{\frac{R^2}{\sqrt{R^2-(r')^2 e^{\frac{2ct'}{R}}}}} \\
&= r' e^{\frac{ct'}{R}}. \qquad (15.10)
\end{aligned}
$$

We next take a look at the differentials. Beginning with dr we find

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}r'} &= e^{\frac{ct'}{R}} + \frac{cr'e^{\frac{ct'}{R}}}{R}\frac{\mathrm{d}t'}{\mathrm{d}r'} \\ \Rightarrow dr &= e^{\frac{ct'}{R}}dr' + \frac{cr'e^{\frac{ct'}{R}}}{R}dt'. \end{aligned} \tag{15.11}$$

And from this we get dr^2 of

$$dr^2 = e^{\frac{2ct'}{R}}(dr')^2 + \frac{2cr'e^{\frac{2ct'}{R}}}{R}dr'dt' + \frac{c^2(r')^2e^{\frac{2ct'}{R}}}{R^2}(dt')^2. \tag{15.12}$$

And for dt we find

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}t'} &= 1 - \frac{R}{2c}\left(1 - \frac{(r')^2e^{\frac{2ct'}{R}}}{R^2}\right)^{-1}\left(-\frac{2r'e^{\frac{2ct'}{R}}}{R^2}\frac{\mathrm{d}r'}{\mathrm{d}t'} - \frac{(r')^2e^{\frac{2ct'}{R}}}{R^2}\left(\frac{2c}{R}\right)\right) \\ \Rightarrow dt &= dt' + \left(\frac{R^2-(r')^2e^{\frac{2ct'}{R}}}{R^2}\right)^{-1}\left(\frac{r'e^{\frac{2ct'}{R}}}{cR}dr' + \frac{(r')^2e^{\frac{2ct'}{R}}}{R^2}dt'\right) \\ \Rightarrow &= dt' + \frac{\frac{r'e^{\frac{2ct'}{R}}}{cR}dr' + \frac{(r')^2e^{\frac{2ct'}{R}}}{R^2}dt'}{\frac{R^2-(r')^2e^{\frac{2ct'}{R}}}{R^2}} \\ &= dt' + \frac{Rr'e^{\frac{2ct'}{R}}}{c\left(R^2-(r')^2e^{\frac{2ct'}{R}}\right)}dr' + \frac{(r')^2e^{\frac{2ct'}{R}}}{R^2-(r')^2e^{\frac{2ct'}{R}}}dt' \\ &= \frac{Rr'e^{\frac{2ct'}{R}}}{c\left(R^2-(r')^2e^{\frac{2ct'}{R}}\right)}dr' + \left(1 + \frac{(r')^2e^{\frac{2ct'}{R}}}{R^2-(r')^2e^{\frac{2ct'}{R}}}\right)dt'. \end{aligned} \tag{15.13}$$

This gives the slightly more complex form for dt^2 as

$$
\begin{aligned}
dt^2 &= \left(\frac{Rr' e^{\frac{2ct'}{R}}}{c\left(R^2 - (r')^2 e^{\frac{2ct'}{R}}\right)} dr' + \left(1 + \frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2 - (r')^2 e^{\frac{2ct'}{R}}}\right) dt' \right)^2 \\
&= \frac{R^2 (r')^2 e^{\frac{4ct'}{R}}}{\left(cR^2 - c(r')^2 e^{\frac{2ct'}{R}}\right)^2} (dr')^2 \\
&\quad + 2\left[\left(\frac{Rr' e^{\frac{2ct'}{R}}}{c\left(R^2 - (r')^2 e^{\frac{2ct'}{R}}\right)} \right) \left(1 + \frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2 - (r')^2 e^{\frac{2ct'}{R}}}\right) \right] dr' dt' \\
&\quad + \left(1 + \frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2 - (r')^2 e^{\frac{2ct'}{R}}}\right)^2 (dt')^2 \\
&= \frac{R^2 (r')^2 e^{\frac{4ct'}{R}}}{\left(cR^2 - c(r')^2 e^{\frac{2ct'}{R}}\right)^2} (dr')^2 + \frac{2R^3 r' e^{\frac{2ct'}{R}}}{c\left(R^2 - (r')^2 e^{\frac{2ct'}{R}}\right)^2} dr' dt' + \frac{R^4}{\left(R^2 - (r')^2 e^{\frac{2ct'}{R}}\right)^2} (dt')^2
\end{aligned} \tag{15.14}
$$

So we have

$$
\begin{aligned}
dt^2 &= \frac{R^2 (r')^2 e^{\frac{4ct'}{R}}}{\left(cR^2 - c(r')^2 e^{\frac{2ct'}{R}}\right)^2} (dr')^2 + \frac{2R^3 r' e^{\frac{2ct'}{R}}}{c\left(R^2 - (r')^2 e^{\frac{2ct'}{R}}\right)^2} dr' dt' + \frac{R^4}{\left(R^2 - (r')^2 e^{\frac{2ct'}{R}}\right)^2} (dt')^2 \\
dr^2 &= e^{\frac{2ct'}{R}} (dr')^2 + \frac{2cr' e^{\frac{2ct'}{R}}}{R} dr' dt' + \frac{c^2 (r')^2 e^{\frac{2ct'}{R}}}{R^2} (dt')^2 \\
d\theta^2 &= d\theta'^2 \\
d\phi^2 &= d\phi'^2
\end{aligned} \tag{15.15}
$$

To summarize, we starting with the Lamaître-Robertson transformation of

$$r' = \frac{r}{\sqrt{1-\frac{r^2}{R^2}}} e^{-\frac{ct}{R}}, \qquad t' = t + \frac{R}{2c} ln \left(1 - \frac{r^2}{R^2}\right), \qquad \theta' = \theta, \qquad \phi' = \phi. \tag{15.16}$$

We then found the inverse relationship is given by

$$t = t' - \frac{R}{2c} \log\left(1 - \frac{(r')^2 e^{\frac{2ct'}{R}}}{R^2}\right), \qquad r = r' e^{\frac{ct'}{R}}, \qquad \theta = \theta', \qquad \phi = \phi' = \phi; \tag{15.17}$$

with the differentials of

$$\begin{aligned}
dt^2 &= \frac{R^2 (r')^2 e^{\frac{4ct'}{R}}}{\left(cR^2 - c(r')^2 e^{\frac{2ct'}{R}}\right)^2} (dr')^2 + \frac{2R^3 r' e^{\frac{2ct'}{R}}}{c\left(R^2 - (r')^2 e^{\frac{2ct'}{R}}\right)^2} dr' dt' + \frac{R^4}{\left(R^2 - (r')^2 e^{\frac{2ct'}{R}}\right)^2} (dt')^2 \\
dr^2 &= e^{\frac{2ct'}{R}} (dr')^2 + \frac{2cr' e^{\frac{2ct'}{R}}}{R} dr' dt' + \frac{c^2 (r')^2 e^{\frac{2ct'}{R}}}{R^2} (dt')^2 \\
d\theta^2 &= d\theta'^2 \\
d\phi^2 &= d\phi'^2
\end{aligned} \tag{15.18}$$

So, all that is needed to transform a given metric using the Lamaître-Robertson transformation, or its inverse, is to plug in these quantities and reduce.

Dimensionless Units

We have been considering M_{bh} as the mass of the black hole. However, we very often see terms like $\frac{M_{bh}}{re^{\frac{ct}{R}}}$. It is clear that the units of measure for distance and time cancel out in the $e^{\frac{ct}{R}}$ term, but the units for $\frac{M_{bh}}{r}$ are mass over distance (for example kg/m). In order for us to compare the size of this quantity against others it will be convenient to work with dimensionless units. So, let us consider the transformation $M_{bh} \to \frac{GM_{bh}}{c^2}$, where as usual, c represents the speed of light and G is the gravitational constant known to be approximately $6.674 \times 10^{-11} m^3 \cdot kg^{-1} \cdot s^{-1}$. Now the term $\frac{GM_{bh}}{c^2 r}$. We will often denote this term using the symbol λ.

d'Alembert's Covariant Wave Operator

The wave equation

$$\frac{\partial^2}{\partial t^2} u = c^2 \triangle u, \tag{15.19}$$

describes the propagation of waves in space where $u = u(t,x,y,z)$ and x,y,z refer to the typical spacial coordinates. This equation can be rewritten as

$$\triangle u - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u = 0 \tag{15.20}$$

representing a homogeneous partial differential wave equation. However, it is also possible to consider the wave equation in the presence of a force or source function f by

$$\triangle u - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u = f. \tag{15.21}$$

We can then think of the left hand side of these equations as some operator acting on the function u that provides us with the equation for wave propagation. Such an operator is the covariant d'Alembert's box operator.

Definition 15.7 (Covariant D'Alembert's Box Operator) *The covariant D'Alembert's box operator (or wave operator) is a second-order differential operator that is represented in Cartesian form as*

$$\Box u = \triangle u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.$$

In its general curvilinear form, for a function ψ, the D'Alembert operator can determined by the given metric g_{ij} by

$$\Box_g \psi = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^j} \left(\sqrt{|g|} g^{ik} \frac{\partial \psi}{\partial x^k} \right), \tag{15.22}$$

where g is a $(0,2)$-tensor, and $|g|$ represents the determinant of the matrix $[g^{ik}]$.

So for the indicies i and k ranging through our symmetric metric tensor, g_{ik}, we can find the solution in symbolic form as

$$\begin{aligned}
\Box_g \psi = & \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial t} \left(\sqrt{|g|} g^{tt} \frac{\partial \psi}{\partial t} \right) + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial t} \left(\sqrt{|g|} g^{tr} \frac{\partial \psi}{\partial r} \right) \\
& + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial r} \left(\sqrt{|g|} g^{rt} \frac{\partial \psi}{\partial t} \right) + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial r} \left(\sqrt{|g|} g^{rr} \frac{\partial \psi}{\partial r} \right) \\
& + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \theta} \left(\sqrt{|g|} g^{\theta\theta} \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \phi} \left(\sqrt{|g|}\, g^{\phi\phi} \frac{\partial \psi}{\partial \phi} \right) \\
= & \frac{1}{\sqrt{|g|}} \left[\frac{\partial}{\partial t} \left(\sqrt{|g|} g^{tt} \frac{\partial \psi}{\partial t} \right) + \frac{\partial}{\partial t} \left(\sqrt{|g|} g^{tr} \frac{\partial \psi}{\partial r} \right) \right] && (15.23a) \\
& + \frac{1}{\sqrt{|g|}} \left[\frac{\partial}{\partial r} \left(\sqrt{|g|} g^{rt} \frac{\partial \psi}{\partial t} \right) + \frac{\partial}{\partial r} \left(\sqrt{|g|} g^{rr} \frac{\partial \psi}{\partial r} \right) \right] && (15.23b) \\
& + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \theta} \left(\sqrt{|g|} g^{\theta\theta} \frac{\partial \psi}{\partial \theta} \right) && (15.23c) \\
& + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \phi} \left(\sqrt{|g|} g^{\phi\phi} \frac{\partial \psi}{\partial \phi} \right). && (15.23d)
\end{aligned}$$

Taking this equation line-by-line, we can expand it out using the product rule beginning with part (a) of our equation.

