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True Integer Valued Autoregressive Model with Skellam Distribution

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Abstract: In the present article, we introduce a new true integer valued autoregressive model of order one TPDINAR(1) for data sets on $\mathbb{Z}$ and either positive or negative correlations based on the Poisson difference (Skellam) marginal distribution and using a random walk variable (I_t). Properties of the model are derived. We consider several methods for estimating the unknown parameters of the model, and their properties are discussed. Simulations are carried out for the performance of these estimators for illustrative purposes. Finally, the analysis of real life time series data is presented, and its performance is compared to two different INAR(1) models that may also be used over the observed data.

Keywords: True Integer Valued process, autoregressive models, Skellam distribution, negative autocorrelations

1 Introduction

Integer valued time series models have received considerable attention in the literature in the last years. This is partially due to the occurrence in many real life situations and application fields. In each of these examples an element of the process at time t can be either the existence of an element of the process at previous times, or an arrival (innovation) sequence which has a specific discrete distribution. The majority of non-negative integer valued time series models are based on thinning operators of Steutel and van Harn [1] and several marginal distributions.

In many applications in real life we may face time series data with positive and negative integer values and positive and negative autocorrelation functions. Some of these data are also obtained when the difference operator is applied to a non-stationary count data. By using the signed binomial thinning operator and focusing on the case where the innovation has the Skellam distribution, Karlis and Anderson [2] demonstrated the ZINAR process as an expansion of the INAR model (difference between two independent Poisson processes).According to Freeland [3], TINAR(1) is the difference between two INAR processes, which requires observation of the two processes. Barreto-Souza and Bourguignon [4] introduced a stationary first order integer valued autoregressive process on $\mathbb{Z}$ with skew discrete Laplace marginal (difference between two independent geometric processes), named a skew true INAR(1) model (STINAR). Alzaid and Omair [5] defined the extended binomial thinning operator and studied the Poisson difference integer valued autoregressive model of order one. For a review, see for example, ([6], [7]).

The rest of the paper is arranged as follows: Section 2 includes a new true Skellam integer valued autoregressive model of order one (denoted by TPDINAR(1)) using a random walk variable $I_t(\alpha)$. The properties of this model are also considered, and we have proved that the TPDINAR(1) process is second order stationary and strictly stationary in this section. In Section 3, the estimation of the model parameters is considered by using three methods of estimation, conditional least square method, Yule Walker method, and conditional maximum likelihood method. And in section 4, we use real life time series data of annual increases in the Swedish population (per thousand people) over the 1750–1849 century [4] to apply these methods.

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2 TPDINAR(1)

A Skellam process is defined as

$$X(t) = N_{(1)}(t) - N_{(2)}(t), t \geq 0$$

where $N_{(1)}(t), t \geq 0$ and $N_{(2)}(t), t \geq 0$ are two independent Poisson processes with parameters $\mu_1 > 0$ and $\mu_2 > 0$, respectively.

The probability mass function of $X(t)$ is given by Alzaid, and Omair [8] as

$$P(X(t) = x) = e^{-(\mu_1+\mu_2)}(\mu_1\mu_2)^{x/2}B_{|x|}(2\sqrt{\mu_1\mu_2}),$$

for all $x \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}, \mu_1, \mu_2 > 0$ where $B_r(y)$ is the modified Bessel function of the first kind see (Abramowitz and Stegun [9]) given by

$$B_r(y) = (\frac{y}{2})^r \sum_{k=0}^{\infty} \frac{(\frac{y^2}{4})^k}{k!(r+k)!}.$$

The mean and the variance of $X(t)$ are

$$E(X(t)) = \mu_1 - \mu_2 = \mu_m \text{ and } Var(X(t)) = \mu_1 + \mu_2 = \mu_p.$$

The moment generating function (MGF) of the Skellam process is

$$M_X(u) = e^{-(\mu_1+\mu_2)+\mu_1 e^u + \mu_2 e^{-u}},$$

and its moments are

$$m_1 = E[\varepsilon_t] = \mu_m.$$
$$m_2 = E[\varepsilon_t^2] = \mu_m^2 + \mu_p.$$
$$m_3 = E[\varepsilon_t^3] = \mu_m(1 + 3\mu_p + \mu_m^2).$$
$$m_4 = E[\varepsilon_t^4] = \mu_m^2(4 + 6\mu_p + \mu_m^2) + 3\mu_m^3 + \mu_p.$$

Now, we present a new integer valued autoregressive process of order 1, which can accommodate both positive and negative autocorrelations and can handle both negative and positive integer valued time series. TPDINAR(1), or True Poisson Difference INteger valued AutoRegressive of order 1, is the name of this process.

Definition 1.*Let $\{\varepsilon_t\}$ be a sequence of i.i.d. random variables with the Poisson difference (Skellam) distribution PD(μ_1, μ_2). The TPDINAR (1) process $\{Z_t\}$ is defined by*

$$Z_t = I_t(\alpha)Z_{t-1} + \varepsilon_t, \tag{1}$$

where $I_t(\alpha)$, Z_{t-1}, and ε_t are independent random variables and $I_t(\alpha)$ be a random variable, defined for any $\alpha \in (0,1)$, as

$$I_t(\alpha) = U_t(\alpha) - 1 = \begin{cases} -1 & \text{for prob. } (1-\alpha)^2 \\ 0 & \text{for prob. } 2\alpha(1-\alpha) \\ 1 & \text{for prob. } \alpha^2 \end{cases},$$

and $U_t(\alpha)$ be i.i.d. binomial random variables $Bin(2,\alpha)$ for each given t so that,

$$E[I_t^k(\alpha)] = \begin{cases} 2\alpha - 1 & k = 1,3,5,... \\ 1 - 2\alpha(1-\alpha) & k = 2,4,6,... \end{cases} \quad \textit{and} \quad var(I_t(\alpha)) = 2\alpha(1-\alpha).$$

Since there is movement in both directions instead of just rising, this process differs slightly from counting processes. Thus, this process is similar to a random walk with discrete steps of size one and Skellam innovations. This model is suitable to fit the change in a commodity's price, stock or other applications in finance and economics. The sample path of the process (1) having $PD(\mu_1, \mu_2)$ as marginal with several and different values of μ_1 and μ_2 are plotted in Figure 1.

