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Residual and Past Entropies of Concomitants from Lai And Xie Extensions of Case-II of Generalized Order Statistics and its Dual

A. Gamal[1,*], *S. M. EL-Arishy*[2], *L. S. Diab* [2,3] *and Mohamed Said Mohamed*[4]

[1]Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt
[2]Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo, Egypt
[3]Department of Mathematics & Statistics, College of Science 323, King Abdullah City for Female Students, Imam Mohammad Ibn Saud Islamic University, Riyadh, Saudi Arabia
[4]Department of Mathematics, Faculty of Education, Ain Shams University, Cairo 11341, Egypt

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Abstract: In this article, we consider a new extensions of Morgenstern family is Lai and Xie extensions and discuss their concomitants for case-II of generalized order statistics and case-II of dual generalized order statistics. Additionally, recurrence relation between moments is found for the recommended models. We have also derived the expression for the joint distribution of concomitants for case-II of generalized order statistics and its dual. The residual and past entropies are shown last.

Keywords: Concomitants; Moments; Joint distribution.

1 Introduction

Kamps [1] provides generalized order statistics (GOS), which called as case-I of GOS. The second GOS model that Kamps and Cramer [2] is developed, case-II of GOS, in which the parameters are pairwise different. However, the concept of lower GOS was created by Pawlas and Szynal [3] and afterwards by Burkschat et al. [4]. Dual generalized order statistics (DGOS) is how they referred to it.

For $f \geq 1$, $q \in \mathbb{N}$, $z_1, \ldots, z_{q-1} \in \mathbb{R}$, $1 \leq r \leq q-1$, $Z_r = \sum_{j=r}^{q-1} z_j$, and let $\tilde{z} = (z_1, \ldots, z_{q-1})$, their are the following cases:
Case-II of GOS: If $\lambda_i \neq \lambda_j$, $i, j = 1, 2, \ldots, q$ and $i \neq j$, the pdf of $U_{(r,q,\tilde{z},f)}$ was introduced by [2] as follows:

$$g_{(r,q,\tilde{z},f)}(u) = m_{r-1} \sum_{i=1}^{r} a_i(r) \left(1 - G_U(u)\right)^{\lambda_i - 1} g_U(u), \tag{1}$$

where $\lambda_i = f + q - i + Z_i > 0$,

$$a_i(r) = \prod_{j=1, i \neq j}^{r} \frac{1}{\lambda_j - \lambda_i}, \lambda_j \neq \lambda_i, 1 \leq i \leq r \leq q,$$

and $m_{r-1} = \prod_{j=1}^{r} \lambda_j$. The joint pdf of $U_{(r,q,\tilde{z},f)}$ and $U_{(s,q,\tilde{z},f)}$ is given by:

$$\begin{aligned} g_{(r,s,q,\tilde{z},f)}(u_1,u_2) = m_{s-1} &\left[\sum_{l=1+r}^{s} a_l^{(r)}(s) \left(\frac{1 - G_U(u_2)}{1 - G_U(u_1)} \right)^{\lambda_l} \right] \\ &\times \left[\sum_{i=1}^{r} a_i(r) \left(1 - G_U(u_1)\right)^{\lambda_i} \right] \frac{g_U(u_1) g_U(u_2)}{(1 - G_U(u_1))(1 - G_U(u_2))}, \end{aligned} \tag{2}$$

* Corresponding author e-mail: agt11@fayoum.edu.eg

where $u_1 < u_2$, $1 \le r < s \le q$, $1 \le i \le q$,

$$a_l^{(r)}(s) = \prod_{j=1+r, l\neq j}^{s} \frac{1}{\lambda_j - \lambda_l}, \lambda_j \neq \lambda_l, r+1 \le l \le s \le q,$$

$$a_l(s) = a_l^{(0)}(s), 1 \le r \le q.$$

Case-II of DGOS: When $\lambda_i \neq \lambda_j$, $i, j = 1, 2, \ldots, q-1$, in this case, the pdf of $U_{d(r,q,\tilde{z},f)}$ is defined by, see [5]:

$$g_{d(r,q,\tilde{z},f)}(u) = m_{r-1} \sum_{i=1}^{r} a_i(r) \left(G_U(u)\right)^{\lambda_i - 1} g_U(u), \tag{3}$$

where $a_i(r) = \prod_{j=1, j\neq i}^{r} \frac{1}{\lambda_j - \lambda_i}$, $1 \le r \le q$ and $\lambda_i = f + q - i + Z_i > 0$. The joint pdf of $U_{d(r,q,\tilde{z},f)}$ and $U_{d(s,q,\tilde{z},f)}$ is given by:

$$g_{d(r,s,q,\tilde{z},f)}(u_1, u_2) = m_{s-1} \left[\sum_{l=1+r}^{s} a_l^{(r)}(s) \left(\frac{G_U(u_2)}{G_U(u_1)}\right)^{\lambda_l}\right] \left[\sum_{i=1}^{r} a_i(r) \left(G_U(u_1)\right)^{\lambda_i}\right] \frac{g_U(u_1) g_U(u_2)}{G_U(u_1) G_U(u_2)}, \tag{4}$$

where $u_1 < u_2$, $1 \le r < s \le q$, $1 \le i \le q$, $a_l^{(r)}(s) = \prod_{j=r+1, j\neq l}^{s} \frac{1}{\lambda_j - \lambda_l}$ and $a_l(s) = a_l^{(0)}(s)$.

We have the probability density function (pdf) and cdf of the concomitant of case-II GOS $P_{(r,q,\tilde{z},f)}$, $1 \le r \le q$, as:

$$g_{[r,q,\tilde{z},f]}(p) = \int_{-\infty}^{\infty} g_{(r,q,\tilde{z},f)}(u) g_{P|U}(p \mid u) du, \tag{5}$$

and

$$G_{[r,q,\tilde{z},f]}(p) = \int_{-\infty}^{\infty} g_{(r,q,\tilde{z},f)}(u) G_{P|U}(p \mid u) du, \tag{6}$$

where the pdf of $U_{(r,q,\tilde{z},f)}$ is $g_{(r,q,\tilde{z},f)}(u)$ described in (1).

We have pdf and cdf of the concomitant of case-II $DGOS$ $P_{d[r,q,\tilde{z},f]}$, $1 \le r \le q$, as:

$$g_{d[r,q,\tilde{z},f]}(p) = \int_{-\infty}^{\infty} g_{d(r,q,\tilde{z},f)}(u) g_{P|U}(p \mid u) du, \tag{7}$$

and

$$G_{d[r,q,\tilde{z},f]}(p) = \int_{-\infty}^{\infty} g_{d(r,q,\tilde{z},f)}(u) G_{P|U}(p \mid u) du, \tag{8}$$

where the pdf of $U_{d(r,q,\tilde{z},f)}$ is $g_{d(r,q,\tilde{z},f)}(u)$ described in (3).

Numerous works on the concomitants of the GOS and DGOS models may be found in the literature. Researchers like [6, 7, 8, 9, 10] are included in this group. On the other hand, Mohie El-Din et al. [11] who have researched the GOS concomitants from the Farlie- Gumbel-Morgenstern family (FGM) distributions where $\gamma_i \neq \gamma_j, i \neq j, i, j = 1, 2, \ldots, n-1$.

Ebrahimi [12] defined the uncertainty of residual lifetime distributions as follows:

$$\zeta(P;t) = \ln \overline{G}_P(t) - \frac{1}{\overline{G}_P(t)} \int_t^{\infty} g_P(p) \ln g_P(p) dp, \tag{9}$$

Di Crescenzo and Longobardi [13] introduced past entropy over $(0,t)$, where P denotes the lifetime of an item, defined as:

$$\overline{\zeta}(P;t) = \ln G_P(t) - \frac{1}{G_P(t)} \int_0^{t} g_P(p) \ln g_P(p) dp, \tag{10}$$

where $\frac{g_P(p)}{G_P(p)}$ is the reversed hazard rate of P.

A parameter μ, and the marginal distribution functions $G_U(u)$ and $G_P(p)$ is described by FGM. By adding further parameters, Lai and Xie [14] examined the bivariate FGM distribution as broader. They suggested pdf as

$$g(u,p) = g_U(u)g_P(p)(1+\mu[b-(b+a)G_U(u)][b-(b+a)G_P(p)]G_U(u)^{b-1}G_P(p)^{b-1}\overline{G}_U(u)^{a-1}\overline{G}_P(p)^{a-1}), \tag{11}$$

for $0\leq \mu \leq 1$, and $a,b \geq 1$. The cdf is given as

$$G(u,p) = G_U(u)G_P(p)+\mu G_U(u)^b G_P(p)^b \overline{G}_U(u)^a \overline{G}_P(p)^a. \tag{12}$$

According to [15], μ satisfying a wider range where

$$\min\{\frac{1}{[C^+(a,b)]^2}, \frac{1}{[C^-(a,b)]^2}\} \leq \mu \leq \frac{1}{C^+(a,b)C^-(a,b)},$$

where C^+ and C^- are functions of a and b.