$$\begin{aligned}
\frac{1}{\sqrt{|g|}}\left[\frac{\partial}{\partial t}\left(\sqrt{|g|}g^{tt}\frac{\partial\psi}{\partial t}\right)+\frac{\partial}{\partial t}\left(\sqrt{|g|}g^{tr}\frac{\partial\psi}{\partial r}\right)\right] &= \frac{1}{\sqrt{|g|}}\left[\frac{\partial}{\partial t}\left(\sqrt{|g|}\left(g^{tt}\frac{\partial\psi}{\partial t}+g^{rt}\frac{\partial\psi}{\partial r}\right)\right)\right] \\
&= \frac{1}{\sqrt{|g|}}\left[\left(\frac{\partial}{\partial t}\sqrt{|g|}\right)\left(g^{tt}\frac{\partial\psi}{\partial t}+g^{rt}\frac{\partial\psi}{\partial r}\right)\right. \\
&\quad +\sqrt{|g|}\left(\frac{\partial g^{tt}}{\partial t}\frac{\partial\psi}{\partial t}+g^{tt}\frac{\partial}{\partial t}\frac{\partial\psi}{\partial t}\right. \\
&\quad \left.\left.+\frac{\partial g^{rt}}{\partial t}\frac{\partial\psi}{\partial r}+g^{rt}\frac{\partial}{\partial t}\frac{\partial\psi}{\partial r}\right)\right] \\
&= \frac{\partial\sqrt{|g|}}{\partial t}\frac{g^{tt}}{\sqrt{|g|}}\frac{\partial\psi}{\partial t}+\frac{\partial\sqrt{|g|}}{\partial t}\frac{g^{rt}}{\sqrt{|g|}}\frac{\partial\psi}{\partial r} \\
&\quad +\frac{\partial g^{tt}}{\partial t}\frac{\partial\psi}{\partial t}+g^{tt}\frac{\partial^2\psi}{\partial t^2} \\
&\quad +\frac{\partial g^{rt}}{\partial t}\frac{\partial\psi}{\partial r}+g^{rt}\frac{\partial^2\psi}{\partial r\partial t} \\
&= g^{tt}\frac{\partial^2\psi}{\partial t^2}+\left(\frac{\partial\sqrt{|g|}}{\partial t}\frac{g^{tt}}{\sqrt{|g|}}+\frac{\partial g^{tt}}{\partial t}\right)\frac{\partial\psi}{\partial t} \\
&\quad +\left(\frac{\partial\sqrt{|g|}}{\partial t}\frac{g^{rt}}{\sqrt{|g|}}+\frac{\partial g^{rt}}{\partial t}\right)\frac{\partial\psi}{\partial r}+g^{rt}\frac{\partial^2\psi}{\partial r\partial t}.
\end{aligned} \tag{15.24}$$

Then for (b) we get

$$\begin{aligned}
\frac{1}{\sqrt{|g|}}\left[\frac{\partial}{\partial r}\left(\sqrt{|g|}g^{rt}\frac{\partial\psi}{\partial t}\right)+\frac{\partial}{\partial r}\left(\sqrt{|g|}g^{rr}\frac{\partial\psi}{\partial r}\right)\right] &= \frac{1}{\sqrt{|g|}}\left[\frac{\partial}{\partial r}\left(\sqrt{|g|}\left(g^{rt}\frac{\partial\psi}{\partial t}+g^{rr}\frac{\partial\psi}{\partial r}\right)\right)\right] \\
&= \frac{1}{\sqrt{|g|}}\left[\left(\frac{\partial}{\partial r}\sqrt{|g|}\right)\left(g^{rt}\frac{\partial\psi}{\partial t}+g^{rr}\frac{\partial\psi}{\partial r}\right)\right. \\
&\quad+\sqrt{|g|}\left(\frac{\partial g^{rt}}{\partial r}\frac{\partial\psi}{\partial t}+g^{rt}\frac{\partial}{\partial r}\frac{\partial\psi}{\partial t}\right. \\
&\quad\left.\left.+\frac{\partial g^{rr}}{\partial r}\frac{\partial\psi}{\partial r}+g^{rr}\frac{\partial}{\partial r}\frac{\partial\psi}{\partial r}\right)\right] \\
&= \frac{\partial\sqrt{|g|}}{\partial r}\frac{g^{rt}}{\sqrt{|g|}}\frac{\partial\psi}{\partial t}+\frac{\partial\sqrt{|g|}}{\partial r}\frac{g^{rr}}{\sqrt{|g|}}\frac{\partial\psi}{\partial r} \\
&\quad+\frac{\partial g^{rt}}{\partial r}\frac{\partial\psi}{\partial t}+g^{rt}\frac{\partial^2\psi}{\partial r\partial t} \\
&\quad+\frac{\partial g^{rr}}{\partial r}\frac{\partial\psi}{\partial r}+g^{rr}\frac{\partial^2\psi}{\partial r^2} \\
&= g^{rr}\frac{\partial^2\psi}{\partial r^2}+\left(\frac{\partial\sqrt{|g|}}{\partial r}\frac{g^{rr}}{\sqrt{|g|}}+\frac{\partial g^{rr}}{\partial r}\right)\frac{\partial\psi}{\partial r} \\
&\quad+\left(\frac{\partial\sqrt{|g|}}{\partial r}\frac{g^{rt}}{\sqrt{|g|}}+\frac{\partial g^{rt}}{\partial r}\right)\frac{\partial\psi}{\partial t}+g^{rt}\frac{\partial^2\psi}{\partial r\partial t}.
\end{aligned}\tag{15.25}$$

Similarly for (c) we get

$$\begin{aligned}
\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial\theta}\left(\sqrt{|g|}g^{\theta\theta}\frac{\partial\psi}{\partial\theta}\right) &= \frac{g^{\theta\theta}}{\sqrt{|g|}}\frac{\partial\sqrt{|g|}}{\partial\theta}\frac{\partial\psi}{\partial\theta}+\frac{\partial g^{\theta\theta}}{\partial\theta}\frac{\partial\psi}{\partial\theta}+g^{\theta\theta}\frac{\partial^2\psi}{\partial\theta^2} \\
&= \left(\frac{g^{\theta\theta}}{\sqrt{|g|}}\frac{\partial\sqrt{|g|}}{\partial\theta}+\frac{\partial g^{\theta\theta}}{\partial\theta}\right)\frac{\partial\psi}{\partial\theta}+g^{\theta\theta}\frac{\partial^2\psi}{\partial\theta^2}.
\end{aligned}\tag{15.26}$$

And finally for (d) we have

$$\begin{aligned}
\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial\phi}\left(\sqrt{|g|}g^{\phi\phi}\frac{\partial\psi}{\partial\phi}\right) &= \frac{g^{\phi\phi}}{\sqrt{|g|}}\frac{\partial\sqrt{|g|}}{\partial\phi}\frac{\partial\psi}{\partial\phi}+\frac{\partial g^{\phi\phi}}{\partial\phi}\frac{\partial\psi}{\partial\phi}+g^{\phi\phi}\frac{\partial^2\psi}{\partial\phi^2} \\
&= \left(\frac{g^{\phi\phi}}{\sqrt{|g|}}\frac{\partial\sqrt{|g|}}{\partial\phi}+\frac{\partial g^{\phi\phi}}{\partial\phi}\right)\frac{\partial\psi}{\partial\phi}+g^{\phi\phi}\frac{\partial^2\psi}{\partial\phi^2}.
\end{aligned}\tag{15.27}$$

Next, we collect the common terms to write $\Box_g \psi$ in the form

$$\Box_{dS}\psi = A_1\frac{\partial^2\psi}{\partial t^2}+A_2\frac{\partial\psi}{\partial t}+A_3\frac{\partial^2\psi}{\partial t\partial r}+A_4\frac{\partial\psi}{\partial r}+A_5\frac{\partial^2\psi}{\partial r^2}+A_6\frac{\partial\psi}{\partial\theta}+A_7\frac{\partial^2\psi}{\partial\theta^2}+A_8\frac{\partial\psi}{\partial\phi}+A_9\frac{\partial^2\psi}{\partial\phi^2}. \tag{15.28}$$

Wave Propagation in a Black Hole Toy model

Yagdjian [10] in evaluating toy model for a spacetime with a stationary black hole represented by metric

$$g_{Toy} = \begin{pmatrix} -c^2\left(1-\frac{2Gm}{c^2r}\right) & 0 & 0 & 0 \\ 0 & e^{\frac{2ct}{R}}\left(1-\frac{2Gm}{c^2r}\right)^{-1} & 0 & 0 \\ 0 & 0 & e^{\frac{2ct}{R}}r^2 & 0 \\ 0 & 0 & 0 & e^{\frac{2ct}{R}}r^2\sin^2\theta \end{pmatrix}, \tag{15.29}$$

has evaluated the covariant wave equation to be

$$\begin{gathered}\Box_{gToy}\psi = \frac{\partial^2\psi}{\partial t^2}+\frac{3c}{R}\frac{\partial\psi}{\partial t}-\frac{c^2}{e^{\frac{2ct}{R}}}\left((1-2\lambda)^2\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}(1-\lambda)(1-2\lambda)\frac{\partial\psi}{\partial r}+\right. \\ \left.(1-2\lambda)\frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial\theta^2}+\cot\theta\frac{\partial\psi}{\partial\theta}+\csc^2\theta\frac{\partial^2\psi}{\partial\phi^2}\right)\right),\end{gathered} \tag{15.30}$$

where $\lambda = \frac{GM_{bh}}{c^2r}$ represents the dimensionless mass term.

This wave equation approaches that of the wave equation in expanding $FLWR$ spacetime as the mass term vanishes (i.e., $\lambda\to 0$) represented by

$$\begin{gathered}\Box_{FLWR}\psi = \frac{\partial^2\psi}{\partial t^2}+\frac{3c}{R}\frac{\partial\psi}{\partial t}-\frac{c^2}{e^{\frac{2ct}{R}}}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}+\right. \\ \left.\frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial\theta^2}+\cot\theta\frac{\partial\psi}{\partial\theta}+\csc^2\theta\frac{\partial^2\psi}{\partial\phi^2}\right)\right),\end{gathered} \tag{15.31}$$

So this provides one way of evaluating a stationary blackhole in an expanding space.

A Time Evolving Transform for the de Sitter Metric

In the last chapter, our derivation of the de Sitter metric involved the assumption that the spacetime metric was static. However, in this representation the reference frame was non-inertial. In fact, a quick computation of the dynamic potentials, will show that a free particle is acted on by a gravitational force. [6] This can be traced to the finite allowable region for r where $r < R$. Also, this can be avoided by a change of coordinates to a non-static reference frame. The particular coordinate chart we will investigate is a result of the Lemaître-Robertson transformation introduced above.

Performing this coordinate change on the de-Sitter line-element results in

$$ds^2 = -c^2\left(1-\frac{r^2}{R^2}\right)dt^2 + \left(1-\frac{r^2}{R^2}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{15.32}$$

$$= -c^2(dt')^2 + e^{\frac{2ct'}{R}}\left[\left(dr'\right)^2 + r'\left(\left(d\theta'\right)^2 + \sin\theta'\left(d\phi'\right)^2\right)\right], \tag{15.33}$$

which we can clean this up a bit by simply dropping the primes to get

$$ds^2 = -c^2dt^2 + e^{\frac{2ct}{R}}\left[dr^2 + r\left(d\theta^2 + \sin\theta d\phi^2\right)\right]. \tag{15.34}$$

Here the quantity within the square brackets is just the euclidean three sphere, and thus we have a warped product on normal Euclidean space represented by the line element

$$ds^2 = -c^2dt^2 + e^{\frac{2ct}{R}}\left(dx^2 + dy^2 + dz^2\right), \tag{15.35}$$

with scale factor of $e^{\frac{ct}{R}}$.

Now, in our new coordinates, we have changed from the bounded $r < R$ to an unbounded value for $r' \in (-\infty, \infty)$. This has the added benefit, that the singularity occurring when $r = R$ has been removed from the finite domain; that is we have transformed away from the cosmological horizon singularity. However, although we are now working within an universe without a restricted radius, this does not mean that we can evaluate events for $r > R$. In fact, it can be shown that due to the isotropic nature of the speed of light, light will only reach an observer $r = 0$ in the case of $r < R$. [6]

To summarize, we have found a Robertson-Walker form for the de Sitter metric. That is, we have moved from a static and non-inertial coordinate reference frame in de Sitter space into a coordinate reference frame that is both evolving and inertial.