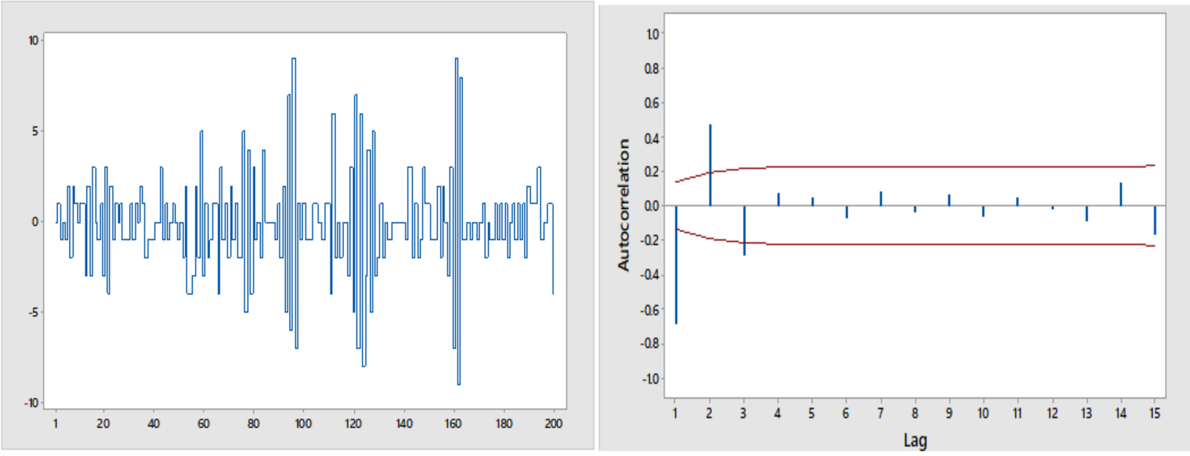

Fig.1(a) Series(1) and its autocorrelations with $\alpha = 0.2$, $\mu_1 = .8$ and $\mu_2 = 1$.

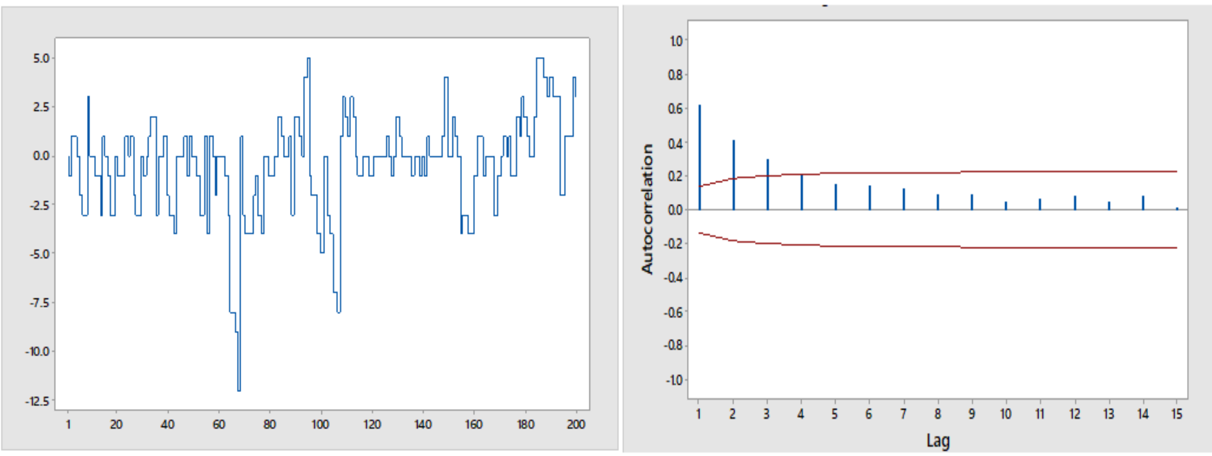

Fig.1(b) Series(2) and its autocorrelations with $\alpha = 0.8$, $\mu_1 = .8$ and $\mu_2 = 1$.

The mean, variance and the autocorrelation functions of stationay $\{Z_t\}$ are given by

$$E[Z_t] = \mu_Z = \frac{\mu_m}{2(1-\alpha)}, \tag{2}$$

$$\gamma_0 = E[Z_t^2] = \frac{2\alpha\mu_m^2 + 2\mu_p(1-\alpha)}{4\alpha(1-\alpha)^2}, \tag{3}$$

$$G_0 = Var(Z_t) = \frac{\alpha\mu_m^2 + 2\mu_p(1-\alpha)}{4\alpha(1-\alpha)^2}, \text{ and} \tag{4}$$

$$\rho_k = (2\alpha - 1)^k \quad , k \geq 0. \tag{5}$$

Clearly, for $0 < \alpha < 0.5$, ρ_k is negative for odd k and positive otherwise.

We now investigate the second order *stationarity of the process* (1) *with* $0 < \alpha < 1$.

Consider the process (1) with $0 < \alpha < 1$

$$
\begin{aligned}
E\left[Z_t\right] &= E[I_t(\alpha)Z_{t-1} + \varepsilon_t] \\
&= E[I_t(\alpha)I_{t-1}(\alpha)Z_{t-2} + I_t(\alpha)\varepsilon_{t-1} + \varepsilon_t \\
&= E\left[\prod_{j=1}^{t} I_j(\alpha)Z_0 + \sum_{k=1}^{t-1}\left(\prod_{j=k+1}^{t} I_j(\alpha)\right)\varepsilon_k + \varepsilon_t\right] \\
&= E\left[\eta_1 Z_0 + \sum_{k=1}^{t-1}\eta_{k+1}\varepsilon_k + \varepsilon_t\right] \\
&= (2\alpha-1)^t E[Z_0] + \sum_{k=1}^{t-1}(2\alpha-1)^{t-k}\mu_m + \mu_m \\
&= (2\alpha-1)^t E[Z_0] + \frac{(2\alpha-1)(1-(2\alpha-1)^{t-1})}{1-(2\alpha-1)}\mu_m + \mu_m \\
&= (2\alpha-1)^t\left[E[Z_0] - \frac{\mu_m}{2(1-\alpha)}\right] + \frac{\mu_m}{2(1-\alpha)}
\end{aligned}
$$

where $\eta_l = \prod_{j=l}^{t} I_j(\alpha)$ and by virtue of the independence of $I_t(\alpha)'s$ we have $E[\eta_l] = E[I_t(\alpha)]^{t-l+1} = (2\alpha-1)^{t-l+1}$.