Furthermore, the conditional pdf and cdf are:

$$g_{P|U}(p\mid u) = g_P(p)(1+\mu[b-(b+a)G_U(u)][b-(b+a)G_P(p)]G_U(u)^{b-1}G_P(p)^{b-1}\overline{G}_U(u)^{a-1}\overline{G}_P(p)^{a-1}), \tag{13}$$

$$G_{P|U}(p\mid u) = G_P(p)+\mu G_U(u)^{b-1}G_P(p)^b \overline{G}_U(u)^a \overline{G}_P(p)^a. \tag{14}$$

2 Concomitants of case-II GOS and its dual

The pdf and cdf for Lai and Xie extension of concomitants in case-II GOS and its dual are presented in the following theorems:

2.1 Case-II GOS:

Theorem 2.1. Utilizing (1), (13), (14) in (5) and (6), the pdf and cdf of the concomitant $P_{[r,q,\tilde{z},f]}$, of r-th case-II GOS from Lai and Xie extension are given as:

$$g_{[r,q,\tilde{z},f]}(p) = g_P(p)\left[1+\Omega^*_{[r,q,\tilde{z},f]}\mu\left(b-(b+a)G_P(p)\right)\overline{G}_P(p)^{a-1}G_P(p)^{b-1}\right], \tag{15}$$

$$G_{[r,q,\tilde{z},f]}(p) = G_P(p)\left[1+\tau^*_{[r,q,\tilde{z},f]}\mu G_P(p)^{b-1}\overline{G}_P(p)^a\right], \tag{16}$$

where

$$\Omega^*_{[r,q,\tilde{z},f]} = m_{r-1}\sum_{i=1}^{r} a_i(r)(\lambda_i+a-2)!\left[\frac{b!(b+a)}{(b+\lambda_i+a-1)!}-\frac{(b-1)!b}{(b+\lambda_i+a-2)!}\right],$$

and

$$\tau^*_{[r,q,\tilde{z},f]} = m_{r-1}\sum_{i=1}^{r} a_i(r)\frac{(\lambda_i+a-1)!(b-1)!}{(\lambda_i+a+b-1)!}.$$

Proof. From $g_{(r,q,\tilde{z},f)}(u)$ the pdf of case-II $GOS\ U_{[r,q,\tilde{z},f]}$ defined in (1) and (13), the pdf of the concomitant of r-th case-II GOS, $P_{[r,q,\tilde{z},f]}$, is:

$$\begin{aligned} g_{[r,q,\tilde{z},f]}(p) = {} & g_P(p)+\mu\left(b-(b+a)G_P(p)\right)G_P(p)^{b-1}g_P(p)\overline{G}_P(p)^{-1+a} \\ & \int_{-\infty}^{\infty} g_{(r,q,\tilde{z},f)}(u)\{bG_U(u)^{b-1}-(b+a)G_U(u)^b\}\overline{G}_U(u)^{-1+a}du \\ = {} & g_P(p)+\mu\left(b-(b+a)G_P(p)\right)G_P(p)^{b-1}g_P(p)\overline{G}_P(p)^{-1+a}m_{r-1}\sum_{i=1}^{r}a_i(r) \\ & \times\int_{-\infty}^{\infty}\{bG_U(u)^{b-1}-(b+a)G_U(u)^b\}\overline{G}_U(u)^{\lambda_i-2+a}g_U(u)du \\ & let\ \ x=\overline{G}_U(u),\ \ then\ \ we\ \ have \\ = {} & g_P(p)+\mu\left(b-(b+a)G_P(p)\right)G_P(p)^{b-1}g_P(p)\overline{G}_P(p)^{-1+a}m_{r-1}\sum_{i=1}^{r}a_i(r) \\ & \times\int_0^1\{(b+a)x^{\lambda_i+a-2}(-x+1)^b-bx^{\lambda_i-2+a}(-x+1)^{b-1}\}dx. \end{aligned}$$

We can prove cdf of case-II GOS in the same way.

2.2 Case-II DGOS:

Theorem 2.2. Utilizing (3), (13),(14) in (7) and (8), the pdf and cdf of the concomitant $P_{[r,q,\tilde{z},f]}$, of r-th case-II $DGOS$ for Lai and Xie extension are given as:

$$g_{d[r,q,\tilde{z},f]}(p)=g_P(p)\left[1+\Omega^*_{d[r,q,\tilde{z},f]}\mu\left(b-(b+a)G_P(p)\right)\overline{G}_P(p)^{a-1}G_P(p)^{b-1}\right], \tag{17}$$

$$G_{d[r,q,\tilde{z},f]}(p)=G_P(p)\left[1+\tau^*_{d[r,q,\tilde{z},f]}\mu G_P(p)^{b-1}\overline{G}_P(p)^a\right], \tag{18}$$

where

$$\Omega^*_{d[r,q,\tilde{z},f]}=-am_{r-1}\sum_{i=1}^{r}a_i(r)\sum_{\varepsilon=1}^{a-1}\frac{(-1)^{\varepsilon}\binom{a-1}{\varepsilon}}{(\lambda_i+\varepsilon+b-1)}, \tag{19}$$

and

$$\tau^*_{d[r,q,\tilde{z},f]}=m_{r-1}\sum_{i=1}^{r}a_i(r)\sum_{\varepsilon=1}^{a}\frac{(-1)^{\varepsilon}\binom{a}{\varepsilon}}{(\lambda_i+\varepsilon+b-1)}. \tag{20}$$

3 Moment of Concomitants for case-II GOS and its dual

3.1 Case-II GOS:

From the results of the last part, we can write the pdf of Case-II GOS, $P_{[r,q,\tilde{z},f]}$ as follows

$$\begin{aligned} g_{[r,q,\tilde{z},f]}(p) &= g_P(p)\left[1+\Omega^*_{[r,q,\tilde{z},f]}\mu\,(b-(b+a)G_P(p))\,G_P(p)^{b-1}\overline{G}_P(p)^{a-1}\right] \\ &= g_P(p)\left[1+\Omega^*_{[r,q,\tilde{z},f]}\mu\,(b-(b+a)G_P(p))\sum_{j=0}^{a-1}\binom{a-1}{j}(-1)^j G_P(p)^j G_P(p)^{b-1}\right] \\ &= g_P(p)+b\Omega^*_{[r,q,\tilde{z},f]}\mu\sum_{j=0}^{a-1}\rho_1(j+b)G_P(p)^{j+b-1}g_P(p) \\ &\quad -(a+b)\Omega^*_{[r,q,\tilde{z},f]}\mu\sum_{j=0}^{a-1}\rho_2(j+b+1)G_P(p)^{j+b}g_P(p) \\ &= g_P(p)+\Omega^*_{[r,q,\tilde{z},f]}\mu\left[b\sum_{j=0}^{a-1}\rho_1 g_{V_1}(p)-(a+b)\sum_{j=0}^{a-1}\rho_2 g_{V_2}(p)\right], \end{aligned} \tag{21}$$

where

$$\rho_1=\frac{(-1)^j\binom{-1+a}{j}}{b+j},\rho_2=\frac{(-1)^j\binom{-1+a}{j}}{b+1+j}, \tag{22}$$

$$g_{V_1}(p)=(j+b)g_P(p)G_P(p)^{b-1+j}, g_{V_2}(p)=(b+1+j)g_P(p)G_P(p)^{j+b}, \tag{23}$$

$$V_1\sim G_P(p)^{j+b-1}, V_2\sim G_P(p)^{j+b}, \tag{24}$$

Hence, the moment generating function (mgf) of $P_{[r,q,\tilde{z},f]}$ is

$$M_{[r,q,\tilde{z},f]}(t)=M_P(t)+\Omega^*_{[r,q,\tilde{z},f]}\mu\left[b\sum_{j=0}^{a-1}\rho_1 M_{V_1}(t)-(a+b)\sum_{j=0}^{a-1}\rho_2 M_{V_2}(t)\right]. \tag{25}$$

Theorem 3.1. For $2\le r\le q$, the recurrence relation for two moments from Lai and Xie extension of concomitants of case-II GOS given as:

$$\begin{aligned} M_{[r,q,\tilde{z},f]}(t)-M_{[r-1,q,\tilde{z},f]}(t) &= m_{r-2}\mu\left[b\sum_{j=0}^{a-1}\rho_1 M_{V_1}(t)-(a+b)\sum_{j=0}^{a-1}\rho_2 M_{V_2}(t)\right] \\ &\quad\times\left[\sum_{i=0}^{r}\lambda_i a_i(r)(\lambda_i+a-2)!\{\frac{b!(a+b)}{(\lambda_i+a-1+b)!}-\frac{(b-1)!b}{(\lambda_i+a-2+b)!}\}\right]. \end{aligned} \tag{26}$$

