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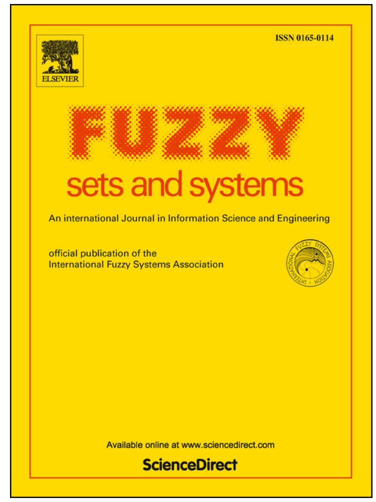

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Abstract

This paper is the natural extension of Fuzzy Closure Structures as Formal Concepts. In this paper we take into consideration the concept of closure system which is not dealt with in the previous one. Hence, a connection must be found between fuzzy ordered sets and a crisp ordered set. This problem is two-fold, the core of the fuzzy orders can be considered in order to complete the ensemble, or the crisp order can be fuzzified. Both ways are studied in the paper. The most interesting result is, similarly to the previous paper, that closure systems are formal concepts of these Galois connections as well.

Keywords: Closure system, Galois connection, Fuzzy lattice

1. Introduction

Closure systems, also called Moore families, were introduced by E. H. Moore in 1910 [16]. They play a major role in computer science and both pure and applied mathematics [11]. The extension to the fuzzy framework of closure systems has been approached from several distinct perspectives in the literature, to cite a few we mention the following [2, 9, 12, 15, 17, 19]. In this work, we will use the definition introduced in [19], which extends closure systems as meet-subsemilattices in the framework of complete fuzzy lattices. This extension is done in two levels, first as crisp sets and later as fuzzy sets, both these notions will be used in the paper. The counterpart of closure systems, the so-called closure operators, have also been extended to the fuzzy setting, and most authors use the same definition, i.e., a mapping that is inflationary, isotone and idempotent. Fuzzy closure operators were defined in [2, 6] and they appear naturally in different areas of fuzzy logic and its applications.

Besides closure structures, the main notion in this paper are Galois connections and formal concepts. Galois connections seem to be ubiquitous, they appear in several mathematical theories and in plenty of instances in the theory of relations [21]. For instance, it is well-known that the derivation operators of Formal Concept Analysis form a Galois connection [13]. Therefore, the research on Galois connections complements that on FCA. The extension of this notion to the fuzzy framework was introduced by Bělohlávek [1], the so-called Galois condition, which is an "if and only if" in the crisp case, is substituted by an equality of the fuzzy preorders. This extension provided a way to study Fuzzy Formal Concept Analysis.

In this paper, we continue the study started in [18]. In that paper, the framework was a complete $\mathbb{L}$-fuzzy lattice (A, ρ), where the infimum and the supremum are denoted by $\sqcap$ and $\sqcup$, respectively and $\mathbb{L}$ is a complete residuated lattice. Lattice type fuzzy orders were originally introduced by Bělohlávek [5]. There are other definitions of lattice type orders in the literature, e.g., [22], but this is also the case concerning fuzzy orders. The notion of fuzzy order is scattered through the literature; many distinct definitions have appeared since the original one by Zadeh, but there is no exhaustive analysis or comparison among them. The one used in this paper follows the spirit of Bodenhofer's definition [10].

The topic of [18] was the study of the mappings that relate one closure structure to the other. For example, if $\Phi \in L^A$ is a fuzzy closure system, then the mapping $\widehat{\mathsf{c}}(\Phi)\colon A \to A$ defined as $\widehat{\mathsf{c}}(\Phi)(a) = \bigsqcap(a^\rho \otimes \Phi)$ is a closure operator. This allows us to define $\widehat{\mathsf{c}}\colon L^A \to A^A$ as $\Phi \mapsto \widehat{\mathsf{c}}(\Phi)$ for all $\Phi \in L^A$. Similarly, for a closure operator $\mathsf{c}\colon A \to A$, the mapping $\widetilde{\Psi}\colon A^A \to L^A$ maps c to a fuzzy closure system $\widetilde{\Psi}(\mathsf{c})$, defined by $\widetilde{\Psi}(\mathsf{c})(a) = \rho(\mathsf{c}(a), a)$. One of the main results in [18] proved that the pair $(\widehat{\mathsf{c}}, \widetilde{\Psi})$ is a fuzzy Galois connection between (L^A, S) and $(\mathrm{Isot}(A^A), \tilde{\rho})$. Furthermore, it is proved that any pair of closure structures (Φ, c) is a formal concept of the Galois connection. This problem is also studied for fuzzy closure relations, hence studying two additional Galois connections, one between $(\mathrm{Isot}(A^A), \widetilde{\rho})$ and $(\mathrm{IsotTot}(L^{A\times A}), \hat{\rho})$ and another one between (L^A, S) and $(\mathrm{IsotTot}(L^{A\times A}), \hat{\rho})$.

The aim of this paper is to insert crisp closure systems in the problem. Notice that there are different approaches to this addition since crisp sets form a classical lattice $(2^A, \subseteq)$, whereas the three sets mentioned above are endowed with fuzzy relations. The approach in this paper is two-fold. First, we examine the existence of crisp Galois connections between the crisp lattice $(2^A, \subseteq)$ and the three sets of the previous paragraph endowed with the 1-cut of their fuzzy relations. Thus, we turn this problem into a fully crisp problem and study the existence of Galois connections and the behavior of the sets of fixed points. The second approach would be considering the fuzzification of the crisp order and study the existence of fuzzy Galois connections there. Surprisingly, there is a fuzzy Galois connection between $(2^A, S)$ and $(\mathrm{Isot}(A^A), \tilde{\rho})$ if and only if (A, ρ) is a crisp lattice, that is, $\rho(a, b) \in \{0, 1\}$ for all $a, b \in A$.

The outline of the paper is as follows. First, a section of preliminaries to recall already known results that are useful to understand the paper better. The next section summarizes the framework of the problem, the lattices, posets and preposets in consideration, in addition to the results that are derived straightforwardly from [18] and taking the 1-cut of the fuzzy relations. The following section proves that the mappings we are considering form a Galois connection between $(2^A, \subseteq)$ and $(\mathrm{Isot}(A^A), \tilde{\rho}^1)$ and closure systems and closure operators are indeed related to the formal concepts. Later it is proved that there is not a fuzzy adjunction between $(2^A, \subseteq)$ and $(L^A, \subseteq)$. However, if we restrict to the set of extensional fuzzy sets, then there is an adjunction between $(2^A, \subseteq)$ and $(\mathrm{Ext}(L^A), \subseteq)$ and again closure systems and fuzzy closure systems are related to the sets of formal concepts. Actually, under the most common set of condition in applications, namely linear order of truth values and finite complete fuzzy lattice, we have characterized the fixed points of this adjunction. The next section examines whether we could solve this problem in the fuzzy framework, that is, fuzzifying the crisp subsethood order $\subseteq$. The answer is negative, since we prove that the fuzzy Galois connection exists if and only if the complete fuzzy lattice (A, ρ) is a crisp lattice. Last, there is a section of conclusions and further work where the results are discussed and some hints of future research lines are shown.

