Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/130201

Transient Analysis of Two- Class Priority Queuing System

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Received: 17 Aug. 2023, Revised: 7 Sep. 2023, Accepted: 27 Sep. 2023.
Published online: 1 Mar. 2024.

Abstract: A transient solution of Two-class priority queuing system was discussed in this paper, the first class with high priority and the second one with low priority and the capacity of the system is infinite. Various performance measures are examined. Some numerical analyses are discussed and some special cases are deduced and confirmed the results.

Keywords: Queuing Systems, Transient Solution, Two -class Priority, Numerical Analysis.

1 Introduction

Queueing theory is a very fascinating field of science, with wide applications. Telecommunication, traffic engineering, waiting in shops, factories, services in offices and hospitals can be designed and analyzed as queueing models. The process of priority occurs in many situations of our life, particularly when discriminatory handling is granted to specific kinds of persons. The mechanism of priority process is a precious scheduling technique that allows messages of different classes to receive different quality of service. Queuing model under preemptive resume priority has been widely investigated by a lot of researchers; for example Cobham (1957) [1] was the initiator who proposed the non-preemptive priority model according to Poisson distribution, the holding and input time subject to exponential distribution and investigated them for both single and multi-channel cases. Barry (1956) [13] and Stephan (1958) [9] investigated the model of preemptive priority system model with double priorities with Poisson distribution input and exponential distribution for the holding time. Many efforts have been made to calculate an easy format for the transient state probabilities for a single server queue M/M/1/N. Dieter, Herwig, SMACS Research Group (2005) [12], afford a transient analysis of the system contents in a two-class priority queue with single-slot service times. Tarabia (2007a) [2] presents a two-class priority M/M/1 queueing model, with two kinds of customers; the first kind of customers have preemptive priority resume discipline and are served with law of first-come, first served (FCFS) with finite capacity, the second kind of customers has low priority and infinite waiting space. He obtained explicit formulas for the generating functions in terms of the second kind polynomial of Chebyshev for the steady state case.

Tarabia (2007b) [3] based on the generating function technique; he investigated a two-class single-server preemptive priority queueing system model; each class with arrivals according to a Poisson distribution with times of service follows exponential distribution. New expressions which are free from any integral or special functions for the mean value of the length of queue and the steady-state joint distribution of the both number of high and low priority customers in this model are obtained. The analysis is based on the technique of generating function expansion. Walraevens et. al. (2005) [12] analyzes transient state case of the model contents in a double-class priority queue with single-slot service times. They obtain a generating function formula for the transient state contents of both classes. Also, they calculate some of the performance measures using this generating function. Bruneel et. al. (2017) [11] study a discrete-time queueing system with two kinds of customers and two servers, one for each customer kind. Also, they assumed that the service times distribution of both kinds of customers are independent, geometrically distributed random variables. Tarbia (2002) [4] present non- empty M/M/1/∞ queue and in (2011) [5] study steady state analysis of an M/M/1 queue with balking, catastrophes, server failures and repairs. Recently, Kumar and Sharma (2021) [15], discuss transient analysis of a Markovian queuing model with multiple-heterogeneous servers, and customer’s impatience, and obtain its time dependent solution.

Tarabia et al. (2022) [6], present the transient solution for two-class priority queuing system model with both balking and with existence of catastrophes in infinite system capacity and give some special cases that confirm the given formulas.

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After this introduction, the paper ordered as follows; in Section 2, we declare the model description with the assumption and several statements that used in the paper. In Section 3, the generating function was obtained. Some various performance measures are calculated in Section 4. Some special cases are obtained in Section 5. Finally, in section 6 some numerical analysis is carried out.

2 Model Demonstrations

In the proposed model, a single server queuing model is considered that serving two types of incoming customers say class-1 and class-2, each type has its own particular lane with independent arrival process for both types. A higher priority is dedicated to class-1 and a low priority to class-2. Suppose that the service law for each class is *First In First Out* (FIFO) and the priority of the system considered as preemptive resumed, i.e. if new customers join high priority lane during the service of a low priority customer, then the service of low priority customer is stopped and will be proceeded again when there is no high priority customers in the system. It is easy to see that the given system can be modeled by Markov process $\{(X_1(t), X_2(t)), t \geq 0\}$, $X_i(t)$, $i = 1,2$ denotes the number of customer's in the high and low class respectively and we assume that the system capacity is infinite. We denote $p_{i,j}(t)$={the probabilities of the system where the first class is at the state i and the second class is at state j , and all of them are infinite}. The inter-arrival and service times are taken to be negative exponentially distributed with mean$1/\lambda_1$and$1/\mu_1$ respectively for the first class and the inter-arrival and service times are taken to be negative exponentially distributed with mean $1/\lambda_2$ and$1/\mu_2$ respectively for the second class. From all the previous assumptions and declarations the consequence stochastic of behavior as set of what essentially are forward equations of Chapman-Kolmogorov which can be written as:

$$\frac{dp_{0,0}(t)}{dt} = -(\lambda_1 + \lambda_2)p_{0,0}(t) + \mu_1 p_{1,0}(t) + \mu_2 p_{0,1}(t) \tag{1}$$

$$\begin{aligned} \frac{dp_{0,j}(t)}{dt} &= -(\lambda_1 + \lambda_2 + \mu_2)p_{0,j}(t) + \mu_1 p_{1,j}(t) + \mu_2 p_{0,j+1}(t) \\ &\quad + \lambda_2 p_{0,j-1}(t) \qquad , j \geq 1 \end{aligned} \tag{2}$$

$$\frac{dp_{i,0}(t)}{dt} = -(\lambda_1 + \lambda_2 + \mu_1)p_{i,0}(t) + \mu_1 p_{i+1,0}(t) + \lambda_1 p_{i-1,0}(t) \, , i \ \geq 1 \tag{3}$$

$$\begin{aligned} \frac{dp_{i,j}(t)}{dt} &= -(\lambda_1 + \lambda_2 + \mu_1)p_{i,j}(t) + \mu_1 p_{i+1,j}(t) + \lambda_1 p_{i-1,j}(t) \\ &\quad + \lambda_2 p_{i,j-1}(t) \qquad i \, , j \geq 1 \end{aligned} \tag{4}$$

