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ORIGINAL PAPER

Statistical structures arising in null submanifolds

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Abstract
We show a link between affine differential geometry and null submanifolds in a semi-Riemannian manifold via statistical structures. Once a rigging for a null submanifold is fixed, we can construct a semi-Riemannian metric on it. This metric and the induced connection constitute a statistical structure on the null submanifold in some cases. We study the statistical structures arising in this way. We also construct statistical structures on a null hypersurface in the Lorentz–Minkowski space using the null second fundamental form. This extends the classical construction to the null case.

Keywords Null submanifolds · Statistical structures · Dual connections · Rigging technique · Blaschke metric

Mathematics Subject Classification 53B05 · 53B12 · 53B3

1 Introduction

Affine differential geometry deals with the study of an affine connection on a manifold, which a priori is not the Levi–Civita connection of a metric. On the other hand, a submanifold in a semi-Riemannian manifold is called null if the induced metric tensor is degenerate at every point. Thus, it does not inherit an useful metric, but it inherits an affine connection once some arbitrary choices are made. This suggests that we should study null submanifolds using the affine geometry techniques. However, affine differential geometry and null submanifolds have been developed independently and there is not much interplay between them, despite their similarities. One of the reasons why affine geometry has not been applying systematically

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to null submanifolds is that the second fundamental form of a null submanifold is always degenerate. This impedes to define a fundamental concept in affine differential geometry as the Blaschke normal to the submanifold.

A statistical structure is another important concept in affine differential geometry. It is basically formed by an affine symmetric connection $\bar{\nabla}^s$, called statistical connection, and a semi-Riemannian metric g such that $\bar{\nabla}^s g$ is totally symmetric. This name is due to the fact that they appear naturally in probability and statistical inference. Indeed, if we have a parametrized probability density, then we can construct, under suitable conditions, a Riemannian metric called the Fisher metric and an affine connection which constitute a statistical structure. This provides a surprising link between probability and Riemannian geometry, [3]. In the context of affine geometry, the statistical structures have often been called Codazzi structures.

In this paper we relate the affine differential geometry theory and the null submanifolds theory. To handle a null submanifold we need to fix a rigging, which is a vector field transverse to the orthogonal complement to the radical of the submanifold. We show that if there is a rigging for the null submanifold with certain properties, then it gives rise to a statistical structure on the null submanifold.

Due to the great variety of terminology and ambiguity in affine differential geometry and null submanifolds, we have included preliminaries sections on both topics for the sake of readability of this paper. In Sect. 2 we introduce basic facts about null submanifolds and the rigging technique used to handle them. In particular, we show how to induce a semi-Riemannian metric on a null submanifold, which is called rigged metric. In Sect. 3 we do the same with affine geometry and statistical structures. We fix the terminology, review some basic facts and study some special statistical structures that will appear throughout this paper. These special types of statistical structures are constructed from one or two vector fields. This is why we call them ξ-statistical structure or (ξ, E)-statistical structure, being ξ and E vector fields. Conditions to being conjugate symmetric, statistical curvature symmetric or geodesically complete are given in the last part of Sect. 3.

If we have a semi-Riemannian manifold furnished with a statistical structure, then the statistical structure is inherited in a nice way by their nondegenerate submanifolds. In the case of a null submanifold the statistical structure is not inherited due to the degeneracy of the induced metric tensor. Nevertheless, under certain conditions, the induced connection from the ambient statistical connection and the rigged metric constitute a statistical structure on the null submanifold, Theorem 2. In Sect. 4 we study the constructed statistical structures on a null submanifold by this way and we show that ξ-statistical and (ξ, E)-statistical structures appear as fundamental pieces of them.