2. Preliminaries

The main structure used in this paper is going to be that of Galois connection in the fuzzy framework. Therefore, here we present some of the concepts in the fuzzy setting necessary to follow the paper.

The general framework throughout the paper is going to be a complete residuated lattice $\mathbb{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$. For the properties of complete residuated lattices we refer the reader to [4, Chapter 2].

Given a $\mathbb{L}$-fuzzy set $X \in L^A$, the 1-cut of X, denoted by X^1, is the crisp set $\{a \in A \mid X(a) = 1\}$. Equivalently, we will consider it as the fuzzy set whose characteristic mapping is $X^1(a) = 1$ if $X(a) = 1$ and 0 otherwise. Given a fuzzy relation, or $\mathbb{L}$-relation, μ between A and B, i.e., a crisp mapping $\mu\colon A \times B \to L$, and $a \in A$, the *afterset* a^μ is the fuzzy set $a^\mu\colon B \to L$ given by $a^\mu(b) = \mu(a, b)$. A fuzzy relation μ is said to be *total* if, for all $a \in A$, the aftersets a^μ are normal fuzzy sets, i.e., there exists $x \in A$ such that $a^\kappa(x) = 1$.

For ρ being a binary $\mathbb{L}$-relation in A, we say that

- ρ is *reflexive* if $\rho(x, x) = 1$ for all $x \in A$.
- ρ is *symmetric* if $\rho(x, y) = \rho(y, x)$ for all $x, y \in A$.
- ρ is *antisymmetric* if $\rho(x, y) \otimes \rho(y, x) = 1$ implies $x = y$ for all $x, y \in A$.
- ρ is *transitive* if $\rho(x, y) \otimes \rho(y, z) \leq \rho(x, z)$ for all $x, y, z \in A$.

Notice that this definition of antisymmetry differs from the original one by Zadeh. There is a wide variety of distinct definitions of fuzzy order in the literature, a nice survey on this topic can be found in [7, 8]. In particular, Bodenhofer's definition of fuzzy order needs a fuzzy preorder ρ and a fuzzy similarity relation $\approx$. For a single fuzzy preorder, several similarities can be considered. The approach used in this paper follows the idea given by Bodenhofer [10, Section 5] where the equality relation $\approx$ is defined by the preorder ρ.

Definition 1. Given a non-empty set A and a binary $\mathbb{L}$-relation ρ on A, the pair (A, ρ) is said to be a

- *fuzzy preposet* if ρ is a *fuzzy preorder*, i.e. if ρ is reflexive and transitive;
- *fuzzy poset* if ρ is a *fuzzy order*, i.e. if ρ is reflexive, antisymmetric and transitive.

A typical example of fuzzy poset is (L^A, S), for any set A. If (A, ρ) is a fuzzy poset, we will also use the so-called *full fuzzy powering* $\rho_\propto$, which is the fuzzy relation on L^A defined as follows: for all $X, Y \in L^A$,

$$\rho_\propto(X, Y) = \bigwedge_{x,y \in A} (X(x) \otimes Y(y)) \rightarrow \rho(x, y).$$

Even though the relation $\rho_\propto$ is not a preorder in general, it satisfies a sort of transitivity.

Theorem 2. *Let (A, ρ) be a fuzzy poset and $X, Y, Z \in L^A$. If Y is normal then,*

$$\rho_\propto(X, Y) \otimes \rho_\propto(Y, Z) \leq \rho_\propto(X, Z).$$

The definition of infimum and supremum used throughout the paper is the standard one in the fuzzy framework, originally introduced by Bělohlávek in [5], we write it out to ease the reading of the paper.

Definition 3. Let (A, ρ) be a fuzzy poset and $X \in L^A$. The down-cone (resp. up-cone) of X is defined as a fuzzy set with the following membership function.

$$X_\rho(x) = \bigwedge_{a \in A} X(a) \to \rho(x, a) \left(\text{resp. } X^\rho(x) = \bigwedge_{a \in A} X(a) \to \rho(a, x)\right).$$

Definition 4. Let (A, ρ) be a fuzzy poset and $X \in L^A$. An element $a \in A$ is said to be *infimum* (resp. *supremum*) of X if the following conditions hold:

1. $X_\rho(a) = 1$ (resp. $X^\rho(a) = 1$).
2. $X_\rho(x) \leq \rho(x, a)$ (resp. $X^\rho(x) \leq \rho(a, x)$), for all $x \in A$.

Hereinafter, suprema and infima in A will be denoted by $\sqcup$ and $\sqcap$, respectively. As a straightforward consequence we have that, if $a = \bigsqcap X$, then $X \subseteq a^\rho$.

Theorem 5. *An element $a \in A$ is infimum (resp. supremum) of $X \in L^A$ if and only if, for all $x \in A$,*

$$\rho(x, a) = X_\rho(x) \qquad (\textit{resp. } \rho(a, x) = X^\rho(x)).$$

Definition 6. Let (A, ρ) be a fuzzy poset. The couple (A, ρ) is said to be a complete fuzzy lattice if $\bigsqcap X$ and $\bigsqcup X$ exist for all $X \in L^A$.

The last definition was originally introduced by Bělohlávek under the name *completely lattice* $\mathbb{L}$*-ordered set*, and has also been used by Zhang and Fan [24] under the name $\mathbb{L}$*-fuzzy complete lattice* or Konečny [14] under the name *fuzzy complete lattice*.

Fuzzy closure operators and systems were first introduced in the fuzzy framework by Bělohlávek in [3]. The definition of fuzzy closure operator used in this paper is the original one, used also in [2, 3, 4], i.e., a mapping $\mathsf{c}\colon A \to A$ that is inflationary, isotone and idempotent. On the other hand, we will consider fuzzy closure systems on arbitrary complete fuzzy lattices, not necessarily on the powerset, as defined in [19], where they are extensional hulls of crisp sets which are closure systems.

Every fuzzy order induces a symmetric relation, called symmetric kernel relation.

Definition 7. Given a fuzzy poset (A, ρ), the *symmetric kernel relation* is defined as $\approx\colon A \times A \to L$ where $(x \approx y) = \rho(x, y) \otimes \rho(y, x)$ for all $x, y \in A$.

The notion of extensionality was introduced in the very beginning of the study of fuzzy sets. It has also been called compatibility (with respect to the similarity relation) in the literature.

Definition 8. A fuzzy set $X \in L^A$ is said to be extensional or compatible with respect to $\approx$ if it satisfies $X(x) \otimes (x \approx y) \le X(y)$, for all $x, y \in A$.