So, in order the mean of $\{Z_t\}$ to be free of t, we must have $E[Z_0] = \frac{\mu_m}{2(1-\alpha)}$ and in this case $E[Z_t] = \frac{\mu_m}{2(1-\alpha)} = E[Z_0]$.

$$
\begin{aligned}
E\left[Z_t^2\right] &= E[(I_t(\alpha)Z_{t-1} + \varepsilon_t)^2] \\
&= E[I_t^2 Z_{t-1}^2 + 2I_t\varepsilon_t Z_{t-1} + \varepsilon_t^2] \\
&= E[I_t^2[I_{t-1}^2 Z_{t-2}^2 + 2I_{t-1}\varepsilon_{t-1}Z_{t-2} + \varepsilon_{t-1}^2] + 2I_t\varepsilon_t Z_{t-1} + \varepsilon_t^2] \\
&= E\left[\begin{array}{c}\left(I_t^2 I_{t-1}^2\cdots I_1^2\right)Z_0^2 + 2\left(I_t^2 I_{t-1}^2\cdots I_2^2\right)I_1\varepsilon_1 Z_0 + 2\left(I_t^2 I_{t-1}^2\cdots I_3^2\right)I_2\varepsilon_2 Z_1 \\ +\cdots+2I_t^2 I_{t-1}\varepsilon_{t-1}Z_{t-2} + 2I_t\varepsilon_t Z_{t-1} \\ +\left(I_t^2 I_{t-1}^2\cdots I_2^2\right)\varepsilon_1^2 + \left(I_t^2 I_{t-1}^2\cdots I_3^2\right)\varepsilon_2^2 + \cdots + I_t^2\varepsilon_{t-1}^2 + \varepsilon_t^2\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \beta^t E[Z_0^2] + \sum_{k=1}^{t-1}\beta^{t-k}\frac{(2\alpha-1)}{(1-\alpha)}\mu_m^2 + \frac{(2\alpha-1)}{(1-\alpha)}\mu_m^2 \\
&\quad + \sum_{k=1}^{t-1}\beta^{t-k}(\mu_p + \mu_m^2) + (\mu_p + \mu_m^2). \\
&= \beta^t E[Z_0^2] + \frac{\beta(1-\beta^{t-1})}{1-\beta}\frac{(2\alpha-1)}{(1-\alpha)}\mu_m^2 + \frac{(2\alpha-1)}{(1-\alpha)}\mu_m^2 \\
&\quad + \frac{\beta(1-\beta^{t-1})}{1-\beta}(\mu_p + \mu_m^2) + (\mu_p + \mu_m^2) \\
&= \beta^t\left(E[Z_0^2] - \frac{2\alpha\mu_m^2 + 2(1-\alpha)\mu_p}{4\alpha(1-\alpha)^2}\right) + \frac{2\alpha\mu_m^2 + 2(1-\alpha)\mu_p}{4\alpha(1-\alpha)^2}
\end{aligned}
$$

where $\beta = E[I_t^2(\alpha)] = (1-2\alpha+2\alpha^2)$ and $E[\eta_l^2] = E[I_t^2(\alpha)]^{t-l+1} = (1-2\alpha+2\alpha^2)^{t-l+1} = \beta^{t-l+1}$.

$$
\begin{aligned}
Var(Z_t) &= \beta^t\left((E[Z_0^2] - (E[Z_0])^2) + \frac{\mu_m^2}{4(1-\alpha)^2} - \frac{2\alpha\mu_m^2 + 2(1-\alpha)\mu_p}{4\alpha(1-\alpha)^2}\right) \\
&\quad + \frac{2\alpha\mu_m^2 + 2(1-\alpha)\mu_p}{4\alpha(1-\alpha)^2} - \frac{\mu_m^2}{4(1-\alpha)^2} \\
&= \beta^t\left(Var(Z_0) - \frac{\alpha\mu_m^2 + 2(1-\alpha)\mu_p}{4\alpha(1-\alpha)^2}\right) + \frac{\alpha\mu_m^2 + 2(1-\alpha)\mu_p}{4\alpha(1-\alpha)^2}
\end{aligned}
$$

So, in order the variance of $\{Z_t\}$ to be free of t, we must have

$$
Var(Z_0) = \frac{\alpha\mu_m^2 + 2(1-\alpha)\mu_p}{4\alpha(1-\alpha)^2} \text{ and in this case } Var(Z_t) = \frac{2\alpha\mu_m^2 + 2(1-\alpha)\mu_p}{4\alpha(1-\alpha)^2} = Var(Z_0). \tag{6}
$$

Similarly we can prove that $cov(Z_t, Z_{t-k})$ is free of t.

Since $0 < \alpha < 1$, the mean, variance and autocovariances of $\{Z_t\}$ are free of t, then the process TPDINAR(1) is second order stationary.

Theorem 1 *The TPDINAR(1) process is strictly stationary and ergodic.*

*Proof.*The process $\{Z_t\}$ given by Definition 1, can be written as

$$Z_t = \begin{cases} -Z_{t-1} + \varepsilon_t & \text{for prob. } (1-\alpha)^2 \\ \varepsilon_t & \text{for prob. } 2\alpha(1-\alpha) \\ Z_{t-1} + \varepsilon_t & \text{for prob. } \alpha^2 \end{cases}$$

From which it is obvious that the Markovian property is conducted. Based on Definition 3.1 and since $\{\varepsilon_t\}$ is an i.i.d. sequence, the conditional distribution of Z_{t+1} given Z_t and Z_{t+s+1} given Z_{t+s} are equal for any $t, s \in \mathbb{N}$. This, with the assumption that the random variables are equally distributed, and the Markovian property of the process implies that the joint distribution of $(Z_1, Z_2, \ldots, Z_t)$ and $(Z_{k+1}, Z_{k+2}, \ldots, Z_{k+t})$ are equal for any $t, s \in \mathbb{N}$, therefore the process $\{Z_t\}$ is strictly stationary. The ergodic property of the process can be obtained by the same way of Ristić and Nastic [10].

