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ORIGINAL PAPER

Hilbert-type operator induced by radial weight on Hardy spaces

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Abstract

We consider the Hilbert-type operator defined by

$$H_\omega(f)(z)=\int_0^1 f(t)\left(\frac{1}{z}\int_0^z B_t^\omega(u)\,du\right)\omega(t)dt,$$

where $\{B_\zeta^\omega\}_{\zeta\in\mathbb{D}}$ are the reproducing kernels of the Bergman space A^2_ω induced by a radial weight ω in the unit disc $\mathbb{D}$. We prove that H_ω is bounded on the Hardy space H^p, $1<p<\infty$, if and only if

$$\sup_{0\le r<1}\frac{\widehat{\omega}(r)}{\widehat{\omega}\left(\frac{1+r}{2}\right)}<\infty, \tag{†}$$

and

$$\sup_{0<r<1}\left(\int_0^r\frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}}\left(\int_r^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}dt\right)^{\frac{1}{p'}}<\infty,$$

where $\widehat{\omega}(r)=\int_r^1\omega(s)\,ds$. We also prove that $H_\omega:H^1\to H^1$ is bounded if and only if (†) holds and

$$\sup_{r\in[0,1)}\frac{\widehat{\omega}(r)}{1-r}\left(\int_0^r\frac{ds}{\widehat{\omega}(s)}\right)<\infty.$$

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As for the case $p = \infty$, H_ω is bounded from H^∞ to BMOA, or to the Bloch space, if and only if (†) holds. In addition, we prove that there does not exist radial weights ω such that $H_\omega : H^p \to H^p$, $1 \le p < \infty$, is compact and we consider the action of H_ω on some spaces of analytic functions closely related to Hardy spaces.

Keywords Hilbert operator · Hardy space · Bergman reproducing kernel · Radial weight

Mathematics Subject Classification 47G10 · 30H10

1 Introduction

For $0 < p < \infty$, let $L^p_{[0,1)}$ be the Lebesgue space of measurable functions such that

$$\|f\|^p_{L^p_{[0,1)}} = \int_0^1 |f(t)|^p \, dt < \infty,$$

and let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The Hardy space H^p consists of $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \max_{0\le\theta\le 2\pi} |f(re^{i\theta})|.$$

For a nonnegative function $\omega \in L^1_{[0,1)}$, the extension to $\mathbb{D}$, defined by $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, is called a radial weight. Let A^2_ω denote the weighted Bergman space of $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|^2_{A^2_\omega} = \int_{\mathbb{D}} |f(z)|^2 \omega(z)\, dA(z) < \infty$, where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized area measure on $\mathbb{D}$. Throughout this paper we assume $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\, ds > 0$ for all $z \in \mathbb{D}$, for otherwise $A^2_\omega = \mathcal{H}(\mathbb{D})$.

The Hilbert matrix is the infinite matrix whose entries are $h_{n,k} = (n + k + 1)^{-1}$, $k, n \in \mathbb{N} \cup \{0\}$. It can be viewed as an operator on spaces of analytic functions, by its action on the Taylor coefficients

$$\widehat{f}(n) \mapsto \sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n + k + 1}, \quad n \in \mathbb{N} \cup \{0\},$$

called the Hilbert operator. That is, if $f(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^k \in \mathcal{H}(\mathbb{D})$

$$H(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n + k + 1}\right) z^n, \tag{1.1}$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

The Hilbert operator H is bounded on Hardy spaces H^p if and only if $1 < p < \infty$ [4]. A proof of this result can be obtained using the following integral representation, valid for any $f \in H^1$,

$$H(f)(z) = \int_0^1 f(t) \frac{1}{1-tz}\, dt. \tag{1.2}$$

Going further, the formula (1.2) has been employed to solve a good number of questions in operator theory related to the boundedness, the operator norm and the spectrum of the Hilbert operator on classical spaces of analytic functions [1, 3, 5, 24]. During the last decades several generalizations of the Hilbert operator have attracted a considerable amount of attention [9, 11, 24, 26]. We will focus on the following, introduced in [26]. For a radial weight ω, we consider the Hilbert-type operator

$$H_\omega(f)(z) = \int_0^1 f(t) \left(\frac{1}{z} \int_0^z B_t^\omega(\zeta)\, d\zeta \right) \omega(t)dt,$$

where $\{B_z^\omega\}_{z\in\mathbb{D}} \subset A^2_\omega$ are the Bergman reproducing kernels of A^2_ω. The choice $\omega = 1$ gives the integral representation (1.2) of the classical Hilbert operator, therefore it is natural to think of the features of a radial weight ω so that H_ω has some of the nice properties of the (classical) Hilbert operator. In this paper, among other results, we describe the radial weights ω such that the Hilbert-type operator H_ω is bounded on H^p, $1 \le p < \infty$.

In order to state our results some more notation is needed. For $0 < p < \infty$, the Dirichlet-type space D^p_{p-1} is the space of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{D^p_{p-1}} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-1}\, dA(z) < \infty,$$

and the Hardy–Littlewood space $HL(p)$ consists of the $f(z) = \sum_{n=0}^\infty \widehat{f}(n) z^n \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{HL(p)} = \sum_{n=0}^\infty |\widehat{f}(n)|^p (n+1)^{p-2} < \infty.$$

We will also consider the space $H(\infty, p) = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|^p_{H(\infty,p)} = \int_0^1 M^p_\infty(r, f)\, dr < \infty\}$. These spaces satisfy the well-known inclusions

$$D^p_{p-1} \subset H^p \subset HL(p), \quad 0 < p \le 2, \tag{1.3}$$

$$HL(p) \subset H^p \subset D^p_{p-1}, \quad 2 \le p < \infty, \tag{1.4}$$

and

$$H^p \subset H(\infty, p), \quad D^p_{p-1} \subset H(\infty, p), \quad 0 < p < \infty. \tag{1.5}$$

See [6, 7, 14] for proofs of (1.3) and (1.4), and [27, p. 127] and [8, Lemma 4] for a proof of (1.5).

The Bergman reproducing kernels, induced by a radial weight ω, can be written as $B_z^\omega(\zeta) = \sum \overline{e_n(z)} e_n(\zeta)$ for each orthonormal basis $\{e_n\}$ of A^2_ω, and therefore using the basis induced by the normalized monomials,

$$B_z^\omega(\zeta) = \sum_{n=0}^\infty \frac{(\overline{z}\zeta)^n}{2\omega_{2n+1}}, \quad z, \zeta \in \mathbb{D}. \tag{1.6}$$

Here ω_{2n+1} are the odd moments of ω, and in general from now on we write $\omega_x = \int_0^1 r^x \omega(r)\,dr$ for all $x \geq 0$. A radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if $\widehat{\omega}(r) \leq C\widehat{\omega}(\frac{1+r}{2})$ for some constant $C = C(\omega) > 1$ and all $0 \leq r < 1$. If there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \geq C\widehat{\omega}\left(1 - \frac{1-r}{K}\right)$ for all $0 \leq r < 1$, then $\omega \in \check{\mathcal{D}}$. Further, we write $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$ for short. Recall that $\omega \in \mathcal{M}$ if there exist constants $C = C(\omega) > 1$ and $K = K(\omega) > 1$ such that $\omega_x \geq C\omega_{Kx}$ for all $x \geq 1$. It is known that $\check{\mathcal{D}} \subset \mathcal{M}$ [23, Proof of Theorem 3] but $\check{\mathcal{D}} \subsetneq \mathcal{M}$ [23, Proposition 14]. However, [23, Theorem 3] ensures that $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}} = \widehat{\mathcal{D}} \cap \mathcal{M}$. These classes of weights arise in meaningful questions concerning radial weights and classical operators, such as the differentiation operator $f^{(n)}$ or the Bergman projection $P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B_z^\omega(\zeta)}\,\omega(\zeta)dA(\zeta)$ [23]. We will also deal with the sublinear Hilbert-type operator

$$\widetilde{H_\omega}(f)(z) = \int_0^1 |f(t)| \left(\frac{1}{z}\int_0^z B_t^\omega(\zeta)d\zeta\right) \omega(t)\,dt.$$

If $X, Y \subset \mathcal{H}(\mathbb{D})$ are normed vector spaces, and T is a sublinear operator, we denote $\|T\|_{X\to Y} = \sup_{\|f\|_X \leq 1} \|T(f)\|_Y$.

Theorem 1 *Let ω be a radial weight and $1 < p < \infty$. Let $X_p, Y_p \in \{H(\infty, p), H^p, D^p_{p-1}, HL(p)\}$ and $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then the following statements are equivalent:*

(i) *$T : X_p \to Y_p$ is bounded;*

(ii) *$\omega \in \mathcal{D}$ and $M_p(\omega) = \sup_{N\in\mathbb{N}} \left(\sum_{n=0}^N \frac{1}{(n+1)^2\omega_{2n+1}^p}\right)^{\frac{1}{p}} \left(\sum_{n=N}^\infty \omega_{2n+1}^{p'}(n+1)^{p'-2}\right)^{\frac{1}{p'}} < \infty$;*

(iii) *$\omega \in \widehat{\mathcal{D}}$ and $M_p(\omega) = \sup_{N\in\mathbb{N}} \left(\sum_{n=0}^N \frac{1}{(n+1)^2\omega_{2n+1}^p}\right)^{\frac{1}{p}} \left(\sum_{n=N}^\infty \omega_{2n+1}^{p'}(n+1)^{p'-2}\right)^{\frac{1}{p'}} < \infty$;*

(iv) *$\omega \in \widehat{\mathcal{D}}$ and $M_{p,c}(\omega) = \sup_{0<r<1} \left(\int_0^r \frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}} \left(\int_r^1 \left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'} dt\right)^{\frac{1}{p'}} < \infty$.*

The proof of (i)⇒(iii) of Theorem 1 has two steps. Firstly, we prove that $\omega \in \widehat{\mathcal{D}}$, and later on the condition $M_p(\omega) < \infty$ is obtained by using polynomials of the form $f_{N,M}(z) = \sum_{k=N}^M \omega_{2k}^\alpha (k+1)^\beta z^k$, $N, M \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$ as test functions. Then, we see that any radial weight ω satisfying the condition $M_p(\omega) < \infty$, belongs to $\mathcal{M}$. This proves (ii)⇔(iii). The proof of (iii)⇔(iv) is a calculation based on known descriptions of the class $\widehat{\mathcal{D}}$ [21, Lemma 2.1]. Finally, we prove (iv)⇒(i) which is the most involved implication in the proof of Theorem 1. In order to obtain it, we merge techniques coming from complex and harmonic analysis, such as a very convenient description of the class $\mathcal{D}$, see Lemma 14 below, precise estimates of the integral means of order p of the derivative of the kernels $K_u^\omega(z) = \frac{1}{z}\int_0^z B_u^\omega(z)\,du$, decomposition norm theorems and classical weighted inequalities for Hardy operators.

Observe that both, the discrete condition $M_p(\omega) < \infty$ and its continuous version $M_{p,c}(\omega) < \infty$, are used in the proof of Theorem 1. The first one follows from (i), and the condition $M_{p,c}(\omega) < \infty$ is employed to prove that $T : X_p \to Y_p$ is bounded.

As for the case $p = 1$ we obtain the following result.

Theorem 2 *Let ω be a radial weight, $X_1, Y_1 \in \{H(\infty, 1), H^1, D_0^1, HL(1)\}$ and $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then the following statements are equivalent:*

(i) *$T : X_1 \to Y_1$ is bounded;*

(ii) *$\omega \in \widehat{\mathcal{D}}$ and the measure μ_ω defined as $d\mu_\omega(z) = \omega(z)\left(\int_0^{|z|} \frac{ds}{\widehat{\omega}(s)}\right)\chi_{[0,1)}(z)\,dA(z)$ is a 1-Carleson measure for X_1;*

(iii) *$\omega \in \widehat{\mathcal{D}}$ and satisfies the condition*

$$M_{1,c}(\omega) = \sup_{a\in[0,1)} \frac{1}{1-a}\int_a^1 \omega(t)\left(\int_0^t \frac{ds}{\widehat{\omega}(s)}\right)dt < \infty;$$

(iv) *$\omega \in \mathcal{D}$ and satisfies the condition $M_{1,c}(\omega) < \infty$;*

(v) *$\omega \in \widehat{\mathcal{D}}$ and satisfies the condition $M_{1,d}(\omega) = \sup_{a\in[0,1)} \frac{\widehat{\omega}(a)}{1-a}\left(\int_0^a \frac{ds}{\widehat{\omega}(s)}\right) < \infty$;*

(vi) *$\omega \in \widehat{\mathcal{D}}$ and satisfies the condition*

$$M_1(\omega) = \sup_{N\in\mathbb{N}} (N+1)\omega_{2N} \sum_{k=0}^{N} \frac{1}{(k+1)^2\omega_{2k}} < \infty.$$

We recall that given a Banach space (or a complete metric space) X of analytic functions on $\mathbb{D}$, a positive Borel measure μ on $\mathbb{D}$ is called a q-Carleson measure for X if the identity operator $I_d : X \to L^q(\mu)$ is bounded. Carleson provided a geometric description of p-Carleson measures for Hardy spaces H^p, $0 < p < \infty$, [6, Chapter 9]. These measures are called classical Carleson measures. The proof of Theorem 2 uses characterizations of Carleson measures for X_1-spaces, universal Cesàro basis of polynomials and some of the main ingredients of the proofs of Theorem 1 and [26, Theorem 2].

Concerning the classes of radial weights $\widehat{\mathcal{D}}$ and $M_{p,c} = \{\omega : M_{p,c}(\omega) < \infty\}$, $1 \le p < \infty$, a standard weight, $\omega(z) = (1-|z|)^\beta$, $\beta > -1$, satisfies the condition $M_{p,c}(\omega) < \infty$ if and only if $\beta > \frac{1}{p} - 1$, so $H_\omega : H^p \to H^p$ is bounded if and only if $\beta > \frac{1}{p} - 1$. Moreover, a calculation shows that the exponential type weight $\omega(r) = \exp\left(-\frac{1}{1-r}\right) \in M_{p,c}$ for any $p \in [1,\infty)$, but $\omega \notin \widehat{\mathcal{D}}$, see [28, Example 3.2] for further details. So, $\widehat{\mathcal{D}}$ and $M_{p,c}$ are not included in each other.

The study of the radial weights ω such that $H_\omega : H^p \to H^p$ is bounded, has been previously considered in [26]. Indeed, Theorem 2 improves [26, Theorem 2], by removing the initial hypothesis $\omega \in \widehat{\mathcal{D}}$. On the other hand, [26, Theorem 3] describes the weights $\omega \in \widehat{\mathcal{D}}$ such that $H_\omega : L^p_{[0,1)} \to H^p$ is bounded, and consequently gives a sufficient condition for the boundedness of $H_\omega : H^p \to H^p$, $1 < p < \infty$. The following improvement of [26, Theorem 3] is a byproduct of Theorem 1.

Corollary 3 *Let ω be a radial weight and $1 < p < \infty$. Let $Y_p \in \{H(\infty,p), H^p, D^p_{p-1}, HL(p)\}$ and $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then the following statements are equivalent:*

(i) *$T : L^p_{[0,1)} \to Y_p$ is bounded;*

(ii) *$\omega \in \mathcal{D}$ and satisfies the condition*

$$m_p(\omega) = \sup_{0<r<1}\left(1 + \int_0^r \frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}}\left(\int_r^1 \omega(t)^{p'}\,dt\right)^{\frac{1}{p'}} < \infty;$$

(iii) *$\omega \in \widehat{\mathcal{D}}$ and satisfies the condition $m_p(\omega) < \infty$.*

In relation to an analogous result to Corollary 3 for $p = 1$, Theorem 26 below shows that the radial weights such that $T : L^1_{[0,1)} \to Y_1$ is bounded, where $Y_1 \in \{H(\infty,1), H^1, D^1_0, HL(1)\}$ and $T \in \{H_\omega, \widetilde{H_\omega}\}$, are the weights $\omega \in \mathcal{D}$ such that $m_1(\omega) = \operatorname{ess\,sup}_{t\in[0,1)} \omega(t)\left(1 + \int_0^t \frac{ds}{\widehat{\omega}(s)}\right) < \infty$.