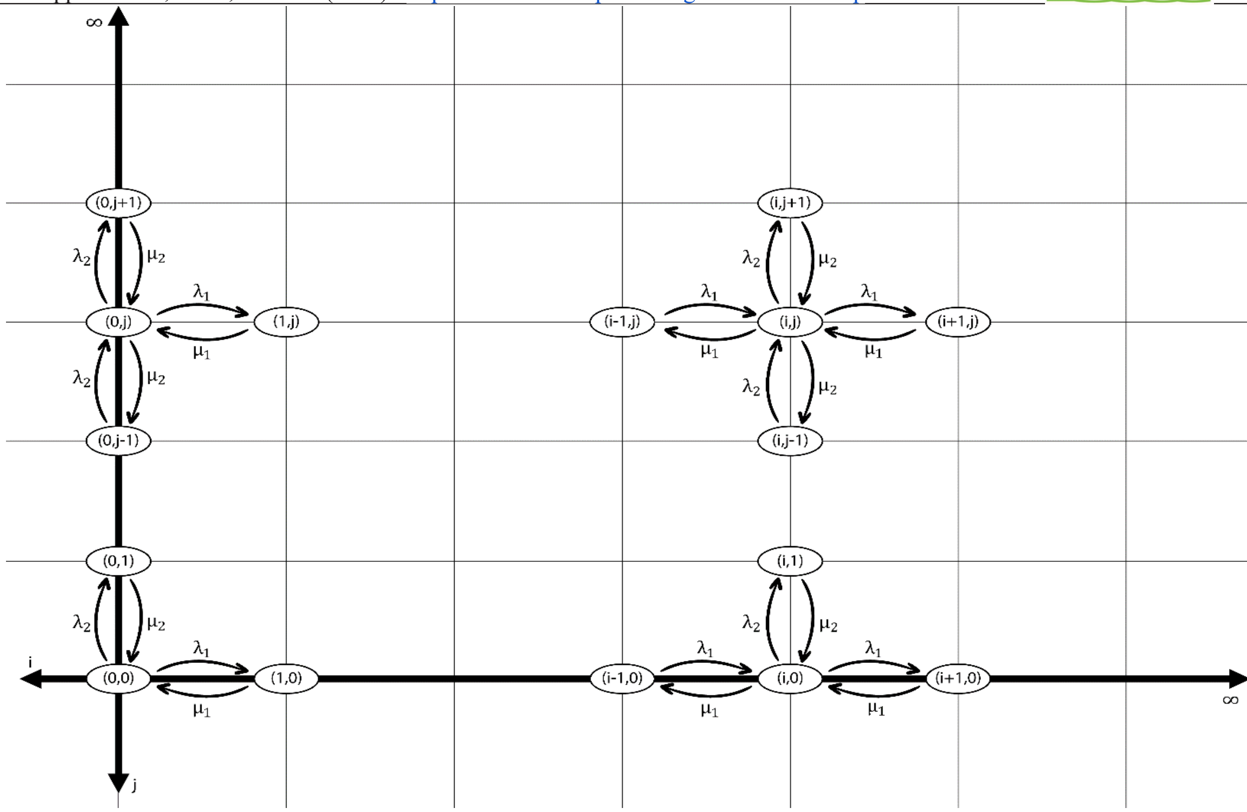

Fig. 1 Transient - state of two-class priority with infinite capacity

3 The transient solution of the model

Theorem (1):

The probabilities of the transient solution of two-class priority queueing system with infinite capacity are:

$$p_{i,j}(t) = [\int_0^t (i+1)\beta^i \frac{I_{i+1}(\propto u)}{u} e^{(\lambda_1+\lambda_2+\mu_1)u}\, du \; + \int_0^t (-1)\frac{I_{-1}(\propto(t-u))}{(t-u)} e^{-(\lambda_1+\mu_1)(t-u)}$$

$$.\,(i+1)\beta^{(i+1)} \frac{I_{i+1}(\propto u)}{u} e^{(\lambda_1+\lambda_2+\mu_1)u} du] \frac{(\lambda_2)^j}{j!}, i,j \geq 1. \qquad (5)$$

Where $\alpha = 2\sqrt{\lambda_1\mu_1}$, $\beta = \sqrt{\frac{\lambda_1}{\mu_1}}$, $I_n(.)$ is the first kind modified Bessel function.

Proof:

The transient solution of the two-class priority queuing system can be obtained by using the generating function:

$$G_i(s,t) = \sum_{j=0}^{\infty} p_{i,j}(t)s^j \quad , \quad \frac{\partial G_i(s,t)}{\partial t} = \sum_{j=0}^{\infty} \frac{dp_{i,j}(t)}{dt} s^j \;.\, 0 \leq i \leq \infty.$$

Using Eq. (3) and Eq. (4) we get:

$$\sum_{j=0}^{\infty}\frac{dp_{i,j}(t)}{dt}s^j = -(\lambda_1+\lambda_2+\mu_1)p_{i,0}(t)+\mu_1 p_{i+1,0}(t)+\lambda_1 p_{i-1,0}(t) \quad -(\lambda_1+\lambda_2+\mu_1)\sum_{j=1}^{\infty}p_{i,j}(t)s^j$$

$$+\mu_1\sum_{j=1}^{\infty}p_{i+1,j}(t)s^j$$

$$+\lambda_1\sum_{j=1}^{\infty}p_{i-1,j}(t)s^j+\lambda_2\sum_{j=1}^{\infty}p_{i,j-1}(t)s^j$$

$$= -(\lambda_1+\lambda_2+\mu_1)\sum_{j=0}^{\infty}p_{i,j}(t)s^j+\lambda_1\sum_{j=0}^{\infty}p_{i-1,j}(t)s^j+\mu_1\sum_{j=0}^{\infty}p_{i+1,j}(t)s^j+\lambda_2 s\sum_{j=1}^{\infty}p_{i,j-1}(t)\, s^{j-1}$$

$$= -(\lambda_1+\lambda_2+\mu_1)\ G_i(s,t)+\lambda_1 G_{i-1}(s,t)+\mu_1 G_{i+1}(s,t)+\lambda_2 s G_i(s,t) \qquad ,1\le i$$

$$\frac{\partial G_i(s,t)}{\partial t} = [-(\lambda_1+\lambda_2+\mu_1)\ +\lambda_2 s\ ]G_i(s,t)+\lambda_1 G_{i-1}(s,t)$$

$$+\mu_1 G_{i+1}(s,t) \qquad ,1\le i \quad . \tag{6}$$

By using the Laplace transform:

$$zG_i^{\ *}(s,z)-G_i(s,0)$$

$$=[-(\lambda_1+\lambda_2+\mu_1)\ +\lambda_2 s\ ]G_i^{\ *}(s,z)+\lambda_1 G_{i-1}^{\ *}(s,z)$$

$$+\mu_1 G_{i+1}^{\ *}(s,z) \qquad ,1\le i\ .$$

Since $p_{0,0}(0)=1$, $p_{i,j}(0)=0, i\neq j\neq 0$, then

$G_0(s,0)=\sum_{j=0}^{\infty}p_{0,j}(0)\ s^j = 1, otherwise\ equal\ zero$. Then

$$[z+(\lambda_1+\lambda_2+\mu_1)-\lambda_2 s\ ]G_i^{\ *}(s,z)$$

$$-\lambda_1 G_{i-1}^{\ *}(s,z)-\mu_1 G_{i+1}^{\ *}(s,z)\ =0\ ,1\le i \tag{7}$$

Solving equation (7) we get

$$G_i^{\ *}(s,z)=A_1 m_1^i+A_2 m_2^i \ .$$

where

$$m_1,m_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4\lambda_1\mu_1}}{2\mu_1}$$

Since $\sum_{i=0}^{\infty}G_i(1,t)=1$, then $\sum_{i=0}^{\infty}G_i^{\ *}(1,z)=\frac{1}{z}$,

$b_{at\ s=1}=z+(\lambda_1+\lambda_2+\mu_1)-\lambda_2=z+\lambda_1+\mu_1$.

Then we accept $m_1=\frac{-b-\sqrt{b^2-4ac}}{2a}$ and defused $m_2=\frac{-b+\sqrt{b^2-4ac}}{2a}$ that m_2 increasing and $\sum_{i=0}^{\infty}G_i^{\ *}(1,z)=\frac{1}{z}=$ $A_1\sum_{i=0}^{\infty}m_1^i = A_1\sum_{i=0}^{\infty}[\frac{-b-\sqrt{b^2-4\lambda_1\mu_1}}{2\mu_1}]^i$

$$\sum_{i=0}^{\infty}G_i^{\ *}(1,z)=A_1\sum_{i=0}^{\infty}m_1^i=\frac{A_1}{1-m_1}=\frac{1}{z}\rightarrow A_1=\frac{1-m_1}{z}$$

$$A_1 = \frac{1 - \left\{\frac{-b(1) - \sqrt{b^2(1) - 4\lambda_1\mu_1}}{2\mu_1}\right\}}{z}$$

$$= \frac{1}{z} + \frac{(z + \lambda_1 + \mu_1) + \sqrt{(z + \lambda_1 + \mu_1)^2 - 4\lambda_1\mu_1}}{2z\mu_1}.$$

$$G_i^*(s, z) = A_1[\frac{-b - \sqrt{b^2 - 4\lambda_1\mu_1}}{2\mu_1}]^i, b = [z + (\lambda_1 + \lambda_2 + \mu_1) - \lambda_2 s].$$

Suppose $\alpha = 2\sqrt{\lambda_1\mu_1}$, $\beta = \sqrt{\frac{\lambda_1}{\mu_1}}$ then $\alpha\beta = 2\lambda_1, \frac{\alpha}{\beta} = 2\mu_1$.

$G_i^*(s, z) = [\frac{1}{z} + \frac{(z+\lambda_1+\mu_1)+\sqrt{(z+\lambda_1+\mu_1)^2-\alpha^2}}{2z\mu_1}]$.

$$[\frac{-[z + (\lambda_1 + \lambda_2 + \mu_1) - \lambda_2 s] - \sqrt{[z + (\lambda_1 + \lambda_2 + \mu_1) - \lambda_2 s]^2 - \alpha^2}}{2\mu_1}]^i.$$

Suppose $a = (z + \lambda_1 + \mu_1)$, $c = -[z + (\lambda_1 + \lambda_2 + \mu_1) - \lambda_2 s]$ then $G_i^*(s, z)$ can be rewritten as:

$$G_i^*(s, z) = \frac{\beta^i}{z}[\frac{c-\sqrt{c^2-\alpha^2}}{\alpha}]^i + \frac{\beta^{i+1}}{z}\left[\frac{a-\sqrt{a^2-\alpha^2}}{\alpha}\right]^{-1} . [\frac{c-\sqrt{c^2-\alpha^2}}{\alpha}]^i \quad (8)$$

On inversion we have an explicit expression of $G_i(s,t)$ by using first kind of the modified Bessel function $I_n(.)$ and the Bessel function properties, we get

$$G_i(s, t) = \int_0^t (i + 1)\beta^i \frac{I_{i+1}(\alpha u)}{u} e^{-(\lambda_2 s-(\lambda_1+\lambda_2+\mu_1))u} du$$

$$+ \int_0^t (-1)\frac{I_{-1}(\alpha(t - u))}{(t - u)} e^{-(\lambda_1+\mu_1)(t-u)}$$

$$.(i + 1)\beta^{(i+1)} \frac{I_{i+1}(\alpha(t-u))}{(t-u)} e^{-(\lambda_2 s-(\lambda_1+\lambda_2+\mu_1))u} du, i \geq 1 .$$

Since $G_i(s,t) = \sum_{j=0}^{\infty} p_{i,j}(t)s^j$ then by taking the expansion of s^j on both sides we have:

$$p_{i,j}(t) = [\int_0^t (i + 1)\beta^i \frac{I_{i+1}(\alpha u)}{u} e^{(\lambda_1+\lambda_2+\mu_1)u} du \qquad + \int_0^t (-1)\frac{I_{-1}(\alpha(t-u))}{(t-u)} e^{-(\lambda_1+\mu_1)(t-u)}$$

$$.(i + 1)\beta^{(i+1)} \frac{I_{i+1}(\alpha u)}{u} e^{(\lambda_1+\lambda_2+\mu_1)u} du] \frac{(\lambda_2)^j}{j!}, i, j \geq 1. \quad (9)$$

It’s the probabilities of the transient-state of two-class priority model of queueing system with infinite capacity where the first class and second class are more than one.