2.1 Conditional statistical measures

The conditional statistical measures of a stationary TPDINAR(1) process are derived as follows: From Equation (1) and because of the stationarity of the process and the independence of $I_t(\alpha)'s$, and $\varepsilon_t's$, the one step ahead conditional mean and variance are calculated as,

$$E[Z_t|Z_{t-1}] = E[(I_t(\alpha)Z_{t-1} + \varepsilon_t)|Z_{t-1}] = (2\alpha - 1)Z_{t-1} + \mu_m.$$

and

$$Var[Z_t|Z_{t-1}] = Var[(I_t(\alpha)Z_{t-1} + \varepsilon_t)|Z_{t-1}] = 2\alpha(1-\alpha)Z_{t-1}^2 + \mu_p.$$

Theorem 2 *Consider that Z_t be stationary process and when k approaches very large values ($k \to \infty$), the conditional mean and the conditional variance of the process $k+1$ steps ahead returns to the unconditional mean and variance of the process (1).*

*Proof.*Repeating the recursive substitution $t+k$ times of Equation (1) , we obtain

$$\begin{aligned} Z_{t+k} &= I_{t+k}(\alpha)Z_{t+k-1} + \varepsilon_{t+k} \\ &= I_{t+k}(\alpha)I_{t+k-1}(\alpha)Z_{t+k-2} + I_{t+k}(\alpha)\varepsilon_{t+k-1} + \varepsilon_{t+k} \\ &= I_{t+k}(\alpha)I_{t+k-1}(\alpha)I_{t+k-2}(\alpha)Z_{t+k-3} + I_{t+k}(\alpha)I_{t+k-1}(\alpha)\varepsilon_{t+k-2} + I_{t+k-1}(\alpha)\varepsilon_{t+k-1} + \varepsilon_{t+k} \\ &\;\;\vdots \\ &= \zeta_{k+1}Z_{t-1} + \sum_{j=0}^{k-1}\zeta_{k-j}\varepsilon_{t+j} + \varepsilon_{t+k}. \end{aligned} \tag{7}$$

where $\zeta_m = \prod_{j=k-m+1}^{k} I_{t+j}(\alpha)$, $m = 1, 2, \ldots, k+1$.

Note that $E[\zeta_m] = E[I_t(\alpha)]^m = (2\alpha - 1)^m$. and $E[\zeta_m^2] = E[I_t^2(\alpha)]^m = \beta^m$.

From the expected value properties and given that $\{I_t(\alpha)\}$ and $\{\varepsilon_t\}$ are two independent i.i.d. sequences of random variables, one has

$$E[Z_{t+k}|Z_{t-1}] = (2\alpha - 1)^{k+1}Z_{t-1} + \frac{1-(2\alpha-1)^{k+1}}{2(1-\alpha)}\mu_m.$$

when $k \to \infty$, we find that

$$E[Z_{t+k}|Z_{t-1}] \to \frac{\mu_m}{2(1-\alpha)}$$

which is the unconditional mean of the process (1) given by Equation (2). That is, when k approaches very large values, the conditional mean of the process $k+1$ steps ahead returns to the unconditional mean of the process.

To find the conditional variance $k+1$ steps ahead, we first evaluate the conditional expectation of Z_{t+k}^2.

Squaring both sides of Equation (7) and taking the conditional expectation, we obtain

$$E[Z_{t+k}^2|Z_{t-1}] = \beta^{k+1}Z_{t-1}^2 + \left(1+\sum_{j=1}^{k}\beta^j\right)(\mu_p+\mu_m^2)+2\left(\sum_{j=1}^{k}\nu^j\beta^{k-j}\right)\mu_m Z_{t-1} + 2\left[\left(\sum_{j=1}^{k}\nu^j\beta^{k-j}\right)+\left(\sum_{j=1}^{k}\nu^j\beta^{k-j}\right)+\ldots+\nu\right]\mu_m^2$$

$$= \beta^{k+1}Z_{t-1}^2 + 2\left(\sum_{j=1}^{k}\nu^j\beta^{k-j+1}\right)\mu_m Z_{t-1} + \frac{1-\beta^{k+1}}{1-\beta}(\mu_p+\mu_m^2)+2\left[\sum_{l=0}^{k-l}\sum_{j=1}^{k}\nu^j\beta^{k-j-l}\right]\mu_m^2$$

$$= \beta^{k+1}Z_{t-1}^2 + \nu\beta^k\left(\frac{1-(\nu/\beta)^k}{2(1-\alpha)^2}\right)\mu_m Z_{t-1} + \frac{1-\beta^{k+1}}{1-\beta}(\mu_p+\mu_m^2)+\frac{2\nu}{\beta-\nu}\left[\frac{\beta-\beta^k}{1-\beta}-\frac{\nu-\nu^k}{1-\nu}\right]$$

where $\nu = E[I_t(\alpha)] = (2\alpha-1)$ and $\beta<1$ (note that $(\nu/\beta)^k \to 0$, as $k\to\infty$ since $\nu<\beta$). When $k\to\infty$, we find that

$$E[Z_{t+k}^2|Z_{t-1}] \to \frac{1}{1-\beta}(\mu_p+\mu_m^2)+\frac{2\nu}{\beta-\nu}\left[\frac{\beta}{1-\beta}-\frac{\nu}{1-\nu}\right] = \frac{\mu_p}{2\alpha(1-\alpha)}+\frac{\mu_m^2}{2(1-\alpha)^2}$$

which is the unconditional expectation of Z_t^2 given by Equation (7).

Theorem 3 *The MGF(Moment Generating Function) of Z_t is given by*

$$M_Z(u) = \frac{\phi_u[2(1-\beta)-2\nu\phi_{-u}]+\psi_u}{2-(\beta+\nu)[\phi_u+\phi_{-u}]+\psi_u},$$

where $\psi_u = 2\nu\beta e^{-2(\mu_1+\mu_2)+\mu_1(e^u+e^{-u})+\mu_2(e^u+e^{-u})}$ *and* $\phi_u = e^{-(\mu_1+\mu_2)+\mu_1 e^u+\mu_2 e^{-u}}$.

*Proof.*The MGF of Z_t is given by

$$\begin{aligned} M_Z(u) &= E[e^{uZ_t}] = E[e^{u(I_t(\alpha)Z_{t-1}+\varepsilon_t)}] \\ &= M_\varepsilon(u)E[(1-\alpha)^2e^{-uZ_{t-1}}+2\alpha(1-\alpha)+\alpha^2e^{uZ_{t-1}}] \\ &= M_\varepsilon(u)[(1-\alpha)^2M_Z(-u)+2\alpha(1-\alpha)+\alpha^2M_Z(u)] \\ [1-\alpha^2M_\varepsilon(u)]M_Z(u) &= (1-\alpha)^2M_\varepsilon(u)M_Z(-u)+2\alpha(1-\alpha)M_\varepsilon(u) \end{aligned}$$

$$M_Z(u) = \frac{(1-\alpha)^2M_\varepsilon(u)M_Z(-u)+2\alpha(1-\alpha)M_\varepsilon(u)}{1-\alpha^2M_\varepsilon(u)}. \tag{8}$$