In view of the above findings, we compare the conditions $M_{p,c}(\omega)<\infty$, $M_{p,d}(\omega)<\infty$ and $m_p(\omega)<\infty$ in order to put the boundedness of $T: X_p\to Y_p$ alongside the boundedness of $T: L^p_{[0,1)}\to Y_p$, where $X_p, Y_p\in\{H(\infty,p), H^p, D^p_{p-1}, HL(p)\}$ and $T\in\{H_\omega, \widetilde{H_\omega}\}$ for $1\le p<\infty$. Bearing in mind (1.5), it is clear that the condition $m_p(\omega)<\infty$ implies that $M_{p,c}(\omega)<\infty$, for any weight $\omega\in\widehat{\mathcal{D}}$. Moreover, observe that $M_{p,c}(\omega)<\infty$ if and only if

$$\sup_{0<r<1}\left(1+\int_0^r\frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}}\left(\int_r^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}dt\right)^{\frac{1}{p'}}<\infty,\quad\text{when } 1<p<\infty$$

and $\sup_{a\in[0,1)}\frac{\widehat{\omega}(a)}{1-a}\left(1+\int_0^a\frac{ds}{\widehat{\omega}(s)}\right)<\infty$ if and only if $M_{1,d}(\omega)<\infty$. So, the conditions $M_{p,c}(\omega)<\infty$ and $m_p(\omega)<\infty$, are equivalent for any $1\le p<\infty$ whenever ω satisfies the pointwise inequality

$$\omega(t)\lesssim\frac{\widehat{\omega}(t)}{1-t},\quad t\in[0,1),\tag{1.7}$$

and $\omega\in\widehat{\mathcal{D}}$. The condition (1.7) implies restrictions on the decay and on the regularity of the weight, in fact if ω fulfills (1.7) then ω cannot decrease rapidly and cannot oscillate strongly. For instance, the exponential type weight $\omega(r)=\exp\left(-\frac{1}{1-r}\right)$, which is a prototype of rapidly decreasing weight (see [18]), has the property

$$\widehat{\omega}(r)\asymp\omega(r)(1-r)^2,\quad 0\le r<1,$$

so it does not satisfy (1.7). On the other hand, any regular or rapidly increasing weight satisfies (1.7). Regular and rapidly increasing weights are large subclasses of $\widehat{\mathcal{D}}$, see [25, Section 1.2] for the definitions and examples of these classes of radial weights. However, we construct in Corollaries 19 and 28 weights $\omega\in\mathcal{D}$ with a strong oscillatory behaviour so that $M_{p,c}(\omega)<\infty$ and $m_p(\omega)=\infty$, and consequently they do not satisfy (1.7).

With the aim of discussing some results concerning the case $p=\infty$, we recall that the space BMOA consists of those functions in the Hardy space H^1 that have bounded mean oscillation on the boundary of $\mathbb{D}$ [10], and the Bloch space $\mathcal{B}$ is the space of all analytic functions on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)\,|f'(z)|<\infty.$$

We also consider the space $HL(\infty)$ of the $f(z)=\sum_{n=0}^\infty\widehat{f}(n)z^n\in\mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{HL(\infty)}=\sup_{n\in\mathbb{N}\cup\{0\}}(n+1)\left|\widehat{f}(n)\right|<\infty.$$

The following chain of inclusions hold [10]

$$HL(\infty)\subsetneq\text{BMOA}\subsetneq\mathcal{B}.\tag{1.8}$$

If ω is a radial weight

$$H_\omega(1)(x)=\sum_{n=0}^\infty\frac{\omega_n}{2\omega_{2n+1}(n+1)}x^n\ge\frac{1}{2x}\sum_{n=0}^\infty\frac{x^{n+1}}{n+1}=\frac{1}{2x}\log\left(\frac{1}{1-x}\right),\quad x\in(0,1),$$

so H_ω is not bounded on H^∞. As for the classical Hilbert matrix H, it is bounded from H^∞ to $BMOA$ [13, Theorem 1.2]. So, it is natural wondering about the radial weights such that $H_\omega: H^\infty\to\text{BMOA}$ is bounded. The next result answers this question.

Theorem 4 *Let ω be a radial weight and let $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then, the following statements are equivalent:*

(i) $T : H^\infty \to HL(\infty)$ *is bounded;*
(ii) $T : H^\infty \to \mathrm{BMOA}$ *is bounded;*
(iii) $T : H^\infty \to \mathcal{B}$ *is bounded;*
(iv) $\omega \in \widehat{\mathcal{D}}$.

The equivalence (iii)⇔(iv) was proved in [26, Theorem 1], so our contribution in Theorem 4 consists on proving the rest of equivalences.

Bearing in mind Theorems 1, 2 and 4, we deduce that $T \in \{H_\omega, \widetilde{H_\omega}\}$ is bounded from H^∞ to $HL(\infty)$ if $T : X_p \to Y_p$ is bounded, where $X_p, Y_p \in \{H(\infty, p), H^p, D^p_{p-1}, HL(p)\}$, $1 \le p < \infty$. We prove that this is a general phenomenon for Hilbert-type operators and parameters $1 \le q < p$.

Theorem 5 *Let ω be radial weight, $T \in \{H_\omega, \widetilde{H_\omega}\}$ and $1 \le q < p < \infty$. Further, let $X_q, Y_q \in \{H^q, D^q_{q-1}, HL(q), H(\infty, q)\}$ and $X_p, Y_p \in \{H^p, D^p_{p-1}, HL(p), H(\infty, p)\}$. If $T : X_q \to Y_q$ is bounded, then $T : X_p \to Y_p$ is bounded.*

We also prove that that there does not exist radial weights ω such that $H_\omega : X_p \to Y_p$ is compact, where $X_p, Y_p \in \{H^p, D^p_{p-1}, HL(p), H(\infty, p)\}$ and $1 \le p < \infty$, neither radial weights such that $H_\omega : H^\infty \to \mathcal{B}$ is compact, see Theorems 22, 31, 34 below.

The letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \le Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$ and say that a and b are comparable. We remark that if a or b are quantities which depends on a radial weight ω, the constant C such that $a \lesssim b$ or $a \gtrsim b$ may depend on ω but it does not depend on a neither on b.

The rest of the paper is organized as follows. Section 2 is devoted to prove some auxiliary results. We prove Theorem 1 and Corollary 3 in Sect. 3, and Theorem 2 is proved in Sect. 4. Section 5 contains a proof of Theorem 4 and Theorem 5 is proved in Sect. 6 together with some reformulations of the condition $M_{p,c}(\omega) < \infty$.

2 Preliminary results

In this section, we will prove some convenient preliminary results which will be repeatedly used throughout the rest of the paper. The first auxiliary lemma contains several characterizations of upper doubling radial weights. For a proof, see [21, Lemma 2.1].

Lemma 6 *Let ω be a radial weight on $\mathbb{D}$. Then, the following statements are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;
(ii) *There exist $C = C(\omega) \ge 1$ and $\beta_0 = \beta_0(\omega) > 0$ such that*

$$\widehat{\omega}(r) \le C \left(\frac{1-r}{1-t}\right)^\beta \widehat{\omega}(t), \quad 0 \le r \le t < 1;$$

for all $\beta \ge \beta_0$.

(iii)

$$\int_0^1 s^x\omega(s)ds \asymp \widehat{\omega}\left(1-\frac{1}{x}\right), \quad x\in[1,\infty);$$

(iv) *There exists* $C=C(\omega)>0$ *and* $\beta=\beta(\omega)>0$ *such that*

$$\omega_x \le C\left(\frac{y}{x}\right)^\beta \omega_y, \quad 0<x\le y<\infty;$$

(v) $\widehat{\mathcal{D}}(\omega)=\sup_{n\in\mathbb{N}}\frac{\omega_n}{\omega_{2n}}<\infty$.

We will also use the following characterizations of the class $\check{\mathcal{D}}$, see [23, (2.27)].

Lemma 7 *Let* ω *be a radial weight. The following statements are equivalent:*

(i) $\omega\in\check{\mathcal{D}}$;
(ii) *There exist* $C=C(\omega)>0$ *and* $\alpha_0=\alpha_0(\omega)>0$ *such that*

$$\widehat{\omega}(s)\le C\left(\frac{1-s}{1-t}\right)^\alpha \widehat{\omega}(t), \quad 0\le t\le s<1$$

for all $0<\alpha\le\alpha_0$;
(iii) *There exist* $K=K(\omega)>1$ *and* $C=C(\omega)>0$ *such that*

$$\int_r^{1-\frac{1-r}{K}} \omega(s)ds \ge C\widehat{\omega}(r), \quad 0\le r<1. \tag{2.1}$$

Embedding relations among spaces $X_p, Y_p\in\{H^p, D^p_{p-1}, HL(p), H(\infty,p)\}$ are quite useful in the study of operators acting on them. In particular, we recall that

$$\|f\|_{H(\infty,p)}\le C_p\|f\|_{X_p}, \quad 0<p<\infty, \tag{2.2}$$

for $X_p\in\{H^p, D^p_{p-1}\}$, see [27, p. 127] and [8, Lemma 4].

This inequality is no longer true for $X_p=HL(p)$ if $0<p<1$. In fact, take $f(z)=\sum_{n=0}^\infty 2^{\frac{n}{p}}z^{2^n}$. A calculation shows that $f\in HL(p)$, if $0<p<1$. However, using [15, Theorem 1],

$$\|f\|^p_{H(\infty,p)}=\int_0^1\left(\sum_{n=0}^\infty 2^{\frac{n}{p}}s^{2^n}\right)^p ds \asymp \sum_{n=0}^\infty 1=\infty.$$

Our following result extends the inequality (2.2) to $X_p=HL(p)$ and $1\le p<\infty$.

Lemma 8 *Let* $1\le p<\infty$. *Then, there is* $C_p>0$ *such that*

$$\|f\|_{H(\infty,p)}\le C_p\|f\|_{X_p}, \quad f\in\mathcal{H}(\mathbb{D}),$$

where $X_p\in\{H^p, D^p_{p-1}, HL(p)\}$.

Proof By (2.2) it is enough to prove the inequality for $X_p = HL(p)$. By [15, Theorem 1] and Hölder's inequality

$$\begin{aligned}\int_0^1 M_\infty^p(t,f)\,dt &\le \int_0^1 \left(\sum_{n=0}^\infty |\widehat{f}(n)|t^n\right)^p dt\\ &\lesssim |\widehat{f}(0)|^p + \sum_{n=0}^\infty 2^{-n}\left(\sum_{k=2^n}^{2^{n+1}-1} |\widehat{f}(k)|\right)^p\\ &\le |\widehat{f}(0)|^p + \sum_{n=0}^\infty 2^{n(p-2)}\sum_{k=2^n}^{2^{n+1}-1} |\widehat{f}(k)|^p\\ &\lesssim |\widehat{f}(0)|^p + \sum_{n=0}^\infty \sum_{k=2^n}^{2^{n+1}-1} (k+1)^{p-2}|\widehat{f}(k)|^p = \|f\|_{HL(p)}^p.\end{aligned}$$

This finishes the proof. □

For $0 < p < \infty$ and ω a radial weight, let $L^p_{\omega,[0,1)}$ be the Lebesgue space of measurable functions such that

$$\|f\|^p_{L^p_{\omega,[0,1)}} = \int_0^1 |f(t)|^p \omega(t)\,dt < \infty.$$

Next, we will prove that the sublinear operator $\widetilde{H_\omega}$ does not distinguish the norm of the spaces $H(\infty,p)$, $HL(p)$, D^p_{p-1}, H^p, when $1 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$.

Lemma 9 *Let $\omega \in \widehat{\mathcal{D}}$, $1 < p < \infty$ and $X_p, Y_p \in \{H(\infty,p), HL(p), D^p_{p-1}, H^p\}$. Then,*

$$\|\widetilde{H_\omega}(f)\|_{X_p} \asymp \|\widetilde{H_\omega}(f)\|_{Y_p}, \quad f \in L^1_{\omega,[0,1)}.$$

Proof Here and on the following, let us denote $I(n) = \{k \in \mathbb{N} : 2^n \le k < 2^{n+1}\}$, $n \in \mathbb{N}\cup\{0\}$. By Lemma 6

$$\omega_{2^{n+2}} \asymp \omega_{2k+2} \asymp \omega_{2k} \asymp \omega_{2^n}, \quad \text{for any} n \in \mathbb{N}\cup\{0\} \text{and} k \in I(n). \tag{2.3}$$

The above equivalences and [15, Theorem 1], yield

$$\begin{aligned}\|\widetilde{H_\omega}(f)\|^p_{H(\infty,p)} &\asymp \sum_{n=0}^\infty 2^{-n}\left(\sum_{k\in I(n)} \frac{\int_0^1 |f(t)|t^k\omega(t)\,dt}{(k+1)\omega_{2k+1}}\right)^p + \left(\int_0^1 |f(t)|\omega(t)\,dt\right)^p\\ &\asymp \sum_{n=0}^\infty 2^{-n}\left(\frac{\int_0^1 |f(t)|t^{2^n}\omega(t)\,dt}{\omega_{2^{n+1}}}\right)^p + \left(\int_0^1 |f(t)|\omega(t)\,dt\right)^p\\ &\asymp \|\widetilde{H_\omega}(f)\|^p_{HL(p)}, \quad f \in L^1_{\omega,[0,1)}.\end{aligned}$$

This, together with [26, Lemma 8], finishes the proof. □

3 Hilbert-type operators acting on X_p-spaces, $1 < p < \infty$

3.1 Necessity part of Theorem 1

We begin this section with the construction of appropriate families of test functions to be used in the proof of Theorem 1. To do this, some notation and previous results are needed. Let $g(z) = \sum_{k=0}^{\infty} \widehat{g}(k)z^k \in \mathcal{H}(\mathbb{D})$, and denote $\Delta_n g(z) = \sum_{k\in I(n)} \widehat{g}(k)z^k$. In the particular case $g(z) = \frac{1}{1-z}$, we simply write $\Delta_n(z) = \Delta_n(g)(z) = \sum_{k\in I(n)} z^k$. We recall that

$$\|\Delta_n\|_{H^p} \asymp 2^{n(1-1/p)}, \quad n \in \mathbb{N}\cup\{0\}, \quad 1 < p < \infty, \tag{3.1}$$

see [2, Lemma 2.7].

For any $n_1,\ n_2 \in \mathbb{N}\cup\{0\}$, $n_1 < n_2$, write $S_{n_1,n_2} g(z) = \sum_{k=n_1}^{n_2-1} \widehat{g}(k)z^k$. The next known result can be proved mimicking the proof of [13, Lemma 3.4] (see also [24, Lemma E]), that is, by summing by parts and using the M. Riesz projection theorem.

Lemma 10 *Let $1 < p < \infty$ and $\lambda = \{\lambda_k\}_{k=0}^{\infty}$ be a positive and monotone sequence. Let $g(z) = \sum_{k=0}^{\infty} b_k z^k$ and $(\lambda g)(z) = \sum_{k=0}^{\infty} \lambda_k b_k z^k$.*

(a) *If $\{\lambda_k\}_{k=0}^{\infty}$ is nondecreasing, there exists a constant $C > 0$ such that*

$$C^{-1}\lambda_{n_1}\|S_{n_1,n_2} g\|_{H^p} \le \|S_{n_1,n_2}(\lambda g)\|_{H^p} \le C\lambda_{n_2}\|S_{n_1,n_2} g\|_{H^p}.$$

(b) *If $\{\lambda_k\}_{k=0}^{\infty}$ is nonincreasing, there exists a constant $C > 0$ such that*

$$C^{-1}\lambda_{n_2}\|S_{n_1,n_2} g\|_{H^p} \le \|S_{n_1,n_2}(\lambda g)\|_{H^p} \le C\lambda_{n_1}\|S_{n_1,n_2} g\|_{H^p}.$$

Lemma 11 *Let $\omega \in \widehat{\mathcal{D}}$, $1 < p < \infty$, $\alpha, \beta \in \mathbb{R}$ and $M, N \in \mathbb{N}\cup\{0\}$ such that $0 \le N < 4N+1 \le M$. Let us consider the function*

$$f_{N,M}(z) = \sum_{k=N}^{M} \omega_{2k}^{\alpha}(k+1)^{\beta} z^k.$$

Then,

$$\|f_{N,M}\|_{HL(p)} \asymp \|f_{N,M}\|_{H^p} \asymp \|f_{N,M}\|_{D^p_{p-1}}, \tag{3.2}$$

where the constants involved do not depend on M or N. In particular, if $\alpha = 0$ then (3.2) holds for any radial weight.