Theorem (2)

The probabilities of the two-class priority queueing system with infinite capacity at the first point (*i=0*) is

$$p_{0,n}(t) = \varphi^n I_n(\omega t)e^{-(\lambda_1+\lambda_2+\mu_2)t}$$

$$+\mu_1\varphi^n \int_0^t e^{-(\lambda_1+\lambda_2+\mu_2)u} p_{1,n}(u) I_n(\omega(t - u))du$$

$$+\mu_2\varphi^n \int_o^t e^{[-(\lambda_1+\lambda_2+\mu_2)]u}\{I_n\big(\omega(t-u)\big)$$

$$-\varphi\, I_{n+1}\big(\omega(t-u)\big)\}p_{\,0,0}(u)du,\ \ n\geq 0\,. \qquad (10)$$

Where $\omega = 2\sqrt{\lambda_2\mu_2}$, $\varphi = \sqrt{\lambda_2/\mu_2}$, $I_n(.)$ is the modified Bessel function of the first kind.

Proof:

The transient solution of the two-class priority queuing system can be obtained by using the generating function:

$G_i(s,t) = \sum_{j=0}^{\infty} p_{i,j}(t)s^j$, $\frac{\partial G_i(s,t)}{\partial t} = \sum_{j=0}^{\infty}\frac{dp_{i,j}(t)}{dt}s^j$ $.0 \leq i \leq \infty$.

Using Eq. (1) and Eq. (2) we get:

$$\frac{\partial G_0(s,t)}{\partial t} = \sum_{j=0}^{\infty}\frac{dp_{0,j}(t)}{dt}s^j$$

$$= -(\lambda_1+\lambda_2)p_{0,0}(t) + \mu_1 p_{1,0}(t) + \mu_2 p_{0,1}(t) - (\lambda_1+\lambda_2+\mu_2)\sum_{j=1}^{\infty}p_{0,j}(t)s^j + \mu_1\sum_{j=1}^{\infty}p_{1,j}(t)s^j$$

$$+\mu_2\sum_{j=1}^{\infty}p_{0,j+1}(t)s^j + \lambda_2\sum_{j=1}^{\infty}p_{0,j-1}(t)s^j$$

$$= \left[-(\lambda_1+\lambda_2)p_{0,0}(t) - (\lambda_1+\lambda_2+\mu_2)\sum_{j=1}^{\infty}p_{0,j}(t)s^j\right]$$

$$+\left[\mu_1 p_{1,0}(t) + \mu_1\sum_{j=1}^{\infty}p_{1,j}(t)s^j\right]$$

$$+[\mu_2 p_{0,1}(t) + \mu_2\sum_{j=1}^{\infty}p_{0,j+1}(t)s^j\,] + \lambda_2\sum_{j=1}^{\infty}p_{0,j-1}(t)s^j$$

$$=\left[-(\lambda_1+\lambda_2+\mu_2)\sum_{j=0}^{\infty}p_{0,j}(t)s^j + \mu_2 p_{0,0}(t)\right] + \left[\mu_1\sum_{j=0}^{\infty}p_{1,j}(t)s^j\right]$$

$$+\left[\frac{\mu_2}{s}p_{0,1}(t)s + \frac{\mu_2}{s}\sum_{j=1}^{\infty}p_{0,j+1}(t)s^{j+1}\right] + \lambda_2 s\sum_{j=1}^{\infty}p_{0,j-1}(t)s^{j-1}$$

$$\frac{\partial G_0(s,t)}{\partial t} = \left[-(\lambda_1+\lambda_2+\mu_2) + \frac{\mu_2}{s} + \lambda_2 s\right]G_0(s,t) + \mu_1 G_1(s,t)$$

$$+\mu_2 p_{0,0}(t)\left[1-\tfrac{1}{s}\right]. \qquad (11)$$

$$G_0(s,t) = G_0(s,0)e^{\left[-(\lambda_1+\lambda_2+\mu_2)+\frac{\mu_2}{s}+\lambda_2 s\right]t}$$

$$+\mu_1\int_0^t e^{\left[-(\lambda_1+\lambda_2+\mu_2)+\frac{\mu_2}{s}+\lambda_2 s\right]u}G_1(s,u)du$$

$$+\mu_2\left[1-\tfrac{1}{s}\right]\int_o^t e^{\left[-(\lambda_1+\lambda_2+\mu_2)+\frac{\mu_2}{s}+\lambda_2 s\right]u}p_{0,0}(u)\,du\ ,$$

Since $G_0(s,0) = 1$, suppose $\omega = 2\sqrt{\lambda_2\mu_2}$, $\varphi = \sqrt{\lambda_2/\mu_2}$ using the function of first kind modified Bessel with the use of Bessel function properties, we get

$$G_0(s,t)=\sum_{j=0}^{\infty}p_{0,n}(t)s^j=e^{-(\lambda_1+\lambda_2+\mu_2)t}\sum_{n=-\infty}^{\infty}I_n(\omega t)(\varphi s)^n$$

$$+\mu_1\int_0^t e^{-(\lambda_1+\lambda_2+\mu_2)u}\sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}p_{1,n}(u)s^j\,I_n(\omega u)(\varphi)^n s^n du$$

$$+\mu_2\left[1-\frac{1}{s}\right]\int_o^t e^{-(\lambda_1+\lambda_2+\mu_2)u}p_{0,0}(u)\sum_{n=-\infty}^{\infty}I_n(\omega u)(\varphi s)^n\,du.$$

By comparing the coefficient of s^n on both sides we have:

$$p_{0,n}(t)=\varphi^n I_n(\omega t)e^{-(\lambda_1+\lambda_2+\mu_2)t}$$

$$+\mu_1\varphi^n\int_0^t e^{-(\lambda_1+\lambda_2+\mu_2)u}\,p_{1,n}(u)\,I_n\big(\omega(t-u)\big)d\,u\qquad+\mu_2\varphi^n\int_o^t e^{[-(\lambda_1+\lambda_2+\mu_2)]u}\{I_n\big(\omega(t-u)\big)$$

$$-\varphi\,I_{n+1}\big(\omega(t-u)\big)\}p_{\,0,0}(u)du,\;\; n\geq 0\,.$$

From $p_{0,n}(t)$ we have the probabilities of the low priority of the second kind. Its probabilities are the same as Krishna (2000) [7].