Replace each u in Equation (8) by $-u$ we obtain

$$M_Z(-u) = \frac{(1-\alpha)^2M_\varepsilon(-u)M_Z(u)+2\alpha(1-\alpha)M_\varepsilon(-u)}{1-\alpha^2M_\varepsilon(-u)},$$

substituting in Equation (8), we obtain

$$M_Z(u) = \frac{(1-\alpha)^2M_\varepsilon(u)}{1-\alpha^2M_\varepsilon(u)}\,\frac{(1-\alpha)^2M_\varepsilon(-u)M_Z(u)+2\alpha(1-\alpha)M_\varepsilon(-u)}{1-\alpha^2M_\varepsilon(-u)}+\frac{2\alpha(1-\alpha)M_\varepsilon(u)}{1-\alpha^2M_\varepsilon(u)}$$

$$\begin{aligned} [(1-\alpha^2M_\varepsilon(u))(1-\alpha^2M_\varepsilon(-u))]M_Z(u) &= (1-\alpha)^4M_\varepsilon(u)M_\varepsilon(-u)M_Z(u)+(1-\alpha)^2 2\alpha(1-\alpha) \\ &\quad M_\varepsilon(u)M_\varepsilon(-u)+[(1-\alpha^2M_\varepsilon(-u))](2\alpha(1-\alpha)M_\varepsilon(u)) \end{aligned}$$

$$\begin{aligned} &[(1-\alpha^2M_\varepsilon(u))(1-\alpha^2M_\varepsilon(-u))-(1-\alpha)^4M_\varepsilon(u)M_\varepsilon(-u)]M_Z(u) \\ &= (1-\alpha)^2 2\alpha(1-\alpha)M_\varepsilon(u)M_\varepsilon(-u)+[(1-\alpha^2M_\varepsilon(-u))](2\alpha(1-\alpha)M_\varepsilon(u)). \end{aligned}$$

Calculations are done, and the results are

$$\begin{aligned}
M_Z(u) &= \frac{(1-\beta)M_\varepsilon(u)[1-\nu M_\varepsilon(-u)]}{1-\frac{(\beta+\nu)}{2}[M_\varepsilon(u)+M_\varepsilon(-u)]+\nu\beta M_\varepsilon(u)M_\varepsilon(-u)} \\
&= \frac{2(1-\beta)M_\varepsilon(u)[1-\nu M_\varepsilon(-u)]}{2-(\beta+\nu)[M_\varepsilon(u)+M_\varepsilon(-u)]+2\nu\beta M_\varepsilon(u)M_\varepsilon(-u)} \\
&= \frac{\phi_u[2(1-\beta)-2\nu\phi_{-u}]+\psi_u}{2-(\beta+\nu)[\phi_u+\phi_{-u}]+\psi_u}.
\end{aligned}$$

3 Parameter Estimation

Assume that the stationary TPDINAR (1) procedure has produced n+1 observations. Three parameters in the TPDINAR (1) process need to be estimated: α, μ_1, and μ_2. In this part, conditional least squares method, the Yule Walker method, and conditional maximum likelihood approach will all be discussed.

3.1 Conditional least squares

Let's think about estimating the unknown parameters, α, μ_1 and μ_2 using conditional least squares (CLS). Considering the conditional expectation $E[Z_t|Z_{t-1}] = (2\alpha-1)Z_{t-1}+\mu_m$, then there are two normal equations only in $\hat{\alpha}$ and $\hat{\mu}_m$. Thus, we'll employ conditional least squares in two steps. First, we find the CLS estimators $\hat{\alpha}$ and $\hat{\mu}_m$, and next, we find the CLS estimator $\hat{\mu}_p$. From the CLS $\hat{\mu}_m = (\hat{\mu}_1-\hat{\mu}_2)$ and $\hat{\mu}_p = (\hat{\mu}_1+\hat{\mu}_2)$ we obtain the CLS estimators $\hat{\mu}_1$ and $\hat{\mu}_2$.

Theorem 4 *The CLS estimators are*

$$\hat{\alpha} = \frac{1}{2}\left[1+\frac{\sum_{t=2}^N Z_t Z_{t-1} - \frac{1}{N-1}(\sum_{t=2}^N Z_t)(\sum_{t=2}^N Z_{t-1})}{\sum_{t=2}^N Z_t^2 - \frac{1}{N-1}(\sum_{t=2}^N Z_{t-1})^2}\right],$$

$$\hat{\mu}_m = \frac{1}{N-1}\left[\sum_{t=2}^N Z_t - (2\hat{\alpha}-1)\sum_{t=2}^N Z_{t-1}\right].$$

*Proof.*In the CLS method we minimize the quadratic function

$$\begin{aligned}
Q &= \sum_{t=2}^N [Z_t - E[Z_t|Z_{t-1}]]^2 \\
&= \sum_{t=2}^N [Z_t - (2\alpha-1)Z_{t-1} - \mu_m]^2 \qquad (9)
\end{aligned}$$

solving $\frac{\partial Q}{\partial \alpha} = 0$ and $\frac{\partial Q}{\partial \mu_m} = 0$, we obtain

$$\hat{\alpha} = \frac{1}{2}\left[1+\frac{\sum_{t=2}^N Z_t Z_{t-1} - \frac{1}{N-1}(\sum_{t=2}^N Z_t)(\sum_{t=2}^N Z_{t-1})}{\sum_{t=2}^N Z_t^2 - \frac{1}{N-1}(\sum_{t=2}^N Z_{t-1})^2}\right] \qquad (10)$$

$$\hat{\mu}_m = \frac{1}{N-1}\left[\sum_{t=2}^N Z_t - (2\alpha-1)\sum_{t=2}^N Z_{t-1}\right]. \qquad (11)$$

Next, we will derive the asymptotic properties of the previously obtained CLS estimators of the unknown parameters α and μ_m.

The following theorem establishes the estimators' consistency and asymptotic distribution.