Proof Firstly, let us show that for all $N, M \in \mathbb{N}$, $M > N$,

$$\|f_{2^N+1,2^M}\|_{D^p_{p-1}} \asymp \|f_{2^N+1,2^M}\|_{HL(p)}. \tag{3.3}$$

[16, Theorem 2.1(b)] (see also [20, 7.5.8]), Lemma 10, (2.3) and (3.1) implies

$$\begin{aligned}\|f_{2^N+1,2^M}\|^p_{D^p_{p-1}} &\asymp \sum_{n=N}^{M-1} 2^{-np}\left\|\sum_{k\in I(n)} \omega^\alpha_{2k+2}(k+2)^\beta(k+1)z^k\right\|^p_{H^p}\\ &\asymp \sum_{n=N}^{M-1} 2^{np\beta}\omega^{p\alpha}_{2^{n+1}}\|\Delta_n\|^p_{H^p}\\ &\asymp \sum_{n=N}^{M-1} 2^{n(p\beta+p-1)}\omega^{p\alpha}_{2^{n+1}}\\ &\asymp \sum_{k=2^N+1}^{2^M}(k+1)^{p\beta+p-2}\omega^{p\alpha}_{2k} = \|f_{2^N+1,2^M}\|^p_{HL(p)},\end{aligned}$$

A similar calculation shows that

$$\|f_{2^{N+1}+1,2^M}\|_{D^p_{p-1}} \asymp \|f_{2^N+1,2^{M+1}}\|_{D^p_{p-1}}, \quad M > N+1. \tag{3.4}$$

Next, if $N > 2$, there is $N^\star, M^\star \in \mathbb{N}$ such that $2^{N^\star} \le N-1 < 2^{N^\star+1}$ and $2^{M^\star} \le M-1 < 2^{M^\star+1}$, so $N^\star+1 < M^\star$. Then, by [16, Theorem 2.1(b)] and the boundedness of the Riesz projection, (3.3) and (3.4)

$$\begin{aligned}\|f_{N,M}\|^p_{D^p_{p-1}} &\asymp 2^{-pN^\star}\left\|\sum_{k=N-1}^{2^{N^\star+1}-1}(k+1)\widehat{f}_{N,M}(k+1)z^k\right\|^p_{H^p} + \|f_{2^{N^\star+1}+1,2^{M^\star}}\|^p_{D^p_{p-1}}\\ &\quad + 2^{-pM^\star}\left\|\sum_{k=2^{M^\star}}^{M-1}(k+1)\widehat{f}_{N,M}(k+1)z^k\right\|^p_{H^p}\\ &\lesssim \|f_{2^{N^\star}+1,2^{M^\star+1}}\|^p_{D^p_{p-1}}\\ &\asymp \|f_{2^{N^\star+1}+1,2^{M^\star}}\|^p_{D^p_{p-1}} \asymp \|f_{2^{N^\star+1}+1,2^{M^\star}}\|^p_{HL(p)} \lesssim \|f_{N,M}\|^p_{HL(p)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|f_{N,M}\|^p_{D^p_{p-1}} &\gtrsim \|f_{2^{N^\star+1}+1,2^{M^\star}}\|^p_{D^p_{p-1}} \asymp \|f_{2^{N^\star}+1,2^{M^\star+1}}\|^p_{D^p_{p-1}}\\ &\asymp \|f_{2^{N^\star}+1,2^{M^\star+1}}\|^p_{HL(p)} \ge \|f_{N,M}\|^p_{HL(p)}.\end{aligned}$$

Then, bearing in mind (1.3) and (1.4), we obtain $\|f_{N,M}\|_{HL(p)} \asymp \|f_{N,M}\|_{H^p} \asymp \|f_{N,M}\|_{D^p_{p-1}}$ for each $N > 2$.

If $N \in \{0,1,2\}$, the previous argument together with minor modifications implies (3.2). This finishes the proof. □

Now we are ready to prove the necessity part of Theorem 1.

Proposition 12 *Let ω be a radial weight and $1 < p < \infty$. If $X_p, Y_p \in \{H(\infty,p), H^p, D^p_{p-1}, HL(p)\}$, $T \in \{H_\omega, \widetilde{H_\omega}\}$, and $T : X_p \to Y_p$ is a bounded operator. Then, $\omega \in \mathcal{D}$ and*

$$M_p(\omega) = \sup_{N\in\mathbb{N}}\left(\sum_{n=0}^{N}\frac{1}{(n+1)^2\omega^p_{2n+1}}\right)^{\frac{1}{p}}\left(\sum_{n=N}^{\infty}\omega^{p'}_{2n+1}(n+1)^{p'-2}\right)^{\frac{1}{p'}} < \infty. \tag{3.5}$$

Proof In order to obtain both conditions, $\omega \in \mathcal{D}$ and $M_p(\omega) < \infty$, we are going to work with families of test functions constructed in Lemma 11. Since they have non-negative Maclaurin coefficients, it is enough to prove the result for $T = H_\omega$. Take $f \in \mathcal{H}(\mathbb{D})$ such that $\widehat{f}(n) \geq 0$ for all $n \in \mathbb{N}$.

First Step. We will prove that $\omega \in \widehat{\mathcal{D}}$. By Lemma 8, it is enough to deal with the case $Y_p = H(\infty, p)$.

Observe that $M_\infty(r, H_\omega(f)) = \sum_{n=0}^\infty \frac{1}{2(n+1)\omega_{2n+1}} \left(\sum_{k=0}^\infty \widehat{f}(k)\omega_{n+k}\right) r^n$. Now, consider the test functions $f_N(z) = \sum_{n=0}^N \frac{1}{(n+1)^{1-\frac{1}{p-1}}} z^n$, $N \in \mathbb{N}$. Given that $\sum_{k=0}^N \frac{1}{(k+1)^{1-\frac{1}{p-1}}} \asymp (N+1)^{\frac{1}{p-1}}$,

$$\begin{aligned} M_\infty(r, H_\omega(f_N)) &\geq \sum_{n=6N}^{7N} \frac{1}{2(n+1)\omega_{2n+1}} \left(\sum_{k=0}^N \widehat{f_N}(k)\omega_{n+k}\right) r^n \\ &\gtrsim \sum_{n=6N}^{7N} \frac{\omega_{n+N}}{(n+1)\omega_{2n+1}} \left(\sum_{k=0}^N \frac{1}{(k+1)^{1-\frac{1}{p-1}}}\right) r^n \\ &\gtrsim (N+1)^{\frac{1}{p-1}} \frac{\omega_{8N}}{\omega_{12N}} r^{7N}, \quad N \in \mathbb{N}, \quad 0 \leq r < 1. \end{aligned}$$

So,

$$\|H_\omega(f_N)\|_{H(\infty,p)}^p \gtrsim (N+1)^{\frac{1}{p-1}} \left(\frac{\omega_{8N}}{\omega_{12N}}\right)^p, \quad N \in \mathbb{N}.$$

By Lemmas 11 and 8,

$$\|f_N\|_{X_p}^p \lesssim \|f_N\|_{HL(p)}^p = \sum_{n=0}^N \frac{1}{(n+1)^{1-\frac{1}{p-1}}} \asymp (N+1)^{\frac{1}{p-1}}.$$

Consequently,

$$(N+1)^{\frac{1}{p-1}} \left(\frac{\omega_{8N}}{\omega_{12N}}\right)^p \lesssim \|H_\omega(f_N)\|_{H(\infty,p)}^p \lesssim \|f_N\|_{X_p}^p \lesssim (N+1)^{\frac{1}{p-1}}, \quad N \in \mathbb{N}.$$

Therefore, there is $C = C(\omega, p)$ such that $\omega_{8N} \leq C\omega_{12N}, \quad N \in \mathbb{N}$. From now on, for each $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the biggest integer $\leq x$. For any $x \geq 120$, take $N \in \mathbb{N}$ such that $8N \leq x < 8N + 8$, and then

$$\omega_x \leq \omega_{8N} \leq C\omega_{12N} \leq C\omega_{8\lfloor\frac{3N}{2}\rfloor} \leq C^2\omega_{12\lfloor\frac{3N}{2}\rfloor} \leq C^2\omega_{18N-12} \leq C^2\omega_{16N+16} \leq C^2\omega_{2x}.$$

So, $\omega \in \widehat{\mathcal{D}}$ by Lemma 6.

Second Step. We will prove that $M_p(\omega) < \infty$.

Case $\boldsymbol{Y_p = HL(p)}$. Set an arbitrary $N \in \mathbb{N}$. Then, bearing in mind that $\{\omega_k\}_{k=0}^\infty$ is decreasing,

$$\begin{aligned} \left(\sum_{n=0}^N \frac{1}{(n+1)^2\omega_{2n+1}^p}\right)\left(\sum_{k=N}^\infty \widehat{f}(k)\omega_{2k+1}\right)^p &\leq \sum_{n=0}^\infty \frac{1}{(n+1)^2\omega_{2n+1}^p}\left(\sum_{k=0}^\infty \widehat{f}(k)\omega_{n+k}\right)^p \\ &\lesssim \|H_\omega\|_{X_p\to HL(p)}^p \|f\|_{X_p}^p. \end{aligned} \tag{3.6}$$

Take $M, N \in \mathbb{N}$, $M > 4N+1$, and consider the family of test polynomials

$$f_{N,M}(z) = \sum_{k=N}^{M} \omega_{2k+1}^{p'-1}(k+1)^{p'-2} z^k, \ z \in \mathbb{D}. \tag{3.7}$$

Then, Lemmas 8 and 11 yield

$$\sum_{k=N}^{M} \omega_{2k+1}^{p'}(k+1)^{p'-2} = \|f_{N,M}\|_{HL(p)}^p \gtrsim \|f_{N,M}\|_{X_p}^p$$

where the constants do not depend on M or N.

So, testing this family of functions in (3.6), there exists $C = C(p, \omega) > 0$ such that

$$\left(\sum_{n=0}^{N} \frac{1}{(n+1)^2 \omega_{2n+1}^p}\right)\left(\sum_{k=N}^{M} \omega_{2k+1}^{p'}(k+1)^{p'-2}\right)^{p-1} \leq C, \quad \text{for any} M, N \in \mathbb{N}, M > 4N+1.$$

By letting $M \to \infty$, and taking the supremum in $N \in \mathbb{N}$, (3.5) holds.

Case $Y_p \in \{H(\infty, p), H^p, D_{p-1}^p\}$. Let $f_{N,M}$ be the functions defined in (3.7), then $H_\omega(f_{N,M}) = \widetilde{H_\omega}(f_{N,M})$. This together with the fact that $\omega \in \widehat{\mathcal{D}}$ and Lemma 9, yields

$$\|H_\omega(f_{N,M})\|_{Y_p} \asymp \|H_\omega(f_{N,M})\|_{HL(p)}$$

where the constants in the inequalities do not depend on M or N.

Therefore, using Lemmas 8, 9 and 11, there exists $C = C(p, \omega) > 0$ such that

$$\|H_\omega(f_{N,M})\|_{HL(p)} \leq C\|f_{N,M}\|_{HL(p)}.$$

So, arguing as in the case $Y_p = HL(p)$, we obtain $M_p(\omega) < \infty$.

Third Step. We will prove that the condition $M_p(\omega) < \infty$ implies that $\omega \in \mathcal{M}$. Indeed, set $K, M \in \mathbb{N}$, $K, M > 1$ and $N \in \mathbb{N}$. By (3.5),

$$\begin{aligned}\infty > M_p(\omega) &\geq \left(\sum_{j=N}^{KN} \frac{1}{(j+1)^2 \omega_{2j+1}^p}\right)^{\frac{1}{p}} \left(\sum_{j=KN}^{(K+M)N-1} \omega_{2j+1}^{p'}(j+1)^{p'-2}\right)^{\frac{1}{p'}} \\ &\geq \frac{\omega_{2(K+M)N}}{\omega_{2N}} \left(\sum_{j=N}^{KN} \frac{1}{(j+1)^2}\right)^{\frac{1}{p}} \left(\sum_{j=KN}^{(K+M)N-1} (j+1)^{p'-2}\right)^{\frac{1}{p'}},\end{aligned}$$

So, there is $C = C(p) > 0$ such that

$$\omega_{2N} \geq \omega_{2(K+M)N} \frac{1}{M_p(\omega)} C\left((K+M)^{p'-1} - K^{p'-1}\right)^{\frac{1}{p'}}, \text{ for all} N \in \mathbb{N}. \tag{3.8}$$

Now, fix $K > 1$ and take $M \in \mathbb{N}$ large enough such that

$$\frac{1}{M_p(\omega)} C\left((K+M)^{p'-1} - K^{p'-1}\right)^{\frac{1}{p'}} = C(K, M, p, \omega) > 1.$$

Let $x \geq 1$ and take $N \in \mathbb{N}$ such that $2N - 2 \leq x < 2N$. Then, by (3.8)

$$\begin{aligned}\omega_x &\geq \omega_{2N} \geq C(K, M, p, \omega)\omega_{2(K+M)N} \geq C(K, M, p, \omega)\omega_{(K+M)x+2(K+M)} \\ &\geq C(K, M, p, \omega)\omega_{3(K+M)x},\end{aligned}$$

so $\omega \in \mathcal{M}$. Since $\omega \in \widehat{\mathcal{D}}$, [23, Theorem 3] yields $\omega \in \mathcal{D}$. The proof is finished. □

3.2 Sufficiency part of Theorem 1

For the purpose of proving Theorem 1 we need some additional preparations. In particular, we aim for reformulating the necessary discrete condition on the moments of the radial weight ω, $M_p(\omega)<\infty$, as a continuous inequality in terms of $\widehat{\omega}(r)$. Observe that a radial weight ω satisfies the condition

$$K_{p,c}(\omega)=\sup_{0<r<1}\left(1+\int_0^r\frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}}\left(\int_r^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}dt\right)^{\frac{1}{p'}}<\infty$$

if and only if $M_{p,c}(\omega)<\infty$. This fact will be used repeatedly throughout the paper.

Lemma 13 *Let $1<p<\infty$ and $\omega\in\widehat{\mathcal{D}}$. Set*

$$K_{p,c}(\omega)=\sup_{0<r<1}\left(1+\int_0^r\frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}}\left(\int_r^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}dt\right)^{\frac{1}{p'}}.$$

Then,

$$\int_0^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}dt\asymp\sum_{k=0}^{\infty}\omega_{2k+1}^{p'}(k+1)^{p'-2}$$

and

$$M_p(\omega)\asymp K_{p,c}(\omega).$$

Proof Let $0<r<1$ and set $N\in\mathbb{N}$ such that $1-\frac{1}{N}\le r<1-\frac{1}{N+1}$. Then, by using Lemma 6,

$$\begin{aligned}\sum_{k=0}^{N}\frac{1}{(k+1)^2\omega_{2k+1}^p}&\asymp\sum_{k=0}^{N}\frac{1}{(k+1)^2\widehat{\omega}\left(1-\frac{1}{k+1}\right)^p}\gtrsim 1+\sum_{k=1}^{N}\int_k^{k+1}\frac{1}{x^2\widehat{\omega}\left(1-\frac{1}{x}\right)^p}dx\\&=1+\int_0^{1-\frac{1}{N+1}}\frac{1}{\widehat{\omega}(s)^p}ds\ge 1+\int_0^r\frac{1}{\widehat{\omega}(s)^p}ds.\end{aligned}$$

In addition, by Lemma 6 again,

$$\begin{aligned}\int_r^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}dt&\le\int_{1-\frac{1}{N}}^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}dt=\sum_{k=N}^{\infty}\int_{1-\frac{1}{k}}^{1-\frac{1}{k+1}}\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}dt\\&\le\sum_{k=N}^{\infty}\widehat{\omega}\left(1-\frac{1}{k}\right)^{p'}\int_{1-\frac{1}{k}}^{1-\frac{1}{k+1}}\frac{1}{(1-t)^{p'}}dt\lesssim\sum_{k=N}^{\infty}\omega_{2k+1}^{p'}(k+1)^{p'-2}.\end{aligned}\tag{3.9}$$

Therefore, $K_{p,c}(\omega)\lesssim M_p(\omega)$.