Time Evolving Transformation of the Schwarzschild-de Sitter Metric

We have already seen that the Schwarzschild metric with cosmological constant has the metric

$$ds^2 = \left(1 - \frac{2M_{bh}}{r} - \frac{r^2}{R^2}\right) dt^2 + \left(1 - \frac{2M_{bh}}{r} - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \qquad (15.36)$$

We now seek to find a time evolving coordinate reference frame in this spacetime.

LR Transform of SdS Metric

To apply the LR transformation we will first look at the dt^2 term.

$$
\begin{aligned}
&-c^2\left(1-\frac{2M_{bh}}{r}-\frac{r^2}{R^2}\right)dt^2\\
&\quad\Rightarrow -c^2\left(1-\frac{2M_{bh}}{re^{\frac{ct'}{R}}}-\frac{\left(re^{\frac{ct}{R}}\right)^2}{R^2}\right)\\
&\quad\left(\frac{R^2r^2e^{\frac{4ct}{R}}}{\left(cR^2-cr^2e^{\frac{2ct}{R}}\right)^2}dr^2+\frac{2R^3re^{\frac{2ct}{R}}}{c\left(R^2-r^2e^{\frac{2ct}{R}}\right)^2}drdt+\frac{R^4}{\left(R^2-r^2e^{\frac{2ct}{R}}\right)^2}dt^2\right)\\
&=\left(\frac{c^2e^{-\frac{ct}{R}}R^2\left(2M_{bh}R^2-e^{\frac{ct}{R}}R^2r+e^{\frac{3ct}{R}}(r)^3\right)}{r\left(R^2-e^{\frac{2ct}{R}}r^2\right)^2}\right)dt^2\\
&\quad+\left(\frac{2cRe^{\frac{ct'}{R}}\left(2R^2M_{bh}-R^2re^{\frac{ct}{R}}+r^3e^{\frac{3ct}{R}}\right)}{\left(R^2-r^2e^{\frac{2ct}{R}}\right)^2}\right)dtdr\\
&\quad+\left(\frac{re^{\frac{3ct}{R}}\left(-R^2re^{\frac{ct}{R}}+r^3e^{\frac{3ct}{R}}+2M_{bh}R^2\right)}{\left(R^2-r^2e^{\frac{2ct}{R}}\right)^2}\right)dr^2. \qquad (15.37)
\end{aligned}
$$

We turn next to the dr^2 term.

$$
\begin{aligned}
&\left(1-\frac{2M_{bh}}{r}-\frac{r^2}{R^2}\right)^{-1}dr^2\\
&\quad\Rightarrow\left(1-\frac{2e^{\frac{ct}{R}}M_{bh}}{r}-\frac{r^2e^{\frac{2ct}{R}}}{R^2}\right)^{-1}\left(e^{\frac{2ct}{R}}dr^2+\frac{2cre^{\frac{2ct}{R}}}{R}drdt+\frac{c^2r^2e^{\frac{2ct}{R}}}{R^2}dt^2\right)\\
&\quad=\frac{c^2r^3e^{\frac{3ct}{R}}}{R^2re^{\frac{ct}{R}}-r^3e^{\frac{3ct}{R}}-2M_{bh}R^2}dt^2\\
&\qquad+\frac{2cRr^2e^{\frac{3ct}{R}}}{R^2re^{\frac{ct}{R}}-r^3e^{\frac{3ct}{R}}-2M_{bh}R^2}dtdr\\
&\qquad+\frac{R^2re^{\frac{3ct}{R}}}{R^2re^{\frac{ct}{R}}-r^3e^{\frac{3ct}{R}}-2M_{bh}R^2}dr^2. \qquad (15.38)
\end{aligned}
$$

The $d\theta^2$ and $d\phi^2$ terms translate straightforwardly.

$$r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 = (r')^2 e^{\frac{2ct'}{R}} (d\theta')^2 + (r')^2 e^{\frac{2ct'}{R}} \sin^2(\theta')(d\phi')^2. \tag{15.39}$$

Next, we have only to add the terms together to obtain the following line element.

$$\begin{aligned} ds^2 = &\left(\frac{c^2 R^2 e^{\frac{-ct}{R}} \left(r^3 e^{\frac{3ct}{R}} - rR^2 e^{\frac{ct}{R}} + 2M_{bh} R^2 \right)}{r \left(R^2 - r^2 e^{\frac{2ct}{R}} \right)^2} - \frac{c^2 r^3 e^{\frac{3ct}{R}}}{r^3 e^{\frac{3ct}{R}} - rR^2 e^{\frac{ct}{R}} + 2M_{bh} R^2} \right) dt^2 \\ &+ \left(\frac{2cRe^{\frac{ct}{R}} \left(r^3 e^{\frac{3ct}{R}} - rR^2 e^{\frac{ct}{R}} + 2M_{bh} R^2 \right)}{\left(R^2 - r^2 e^{\frac{2ct}{R}} \right)^2} - \frac{2cr^2 R e^{\frac{3ct}{R}}}{r^3 e^{\frac{3ct}{R}} - rR^2 e^{\frac{ct}{R}} + 2M_{bh} R^2} \right) dtdr \\ &+ \left(\frac{re^{\frac{3ct}{R}} \left(r^3 e^{\frac{3ct}{R}} - rR^2 e^{\frac{ct}{R}} + 2M_{bh} R^2 \right)}{\left(R^2 - r^2 e^{\frac{2ct}{R}} \right)^2} - \frac{rR^2 e^{\frac{3ct}{R}}}{r^3 e^{\frac{3ct}{R}} - rR^2 e^{\frac{ct}{R}} + 2M_{bh} R^2} \right) dr^2 \\ &+ r^2 e^{\frac{2ct}{R}} d\theta^2 + r^2 e^{\frac{2ct}{R}} \sin^2\theta d\phi^2. \end{aligned} \tag{15.40}$$

Relation to de Sitter spacetime

Our original Schwarzschild-de Sitter metric approached de Sitter space as the black hole mass approached zero, and so we are interested in how the this coordinate reference frame behaves under the same conditions. It is possible, by simple algebraic manipulation to rewrite the above metric emphasizing the impacts of the black hole mass as

$$
\begin{aligned}
ds^2 = &-c^2\left(\frac{r^2e^{\frac{ct}{R}}\left(r^2e^{\frac{2ct}{R}}-R^2\right)^3-4M_{bh}r^3R^4e^{\frac{2ct}{R}}-4M_{bh}^2R^6e^{\frac{-ct}{R}}+4M_{bh}rR^6}{r^2e^{\frac{ct}{R}}\left(r^2e^{\frac{2ct}{R}}-R^2\right)^3-4M_{bh}r^3R^4e^{\frac{2ct}{R}}+2M_{bh}r^5R^2e^{\frac{4ct}{R}}+2M_{bh}rR^6}\right)dt^2\\
&+8ce^{\frac{ct}{R}}M_{bh}R^3\left(\frac{re^{\frac{ct}{R}}\left(R^2-r^2e^{\frac{2ct}{R}}\right)-M_{bh}R^2}{re^{\frac{ct}{R}}\left(R^2-r^2e^{\frac{2ct}{R}}\right)+4M_{bh}r^2R^4e^{\frac{2ct}{R}}-2M_{bh}r^4R^2e^{\frac{4ct}{R}}-2M_{bh}R^6}\right)dtdr\\
&+e^{\frac{2ct}{R}}\left(\frac{re^{\frac{ct}{R}}\left(r^2e^{\frac{2ct}{R}}-R^2\right)^3-4M_{bh}r^2R^4e^{\frac{2ct}{R}}+4M_{bh}^2rR^4e^{\frac{ct}{R}}+4M_{bh}r^4R^2e^{\frac{4ct}{R}}}{re^{\frac{ct}{R}}\left(r^2e^{\frac{2ct}{R}}-R^2\right)^3-4M_{bh}r^2R^4e^{\frac{2ct}{R}}+2M_{bh}R^6+2M_{bh}r^4R^2e^{\frac{4ct}{R}}}\right)dr^2\\
&+r^2e^{\frac{2ct}{R}}d\theta^2+r^2e^{\frac{2ct}{R}}\sin^2\theta d\phi^2.
\end{aligned}
\tag{15.41}
$$

In this form the effects of a negligible black hole mass become evident. Taking the black hole mass of zero we simply have

$$
ds^2 = -cdt^2+e^{\frac{2ct}{R}}\left(dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right). \tag{15.42}
$$

This is just as we saw for the de-Sitter metric under the LR transformation. That is, for $\frac{2m}{r} \ll 1$ where the mass of the black hole is considered negligible in relation to the 'radius' of the observer, the Schwarszchild-de Sitter spacetime approximates the isometric and time evolving $FLRW$ spacetime.

Relation to vacuum Schwarzschild spacetime

A similar type of manipulation would show that as R tends toward infinity, we obtain the Schwarzchild metric without the cosmological constant term. That is for $R \gg r$ the cosmological constant can be ignored, and we approximate the Schwarzschild vacuum solution as

$$
\lim_{R\to\infty} ds^2 = -c^2\left(1-\frac{2M_{bh}}{r}\right)dt^2+\left(1-\frac{2M_{bh}}{r}\right)^{-1}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2. \tag{15.43}
$$

So, we see that with the ‘radius’ of the observer being much closer to the point mass than the cosmological horizon, we may consider simply the Schwarzschild vacuum solution.

Domain of Existence

We have found the domain of existence for the Schwarzschild-de Sitter system, but now need to update it based upon our coordinate transformations. In the transformed coordinates, we now have the g_{00} term

$$\begin{aligned} g_{00} &= \frac{c^2R^2\left(r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2\right)}{re^{\frac{ct}{R}}\left(R^2 - r^2e^{\frac{2ct}{R}}\right)^2} - \frac{c^2r^3e^{\frac{3ct}{R}}}{r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2} && (15.44)\\ &= \frac{c^2R^2\left(r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2\right)^2 - c^2r^4e^{\frac{4ct}{R}}\left(R^2 - r^2e^{\frac{2ct}{R}}\right)^2}{re^{\frac{ct}{R}}\left(R^2 - r^2e^{\frac{2ct}{R}}\right)^2\left(r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2\right)} && (15.45)\\ &= \frac{\left[cR\left(r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2\right)\right]^2 - \left[cr^2e^{\frac{2ct}{R}}\left(R^2 - r^2e^{\frac{2ct}{R}}\right)\right]^2}{re^{\frac{ct}{R}}\left(R^2 - r^2e^{\frac{2ct}{R}}\right)^2\left(r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2\right)} && (15.46)\\ &= \frac{\left[cR\left(r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2\right) \pm cr^2e^{\frac{2ct}{R}}\left(R^2 - r^2e^{\frac{2ct}{R}}\right)\right]}{re^{\frac{ct}{R}}\left(R^2 - r^2e^{\frac{2ct}{R}}\right)^2\left(r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2\right)}. && (15.47) \end{aligned}$$

From the denominator, we get

$$\begin{cases} r & \neq 0 \\ r & \neq Re^{\frac{-ct}{R}} \\ r & \neq A, \end{cases} \tag{15.48}$$

where A represents the three roots of $\left(r^3 + 2M_{bh} - rR^2e^{\frac{2ct}{R}}\right)$ as $r \to re^{\frac{ct}{R}}$.

Physical Singularities

As a result of our physical singularities we have the following restrictions

$$\begin{cases} r \neq 0 \\ R \neq 0 \end{cases} . \tag{15.49}$$

Since R is not altered by the coordinate transform, this singularity remains in place as is. However, we transformed $r \to e^{\frac{ct'}{R}} r'$. So for $r \neq 0$ we get

$$e^{\frac{ct'}{R}} r' \neq 0. \tag{15.50}$$

Further, since the exponent is never zero, we are left with only $r' \neq 0$. So, if we drop the prime notation, we are left where we started with the physical singularities as r or R at zero.