Theorem 5 *The CLS estimators $\hat{\alpha}$ and $\hat{\mu}_m$ given by Equations (10) and (11) of a stationary TPDINAR(1) process have the following asymptotic distribution*

$$\sqrt{N}\begin{pmatrix} \hat{\alpha}-\alpha \\ \hat{\mu}_m-\mu_m \end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, [D]\right).$$

*Proof.*Let $Z_1, Z_2, \ldots, Z_N$ be a sample of the TPDINAR(1) defined by Equation (1). First we will check whether the regularity conditions of Theorems 3.1 and 3.2 of (Klimko and Nelson [11]) are satisfied. Denote $\theta = (\theta_1, \theta_2) = (\alpha, \mu_m)$ and $g(\theta, Z_{t-1}) = E[Z_t|Z_{t-1}] = (2\alpha-1)Z_{t-1} + \mu_m$. Since $\frac{\partial g}{\partial \alpha} = 2Z_{t-1}$, $\frac{\partial g}{\partial \mu_m} = 1$, and $\frac{\partial^2 g}{\partial \alpha^2} = \frac{\partial^2 g}{\partial \alpha \partial \mu_m} = \frac{\partial^2 g}{\partial \mu_m^2} = 0$, then all conditions on [11] are satisfied. From Theorem 3.2 of Klimko and Nelson, we obtain

$$D = V^{-1} W V^{-1}.$$

where

$$\begin{aligned} V &= E\left[\begin{bmatrix} \left(\frac{\partial g}{\partial \alpha}\right)^2 & \left(\frac{\partial g}{\partial \alpha}\right)\left(\frac{\partial g}{\partial \mu}\right) \\ \left(\frac{\partial g}{\partial \alpha}\right)\left(\frac{\partial g}{\partial \mu}\right) & \left(\frac{\partial g}{\partial \mu}\right)^2 \end{bmatrix}\right] \\ &= E\left[\begin{bmatrix} 4Z_{t-1}^2 & 2Z_{t-1} \\ 2Z_{t-1} & 1 \end{bmatrix}\right] \\ &= \begin{bmatrix} 4\gamma_0 & 2\mu_Z \\ 2\mu_Z & 1 \end{bmatrix}. \end{aligned}$$

$|V| = 4\left(\gamma_0 - \mu_Z^2\right) = 4Var(Z_t) = 4g_0$, thus

$$V^{-1} = \frac{1}{4g_0}\begin{bmatrix} 1 & -2\mu_Z \\ -2\mu_Z & 4\gamma_0 \end{bmatrix}.$$

and

$$\begin{aligned} W &= E\left[Var[Z_t|Z_{t-1}]\begin{bmatrix} \left(\frac{\partial g}{\partial \alpha}\right)^2 & \left(\frac{\partial g}{\partial \alpha}\right)\left(\frac{\partial g}{\partial \mu}\right) \\ \left(\frac{\partial g}{\partial \alpha}\right)\left(\frac{\partial g}{\partial \mu}\right) & \left(\frac{\partial g}{\partial \mu}\right)^2 \end{bmatrix}\right] \\ &= E\left[\left(2\alpha(1-\alpha)Z_{t-1}^2 + \mu_p\right)\begin{bmatrix} 4Z_{t-1}^2 & 2Z_{t-1} \\ 2Z_{t-1} & 1 \end{bmatrix}\right] \\ &= (1-\beta)E\left[\begin{bmatrix} 4Z_{t-1}^4 & 2Z_{t-1}^3 \\ 2Z_{t-1}^3 & Z_{t-1}^2 \end{bmatrix}\right] + (m_2 - m_1^2)E\left[\begin{bmatrix} 4Z_{t-1}^2 & 2Z_{t-1} \\ 2Z_{t-1} & 1 \end{bmatrix}\right]. \end{aligned}$$

After some calculations, we obtain

$$E[Z_{t-1}^3] = \mu_3 = \frac{3\gamma_0 m_1 + F_1}{1-\nu},$$

$$E[Z_{t-1}^4] = \mu_4 = \frac{6\gamma_0 m_2 + F_2}{1-\beta},$$

where $F_1 = 4m_1^3 - 3m_1m_2 + m_1 + 3(m_2 - 2m_1)\mu_Z$, $F_2 = m_1[(1-3\nu)\mu_3 + A - 3\gamma_0 m_1]$, and $A = 5m_1^3 - 3m_1m_2 - 1 + (4 - 3(m_2 - m_1^2) - 5m_1^2)\mu_Z$.

Then

$$W = \begin{bmatrix} \gamma_0(28m_2 - 4m_1) + 4F_2 & 2\frac{(1-\beta)}{(1-\nu)}(3\gamma_0 m_1 + F_1 + 2(m_2 - m_1^2)\mu_Z) \\ 2\frac{(1-\beta)}{(1-\nu)}(3\gamma_0 m_1 + F_1 + 2(m_2 - m_1^2)\mu_Z) & (1-\beta)\gamma_0 + (m_2 - m_1^2) \end{bmatrix}.$$

The second stage is when the unknown parameter μ_p is estimated. Define the random variable V_t as

$$V_t = [Z_t - E[Z_t|Z_{t-1}]]^2 = [Z_t - (2\alpha-1)Z_{t-1} - \mu_m]^2.$$

Note that $E[V_t|Z_{t-1}] = Var(Z_t|Z_{t-1}) = 2\alpha(1-\alpha)Z_{t-1}^2 + \mu_p$.

Now, the CLS estimator of μ_p is obtained by minimizing the quadratic function

$$\begin{aligned} Q &= \sum_{t=2}^{N}[V_t - E[V_t|Z_{t-1}]]^2 \\ &= \sum_{t=2}^{N}\left[V_t - 2\alpha(1-\alpha)Z_{t-1}^2 - \mu_p\right]^2. \end{aligned}$$

By solving $\frac{\partial Q_p}{\partial \mu_p} = 0$, we obtain

$$\hat{\mu}_p = \frac{1}{N-1}\sum_{t=2}^{n}[Z_t - (2\alpha-1)Z_{t-1} - \mu_m]^2 - \frac{2\alpha(1-\alpha)}{N-1}\sum_{t=2}^{n} Z_{t-1}^2, \quad (12)$$

solving Equation (11) and Equation (12) we obtain the CLS of $\hat{\mu}_1$ and $\hat{\mu}_2$ as

$$\hat{\mu}_1 = \frac{1}{2}(\hat{\mu}_m + \hat{\mu}_p), \text{ and } \hat{\mu}_2 = \frac{1}{2}(\hat{\mu}_p - \hat{\mu}_m).$$

3.2 Yule Walker method

The Yule Walker (YW) equation can be solved by substituting the sample autocorrelation function for $\hat{\rho}_1$ to obtain an estimation for α.

$$\hat{\rho}_1 = (2\hat{\alpha} - 1)$$

then $\hat{\alpha} = \frac{\hat{\rho}_1+1}{2}$, and we can use the following two equations to obtain μ_1 and μ_2 estimators

$$\mu_Z = \frac{\mu_m}{2(1-\alpha)}.$$

$$G_0 = \frac{\alpha\mu_m^2 + 2\mu_p(1-\alpha)}{4\alpha(1-\alpha)^2},$$

thus,

$$\hat{\mu}_m = 2(1-\hat{\alpha})\mu_Z,$$
$$\hat{\mu}_p = \frac{g_0(4\hat{\alpha}(1-\hat{\alpha})^2) - \hat{\mu}_m\hat{\alpha}}{2(1-\hat{\alpha})}$$

since

$$\mu_m + \mu_p = 2\mu_1$$

then

$$\hat{\mu}_1 = \frac{\hat{\mu}_m + \hat{\mu}_p}{2}, \text{ and } \hat{\mu}_2 = \hat{\mu}_1 - \hat{\mu}_m.$$