Conversely, in order to obtain the reverse inequality, a similar argument to (3.9) yields

$$\sum_{k=0}^{N}\frac{1}{(k+1)^2\omega_{2k+1}^p}\asymp 1+\sum_{k=1}^{N}\frac{1}{(k+1)^2\widehat{\omega}\left(1-\frac{1}{k}\right)^p}\lesssim 1+\int_0^{1-\frac{1}{N+1}}\frac{1}{\widehat{\omega}(s)^p}ds.$$

Now, on the one hand, if $r \le \frac{1}{2}$ then $\sum_{k=0}^{N} \frac{1}{(k+1)^2\omega_{2k+1}^p} \lesssim 1 \le 1 + \int_0^r \frac{1}{\widehat{\omega}(s)^p}ds$. On the other hand, if $\frac{1}{2} \le r < 1$,

$$\begin{aligned}\sum_{k=0}^{N} \frac{1}{(k+1)^2\omega_{2k+1}^p} &\lesssim 1 + \int_0^r \frac{1}{\widehat{\omega}(s)^p}ds + \int_{1-\frac{1}{N}}^{1-\frac{1}{N+1}} \frac{1}{\widehat{\omega}(s)^p}ds \\ &\lesssim 1 + \int_0^r \frac{1}{\widehat{\omega}(s)^p}ds + \frac{1}{N\widehat{\omega}\left(1-\frac{1}{N+1}\right)^p}\end{aligned}$$

So, Lemma 6 yields

$$\begin{aligned}\sum_{k=0}^{N} \frac{1}{(k+1)^2\omega_{2k+1}^p} &\lesssim 1 + \int_0^r \frac{1}{\widehat{\omega}(s)^p}ds + \frac{1-r}{\widehat{\omega}\,(2r-1)^p} \\ &\asymp 1 + \int_0^r \frac{1}{\widehat{\omega}(s)^p}ds + \int_{2r-1}^{r} \frac{1}{\widehat{\omega}(s)^p}ds \\ &\lesssim 1 + \int_0^r \frac{1}{\widehat{\omega}(s)^p}ds, \quad \frac{1}{2} \le r < 1.\end{aligned}$$

Next,

$$\begin{aligned}\sum_{k=N}^{\infty} \omega_{2k+1}^{p'}(k+1)^{p'-2} &\asymp \sum_{k=N}^{\infty} \widehat{\omega}\left(1-\frac{1}{k+2}\right)^{p'}(k+1)^{p'} \int_{1-\frac{1}{k+1}}^{1-\frac{1}{k+2}} dt \\ &\lesssim \int_{1-\frac{1}{N+1}}^{1} \left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'} dt \le \int_r^1 \left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'} dt,\end{aligned} \tag{3.10}$$

and consequently, $M_p(\omega) \lesssim K_{p,c}(\omega)$. Finally, (3.9) and (3.10) imply

$$\int_0^1 \left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'} dt \asymp \sum_{k=0}^{\infty} \omega_{2k+1}^{p'}(k+1)^{p'-2}.$$

This finishes the proof. □

We will also need the following description of the class $\mathcal{D}$.

Lemma 14 *Let ω be a radial weight. Then the following conditions are equivalent:*

(i) $\omega \in \mathcal{D}$;
(ii) *The function defined as* $\widetilde{\omega}(r) = \frac{\widehat{\omega}(r)}{1-r}$, $0 \le r < 1$, *is a radial weight and satisfies*

$$\widehat{\omega}(r) \asymp \widehat{\widetilde{\omega}}(r), \quad 0 \le r < 1.$$

Proof (i)⇒(ii). By Lemma 7, there is $\alpha > 0$ such that

$$\int_r^1 \widetilde{\omega}(s)\,ds \lesssim \frac{\widehat{\omega}(r)}{(1-r)^\alpha} \int_r^1 (1-s)^{\alpha-1}\,ds \lesssim \widehat{\omega}(r), \quad 0 \le r < 1,$$

which, in particular, implies that $\widetilde{\omega}$ is a radial weight. On the other hand, by Lemma 6, there is $\beta > 0$ such that

$$\int_r^1 \widetilde{\omega}(s)\,ds \gtrsim \frac{\widehat{\omega}(r)}{(1-r)^\beta} \int_r^1 (1-s)^{\beta-1}\,ds \gtrsim \widehat{\omega}(r), \quad 0 \le r < 1.$$

Reciprocally, if (ii) holds, there are $C_1, C_2 > 0$ such that

$$C_1\widehat{\omega}(r) \le \widehat{\widehat{\omega}}(r) \le C_2\widehat{\omega}(r), \quad 0 \le r < 1.$$

So, for any $K > 1$,

$$\widehat{\omega}(r) \ge \frac{1}{C_2}\int_r^{1-\frac{1-r}{K}} \widetilde{\omega}(s)\,ds \ge \frac{\log K}{C_2}\widehat{\omega}\left(1-\frac{1-r}{K}\right), \quad 0 \le r < 1.$$

Therefore, taking K such that $\frac{\log K}{C_2} > 1$, $\omega \in \check{\mathcal{D}}$.

Moreover, for any $K > 1$

$$\widehat{\omega}(r) \le \frac{1}{C_1}\widehat{\widehat{\omega}}(r) \le \frac{\log K}{C_1}\widehat{\omega}(r) + \frac{1}{C_1}\int_{1-\frac{1-r}{K}}^1 \widetilde{\omega}(s)\,ds \quad 0 \le r < 1.$$

If $\frac{\log K}{C_1} < 1$, then

$$\widehat{\omega}(r) \le \frac{C_2}{1-\frac{\log K}{C_1}}\widehat{\omega}\left(1-\frac{1-r}{K}\right), \quad 0 \le r < 1.$$

So, $\omega \in \widehat{\mathcal{D}}$. This finishes the proof. □

The previous lemma may be used to prove that a differentiable non-decreasing function $h : [0, 1) \to [0, \infty)$ belongs to $L^p_{\omega,[0,1)}$ if and only if it belongs to $L^p_{\widetilde{\omega},[0,1)}$. This result is essential for our purposes. In particular, bearing in mind Lemma 14 and two integration by parts,

$$\int_0^1 h(t)\omega(t)\,dt \lesssim h(0)\widehat{\omega}(0) + \int_0^1 h(t)\widetilde{\omega}(t)\,dt, \tag{3.11}$$

for any differentiable non-decreasing function $h : [0, 1) \to [0, \infty)$.

Bearing in mind Lemma 13, our next result ensures that the Hilbert-type operators H_ω and $\widetilde{H_\omega}$ are well defined on $X_p \in \{H(\infty, p), H^p, D^p_{p-1}, HL(p)\}$, $1 < p < \infty$ when $\omega \in \mathcal{D}$ and $M_p(\omega) < \infty$.

Lemma 15 *Let $\omega \in \mathcal{D}$ and $1 < p < \infty$ such that $\int_0^1 \left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'} dt < \infty$.*

Then

$$\int_0^1 M_\infty(t, f)\omega(t)\,dt \lesssim \|f\|_{H(\infty,p)}\left(\int_0^1 \left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'} dt\right)^{1/p'}, \quad f \in \mathcal{H}(\mathbb{D}).$$

In particular, $T(f) \in \mathcal{H}(\mathbb{D})$ for any $f \in X_p$, where $X_p \in \{H(\infty, p), H^p, D^p_{p-1}, HL(p)\}$ and $T \in \{H_\omega, \widetilde{H_\omega}\}$.

Proof By (3.11)

$$\int_0^1 M_\infty(t, f)\omega(t)\,dt \le |f(0)|\widehat{\omega}(0) + \int_0^1 M_\infty(t, f)\widetilde{\omega}(t)\,dt. \tag{3.12}$$

Then, by Hölder's inequality

$$\int_0^1 M_\infty(t,f)\omega(t)\,dt \le |f(0)|\widehat{\omega}(0) + \left(\int_0^1 M_\infty^p(t,f)\,dt\right)^{1/p}\left(\int_0^1 \widetilde{\omega}(t)^{p'}\,dt\right)^{1/p'}$$
$$\lesssim \|f\|_{H(\infty,p)}\left(\int_0^1 \widetilde{\omega}(t)^{p'}\,dt\right)^{1/p'} < \infty, \quad f \in H(\infty,p).$$

Joining the above chain of inequalities with Lemma 8, the proof is finished. □

Next, for $p, q > 0$ and $\alpha > -1$, let $H^1(p,q,\alpha)$ denote the space of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^1(p,q,\alpha)} = \left(|f(0)|^p + \int_0^1 M_q^p(r,f')(1-r)^\alpha dr\right)^{\frac{1}{p}} < \infty.$$

It is worth mentioning that $H^1(p,p,p-1) = D^p_{p-1}$.

The following inequality will be used in the proof of Theorem 1. It was proved in [16, Corollary 3.1].

Lemma 16 *Let $1 < q < p < \infty$. Then,*

$$\|f\|_{H^p} \lesssim \|f\|_{H^1\left(p,q,p\left(1-\frac{1}{q}\right)\right)}, \quad f \in \mathcal{H}(\mathbb{D}).$$

Now, we are ready to prove the main result of this section.

Proof of Theorem 1 The implication (i)⇒(ii) was proved in Proposition 12. The implication (ii)⇒(iii) is clear, and (iii)⇒(ii) follows from the third step in the proof of Proposition 12. On the other hand, bearing in mind that $M_{p,c}(\omega) < \infty$ if and only if $K_{p,c}(\omega) < \infty$, (iii)⇔(iv) follows from Lemma 13. Then, it is enough to prove (ii)⇒(i).
(ii)⇒(i).

First Step. We will prove the inequality

$$\|T(f)\|_{Y_p} \lesssim \|f\|_{X_p} + \|\widetilde{H}_\omega(f)\|_{Y_p}, \quad f \in X_p. \tag{3.13}$$

By Lemmas 8 and 16, it is enough to prove

$$\|H_\omega(f)\|_{H^1\left(p,q,p\left(1-\frac{1}{q}\right)\right)} \lesssim \|f\|_{X_p} + \|\widetilde{H}_\omega(f)\|_{Y_p}, \quad 1 < q, p < \infty, \quad f \in X_p. \tag{3.14}$$

Let $f \in X_p$. Then, Lemmas 13 and 15 ensure that $H_\omega(f) \in \mathcal{H}(\mathbb{D})$. By [16, Theorem 2.1]

$$\|H_\omega(f)\|^p_{H^1\left(p,q,p\left(1-\frac{1}{q}\right)\right)} \asymp |H_\omega(f)(0)|^p + |H_\omega(f)'(0)|^p$$
$$+ \sum_{n=0}^\infty 2^{-n\left(p\left(1-\frac{1}{q}\right)+1\right)}\|\Delta_n(H_\omega(f))'\|^p_{H^q}. \tag{3.15}$$

Due to

$$(H_\omega(f))'(z) = \sum_{n=0}^\infty \frac{n+1}{2(n+2)\omega_{2n+3}}\left(\int_0^1 f(t)t^{n+1}\omega(t)dt\right)z^n,$$

and using the proof of [8, Lemma 7], Lemma 10 and (3.1),

$$\|\Delta_n(H_\omega(f))'\|_{H^q}^p \lesssim \frac{\left(\int_0^1 t^{2^{n-2}+1}|f(t)|\omega(t)dt\right)^p}{\omega_{2^{n+2}+3}^p} 2^{np\left(1-\frac{1}{q}\right)}, \quad n \geq 3.$$

Hence, by using Lemma 6,

$$\begin{aligned}\sum_{n=3}^{\infty} 2^{-n\left(p\left(1-\frac{1}{q}\right)+1\right)} \|\Delta_n(H_\omega(f))'\|_{H^q}^p &\lesssim \sum_{n=3}^{\infty} \frac{2^{-2n}}{\omega_{2^{n+2}}^p} \sum_{k=2^{n-3}}^{2^{n-2}} \left(\int_0^1 t^k |f(t)|\omega(t)dt\right)^p \\ &\lesssim \sum_{k=1}^{\infty} \frac{\left(\int_0^1 t^k|f(t)|\omega(t)dt\right)^p}{\omega_{2k+1}^p (k+1)^2} \lesssim \|\widetilde{H}_\omega(f)\|_{HL(p)}^p.\end{aligned} \tag{3.16}$$

In addition, by Lemmas 8 and 15

$$|H_\omega(f)(0)|^p + |H_\omega(f)'(0)|^p + \sum_{n=0}^{2} 2^{-np} \|\Delta_n(H_\omega(f))'\|_{H^p}^p \lesssim \|f\|_{X_p}^p. \tag{3.17}$$

Therefore, by putting together (3.15), (3.16) and (3.17)

$$\|H_\omega(f)\|_{H^1\left(p,q,p\left(1-\frac{1}{q}\right)\right)}^p \lesssim \|f\|_{X_p}^p + \|\widetilde{H}_\omega(f)\|_{HL(p)}^p.$$

The above inequality, together with Lemma 9, yields (3.14).

Second Step. We will prove the inequality

$$\|\widetilde{H}_\omega(f)\|_{D_{p-1}^p} \lesssim \|f\|_{H(\infty,p)}, \quad f \in H(\infty, p). \tag{3.18}$$

We denote by

$$G_t^\omega(z) = \frac{d}{dz}\left(\frac{1}{z}\int_0^z B_t^\omega(\zeta)d\zeta\right). \tag{3.19}$$

By [26, Lemma B]

$$M_p(r, G_t^\omega) \lesssim \left(1 + \int_0^{rt} \frac{ds}{\widehat{\omega}(s)(1-s)^p}\right)^{1/p} \lesssim \frac{1}{\widehat{\omega}(rt)(1-rt)^{1-\frac{1}{p}}}, \quad 0 \leq r, t < 1,$$

which together with Minkowski's inequality yields

$$\begin{aligned}\|\widetilde{H}_\omega(f)\|_{D_{p-1}^p}^p &\lesssim |\widetilde{H}_\omega(f)(0)|^p + \int_0^1 \left(\int_0^1 |f(t)|\omega(t) M_p(r, G_t^\omega)dt\right)^p (1-r)^{p-1}\, dr \\ &\lesssim |\widetilde{H}_\omega(f)(0)|^p + \int_0^1 \left(\int_0^1 \frac{|f(t)|\omega(t)}{\widehat{\omega}(rt)(1-rt)^{1-\frac{1}{p}}}dt\right)^p (1-r)^{p-1}\, dr \\ &\leq |\widetilde{H}_\omega(f)(0)|^p + \int_0^1 \left(\int_0^1 \frac{M_\infty(t,f)\omega(t)}{\widehat{\omega}(rt)(1-rt)^{1-\frac{1}{p}}}dt\right)^p (1-r)^{p-1}\, dr.\end{aligned}$$

Now, by (3.11)

$$\|\widetilde{H}_\omega(f)\|^p_{D^p_{p-1}} \lesssim |\widetilde{H}_\omega(f)(0)|^p + |f(0)|^p + \int_0^1 \left(\int_0^1 \frac{M_\infty(t,f)}{\widehat{\omega}(rt)(1-rt)^{1-\frac{1}{p}}} \frac{\widehat{\omega}(t)}{1-t} dt \right)^p (1-r)^{p-1}\, dr. \quad (3.20)$$

Next, by Lemma 13, $M_{p,c}(\omega) < \infty$ holds, so [17, Theorem 2] yields

$$\begin{aligned} &\int_0^1 \left(\int_r^1 \frac{M_\infty(t,f)}{\widehat{\omega}(rt)(1-rt)^{1-\frac{1}{p}}} \frac{\widehat{\omega}(t)}{1-t} dt \right)^p (1-r)^{p-1}\, dr \\ &\asymp \int_0^1 \left(\int_r^1 M_\infty(t,f) \frac{\widehat{\omega}(t)}{1-t} dt \right)^p \frac{1}{\widehat{\omega}(r)^p}\, dr \\ &\lesssim \|f\|^p_{H(\infty,p)} \end{aligned} \quad (3.21)$$

On the other hand, by [17, Theorem 1],

$$\begin{aligned} &\int_0^1 \left(\int_0^r \frac{M_\infty(t,f)}{\widehat{\omega}(rt)(1-rt)^{1-\frac{1}{p}}} \frac{\widehat{\omega}(t)}{1-t} dt \right)^p (1-r)^{p-1}\, dr \\ &\asymp \int_0^1 \left(\int_0^r \frac{M_\infty(t,f)}{(1-t)^{2-\frac{1}{p}}} dt \right)^p (1-r)^{p-1}\, dr \\ &\lesssim \|f\|^p_{H(\infty,p)}, \end{aligned} \quad (3.22)$$

where in the last inequality we have used that $\sup_{0<r<1} \left(\int_r^1 (1-t)^{p-1} \right)^{\frac{1}{p}} \left(\int_0^r (1-t)^{-1-p'} \right)^{\frac{1}{p'}} < \infty$. So, joining Lemma 15, (3.20), (3.21) and (3.22), we get (3.18).