Proof. Using the following sentence $m_{r-2}=\frac{m_{r-1}}{\lambda_r}$, $(\lambda_r-\lambda_i)\,a_i(r)=a_i(r-1)$.

$$\begin{aligned} \Omega^*_{[r,q,\tilde{z},f]}-\Omega^*_{[r-1,q,\tilde{z},f]} &= m_{r-1}\sum_{i=1}^{r}a_i(r)(\lambda_i+a-2)!\{\frac{b!(a+b)}{(b+\lambda_i+a-1)!}-\frac{(b-1)!b}{(b+\lambda_i+a-2)!}\} \\ &\quad -m_{r-2}\sum_{i=1}^{-1+r}a_i(-1+r)(\lambda_i+a-2)!\{\frac{b!(a+b)}{(b+\lambda_i+a-1)!}-\frac{(b-1)!b}{(b+\lambda_i+a-2)!}\} \\ &= m_{r-1}a_r(r)(\lambda_r+a-2)!\{\frac{b!(a+b)}{(b+\lambda_r+a-1)!}-\frac{(b-1)!b}{(b+\lambda_r+a-2)!}\} \\ &\quad +m_{r-2}\sum_{i=1}^{r-1}\lambda_i a_i(r-1)(\lambda_i+a-2)!\{\frac{b!(a+b)}{(b+\lambda_i+a-1)!}-\frac{(b-1)!b}{(b+\lambda_i+a-2)!}\}. \end{aligned}$$

3.2 Case-II DGOS:

From case-II *DGOS* and Lai and Xie extension ,we can write the pdf of $P_{[r,q,\tilde{z},f]}$ as:

$$g_{d[r,q,\tilde{z},f]}(p) = g_P(p) + \Omega^*_{d[r,q,\tilde{z},f]}\mu\left[b\sum_{j=0}^{a-1}\rho_1 g_{V_1}(p) - (a+b)\sum_{j=0}^{a-1}\rho_2 g_{V_2}(p)\right], \tag{27}$$

where $\rho_1, \rho_2, g_{V_1}(p), g_{V_2}(p), V_1$ and V_2 are defined in (22),(23) and (24)
Moreover, we get mgf of $P_{[r,q,\tilde{z},f]}$ as:

$$M_{[r,q,\tilde{z},f]}(t) = M_P(t) + \Omega^*_{d[r,q,\tilde{z},f]}\mu\left[b\sum_{j=0}^{a-1}\rho_1 M_{V_1}(t) - (a+b)\sum_{j=0}^{a-1}\rho_2 M_{V_2}(t)\right]. \tag{28}$$

Theorem 3.2. For $2 \leq r \leq q$, and recurrence relation for two moments from Lai and Xie extension of concomitants of case-II *DGOS* given as:

$$\begin{aligned} M_{[r,q,\tilde{z},f]}(t) - M_{[r-1,q,\tilde{z},f]}(t) &= m_{r-2}\mu\left[b\sum_{j=0}^{a-1}\rho_1 M_{V_1}(t) - (a+b)\sum_{j=0}^{a-1}\rho_2 M_{V_2}(t)\right] \\ &\times\sum_{\varepsilon=0}^{a-1}\binom{a-1}{\varepsilon}(-1)^{\varepsilon}\sum_{i=0}^{r}\frac{\lambda_i a_i(r)}{\lambda_i+\varepsilon+b-1}. \end{aligned} \tag{29}$$

4 Joint Distribution of Two Concomitants

4.1 Case-II GOS:

We can write the joint pdf of $P_{[r,q,\tilde{z},f]}$ and $P_{[s,q,\tilde{z},f]}$, $r < s$, as:

$$\begin{aligned} g_{(r,s,q,\tilde{z},f)}(p_1,p_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{u_2} g_{(r,q,\tilde{z},f)}(u_1,u_2) g_{P|U}(p_1|u_1) g_{P|U}(p_2|u_2)du_1du_2 \\ &= g_P(p_1)g_P(p_2)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2}\{1+\mu^2 D_{P_1}D_{P_2}[b^2(G_U(u_1)G_U(u_2))^{b-1}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1} \\ &\quad - b(a+b)G_U(u_1)^b G_U(u_2)^{b-1}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1} \\ &\quad - b(a+b)G_U(u_1)^{b-1}G_U(u_2)^b(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1} \\ &\quad + (a+b)^2 G_U(u_1)^b G_U(u_2)^b(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1}] \\ &\quad + \mu D_{P_1}[bG_U(u_1)^{b-1}\overline{G}_U(u_1)^{a-1} - (a+b)G_U(u_1)^b\overline{G}_U(u_1)^{a-1}] \\ &\quad + \mu D_{P_2}[bG_U(u_2)^{b-1}\overline{G}_U(u_2)^{a-1} - (a+b)G_U(u_2)^b\overline{G}_U(u_2)^{a-1}]\} \\ &\quad \times\sum_{l=1+r}^{s}a_l^{(r)}(s)\sum_{i=1}^{r}a_i(r)\overline{G}_U(u_1)^{\lambda_i-\lambda_l-1}\overline{G}_U(u_2)^{\lambda_l-1}g_U(u_1)g_U(u_2)du_1du_2 \end{aligned} \tag{30}$$

where

$$\begin{aligned} D_{P_1} &= (b-(a+b)G_P(p_1))G_P(p_1)^{b-1}\overline{G}_P(p_1)^{a-1}, \\ D_{P_2} &= (b-(a+b)G_P(p_2))G_P(p_2)^{b-1}\overline{G}_P(p_2)^{a-1}. \end{aligned} \tag{31}$$

Hence, we have

$$J_1 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty}\int_{-\infty}^{u_2} (G_U(u_1)G_U(u_2))^{b-1}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1}$$
$$\times (1-G_U(u_1))^{\lambda_i-\lambda_l-1}(1-G_U(u_2))^{\lambda_l-1} g_U(u_1)g_U(u_2)du_1du_2$$
$$= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{\lambda_i+a-2-\lambda_l} \binom{\lambda_i-\lambda_l+a-2}{j_1}\frac{(-1)^{j_1}}{b+j_1}$$
$$\times \int_{-\infty}^{\infty} G_U(u_2)^{j_1+2b-1}(1-G_U(u_2))^{\lambda_l+a-2} g_U(u_2)du_2$$
$$= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{\lambda_i+a-2-\lambda_l} \binom{\lambda_i-\lambda_l+a-2}{j_1}\frac{(-1)^{j_1}}{b+j_1}\beta(\lambda_l+a-1, 2b+j_1),$$

$$J_2 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G_U(u_1)^b G_U(u_2)^{b-1}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1}$$
$$\times (1-G_U(u_1))^{\lambda_i-\lambda_l-1}(1-G_U(u_2))^{\lambda_l-1} g_U(u_1)g_U(u_2)du_1du_2$$
$$= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{\lambda_i+a-2-\lambda_l} \binom{\lambda_i-\lambda_l+a-2}{j_1}\frac{(-1)^{j_1}}{b+j_1+1}\beta(\lambda_l+a-1, 2b+j_1+1),$$

$$J_3 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G_U(u_1)^{b-1} G_U(u_2)^{b}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1}$$
$$\times (1-G_U(u_1))^{\lambda_i-\lambda_l-1}(1-G_U(u_2))^{\lambda_l-1} g_U(u_1)g_U(u_2)du_1du_2$$
$$= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{\lambda_i+a-2-\lambda_l} \binom{\lambda_i-\lambda_l+a-2}{j_1}\frac{(-1)^{j_1}}{b+j_1}\beta(\lambda_l+a-1, 2b+j_1+1),$$

$$J_4 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G_U(u_1)^{b} G_U(u_2)^{b}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1}$$
$$\times (1-G_U(u_1))^{\lambda_i-\lambda_l-1}(1-G_U(u_2))^{\lambda_l-1} g_U(u_1)g_U(u_2)du_1du_2$$
$$= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{\lambda_i+a-2-\lambda_l} \binom{\lambda_i-\lambda_l+a-2}{j_1}\frac{(-1)^{j_1}}{b+j_1-1}\beta(\lambda_l+a-1, 2b+j_1+2),$$

$$J_5 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G_U(u_1)^{b-1}\overline{G}_U(u_1)^{a-1}$$
$$\times (1-G_U(u_1))^{\lambda_i-\lambda_l-1}(1-G_U(u_2))^{\lambda_l-1} g_U(u_1)g_U(u_2)du_1du_2$$
$$= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{\lambda_i+a-2-\lambda_l} \binom{\lambda_i-\lambda_l+a-2}{j_1}\frac{(-1)^{j_1}}{b+j_1}\beta(\lambda_l, b+j_1+1),$$