Where $\gamma=0\,,\mu_2=\mu,\varphi^n=\beta^n,\mu_1=0.$

At n=0:

$$P_{0,0}(t)=I_0(\omega t)e^{-(\lambda_1+\lambda_2+\mu_2)t}+\mu_1\int_0^t e^{-(\lambda_1+\lambda_2+\mu_2)u}p_{1,0}(u)I_0\big(\omega(t-u)\big)du$$

$$+\mu_2\int_o^t e^{-(\lambda_1+\lambda_2+\mu_2)u}\big[\,I_0\big(\omega(t-u)\big)-\varphi\,I_1\big(\omega(t-u)\big)\big]p_{0,0}(u)du. \tag{12}$$

Obtaining $P_{0,0}(t)$:

Using Laplace transform to equation (12), with $r=(z+\lambda_1+\lambda_2+\mu_2)$then

$$P_{0,0}{}^*(z)=\frac{1+\mu_1 p_{1,0}{}^*(z)}{z\sqrt{r^2-\omega^2}}+\mu_2 P_{0,0}{}^*(z)\{\frac{1}{z\sqrt{r^2-\omega^2}}-\varphi[\frac{\left[r-\sqrt{r^2-\omega^2}\right]}{\omega\sqrt{r^2-\omega^2}}]\}$$

$$P_{0,0}{}^*(z)=\frac{1}{\mu_2}\sum_{n=1}^{\infty}\frac{1}{z}\varphi^{-(n+1)}\frac{\left[r-\sqrt{r^2-\omega^2}\right]^{n+1}}{\omega^{n+1}}$$

$$-\frac{1}{\mu_2}.\frac{1}{z}\sum_{n=1}^{\infty}\frac{\left[r-\sqrt{r^2-\omega^2}\right]^n}{\omega^n}+\frac{\mu_1}{\mu_2}p_{1,0}{}^*(z)\sum_{n=1}^{\infty}\frac{1}{z}.\frac{\left[r-\sqrt{r^2-\omega^2}\right]^n}{\omega^n}$$

On inversion we have an explicit expression of $P_{0,0}(t)$:

$$P_{0,0}(t)=\frac{1}{\mu_2}\int_0^t\sum_{n=1}^{\infty}(n+1)\varphi^{-(n+1)}\frac{I_{n+1}(\omega u)}{\omega^{n+1}}e^{-(\lambda_1+\lambda_2+\mu_2)u}du$$

$$-\frac{1}{\mu_2}\int_0^t\sum_{n=1}^{\infty}n\varphi^{-n}\frac{I_n(\omega u)}{\omega^n}e^{-(\lambda_1+\lambda_2+\mu_2)u}du$$

$$+\frac{\mu_1}{\mu_2}p_{1,0}(t)\int_0^t\sum_{n=1}^{\infty}n\varphi^{-n}\frac{I_n(\omega u)}{\omega^n}e^{-(\lambda_1+\lambda_2+\mu_2)u}du. \tag{13}$$

Obtaining$p_{0,1}(t)$:

By using the generating function:

$G_i(s,t)=\sum_{j=0}^{\infty}p_{i,j}(t)s^j$, $\frac{\partial G_i(s,t)}{\partial t}=\sum_{j=0}^{\infty}\frac{dp_{i,j}(t)}{dt}s^j$ $.0\le i\le\infty.$

Using Eq. (1) and Eq. (3) we get:

$$\sum_{i=0}^{\infty}\frac{dp_{i,0}(t)}{dt}s^0$$

$$=-(\lambda_1+\lambda_2)p_{0,0}(t)+\mu_1p_{1,0}(t)+\mu_2p_{0,1}(t)-(\lambda_1+\lambda_2+\mu_1)\sum_{i=1}^{\infty}p_{i,0}(t)$$

$$+\mu_1\sum_{i=1}^{\infty}p_{i+1,0}(t)+\lambda_1\sum_{i=1}^{\infty}p_{i-1,0}(t)$$

$$=-(\lambda_1+\lambda_2+\mu_1)\sum_{i=0}^{\infty}p_{i,0}(t)+\mu_1p_{0,0}(t)+\mu_2p_{0,1}(t)+\mu_1\sum_{i=0}^{\infty}p_{i,0}(t)$$

$$+\lambda_1\sum_{i=0}^{\infty}p_{i,0}(t)-\mu_1p_{0,0}(t)$$

$$0=-\lambda_2\sum_{i=0}^{\infty}p_{i,0}(t)+\mu_2p_{0,1}(t)\rightarrow\lambda_2=\mu_2p_{0,1}(t)$$

$$p_{0,1}(t)=\frac{\lambda_2}{\mu_2}=\rho_2 \tag{14}$$

The probabilities of the low priority of the second class where one customer in the system.

The transient solution for the first-class with high priority where $i\ge 0$:

For the probabilities of the first-class with high priority can be calculated as:

$G_j(s,t)=\sum_{i=0}^{\infty}p_{i,j}(t)s^i$, $\frac{\partial G_j(s,t)}{\partial t}=\sum_{i=0}^{\infty}\frac{dp_{i,j}(t)}{dt}s^i$ $.0\le i\le\infty.$

Using Eq. (1) and Eq. (3) we get:

$$\frac{\partial G_0(s,t)}{\partial t}=-(\lambda_1+\lambda_2)p_{0,0}(t)+\mu_1p_{1,0}(t)+\mu_2p_{0,1}(t)$$

$$-(\lambda_1+\lambda_2+\mu_1)\sum_{i=1}^{\infty}p_{i,0}(t)s^i+\mu_1\sum_{i=1}^{\infty}p_{i+1,0}(t)s^i+\lambda_1\sum_{i=1}^{\infty}p_{i-1,0}(t)s^i$$

$$\frac{\partial G_0(s,t)}{\partial t}=-(\lambda_1+\lambda_2+\mu_1)G_0(s,t)+\mu_1p_{0,0}(t)+\mu_2p_{0,1}(t)+\frac{\mu_1}{s}G_0(s,t)$$

$$+\lambda_1sG_0(s,t)-\frac{\mu_1}{s}p_{0,0}(t)$$

$$\frac{\partial G_0(s,t)}{\partial t}=\left[-(\lambda_1+\lambda_2+\mu_1)+\frac{\mu_1}{s}+\lambda_1s\right]G_0(s,t)+[\mu_1-\frac{\mu_1}{s}]p_{0,0}(t)+\mu_2p_{0,1}(t)$$

$$\frac{\partial G_0(s,t)}{\partial t} - \left[-(\lambda_1+\lambda_2+\mu_1)+\frac{\mu_1}{s}+\lambda_1 s\right]G_0(s,t) = [\mu_1-\frac{\mu_1}{s}]p_{0,0}(t)+\mu_2 p_{0,1}(t)$$

$$G_0(s,t) = G_0(s,0)e^{-\left[(\lambda_1+\lambda_2+\mu_1)-\frac{\mu_1}{s}-\lambda_1 s\right]t}$$

$$+\left[\mu_1-\frac{\mu_1}{s}\right]\int_0^t e^{-\left[(\lambda_1+\lambda_2+\mu_1)-\frac{\mu_1}{s}-\lambda_1 s\right]u}p_{0,0}(u)du$$

$$+\mu_2\int_0^t e^{-\left[(\lambda_1+\lambda_2+\mu_1)-\frac{\mu_1}{s}-\lambda_1 s\right]u}p_{0,1}(u)du\ .$$