Restrictions from the Metric

So, here we have eight roots from the numerator of g_{00}. Although, these roots involve similar patterns, they do involve nested radicands. In order to simply state the roots, we let

$$A = 512e^{\frac{12ct}{R}}m^3R^9 + 107e^{\frac{12ct}{R}}m^2R^{10} - 16e^{\frac{12ct}{R}}mR^{11} \tag{15.51}$$

$$B = \frac{11}{12}e^{\frac{-2ct}{R}}R^2 + \frac{4e^{\frac{-8ct}{R}}\left(e^{\frac{4ct}{R}}R^4 - 6e^{\frac{4ct}{R}}mR^3\right)}{3\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{A} + 8e^{\frac{-6ct}{R}}R^6 - 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}}} \tag{15.52}$$

$$+\frac{1}{3}\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{A} + 8e^{\frac{-6ct}{R}}R^6 - 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}} \tag{15.53}$$

$$AB1 = \frac{11}{6}e^{\frac{-2ct}{R}}R^2 - \frac{4e^{\frac{-8ct}{R}}\left(e^{\frac{4ct}{R}}R^4 - 6e^{\frac{4ct}{R}}mR^3\right)}{3\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{A} + 8e^{\frac{-6ct}{R}}R^6 - 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}}} \tag{15.54}$$

$$-\frac{1}{3}\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{A} + 8e^{\frac{-6ct}{R}}R^6 - 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}} \tag{15.55}$$

$$-\frac{3e^{\frac{-3ct}{R}}R^3}{4\sqrt{B}} \tag{15.56}$$

$$AB2 = \frac{11}{6}e^{\frac{-2ct}{R}}R^2 - \frac{4e^{\frac{-8ct}{R}}\left(e^{\frac{4ct}{R}}R^4 - 6e^{\frac{4ct}{R}}mR^3\right)}{3\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{A} + 8e^{\frac{-6ct}{R}}R^6 - 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}}} \tag{15.57}$$

$$-\frac{1}{3}\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{A} + 8e^{\frac{-6ct}{R}}R^6 - 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}} \tag{15.58}$$

$$+\frac{3e^{\frac{-3ct}{R}}R^3}{4\sqrt{B}}, \tag{15.59}$$

and

$$\alpha = -512e^{\frac{12ct}{R}}m^3R^9 + 107e^{\frac{12ct}{R}}m^2R^{10} + 16e^{\frac{12ct}{R}}mR^{11} \tag{15.60}$$

$$\beta = \frac{11}{12}e^{\frac{-2ct}{R}}R^2 + \frac{4e^{\frac{-8ct}{R}}\left(e^{\frac{4ct}{R}}R^4 + 6e^{\frac{4ct}{R}}mR^3\right)}{3\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{\alpha} + 8e^{\frac{-6ct}{R}}R^6 + 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}}} \tag{15.61}$$

$$+ \frac{1}{3}\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{\alpha} + 8e^{\frac{-6ct}{R}}R^6 + 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}} \tag{15.62}$$

$$\alpha\beta 1 = \frac{11}{6}e^{\frac{-2ct}{R}}R^2 - \frac{4e^{\frac{-8ct}{R}}\left(e^{\frac{4ct}{R}}R^4 + 6e^{\frac{4ct}{R}}mR^3\right)}{3\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{\alpha} + 8e^{\frac{-6ct}{R}}R^6 + 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}}} \tag{15.63}$$

$$- \frac{1}{3}\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{\alpha} + 8e^{\frac{-6ct}{R}}R^6 + 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}} \tag{15.64}$$

$$- \frac{3e^{\frac{-3ct}{R}}R^3}{4\sqrt{\beta}} \tag{15.65}$$

$$\alpha\beta 2 = \frac{11}{6}e^{\frac{-2ct}{R}}R^2 - \frac{4e^{\frac{-8ct}{R}}\left(e^{\frac{4ct}{R}}R^4 + 6e^{\frac{4ct}{R}}mR^3\right)}{3\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{\alpha} + 8e^{\frac{-6ct}{R}}R^6 + 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}}} \tag{15.66}$$

$$- \frac{1}{3}\left(3\sqrt{3}e^{\frac{-12ct}{R}}\sqrt{\alpha} + 8e^{\frac{-6ct}{R}}R^6 + 99e^{\frac{-6ct}{R}}mR^5\right)^{\frac{1}{3}} \tag{15.67}$$

$$+ \frac{3e^{\frac{-3ct}{R}}R^3}{4\sqrt{\beta}}. \tag{15.68}$$

With these expressions defined, our domain restrictions can be simply written as

$$r \neq \begin{cases} \frac{R}{4e^{\frac{ct}{R}}} \pm \frac{\sqrt{AB1}}{2} + \frac{\sqrt{B}}{2} \\ \frac{R}{4e^{\frac{ct}{R}}} \pm \frac{\sqrt{AB2}}{2} - \frac{\sqrt{B}}{2} \\ \frac{R}{4e^{\frac{ct}{R}}} \pm \frac{\sqrt{\alpha\beta 1}}{2} + \frac{\sqrt{\beta}}{2} \\ \frac{R}{4e^{\frac{ct}{R}}} \pm \frac{\sqrt{\alpha\beta 2}}{2} - \frac{\sqrt{\beta}}{2} \end{cases}. \tag{15.69}$$

Wave Propagation In the Presence of a Single Black Hole

We have already established that the Schwarzschild-de Sitter metric in Lamaître-Robertson coordinates is represented by

$$\begin{aligned} ds^2 = & \left(\frac{c^2R^2e^{\frac{-ct}{R}}\left(r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2\right)}{r\left(R^2 - r^2e^{\frac{2ct}{R}}\right)^2} - \frac{c^2r^3e^{\frac{3ct}{R}}}{r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2} \right) dt^2 \\ & + \left(\frac{2cRe^{\frac{ct}{R}}\left(r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2\right)}{\left(R^2 - r^2e^{\frac{2ct}{R}}\right)^2} - \frac{2cr^2Re^{\frac{3ct}{R}}}{r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2} \right) dtdr \\ & + \left(\frac{re^{\frac{3ct}{R}}\left(r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2\right)}{\left(R^2 - r^2e^{\frac{2ct}{R}}\right)^2} - \frac{rR^2e^{\frac{3ct}{R}}}{r^3e^{\frac{3ct}{R}} - rR^2e^{\frac{ct}{R}} + 2M_{bh}R^2} \right) dr^2 \\ & + r^2e^{\frac{2ct}{R}}d\theta^2 + r^2e^{\frac{2ct}{R}}\sin^2(\theta)d\phi^2. \end{aligned} \tag{15.70}$$

However, we are now interested in evaluating the dimensionless unit form. So we begin with the original Schwarzschild-de Sitter metric and update for the dimensionless units introduced earlier. Our starting point is then

$$g_{ij} = \begin{pmatrix} -c^2\left(1 - \frac{2Gm}{c^2r} - \frac{r^2}{R^2}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2Gm}{c^2r} - \frac{r^2}{R^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{pmatrix}. \tag{15.71}$$

Then applying our transformation we obtain the line element

$$ds^2 = \left(\frac{R^2 e^{-\frac{ct}{R}} \left(c^2 r e^{\frac{ct}{R}} \left(r^2 e^{\frac{2ct}{R}} - R^2 \right) + 2GmR^2 \right)}{r \left(R^2 - r^2 e^{\frac{2ct}{R}} \right)^2} - \frac{c^4 r^3 e^{\frac{3ct}{R}}}{c^2 r e^{\frac{ct}{R}} \left(r^2 e^{\frac{2ct}{R}} - R^2 \right) + 2GmR^2} \right) dt^2 + \tag{15.72}$$

$$\left(\frac{2Re^{\frac{ct}{R}} \left(c^2 r e^{\frac{ct}{R}} \left(r^2 e^{\frac{2ct}{R}} - R^2 \right) + 2GmR^2 \right)}{c \left(R^2 - r^2 e^{\frac{2ct}{R}} \right)^2} - \frac{2c^3 r^2 R e^{\frac{3ct}{R}}}{c^2 r e^{\frac{ct}{R}} \left(r^2 e^{\frac{2ct}{R}} - R^2 \right) + 2GmR^2} \right) dtdr + \tag{15.73}$$

$$\left(\frac{r e^{\frac{3ct}{R}} \left(c^2 r e^{\frac{ct}{R}} \left(r^2 e^{\frac{2ct}{R}} - R^2 \right) + 2GmR^2 \right)}{c^2 \left(R^2 - r^2 e^{\frac{2ct}{R}} \right)^2} - \frac{c^2 r R^2 e^{\frac{3ct}{R}}}{c^2 r e^{\frac{ct}{R}} \left(r^2 e^{\frac{2ct}{R}} - R^2 \right) + 2GmR^2} \right) dr^2 + \tag{15.74}$$

$$r^2 e^{\frac{2ct}{R}} \left(d\theta^2 + \sin^2 \theta d\phi^2 \right). \tag{15.75}$$

And because

$$ds^2 = g_{ij} dx dx, \tag{15.76}$$

we find our symmetric metric tensor g to be

$$g = \begin{pmatrix} g^{tt} & g^{tr} & 0 & 0 \\ g^{rt} & g^{rr} & 0 & 0 \\ 0 & 0 & g^{\theta\theta} & 0 \\ 0 & 0 & 0 & g^{\phi\phi} \end{pmatrix}, \tag{15.77}$$

where each entry is constructed from the coefficients of the line element with the exception of the cross term that is symmetric and half of the coefficient.

$$g^{tr} = g^{rt} = \frac{1}{2}\left(\frac{2Re^{\frac{ct}{R}}\left(c^2re^{\frac{ct}{R}}\left(r^2e^{\frac{2ct}{R}}-R^2\right)+2GmR^2\right)}{c\left(R^2-r^2e^{\frac{2ct}{R}}\right)^2}-\frac{2c^3r^2Re^{\frac{3ct}{R}}}{c^2re^{\frac{ct}{R}}\left(r^2e^{\frac{2ct}{R}}-R^2\right)+2GmR^2}\right). \tag{15.78}$$

The Wave Equation in Schwarzschild-de Sitter Space

As noted above, the covariant d'Alembert's operator is

$$\begin{aligned}\Box_g\psi &= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ik}\frac{\partial\psi}{\partial x^k}\right)\\ &= A_1\frac{\partial^2\psi}{\partial t^2}+A_2\frac{\partial\psi}{\partial t}+A_3\frac{\partial^2\psi}{\partial t\partial r}+A_4\frac{\partial\psi}{\partial r}+A_5\frac{\partial^2\psi}{\partial r^2}+A_6\frac{\partial\psi}{\partial\theta}+A_7\frac{\partial^2\psi}{\partial\theta^2}+A_8\frac{\partial\psi}{\partial\phi}+A_9\frac{\partial^2\psi}{\partial\phi^2},\end{aligned} \tag{15.79}$$

and so we just need to find the coefficients. Using the above equations we can rescale the result by dividing the entire equation by the original A_1 coefficient to obtain the below results.