$$J_6 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G_U(u_1)^{b}\overline{G}_U(u_1)^{a-1}$$
$$\times (1-G_U(u_1))^{\lambda_i-\lambda_l-1}(1-G_U(u_2))^{\lambda_l-1} g_U(u_1)g_U(u_2)du_1du_2$$
$$= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{\lambda_i+a-2-\lambda_l} \binom{\lambda_i-\lambda_l+a-2}{j_1}\frac{(-1)^{j_1}}{b+j_1+1}\beta(\lambda_l, b+j_1+1),$$

$$J_7 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G_U(u_2)^{b-1}\overline{G}_U(u_2)^{a-1}$$
$$\times (1-G_U(u_1))^{\lambda_i-\lambda_l-1}(1-G_U(u_2))^{\lambda_l-1} g_U(u_1)g_U(u_2)du_1du_2$$
$$= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\frac{1}{\lambda_i-\lambda_l}(\beta(\lambda_i-1+a, b)-\beta(\lambda_l-1+a, b)),$$

$$J_8 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G_U(u_2)^b \overline{G}_U(u_2)^{a-1} \times (1-G_U(u_1))^{\lambda_i-\lambda_l-1}(1-G_U(u_2))^{\lambda_l-1} g_U(u_1)g_U(u_2)du_1du_2 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\frac{1}{\lambda_i-\lambda_l}(\beta(\lambda_i-1+a,1+b)-\beta(\lambda_l-1+a,1+b)).$$

From the previous results, we have pdf as:

$$\begin{aligned} g_{(r,s,q,\tilde{z},f)}(p_1,p_2) &= g_P(p_1)g_P(p_2)\{1+\mu^2 D_{P_1}D_{P_2}[bJ_1-b(b+a)J_2-b(b+a)J_3+(b+a)^2J_4] \\ &\quad +\mu D_{P_1}[bJ_5-(b+a)J_6]+\mu D_{P_2}[bJ_7-(b+a)J_8]\} \\ &= g_P(p_1)g_P(p_2)+\eta_1\eta_2\mu^2[bJ_1-b(b+a)(J_2+J_3)+(b+a)^2J_4] \\ &\quad +\mu\eta_1 g_P(p_2)[bJ_5-(b+a)J_6]+\mu\eta_2 g_P(p_1)[bJ_7-(b+a)J_8], \end{aligned} \tag{32}$$

where

$$\eta_1 = b\Sigma_{j=0}^{a-1}\rho_1 g_{V_1}(p_1)-(a+b)\Sigma_{j=0}^{a-1}\rho_2 g_{V_2}(p_1), \eta_2 = b\Sigma_{j=0}^{a-1}\rho_1 g_{V_1}(p_2)-(a+b)\Sigma_{j=0}^{a-1}\rho_2 g_{V_2}(p_2), \tag{33}$$

$\rho_1,\rho_2,g_{V_1}(p),g_{V_2}(p),V_1$ and V_2 are defined in (22),(23) and (24).
Let concomitants $V_{[r,q,\tilde{z},f]}$ and $V_{[s,q,\tilde{z},f]}$, $r<s$, then we can write cdf as:

$$\begin{aligned} G_{(r,s,q,\tilde{z},f)}(p_1,p_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{u_2} g_{(r,s,q,\tilde{z},f)}(u_1,u_2)G_{P|U}(p_1|u_1)G_{P|U}(p_2|u_2)du_1du_2 \\ &= G_P(p_1)G_P(p_2)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2}\{1+\alpha^2 D_{P_1}^{*}D_{P_2}^{*}(G_U(u_1)G_U(u_2))^{b-1}(\overline{G}_U(u_1)\overline{G}_U(u_2))^a \\ &\quad +\mu D_{P_1}^{*}G_U(u_1)^{b-1}\overline{G}_U(u_1)^a+\mu D_{P_2}^{*}G_U(u_2)^{b-1}\overline{G}_U(u_2)^a\} \\ &\quad \times\sum_{l=1+r}^{s}a_l^{(r)}(s)\sum_{i=1}^{r}a_i(r)\overline{G}_U(u_1)^{\gamma_i-\gamma_l-1}\overline{G}_U(u_2)^{\gamma_l-1}g_U(u_1)g_U(u_2)du_1du_2, \end{aligned} \tag{34}$$

where

$$\begin{aligned} D_{P_1}^{*} &= G_P(p_1)^{b-1}\overline{G}_P(p_1)^a, \\ D_{P_2}^{*} &= G_P(p_2)^{b-1}\overline{G}_P(p_2)^a. \end{aligned} \tag{35}$$

Hence, we have

$$J_1^{*} = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2}(G_U(u_1)G_U(u_2))^{b-1}(\overline{G}_U(u_1)\overline{G}_U(u_2))^a \times(1-G_U(u_1))^{\lambda_i-\lambda_l-1}(1-G_U(u_2))^{\lambda_l-1}g_U(u_1)g_U(u_2)du_1du_2 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{\lambda_i+a-1-\lambda_l}\binom{\lambda_i-\lambda_l+a-1}{j_1}\frac{(-1)^{j_1}}{b+j_1}\beta(\gamma_l+a,b+j_1+1),$$

$$J_2^{*} = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2}G_U(u_1)^{b-1}\overline{G}_U(u_1)^a \times(1-G_U(u_1))^{\lambda_i-\lambda_l-1}(1-G_U(u_2))^{\lambda_l-1}g_U(u_1)g_U(u_2)du_1du_2 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{\lambda_i+a-1-\lambda_l}\binom{\lambda_i-\lambda_l+a-1}{j_1}\frac{(-1)^{j_1}}{b+j_1}\beta(\lambda_l,b+j_1+1),$$

$$J_3^{*} = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2}G_U(u_2)^{b-1}\overline{G}_U(u_2)^a \times(1-G_U(u_1))^{\lambda_i-\lambda_l-1}(1-G_U(u_2))^{\lambda_l-1}g_U(u_1)g_U(u_2)du_1du_2 = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\frac{1}{\lambda_i-\lambda_l}(\beta(\lambda_i+a,b)-\beta(\lambda_l+a,b)).$$

Then we have cdf as:

$$G_{(r,s,q,\tilde{z},f)}(p_1,p_2) = G_P(p_1)G_P(p_2)\{1+\mu^2 D^*_{P_1} D^*_{P_2} J^*_1 + \mu D^*_{P_1} J^*_2 + \mu D^*_{P_2} J^*_3\}. \tag{36}$$

The product moment of $P_{[r,q,\tilde{z},f]}, P_{[s,q,\tilde{z},f]}$ as $M_{[r,s,q,\tilde{z},f]}(t_1,t_2)$ is simply acquired from (32) as:

$$\begin{aligned} M_{[r,s,q,\tilde{z},f]}(t_1,t_2) &= M_{P_1}(t_1)M_{P_2}(t_2) + \mu^2\eta^*_1\eta^*_2[bJ_1 - b(a+b)(J_2+J_3)+(a+b)^2 J_4]) \\ &\quad + \mu\eta^*_1 M_{P_2}(t_2)[bJ_5-(a+b)J_6] + \mu\eta^*_2 M_{P_1}(t)[bJ_7-(a+b)J_8], \end{aligned} \tag{37}$$

where

$$\eta^*_1 = b\Sigma_{j=0}^{-1+a}\rho_1 M_{V_1 p_1}(t_1) - (b+a)\Sigma_{j=0}^{-1+a}\rho_2 M_{V_2 p_1}(t_1), \eta^*_2 = b\Sigma_{j=0}^{-1+a}\rho_1 M_{V_1 p_2}(t_2) - (b+a)\Sigma_{j=0}^{-1+a}\rho_2 M_{V_2 p_2}(t_2), \tag{38}$$

where ρ_1, ρ_2, V_1 and V_2 are defined in (22) and (24).