Since $G_0(s,0)=1$,Suppose $\alpha = 2\sqrt{\lambda_1\mu_1}$,$\beta=\sqrt{\frac{\lambda_1}{\mu_1}}$ then

$$G_0(s,t)=\sum_{i=0}^{\infty}p_{i,0}(t)s^i = e^{-(\lambda_1+\lambda_2+\mu_1)t}\sum_{n=-\infty}^{\infty}I_n(\alpha t)\,(\beta s)^n$$

$$+\left[\mu_1-\frac{\mu_1}{s}\right]\int_0^t e^{-(\lambda_1+\lambda_2+\mu_1)u}\sum_{n=-\infty}^{\infty}I_n(\alpha(t-u))\,(\beta s)^n p_{0,0}(u)du$$

$$+\mu_2\int_0^t e^{-(\lambda_1+\lambda_2+\mu_1)u}\sum_{n=-\infty}^{\infty}I_n(\alpha(t-u))\,(\beta s)^n p_{0,1}(u)du.$$

By comparing the coefficient of s^n on both sides

$$p_{n,0}(t) = I_n(\alpha t)\beta^n e^{-(\lambda_1+\lambda_2+\mu_1)t}$$

$$+\mu_1\int_0^t e^{-(\lambda_1+\lambda_2+\mu_1)u}\beta^n p_{0,0}(u)[I_n(\alpha(t-u))-I_{n-1}(\alpha(t-u))]du$$

$$+\mu_2\int_0^t e^{-(\lambda_1+\lambda_2+\mu_1)u}\sum_{n=-\infty}^{\infty}I_n(\alpha(t-u))\,\beta^n p_{0,1}(u)du\ ,n\geq 0\ . \tag{15}$$

The probabilities of the first–class with high priority when there is no customer in the second kind.

At n=1:

$$p_{1,0}(t) = I_1(\alpha t)\beta e^{-(\lambda_1+\lambda_2+\mu_1)t}$$

$$+\mu_1\int_0^t e^{-(\lambda_1+\lambda_2+\mu_1)u}\beta p_{0,0}(u)[I_1(\alpha(t-u))-I_0(\alpha(t-u))]du$$

$$+\mu_2\int_0^t e^{-(\lambda_1+\lambda_2+\mu_1)u}\sum_{n=-\infty}^{\infty}I_1(\alpha(t-u))\,\beta p_{0,1}(u)du\,.$$

From Eq. (13)

$$p_{1,0}(t) = I_1(\alpha t)\beta e^{-(\lambda_1+\lambda_2+\mu_1)t}$$

$$+\mu_1\int_0^t e^{-(\lambda_1+\lambda_2+\mu_1)u}\beta p_{0,0}(u)[I_1(\alpha(t-u))-I_0(\alpha(t-u))]du$$

$$+\mu_2\int_0^t e^{-(\lambda_1+\lambda_2+\mu_1)u}\sum_{n=-\infty}^{\infty} I_1(\alpha(t-u))\,\beta\frac{\lambda_2}{\mu_2}du$$

$$p_{1,0}(t)=I_1(\alpha t)\beta e^{-(\lambda_1+\lambda_2+\mu_1)t}$$

$$+\mu_1\int_0^t e^{-(\lambda_1+\lambda_2+\mu_1)u}\beta p_{0,0}(u)[I_1\big(\alpha(t-u)\big)-I_0\big(\alpha(t-u)\big)]du$$

$$+\lambda_2\int_0^t e^{-(\lambda_1+\lambda_2+\mu_1)u}I_1\big(\alpha(t-u)\big)\beta du. \tag{16}$$

By substituting in Eq. (13):

$$P_{0,0}(t)=\frac{1}{\mu_2}\int_0^t\sum_{n=1}^{\infty}(n+1)\varphi^{-(n+1)}\frac{I_{n+1}(\omega u)}{\omega^{n+1}}e^{-(\lambda_1+\lambda_2+\mu_2)u}du$$

$$-\frac{1}{\mu_2}\int_0^t\sum_{n=1}^{\infty}n\varphi^{-n}\frac{I_n(\omega u)}{\omega^n}e^{-(\lambda_1+\lambda_2+\mu_2)u}du$$

$$+\frac{\mu_1}{\mu_2}p_{1,0}(t)\int_0^t\sum_{n=1}^{\infty}n\varphi^{-n}\frac{I_n(\omega u)}{\omega^n}e^{-(\lambda_1+\lambda_2+\mu_2)u}du$$

We suppose that $L=(z+\lambda_1+\lambda_2+\mu_1)$, $r=(z+\lambda_1+\lambda_2+\mu_2)$. By using Laplace transform then

$$p_{0,0}^*(z)-\frac{{\mu_1}^2}{\mu_2}.\sum_{n=1}^{\infty}n\varphi^{-n}\frac{\left[L-\sqrt{L^2-\omega^2}\right]^{n+1}}{\omega^{n+1}\sqrt{L^2-\omega^2}}.\beta p_{0,0}^*(z)\left[\frac{\left[L-\sqrt{L^2-\omega^2}\right]}{\omega\sqrt{L^2-\omega^2}}-\frac{1}{\sqrt{L^2-\omega^2}}\right]$$

$$=\frac{1}{\mu_2}\sum_{n=1}^{\infty}(n+1)\varphi^{-(n+1)}\frac{\left[r-\sqrt{r^2-\omega^2}\right]^{n+2}}{\omega^{n+2}\sqrt{r^2-\omega^2}}$$

$$+\frac{\mu_1}{\mu_2}\sum_{n=1}^{\infty}n\varphi^{-n}\frac{\left[r-\sqrt{r^2-\omega^2}\right]^{n}}{\omega^{n}\sqrt{r^2-\omega^2}}\beta\{\frac{\left[L-\sqrt{L^2-\omega^2}\right]}{\omega\sqrt{L^2-\omega^2}}+\lambda_2\frac{\left[L-\sqrt{L^2-\omega^2}\right]}{\omega\sqrt{L^2-\omega^2}}\}$$