$$
\begin{aligned}
A_1 &= 1 \\
A_2 &= \left(\frac{\partial \sqrt{|g|}}{\partial t} \frac{g^{tt}}{\sqrt{|g|}} + \frac{\partial g^{tt}}{\partial t} + \frac{\partial \sqrt{|g|}}{\partial r} \frac{g^{rt}}{\sqrt{|g|}} + \frac{\partial g^{rt}}{\partial r} \right) \left(g^{tt}\right)^{-1} \\
A_3 &= 2g^{rt} \left(g^{tt}\right)^{-1} \\
A_4 &= \left(\frac{\partial \sqrt{|g|}}{\partial r} \frac{g^{rr}}{\sqrt{|g|}} + \frac{\partial g^{rr}}{\partial r} + \frac{\partial \sqrt{|g|}}{\partial t} \frac{g^{rt}}{\sqrt{|g|}} + \frac{\partial g^{rt}}{\partial t} \right) \left(g^{tt}\right)^{-1} \\
A_5 &= g^{rr} \left(g^{tt}\right)^{-1} \\
A_6 &= \left(\frac{g^{\theta\theta}}{\sqrt{|g|}} \frac{\partial \sqrt{|g|}}{\partial \theta} + \frac{\partial g^{\theta\theta}}{\partial \theta} \right) \left(g^{tt}\right)^{-1} \\
A_7 &= g^{\theta\theta} \left(g^{tt}\right)^{-1} \\
A_8 &= \left(\frac{g^{\phi\phi}}{\sqrt{|g|}} \frac{\partial \sqrt{|g|}}{\partial \phi} + \frac{\partial g^{\phi\phi}}{\partial \phi} \right) \left(g^{tt}\right)^{-1} \\
A_9 &= g^{\phi\phi} \left(g^{tt}\right)^{-1}
\end{aligned}
\tag{15.80}
$$

Letting $\lambda = \frac{Gm}{c^2 r}$ represent the mass term, and $\alpha = \left(\frac{r}{R}\right)$ represent the radial ratio we can rewrite this as

$$
\begin{aligned}
A_1 &= 1\\
A_2 &= \frac{3c}{R}\left(1+\frac{\lambda\left(2\alpha^2 e^{\frac{2ct}{R}}+5\alpha e^{\frac{ct}{R}}+4\lambda-5\right)}{3\left(-\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}+\alpha(1-2\lambda)e^{\frac{ct}{R}}+1\right)}+\frac{\lambda\left(2\alpha^2 e^{\frac{2ct}{R}}-5\alpha e^{\frac{ct}{R}}+4\lambda-5\right)}{3\left(\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1\right)}\right)\\
A_3 &= 8c\alpha\lambda\frac{\alpha^2 e^{\frac{2ct}{R}}+\lambda-1}{1-\alpha^2 e^{\frac{2ct}{R}}\left(\alpha^4 e^{\frac{4ct}{R}}+4\lambda\left(\alpha^2 e^{\frac{2ct}{R}}+\lambda-1\right)-3\alpha^2 e^{\frac{2ct}{R}}+3\right)}\\
A_4 &= -\frac{2c^2}{re^{\frac{2ct}{R}}}\left(1-\frac{\lambda}{2}\frac{5e^{\frac{ct}{R}}\alpha+2\lambda-3}{\alpha^3 e^{\frac{3ct}{R}}+\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}-1}-\frac{\lambda}{2}\frac{5e^{\frac{ct}{R}}\alpha-2\lambda+3}{\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}\right)\\
A_5 &= -\frac{c^2}{e^{\frac{2ct}{R}}}\left(1+2\lambda\frac{\alpha e^{\frac{ct}{R}}+\lambda-1}{\alpha^3\left(-e^{\frac{3ct}{R}}\right)-\alpha^2 e^{\frac{2ct}{R}}+\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}+2\lambda\frac{\alpha\left(-e^{\frac{ct}{R}}\right)+\lambda-1}{\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}\right)\\
A_6 &= -\frac{c^2\cot\theta}{r^2 e^{\frac{2ct}{R}}}\left(1-\lambda\frac{1-\alpha e^{\frac{ct}{R}}}{\alpha^3\left(-e^{\frac{3ct}{R}}\right)-\alpha^2 e^{\frac{2ct}{R}}+\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}-\lambda\frac{\alpha e^{\frac{ct}{R}}+1}{\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}\right)\\
A_7 &= -\frac{c^2}{r^2 e^{\frac{2ct}{R}}}\left(1-\lambda\frac{1-\alpha e^{\frac{ct}{R}}}{\alpha^3\left(-e^{\frac{3ct}{R}}\right)-\alpha^2 e^{\frac{2ct}{R}}+\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}-\lambda\frac{\alpha e^{\frac{ct}{R}}+1}{\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}\right)\\
A_8 &= 0\\
A_9 &= -\frac{c^2}{r^2\sin^2\theta e^{\frac{2ct}{R}}}\left(1-\lambda\frac{1-\alpha e^{\frac{ct}{R}}}{\alpha^3\left(-e^{\frac{3ct}{R}}\right)-\alpha^2 e^{\frac{2ct}{R}}+\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}-\lambda\frac{\alpha e^{\frac{ct}{R}}+1}{\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}\right).
\end{aligned}
\tag{15.81}
$$

Comparision of the Covariant Wave Equation in the Different Spacetimes

Our full covariant wave equation in $SdS-LR$ space time is

$$
\begin{aligned}
\Box_{SdS-LR}\psi = {} & \frac{\partial^2\psi}{\partial t^2} \\
& + \frac{3c}{R}\left(1+\frac{\lambda\left(2\alpha^2 e^{\frac{2ct}{R}}+5\alpha e^{\frac{ct}{R}}+4\lambda-5\right)}{3\left(-\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}+\alpha(1-2\lambda)e^{\frac{ct}{R}}+1\right)}+\frac{\lambda\left(2\alpha^2 e^{\frac{2ct}{R}}-5\alpha e^{\frac{ct}{R}}+4\lambda-5\right)}{3\left(\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1\right)}\right)\frac{\partial\psi}{\partial t} \\
& + 8c\alpha\lambda\frac{\alpha^2 e^{\frac{2ct}{R}}+\lambda-1}{1-\alpha^2 e^{\frac{2ct}{R}}\left(\alpha^4 e^{\frac{4ct}{R}}+4\lambda\left(\alpha^2 e^{\frac{2ct}{R}}+\lambda-1\right)-3\alpha^2 e^{\frac{2ct}{R}}+3\right)}\frac{\partial^2\psi}{\partial t\partial r} \\
& + -\frac{2c^2}{re^{\frac{2ct}{R}}}\left(1-\frac{\lambda}{2}\frac{5e^{\frac{ct}{R}}\alpha+2\lambda-3}{\alpha^3 e^{\frac{3ct}{R}}+\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}-1}-\frac{\lambda}{2}\frac{5e^{\frac{ct}{R}}\alpha-2\lambda+3}{\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}\right)\frac{\partial\psi}{\partial r} \\
& + -\frac{c^2}{e^{\frac{2ct}{R}}}\left(1+2\lambda\frac{\alpha e^{\frac{ct}{R}}+\lambda-1}{\alpha^3\left(-e^{\frac{3ct}{R}}\right)-\alpha^2 e^{\frac{2ct}{R}}+\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}+2\lambda\frac{\alpha\left(-e^{\frac{ct}{R}}\right)+\lambda-1}{\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}\right)\frac{\partial^2\psi}{\partial r^2} \\
& + -\frac{c^2\cot\theta}{r^2 e^{\frac{2ct}{R}}}\left(1-\lambda\frac{1-\alpha e^{\frac{ct}{R}}}{\alpha^3\left(-e^{\frac{3ct}{R}}\right)-\alpha^2 e^{\frac{2ct}{R}}+\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}-\lambda\frac{\alpha e^{\frac{ct}{R}}+1}{\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}\right)\frac{\partial\psi}{\partial\theta} \\
& + -\frac{c^2}{r^2 e^{\frac{2ct}{R}}}\left(1-\lambda\frac{1-\alpha e^{\frac{ct}{R}}}{\alpha^3\left(-e^{\frac{3ct}{R}}\right)-\alpha^2 e^{\frac{2ct}{R}}+\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}-\lambda\frac{\alpha e^{\frac{ct}{R}}+1}{\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}\right)\frac{\partial^2\psi}{\partial\theta^2} \\
& + -\frac{c^2}{r^2\sin^2\theta e^{\frac{2ct}{R}}}\left(1-\lambda\frac{1-\alpha e^{\frac{ct}{R}}}{\alpha^3\left(-e^{\frac{3ct}{R}}\right)-\alpha^2 e^{\frac{2ct}{R}}+\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}-\lambda\frac{\alpha e^{\frac{ct}{R}}+1}{\alpha^3 e^{\frac{3ct}{R}}-\alpha^2 e^{\frac{2ct}{R}}-\alpha(1-2\lambda)e^{\frac{ct}{R}}+1}\right)\frac{\partial^2\psi}{\partial\phi^2}.
\end{aligned}
\tag{15.82}
$$

We saw in our introduction that the covariant wave equation in the black hole toy model approached that of the $FLWR$ spacetime as the mass term disappeared. By setting $\lambda = 0$ in our equation above, we find that the same is true for the covariant wave equation in $SdS-LR$ space. However our $SdS-LR$ wave equation differs from the wave equation in the toy model if we are neglecting the boundary of the universe. If we let the radius of the universe be infinite, or rather $\alpha \to 0$, we end up with

$$
\begin{aligned}
\Box_{SdS-LR}\big|_{\alpha\to 0}\,\psi = {} & \frac{\partial^2\psi}{\partial t^2} + \frac{3c}{R}\left(1 - \frac{10}{3}\lambda + \frac{8}{3}\lambda^2\right)\frac{\partial\psi}{\partial t} \\
& - \frac{c^2}{e^{\frac{2ct}{R}}}\left((1-2\lambda)^2\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}(1-\lambda)(1-2\lambda)\frac{\partial\psi}{\partial r}\right. \\
& \left. + \frac{1-2\lambda}{r^2}\left(\frac{\partial^2\psi}{\partial\theta^2} + \cot\theta\frac{\partial\psi}{\partial\theta} + \csc^2\theta\frac{\partial^2\psi}{\partial\phi^2}\right)\right).
\end{aligned}
\tag{15.83}
$$

When comparing this against the result for the toy model we find an additional factor of $\left(1 - \frac{10}{3}\lambda + \frac{8}{3}\lambda^2\right)$ with the $\frac{\partial\psi}{\partial t}$ term, but the comparison of the other factors are a match.

Chapter Summary

The main result of this section is that the Schwarzschild-de Sitter spacetime under the LR transformation:

a. Approximates the evolving spacetime given by Robertson-Walker spacetime with scale factor $e^{\frac{ct}{R}}$ when the ‘radius’ of the observer is significantly far away from the point mass,
b. Approximates the Schwarzschild vacuum solution when the ‘radius’ of the universe is much greater than the ‘radius’ of the observer, and
c. Represents a black hole solution within an expanding universe

In the next chapter we will take a look at a new metric for a black hole in an expanding universe.

CHAPTER XVI

NEW FORM OF S-DS METRIC

Conceptual Introduction

We have looked at two metrics for a black hole in an expanding universe. The first one we looked at was the familiar *SdS* metric under the *LR* transformation. The second metric was the toy model explored by Yagdjian [10]. The first metric was initially in static coordinates, but the *LR* transformation moved into a time evolving reference frame. The second model has the expansion term, but considers a stationary black hole. Here we will see what happens if we expand upon the toy model for an evolving black hole, and then try to change the coordinates under the inverse of the *LR* transformation. We find another metric for a black hole inside of an expanding universe, but this time we have an additional boundary for our universe. We can consider this black hole universe as an immersion into the larger *FLRW* expanding universe.

We begin this section by a review of the static spacetimes we have considered and how they relate to each other in limit cases. We will then do the same regarding time evolving spacetimes. From here we will proceed to derive the new metric. Then we will solve Einstein's equation with this new metric, consider the limit cases for this solution, and then proceed to evaluate the covariant wave equation in this background space.

Prerequisites

Static Spacetimes

We have seen the Schwarzshild metric, S, which is represented by the line element

$$ds^2 = -c^2\left(1-\frac{2GM_{bh}}{c^2 r}\right)dt^2 + \left(1-\frac{2GM_{bh}}{c^2 r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2 d\phi^2\right), \tag{16.1}$$

and containing the event horizon contained at $r_h = \frac{2GM_{bh}}{c^2}$. Some additional important assumptions for this spacetime include that it is a static and spherically symmetric vacuum spacetime.