4.2 *Case-II of DGOS:*

We can write the joint pdf of $P_{[r,q,\tilde{z},f]}$ and $P_{[s,q,\tilde{z},f]}$, $r<s$, as:

$$\begin{aligned} g_{d(r,s,q,\tilde{z},f)}(p_1,p_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{u_2} g_{d(r,s,q,\tilde{z},f)}(u_1,u_2) g_{P|U}(p_1|u_1) g_{P|U}(p_2|u_2) du_1 du_2 \\ &= g_P(p_1)g_P(p_2)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2}\{1+\mu^2 D_{P_1}D_{P_2}[b^2(G_U(u_1)G_U(u_2))^{b-1}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1} \\ &\quad - b(a+b)G_U(u_1)^b G_U(u_2)^{b-1}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1} \\ &\quad - b(a+b)G_U(u_1)^{b-1} G_U(u_2)^{b}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1} \\ &\quad + (a+b)^2(G_U(u_1)G_U(u_2))^b(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1}] \\ &\quad + \mu D_{P_1}[bG_U(u_1)^{b-1}\overline{G}_U(u_1)^{a-1} - (a+b)G_U(u_1)^b\overline{G}_U(u_1)^{a-1}] \\ &\quad + \mu D_{P_2}[bG_U(u_2)^{b-1}\overline{G}_U(u_2)^{a-1} - (a+b)G_U(u_2)^b\overline{G}_U(u_2)^{a-1}]\} \\ &\quad \times \sum_{l=1+r}^{s} a_l^{(r)}(s)\sum_{i=1}^{r} a_i(r) G_U(u_1)^{\lambda_i-\lambda_l-1} G_U(u_2)^{\lambda_l-1} g_U(u_1) g_U(u_2) du_1 du_2, \end{aligned} \tag{39}$$

where $D_{P_1} and D_{P_2}$ are defined in (31).

Hence, we have

$$\begin{aligned} J_{d_1} &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2}(G_U(u_1)G_U(u_2))^{b-1}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1} \\ &\quad G_U(u_1)^{\lambda_i-\lambda_l-1}G_U(u_2)^{\lambda_l-1}g_U(u_1)g_U(u_2)du_1du_2 \\ &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{-1+a}\binom{a-1}{j_1}\frac{(-1)^{j_1}}{b+j_1+\lambda_i-\lambda_l-1}\beta(a,2b+j_1+\gamma_i-2), \end{aligned}$$

$$\begin{aligned} J_{d_2} &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G_U(u_1)^b G_U(u_2)^{b-1}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1} \\ &\quad \times G_U(u_1)^{\lambda_i-\lambda_l-1}G_U(u_2)^{\lambda_l-1}g_U(u_1)g_U(u_2)du_1du_2 \\ &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{-1+a}\binom{a-1}{j_1}\frac{(-1)^{j_1}}{b+j_1+\lambda_i-\lambda_l}\beta(a,2b+j_1+\lambda_i-1), \end{aligned}$$

$$\begin{aligned} J_{d_3} &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G_U(u_1)^{b-1} G_U(u_2)^{b}(\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1} \\ &\quad \times G_U(u_1)^{\lambda_i-\lambda_l-1}G_U(u_2)^{\lambda_l-1}g_U(u_1)g_U(u_2)du_1du_2 \\ &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{-1+a}\binom{a-1}{j_1}\frac{(-1)^{j_1}}{b+j_1+\lambda_i-\lambda_l-1}\beta(a,2b+j_1+\lambda_i-1), \end{aligned}$$

$$
\begin{aligned}
J_{d_4} &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty} \int_{-\infty}^{u_2} (G_U(u_1)G_U(u_2))^b (\overline{G}_U(u_1)\overline{G}_U(u_2))^{a-1} \\
&\quad \times G_U(u_1)^{\lambda_i-\lambda_l-1} G_U(u_2)^{\lambda_l-1} g_U(u_1) g_U(u_2) du_1 du_2 \\
&= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \Sigma_{j_1=0}^{-1+a} \binom{a-1}{j_1} \frac{(-1)^{j_1}}{b+j_1+\lambda_i-\lambda_l} \beta(a, 2b+j_1+\lambda_i),
\end{aligned}
$$

$$
\begin{aligned}
J_{d_5} &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty} \int_{-\infty}^{u_2} G_U(u_1)^{b-1} \overline{G}_U(u_1)^{a-1} G_U(u_1)^{\lambda_i-\lambda_l-1} G_U(u_2)^{\lambda_l-1} g_U(u_1) g_U(u_2) du_1 du_2 \\
&= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \Sigma_{j_1=0}^{-1+a} \binom{a-1}{j_1} \frac{(-1)^{j_1}}{(b+j_1+\lambda_i-\lambda_l-1)(\lambda_i+b+j_1-1)},
\end{aligned}
$$

$$
\begin{aligned}
J_{d_6} &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty} \int_{-\infty}^{u_1} G_U(u_1)^{b} \overline{G}_U(u_1)^{a-1} G_U(u_1)^{\lambda_i-\lambda_l-1} G_U(u_2)^{\lambda_l-1} g_U(u_1) g_U(u_2) du_1 du_2 \\
&= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \Sigma_{j_1=0}^{-1+a} \binom{a-1}{j_1} \frac{(-1)^{j_1}}{(b+j_1+\lambda_i-\lambda_l)(\lambda_i+b+j_1)},
\end{aligned}
$$

$$
\begin{aligned}
J_{d_7} &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty} \int_{-\infty}^{u_2} G_U(u_2)^{b-1} \overline{G}_U(u_2)^{a-1} G_U(u_1)^{\lambda_i-\lambda_l-1} G_U(u_2)^{\lambda_l-1} g_U(u_1) g_U(u_2) du_1 du_2 \\
&= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \frac{1}{\lambda_i-\lambda_l} \beta(a, \lambda_i+b-1),
\end{aligned}
$$

$$
\begin{aligned}
J_{d_8} &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty} \int_{-\infty}^{u_2} G_U(u_2)^{b} \overline{G}_U(u_2)^{a-1} G_U(u_1)^{\lambda_i-\lambda_l-1} G_U(u_2)^{\lambda_l-1} g_U(u_1) g_U(u_2) du_1 du_2 \\
&= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \frac{1}{\lambda_i-\lambda_l} \beta(a, \lambda_i+b).
\end{aligned}
$$

Then we can write pdf as:

$$
\begin{aligned}
g_{d(r,s,q,\tilde{z},f)}(p_1,p_2) &= g_P(p_1) g_P(p_2) \{1 + \mu^2 D_{P_1} D_{P_2} [bJ_{d_1} - b(b+a)J_{d_2} - b(b+a)J_{d_3} + (b+a)^2 J_{d_4}] \\
&\quad + \mu D_{P_1} [bJ_{d_5} - (b+a)J_{d_6}] + \mu D_{P_2} [bJ_{d_7} - (b+a)J_{d_8}]\} \\
&= g_P(p_1) g_P(p_2) + \eta_1 \eta_2 \mu^2 [bJ_{d_1} - b(b+a)(J_{d_2} + J_{d_3}) + (b+a)^2 J_{d_4}] \\
&\quad + \mu \eta_1 g_P(p_2) [bJ_{d_5} - (b+a)J_{d_6}] + \mu \eta_2 g_P(p_1) [bJ_{d_7} - (b+a)J_{d_8}],
\end{aligned}
\tag{40}
$$

where η_1, η_2 are defined in (33).
Let concomitants $P_{[r,q,\tilde{z},f]}$ and $P_{[s,q,\tilde{z},f]}$, $r < s$, then we can write cdf as:

$$
\begin{aligned}
G_{d(r,s,q,\tilde{z},f)}(p_1,p_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{u_2} g_{d(r,s,q,\tilde{z},f)}(u_1,u_2) G_{P|U}(p_1|u_1) G_{P|U}(p_2|u_2) du_1 du_2 \\
&= G_P(p_1) G_P(p_2) \int_{-\infty}^{\infty} \int_{-\infty}^{u_2} \{1 + \mu^2 D^*_{P_1} D^*_{P_2} (G_U(u_1)G_U(u_2))^{b-1} (\overline{G}_U(u_1)\overline{G}_U(u_2))^a \\
&\quad + \mu D^*_{P_1} G_U(u_1)^{b-1} \overline{G}_U(u_1)^a + \mu D^*_{P_2} G_U(u_2)^{b-1} \overline{G}_U(u_2)^a\} \\
&\quad \times \sum_{l=1+r}^{s} a_l^{(r)}(s) \sum_{i=1}^{r} a_i(r) G_U(u_1)^{\lambda_i-\lambda_l-1} G_U(u_2)^{\lambda_l-1} g_U(u_1) g_U(u_2) du_1 du_2
\end{aligned}
\tag{41}
$$

where $D^*_{P_1} and D^*_{P_2}$ are defined in (35). Hence, we have

$$
\begin{aligned}
J^*_{d_1} &= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \int_{-\infty}^{\infty} \int_{-\infty}^{u_2} (G_U(u_1)G_U(u_2))^{b-1} (\overline{G}_U(u_1)\overline{G}_U(u_2))^a \\
&\quad \times G_U(u_1)^{\lambda_i-\lambda_l-1} G_U(u_2)^{\lambda_l-1} g_U(u_1) g_U(u_2) du_1 du_2 \\
&= \Sigma_{l=1+r}^{s} a_l^{(r)}(s) \Sigma_{i=1}^{r} a_i(r) \Sigma_{j_1=0}^{a-1} \binom{a-1}{j_1} \frac{(-1)^{j_1}}{b+j_1+\lambda_i-\lambda_l-1} \beta(a+1, 2b+j_1+\lambda_i-2),
\end{aligned}
$$

$$J_{d_2}^* = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G_U(u_1)^{b-1}\overline{G}_U(u_1)^a G_U(u_1)^{\lambda_i-\lambda_l-1} G_U(u_2)^{\lambda_l-1} g_U(u_1) g_U(u_2) du_1 du_2$$
$$= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\Sigma_{j_1=0}^{a}\binom{a}{j_1}\frac{(-1)^{j_1}}{(b+j_1+\lambda_i-\lambda_l-1)(\lambda_i+b+j_1-1)},$$

$$J_{d_3}^* = \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G_U(u_2)^{b-1}\overline{G}_U(u_2)^a G_U(u_1)^{\lambda_i-\lambda_l-1} G_U(u_2)^{\lambda_l-1} g_U(u_1) g_U(u_2) du_1 du_2$$
$$= \Sigma_{l=1+r}^{s} a_l^{(r)}(s)\Sigma_{i=1}^{r} a_i(r)\frac{1}{\lambda_i-\lambda_l}\beta(a+1,\lambda_i-1+b).$$