Definition 9. Given a fuzzy set $X \in L^A$, the extensional hull of X, denoted by $X^{\approx}$, is the smallest extensional set that contains X. Its explicit formula is the following:

$$X^{\approx}(x) = \bigvee_{a \in A} (X(a) \otimes (a \approx x)).$$

Remark 1. For a crisp set $X \subseteq A$, the expression of the extensional hull is simplified since we have

$$X^{\approx}(x) = \bigvee_{a \in A} (X(a) \otimes (a \approx x)) = \bigvee_{a \in X} (a \approx x).$$

We focus now on closure structures in the fuzzy framework. The definition of fuzzy closure operator in the fuzzy setting is the one used in [2, 4].

Definition 10. Given a fuzzy poset (A, ρ), a mapping $\mathsf{c}\colon A \to A$ is said to be a *closure operator* on $\mathbb{A}$ if the following conditions hold:

1. $\rho(a, b) \le \rho(\mathsf{c}(a), \mathsf{c}(b))$, for all $a, b \in A$ (isotonicity)
2. $\rho(a, \mathsf{c}(a)) = 1$, for all $a \in A$ (inflationarity)
3. $\rho(\mathsf{c}(\mathsf{c}(a)), \mathsf{c}(a)) = 1$, for all $a \in A$ (idempotency)

An element $q \in A$ is said to be *closed* for c if $\rho(\mathsf{c}(q), q) = 1$.

The counterpart of closure operators is closure systems, there are several approaches to defining this concept in the fuzzy framework, originally introduced on the fuzzy powerset lattice by Bělohlávek [2]. The extension to arbitrary complete fuzzy lattices was introduced in [19] and is the one used here.

Definition 11. Let (A, ρ) be a complete fuzzy lattice. A crisp set $\mathcal{F} \subseteq A$ is said to be a closure system if $\bigsqcap X \in \mathcal{F}$ for all $X \in L^{\mathcal{F}}$.

This definition of closure system in the fuzzy framework maintains the one-to-one relation of the crisp case [17].

Theorem 12. *Let (A, ρ) be a complete fuzzy lattice. The following assertions hold:*

1. *If c is a closure operator on (A, ρ), the crisp set $\mathcal{F}_{\mathsf{c}}$ defined as $\{a \in A \mid \mathsf{c}(a) = a\}$ is a closure system.*
2. *If $\mathcal{F}$ is a closure system, the mapping $\mathsf{c}_{\mathcal{F}} \colon A \to A$ defined as $\mathsf{c}_{\mathcal{F}}(a) = \bigsqcap(a^{\rho} \otimes \mathcal{F})$ is a closure operator on (A, ρ).*
3. *If $\mathsf{c} \colon A \to A$ is a closure operator on (A, ρ), then $\mathsf{c}_{\mathcal{F}_{\mathsf{c}}} = \mathsf{c}$.*
4. *If $\mathcal{F}$ is a closure system, then $\mathcal{F} = \mathcal{F}_{\mathsf{c}_{\mathcal{F}}}$.*

Remark 2. Notice that the set $\mathcal{F}_{\mathsf{c}}$ can be defined as $\{a \in A \mid \rho(\mathsf{c}(a), a) = 1\}$ since, as c is inflationary, if $\rho(\mathsf{c}(a), a) = 1$ we would also have $\rho(a, \mathsf{c}(a)) = 1$ and then $\mathsf{c}(a) = a$ by antisymmetry.

Since closure systems are crisp structures with certain fuzzy properties, it is natural to wonder whether a fuzzy structure can be defined. An affirmative answer was found in [19].

Definition 13. Let (A, ρ) be a complete fuzzy lattice. A fuzzy set $\Phi \in L^A$ is said to be a fuzzy closure system if Φ^1 is a closure system and Φ is the extensional hull of Φ^1.

This definition of fuzzy closure system maintains the well-known one-to-one relationship between closure systems and closure operators [19].

Theorem 14. *Let (A, ρ) be a complete fuzzy lattice. The following assertions hold:*

1. *If* c *is a closure operator on* (A, ρ)*, the fuzzy set* Φ_{c} *defined as* $\Phi_{\mathsf{c}}(a) = \rho(\mathsf{c}(a), a)$ *is a fuzzy closure system.*
2. *If* Φ *is a fuzzy closure system, the mapping* $\mathsf{c}_\Phi\colon A \to A$ *defined as* $\mathsf{c}_\Phi(a) = \bigsqcap(a^\rho \otimes \Phi)$ *is a closure operator on* (A, ρ)*.*
3. *If* $\mathsf{c}\colon A \to A$ *is a closure operator on* (A, ρ)*, then* $\mathsf{c}_{\Phi_{\mathsf{c}}} = \mathsf{c}$*.*
4. *If* Φ *is a fuzzy closure system on* (A, ρ)*, then* $\Phi = \Phi_{\mathsf{c}_\Phi}$*.*

In [20], the mapping $f(x) = \bigsqcap(x^\rho \otimes \Phi)$ was thoroughly studied. In particular we have the following result.

Proposition 15. *Let* (A, ρ) *be a complete fuzzy lattice and* c *be a closure operator on* $\mathbb{A}$*. For* $\Phi_{\mathsf{c}} \in L^A$*, defined as* $\Phi_{\mathsf{c}}(a) = \rho(\mathsf{c}(a), a)$*, we have that* $(\Phi_{\mathsf{c}})^1 = \mathcal{F}_{\mathsf{c}}$ *is the closure system associated to* c *and, for all* $x \in A$*,*

$$\bigsqcap(x^\rho \otimes \Phi_{\mathsf{c}}) = \bigsqcap(x^\rho \cap \mathcal{F}_{\mathsf{c}}) \tag{1}$$

In addition, for all $a \in A$*, one has*

$$\Phi_{\mathsf{c}}(a) = \bigvee_{x \in \mathcal{F}_{\mathsf{c}}} (a \approx x) \tag{2}$$

Fuzzy Galois connections are a main concept in this paper as well. Let us recall the definition.

Definition 16 ([23])**.** Let (A, ρ_A) and (B, ρ_B) be fuzzy posets, $f\colon A \to B$ and $g\colon B \to A$ be two mappings.

- The pair (f, g) is called an *isotone fuzzy Galois connection* or *fuzzy adjunction between* (A, ρ_A) *and* (B, ρ_B), denoted by $(f, g) : (A, \rho_A) \rightleftharpoons (B, \rho_B)$, if
$$\rho_A(g(b), a) = \rho_B(b, f(a)) \quad \text{for all } a \in A \text{ and } b \in B.$$
- The pair (f, g) is called a *fuzzy Galois connection between* (A, ρ_A) *and* (B, ρ_B), denoted by $(f, g) : (A, \rho_A) \rightharpoonup\!\!\!\!\leftharpoondown (B, \rho_B)$, if
$$\rho_A(a, g(b)) = \rho_B(b, f(a)) \quad \text{for all } a \in A \text{ and } b \in B.$$

A *fixed point*, also called *fixed pair* or *formal concept*, of a fuzzy Galois connection (f, g) is a couple $(a, b) \in A \times B$ such that $f(a) = b$ and $g(b) = a$.