We have also seen the de Sitter metric, dS, represented by the line element

$$ds^2 = -c^2\left(1 - \frac{r^2}{R^2}\right)dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2 d\phi^2\right), \tag{16.2}$$

and containing the cosmological horizon when $r_h = R$. This spacetime is also static and spherically symmetric, but assumed to have a homogeneous matter distribution where the stress-energy is modeled as a perfect fluid.

We then saw a combined Schwarzschild-de Sitter spacetime, SdS, represented by the line element

$$ds^2 = -c^2\left(1 - \frac{2GM_{bh}}{c^2 r} - \frac{r^2}{R^2}\right)dt^2 + \left(1 - \frac{2GM_{bh}}{c^2 r} - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2 d\phi^2\right), \tag{16.3}$$

that contains both the event horizon of the Schwarzschild spacetime and the cosmological horizon of the de Sitter spacetime. In deriving this spacetime we considered the extended Einstein equation of

$$R_{ij} - \frac{1}{2} + \Lambda g_{ij} = 0 \tag{16.4}$$

where the stress-energy tensor is zero in accordance with the vacuum solution of the Schwarzschild spacetime and the cosmological constant $\Lambda = \frac{3}{R^2}$ representing the vacuum energy density found through the de Sitter assumption. This space time comes with the interesting feature that that $SdS \to dS$ as $M_{bh} \to 0$ but $SdS \to S$ as $\frac{r}{R} \to 0$.

Evolving Spacetimes

The above spacetimes were originally represented in static coordinates. However, we saw that the dS spacetime, under the Lemaître-Robertson transformation, LR, resulted in the expanding $FLWR$ spacetime with the scale factor $e^{\frac{ct}{R}}$,

$$ds^2_{(dS)} \underset{LR}{\longrightarrow} dt^2 + e^{\frac{ct}{R}} \left(dr^2 + r^2 \left(d\theta^2 + \sin^2 d\phi^2\right)\right). \tag{16.5}$$

This particular coordinate transformation had the added benefit of removing the singularity from $r = R$, and making the metric valid over the entire domain.

However, there is no known transformation of the SdS metric that results in such a nice warped product, and our progress can be summarized in the below diagram.

$$\begin{tikzcd}
 & dS \arrow[r, "LR"] & FLRW \arrow[dr, "\frac{c}{R} \to 0"] & \\
SdS \arrow[ur, "M_{bh} \to 0"] \arrow[dr, "\left(\frac{r}{R}\right)^2 \to 0"'] & & & Minkowski \\
 & S \arrow[urr, "M_{bh} \to 0"'] & &
\end{tikzcd}$$

Figure 16.1: Spacetime Metric Transformations

Here our upper path provides us with a time evolving reference in the $FLRW$ space, but no black hole. The Lower path maintains the presence of a black hole in the S space, but has no representation in a time evolving reference.

Derivation of a New Form of Schwarzschild-de Sitter Metric

Here we will consider a stationary black hole inserted into an expanding universe. Beginning with the manifold $\widetilde{M}$ of $FLRW$ spacetime with scale factor e^{Ht}

$$ds^2 = dt^2 + e^{2Ht} \left(dr^2 + r^2 \left(d\theta^2 + \sin^2 d\phi^2\right)\right), \tag{16.6}$$

where H is the Hubble constant representing the rate of expansion within $\widetilde{M}$. We can use the immersed manifold M defined by the metric

$$ds^2 = -c^2\left(1-\frac{2GM_{bh}}{c^2r}\right)dt^2 + e^{2Ht}\left(\left(1-\frac{2GM_{bh}}{c^2r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2 d\phi^2\right)\right), \quad (16.7)$$

to evaluate a stationary black hole immersed within an expanding spacetime. That is, in this model we get a stationary Schwarzschild radius inside of an expanding universe with the same scale factor as in $\widetilde{M}$. This particular metric was explored by Yagdjian [10], and we will use those results in our comparisons later.

Next, let us consider a change to this toy model so that the Schwarzschild radius is impacted by the expansion and thus making it non-stationary. In particular, we are looking at a time evolving black hole immersed within an expanding spacetime, and represented by the line element

$$ds^2 = -c^2\left(1-\frac{2GM_{bh}}{c^2re^{\frac{ct}{R}}}\right)dt^2 + e^{2Ht}\left(\left(1-\frac{2GM_{bh}}{c^2re^{\frac{ct}{R}}}\right)^{-1}dr^2 + r^2e^{\frac{2ct}{R}}\left(d\theta^2+\sin^2 d\phi^2\right)\right), \quad (16.8)$$

where here again H is the Hubble constant, and can be considered as the speed of light divided by the boundary of $\widetilde{M}$ (or the radius of the universe).

We have seen how the *LR* transformation can be used to switch from static coordinates to evolving coordinates. Imagine then that these new coordinates are *LR* coordinates. With the variables above replaced by their primed counter parts, then the pre-image is obtained by the transformation,

$$\begin{cases} r' & \to \frac{r}{\sqrt{1-\frac{r^2}{R^2}}}e^{\frac{-ct}{R}}, \\ t' & \to t+\frac{R}{2c}\log\left(1-\frac{r^2}{R^2}\right) \\ \theta & \to \theta \\ \phi & \to \phi \end{cases}, \quad (16.9)$$

and we get the new line element, $SdS^{(New)}$

$$ds^2 = -c^2\left(1-\frac{2GM_{bh}}{c^2 r}\right)dt^2 + e^{2Ht}\left(1-\frac{r^2}{R^2}\right)^{\frac{HR}{c}}\left(\left(1-\frac{2GM_{bh}}{c^2 r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right). \tag{16.10}$$

Here we have the e^{Ht} scale factor representing the expansion from the principle manifold $\widetilde{M}$, and the terms M_{bh}, r, and R as representative features of the immersed manifold M. This provides us with a different form for an expanding spacetime, represented by the scale factor e^{Ht}, that maintains both the event horizon, in $\frac{2GM_{bh}}{c^2 r}$, and a cosmological horizon, represented in $\left(\frac{r}{R}\right)$, and provides a new boundary R in M less than R' in $\widetilde{M}$.

Notice here that as R from M approaches R' from $\widetilde{M}$, that is R from our model approaches the actual boundary of the universe, our line element simplifies to

$$ds^2 = -c^2\left(1-\frac{2GM_{bh}}{c^2 r}\right)dt^2 + e^{\frac{2ct}{R}}\left(1-\frac{r^2}{R^2}\right)\left(\left(1-\frac{2GM_{bh}}{c^2 r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right). \tag{16.11}$$

Further, regardless of our value of R in M compared to $\widetilde{M}$, if we take $R \gg r$, then our line element reduces to

$$ds^2 = -c^2\left(1-\frac{2GM_{bh}}{c^2 r}\right)dt^2 + e^{2Ht}\left(\left(1-\frac{2GM_{bh}}{c^2 r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2 d\phi^2\right)\right), \tag{16.12}$$

which is precisely our toy model from above, $S^{(Toy)}$.

On one hand, if we then take $M_{bh} \to 0$ too, we end up in the (dS) spacetime under the LR coordinate transformation where a further assumption of $H = 0$ brings us into flat *Minkowski* space. But on the other hand if we leave the mass alone, by setting $H = 0$, wen end up in S spacetime where the further setting of mass to zero brings us to flat *Minkowski* space once again.

This can be summarized as follows:

$$\begin{tikzcd}
SdS^{(New)} \arrow[dr, "\left(\frac{r}{R}\right) \to 0"'] & & & dS \arrow[dll, "LR"] \\
& S^{(Toy)} \arrow[r, "M_{bh} \to 0"'] \arrow[d, "H \to 0"] & FLWR \arrow[d, "H \to 0"] & \\
& S \arrow[r, "M_{bh} \to 0"'] & Minkowski &
\end{tikzcd}$$

Figure 16.2: New Spacetime Metric Transformations

With this establishment of this new metric we can begin our evaluation by looking at the solution to Einstein's field equations.

Curvature of $(SdS^{(New)})$

To find the solution to the Einstein equations we follow the typical procedure of identifying the Christoffel symbols, the curvature tensors, and then Einstein's tensor.

Christoffel Symbols

The non-zero Christoffel Symbols are,

$$
\begin{aligned}
\Gamma^0_{10} &= \frac{G\text{Mbh}}{r\left(c^2 r - 2G\text{Mbh}\right)},\\
\Gamma^0_{11} &= \frac{c^2 H r^2 e^{2Ht}\left(1-\frac{r^2}{R^2}\right)^{\frac{HR}{c}}}{\left(c^2 r - 2G\text{Mbh}\right)^2},\\
\Gamma^0_{22} &= \frac{H r^3 e^{2Ht}\left(1-\frac{r^2}{R^2}\right)^{\frac{HR}{c}}}{c^2 r - 2G\text{Mbh}},\\
\Gamma^0_{33} &= \frac{H r^3 \sin^2(\theta) e^{2Ht}\left(1-\frac{r^2}{R^2}\right)^{\frac{HR}{c}}}{c^2 r - 2G\text{Mbh}},\\
\Gamma^1_{00} &= \frac{G\text{Mbh}e^{-2Ht}\left(1-\frac{r^2}{R^2}\right)^{-\frac{HR}{c}}}{r^2} - \frac{2G^2\text{Mbh}^2 e^{-2Ht}\left(1-\frac{r^2}{R^2}\right)^{-\frac{HR}{c}}}{c^2 r^3},\\
\Gamma^1_{01} &= \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = H,\\
\Gamma^1_{11} &= \frac{G\text{Mbh}}{2G\text{Mbh}r - c^2 r^2} + \frac{HrR}{cr^2 - cR^2},\\
\Gamma^1_{22} &= \frac{2GH\text{Mbh}r^2 R}{c^3\left(r^2-R^2\right)} + \frac{2G\text{Mbh}r^2}{c^2\left(r^2-R^2\right)} - \frac{2G\text{Mbh}R^2}{c^2\left(r^2-R^2\right)} - \frac{Hr^3 R}{c\left(r^2-R^2\right)} + \frac{rR^2}{r^2-R^2} - \frac{r^3}{r^2-R^2},\\
\Gamma^1_{33} &= \frac{2GH\text{Mbh}r^2 R\sin^2(\theta)}{c^3\left(r^2-R^2\right)} + \frac{2G\text{Mbh}r^2\sin^2(\theta)}{c^2\left(r^2-R^2\right)} - \frac{2G\text{Mbh}R^2\sin^2(\theta)}{c^2\left(r^2-R^2\right)} - \frac{Hr^3 R\sin^2(\theta)}{c\left(r^2-R^2\right)} + \frac{rR^2\sin^2(\theta)}{r^2-R^2} - \frac{r^3\sin^2(\theta)}{r^2-R^2},\\
\Gamma^2_{12} &= \frac{Hr^2 R}{cr^3 - crR^2} - \frac{cR^2}{cr^3 - crR^2} + \frac{cr^2}{cr^3 - crR^2},\\
\Gamma^2_{33} &= \sin(\theta)(-\cos(\theta)),\\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{Hr^2 R}{cr^3 - crR^2} - \frac{cR^2}{cr^3 - crR^2} + \frac{cr^2}{cr^3 - crR^2},\\
\Gamma^3_{23} &= \Gamma^3_{32} = \cot\theta.
\end{aligned}
\tag{16.13}
$$

Ricci Tensor

The above Christoffel symbols provide us with the Ricci Tensor components,

$$
\begin{aligned}
R_{00} &= -c^2(1-2\lambda)\left(\frac{3H^2}{c^2(1-2\lambda)}+\frac{H}{Rc}\frac{\lambda}{e^{2Ht}\left(1-\alpha^2\right)^{1+\frac{HR}{c}}}\right),\\
R_{01} = R_{10} &= \frac{2H\lambda}{r(1-2\lambda)},\\
R_{11} &= \frac{e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}}}{1-2\lambda}\left(\frac{3H^2}{c^2(1-2\lambda)}+\frac{H\left(4-\left(\alpha^2+7\right)\lambda\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right),\\
R_{22} &= r^2e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}}\left(\frac{3H^2}{c^2(1-2\lambda)}+\frac{H\left(2\left(-\left(\alpha^2(1-\lambda)\right)-3\lambda+2\right)-\frac{H(1-2\lambda)r^2}{cR}\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right),\\
R_{33} &= r^2e^{2Ht}\sin^2\theta\left(1-\alpha^2\right)^{\frac{HR}{c}}\left(\frac{3H^2}{c^2(1-2\lambda)}+\frac{H\left(2\left(-\left(\alpha^2(1-\lambda)\right)-3\lambda+2\right)-\frac{H(1-2\lambda)r^2}{cR}\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right),
\end{aligned}
\tag{16.14}
$$

where $\alpha = \frac{r}{R}$ and $\lambda = \frac{GM_{bh}}{c^2r}$.