$$\frac{p_{0,0}^*(z)}{\sum_{n=1}^{\infty}n\varphi^{-n}\frac{\left[L-\sqrt{L^2-\omega^2}\right]^{n+1}}{\omega^{n+1}\sqrt{L^2-\omega^2}}}-\frac{{\mu_1}^2}{\mu_2}.\beta p_{0,0}^*(z)\left[\frac{\left[L-\sqrt{L^2-\omega^2}\right]}{\omega\sqrt{L^2-\omega^2}}-\frac{1}{\sqrt{L^2-\omega^2}}\right]$$

$$p_{0,0}^*(z)=\frac{\frac{1}{\mu_2}\left(-\varphi\frac{\left[L-\sqrt{L^2-\omega^2}\right]}{\omega\sqrt{L^2-\omega^2}}\right)+\frac{\mu_1}{\mu_2}\beta\{1+\lambda_2\}}{\{\frac{1}{\sum_{n=2}^{\infty}n\varphi^{-n}\frac{\left[L-\sqrt{L^2-\omega^2}\right]^{n+1}}{\omega^{n+1}\sqrt{L^2-\omega^2}}}-\frac{{\mu_1}^2}{\mu_2}.\beta\left[1-\frac{\left[L-\sqrt{L^2-\omega^2}\right]}{\omega}\right]\}}$$

By using the inverse Laplace we have

$$P_{0,0}(t)=\frac{1}{\mu_2}\int_0^t\sum_{n=1}^{\infty}(n+1)\varphi^{-(n+1)}\frac{I_{n+1}(\omega u)}{\omega^{n+1}}e^{-(\lambda_1+\lambda_2+\mu_2)u}du$$

$$+\frac{\mu_1}{\mu_2}\beta\{1+\lambda_2\}\frac{1}{\mu_2}\int_0^t\sum_{n=1}^{\infty}(n+1)\varphi^{-(n+1)}\frac{I_{n+1}(\omega u)}{\omega^{n+1}}e^{-(\lambda_1+\lambda_2+\mu_2)u}du$$

4 Various Performance Measures

- The marginal probabilities for the high priority class, $p_{i,.}$, can evaluated as follows:

$$p_{i,.} = \sum_{j=0}^{\infty} p_{i,j}(t) = G_i(1,t)$$

$$p_{i,.} = \int_0^t (i+1)\beta^i \frac{I_{i+1}(\propto u)}{u} e^{(\lambda_1+\mu_1)u} du$$

$$+\int_0^t (-1)\frac{I_{-1}(\propto(t-u))}{(t-u)} e^{-(\lambda_1+\mu_1)(t-u)}.(i+1)\beta^{(i+1)}\frac{I_{i+1}(\propto(t-u))}{(t-u)} e^{(\lambda_1+\mu_1)u} du, \quad (17)$$

and $$p_{i,0}(t) = G_i(0,t) = p_{i,0}(t) + \sum_{j=1}^{\infty} p_{i,j}(t)(0)^j$$

$$p_{i,0}(t) = \int_0^t (i+1)\beta^i \frac{I_{i+1}(\propto u)}{u} e^{(\lambda_1+\lambda_2+\mu_1)u} du$$

$$+\int_0^t (-1)\frac{I_1(\propto(t-u))}{(t-u)} e^{-(\lambda_1+\mu_1)(t-u)}.(i+1)\beta^{(i+1)}\frac{I_{i+1}(\propto(t-u))}{(t-u)} e^{(\lambda_1+\lambda_2+\mu_1)u} du.$$

- The expectation for the high priority class can be calculated as

$$E_i(y) = \frac{\partial G_i(s,t)}{\partial s}|_{s=1} = \sum_{j=0}^{\infty} j\, p_{i,j}(t)$$

$$= -(\lambda_1+\mu_1)G_i(1,t) + \lambda_1 G_{i-1}(1,t) + \mu_1 G_{i+1}(1,t)$$

$$= -(\lambda_1+\mu_1)p_{i,.}(t) + \lambda_1 p_{i-1,.}(t) + \mu_1 p_{i+1,.}(t) \quad ,1 \le i$$

From Eq. (17) we get $-(\lambda_1+\mu_1)p_{i,.}$, $\lambda_1 p_{i-1,.}(t)$ and $\mu_1 p_{i+1,.}(t)$, then

$$E_i(y) = \frac{\partial G_i(1,t)}{\partial s} = \int_0^t (i+1)\beta^i \frac{I_{i+1}(\propto u)}{u} . \frac{e^{(\lambda_1+\mu_1)u}}{-\lambda_2} du$$

$$+\int_0^t (-1)\frac{I_{-1}(\propto(t-u))}{(t-u)} e^{-(\lambda_1+\mu_1)(t-u)}.(i+1)\beta^{(i+1)}\frac{I_{i+1}(\propto(t-u))}{(t-u)} . \frac{e^{(\lambda_1+\mu_1)u}}{-\lambda_2} du, i \ge 1 \quad (18)$$

which is the same mean queue of M/M/1 [8], [14].

- The mean length of the queue for the class of high priority

$$E(N_1) = \sum_{n=0}^{\infty} n p_{n,.} = \sum_{n=0}^{\infty} n\rho_1{}^n(1-\rho_1)$$

$$= (1-\rho_1)\rho_1 \sum_{n=0}^{\infty} \frac{d}{d\rho_1}\rho_1{}^n = (1-\rho_1)\rho_1 \frac{d}{d\rho_1}\sum_{n=0}^{\infty}\rho_1{}^n$$

$$= (1-\rho_1)\rho_1 \frac{d}{d\rho_1}\frac{1}{(1-\rho_1)} = (1-\rho_1)\rho_1 \frac{1}{(1-\rho_1)^2} = \frac{\rho_1}{(1-\rho_1)}, \quad (19)$$

which agree with mean queue length of M/M/1 [8, 14].

5 Special Cases

5.1 Special cases for the first class with high priority:

We can discuss some special cases for the first class with high priority:

1- If $j = 0$, $\lambda_2 = \mu_2 = 0$ then

$$\omega = \varphi = 0, P_{0,0}(t) = 0 \; for \; j(the \; second \; class \; of \; priority), \alpha = 2\sqrt{\lambda_1\mu_1} \;, \beta = \sqrt{\frac{\lambda_1}{\mu_1}},$$

$$p_i(t) = \{\int_0^t (i+1)\beta^i \frac{I_{i+1}(\propto u)}{u} e^{(\lambda_1+\mu_1)u} du \;\; + \int_0^t (-1) \frac{I_{-1}(\propto (t-u))}{(t-u)} e^{-(\lambda_1+\mu_1)(t-u)}.$$

$$(i+1)\beta^{(i+1)} \frac{I_{i+1}(\propto u)}{u} e^{(\lambda_1+\mu_1)u}\}, i \geq 1.$$

which represent the probabilities of the first class with high priority.