Third Step. Since $\omega \in \widehat{\mathcal{D}}$, by Lemma 9

$$\|\widetilde{H}_\omega(f)\|_{D^p_{p-1}} \asymp \|\widetilde{H}_\omega(f)\|_{Y_p}, \quad f \in X_p,$$

for $Y_p \in \{H(\infty,p), H^p, D^p_{p-1}, HL(p)\}$. This, together with (3.13), (3.18) and Lemma 8 implies

$$\begin{aligned} \|T(f)\|_{Y_p} &\lesssim \|f\|_{X_p} + \|\widetilde{H}_\omega(f)\|_{Y_p} \\ &\asymp \|f\|_{X_p} + \|\widetilde{H}_\omega(f)\|_{D^p_{p-1}} \\ &\lesssim \|f\|_{X_p} + \|f\|_{H(\infty,p)} \\ &\lesssim \|f\|_{X_p}, \quad f \in X_p. \end{aligned}$$

This finishes the proof. □

3.3 $H_\omega : X_p \to Y_p$ versus $H_\omega : L^p_{[0,1)} \to Y_p$, $1 < p < \infty$

An additional byproduct of Theorem 1 is the following improvement of [26, Theorem 3].

Proof of Corollary 3 (i)⇒(ii). By Lemma 8, $T : Y_p \to Y_p$ is bounded, and so by Theorem 1, $\omega \in \mathcal{D}$. Next, by the proofs of [26, Theorems 3 and 4] we obtain $m_p(\omega) < \infty$.

(ii)⇒(iii) is clear. Finally, (iii)⇒(i) follows from Lemma 8 and [26, Theorem 3].

Putting together Lemma 8, Theorem 1 and Corollary 3, we deduce the following result.

Corollary 17 *Let ω be a radial weight and $1 < p < \infty$. Let $X_p, Y_p \in \{H(\infty, p), H^p, D^p_{p-1}, HL(p)\}$ and let $T \in \{H_\omega, \widetilde{H_\omega}\}$. If there exists $C > 0$ such that*

$$\omega(t) \leq C\frac{\widehat{\omega}(t)}{1-t} \text{for all} 0 \leq t < 1. \tag{3.23}$$

Then, the following statements are equivalent:

(i) $T : L^p_{[0,1)} \to Y_p$ *is bounded;*
(ii) $T : X_p \to Y_p$ *is bounded;*
(iii) $\omega \in \mathcal{D}$ *and* $M_{p,c}(\omega) < \infty$*;*
(iv) $\omega \in \widehat{\mathcal{D}}$ *and* $M_{p,c}(\omega) < \infty$*;*
(v) $\omega \in \mathcal{D}$ *and* $m_p(\omega) < \infty$*;*
(vi) $\omega \in \widehat{\mathcal{D}}$ *and* $m_p(\omega) < \infty$*.*

Proof The implication (i)⇒(ii) follows from Lemma 8, and (ii)⇔(iii)⇔(iv) follows from Theorem 1. Next, (iii)⇒(v) is a byproduct of the hypothesis (3.23). The equivalence (v)⇔(vi) and the implication (vi)⇒(i) have been proved in Corollary 3. This finishes the proof. □

Next, we will prove that there are weights $\omega \in \mathcal{D}$, such that $M_{p,c}(\omega) < \infty$ and $m_p(\omega) = \infty$, so in particular they do not satisfy (3.23). Consequently, the boundedness of the operator $H_\omega : L^p_{[0,1)} \to Y_p$ is not equivalent to the boundedness of the the operator $H_\omega : X_p \to Y_p$, where $X_p, Y_p \in \{H(\infty, p), H^p, D^p_{p-1}, HL(p)\}$. With this aim we prove the next result, which shows that despite its innocent looking condition, the class $\mathcal{D}$ has in a sense a complex nature.

Lemma 18 *Let $1 < p < \infty$ and $\nu \in \mathcal{D}$. Then, there exists $\omega \in \mathcal{D}$ such that*

$$\widehat{\omega}(t) \asymp \widehat{\nu}(t), \quad t \in [0, 1),$$

$\omega \in L^{p'}_{[0,r_0]}$ for any $r_0 \in (0, 1)$ and $\omega \notin L^{p'}_{[0,1)}$.

Proof By Lemma 14, $\widetilde{\nu} \in \mathcal{D}$. So, we can choose $K > 1$ so that $\widetilde{\nu}$ satisfies (2.1). Next, consider the sequences $r_n = 1 - \frac{1}{K^n}$, $t_n = r_n + a_n$, with

$$0 < a_n < \min\left(r_{n+1} - r_n, \frac{\left(\widehat{\widetilde{\nu}}(r_n)\right)^p}{(n+1)^{p-1}}\right), \quad n \in \mathbb{N} \cup \{0\}.$$

Let

$$\omega(t) = \sum_{n=0}^{\infty} h_n \chi_{[r_n,t_n]}(t)\widetilde{\nu}(t), \ t \in [0, 1), \quad \text{where} \quad h_n = \frac{\widehat{\widetilde{\nu}}(r_n) - \widehat{\widetilde{\nu}}(r_{n+1})}{\widehat{\widetilde{\nu}}(r_n) - \widehat{\widetilde{\nu}}(t_n)}, \ n \in \mathbb{N} \cup \{0\}.$$

Observe that the sequence $\{h_n\}_{n=0}^{\infty}$ is well-defined because

$$\widehat{\widetilde{\nu}}(r_n) - \widehat{\widetilde{\nu}}(t_n) = \int_{r_n}^{t_n} \frac{\widehat{\nu}(s)}{1-s}\, ds \geq \widehat{\nu}(t_n) \log\left(1 + \frac{a_n}{1-t_n}\right) > 0, \ n \in \mathbb{N} \cup \{0\}.$$

Moreover, ω is non-negative and

$$\begin{aligned}\int_0^1 \omega(t)\, dt &= \sum_{n=0}^{\infty} h_n \left(\widehat{\widetilde{\nu}}(r_n) - \widehat{\widetilde{\nu}}(t_n)\right) \\ &= \sum_{n=0}^{\infty} \left(\widehat{\widetilde{\nu}}(r_n) - \widehat{\widetilde{\nu}}(r_{n+1})\right) = \widehat{\widetilde{\nu}}(0) = \int_0^1 \widetilde{\nu}(t)\, dt \asymp \int_0^1 \nu(t)\, dt < \infty,\end{aligned}$$

where in the last equivalence we have used Lemma 14.

Next, take $t \in [0,1)$ and $N \in \mathbb{N} \cup \{0\}$ such that $r_N \le t < r_{N+1}$. By Lemmas 14 and 6,

$$\widehat{\omega}(t) \le \widehat{\omega}(r_N) = \sum_{n=N}^{\infty} \left(\widehat{\widehat{\nu}}(r_n) - \widehat{\widehat{\nu}}(r_{n+1})\right) = \widehat{\widehat{\nu}}(r_N) \asymp \widehat{\nu}(r_N) \lesssim \widehat{\nu}(t) \quad \text{and}$$

$$\widehat{\omega}(t) \ge \widehat{\omega}(r_{N+1}) = \sum_{n=N+1}^{\infty} \left(\widehat{\widehat{\nu}}(r_n) - \widehat{\widehat{\nu}}(r_{n+1})\right) = \widehat{\widehat{\nu}}(r_{N+1}) \asymp \widehat{\nu}(r_{N+1}) \gtrsim \widehat{\nu}(t),$$

so $\widehat{\omega}(t) \asymp \widehat{\nu}(t)$ and hence $\omega \in \mathcal{D}$.

It is clear that $\omega \in L^{p'}_{[0,r_0]}$ for any $r_0 \in (0,1)$, so it only remains to prove that $\omega \notin L^{p'}_{[0,1)}$. Bearing mind (2.1), we get that

$$h_n \asymp \frac{\widehat{\widehat{\nu}}(r_n)}{\widehat{\widehat{\nu}}(r_n) - \widehat{\widehat{\nu}}(t_n)}, \quad N \in \mathbb{N} \cup \{0\}.$$

This, together with Lemma 6 and Hölder's inequality, implies

$$\int_0^1 \omega(t)^{p'}\,dt = \sum_{n=0}^{\infty} h_n^{p'} \int_{r_n}^{t_n} \left(\frac{\widehat{\nu}(t)}{1-t}\right)^{p'} dt \asymp \sum_{n=0}^{\infty} \left(\widehat{\widehat{\nu}}(r_n)\right)^{p'} \frac{\int_{r_n}^{t_n} \left(\frac{\widehat{\nu}(t)}{1-t}\right)^{p'} dt}{\left(\int_{r_n}^{t_n} \frac{\widehat{\nu}(t)}{1-t}\,dt\right)^{p'}}$$

$$\ge \sum_{n=0}^{\infty} \left(\frac{\widehat{\widehat{\nu}}(r_n)}{(t_n - r_n)^{1/p}}\right)^{p'} = \sum_{n=0}^{\infty} \left(\frac{\widehat{\widehat{\nu}}(r_n)}{a_n^{1/p}}\right)^{p'} \ge \sum_{n=0}^{\infty} (n+1) = \infty.$$

□

Corollary 19 *Let $1 < p < \infty$ and $X_p, Y_p \in \{H(\infty,p), H^p, D^p_{p-1}, HL(p)\}$. For each radial weight ν such that $Q : L^p_{[0,1)} \to Y_p$ is bounded, where $Q \in \{H_\nu, \widetilde{H_\nu}\}$, there is a radial weight ω such that*

$$\widehat{\omega}(t) \asymp \widehat{\nu}(t), \quad t \in [0,1),$$

$\omega \in L^{p'}_{[0,r_0]}$ for any $r_0 \in (0,1)$, $T : X_p \to Y_p$ is bounded and $T : L^p_{[0,1)} \to Y_p$ is not bounded. Here $T \in \{H_\omega, \widetilde{H_\omega}\}$.

Proof Since $Q : L^p_{[0,1)} \to Y_p$ is bounded, by Theorem 1, $\nu \in \mathcal{D}$ and $M_{p,c}(\nu) < \infty$. Now, by Lemma 18 there is a radial weight ω such that $\widehat{\omega}(t) \asymp \widehat{\nu}(t)$, $\omega \in L^{p'}_{[0,r_0]}$ for any $r_0 \in (0,1)$ and $\omega \notin L^{p'}_{[0,1)}$. So, $m_p(\omega) = \infty$ and by Corollary 3, $T : L^p_{[0,1)} \to Y_p$ is not bounded. Moreover, $\omega \in \mathcal{D}$ and $M_{p,c}(\omega) < \infty$ because ν satisfies both properties, so Theorem 1 yields $T : X_p \to Y_p$ is bounded. □

3.4 Compactness of Hilbert-type operators on X_p-spaces. Case $1 < p < \infty$

For X, Y two Banach spaces, a sublinear operator $L : X \to Y$ is said to be compact provided $L(A)$ has compact closure for any bounded set $A \subset X$. Once it has been understood the radial weights ω such that $H_\omega : X_p \to Y_p$ is bounded, $X_p, Y_p \in \{H(\infty,p), D^p_{p-1}, H^p, HL(p)\}$, $1 < p < \infty$, it is natural to consider the analogous problem, replacing boundedness by compactness. Theorem 22 in this section answers this question, but firstly we need some previous results.

Lemma 20 *Let $1 < p < \infty$ and $\omega \in \mathcal{D}$ such that $\|\widetilde{\omega}\|_{L^{p'}_{[0,1)}} < \infty$. Let $\{f_k\}_{k=0}^{\infty} \subset X_p \in \{H(\infty, p), D^p_{p-1}, H^p, HL(p)\}$ such that $\sup_{k\in\mathbb{N}} \|f_k\|_{X_p} < \infty$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. Then the following statements hold:*

(i) $\int_0^1 |f_k(t)|\omega(t)dt \to 0$ *when* $k \to \infty$.
(ii) *If* $T \in \{H_\omega, \widetilde{H_\omega}\}$*, then* $T(f_k) \to 0$ *uniformly on compact subsets of* $\mathbb{D}$.

Proof (i). Let $\varepsilon > 0$. By hypothesis $\int_0^1 \widetilde{\omega}(t)^{p'}dt < \infty$, so there exists $0 < \rho_0 < 1$ such that $\int_{\rho_0}^1 \widetilde{\omega}(t)^{p'}dt < \varepsilon$. Moreover, there exists k_0 such that for every $k \geq k_0$ and $z \in M = \overline{D(0, \rho_0)}$, $|f_k(z)| < \varepsilon$. Then, by Lemma 14, (3.11), and Hölder inequality

$$
\begin{aligned}
\int_0^1 |f_k(t)|\omega(t)dt &\leq |f_k(0)|\widehat{\omega}(0) + \int_0^1 M_\infty(t, f_k)\widetilde{\omega}(t)dt \\
&\lesssim \int_0^{\rho_0} M_\infty(t, f_k)\widetilde{\omega}(t)dt + \int_{\rho_0}^1 M_\infty(t, f_k)\widetilde{\omega}(t)dt \\
&< \varepsilon \int_0^{\rho_0} \widetilde{\omega}(t)dt + \sup_{k\in\mathbb{N}} \|f_k\|_{H(\infty,p)} \int_{\rho_0}^1 \widetilde{\omega}(t)^{p'}dt \\
&< \varepsilon \left(\int_0^1 \widetilde{\omega}(t)dt + \sup_{k\in\mathbb{N}} \|f_k\|_{H(\infty,p)} \right) = C\varepsilon,
\end{aligned}
$$

where in the last step we have used Lemma 8.

(ii). Let be $M \subset \mathbb{D}$ a compact set and $K_t^\omega(z) = \frac{1}{z}\int_0^z B_t^\omega(u)\,du$. If $z \in M$

$$
|T(f_k)(z)| \leq \int_0^1 |f_k(t)|\,|K_t^\omega(z)|\omega(t)dt \leq \sup_{\substack{z\in M\\ t\in[0,1)}} |K_t^\omega(z)| \int_0^1 |f_k(t)|\omega(t)\,dt.
$$

Since, $M \subset \overline{D(0, \rho_0)}$, for some $\rho_0 \in (0, 1)$, then

$$
\sup_{\substack{z\in M\\ t\in[0,1)}} |K_t^\omega(z)| = \sup_{\substack{z\in M\\ t\in[0,1)}} \left|\sum_{k=0}^{\infty} \frac{t^k z^k}{2(k+1)\omega_{2k+1}}\right| \leq \sum_{k=0}^{\infty} \frac{{\rho_0}^k}{2(k+1)\omega_{2k+1}} = C(\omega, \rho_0) < \infty,
$$

so, by (i), $T(f_k) \to 0$ uniformly on M. This finishes the proof. □

Theorem 21 *Let ω be a radial weight, $1 < p < \infty$, $X_p, Y_p \in \{H(\infty, p), D^p_{p-1}, H^p, HL(p)\}$ and let $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then, the following assertions are equivalent:*

(i) $T : X_p \to Y_p$ *is compact;*
(ii) *For every sequence $\{f_k\}_{k=0}^{\infty} \subset X_p$ such that $\sup_{k\in\mathbb{N}} \|f_k\|_{X_p} < \infty$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$,* $\lim_{k\to\infty} \|T(f_k)\|_{Y_p} = 0$.

Proof (i)⇒(ii). Let $\{f_n\}_{n=0}^{\infty} \subset X_p$ such that $\sup_{n\in\mathbb{N}} \|f_n\|_{X_p} < \infty$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Assume there exist $\varepsilon > 0$ and a subsequence $\{n_k\}_k \subset \mathbb{N}$ such that

$$
\|T(f_{n_k})\|_{Y_p} > \varepsilon, \quad \text{for any} k. \tag{3.24}
$$

Since T is compact, there exists a subsequence $\{n_{k_j}\}_j \subset \mathbb{N}$ and $g \in Y_p$ such that $\lim_{j\to\infty} \|T(f_{n_{k_j}}) - g\|_{Y_p} = 0$. Moreover, Theorem 1 ensures that $\omega \in \mathcal{D}$ and $M_{p,c}(\omega) < \infty$, so $\|\widetilde{\omega}\|_{L^{p'}_{[0,1)}} < \infty$. Therefore Lemma 20, implies that $T(f_{n_{k_j}}) \to 0$ uniformly on compact subsets of $\mathbb{D}$, so $\lim_{j\to\infty} \|T(f_{n_{k_j}})\|_{Y_p} = 0$ which yields a contradiction with (3.24).