We also have the Ricci scalar of:

$$
\mathscr{R} = 4\left(\frac{3H^2}{c^2(1-2\lambda)}-\frac{H\left(\alpha^2(-\lambda)+2\alpha^2-\frac{2H\lambda r^2}{cR}+\frac{Hr^2}{cR}+9\lambda-6\right)}{(cR)\left(2e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right)
\tag{16.15}
$$

We saw that under the assumptions of $HR \to c$ and $\alpha \to 0$, our metric approaches that of the toy model, $S^{(Toy)}$. Under these same assumptions we have the Ricci tensor as,

$$
R_{ij} = \frac{\mathscr{R}}{4}\begin{pmatrix}
-c^2(1-2\lambda)\left(1-\frac{(1-2\lambda)(6-11\lambda)}{3\left(2e^{\frac{2ct}{R}}+6\lambda^2-7\lambda+2\right)}\right) & \frac{(2c\lambda R)\left(1-\frac{(1-2\lambda)(2-3\lambda)}{2e^{\frac{2ct}{R}}+6\lambda^2-7\lambda+2}\right)}{3r} & 0 & 0\\
\frac{(2c\lambda R)\left(1-\frac{6\lambda^2-7\lambda+2}{2e^{\frac{2ct}{R}}+6\lambda^2-7\lambda+2}\right)}{3r} & \frac{e^{\frac{2ct}{R}}\left(\frac{(1-2\lambda)(2-5\lambda)}{3\left(2e^{\frac{2ct}{R}}+6\lambda^2-7\lambda+2\right)}+1\right)}{1-2\lambda} & 0 & 0\\
0 & 0 & r^2e^{\frac{2ct}{R}}\left(\frac{(1-2\lambda)(2-3\lambda)}{3\left(2e^{\frac{2ct}{R}}+6\lambda^2-7\lambda+2\right)}+1\right) & 0\\
0 & 0 & 0 & r^2\sin^2(\theta)e^{\frac{2ct}{R}}\left(\frac{(1-2\lambda)(2-3\lambda)}{3\left(2e^{\frac{2ct}{R}}+6\lambda^2-7\lambda+2\right)}+1\right)
\end{pmatrix},
\tag{16.16}
$$

This is compared to the Ricci Tensor of

$$R_{ij} = \frac{\mathscr{R}}{4} \begin{pmatrix} -c^2(1-2\lambda) & \frac{2c\lambda R}{3r} & 0 & 0 \\ \frac{2c\lambda R}{3r} & \frac{e^{\frac{2ct}{R}}}{1-2\lambda} & 0 & 0 \\ 0 & 0 & r^2 e^{\frac{2ct}{R}} & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta)e^{\frac{2ct}{R}} \end{pmatrix}, \tag{16.17}$$

for the toy model.

Under the assumptions of $H \to \frac{c}{R}$, $\alpha \to 0$, and $\lambda \to 0$, our metric becomes that of expanding $FLWR$ space. We find that the Ricci tensor becomes

$$R_{ij} = \begin{pmatrix} -\frac{3c^2}{R^2} & 0 & 0 & 0 \\ 0 & e^{\frac{2ct}{R}}\left(\frac{4}{R^2 e^{\frac{2ct}{R}}} + \frac{3}{R^2}\right) & 0 & 0 \\ 0 & 0 & r^2 e^{\frac{2ct}{R}}\left(\frac{4}{R^2 e^{\frac{2ct}{R}}} + \frac{3}{R^2}\right) & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta)e^{\frac{2ct}{R}}\left(\frac{4}{R^2 e^{\frac{2ct}{R}}} + \frac{3}{R^2}\right) \end{pmatrix}, \tag{16.18}$$

or broken out in terms of the metric, we have

$$R_{ij} = \left(\Lambda \mathbf{I}_4 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{4}{R^2 e^{\frac{2ct}{R}}} & 0 & 0 \\ 0 & 0 & \frac{4}{R^2 e^{\frac{2ct}{R}}} & 0 \\ 0 & 0 & 0 & \frac{4}{R^2 e^{\frac{2ct}{R}}} \end{pmatrix}\right) g_{ij}^{(FLWR)}, \tag{16.19}$$

where we have the cosmological constant $\Lambda = \frac{3}{R^2}$. So, we can see that as time evolves we begin to approximate Ricci tensor for $FLWR$ space.

Under the assumptions of $H \to \frac{c}{R}$ the Ricci scalar becomes

$$\mathscr{R} \Rightarrow \frac{2\left(\frac{\left(-\left(\alpha^2(2-\lambda)\right)-9\lambda+6\right)e^{-\frac{2ct}{R}}}{\left(1-\alpha^2\right)^3} + \frac{6}{1-2\lambda}\right)}{R^2}. \tag{16.20}$$

Then, if we further have $\alpha \to 0$, the second term simplifies and we are left with a Ricci scalar of

$$\mathscr{R} \Rightarrow \frac{2\left((6-9\lambda)e^{-\frac{2ct}{R}} + \frac{6}{1-2\lambda}\right)}{R^2}. \tag{16.21}$$

Finally, if we let $\lambda \to 0$ as well we get,

$$\mathscr{R} \Rightarrow \frac{12\left(e^{-\frac{2ct}{R}} + 1\right)}{R^2}, \tag{16.22}$$

which as time evolves becomes

$$R \Rightarrow \frac{12}{R^2}. \tag{16.23}$$

Thereby ending up with the same Ricci scalar as the $FLWR$ spacetime.

Solution of Einstein Equations

Now, the Einstein tensor is defined as $G_{ij} = R_{ij} - \frac{1}{2}R$, where we do not yet take into account the cosmological constant. Obviously, as there is a difference in the Ricci tensor between our metric here and the other spacetimes, we will end up with differing values for the Einstein tensor.

Using our curvature calculations above, and the same values for α and λ we find

$$G_{00} = -c^2(1-2\lambda)\left(-\frac{3H^2}{c^2(1-2\lambda)} + \frac{H\left(2\lambda\left(-\alpha^2 - \frac{\alpha^2(HR)}{c} + 5\right)\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)} - \frac{H\left(-2\alpha^2 - \frac{\alpha^2(HR)}{c} + 6\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right), \tag{16.24}$$

$$G_{01} = G_{10} = \frac{2H\lambda}{r(1-2\lambda)}, \tag{16.25}$$

$$G_{11} = \frac{\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}}\right)\left(-\frac{3H^2}{c^2(1-2\lambda)} + \frac{H\left(2\lambda\left(-\alpha^2 - \frac{\alpha^2(HR)}{c} + 1\right)\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)} + \frac{H\left(2\left(\alpha^2-1\right) + \frac{\alpha^2(HR)}{c}\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right)}{1-2\lambda}, \tag{16.26}$$

$$G_{22} = r^2 e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}}\left(-\frac{3H^2}{c^2(1-2\lambda)} + \frac{H\left(\left(\alpha^2+3\right)\lambda\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)} - \frac{2H}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right), \tag{16.27}$$

$$G_{33} = r^2 e^{2Ht}\sin^2\theta\left(1-\alpha^2\right)^{\frac{HR}{c}}\left(-\frac{3H^2}{c^2(1-2\lambda)} + \frac{H\left(\left(\alpha^2+3\right)\lambda\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)} - \frac{2H}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right) \tag{16.28}$$

Recall that the *dS* spacetime was originally modeled as an isotropic homogeneous perfect fluid. Further, the *S* spacetime was modeled as the empty space around a point mass. We expect in certain limit cases, for our model to approximate each of these models, but what are we to make of the energy density for our generic model? We know that the stress energy tensor is proportional to the Einstein tensor, and so lets for the moment consider the scale where the stress energy tensor is represented by the above result.

Notice that we are able to separate out the components of the stress energy tensor as

$$
\begin{aligned}
T_{00} &= \left[-\frac{c^2(1-2\lambda)\left(\frac{\alpha^2H^2(1-2\lambda)}{c^2}-\frac{H\left(-\left(\alpha^2(2-\lambda)\right)-13\lambda+6\right)}{cR}\right)}{e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}}\right]- \\
&\qquad \left[\frac{3H^2}{c^2(1-2\lambda)}+\frac{H\left(\left(\alpha^2+3\right)\lambda\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right]g_{ij} && (16.29) \\
T_{01} = T_{10} &= \left[\frac{2H\lambda}{(1-2\lambda)r}\right]+[0+0]\,g_{ij} && (16.30) \\
T_{11} &= \left[\frac{\left(\frac{\alpha^2H^2(1-2\lambda)}{c^2}-\frac{H\left(-\left(\alpha^2(2-3\lambda)\right)+\lambda+2\right)}{cR}\right)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}}\right)}{(1-2\lambda)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right]- \\
&\qquad \left[\frac{3H^2}{c^2(1-2\lambda)}-\frac{H\left(\left(\alpha^2+3\right)\lambda\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right]g_{ij} && (16.31) \\
T_{22} &= \left[\frac{(2(-H))r^2e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}}}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right]- \\
&\qquad \left[\frac{3H^2}{c^2(1-2\lambda)}-\frac{H\left(\left(\alpha^2+3\right)\lambda\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right]g_{ij} && (16.32) \\
T_{33} &= \left[\frac{(2(-H))r^2\sin^2(\theta)e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}}}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right]- \\
&\qquad \left[\frac{3H^2}{c^2(1-2\lambda)}-\frac{H\left(\left(\alpha^2+3\right)\lambda\right)}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)}\right]g_{ij}. && (16.33)
\end{aligned}
$$

We have seen that this can be viewed as

$$T_{ij} = T_{ij}^{(Matter)} - T_{ij}^{(Density)}g_{ij}. \tag{16.34}$$

This form was used as a way to be able to obtain the cosmological constant in both *SdS* and *dS* spacetimes, but here we are considering a combination of vacuum and non-homogeneous energy density. So, we propose our form is generalized by

$$T_{ij} = \left[T_{ij}^{(Matter)}\right] - \left[T_{ij}^{(HomoDens)} + T_{ij}^{(VacDens)}\right] g_{ij}, \tag{16.35}$$

where the brackets correspond to the separated component form above.