In Sect. 5 we construct statistical structures on some kind of null hypersurfaces in the Lorentz–Minkowski space using their second fundamental form. This extends the classical construction for hypersurfaces in the Euclidean space. By definition, a strongly convex hypersurface is a hypersurface in a Riemannian manifold such that its second fundamental form induces a Riemannian metric on it, which is called the Blaschke metric. In the case of a strongly convex hypersurface in the Euclidean space, the Codazzi equation implies that the induced connection and the Blaschke metric constitute a statistical structure on the hypersurface. This kind of statistical structure has been widely studied and classified. The classical Maschke–Pick–Berwald theorem asserts that if it is trivial, which means that the statistical connection coincides with the Levi–Civita connection, then the hypersurface is a hyperquadric in $\mathbb{R}^n$, [18].

If we consider a null hypersurface in the Lorentz–Minkowski space, then we can not expect to do a similar construction because its second fundamental form is always degenerate. In fact, it is ruled by null straight lines. However, we can construct a Riemannian metric on the null hypersurface from the second fundamental form in those that are screen strongly convex (Definition 6). We call this Riemannian metric rigged-Blaschke metric. For a suitable choice of the rigging for a screen strongly convex null hypersurface, the induced connection and the rigged-Blaschke metric constitute a statistical structure which is never trivial. We show that it is conjugate symmetric if and only if the null hypersurface is contained in a null cone and we also give some examples of this construction in the case of a null cone.

2 Preliminaries on null submanifolds

We review some facts about null submanifolds in a semi-Riemannian manifold. The main references for this section are [4, 6, 8, 16].

Definition 1 A submanifold Σ of a semi-Riemannian manifold (M, g) is called null if $\dim\ T_x\Sigma \cap T_x\Sigma^{\perp} \neq 0$ for all $x \in \Sigma$. Given $r \in \mathbb{N}$, a null submanifold is called r-lightlike if $\dim\ T_x\Sigma \cap T_x\Sigma^{\perp} = r$ for all $x \in \Sigma$.

It is straightforward to check that any null submanifold in a Lorentzian manifold and any null hypersurface in a semi-Riemannian manifold are necessarily 1-lightlike.

Definition 2 Let Σ be a 1-lightlike submanifold in a semi-Riemannian manifold (M, g). A rigging for Σ is a vector field ζ defined in some open set containing Σ such that $\zeta_x \notin \left(T_x\Sigma \cap T_x\Sigma^{\perp}\right)^{\perp}$ for all $x \in \Sigma$.

The above is equivalent to $\zeta_x \notin T_x\Sigma + T_x\Sigma^{\perp}$ for all $x \in \Sigma$. If Σ is a null hypersurface, then $T_x\Sigma^{\perp} \subset T_x\Sigma$ and a rigging is just a vector field transverse to the hypersurface.

From now on, we focus on 1-lightlike submanifolds in semi-Riemannian manifolds and we call them simply null submanifolds. Moreover, we suppose that it always exists a rigging ζ for them.

The screen distribution and the transversal screen distribution are defined by

$$\mathcal{S} = T\Sigma \cap \zeta^{\perp},$$
$$\mathcal{T} = T\Sigma^{\perp} \cap \zeta^{\perp}$$

respectively, which are nondegenerate distributions. The rigged vector field is the unique null vector field $\xi \in \mathfrak{X}(\Sigma)$ such that $g(\zeta, \xi) = 1$ and the transversal null vector field is defined as

$$N = \zeta - \frac{1}{2} g(\zeta, \zeta)\xi, \tag{1}$$

which holds $g(N, N) = 0$ and $g(N, \xi) = 1$. The rigged one-form ω is the one-form on Σ given by $\omega(U) = g(U, N) = g(U, \zeta)$ for all $U \in \mathfrak{X}(\Sigma)$. We have the following decompositions.

$$TM|_{\Sigma} = T\Sigma \oplus \mathcal{T} \oplus span(N), \tag{2}$$
$$T\Sigma = \mathcal{S} \oplus_{orth} span(\xi),$$
$$T\Sigma^{\perp} = \mathcal{T} \oplus_{orth} span(\xi),$$
$$TM|_{\Sigma} = \mathcal{S} \oplus_{orth} \mathcal{T} \oplus_{orth} span(\xi, N). \tag{3}$$

Given $U \in \mathfrak{X}(\Sigma)$, we call $\mathcal{P}_{\mathcal{S}}(U)$ the canonical projection onto $\mathcal{S}$ according to decomposition (3).