(ii)⇒(i). Let $\{f_n\}\subset X_p$ such that $\sup_{n\in\mathbb{N}}\|f_n\|_{X_p}<\infty$. Then, $\{f_n\}$ is uniformly bounded on compact subsets of $\mathbb{D}$. Then, by Montel's Theorem there exists $\{f_{n_k}\}_k$ and $f\in\mathcal{H}(\mathbb{D})$ such that $f_{n_k}\to f$ uniformly on compact subsets of $\mathbb{D}$. Let $g_{n_k}=f_{n_k}-f$, then $g_{n_k}\to 0$ uniformly on compact subsets of $\mathbb{D}$ and $\sup_{k\in\mathbb{N}}\|g_{n_k}\|_{X_p}<\infty$. Therefore, by hypothesis $\lim_{k\to\infty}\|T(g_{n_k})\|_{Y_p}=0$, that is, T is compact. □

Theorem 22 *Let ω be a radial weight, $1<p<\infty$, $X_p, Y_p\in\{H(\infty,p), D^p_{p-1}, H^p, HL(p)\}$, and let $T\in\{H_\omega,\widetilde{H_\omega}\}$. Then, $T: X_p\to Y_p$ is not compact.*

Proof Assume that $T: X_p\to Y_p$ is compact. For each $0<a<1$, set

$$f_a(z)=\left(\frac{1-a^2}{(1-az)^2}\right)^{1/p}=\sum_{n=0}^\infty \widehat{f_a}(n)z^n,\quad z\in\mathbb{D},$$

where $\widehat{f_a}(n)=(1-a^2)^{1/p}\frac{\Gamma(n+2/p)}{n!\Gamma(2/p)}a^n\geq 0$. So, by Stirling's formula

$$\widehat{f_a}(n)\asymp(1-a^2)^{1/p}(n+1)^{2/p-1}a^n,\quad n\in\mathbb{N}\cup\{0\}. \tag{3.25}$$

Consequently, $\|f_a\|_{HL(p)}\asymp 1$, $a\in(0,1)$. Moreover, $\|f_a\|_{H(\infty,p)}\asymp\|f_a\|_{D^p_{p-1}}\asymp\|f_a\|_{H^p}=1$. Furthermore, it is clear that $f_a\to 0$, as $a\to 1$ uniformly on compact subsets of $\mathbb{D}$, and $H_\omega(f_a)=\widetilde{H_\omega}(f_a)$. Since $T: X_p\to Y_p$ is compact, $\omega\in\widehat{\mathcal{D}}$ by Theorem 1. So, Lemma 9 implies that

$$\|T(f_a)\|_{Y_p}\asymp\|T(f_a)\|_{HL(p)},\quad a\in(0,1).$$

Therefore, by using (3.25) we have

$$\begin{aligned}
\|H_\omega(f_a)\|^p_{Y_p}&\gtrsim\|H_\omega(f_a)\|^p_{HL(p)}=\sum_{n=0}^\infty(n+1)^{p-2}\left(\frac{\sum_{k=0}^\infty\widehat{f_a}(k)\omega_{k+n}}{2(n+1)\omega_{2n+1}}\right)^p\\
&\gtrsim(1-a^2)\sum_{n=0}^\infty\frac{1}{(n+1)^2}\left(\sum_{k=0}^\infty(k+1)^{2/p-1}a^k\,\frac{\omega_{k+n}}{\omega_{2n}}\right)^p\\
&\geq(1-a)\sum_{n=0}^\infty\frac{1}{(n+1)^2}\left(\sum_{k=0}^n(k+1)^{2/p-1}a^k\,\frac{\omega_{k+n}}{\omega_{2n}}\right)^p\\
&\geq(1-a)\sum_{n=0}^\infty\frac{1}{(n+1)^2}\left(\sum_{k=0}^n(k+1)^{2/p-1}a^k\right)^p\\
&\geq(1-a)\sum_{n=0}^\infty\frac{a^{pn}}{(n+1)^2}\left(\sum_{k=0}^n(k+1)^{2/p-1}\right)^p\\
&\gtrsim(1-a)\sum_{n=0}^\infty a^{pn}=\frac{1-a}{1-a^p}\gtrsim 1,
\end{aligned}\tag{3.26}$$

so using Theorem 21 we deduce that $T: X_p\to Y_p$ is not a compact operator. □

4 Hilbert type operators acting on X_1-spaces

The first result of this section gives the equivalence of conditions (iii)–(vi) of Theorem 2.

Lemma 23 *Let $\omega \in \widehat{\mathcal{D}}$. Then, the following conditions are equivalent:*

(i) $K_{1,c}(\omega) = \sup_{a\in[0,1)} \frac{1}{1-a}\int_a^1 \omega(t)\left(1+\int_0^t \frac{ds}{\widehat{\omega}(s)}\right)dt < \infty$;

(ii) $K_{1,d}(\omega) = \sup_{a\in[0,1)} \frac{\widehat{\omega}(a)}{1-a}\left(1+\int_0^a \frac{ds}{\widehat{\omega}(s)}\right) < \infty$;

(iii) $M_1(\omega) = \sup_{N\in\mathbb{N}}(N+1)\omega_{2N}\sum_{k=0}^N \frac{1}{(k+1)^2\omega_{2k}} < \infty$.

Moreover,

$$K_{1,c}(\omega) \asymp K_{1,d}(\omega) \asymp M_1(\omega), \tag{4.1}$$

and $\omega \in \check{\mathcal{D}}$ when ω satisfies any of the three previous conditions.

Observe that for any radial weight, $K_{1,c}(\omega) < \infty$ holds if and only if $M_{1,c}(\omega) < \infty$, and analogously $K_{1,d}(\omega) < \infty$ if and only if $M_{1,d}(\omega) < \infty$. This fact will be used repeatedly throughout the paper.

Proof On the one hand,

$$\begin{aligned}\frac{\widehat{\omega}(a)}{1-a}\left(1+\int_0^a \frac{ds}{\widehat{\omega}(s)}\right) &\le \frac{1}{1-a}\int_a^1 \omega(t)\left(1+\int_0^t \frac{ds}{\widehat{\omega}(s)}\right)dt\\ &\le K_{1,c}(\omega), \quad a\in[0,1),\end{aligned}$$

so (i)$\Rightarrow$(ii) and $K_{1,d}(\omega) \lesssim K_{1,c}(\omega)$.

On the other hand assume that (ii) holds. Since $\omega \in \widehat{\mathcal{D}}$, [22, Lemma 3(ii)] (for $\nu(t) = 1$) yields

$$\begin{aligned}\frac{1}{1-a}\int_a^1 \omega(t)\left(1+\int_0^t \frac{ds}{\widehat{\omega}(s)}\right)dt &\le K_{1,d}(\omega)\left(\frac{1}{1-a}\int_a^1 \frac{\omega(t)(1-t)}{\widehat{\omega}(t)}dt\right)\\ &\lesssim K_{1,d}(\omega), \quad 0<a<1,\end{aligned}$$

that is (i) holds and $K_{1,c}(\omega) \lesssim K_{1,d}(\omega)$. Finally, by mimicking the proof of Lemma 13,

$$K_{1,d}(\omega) \asymp M_1(\omega),$$

so (ii)$\Leftrightarrow$(iii) and (4.1) holds.

Next, for any $K > 1$ and $r \in (0,1)$

$$\begin{aligned}M_{1,d}(\omega) &\ge \frac{K\widehat{\omega}\left(1-\frac{1-r}{K}\right)}{1-r}\int_r^{1-\frac{1-r}{K}} \frac{ds}{\widehat{\omega}(s)}\\ &\ge (K-1)\frac{\widehat{\omega}\left(1-\frac{1-r}{K}\right)}{\widehat{\omega}(r)},\end{aligned}$$

that is

$$\widehat{\omega}(r) \ge \frac{K-1}{M_{1,d}(\omega)}\widehat{\omega}\left(1-\frac{1-r}{K}\right), \quad 0<r<1,$$

so taking $K > M_{1,d}(\omega)+1$, we get $\omega \in \check{\mathcal{D}}$. This finishes the proof. □

The following result will be used to prove the equivalence (ii)$\Leftrightarrow$(iii) of Theorem 2.

Proposition 24 *Let μ be a finite positive Borel measure on $[0,1)$ and $X_1 \in \{H(\infty,1), HL(1)\}$. Then μ is a 1-Carleson measure for X_1 if and only if μ is a classical Carleson measure. Moreover,*

$$\|I_d\|_{X_1 \to L^1(\mu)} \asymp \sup_{a\in[0,1)} \frac{\mu([a,1))}{1-a}.$$

Proof If μ is a 1-Carleson measure for X_1, then by (2.2) and (1.3), μ is a 1-Carleson measure for H^1. So, by [6, Theorem 9.3] and its proof, μ is a classical Carleson measure and

$$\sup_{a\in[0,1)} \frac{\mu([a,1))}{1-a} \lesssim \|I_d\|_{X_1 \to L^1(\mu)}.$$

Conversely, if μ is a classical Carleson measure, two integration by parts yield

$$\int_{\mathbb{D}} |f(z)|d\mu(z) = \int_0^1 |f(t)|d\mu(t) \le \int_0^1 M_\infty(t,f)\,d\mu(t) \lesssim \|f\|_{H(\infty,1)} \sup_{a\in[0,1)} \frac{\mu([a,1))}{1-a}.$$

This inequality, together with Lemma 8, finishes the proof. □

We introduce some more notation to prove Theorem 2. For any C^∞-function $\Phi : \mathbb{R} \to \mathbb{C}$ with compact support, define the polynomials

$$W_n^\Phi(z) = \sum_{k\in\mathbb{Z}} \Phi\left(\frac{k}{n}\right) z^k, \quad n \in \mathbb{N}.$$

A particular case of the previous construction is useful for our purposes. Some properties of these polynomials have been gathered in the next lemma, see [12, Section 2] or [19, p. 143–144] for a proof.

Lemma 25 *Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that $\Psi \equiv 1$ on $(-\infty, 1]$, $\Psi \equiv 0$ on $[2,\infty)$ and Ψ is decreasing and positive on $(1,2)$. Set $\psi(t) = \Psi\left(\frac{t}{2}\right) - \Psi(t)$ for all $t \in \mathbb{R}$. Let $V_0(z) = 1 + z$ and*

$$V_n(z) = W_{2^{n-1}}^\psi(z) = \sum_{j=0}^\infty \psi\left(\frac{j}{2^{n-1}}\right) z^j = \sum_{j=2^{n-1}}^{2^{n+1}-1} \psi\left(\frac{j}{2^{n-1}}\right) z^j, \quad n \in \mathbb{N}.$$

Then,

$$f(z) = \sum_{n=0}^\infty (V_n * f)(z), \quad z \in \mathbb{D}, \quad f \in \mathcal{H}(\mathbb{D}), \tag{4.2}$$

and for each $0 < p < \infty$ there exists a constant $C = C(p,\Psi) > 0$ such that

$$\|V_n * f\|_{H^p} \le C\|f\|_{H^p}, \quad f \in H^p, \quad n \in \mathbb{N}. \tag{4.3}$$

In addition

$$\|V_n\|_{H^p} \asymp 2^{n(1-1/p)}, \quad 0 < p < \infty. \tag{4.4}$$

Let us denote $f_r(z) = f(rz)$, $z \in \mathbb{D}$, $r \in (0,1)$. Now we are ready to prove the main theorem of this section.

Proof of Theorem 2 First of all, recall that $M_{1,c}(\omega) < \infty$ if and only if $K_{1,c}(\omega) < \infty$ and analogously $M_{1,d}(\omega) < \infty$ if and only if $K_{1,d}(\omega) < \infty$, so that the equivalences (iii)⇔(iv)⇔(v)⇔(vi) follow from Lemma 23. The equivalence between (ii) and (iii) is a consequence of [6, Theorem 9.3] when $X_1 = H^1$, [29, Theorem 2.1] when $X_1 = D^1_0$ and Proposition 24 when $X_1 \in \{H(\infty,1), HL(1)\}$.

(i)⇒(iii). In order to obtain both conditions, $\omega \in \widehat{\mathcal{D}}$ and $M_{1,c}(\omega) < \infty$, we are going to deal with functions $f \in \mathcal{H}(\mathbb{D})$ such that $\widehat{f}(n) \geq 0$ for all $n \in \mathbb{N}\cup\{0\}$, so it is enough to prove the result for $T = H_\omega$.

First Step. Let us prove $\omega \in \widehat{\mathcal{D}}$. Bearing in mind Lemma 8 and (1.4)

$$\sum_{n=0}^{\infty} \frac{\omega_{n+N}}{(n+1)^2\omega_{2n+1}} \left(\sum_{k=0}^{N} \widehat{f}(k)\right) \leq \|H_\omega(f)\|_{H(\infty,1)} \lesssim \|H_\omega(f)\|_{Y_1} \lesssim \|f\|_{X_1} \lesssim \|f\|_{D^1_0}, \quad N \in \mathbb{N}, \tag{4.5}$$

for any $f \in \mathcal{H}(\mathbb{D})$ such that $\widehat{f}(n) \geq 0$, $n \in \mathbb{N}\cup\{0\}$. Next, for each $N \in \mathbb{N}$, consider the test functions $f_{\alpha,N}(z) = \sum_{k=0}^{N}(k+1)^\alpha z^k$, $\alpha > 0$. Set $M \in \mathbb{N}$ such that $2^M < N \leq 2^{M+1}$. Then, bearing in mind (4.2),

$$(f'_{\alpha,N})_s(z) = \sum_{n=0}^{\infty}(V_n * (f'_{\alpha,N})_s)(z) = \sum_{n=0}^{M}(V_n * (f'_{\alpha,N})_s)(z),$$

which together with [16, Lemma 3.1], [19, Lemma 5.4] and (4.4) gives

$$\begin{aligned}\|f_{\alpha,N}\|_{D^1_0} &\leq \int_0^1 M_1(s, f'_{\alpha,N})ds \leq \sum_{n=0}^{M}\int_0^1 \|V_n * (f'_{\alpha,N})_s\|_{H^1} ds \\ &\lesssim \sum_{n=0}^{M}\int_0^1 s^{2^{n-1}} 2^{n(\alpha+1)}\|V_n\|_{H^1} ds \asymp \sum_{n=0}^{M} 2^{n\alpha} \asymp 2^{M\alpha} \asymp (N+1)^\alpha.\end{aligned}$$

Testing the functions $f_{\alpha,N}$ in (4.5), $\sup_{N\in\mathbb{N}}(N+1)\sum_{n=0}^{\infty}\frac{\omega_{n+N}}{(n+1)^2\omega_{2n+1}} < \infty$. Therefore, there exists $C = C(\omega) > 0$

$$\frac{\omega_{8N}}{\omega_{12N}} \lesssim \frac{\omega_{8N}}{\omega_{12N}}(N+1)\sum_{n=6N}^{7N}\frac{1}{(n+1)^2} \leq (N+1)\sum_{n=6N}^{7N}\frac{\omega_{n+N}}{(n+1)^2\omega_{2n+1}} \leq C.$$

So, arguing as in the first step proof of Proposition 12, $\omega \in \widehat{\mathcal{D}}$.

Second Step. We will prove $M_{1,c}(\omega) < \infty$. Let us consider the test functions $f_a(z) = \frac{1-a^2}{(1-az)^2}$, $a \in (0,1)$. A calculation shows that $\|f_a\|_{D^1_0} \asymp 1$, $a \in (0,1)$. Then, by Lemma 8 and (1.3),

$$\|H_\omega\|_{X_1\to Y_1} \gtrsim \sup_{a\in(0,1)} \|H_\omega(f_a)\|_{Y_1} \gtrsim \sup_{a\in(0,1)} \|H_\omega(f_a)\|_{L^1_{[0,1)}}.$$

Consequently, using that $\omega \in \widehat{\mathcal{D}}$ and mimicking the proof (4.2) of [26, Theorem 2], we get $M_{1,c}(\omega) < \infty$.