Then we can write cdf as:

$$G_{d(r,s,q,\tilde{z},f)}(p_1,p_2) = G_P(p_1)G_P(p_2)\{1+\mu^2 D_{P_1}^* D_{P_2}^* J_{d_1}^* + \mu D_{P_1}^* J_{d_2}^* + \mu D_{P_2}^* J_{d_3}^*\}. \tag{42}$$

The product moment of $P_{[r,q,\tilde{z},f]}, P_{[s,q,\tilde{z},f]}$ as $M_{[r,s,q,\tilde{z},f]}(t_1,t_2)$ is simply acquired from (40) as:

$$M_{[r,s,q,\tilde{z},f]}(t_1,t_2) = M_{P_1}(t_1)M_{P_2}(t_2) + \mu^2\eta_1^*\eta_2^*[bJ_{d_1} - b(a+b)(J_{d_2}+J_{d_3}) + (a+b)^2 J_{d_4}])$$
$$+\mu\eta_1^* M_{P_2}(t_2)[bJ_{d_5} - (a+b)J_{d_6}] + \mu\eta_2^* M_{P_1}(t_1)[aJ_{d_7} - (a+b)J_{d_8}],$$

where η_1^*, η_2^* are defined in (38).

5 Residual and past entropies

An explicit form of the residual and past entropies are get in the coming theorems.

Theorem 5.1. From (9) and (15), then an explicit form of the residual entropy of $P_{[r,q,\tilde{z},f]}$, is:

$$\zeta(P_{[r,q,\tilde{z},f]};t) = \ln\overline{G}_{[r,q,\tilde{z},f]}(t) - \frac{1}{\overline{G}_{[r,q,\tilde{z},f]}(t)}[[\overline{G}_P(t)(\ln\overline{G}_P(t) - \zeta(p;t))] + \mu\Omega_{[r,q,\tilde{z},f]}^*$$
$$[b\psi_{g_1}(p) - (a+b)\psi_{g_2}(p)] + Q(r,\mu,q,\tilde{z},f)], \tag{43}$$

where the residual entropy for P is $\zeta(p;t)$

$$\zeta(p;t) = \ln\overline{G}_P(t) - \frac{1}{\overline{G}_P(t)}\int_t^{\infty} g_P(p)\ln g_P(p)\mathrm{d}p,$$

$$\psi_{g_1}(p) = \int_t^{\infty} g_P(p)G_P(p)^{b-1}\overline{G}_P(p)^{a-1}\ln g_P(p)\mathrm{d}p,$$

and

$$\psi_{g_2}(p) = \int_t^{\infty} g_P(p)G_P(p)^{b}\overline{G}_P(p)^{a-1}\ln g_P(p)\mathrm{d}p,$$

Proof. From (9) and (15) then we get:

$$\zeta(P_{[r,q,\tilde{z},f]};t) = \ln\overline{G}_{[r,q,\tilde{z},f]}(t) - \frac{1}{\overline{G}_{[r,q,\tilde{z},f]}(t)}[\int_t^{\infty} g_P(p)(1+\mu\Omega_{[r,q,\tilde{z},f]}^*(b-(b+a)G_P(p))\overline{G}_P(p)^{a-1}G_P(p)^{b-1})$$
$$\times\ln(g_P(p)(1+\mu\Omega_{[r,q,\tilde{z},f]}^*(b-(b+a)G_P(p))\overline{G}_P(p)^{a-1}G_P(p)^{b-1}))\mathrm{d}p].$$

where

$$Q(r,\mu,q,\tilde{z},f) = \int_t^{\infty} g_P(p)(1+\mu\Omega_{[r,q,\tilde{z},f]}^*(b-(b+a)G_P(p))\overline{G}_P(p)^{a-1}G_P(p)^{b-1})$$
$$\ln(1+\mu\Omega_{[r,q,\tilde{z},f]}^*(b-(b+a)G_P(p))\overline{G}_P(p)^{a-1}G_P(p)^{b-1})\mathrm{d}p,$$

using part integration , we get

$$u = \ln(1 + \mu\Omega^*_{[r,q,\tilde{z},f]}(b - (b+a)G_P(p))\overline{G}_P(p)^{a-1}G_P(p)^{b-1}),$$

then

$$\begin{aligned} du = &\sum_{x=0}^{\infty}(-1)^x(\mu\Omega^*_{[r,q,\tilde{z},f]})^{x+1}(b-(b+a)G_P(p))^x\overline{G}_P(p)^{x(a-1)}G_P(p)^{x(b-1)} \\ &\times g_P(p)[(b-1)(b-(b+a)G_P(p))\overline{G}_P(p)^{a-1}G_P(p)^{b-2} \\ &-(a-1)(b-(b+a)G_P(p))\overline{G}_P(p)^{a-2}G_P(p)^{b-1} - (b+a)G_P(p)^{b-1}\overline{G}_P(p)^{a-1}]\mathrm{d}p, \end{aligned}$$

let,

$$dv = \int g_P(p)(1 + \mu\Omega^*_{[r,q,\tilde{z},f]}(b - (b+a)G_P(p))\overline{G}_P(p)^{a-1}G_P(p)^{b-1})\mathrm{d}p,$$

then

$$v = G_P(p) + \mu\Omega^*_{[r,q,\tilde{z},f]}\sum_{x=0}^{a-1}\binom{a-1}{x}(-1)^x[\frac{bG_P(p)^{x+b}}{x+b} - \frac{(b+a)G_P(p)^{x+1+b}}{x+1+b}].$$

Hence

$$\begin{aligned} Q(r,\mu,q,\tilde{z},f) = uv\,|_{v=t}^{v=\infty} - \int_t^{\infty} vdu = &-(G_P(t) + \mu\Omega^*_{[r,q,\tilde{z},f]}\sum_{x=0}^{a-1}\binom{a-1}{x}(-1)^x[\frac{bG_P(t)^{x+b}}{x+b} - \frac{(b+a)G_P(t)^{x+1+b}}{x+1+b}] \\ &\ln(1+\mu\Omega^*_{[r,q,\tilde{z},f]}(-(b+a)G_P(t)+b)G_P(t)^{b-1}\overline{G}_P(t)^{a-1})) \\ &-\sum_{d=0}^{\infty}(-1)^d(\mu\Omega^*_{[r,q,\tilde{z},f]})^{d+1}[\mu\Omega^*_{[r,q,\tilde{z},f]}\sum_{x=0}^{a-1}\binom{a-1}{x}(-1)^x \\ &[(b-1)(\frac{b}{b+x}\delta_1 - \frac{b+a}{b+1+x}\delta_2) - (a-1)(\frac{b}{b+x}\delta_3 - \frac{b+a}{b+1+x}\delta_4) \\ &-(b+a)(\frac{b}{b+x}\delta_5 - \frac{b+a}{b+1+x}\delta_6)] + (b-1)\delta_7 - (a-1)\delta_8 - (b+a)\delta_9], \end{aligned}$$