2- If $j = 0$, $\lambda_2 = \mu_2 = 0,\; \lambda_1 = \lambda\;, \mu_1 = \mu, \alpha = 2\sqrt{\lambda\mu} \;, \beta = \sqrt{\frac{\lambda}{\mu}},$

$$p_i(t) = \{\int_0^t (i+1)\beta^i \frac{I_{i+1}(\propto u)}{u} e^{(\lambda+\mu)u} du \;\; + \int_0^t (-1) \frac{I_{-1}(\propto (t-u))}{(t-u)} e^{-(\lambda+\mu)(t-u)}.$$

$$(i+1)\beta^{(i+1)} \frac{I_{i+1}(\propto u)}{u} e^{(\lambda+\mu)u}\}, i \geq 1.$$

3- If $j = 0$, $\lambda_2 = \mu_2 = 0,\; \lambda_1 = \mu_1 = \lambda = \mu, \alpha = 2\sqrt{\lambda^2} = 2\lambda \;, \beta = \sqrt{\frac{\lambda}{\lambda}} = 1,$

$$p_i(t) = \{\int_0^t (i+1) \frac{I_{i+1}(\propto u)}{u} e^{2\lambda u} du \;\;\; + \int_0^t (-1) \frac{I_{-1}(\propto (t-u))}{(t-u)} e^{-2\lambda(t-u)}.$$

$$(i+1) \frac{I_{i+1}(\propto u)}{u} e^{2\lambda u}\}, i \geq 1.$$

5.2 Special cases for the second class with low priority:

We can discuss some special cases for the second class with low priority:

$$1 - \;\; At \; i = 0, \lambda_1 = 0, \mu_1 = 0 \;, \; \lambda_2 = \lambda \;, \mu_2 = \mu \;, \vartheta = 2\sqrt{\lambda\mu} \;, \rho = \sqrt{\lambda/\mu}$$

$$p_{0,n}(t) = \rho^n I_n(\vartheta t) e^{-(\lambda+\mu)t}$$

$$+\mu\rho^n \int_o^t e^{-(\lambda+\mu)u} \{I_n\big(\vartheta(t-u)\big) \; -\rho \; I_{n+1}\big(\vartheta(t-u)\big)\} p_{\;0,0}(u) du \,, \;\; n > 0\,.$$

6 Numerical Results

In this section, we will introduce some numerical results to declare the behavior of the transition probabilities; we studied the attitude of the transition probabilities given by *theorem 1*

$$p_{i,j}(t) = [\int_0^t (i+1)\beta^i \frac{I_{i+1}(\propto u)}{u} e^{(\lambda_1+\lambda_2+\mu_1)u} \, du \;\; + \int_0^t (-1) \frac{I_{-1}(\propto (t-u))}{(t-u)} e^{-(\lambda_1+\mu_1)(t-u)}$$

$$.\,(i+1)\beta^{(i+1)} \frac{I_{i+1}(\propto u)}{u} e^{(\lambda_1+\lambda_2+\mu_1)u} du] \frac{(\lambda_2)^j}{j!}, i,j \geq 1.$$

The following figure illustrated the transition probabilities according to various values of the indexed numbers *i* and *j*, (*i* denoted number of customers in the first class and *j* denoted the number of customers in the second class).

It is obvious from the figure that as time goes to infinity, the system model behavior expresses the steady state case, which will be the main interest in our future work.

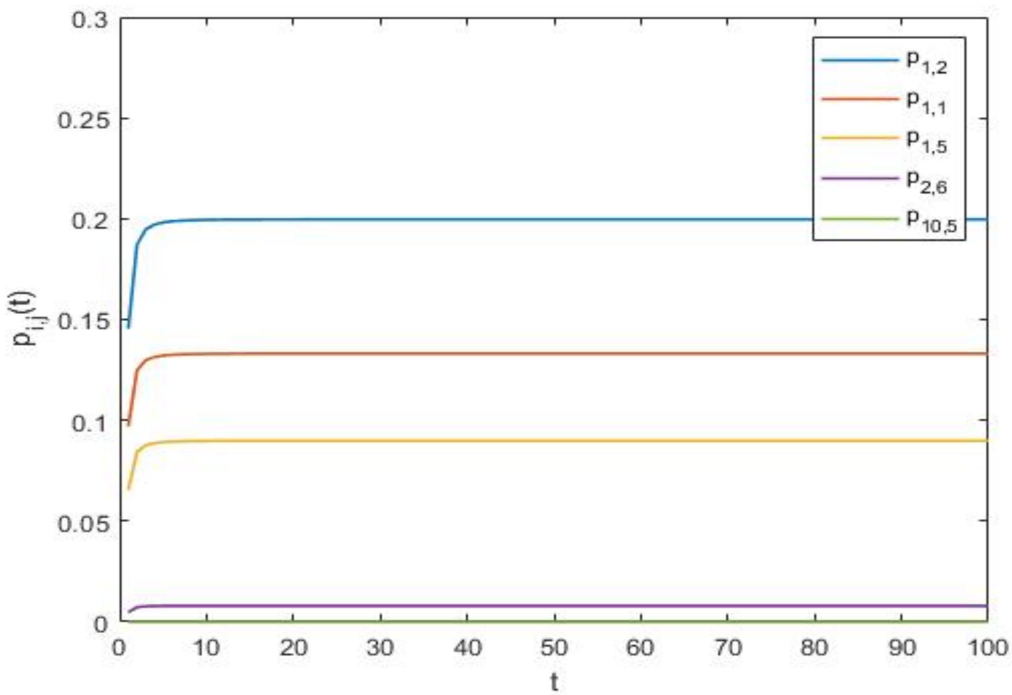

Fig. 2 the transition probabilities with various values of *i* and *j*

7 Conclusions and Future Work

Transient analysis of two-class priority queuing system are calculated, Various performance measures are examined like the marginal probabilities and the expectation for the high priority, some numerical analysis are carried on to explain the behavior of the transient probabilities.

Our future interest is to investigate the same system in the steady state case with some special cases.

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