3. Framework of the proposal

This paper is a sequel of [18], where the setting was a complete fuzzy lattice (A, ρ). In that paper, three fuzzy Galois connections were proved to exist and the relationship among them and the fuzzy closure structures was studied. Thus, we obtained the following.

$$(L^A, S) \xrightarrow{(\widehat{\mathsf{c}},\widetilde{\Psi})} (\mathrm{Isot}(A^A), \widetilde{\rho})$$

$$(L^A, S) \xrightarrow{(\kappa,\widehat{\Psi})} (\mathrm{IsotTot}(L^{A\times A}), \hat{\rho})$$

$$(\mathrm{Isot}(A^A), \widetilde{\rho}) \xrightarrow{(-^1,-^{\approx})} (\mathrm{IsotTot}(L^{A\times A}), \hat{\rho}),$$

where $\mathrm{Isot}(A^A)$ is the set of (crisp) isotone mappings on A, $\mathrm{IsotTot}(L^{A\times A})$ is the set of isotone and total fuzzy relations on A and the fuzzy relations $\widetilde{\rho}, \hat{\rho}$ are defined as follows,

$$\widetilde{\rho}(f_1, f_2) = \bigwedge_{x\in A} \rho(f_1(x), f_2(x)) \text{ for all } f_1, f_2 \in A^A$$
$$\hat{\rho}(\kappa_1, \kappa_2) = \bigwedge_{a\in A} \rho_\propto(a^{\kappa_1}, a^{\kappa_2}) \text{ for all } \kappa_1, \kappa_2 \in L^{A\times A}.$$

The mappings in the diagram are defined as follows,

$$\begin{array}{ll}
\widehat{\mathsf{c}}\colon L^A \to \mathrm{Isot}(A^A) & \widehat{\mathsf{c}}(\Phi) = \mathsf{c}_\Phi \\
\widetilde{\Psi}\colon \mathrm{Isot}(A^A) \to L^A & \widetilde{\Psi}(f) = \Phi_f \\
\kappa\colon L^A \to \mathrm{IsotTot}(L^{A\times A}) & \kappa(\Phi)(a,b) = \bigsqcap(a^\rho \otimes \Phi) \approx b \\
\widehat{\Psi}\colon \mathrm{IsotTot}(L^{A\times A}) \to L^A & \widehat{\Psi}(\mu)(x) = \rho_\propto(x^\mu, x)
\end{array}$$

Results in [18] involve fuzzy closure systems, closure operators and fuzzy closure relations, but closure systems as crisp sets are not considered. Thus, our next step is to study a similar problem with the partially ordered set $(2^A, \subseteq)$. This addition to the problem is not straightforward since $(2^A, \subseteq)$ is a crisp poset, whereas (L^A, S) and $(\mathrm{Isot}(A^A), \widetilde{\rho})$ are complete fuzzy lattices and $(\mathrm{IsotTot}(L^{A\times A}), \hat{\rho})$ is a fuzzy preposet [18]. Thus, there are two possible ways, either consider the 1-cut of the fuzzy relations and study the crisp problem, or consider the fuzzification of the crisp partial order and study the problem in the fuzzy setting. In this paper we will tackle both ways.

First, to study the crisp problem, we have to define the 1-cut of each fuzzy relation. It is well-known that S^1 is Zadeh's inclusion, hence throughout the paper we will follow the classical notation for subsethood $\subseteq$. We will also consider the 1-cut of $\widetilde{\rho}$, we will denote $\widetilde{\rho}^1(f,g) = 1$ as $f \preceq g$, and $\hat{\rho}^1(\kappa_1,\kappa_2) = 1$ will be denoted by $\kappa_1 \sqsubseteq \kappa_2$, that is,

$$f \preceq g \text{ if and only if } \rho(f(a), g(a)) = 1, \text{ for all } a \in A. \tag{3}$$

$$\kappa_1 \sqsubseteq \kappa_2 \text{ if and only if } x^{\kappa_1}(y) \otimes x^{\kappa_2}(z) \leq \rho(y,z) \text{ for all } x,y,z \in A. \tag{4}$$

Observe that the set $(\mathrm{IsotTot}(L^{A\times A}), \sqsubseteq)$ is a preordered set. In addition, $(L^A, \subseteq)$ and $(\mathrm{Isot}(A^A), \preceq)$ are complete lattices by [18, Proposition 13] and [4, Theorem 4.55].

Proposition 17. *The following pairs of mappings form two Galois connections and an adjunction, respectively,*

$$(L^A, \subseteq) \xrightarrow{(\widehat{\mathrm{c}},\widetilde{\Psi})} (\mathrm{Isot}(A^A), \preceq)$$

$$(L^A, \subseteq) \xrightarrow{(\kappa,\widehat{\Psi})} (\mathrm{IsotTot}(L^{A\times A}), \sqsubseteq)$$

$$(\mathrm{IsotTot}(L^{A\times A}), \sqsubseteq) \overset{(-^1,-^{\approx})}{\rightleftharpoons} (\mathrm{Isot}(A^A), \preceq).$$

Proof. All these Galois connections (either isotone or antitone) were proved in the fuzzy setting in [18]. The restriction to the 1-cut of the fuzzy relations maintains the crisp Galois connection. □

The question now is whether it is possible to consider $(2^A, \subseteq)$ in this problem. This is the topic discussed in the next section.

4. Adding the classical powerset to the problem

The main goal of this section is to study the Galois connections in Figure 1 and examine the relationship between their fixed points and fuzzy closure structures. The existence of these two Galois connections will be studied independently in the following subsections.

$$(2^A, \subseteq) \xleftrightarrow{(\widetilde{\mathsf{c}},\mathcal{F})} (\mathrm{Isot}(A^A), \preceq)$$

$$(2^A, \subseteq) \underset{}{\overset{(-^1,-^{\approx})}{\rightleftarrows}} (L^A, \subseteq)$$

Figure 1: Antitone/isotone Galois connections

4.1. Study of the Galois connection between the set of crisp sets and the set of isotone mappings

The following lemma is a technical result. It is presented independently to ease the reading of the rest of the proofs. Since it is a particular case of a more general theorem in [18], we omit the proof.

Lemma 18. *For all $X \in 2^A$, the mapping $\widetilde{\mathsf{c}}(X)$ is inflationary and isotone.*
For all isotone mapping $f\colon A \to A$, the set $\Phi(a) = \rho(f(a), a)$ is an extensional set.