where

$$\begin{aligned} \delta_1 &= \int_t^{\infty}(-(a+b)G_P(p)+b)^{d+1}G_P(p)^{2b+x+(b-1)d-2}\overline{G}_P(p)^{(a-1)(d+1)}g_P(p)\mathrm{d}p \\ &= \sum_{h=0}^{d+1}\binom{d+1}{h}(-(a+b))^h b^{d-h+1}\int_t^{\infty}(1-G_P(p))^{(a-1)(d+1)}G_P(p)^{2b+x+d(b-1)+h-2}g_P(p)\mathrm{d}p \\ &= \sum_{h=0}^{d+1}\binom{d+1}{h}(-(a+b))^h b^{d-h+1}\int_t^{\infty}\sum_{k=0}^{(a-1)(d+1)}(-1)^k\binom{(a-1)(d+1)}{k}G_P(p)^{2b+x+(b-1)d+h+k-2}g_P(p)\mathrm{d}p \\ &= \sum_{h=0}^{d+1}\binom{d+1}{h}(-(a+b))^h b^{d-h+1}\sum_{k=0}^{(a-1)(d+1)}(-1)^k\binom{(a-1)(d+1)}{k}\frac{1-G_P(t)^{2b+x+(b-1)d+h+k-1}}{2b+x+(b-1)d+h+k-1}, \end{aligned}$$

$$\begin{aligned} \delta_2 &= \int_t^{\infty}(-(a+b)G_P(p)+b)^{d+1}G_P(p)^{2b+x+(b-1)d-1}\overline{G}_P(p)^{(a-1)(d+1)}g_P(p)\mathrm{d}p \\ &= \sum_{h=0}^{d+1}\binom{d+1}{h}(-(a+b))^h b^{d-h+1}\sum_{k=0}^{(a-1)(d+1)}(-1)^k\binom{(a-1)(d+1)}{k}\frac{1-G_P(t)^{2b+x+(b-1)d+h+k}}{2b+x+(b-1)d+h+k}, \end{aligned}$$

$$\begin{aligned} \delta_3 &= \int_t^{\infty}(-(a+b)G_P(p)+b)^{d+1}G_P(p)^{2b+x+(b-1)d-1}\overline{G}_P(p))^{(a-2)+(a-1)d}g_P(p)\mathrm{d}p \\ &= \sum_{h=0}^{d+1}\binom{d+1}{h}(-(a+b))^h b^{d-h+1}\sum_{k=0}^{(a-2)+(a-1)d}(-1)^k\binom{(a-2)+(a-1)d}{k}\frac{1-G_P(t)^{2b+x+(b-1)d+h+k}}{2b+x+(b-1)d+h+k}, \end{aligned}$$

$$\delta_4 = \int_t^{\infty} (-(a+b)G_P(p)+b)^{d+1} G_P(p)^{2b+x+(b-1)d} \overline{G}_P(p)^{(a-2)+(a-1)d} g_P(p)\mathrm{d}p$$
$$= \sum_{h=0}^{d+1} \binom{d+1}{h} (-(a+b))^h b^{d-h+1} \sum_{k=0}^{(a-2)+(a-1)d} (-1)^k \binom{(a-2)+(a-1)d}{k} \frac{1-G_P(t)^{2b+x+(b-1)d+h+k+1}}{2b+x+(b-1)d+h+k+1},$$

$$\delta_5 = \int_t^{\infty} (-(a+b)G_P(p)+b)^{d} G_P(p)^{b+x+(d+1)(b-1)} \overline{G}_P(p)^{(a-1)(d+1)} g_P(p)\mathrm{d}p$$
$$= \sum_{h=0}^{d} \binom{d}{h} (-(a+b))^h b^{d-h} \sum_{k=0}^{(a-1)(d+1)} (-1)^k \binom{(a-1)(d+1)}{k} \frac{1-G_P(t)^{b+x+(d+1)(b-1)+h+k+1}}{b+x+(d+1)(b-1)+h+k+1},$$

$$\delta_6 = \int_t^{\infty} (-(a+b)G_P(p)+b)^{d} G_P(p)^{b+p+(d+1)(b-1)+1} \overline{G}_P(p)^{(a-1)(d+1)} g_P(p)\mathrm{d}p$$
$$= \sum_{h=0}^{d} \binom{d}{h} (-(a+b))^h b^{d-h} \sum_{k=0}^{(a-1)(d+1)} (-1)^k \binom{(a-1)(d+1)}{k} \frac{1-G_P(t)^{b+x+(d+1)(b-1)+h+k+2}}{b+x+(d+1)(b-1)+h+k+2},$$

$$\delta_7 = \int_t^{\infty} (-(a+b)G_P(p)+b)^{d+1} G_P(p)^{(d+1)(b-1)} \overline{G}_P(p)^{(d+1)(a-1)} g_P(p)\mathrm{d}p$$
$$= \sum_{h=0}^{d+1} \binom{d+1}{h} (-(a+b))^h b^{d-h+1} \sum_{k=0}^{(a-1)(d+1)} (-1)^k \binom{(a-1)(d+1)}{k} \frac{1-G_P(t)^{(d+1)(b-1)+h+k+1}}{(d+1)(b-1)+h+k+1},$$

$$\delta_8 = \int_t^{\infty} (-(a+b)G_P(p)+b)^{d+1} G_P(p)^{b+(b-1)d} \overline{G}_P(p)^{(a-2)+(a-1)d} g_P(p)\mathrm{d}p$$
$$= \sum_{h=0}^{d+1} \binom{d+1}{h} (-(a+b))^h b^{d-h+1} \sum_{k=0}^{(a-2)+(a-1)d} (-1)^k \binom{(a-2)+(a-1)d}{k} \frac{1-G_P(t)^{b+(b-1)d+h+k+1}}{b+(b-1)d+h+k+1},$$

$$\delta_9 = \int_t^{\infty} (-(a+b)G_P(p)+b)^{d} G_P(p)^{b+(b-1)d} \overline{G}_P(p)^{(d+1)(a-1)} g_P(p)\mathrm{d}p$$
$$= \sum_{h=0}^{d} \binom{d}{h} (-(a+b))^h b^{d-h} \sum_{k=0}^{(d+1)(a-1)} (-1)^k \binom{(d+1)(a-1)}{k} \frac{1-G_P(t)^{b+(b-1)d+h+k+1}}{b+(b-1)d+h+k+1}.$$

Theorem 5.2. From (10) and (15), then an explicit form of the past entropy of $P_{[r,q,\tilde{z},f]}$, is:

$$\overline{\zeta}(P_{[r,q,\tilde{z},f]};t) = \ln G_{[r,q,\tilde{z},f]}(t) - \frac{1}{G_{[r,q,\tilde{z},f]}(t)} \Big[\Big[G_P(t)(\ln G_P(t) - \overline{\zeta}(p;t)) \Big] + \mu \Omega^*_{[r,q,\tilde{z},f]}$$
$$[b\psi_{g_1}(p) - (a+b)\psi_{g_2}(p)] + Q(r,\mu,q,\tilde{z},f) \Big], \tag{44}$$

where

the past entropy for P is $\overline{\zeta}(p;t)$

$$\overline{\zeta}(p;t) = \ln G_P(t) - \frac{1}{G_P(t)} \int_0^t g_P(p) \ln g_P(p)\mathrm{d}p,$$

$$\psi_{g_1}(p) = \int_0^t g_P(p) G_P(p)^{b-1} \overline{G}_P(p)^{a-1} \ln g_P(p)\mathrm{d}p,$$

$$\psi_{g_2}(p) = \int_0^t g_P(p) G_P(p)^{b} \overline{G}_P(p)^{a-1} \ln g_P(p)\mathrm{d}p,$$

and

$$Q(r,\mu,q,\widetilde{z},f)=(G_P(t)+\mu\Omega^*_{[r,q,\widetilde{z},f]}\sum_{x=0}^{a-1}\binom{a-1}{x}(-1)^x[\frac{bG_P(t)^{x+b}}{x+b}-\frac{(b+a)G_P(t)^{x+1+b}}{x+1+b}]$$
$$\ln(1+\mu\Omega^*_{[r,q,\widetilde{z},f]}(-(a+b)G_P(t)+b)\,G_P(t)^{b-1}\overline{G}_P(t)^{a-1}))$$
$$-\sum_{d=0}^{\infty}(-1)^d(\mu\Omega^*_{[r,q,\widetilde{z},f]})^{d+1}[\mu\Omega^*_{[r,q,\widetilde{z},f]}\sum_{x=0}^{a-1}\binom{a-1}{x}(-1)^x$$
$$[(b-1)(\frac{b}{b+x}\delta_1^*-\frac{a+b}{b+x+1}\delta_2^*)-(a-1)(\frac{b}{x+b}\delta_3^*-\frac{b+a}{x+1+b}\delta_4^*)$$
$$-(b+a)(\frac{b}{x+b}\delta_5^*-\frac{a+b}{x+1+b}\delta_6^*)]+(b-1)\delta_7^*-(a-1)\delta_8^*-(b+a)\delta_9^*]$$