In order to validate this we can look first at $T_{ij}^{(HomoDens)}$. The homogeneous matter distribution was presented as a perfect fluid and related to the *dS* spacetime. In our density breakdown we have

$$T_{ij}^{(HomoDens)} = \begin{pmatrix} \frac{H\left(\left(\alpha^2+3\right)\lambda\right)}{(cR)\left(e^{2Ht}(1-\alpha^2)^{\frac{HR}{c}+2}\right)} & 0 & 0 & 0 \\ 0 & -\frac{H\left(\left(\alpha^2+3\right)\lambda\right)}{(cR)\left(e^{2Ht}(1-\alpha^2)^{\frac{HR}{c}+2}\right)} & 0 & 0 \\ 0 & 0 & -\frac{H\left(\left(\alpha^2+3\right)\lambda\right)}{(cR)\left(e^{2Ht}(1-\alpha^2)^{\frac{HR}{c}+2}\right)} & 0 \\ 0 & 0 & 0 & -\frac{H\left(\left(\alpha^2+3\right)\lambda\right)}{(cR)\left(e^{2Ht}(1-\alpha^2)^{\frac{HR}{c}+2}\right)} \end{pmatrix}, \tag{16.36}$$

which is indeed a perfect fluid.

For the vacuum density we have

$$T_{ij}^{(VacDens)} = \begin{pmatrix} \frac{3H^2}{c^2(1-2\lambda)} & 0 & 0 & 0 \\ 0 & \frac{3H^2}{c^2(1-2\lambda)} & 0 & 0 \\ 0 & 0 & \frac{3H^2}{c^2(1-2\lambda)} & 0 \\ 0 & 0 & 0 & \frac{3H^2}{c^2(1-2\lambda)} \end{pmatrix}, \tag{16.37}$$

which is indeed a constant. It does look very similar to the cosmological constant, and in fact if we were to set $H \to \frac{c}{R}$ for the boundary and remove the point mass, it would become the cosmological constant.

Lets now look at a combined density term $T_{ij}^{(Density)}$ with

$$T_{ij}^{(Density)} = \begin{pmatrix} \frac{3H^2}{c^2(2\lambda-1)} - \frac{\left(\alpha^2+3\right)H\lambda e^{-2Ht}\left(1-\alpha^2\right)^{-\frac{HR}{c}}}{(\alpha-1)^2(\alpha+1)^2cR} & 0 & 0 & 0 \\ 0 & \frac{3H^2}{c^2(2\lambda-1)} + \frac{\left(\alpha^2+3\right)H\lambda e^{-2Ht}\left(1-\alpha^2\right)^{-\frac{HR}{c}}}{(\alpha-1)^2(\alpha+1)^2cR} & 0 & 0 \\ 0 & 0 & \frac{3H^2}{c^2(2\lambda-1)} + \frac{\left(\alpha^2+3\right)H\lambda e^{-2Ht}\left(1-\alpha^2\right)^{-\frac{HR}{c}}}{(\alpha-1)^2(\alpha+1)^2cR} & 0 \\ 0 & 0 & 0 & \frac{3H^2}{c^2(2\lambda-1)} + \frac{\left(\alpha^2+3\right)H\lambda e^{-2Ht}\left(1-\alpha^2\right)^{-\frac{HR}{c}}}{(\alpha-1)^2(\alpha+1)^2cR} \end{pmatrix}. \tag{16.38}$$

Now under the assumptions for $\alpha \to 0$ and $H \to \frac{c}{R}$ our metric becomes that of the toy model. In this instance our energy density is

$$T_{ij}^{(Density)} \Rightarrow \begin{pmatrix} \frac{3\left(\frac{1}{2\lambda-1} - \lambda e^{-\frac{2ct}{R}}\right)}{R^2} & 0 & 0 & 0 \\ 0 & \frac{3\left(\lambda e^{-\frac{2ct}{R}} + \frac{1}{2\lambda-1}\right)}{R^2} & 0 & 0 \\ 0 & 0 & \frac{3\left(\lambda e^{-\frac{2ct}{R}} + \frac{1}{2\lambda-1}\right)}{R^2} & 0 \\ 0 & 0 & 0 & \frac{3\left(\lambda e^{-\frac{2ct}{R}} + \frac{1}{2\lambda-1}\right)}{R^2} \end{pmatrix} \tag{16.39}$$

From here we set the mass term to zero for the metric to go to $FLRW$ space. Under these conditions We find the energy density to go to

$$T_{ij}^{(Density)} \Rightarrow \begin{pmatrix} -\frac{3}{R^2} & 0 & 0 & 0 \\ 0 & -\frac{3}{R^2} & 0 & 0 \\ 0 & 0 & -\frac{3}{R^2} & 0 \\ 0 & 0 & 0 & -\frac{3}{R^2} \end{pmatrix} \tag{16.40}$$

Further, if we instead look to check the energy density under the conditions that correspond the metric to approaching that of Schwarzschild we find out selves with a zero energy density..

Thus, our combined energy density in $T_{ij}^{(Density)}$ corresponds in the limits to the energy density for the dS and S spacetimes. Our initial considerations seem valid. We can consider $T_{ij}^{(HomoDens)}$ as the contribution of density from the homogeneous space, and how it relates to the dS spacetime. Likewise, we see how $T_{ij}^{(VacDens)}$ can be though of as the contribution of density from the non-homogeneous contributions of the point mass and how this relates to the vacuum density of the S model. Putting these together, and relating it to our T_{ij} above, we find that the total density is $T_{ij}^{(Density)} = T_{ij}^{(HomoDens)} + T_{ij}^{(VacDens)}$.

Still considering out constant of proportionality between the Einstein tensor and the stress energy tensor as unity, we can add this constant to both sides, to be left with the stress energy tensor for matter, $T_{ij}^{(Matter)}$.

$$T_{ij}^{(Matter)} = \begin{pmatrix} -\frac{c^2(1-2\lambda)\left(\frac{\alpha^2H^2(1-2\lambda)}{c^2} - \frac{H\left(-\left(\alpha^2(2-\lambda)\right)-13\lambda+6\right)}{cR}\right)}{e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}} & \frac{2H\lambda}{(1-2\lambda)r} & 0 & 0 \\ \frac{2H\lambda}{(1-2\lambda)r} & \frac{\left(\frac{\alpha^2H^2(1-2\lambda)}{c^2} - \frac{H\left(-\left(\alpha^2(2-3\lambda)\right)+\lambda+2\right)}{cR}\right)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}}\right)}{(1-2\lambda)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)} & 0 & 0 \\ 0 & 0 & \frac{(2(-H))r^2e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}}}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)} & 0 \\ 0 & 0 & 0 & \frac{(2(-H))r^2\sin^2(\theta)e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}}}{(cR)\left(e^{2Ht}\left(1-\alpha^2\right)^{\frac{HR}{c}+2}\right)} \end{pmatrix} \tag{16.41}$$

If we consider the boundary the same as $\widetilde{M}$ where $H \to \frac{c}{R}$, then a vanishing rate of expansion corresponds to $R \to \infty$. Indeed, under these conditions our stress energy of matter becomes zero, as we would expect.

If we look at flat expanding space under the assumptions of $H \to \frac{c}{R}$, and $\alpha \to 0$ we obtain the stress energy of

$$T_{ij}^{(Matter)} \Rightarrow \begin{pmatrix} \frac{c^2(2\lambda-1)(13\lambda-6)e^{-\frac{2ct}{R}}}{R^2} & \frac{2c\lambda}{rR-2\lambda rR} & 0 & 0 \\ \frac{2c\lambda}{rR-2\lambda rR} & \frac{\lambda+2}{(2\lambda-1)R^2} & 0 & 0 \\ 0 & 0 & -\frac{2r^2}{R^2} & 0 \\ 0 & 0 & 0 & -\frac{2r^2\sin^2(\theta)}{R^2} \end{pmatrix} \tag{16.42}$$

This is a distinctly different spacetime than that of the toy model's

$$T_{ij}^{(Toy)} = \begin{pmatrix} \frac{3c^2}{R^2} & -\frac{2cG\mathrm{Mbh}}{2G\mathrm{Mbh}rR-c^2r^2R} & 0 & 0 \\ -\frac{2cG\mathrm{Mbh}}{2G\mathrm{Mbh}rR-c^2r^2R} & -\frac{3c^4r^2e^{\frac{2ct}{R}}}{R^2\left(c^2r-2G\mathrm{Mbh}\right)^2} & 0 & 0 \\ 0 & 0 & -\frac{3c^2r^3e^{\frac{2ct}{R}}}{R^2(c^2r-2G\mathrm{Mbh})} & 0 \\ 0 & 0 & 0 & -\frac{3c^2r^3\sin^2(\theta)e^{\frac{2ct}{R}}}{R^2(c^2r-2G\mathrm{Mbh})} \end{pmatrix} \tag{16.43}$$

And if we go from here to approximating dS space under $\lambda \to 0$ we get

$$T_{ij}^{(Matter)} \Rightarrow \begin{pmatrix} \frac{6c^2 e^{-\frac{2ct}{R}}}{R^2} & 0 & 0 & 0 \\ 0 & -\frac{2}{R^2} & 0 & 0 \\ 0 & 0 & -\frac{2r^2}{R^2} & 0 \\ 0 & 0 & 0 & -\frac{2r^2\sin^2(\theta)}{R^2} \end{pmatrix}. \tag{16.44}$$

which evolves to

$$T_{ij}^{(Matter)} \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{R^2} & 0 & 0 \\ 0 & 0 & -\frac{2r^2}{R^2} & 0 \\ 0 & 0 & 0 & -\frac{2r^2\sin^2(\theta)}{R^2} \end{pmatrix} \tag{16.45}$$

which again approximates *Minkowski* space since R is very large.

However, if we go from the toy model to approximating S space under the assumption that $H \to 0$, or equivalently that $R \to \infty$, we end up in empty space as expected.

Wave Equation

Continuing the analysis of our metric, we can use the wave equation to identify how massless particles move in this background space.

The corresponding wave equation is,

$$\frac{\partial^2 \psi}{\partial t^2} + 3H\frac{\partial \psi}{\partial t} - c^2 e^{-2Ht}\left(1-\alpha^2\right)^{-\frac{HR}{c}}\left((1-2\lambda)^2\frac{\partial^2 \psi}{\partial r^2}\right. \tag{16.46}$$

$$+\frac{1}{r}\frac{(1-2\lambda)\left(-2\left(1-\alpha^2\right)\lambda+2\left(1-\alpha^2\right)-\frac{\alpha^2(1-2\lambda)(HR)}{c}\right)}{1-\alpha^2}\frac{\partial \psi}{\partial r} \tag{16.47}$$

$$\left.+\frac{(1-2\lambda)}{r^2}\left(\frac{\partial^2 \psi}{\partial \theta^2}+\cot\theta\frac{\partial \psi}{\partial \theta}+\csc^2\theta\frac{\partial^2 \psi}{\partial \phi^2}\right)\right) \tag{16.48}$$

If we then consider $\alpha \to 0$ and $H \to cR$, we get

$$\frac{\partial^2 \psi}{\partial t^2} + \frac{3c}{R}\frac{\partial \psi}{\partial t} - c^2 e^{\frac{2ct}{R}} \left((1-\lambda)^2 \frac{\partial^2 \psi}{\partial r^2} + \frac{(1-\lambda)(2-\lambda)}{r}\frac{\partial \psi}{\partial r} \right. \tag{16.49}$$

$$\left. + \frac{(1-\lambda)}{r2}\left(\frac{\partial^2 \psi}{\partial \theta^2} + \cot\theta \frac{\partial \psi}{\partial \theta} + \csc^2\theta \frac{\partial^2 \psi}{\partial \phi^2} \right) \right), \tag{16.50}$$

which is the same as the toy model introduced above. This again collapses into the wave equation in $FLWR$ in the case of $\lambda = 0$.

Chapter Summary

In this chapter we explored a new metric for a spacetime that contained a black hole and a boundary. We saw how this spacetime related to another black hole toy model as well as the familiar spacetimes of Schwarzschild and de Sitter. We then took a look at the solution to the Einstein field equations using this spacetime's metric and again compared the results to other familiar spacetimes. In particular, we separated out the matter and density portion of the stress energy tensor to see how the matter distribution related to the other spacetimes. We then took a look at the wave equation for wave propagation and found that the wave equation conforms to that of the other spacetimes under the same limit conditions as the metric.

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