Following the spirit in [18], the mappings which form the Galois connections must be the ones relating closure systems and fuzzy closure operators, i.e., $\mathcal{F}(f) = \{a \in A \mid \rho(f(a), a) = 1\}$ and $\widetilde{\mathsf{c}}(X)(a) = \prod(a^\rho \otimes X)$, for any isotone mapping $f \in \mathrm{Isot}(A^A)$ and any set $X \subseteq A$. Indeed, the couple $(\widetilde{\mathsf{c}}, \mathcal{F})$ forms a Galois connection.

Theorem 19. *Let $\widetilde{\mathsf{c}}\colon (2^A, \subseteq) \to (\mathrm{Isot}(A^A), \preceq)$ defined as $\widetilde{\mathsf{c}}(X)(a) = \mathsf{c}_X(a) = \prod(a^\rho \otimes X)$ and $\mathcal{F}\colon (\mathrm{Isot}(A^A), \preceq) \to (2^A, \subseteq)$ given by $\mathcal{F}(f) = \{a \in A \mid \rho(f(a), a) = 1\}$. Then, the couple $(\widetilde{\mathsf{c}}, \mathcal{F})$ is a Galois connection between $(2^A, \subseteq)$ and $(\mathrm{Isot}(A^A), \preceq)$.*

Proof. The mapping $\widetilde{\mathsf{c}}$ is well-defined by Lemma 18.

Assume $X \subseteq \mathcal{F}(f)$, we need to prove $f \preceq \widetilde{\mathsf{c}}(X)$, that is, $\rho(f(a), \widetilde{\mathsf{c}}(X)(a)) = \rho(f(a), \prod(a^\rho \otimes X)) = 1$, for all $a \in A$. By Theorem 5, it suffices to show that $(a^\rho \otimes X)_\rho(f(a)) = 1$. This is trivial for $x \notin X$. Otherwise, since $X \subseteq \mathcal{F}(f)$ and f is an isotone mapping,

$$(a^\rho \otimes X)(x) = \rho(a, x) \leq \rho(f(a), f(x)) = \rho(f(a), f(x)) \otimes \rho(f(x), x) \leq \rho(f(a), x)$$

Therefore, $X \subseteq \mathcal{F}(f)$ implies $f \preceq \widetilde{\mathsf{c}}(X)$.

Conversely, assume $f \preceq \widetilde{\mathsf{c}}(X)$, we need to prove $X \subseteq \mathcal{F}(f)$. Let $x \in X$, it suffices to show that $\rho(f(x), x) = 1$. Since $f \preceq \widetilde{\mathsf{c}}(X)$, in particular we have $\rho(f(x), \bigsqcap(x^\rho \otimes X)) = 1$. Since $\widetilde{\mathsf{c}}(X)(x)$ is an infimum we have $(x^\rho \otimes X)(y) \leq \rho(\widetilde{\mathsf{c}}(X)(x), y)$, for all $y \in A$. Therefore, in the particular case of x, we have $(x^\rho \otimes X)(x) = 1 = \rho(\widetilde{\mathsf{c}}(X)(x), x)$ which by transitivity gives $\rho(f(x), x) = 1$.

Therefore, $f \preceq \widetilde{\mathsf{c}}(X)$ implies $X \subseteq \mathcal{F}(f)$. □

The fixed points of the Galois connection introduced above are studied in the following theorem. As expected, the closure structures are fixed points of the Galois connection. Moreover, all the fixed points are closure structures, in spite of the cases examined in [18].

Theorem 20. *Let $X \in 2^A$ and $f \in \mathrm{Isot}(A^A)$. The following statements are equivalent:*

1. *The couple (X, f) is a fixed point of the Galois connection $(\widetilde{\mathsf{c}}, \mathcal{F})$.*
2. *The crisp set X is a closure system and $\widetilde{\mathsf{c}}(X) = \mathsf{c}_X = f$.*
3. *The mapping f is a closure operator and $\mathcal{F}(f) = \mathcal{F}_f = X$.*

Proof. Items 2 and 3 are equivalent by Theorem 12. Furthermore, the equivalence between 2 and 3 implies directly that the couple (X, f) is a fixed point of the Galois connection. To prove 1 implies 3, assume (X, f) is a fixed point. Then, we have to prove that $f = \mathsf{c}_X$ is a closure operator. By Lemma 18, f is inflationary and isotone.We only have to prove idempotency.

Let $a \in A$, and denote $m = f(a) = \bigsqcap(a^\rho \otimes X)$. Then, by the definition of infimum, $a^\rho \otimes X \subseteq m^\rho$ which implies $a^\rho \otimes X = a^\rho \otimes X \otimes X \subseteq m^\rho \otimes X$ since X is crisp. Taking infima in the last chain of inequalities we get $\rho(\bigsqcap(m^\rho \otimes X), \bigsqcap(a^\rho \otimes X)) = \rho(f(m), m) = 1$. Then, by antisymmetry, we have $f(a) = f(f(a))$. Therefore, f is idempotent and a closure operator. □

The last result shows that every fixed point of the Galois connection is a pair of fuzzy closure structures. This is interesting because this result does not hold in the general fuzzy setting. Recall that in the general fuzzy setting fuzzy closure structures were formal concepts of the Galois connections but there existed formal concepts that were not formed by fuzzy closure structures. Examples of such cases can be found in [18].

4.2. Study of the adjunction between the set of crisp sets and the set of fuzzy sets

Our intended goal would be to prove the adjunction between $(2^A, \subseteq)$ and (L^A, S). Unfortunately, the couple of mappings we have been considering so far, i.e.,

$$\Phi^1 = \{a \in A \mid \Phi(a) = 1\} \text{ and } X^{\approx}(a) = \bigvee_{x \in X} (x \approx a),$$

do not form an adjunction between these two posets. This is illustrated in the next example.

Example 1. Let $L = \{0, 0.5, 1\}$ be the three-valued Łukasiewicz residuated lattice. Consider the complete fuzzy lattice (A, ρ) where the universe set is $A = \{\bot, a, b, c, d, e, \top\}$ and ρ is the fuzzy order defined by the following table:

ρ	$\bot$	a	b	c	d	e	$\top$
$\bot$	1	1	1	1	1	1	1
a	0.5	1	0.5	1	1	1	1
b	0.5	0.5	1	1	1	1	1
c	0.5	0.5	0.5	1	1	1	1
d	0	0.5	0	0.5	1	0.5	1
e	0	0	0.5	0.5	0.5	1	1
$\top$	0	0	0	0.5	0.5	0.5	1

Consider $X = \{a, b\} \subseteq A$ and $\Phi = \{a, b\} \in L^A$. Then, it is clear that $X \subseteq \Phi^1$, but $X^{\approx}(\bot) = (\bot \approx a) \vee (\bot \approx b) = 0.5 \nleq 0 = \Phi(\bot)$. Therefore, $X^{\approx} \nsubseteq \Phi$ and $(-^1, -^{\approx})$ is not an adjunction.