Now let us prove (iv) ⇒ (i). Firstly, observe that the condition $M_{1,c}(\omega) < \infty$ implies $K_{1,c}(\omega) < \infty$ so that $\widetilde{\omega}(t) = \frac{\widehat{\omega}(t)}{1-t}$ is bounded on $[0,1)$. So, using Lemma 8 and (3.11),

$$\int_0^1 M_\infty(t,f)\omega(t)\,dt \lesssim \int_0^1 M_\infty(t,f)\widetilde{\omega}(t)\,dt \lesssim \|f\|_{H(\infty,1)} \lesssim \|f\|_{X_1},$$

that is $H_\omega(f) \in \mathcal{H}(\mathbb{D})$ for any $f \in X_1$. Secondly, by (1.4) and Lemma 8, it is enough to prove the inequality

$$\|H_\omega(f)\|_{D^1_0} \lesssim \|f\|_{H(\infty,1)}, \quad f \in H(\infty,1),$$

to end the proof. Indeed,

$$\begin{aligned}\|H_\omega(f)\|_{D^1_0} &\le \int_0^1 M_1(s, H_\omega(f)')ds \le \int_0^1 \left(\int_0^1 |f(t)|\,\omega(t) M_1(s, G_t^\omega)dt\right) ds \\ &= \int_0^1 |f(t)|\omega(t)\left(\int_0^1 M_1(s, G_t^\omega)ds\right) dt.\end{aligned}$$

Then by (3.19) and [26, Lemma B]

$$M_1(s, G_t^\omega) \asymp 1 + \int_0^{st} \frac{dx}{\widehat{\omega}(x)(1-x)}, \quad 0 \le s, t < 1.$$

Bearing in mind that $M_{1,c}(\omega) < \infty$ implies $K_{1,c}(\omega) < \infty$ and applying Proposition 24, the measure μ_ω defined as $d\mu_\omega(z) = \omega(z)\left(1 + \int_0^{|z|} \frac{ds}{\widehat{\omega}(s)}\right) \chi_{[0,1)}(z)\, dA(z)$ is a 1-Carleson measure for $H(\infty,1)$, so by Tonelli's theorem,

$$\begin{aligned}\|H_\omega(f)\|_{D^1_0} &\lesssim \int_0^1 |f(t)|\,\omega(t)\left(1 + \int_0^1 \left(\int_0^{st} \frac{dx}{\widehat{\omega}(x)(1-x)}\right) ds\right) dt \\ &= \int_0^1 |f(t)|\,\omega(t)\left(1 + \int_0^t \frac{(1-\frac{x}{t})}{\widehat{\omega}(x)(1-x)} dx\right) dt \\ &\le \int_0^1 |f(t)|\,\omega(t)\left(1 + \int_0^t \frac{dx}{\widehat{\omega}(x)}\right) dt \lesssim \|f\|_{H(\infty,1)}. \end{aligned} \tag{4.6}$$

This finishes the proof. □

4.1 $H_\omega : X_1 \to Y_1$ versus $H_\omega : L^1_{[0,1)} \to Y_1$

Firstly, we will study the boundedness of $T : L^1_{[0,1)} \to Y_1$, $T \in \{H_\omega, \widetilde{H_\omega}\}$, $Y_1 \in \{H(\infty,1), H^1, D^1_0, HL(1)\}$.

Theorem 26 *Let ω be a radial weight, let $Y_1 \in \{H(\infty,1), H^1, D^1_0, HL(1)\}$ and let $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then the following statements are equivalent:*

(i) *$T : L^1_{[0,1)} \to Y_1$ is bounded;*

(ii) *$\omega \in \mathcal{D}$ and $m_1(\omega) = \operatorname{ess\,sup}_{t\in[0,1)} \omega(t)\left(1 + \int_0^t \frac{ds}{\widehat{\omega}(s)}\right) < \infty$.*

(iii) *$\omega \in \widehat{\mathcal{D}}$ and $m_1(\omega) = \operatorname{ess\,sup}_{t\in[0,1)} \omega(t)\left(1 + \int_0^t \frac{ds}{\widehat{\omega}(s)}\right) < \infty$.*

Proof (i)⇒(ii). If (i) holds, then $\omega \in \mathcal{D}$ by Lemma 8 and Theorem 2. Next, using Lemma 8 again and making minor modifications in the proof of [26, Theorem 2] we get

$$\|T(f)\|_{Y_1} \gtrsim \|H_\omega(f)\|_{L^1_{[0,1)}} \gtrsim \int_0^1 f(t)\omega(t)\left(1 + \int_0^t \frac{ds}{\widehat{\omega}(s)}\right) dt,$$

for any $f \ge 0$, $f \in L^1_{[0,1)}$. So,

$$\int_0^1 f(t)\omega(t)\left(1 + \int_0^t \frac{ds}{\widehat{\omega}(s)}\right) dt \lesssim \|f\|_{L^1_{[0,1)}}, \quad f \ge 0$$

which implies that $m_1(\omega) < \infty$.

(ii)⇒(iii) is clear.

(iii)⇒(i). If (iii) holds, then $H_\omega(f) \in \mathcal{H}(\mathbb{D})$ for any $f \in L^1_{[0,1)}$ and arguing as in (4.6)

$$\|H_\omega(f)\|_{D^1_0} \lesssim \int_0^1 |f(t)|\,\omega(t)\left(1+\int_0^t \frac{dx}{\widehat{\omega}(x)}\right)dt \lesssim \|f\|_{L^1_{[0,1)}}.$$

This together with (1.3) and Lemma 8 gives that $H_\omega : L^1_{[0,1)} \to Y_1$ is bounded. This finishes the proof. □

Joining Theorems 2, 26 and Lemma 8 we deduce the following.

Corollary 27 *Let ω be a radial weight, $X_1, Y_1 \in \{H(\infty,1), H^1, D^1_0, HL(1)\}$ and let $T \in \{H_\omega, \widetilde{H_\omega}\}$. If ω satisfies the condition (3.23), then the following statements are equivalent:*

(i) *$T : L^1_{[0,1)} \to Y_1$ is bounded;*
(ii) *$T : X_1 \to Y_1$ is bounded;*
(iii) *$\omega \in \widehat{\mathcal{D}}$ and $M_{1,c}(\omega) < \infty$;*
(iv) *$\omega \in \mathcal{D}$ and $M_{1,c}(\omega) < \infty$;*
(v) *$\omega \in \widehat{\mathcal{D}}$ and $m_1(\omega) < \infty$;*
(vi) *$\omega \in \mathcal{D}$ and $m_1(\omega) < \infty$.*

Proof (i)⇒(ii) follows from Lemma 8, and (ii)⇔(iii)⇔(iv) were proved in Theorem 2. Next, since $M_{1,c}(\omega) < \infty$ implies $K_{1,c}(\omega) < \infty$, (iii)⇒(v) is a byproduct of the hypothesis (3.23). Finally, the equivalences (v)⇔(vi)⇔(i) follow from Theorem 26. This finishes the proof. □

A similar comparison between the conditions $M_{1,c}(\omega) < \infty$ and $m_1(\omega) < \infty$, to that made for the conditions $M_{p,c}(\omega) < \infty$ and $m_p(\omega) < \infty$, $1 < p < \infty$, can also be considered. The following result shows that they are not equivalent.

Corollary 28 *Let $X_1, Y_1 \in \{H(\infty,1), H^1, D^1_0, HL(1)\}$. For each radial weigth ν such that $Q : L^1_{[0,1)} \to Y_1$ is bounded, where $Q \in \{H_\nu, \widetilde{H_\nu}\}$, there is a radial weight ω such that*

$$\widehat{\omega}(t) \asymp \widehat{\nu}(t), \quad t \in [0,1),$$

$\omega \in L^\infty_{[0,r_0]}$ for any $r_0 \in (0,1)$, $T : X_1 \to Y_1$ is bounded and $T : L^1_{[0,1)} \to Y_1$ is not bounded. Here $T \in \{H_\omega, \widetilde{H_\omega}\}$.

Proof Since $Q : L^1_{[0,1)} \to Y_1$ is bounded, by Theorem 2, $\nu \in \mathcal{D}$ and $M_{1,c}(\nu) < \infty$. Now, by Lemma 18 and its proof, there is a radial weight ω such that $\widehat{\omega}(t) \asymp \widehat{\nu}(t)$, $\omega \in L^\infty_{[0,r_0]}$ for any $r_0 \in (0,1)$ and $\omega \notin L^\infty_{[0,1)}$. So, $m_1(\omega) = \infty$ and by Theorem 26, $T : L^1_{[0,1)} \to Y_1$ is not bounded. Moreover, $\omega \in \mathcal{D}$ and $M_{1,c}(\omega) < \infty$ because ν satisfies both properties, consequently $T : X_1 \to Y_1$ is bounded. □

4.2 Compactness of Hilbert-type operators on X_1-spaces

Lemma 29 *Let $\omega \in \mathcal{D}$ such that $M_{1,d}(\omega) < \infty$. Let $\{f_k\}_{k=0}^\infty \subset X_1 \in \{H(\infty,1), D^1_0, H^1, HL(1)\}$ such that $\sup_{k\in\mathbb{N}} \|f_k\|_{X_1} < \infty$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. Then the following statements hold:*

(i) *$\int_0^1 |f_k(t)|\omega(t)dt \to 0$ when $k \to \infty$.*
(ii) *If $T \in \{H_\omega, \widetilde{H_\omega}\}$, then $T(f_k) \to 0$ uniformly on compact subsets of $\mathbb{D}$.*

Proof Firstly, let us prove that

$$\lim_{a\to 1^-}\frac{\widehat{\omega}(a)}{1-a}=0. \tag{4.7}$$

Since $M_{1,d}(\omega)<\infty$, then

$$\begin{aligned}\int_0^{\frac{1+a}{2}}\frac{ds}{\widehat{\omega}(s)} &\ge \left(M_{1,d}(\omega)\right)^{-1}\int_0^{\frac{1+a}{2}}\left(\int_0^s\frac{dt}{\widehat{\omega}(t)}\right)\frac{ds}{1-s}\\ &\ge \left(M_{1,d}(\omega)\right)^{-1}\int_{\frac12}^{\frac{1+a}{2}}\left(\int_0^s\frac{dt}{\widehat{\omega}(t)}\right)\frac{ds}{1-s}\\ &\ge \left(M_{1,d}(\omega)\right)^{-1}\left(\int_0^{\frac12}\frac{dt}{\widehat{\omega}(t)}\right)\int_{\frac12}^{\frac{1+a}{2}}\frac{ds}{1-s}\\ &= \left(M_{1,d}(\omega)\right)^{-1}\left(\int_0^{\frac12}\frac{dt}{\widehat{\omega}(t)}\right)\log\frac{1}{1-a},\quad 0<a<1.\end{aligned}$$

So $\lim_{a\to 1^-}\int_0^a\frac{ds}{\widehat{\omega}(s)}=\infty$, and then using again the condition $M_{1,d}(\omega)<\infty$, (4.7) holds.

From now on, the proof follow the lines of Lemma 20. Let $\varepsilon>0$. By (4.7) there exists $0<\rho_0<1$ such that $\widetilde{\omega}(t)<\varepsilon$ for any $t\in[\rho_0,1)$. Moreover, there exists k_0 such that for every $k\ge k_0$ and $z\in M=\overline{D(0,\rho_0)}$, $|f_k(z)|<\varepsilon$. Then, by Lemma 14 and (3.11)

$$\begin{aligned}\int_0^1|f_k(t)|\omega(t)dt &\le |f_k(0)|\widehat{\omega}(0)+\int_0^1 M_\infty(t,f_k)\widetilde{\omega}(t)dt\\ &\lesssim \int_0^{\rho_0}M_\infty(t,f_k)\widetilde{\omega}(t)dt+\int_{\rho_0}^1 M_\infty(t,f_k)\widetilde{\omega}(t)dt\\ &\le \varepsilon\left(\int_0^1\widetilde{\omega}(t)dt+\sup_{k\in\mathbb{N}}\|f_k\|_{X_1}\right)=C\varepsilon,\end{aligned}$$

where in the second to last step we have used Lemma 8.

The proof of (ii) is analogous to that of Lemma 20 so we omit its proof. □

Using the previous lemma and Theorem 2 we obtain the following by mimicking the proof of Theorem 21.

Theorem 30 *Let ω be a radial weight and $X_1, Y_1\in\{H(\infty,1), D_0^1, H^1, HL(1)\}$ and let $T\in\{H_\omega,\widetilde{H_\omega}\}$. Then, the following assertions are equivalent:*

(i) *$T: X_1\to Y_1$ is compact;*

(ii) *For every sequence $\{f_k\}_{k=0}^\infty\subset X_1$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{X_1}<\infty$ and $f_k\to 0$ uniformly on compact subsets of $\mathbb{D}$, $\lim_{k\to\infty}\|T(f_k)\|_{Y_1}=0$.*

Theorem 31 *Let ω be a radial weight and $X_1, Y_1\in\{H(\infty,1), D_0^1, H^1, HL(1)\}$, and let $T\in\{H_\omega,\widetilde{H_\omega}\}$. Then, $T: X_1\to Y_1$ is not compact.*

Proof The proof is analogous to that of Theorem 22, so we provide a sketch of the proof. Assume that $T: X_1\to Y_1$ is compact. For each $0<a<1$, set $f_a(z)=\frac{1-a^2}{(1-az)^2}$, $z\in\mathbb{D}$. A calculation shows that $\sup_{a\in(0,1)}\|f_a\|_{X_1}\asymp 1$ and $f_a\to 0$, as $a\to 1$ uniformly on compact subsets of $\mathbb{D}$. Moreover, since $T(f_a)$ has non-negative Taylor coefficients,

$$\|T(f_a)\|_{Y_1}\gtrsim\|H_\omega(f_a)\|_{H(\infty,1)}\asymp\|H_\omega(f_a)\|_{HL(1)}\gtrsim 1,\quad a\in(0,1),$$

where the last inequality follows taking $p = 1$ in (3.26). So, using Theorem 30 we deduce that $T : X_1 \to Y_1$ is not a compact operator. □

5 Hilbert-type operators acting on H^∞

We will prove a result which includes Theorem 4. With this aim we need some more notation. The space Q_p, $0 \le p < \infty$, consists of those $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{Q_p} = |f(0)|^2 + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^p \, dA(z) < \infty,$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, $z, a \in \mathbb{D}$. If $p > 1$, Q_p coincides with the Bloch space $\mathcal{B}$. The space Q_1 coincides with $BMOA$ (see, e. g., [10, Theorem 5.2]). However, if $0 < p < 1$, Q_p is a proper subspace of $BMOA$ [30]. The space Q_0 reduces to the classical Dirichlet space $\mathcal{D}$.

We recall that

$$\begin{aligned} &Q_p \subsetneq \text{BMOA} \subsetneq \mathcal{B}, \quad 0 < p < 1. \\ &H^\infty \subsetneq \text{BMOA} \subsetneq \mathcal{B}, \end{aligned} \tag{5.1}$$

however if $0 < p < 1$, $H^\infty \not\subset Q_p$, and $Q_p \not\subset H^\infty$, see [30].

Moreover, $HL(\infty) \subsetneq Q_p$. This embedding might have been proved in some previous paper, however we include a short direct proof for the sake of completeness.

Lemma 32 *Let $0 < p \le \infty$, then $HL(\infty) \subsetneq Q_p$ and*

$$\|f\|_{Q_p} \lesssim \|f\|_{HL(\infty)}, \quad f \in \mathcal{H}(\mathbb{D}).$$

Proof Let $f \in HL(\infty)$, then

$$M_2^2(\rho, f') = \sum_{n=1}^{\infty} n^2 |\widehat{f}(n)|^2 \rho^{2n-2} \le \|f\|^2_{HL(\infty)} \sum_{n=1}^{\infty} \rho^{2n-2} = \frac{\|f\|^2_{HL(\infty)}}{1-\rho^2},$$

So for any $0 < p < \infty$,

$$\begin{aligned} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (1 - |\varphi_a(z)|^2)^p |f'(z)|^2 \, dA(z) &\lesssim \sup_{a\in\mathbb{D}} \int_0^1 \left(\frac{(1-|a|)(1-s^2)}{(1-|a|s)^2} \right)^p M_2^2(s, f') \, ds \\ &\le \|f\|^2_{HL(\infty)} \sup_{a\in\mathbb{D}} \int_0^1 \frac{(1-|a|)^p (1-s^2)^{p-1}}{(1-|a|s)^{2p}} \, ds \\ &\lesssim \|f\|^2_{HL(\infty)}, \end{aligned}$$

so $\|f\|_{Q_p} \lesssim \|f\|_{HL(\infty)}$. The lacunary series $f(z) = \sum_{k=0}^{\infty} 2^{-k} \log(k+2) z^{2^k} \in \bigcap_{0<p} Q_p \setminus HL(\infty)$. This finishes the proof. □

Now we will prove the main result of this section, which is an extension of Theorem 4.