3.3 Conditional maximum likelihood method

Consider the conditional maximum likelihood (CML) estimation of the unknown parameters $\hat{\alpha}, \hat{\mu}_1$ and $\hat{\mu}_2$.

$$L(x,\theta) = \prod_{t=1}^{n} P(X_t = x_t | X_{t-1} = x_{t-1})$$

where $L(x,\theta)$ is the conditional likelihood function. Let us denote x_{t-1} by x_1 and x_t by x_2. Then

$$L^*(x,\theta) = \log L(x,\theta) = \sum_{t=1}^{n} \log P(X_t = x_2 | X_{t-1} = x_1)$$

Now;

$$P(X_t = x_2|X_{t-1} = x_1) = \sum P(I_t(\alpha)x_1 = x_2 - k)P(\varepsilon_t = k)$$

for $x_1 = 0$, reduces to

$$P(X_t = x_2|X_{t-1} = 0) = P(\varepsilon_t = x_2)$$

and for $x_1 \neq 0$,

$$\begin{aligned} P(I_t(\alpha)x_1 &= x_2 - k) = P(I_t(\alpha) = \frac{x_2 - k}{x_1}) \\ &= P(U_t = \frac{x_2 - k}{x_1} + 1) \\ &= Binom(\frac{x_2 - k}{x_1} + 1, \alpha, 2) \\ &= \binom{2}{\frac{x_2-k}{x_1} + 1} \alpha^{\frac{x_2-k}{x_1}+1} (1-\alpha)^{1-\frac{(x_2-k)}{x_1}}. \end{aligned}$$

where k has values $x_2 - x_1, x_2, x_2 + x_1$: Numerical maximization of the log-likelihood function (i.e. minimization of $-\log L(x;\theta)$) can be performed using the optimization Matlab toolbox, the function "fminunc" is a minimization function for "$-\log L(x;\theta)$".

3.4 Monte Carlo Results

In this section, we compare the obtained estimators. 500 series of size $N = 100, 300$ and 500 are simulated from the stationary TPDINAR(1) process with parameters $(\alpha, \mu_1, \mu_2) = (0.4, 0.8, 1), (0.7, 0.8, 1), (0.8, 0.8, 1), (0.4, 2, 1), (0.7, 2, 1)$, and $(0.8, 2, 1)$. In each case the mean and the Standard Deviation (SD) of the estimators are calculated. The results are presented in these Tables 1 and 2.

The results show that CLS, YW and CML estimators are close to each other. The mean of each estimated parameter is close to the corresponding true value and the standard deviations are small in all cases.

4 Application of TPDINAR(1)

We now fit the stationary TPDINAR(1) model to a set of real life data in order to show the advantage of using it. We compare here the new TPDINAR(1) model with the STINAR(1) and TINAR(1) models introduced by Barreto-Souza and Bourguignon [4]. Consider the Z_t time series of annual increases in the Swedish population (per thousand people) from 1750 to 1849, as shown in Thomas [12]. Recently, this data collection was applied in [4].

The Swedish population rates series are depicted in this table with some descriptive statistics. Since there are negative integer values in the series, then only the true time series can be used in this case. The following Figures display the time series data and associated sample autocorrelations.

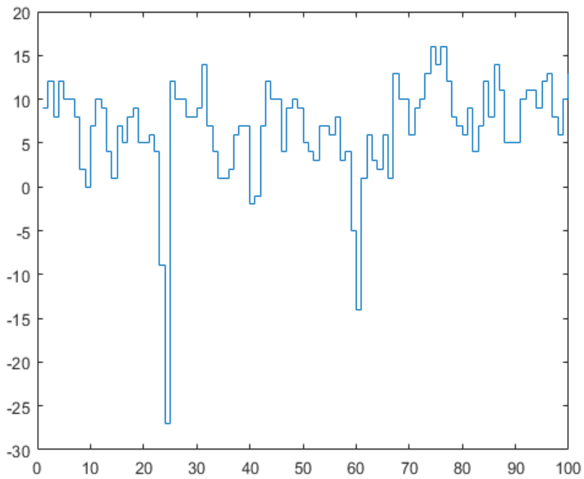

Fig.2(a) The time series plot for the Swedish population rates series (per thousand people) for the century between 1750 - 1849.

Table 1: Mean and SD of estimators for difference estimation methods for TPDINAR(1) with $\mu_1 = 0.8$ and $\mu_2 = 1$.

		CLS			YW			CML		
N	α	$\hat{\alpha}_{CLS}$	$\hat{\mu}_{1CLS}$	$\hat{\mu}_{2CLS}$	$\hat{\alpha}_{YW}$	$\hat{\mu}_{1YW}$	$\hat{\mu}_{2YW}$	$\hat{\alpha}_{CML}$	$\hat{\mu}_{1CML}$	$\hat{\mu}_{2CML}$
	0.4	0.4082	0.6892	0.8898	0.4089	0.7242	0.9242	0.4007	0.7765	0.9712
	SD	0.0395	0.1819	0.1747	0.0394	0.1847	0.1767	0.0365	0.1876	0.1998
100	0.7	0.6818	0.6887	0.9098	0.6805	0.8277	1.0482	0.6676	0.7153	0.9695
	SD	0.0255	0.2010	0.1968	0.0261	0.2011	0.1920	0.0398	0.1550	0.1331
	0.8	0.7530	0.6259	0.8773	0.7444	0.8296	1.0844	0.7496	0.6946	0.9470
	SD	0.0402	0.1742	0.1742	0.0437	0.1729	0.1560	0.0212	0.2089	0.1666
	0.4	0.3950	0.7432	0.9414	0.3954	0.7794	0.9772	0.4042	0.7865	0.9706
	SD	0.0296	0.1631	0.1578	0.0295	0.1613	0.1562	0.0296	0.0838	0.0654
300	0.7	0.7036	0.6807	0.8813	0.7030	0.8201	1.0193	0.7099	0.7707	0.9632
	SD	0.0297	0.1434	0.1410	0.0299	0.1399	0.1371	0.0230	0.0896	0.0936
	0.8	0.7660	0.7560	0.9904	0.7655	0.9932	1.2299	0.7894	0.7648	0.9715
	SD	0.0343	0.2168	0.2292	0.0340	0.2153	0.2280	0.0270	0.0790	0.0872
	0.4	0.3887	0.7618	0.9671	0.3889	0.7998	1.0041	0.3888	0.8155	1.0321
	SD	0.0437	0.0834	0.0880	0.0436	0.0844	0.0885	0.0266	0.0803	0.0883
500	0.7	0.6914	0.7968	0.9970	0.6910	0.9291	1.1288	0.7017	0.7940	0.9953
	SD	0.0278	0.1002	0.0985	0.0277	0.1012	0.1005	0.0212	0.0801	0.0784
	0.8	0.8033	0.7707	0.9662	0.8023	1.0044	1.1990	0.8039	0.7894	0.9987
	SD	0.0427	0.1543	0.1312	0.0435	0.1619	0.1382	0.0258	0.0677	0.0762