where

$$\delta_1^*=\int_0^t(-(a+b)G_P(p)+b)^{d+1}G_P(p)^{2b+x+(b-1)d-2}\overline{G}_P(p)^{(a-1)(d+1)}g_P(p)\mathrm{d}p$$
$$=\sum_{h=0}^{d+1}\binom{d+1}{h}(-(a+b))^hb^{d-h+1}\int_0^t g_P(p)(1-G_P(p))^{(a-1)(d+1)}G_P(p)^{2b+x+(b-1)d+h-2}\mathrm{d}p$$
$$=\sum_{h=0}^{d+1}\binom{d+1}{h}(-(a+b))^hb^{d-h+1}\int_0^t\sum_{k=0}^{(a-1)(d+1)}\binom{(a-1)(d+1)}{k}(-1)^kG_P(p)^{2b+x+(b-1)d+h+k-2}g_P(p)\mathrm{d}p$$
$$=\sum_{h=0}^{d+1}\binom{d+1}{h}(-(a+b))^hb^{d-h+1}\sum_{k=0}^{(a-1)(d+1)}\binom{(a-1)(d+1)}{k}(-1)^k\frac{G_P(t)^{2b+x+(b-1)d+h+k-1}}{2b+x+(b-1)d+h+k-1},$$

$$\delta_2^*=\int_0^t(-(a+b)G_P(p)+b)^{d+1}G_P(p)^{2b+x+(b-1)d-1}\overline{G}_P(p)^{(a-1)(d+1)}g_P(p)\mathrm{d}p$$
$$=\sum_{h=0}^{d+1}\binom{d+1}{h}(-(a+b))^hb^{d-h+1}\sum_{k=0}^{(a-1)(d+1)}\binom{(a-1)(d+1)}{k}(-1)^k\frac{G_P(t)^{2b+x+(b-1)d+h+k}}{2b+x+(b-1)d+h+k},$$

$$\delta_3^*=\int_0^t(-(a+b)G_P(p)+b)^{d+1}G_P(p)^{2b+x+(b-1)d-1}\overline{G}_P(p)^{(a-2)+(a-1)d}g_P(p)\mathrm{d}p$$
$$=\sum_{h=0}^{d+1}\binom{d+1}{h}(-(a+b))^hb^{d-h+1}\sum_{k=0}^{(a-2)+(a-1)d}\binom{(a-2)+(a-1)d}{k}(-1)^k\frac{G_P(t)^{2b+x+(b-1)d+h+k}}{2b+x+(b-1)d+h+k},$$

$$\delta_4^*=\int_0^t(-(a+b)G_P(p)+b)^{d+1}G_P(p)^{2b+x+(b-1)d}\overline{G}_P(p)^{(a-2)+(a-1)d}g_P(p)\mathrm{d}p$$
$$=\sum_{h=0}^{d+1}\binom{d+1}{h}(-(a+b))^hb^{d-h+1}\sum_{k=0}^{(a-2)+(a-1)d}\binom{(a-2)+(a-1)d}{k}(-1)^k\frac{G_P(t)^{2b+x+(b-1)d+h+k+1}}{2b+x+(b-1)d+h+k+1},$$

$$\delta_5^*=\int_0^t(-(a+b)G_P(p)+b)^yG_P(p)^{b+x+(1+y)(b-1)}\overline{G}_P(p)^{(a-1)(1+y)}g_P(p)\mathrm{d}p$$
$$=\sum_{h=0}^{d}\binom{d}{h}(-(a+b))^hb^{d-h}\sum_{k=0}^{(a-1)(d+1)}\binom{(a-1)(d+1)}{k}(-1)^k\frac{G_P(t)^{b+x+(d+1)(b-1)+h+k+1}}{b+x+(d+1)(b-1)+h+k+1},$$

$$\delta_6^*=\int_0^t(-(a+b)G_P(p)+b)^dG_P(p)^{b+x+(d+1)(b-1)+1}\overline{G}_P(p)^{(a-1)(d+1)}g_P(p)\mathrm{d}p$$
$$=\sum_{h=0}^{d}\binom{d}{h}(-(a+b))^hb^{d-h}\sum_{k=0}^{(a-1)(d+1)}\binom{(a-1)(d+1)}{k}(-1)^k\frac{G_P(t)^{b+x+(d+1)(b-1)+h+k+2}}{b+x+(d+1)(b-1)+h+k+2},$$

$$\delta_7^* = \int_0^t (-(a+b)G_P(p)+b)^{d+1} G_P(p)^{(d+1)(b-1)} \overline{G}_P(p)^{(d+1)(a-1)} g_P(p)\mathrm{d}p$$
$$= \sum_{h=0}^{d+1} \binom{d+1}{h} (-(a+b))^h b^{d-h+1} \sum_{k=0}^{(a-1)(d+1)} \binom{(a-1)(d+1)}{k} (-1)^k \frac{G_P(t)^{(d+1)(b-1)+h+k+1}}{(d+1)(b-1)+h+k+1},$$

$$\delta_8^* = \int_0^t (-(a+b)G_P(p)+b)^{d+1} G_P(p)^{b+(b-1)d} \overline{G}_P(p)^{(a-2)+(a-1)d} g_P(p)\mathrm{d}p$$
$$= \sum_{h=0}^{d+1} \binom{d+1}{h} (-(a+b))^h b^{d-h+1} \sum_{k=0}^{(a-2)+(a-1)d} \binom{(a-2)+(a-1)d}{k} (-1)^k \frac{G_P(t)^{b+(b-1)d+h+k+1}}{b+(b-1)d+h+k+1},$$

$$\delta_9^* = \int_0^t (-(a+b)G_P(p)+b)^{d} G_P(p)^{b+(b-1)d} \overline{G}_P(p)^{(d+1)(a-1)} g_P(p)\mathrm{d}p$$
$$= \sum_{h=0}^{d} \binom{d}{h} (-(a+b))^h b^{d-h} \sum_{k=0}^{(d+1)(a-1)} \binom{(d+1)(a-1)}{k} (-1)^k \frac{G_P(t)^{b+(b-1)d+h+k+1}}{b+(b-1)d+h+k+1}.$$

References

[1]U. Kamps. A concept of generalized order statistics. J. Stat. Plan. Inference, 48, 1-23 (1995).
[2] U. Kamps and E. Cramer. On distribution of generalized order statistics. Statistics, 35, 269-280 (2001).
[3]P. Pawlas and D. Szynal. Recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution. Demonstratio Math, 34 (2), 353-358 (2001).
[4] M. Burkschat, E. Cramer and U. Kamps. Dual generalized order statistics. Metron LXI, (1), 13-26 (2003).
[5] Athar, H. and Faizan, M. Moments of lower generalized order statistics from power function distribution and its characterization. International Journal of Statistical Sciences, 11, 125-134 (2011).
[6]Abd Elgawaad, M. A., Alawady, M. A., Barakat, H. M., and Shengwu Xiong. Concomitants of generalized order statistics from Huang-Kotz Farlie-Gumbel-Morgenstern bivariate distribution: some information measures. Buill. Malaays. Math. Sci. Soc, 43, 2627-2645 (2020).
[7]Alawady, M. A., Barakat, H. M., Shengwu Xiong and Abd Elgawad, M. A. On concomitants of dual generalized order statistics from Bairamov-Kotz-Becki Farlie-Gumbel-Morgenstern bivariate distributions. Asian-European J. Math, 14(10), 2150185 (2021).
[8] Barakat, H. M., Nigm, E. M., and Syam, A. H. Concomitants of order statistics and record values from Bairamov–Kotz–Becki–FGM bivariate–generalized exponential distribution. Filomat, 32(9), 3313-3324 (2018).
[9] Beg, M.I. and Ahsanullah, M. Concomitants of generalized order statistics from Farlie-Gumbel-Morgensterm distributions. Statistical Methodology, 5, 1-20 (2008).
[10] Domma, F. and Giordano, S. Concomitants of m-generalized order statistics from generalized Farlie-Gumbel-Morgenstern distribution. Journal of Computational and Applied Mathematics, 294, 413-435 (2016).
[11] Mohie El-Din, M. M., Amein, M. M. and Mohamed, M. S. Concomitants of case-II of generalized order statistics from Farlie-Gumbel-Morgenstern distributions. J. Stat. Appl. Pro., 3(3), 345-353 (2015).
[12]Ebrahimi, N. How to measure uncertainty about residual lifetime. Sankhya, Ser. A, 58, 48-57 (1996).
[13]Di Crescenzo, A. and Longobardi, M. Entropy-based measure of uncertainty in past lifetime distributions. Journal of Applied Probability, 39, 434-440 (2002).
[14]Lai, C.D., Xie, M. A new family of positive quadrant dependent bivariate distributions. Statistics and Probability Letters, 46, 359–364 (2000).
[15] I. Bairamov, S. Kotz and M. Bekci. New generalized Farlie-Gumbel-Morgenstern distributions and concomitants of order statistics. J. of Appl. Stat., 28 (5), 521-536 (2001).