Theorem 33 *Let ω be a radial weight and let $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then, the following statements are equivalent:*

(i) $T : H^\infty \to HL(\infty)$ *is bounded;*
(ii) $T : H^\infty \to Q_p$ *is bounded for* $0 < p < 1$*;*
(iii) $T : H^\infty \to \mathrm{BMOA}$ *is bounded;*
(iv) $T : H^\infty \to \mathcal{B}$ *is bounded;*
(v) $\omega \in \widehat{\mathcal{D}}$*.*

Proof of Theorem 33 By Lemma 6,

$$\begin{aligned}\|H_\omega(f)\|_{HL(\infty)} &= \sup_{k\in\mathbb{N}\cup\{0\}} (k+1)\left|\frac{\int_0^1 f(t)t^k\omega(t)dt}{2(k+1)\omega_{2k+1}}\right| \\ &\le \sup_{k\in\mathbb{N}\cup\{0\}} (k+1)\frac{\int_0^1 |f(t)|t^k\omega(t)dt}{2(k+1)\omega_{2k+1}} \\ &= \|\widetilde{H_\omega}(f)\|_{HL(\infty)} \lesssim \|f\|_{H^\infty},\end{aligned}$$

so (v)$\Rightarrow$ (i). The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) follow from (5.1) and Lemma 32.

The implication (iv)$\Rightarrow$(v) was proved in [26, Theorem 1], and this finishes the proof. □

It is worth mentioning that for $f(z) = \log\frac{1}{1-z} \in HL(\infty)$ and ω a radial weight,

$$\begin{aligned}H_\omega(f)'(x) &= \sum_{n=1}^\infty \frac{n}{2(n+1)\omega_{2n+1}}\left(\sum_{k=1}^\infty \frac{\omega_{n+k}}{k}\right)x^{n-1} \\ &\ge \sum_{n=1}^\infty \frac{n}{2(n+1)}\left(\sum_{k=1}^n \frac{1}{k}\right)x^{n-1} \\ &\gtrsim \sum_{n=1}^\infty \log(n+1)x^{n-1},\end{aligned}$$

so $H_\omega(f) \notin \mathcal{B}$. So, the space H^∞ cannot be replaced by $HL(\infty)$ and by any Q_p space, $0 < p < \infty$, in the statement of Theorem 33. That is, the remaining cases for $p = \infty$, analogous to those of Theorems 1 and 2, which do not appear in Theorem 33, simply do not hold for any radial weight.

Finally, we will prove the analogous result to Theorem 22 for $p = \infty$.

Theorem 34 *Let* ω *be a radial weight and let* $T \in \{H_\omega, \widetilde{H_\omega}\}$*. Then* $T : H^\infty \to Y_\infty$ *is not a compact operator, where* $Y_\infty \in \{Q_p, \mathcal{B}, \mathrm{BMOA}, HL(\infty)\}$*,* $0 < p < 1$.

We need the following result, whose proof can be obtained by mimicking the proof Theorem 21.

Theorem 35 *Let* ω *be a radial weight and* $T \in \{H_\omega, \widetilde{H_\omega}\}$*. Then, the following assertions are equivalent:*

(i) $T : H^\infty \to \mathcal{B}$ *is compact;*
(ii) *For every sequence* $\{f_k\}_{k=0}^\infty \subset H^\infty$ *such that* $\sup_{k\in\mathbb{N}} \|f_k\|_{H^\infty} < \infty$ *and* $f_k \to 0$ *uniformly on compact subsets of* $\mathbb{D}$*,* $\lim_{k\to\infty} \|T(f_k)\|_{\mathcal{B}} = 0$.

Proof of Theorem 34 By (5.1) and Lemma 32, it is enough to prove that $T : H^\infty \to \mathcal{B}$ is not compact. Let consider for every $k \in \mathbb{N}$ the function $f_k(z) = z^k$, $z \in \mathbb{D}$. It is clear that $\|f_k\|_{H^\infty} = 1$ for every $k \in \mathbb{N}$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since

$$T(f_k)(z) = \sum_{n=0}^{\infty} \frac{\int_0^1 f_k(t)t^n\omega(t)\,dt}{2(n+1)\omega_{2n+1}} z^n = \sum_{n=0}^{\infty} \frac{\omega_{n+k}}{2(n+1)\omega_{2n+1}} z^n,$$

for any $k \geq 2$

$$\begin{aligned}
\|T(f_k)\|_{\mathcal{B}} &\geq \sup_{x\in(0,1)} (1-x) \sum_{n=1}^{\infty} \frac{n\omega_{n+k}}{2(n+1)\omega_{2n+1}} x^{n-1} \geq \frac{1}{4} \sup_{x\in(0,1)} (1-x) \sum_{n=1}^{\infty} \frac{\omega_{n+k}}{\omega_{2n}} x^{n-1} \\
&\geq \frac{1}{4} \sup_{x\in(0,1)} (1-x) \sum_{n=k}^{2k} \frac{\omega_{n+k}}{\omega_{2n}} x^{n-1} \geq \frac{1}{4} \sup_{x\in(0,1)} (1-x) \sum_{n=k}^{2k} x^{n-1} \\
&\geq \frac{1}{4} \sup_{x\in(0,1)} (1-x)kx^{2k-1} \geq \frac{1}{8}\left(1-\frac{1}{2k}\right)^{2k-1} \geq \frac{1}{8} \inf_{m\geq 2}\left(1-\frac{1}{m}\right)^m \geq C > 0
\end{aligned}$$

so $\lim_{k\to\infty} \|T(f_k)\|_{\mathcal{B}} \neq 0$ and hence, by Theorem 35, $T : H^\infty \to \mathcal{B}$ is not compact. □

Before ending this section, we briefly compare the action of the Hilbert-type operator H_ω and the Bergman projection

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B_z^\omega(\zeta)}\,\omega(\zeta)dA(\zeta),$$

induced by a radial weight ω. As a consequence of Theorem 33 and [23, Theorem 1], the condition $\omega \in \widehat{\mathcal{D}}$ characterizes the boundedness of the operators $H_\omega : H^\infty \to \mathcal{B}$ and $P_\omega : L^\infty \to \mathcal{B}$. Moreover, $P_\omega : L^\infty \to \mathcal{B}$ is bounded and onto if and only if $\omega \in \mathcal{D}$ [23, Theorem 3]. So, it is natural to think about the radial weights such that the operator $H_\omega : H^\infty \to \mathcal{B}$ is bounded and onto. A straightforward argument proves there is no radial weights such that $H_\omega : H^\infty \to \mathcal{B}$ satisfies both properties: If $H_\omega : H^\infty \to \mathcal{B}$ is bounded, Theorem 33 yields that $H_\omega : H^\infty \to \mathrm{BMOA}$ is also bounded, so if $g \in \mathcal{B} \setminus \mathrm{BMOA}$, e.g. $g(z) = \sum_{k=0}^{\infty} z^{2^k}$, there does not exist $f \in H^\infty$ such that $H_\omega(f) = g$. Consequently, $H_\omega : H^\infty \to \mathcal{B}$ is not surjective.

6 Comparisons and reformulations of the $M_{p,c}$-conditions

In order to prove Theorem 5 we will study the relationship between some of the conditions which describe the boundedness of the Hilbert-type operators H_ω and $\widetilde{H_\omega}$ from X_p to Y_p, and X_q to Y_q.

Theorem 5 Firstly, assume $1 < q < p < \infty$. Since $T : X_q \to Y_q$ is bounded, Theorem 1 yields $\omega \in \widehat{\mathcal{D}}$ and $M_{q,c}(\omega) < \infty$, and as a consequence, $K_{q,c}(\omega) < \infty$. By using Lemma 6,

$$1 + \int_0^r \frac{ds}{\widehat{\omega}(s)^q} \gtrsim \frac{1-r}{\widehat{\omega}(r)^q}, \quad 0 \leq r < 1. \tag{6.1}$$

On the other hand, Hölder's inequality with exponents $x = \frac{q'}{p'} > 1$ and $x' = \frac{x}{x-1}$, implies

$$\left(\int_r^1 \left(\frac{\widehat{\omega}(s)}{1-s}\right)^{p'} ds\right)^{\frac{1}{p'}} \le \left(\int_r^1 \left(\frac{\widehat{\omega}(s)}{1-s}\right)^{q'} ds\right)^{\frac{1}{q'}} (1-r)^{\frac{1}{p'}-\frac{1}{q'}}, \quad 0 \le r < 1. \tag{6.2}$$

Moreover,

$$\left(1 + \int_0^r \frac{1}{\widehat{\omega}(s)^p} ds\right)^{\frac{1}{p}} \lesssim \widehat{\omega}(r)^{\frac{q}{p}-1} \left(1 + \int_0^r \frac{1}{\widehat{\omega}(s)^q} ds\right)^{\frac{1}{p}}, \quad 0 \le r < 1. \tag{6.3}$$

So, the identity $\frac{1}{p'} - \frac{1}{q'} = \frac{1}{q} - \frac{1}{p}$, together with (6.1), (6.2) and (6.3) yield

$$\begin{aligned}
&\left(1 + \int_0^r \frac{1}{\widehat{\omega}(t)^p} dt\right)^{\frac{1}{p}} \left(\int_r^1 \left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'} dt\right)^{\frac{1}{p'}} \\
&\quad \lesssim \left(\frac{1-r}{\widehat{\omega}(r)^q}\right)^{\frac{1}{q}-\frac{1}{p}} \left(1 + \int_0^r \frac{1}{\widehat{\omega}(s)^q} ds\right)^{\frac{1}{p}} \left(\int_r^1 \left(\frac{\widehat{\omega}(s)}{1-s}\right)^{q'} ds\right)^{\frac{1}{q'}} \\
&\quad \lesssim \left(1 + \int_0^r \frac{1}{\widehat{\omega}(s)^q} ds\right)^{\frac{1}{q}} \left(\int_r^1 \left(\frac{\widehat{\omega}(s)}{1-s}\right)^{q'} ds\right)^{\frac{1}{q'}}.
\end{aligned}$$

Consequently $K_{p,c}(\omega) < \infty$, and by Theorem 1, $T : X_p \to Y_p$ is bounded.

Assume that $q = 1$, that is, $T : X_1 \to Y_1$ is bounded. By Theorem 2, $\omega \in \widehat{\mathcal{D}}$ and $M_{1,d}(\omega) < \infty$, so $K_{1,d}(\omega) < \infty$. Then,

$$\begin{aligned}
\int_r^1 \left(\frac{\widehat{\omega}(s)}{1-s}\right)^{p'} ds &\le K_{1,d}^{p'}(\omega) \int_r^1 \left(1 + \int_0^s \frac{1}{\widehat{\omega}(t)} dt\right)^{-p'} ds \\
&\le K_{1,d}^{p'}(\omega)(1-r) \left(1 + \int_0^r \frac{1}{\widehat{\omega}(t)} dt\right)^{-p'} \\
&\lesssim K_{1,d}^{p'}(\omega) \frac{\widehat{\omega}(r)^{p'}}{(1-r)^{p'-1}}, \quad 0 \le r < 1,
\end{aligned}$$

where in the last inequality we have used (6.1) with $q = 1$. Moreover,

$$\begin{aligned}
\left(1 + \int_0^r \frac{1}{\widehat{\omega}(s)^p} ds\right)^{\frac{1}{p}} &\lesssim \frac{1}{\widehat{\omega}(r)^{1-\frac{1}{p}}} \left(1 + \int_0^r \frac{1}{\widehat{\omega}(s)} ds\right)^{\frac{1}{p}} \\
&\le \frac{(1-r)^{\frac{1}{p}}}{\widehat{\omega}(r)} K_{1,d}(\omega)^{1/p}, \quad 0 \le r < 1.
\end{aligned}$$

So, $M_{p,c}(\omega) < \infty$, and by Theorem 1, $T : X_p \to Y_p$ is bounded. This finishes the proof.

Finally, we present two more conditions which characterize the radial weights ω such that $T : X_p \to Y_p$, $1 < p < \infty$, is bounded, where $X_p, Y_p \in \{H(\infty, p), H^p, D^p_{p-1}, HL(p)\}$ and $T \in \{H_\omega, \widetilde{H_\omega}\}$.

Proposition 36 *Let ω be a radial weight and $1 < p < \infty$. Then, the following conditions are equivalent:*

(i) $\omega\in\widehat{\mathcal{D}}$ *and* $K_{p,d}(\omega)=\sup_{0<r<1}\frac{\widehat{\omega}(r)}{(1-r)^{\frac{1}{p}}}\left(1+\int_0^r\frac{1}{\widehat{\omega}(s)^p}\,ds\right)^{\frac{1}{p}}<\infty$;

(ii) $\omega\in\widehat{\mathcal{D}}$ *and* $K_{p,e}(\omega)=\sup_{0<r<1}\frac{(1-r)^{\frac{1}{p}}}{\widehat{\omega}(r)}\left(\int_r^1\left(\frac{\widehat{\omega}(s)}{1-s}\right)^{p'}\,ds\right)^{\frac{1}{p'}}<\infty$;

(iii) $\omega\in\widehat{\mathcal{D}}$ *and* $K_{p,c}(\omega)=\sup_{0<r<1}\left(1+\int_0^r\frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}}\left(\int_r^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}\,dt\right)^{\frac{1}{p'}}<\infty.$

Proof Assume that (i) holds. A calculation shows that $F(r)=(1-r)^{\kappa}\left(1+\int_0^r\frac{ds}{\widehat{\omega}(s)^p}\right)$, with $\kappa=\frac{1}{K_{p,d}^p(\omega)}$, is non-decreasing in [0, 1). So,

$$
\begin{aligned}
\int_r^1\left(\frac{\widehat{\omega}(s)}{1-s}\right)^{p'}\,ds &\le K_{p,d}^{p'}(\omega)\int_r^1\frac{1}{1-s}\left(1+\int_0^s\frac{1}{\widehat{\omega}(t)^p}\,dt\right)^{-\frac{p'}{p}}\,ds\\
&\le K_{p,d}^{p'}(\omega)F(r)^{-\frac{p'}{p}}\int_r^1(1-s)^{\frac{\kappa p'}{p}-1}ds\\
&\lesssim K_{p,d}^{p'+p}(\omega)\left(1+\int_0^r\frac{1}{\widehat{\omega}(t)^p}\,dt\right)^{-\frac{p'}{p}}\\
&\lesssim K_{p,d}^{p'+p}(\omega)\frac{\widehat{\omega}(r)^{p'}}{(1-r)^{p'-1}},\quad 0\le r<1,
\end{aligned}
$$

where in the last inequality we have used (6.1). That is (ii) holds.

Now, assume that (ii) holds. Since $\omega\in\widehat{\mathcal{D}}$

$$
\int_r^1\left(\frac{\widehat{\omega}(s)}{1-s}\right)^{p'}\,ds\gtrsim\frac{\widehat{\omega}(r)^{p'}}{(1-r)^{p'-1}},\quad 0\le r<1.
$$

Moreover, $H(r)=(1-r)^{-\eta}\int_r^1\left(\frac{\widehat{\omega}(s)}{1-s}\right)^{p'}\,ds$, with $\eta=\frac{1}{K_{p,e}^{p'}(\omega)}$, is non-increasing in [0, 1). So,

$$
\begin{aligned}
1+\int_0^r\frac{ds}{\widehat{\omega}(s)^p} &\le 1+K_{p,e}^p(\omega)\int_0^r\frac{1}{1-s}\left(\int_s^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}\,dt\right)^{-\frac{p}{p'}}\,ds\\
&\le 1+K_{p,e}^p(\omega)H(r)^{-\frac{p}{p'}}\int_0^r\frac{1}{(1-s)^{1+\eta\frac{p}{p'}}}ds\\
&\lesssim K_{p,e}^{p+p'}(\omega)\left(\int_r^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}\,dt\right)^{-\frac{p}{p'}}\\
&\lesssim K_{p,e}^{p+p'}(\omega)\frac{1-r}{\widehat{\omega}(r)^p},\quad 0\le r<1,
\end{aligned}
$$

therefore (i) holds.

Next, if (i) holds, then (ii) holds and so it is clear that (iii) holds. Finally, (iii) together with (6.1) implies (ii). This finishes the proof. □

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