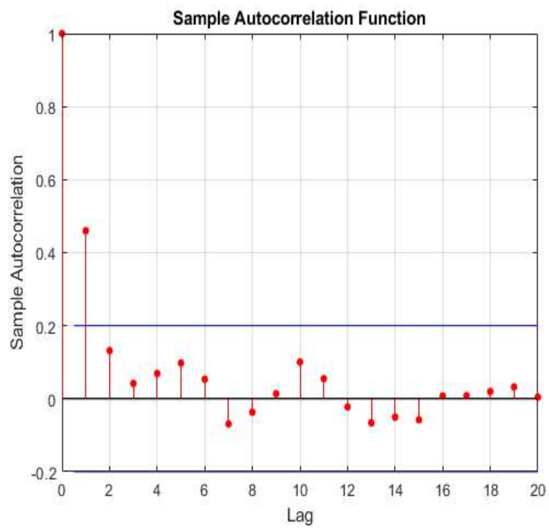

Fig.2(b) The autocorrelation function plot for the Swedish population rates series (per thousand people) for the century between 1750 - 1849.

In Table 4 We give three goodness-of-fit statistics too with the parameter estimates: RMS stands for ”root mean square of differences between observed and predicted values,” MA for ”absolute mean of differences between observed

Table 2: Mean and SD of estimators for difference estimation methods for TPDINAR(1) with $\mu_1 = 2$ and $\mu_2 = 1$.

		CLS			YW			CML		
N	α	$\hat{\alpha}_{CLS}$	$\hat{\mu}_{1CLS}$	$\hat{\mu}_{2CLS}$	$\hat{\alpha}_{YW}$	$\hat{\mu}_{1YW}$	$\hat{\mu}_{2YW}$	$\hat{\alpha}_{CML}$	$\hat{\mu}_{1CML}$	$\hat{\mu}_{2CML}$
	0.4	0.3880	2.2147	1.1491	0.3855	1.8720	0.8215	0.4612	1.9964	0.9301
	SD	0.0785	0.6769	0.5719	0.0477	0.4285	0.5343	0.0717	0.4471	0.2155
100	0.7	0.6558	1.8008	0.6704	0.6539	1.8497	0.7133	0.6809	1.9618	0.9187
	SD	0.0683	0.4606	0.4197	0.0670	0.5248	0.4243	0.0512	0.2018	0.1675
	0.8	0.7652	2.0802	0.8431	0.7807	1.7753	0.6106	0.7488	2.0493	1.1289
	SD	0.0378	0.5898	0.5807	0.0451	0.5780	0.7463	0.0573	0.2651	0.2557
	0.4	0.3992	1.8823	0.8656	0.3996	1.8776	0.8611	0.4025	1.9618	0.9385
	SD	0.0404	0.2801	0.2836	0.0404	0.2810	0.2798	0.0270	0.1717	0.1783
300	0.7	0.6837	1.9238	0.8809	0.6830	1.9371	0.8918	0.6993	1.9916	0.9859
	SD	0.0438	0.4009	0.3728	0.0436	0.4316	0.3755	0.0264	0.2060	0.1731
	0.8	0.7955	1.7207	0.6783	0.7947	1.7571	0.7110	0.7939	2.0153	0.9923
	SD	0.0323	0.3001	0.3988	0.0325	0.2963	0.3039	0.0230	0.1425	0.1286
	0.4	0.4108	2.0188	1.0349	0.4111	2.0106	1.0268	0.4037	2.0039	1.0158
	SD	0.0314	0.2485	0.2483	0.0314	0.2496	0.2451	0.0182	0.1258	0.1344
500	0.7	0.6994	1.9852	0.9879	0.6991	1.9794	0.9808	0.6989	1.9585	0.9660
	SD	0.0388	0.3576	0.3638	0.0388	0.3764	0.3575	0.0261	0.1244	0.1171
	0.8	0.7920	1.8601	0.8467	0.7916	1.8716	0.8559	0.8032	1.9913	1.0109
	SD	0.0265	0.4037	0.4233	0.0266	0.4184	0.4011	0.0179	0.1335	0.1548

Table 3: Descriptive statistics for the Swedish population rates series (per thousand people) for the century between 1750 and 1849.

Minimum	Mean	Median	Variance	$\hat{\rho}_1$	Maximum
-27	6.69	7.5	34.56	0.46	16

Table 4: Estimate of the TPDINAR(1), STINAR(1), and TINAR(1) processes' parameters including the RMS, MA, and MD goodness of fit statistics.

Model	Estimates	RMS	MA	MD
TPDINAR(1)	$\hat{\alpha} = 0.7300$	5.1814	3.3976	2.4126
	$\hat{\mu}_1 = 6.1762$			
	$\hat{\mu}_2 = 2.5636$			
STINAR(1)	$\hat{\alpha} = 0.465$	5.2064	3.4200	2.4381
	$\hat{\mu}_1 = 8.883$			
	$\hat{\mu}_2 = 2.193$			
TINAR(1)	$\hat{\alpha} = 0.465$	5.2064	3.4201	2.4379
	$\hat{\mu}_1 = 11.03$			
	$\hat{\mu}_2 = 7.449$			

and predicted values," and MD for "absolute median of differences between observed and predicted values"; Conditional maximum likelihood parameters estimates and their standard errors are calculated. By comparing these values with the results [4], it is generally believed that the best model to explain the data gives the smallest values for these quantities.

From Table 4 we observe that, based on the goodness-of-fit statistics, our TPDINAR(1) process produces a somewhat better fit to the data than the other two models.

*Remark.*CLS, and YW parameters estimates and their standard errors are close to CML parameter estimates.

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