Even though this pair is not an adjunction in general, restricting to sets that satisfy certain additional properties may work. For instance, every time since the introduction of fuzzy closure systems in the literature [2, 17], the condition of being extensional has been imposed to the definition. In addition, the restriction to the set of extensional fuzzy sets maintains the Galois connections in Proposition 17, since the images of both $\tilde{\Psi}$ and $\hat{\Psi}$ are always extensional sets. The former was proved in [18, Lemma 16] and the latter holds since, for any isotone and total relation $\kappa\colon A \times A \to L$ verifies that

$$\begin{aligned}\hat{\Psi}(\kappa)(x) \otimes (x \approx y) &= \rho_\propto(x^\kappa, x) \otimes \rho(x, y) \otimes \rho(y, x)\\ &\overset{\text{(i)}}{\leq} \rho_\propto(y^\kappa, x^\kappa) \otimes \rho_\propto(x^\kappa, x) \otimes \rho(x, y)\end{aligned}$$

$$\overset{\text{(ii)}}{\leq} \rho_{\propto}(y^{\kappa}, y) = \hat{\Psi}(\kappa)(y),$$

where (i) holds due to isotonicity and (ii) holds by Theorem 2.

Hence, we will work on the set of extensional fuzzy sets of A, denoted by $(\mathrm{Ext}(L^A), S)$. This set maintains many of the good properties of (L^A, S) such as being a complete fuzzy lattice. This is shown in the following result.

Proposition 21. *The couple* $(\mathrm{Ext}(L^A), S)$ *is a fuzzy sublattice of* (L^A, S).

Proof. It is well-known (e.g. see [4]) that (L^A, S) is a complete fuzzy lattice where, given $\Xi\colon L^A \to L$, the infimum and the supremum of Ξ are defined as

$$\big(\bigcap \Xi\big)(a) = \bigwedge_{X \in L^A} \big(\Xi(X) \to X(a)\big) \text{ and, } \big(\bigcup \Xi\big)(a) = \bigvee_{X \in L^A} (\Xi(X) \otimes X(a))$$

Now, let $\Xi\colon \mathrm{Ext}(L^A) \to L$, let us prove that $\bigcap \Xi$ and $\bigcup \Xi$ are extensional as well. Notice that, by hypothesis, $\Xi(X) = 0$, for all $X \notin \mathrm{Ext}(L^A)$. Thus,

$$\bigwedge_{X \in L^A} (\Xi(X) \to X(x)) = \bigwedge_{X \in \mathrm{Ext}(L^A)} (\Xi(X) \to X(x))$$

$$\bigvee_{X \in L^A} (\Xi(X) \otimes X(x)) = \bigvee_{X \in \mathrm{Ext}(L^A)} (\Xi(X) \otimes X(x))$$

First we show infima,

$$\begin{aligned}\left(\bigcap \Xi\right)(x) \otimes (x \approx y) &= \left(\bigwedge_{X \in \mathrm{Ext}(L^A)} (\Xi(X) \to X(x))\right) \otimes (x \approx y)\\ &\overset{\text{(i)}}{\leq} \bigwedge_{X \in \mathrm{Ext}(L^A)} ((\Xi(X) \to X(x)) \otimes (x \approx y))\\ &\overset{\text{(ii)}}{\leq} \bigwedge_{X \in \mathrm{Ext}(L^A)} (\Xi(X) \to (X(x) \otimes (x \approx y)))\\ &\overset{\text{(iii)}}{\leq} \bigwedge_{X \in \mathrm{Ext}(L^A)} (\Xi(X) \to X(y)) = \left(\bigcap \Xi\right)(y),\end{aligned}$$

where (i), (ii) hold due to (2.53) and (2.63) in [4], respectively and (iii) holds due to the extensionality of X and (2.43) in [4]

And now for suprema,

$$\left(\bigcup \Xi\right)(x) \otimes (x \approx y) = \left(\bigvee_{X \in \mathrm{Ext}(L^A)} (\Xi(X) \otimes X(x))\right) \otimes (x \approx y)$$
$$\stackrel{\text{(i)}}{=} \bigvee_{X \in \mathrm{Ext}(L^A)} (\Xi(X) \otimes X(x) \otimes (x \approx y))$$
$$\stackrel{\text{(ii)}}{\leq} \bigvee_{X \in \mathrm{Ext}(L^A)} (\Xi(X) \otimes X(y)) = \left(\bigcup \Xi\right)(y),$$

where (i) holds due to (2.50) in [4] and (ii) holds due to the extensionality of X. □

Moreover, the couple of mappings $(-^1, -^{\approx})$ forms an adjunction between $(2^A, \subseteq)$ and $(\mathrm{Ext}(L^A), \subseteq)$.

Theorem 22. *Let* $-^1 \colon (\mathrm{Ext}(L^A), \subseteq) \to (2^A, \subseteq)$ *and* $-^{\approx} \colon (2^A, \subseteq) \to (\mathrm{Ext}(L^A), \subseteq)$ *defined as*

$$\Phi \mapsto \Phi^1 = \{a \in A \mid \Phi(a) = 1\} \quad \textit{and} \quad X \mapsto X^{\approx}(a) = \bigvee_{x \in X} (x \approx a).$$

Then, the couple $(-^1, -^{\approx})$ *is an adjunction between* $(\mathrm{Ext}(L^A), \subseteq)$ *and* $(2^A, \subseteq)$*.*

Proof. It is clear that the mappings above are well-defined since the extensional hull of a set is always an extensional set.

Let $X \subseteq A$ and $\Phi \in \mathrm{Ext}(L^A)$ be two sets. Assume $X \subseteq \Phi^1$, then we have

$$\Phi(x) = \bigvee_{a \in A} (\Phi(a) \otimes (a \approx x)) \geq \bigvee_{a \in \Phi^1} (a \approx x) \geq \bigvee_{a \in X} (a \approx x) = X^{\approx}(x).$$

Thus, $X^{\approx} \subseteq \Phi$.

Conversely, assume $X^{\approx} \subseteq \Phi$ and let $x \in X$

$$X^{\approx}(x) = \bigvee_{a \in X} (a \approx x) \stackrel{\text{(i)}}{=} 1 \leq \Phi(x),$$

where (i) holds due to reflexivity. Therefore, $x \in \Phi^1$ which implies $X \subseteq \Phi^1$ and concludes the proof. □

As done in the previous case, a study of the fixed points of the adjunction is necessary. This is done in the result below.

Theorem 23. *Let A be a complete fuzzy lattice, then*

1. *Let $\mathcal{F} \subseteq A$ be a closure system, then $(\mathcal{F}^{\approx}, \mathcal{F})$ is a fixed point of the adjunction.*
2. *Let $\Phi \in L^A$ be a fuzzy closure system, then (Φ, Φ^1) is a fixed point of the adjunction.*

Proof. For the first item, let $\mathcal{F}$ be a closure system, then, by Theorem 12, there exists a closure operator $\mathsf{c}\colon A \to A$ such that $\mathcal{F} = \mathcal{F}_{\mathsf{c}} = \{a \in A \mid \mathsf{c}(a) = a\}$. By Proposition 15, the extensional hull of $\mathcal{F}$ is the set $\mathcal{F}^{\approx}(x) = \bigvee_{a\in\mathcal{F}}(x \approx a) = \bigvee_{a\in\mathcal{F}_{\mathsf{c}}}(x \approx \mathsf{c}(a)) = \rho(\mathsf{c}(x), x)$. Therefore,

$$(\mathcal{F}^{\approx})^1 = \{a \in A \mid \rho(\mathsf{c}(a), a) = 1\} = \{a \in A \mid \mathsf{c}(a) = a\} = \mathcal{F}.$$

For the second item, assume that Φ is a fuzzy closure system, by Theorem 14, there is a fuzzy closure operator $\mathsf{c}\colon A \to A$ such that $\Phi(x) = \rho(\mathsf{c}(x), x)$. On the other hand, since $\Phi^1 = \mathcal{F}_{\mathsf{c}}$ and by Proposition 15, we get

$$(\Phi^1)^{\approx}(x) = \bigvee_{a\in\Phi^1}(x \approx a) = \bigvee_{a\in\mathcal{F}_{\mathsf{c}}}(x \approx a) = \rho(\mathsf{c}(x), x) = \Phi(x).$$

Therefore, $(\Phi^1)^{\approx} = \Phi$. $\square$

As expected, fuzzy closure structures are fixed points of the adjunction. However, there are fixed points which are not fuzzy closure structures. This is illustrated in the following example.

Example 2. Let $\mathbb{L}$ be the unit interval with the Łukasiewicz t-norm and residuum, $U = \{u\}$ and the powerset lattice (L^U, S) and consider the set $X = \{\{{}^{u}/_{0.5}\}\} \subseteq L^U$.

$$\begin{aligned}(X^{\approx})(\{{}^{u}/_{\alpha}\}) &= (\{{}^{u}/_{\alpha}\} \approx \{{}^{u}/_{0.5}\}) \\ &= \min\{1, 1-\alpha+0.5\} \otimes \min\{1, 1-0.5+\alpha\} \\ &= \min\{1, 1.5-\alpha\} \otimes \min\{1, 0.5+\alpha\} \\ &= \begin{cases} 1.5-\alpha, & \text{if } \alpha \geq 0.5 \\ 0.5+\alpha, & \text{if } \alpha \leq 0.5 \end{cases}\end{aligned}$$

Thus, $(X^{\approx})(\{{}^{u}/_{\alpha}\}) = 1$ if and only if $\alpha = 0.5$, that is, $(X^{\approx})^1 = X$, the pair $(X, X^{\approx})$ is a fixed point of the adjunction but X is not a closure system because $U \notin X$. Similarly, by $X^{\approx}(U) = 0.5 \neq 1$ we have that $X^{\approx}$ is not a fuzzy closure system.

As shown above, not all the fixed points are (fuzzy) closure systems. The next result gives some conditions to narrow down the set of fixed points of the adjunction.

Proposition 24. *If L is linearly ordered and A is finite then $(X^{\approx}, X)$ is a fixed point of the adjunction for all $X \subseteq A$.*

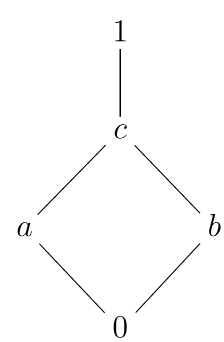
Proof. Assume L is a linearly ordered residuated lattice and A is a finite complete fuzzy lattice. Let $X \subseteq A$, then

$$X^{\approx}(x) = \bigvee_{a \in A} X(a) \otimes (a \approx x) = \bigvee_{a \in X} (a \approx x).$$

Since X is finite and since L is linearly ordered, there exists $x_0 \in X$ such that $(x_0 \approx x) = X^{\approx}(x)$. Consider $x \in (X^{\approx})^1$ then, $X^{\approx}(x) = (x_0 \approx x) = 1$. By antisymmetry we have $x = x_0 \in X$. Thus, $(X^{\approx})^1 \subseteq X$. The converse inclusion is trivial since $X \subseteq X^{\approx}$ implies $X = X^1 \subseteq (X^{\approx})^1$. □

The hypothesis in Proposition 24 is sufficient but not necessary, that is, there are examples of infinite complete fuzzy lattices or with values on a non-linear residuated lattice where $(X^{\approx}, X)$ is a fixed point for all crisp subset X. The following is one of them.

Example 3. Let $\mathbb{L}$ be the Heyting algebra whose Hasse diagram is the following:



Let U be a infinite set and consider the fuzzy powerset lattice (L^U, S). Let $X \subseteq L^U$. Then, for all $\Phi \in L^U$,

$$X^{\approx}(\Phi) = \bigvee_{\Psi \in X} (\Phi \approx \Psi) = \bigvee_{\Psi \in X} (S(\Phi, \Psi) \otimes S(\Psi, \Phi)).$$

Since L is finite, the last supremum is a supremum of a finite number of values. Moreover, in this particular lattice, the supremum equals 1 if and only if one of the terms is 1. Therefore, $X^{\approx}(\Phi) = 1$ if and only if there exists $\Psi_0 \in X$ such that $(\Psi_0 \approx \Phi) = 1$, therefore $\Phi = \Psi_0 \in X$. Thus, we get $(X^{\approx})^1 = X$ even though L is not a chain and L^U is infinite.

Even though the hypothesis in Proposition 24 are not necessary, we can find counterexamples if any of them are dropped. The following examples illustrate some such cases.

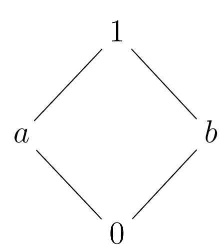
Example 4. Let $\mathbb{L}$ be the unit interval with the Łukasiewicz structure, $U = \{u\}$ and the powerset lattice (L^U, S). Let $Q = \{q_i\}_{i \in \mathbb{N}}$ be an enumeration of the rationals in $(0, 1)$ and consider the set $A = \{\{^{u}/_{q_1}\}, \{^{u}/_{q_2}\}, \{^{u}/_{q_3}\}, \dots\} \subseteq L^U$. Then,

$$A^{\approx}(\{u\}) = \bigvee_{i \in \mathbb{N}} (\{^{u}/_{q_i}\} \approx \{u\}) = \bigvee_{i \in \mathbb{N}} q_i = 1.$$

Therefore, $\{u\} \in (A^{\approx})^1 \smallsetminus A$.

Last example uses a linear L and an infinite fuzzy lattice L^U. One can also find counterexamples of Proposition 24 when both the residuated and the fuzzy lattices are finite but the order in L is non-linear.

Example 5. Let $\mathbb{L}$ be the Heyting algebra whose Hasse diagram is the following:



Let $U = \{u\}$ and consider the fuzzy powerset lattice (L^U, S) and the set $X = \{\{^{u}/_{a}\}, \{^{u}/_{b}\}\}$. Then,

$$(X^{\approx})(\{u\}) = (\{^{u}/_{a}\} \approx \{u\}) \vee (\{^{u}/_{b}\} \approx \{u\})$$

$$= (S(\{{}^{u}\!/_{a}\},\{u\}) \otimes S(\{u\},\{{}^{u}\!/_{a}\})) \vee (S(\{{}^{u}\!/_{b}\},\{x\}) \otimes S(\{u\},\{{}^{u}\!/_{b}\}))$$
$$= a \vee b = 1.$$

Therefore, $\{u\} \in (X^{\approx})^1 \smallsetminus X$.

Proposition 24 gives conditions under which $(X^{\approx}, X)$ is a fixed point for all $X \subseteq A$. We wonder whether an analogous result holds for extensional sets. Unfortunately, it does not hold for all $\Phi \in \mathrm{Ext}(L^A)$, even under the conditions in Proposition 24, as the following example shows.

Example 6. Let (A, ρ) be the complete fuzzy lattice from Example 1, where L is a chain and A is finite. Consider the fuzzy set

$$\Phi = \{{}^{\perp}\!/_{0.5}, {}^{a}\!/_{1}, {}^{b}\!/_{0.5}, {}^{c}\!/_{0.5}, {}^{d}\!/_{1}, {}^{e}\!/_{0.5}, {}^{\top}\!/_{1}\}$$

is extensional, the computation is tedious so it is omitted. Then, $\Phi^1 = \{a, d, \top\}$ and

$$\begin{aligned}(\Phi^1)^{\approx}(b) &= (a \approx b) \vee (d \approx b) \vee (\top \approx b)\\ &= (\rho(a,b) \otimes \rho(b,a)) \vee (\rho(d,b) \otimes \rho(b,d)) \vee (\rho(\top,b) \otimes \rho(b,\top))\\ &= (0.5 \otimes 0.5) \vee (0 \otimes 1) \vee (0 \otimes 1) = 0 \neq 0.5 = \Phi(b).\end{aligned}$$

As shown above, there are extensional sets $\Phi \in (\mathrm{Ext}(L^A), \subseteq)$ such that (Φ, Φ^1) is not a fixed point of the adjunction.

Remark 3. An extensional fuzzy set $\Phi \in \mathrm{Ext}(L^A)$ is a fixed point of the adjunction if and only if $\Phi(a) = \bigvee_{x \in \Phi^1}(a \approx x)$ for all $a \in A$. This result is an immediate consequence of applying the composition of both mappings, nevertheless the explicit formula obtained is remarkable and turns out to be very useful in some proofs.

5. Analysis of the fuzzy setting

In this brief section we will study the behavior of the problem if we had considered the set of crisp subsets as a fuzzy poset. First of all we need to extend to the fuzzy framework the subsethood relation, this is the well-known S operator.

Remark 4. Consider the relation $S\colon 2^A \times 2^A \to L$ defined by

$$S(X,Y) = \bigwedge_{a \in A}(X(a) \to Y(a)) = \begin{cases} 1, & \text{if } X \subseteq Y \\ 0, & \text{otherwise} \end{cases}$$

With this fuzzy relation we have the following result.

Theorem 25. *Let (A,ρ) be a complete fuzzy lattice. Then, the couple $(\widetilde{\mathsf{c}},\mathcal{F})$ is a fuzzy Galois connection between $(2^A,S)$ and $(\mathrm{Isot}(A^A),\widetilde{\rho})$ if and only if ρ is a crisp relation.*

Proof. For the direct implication we need to prove that $\rho(x,y)$ is either 0 or 1 for all $x,y\in A$. Let $a,b\in A$. If $\rho(a,b)=1$ we are done. Let us assume $\rho(a,b)\neq 1$.

Consider now the isotone mapping $f\colon A\to A$ defined as the constant mapping $f(x)=a$ for all $x\in A$ and let $X\in 2^A$ be the singleton $X=\{b\}$. Then we have $b\notin\mathcal{F}(f)$ since $f(b)=a\not\leq b$. Hence $S(X,\mathcal{F}(f))=0$.

On the other hand,

$$\begin{aligned}\widetilde{\rho}(f,\widetilde{\mathsf{c}}(X)) &= \bigwedge_{x\in A}\rho(f(x),\widetilde{\mathsf{c}}(X)(x))\\ &= \bigwedge_{x\in A}\rho(a,\bigsqcap(x^\rho\cap\{b\}))\\ &\overset{\text{(i)}}{=} \bigwedge_{x\in A}(x^\rho\cap\{b\})_\rho(a) = \bigwedge_{x\in A}(\rho(x,b)\to\rho(a,b))\\ &\overset{\text{(ii)}}{\geq} \rho(a,b),\end{aligned}$$

where (i) holds due to Theorem 5 and (ii) holds due to (2.31) in [4] Thus, we can derive the following,

$$\rho(a,b)\leq\bigwedge_{x\in A}\rho(a,\bigsqcap(x^\rho\cap\{b\}))=\widetilde{\rho}(f,\widetilde{\mathsf{c}}(X))\overset{(i)}{=}S(X,\mathcal{F}(f))=0,$$

where (i) holds due to $(\widetilde{\mathsf{c}},\mathcal{F})$ being a fuzzy Galois connection.

For the converse implication, assume (A,ρ) is a crisp poset. Then both $(2^A,\subseteq)$ and $(\mathrm{Isot}(A^A),\preceq)$ are crisp posets and the Galois connection between them was proved in Theorem 19. □

Therefore, in a complete properly-fuzzy lattice, that is, if there exist $a,b\in A$ such that $\rho(a,b)\notin\{0,1\}$, then the couple $(\widetilde{\mathsf{c}},\mathcal{F})$ will never be a fuzzy Galois connection.

6. Conclusions and further work

This paper continues the line of work which initiated in [18], where the mappings that relate fuzzy closure structures are studied from the point of view of fuzzy Galois connections. Here we have introduced the crisp powerset lattice to that same framework. This is remarkable since it is a crisp structure in a fuzzy setting. Thus, there are two possible procedures. We can consider the 1-cut of the fuzzy preorders and study the crisp Galois connections there, or fuzzify the crisp order and study the fuzzy ones. In the former, results are somehow similar to the ones in [18]. The Galois connections exist and the fuzzy closure structures are fixed points of them. However, by analyzing the second procedure, we prove that the mappings form a fuzzy Galois connection if and only if the underlying complete fuzzy lattice is crisp. These results are both surprising and remarkable.

As a prospect of future work, since we know the images of the mappings introduced in [19] are not closure operators and fuzzy closure systems in general, this analysis can be continued and study the nature of its fixed points and examine whether they are interesting for solving some problems as some sort of pre-closure structures.

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Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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