We denote the Levi–Civita connection of g by $\overline{\nabla}$. It can be projected onto Σ using the decomposition (2) in an analogous way as for a nondegenerate submanifold. In fact, given $U, V \in \mathfrak{X}(\Sigma)$ and $X \in \Gamma(\mathcal{S})$ we can write

$$\overline{\nabla}_U V = \nabla_U V + \mathfrak{h}(U, V) + B(U, V)N, \tag{4}$$

$$\overline{\nabla}_U N = -A(U) + \mathfrak{d}(U) + \tau(U)N, \tag{5}$$

where $\nabla_U V, A(U) \in \Gamma(T\Sigma)$ and $\mathfrak{h}(U, V), \mathfrak{d}(U, V) \in \Gamma(\mathcal{T})$. In fact, we have $A(U) \in \Gamma(\mathcal{S})$ since $g(N, N) = 0$. It is easy to check that ∇ is a connection on Σ without torsion, which is called the induced connection. The tensors B and $\mathfrak{h}$ are symmetric and they are called null second fundamental form and screen null second fundamental form respectively.

Moreover, using the decomposition (3) we have

$$\begin{aligned} \nabla_U \xi &= -A^*(U) - \tau(U)\xi, \\ \nabla_U X &= \nabla^*_U X + C(U, X)\xi, \end{aligned} \tag{6}$$

where $A^*(U), \nabla^*_U X \in \Gamma(\mathcal{S})$. The tensor C is symmetric for all $X, Y \in \Gamma(\mathcal{S})$ if and only if $\mathcal{S}$ is integrable. In this case, the induced Levi–Civita connection on the leaves of $\mathcal{S}$ from the ambient is just ∇^*.

The tensors $B, C, \mathfrak{h}, \mathfrak{d}$ and τ are the fundamental tensors of the null submanifold and they play the role of the second fundamental form in the case of a nondegenerate submanifold. Obviously, these tensors and the induced connection ∇ depend on the chosen rigging.

The following relations hold for all $U, V, W \in \mathfrak{X}(\Sigma)$ and $X \in \Gamma(\mathcal{S})$.

$$\begin{aligned} B(U, V) &= g(A^*(U), V) = -g(\overline{\nabla}_U \xi, V), \\ B(U, \xi) &= 0,\ A^*(\xi) = 0, \\ B(A^*(U), V) &= B(U, A^*(V)), \end{aligned} \tag{7}$$

$$\tau(U) = g(\overline{\nabla}_U N, \xi) = g(\overline{\nabla}_U \zeta, \xi), \tag{8}$$

$$\begin{aligned} C(U, X) &= -g(\overline{\nabla}_U N, X) = g(A(U), X), \\ (\nabla_U g)(V, W) &= B(U, V)\omega(W) + B(U, W)\omega(V), \end{aligned} \tag{9}$$

$$\begin{aligned} -2C(U, X) &= d\omega(U, X) + \left(L_\zeta g\right)(U, X) + g(\zeta, \zeta)B(U, X) \\ &= 2g(\overline{\nabla}_U \zeta, X) + g(\zeta, \zeta)B(U, X), \end{aligned} \tag{10}$$

where L_ζ denotes the Lie derivative along ζ. The eigenvalues of $A^* : T\Sigma \to T\Sigma$ are called principal curvatures of Σ. Observe that A^* is self-adjoint and 0 is always a principal curvature with multiplicity at least 1.

If Σ is a null hypersurface, then $\mathcal{T} = 0$ and thus $\mathfrak{h} = \mathfrak{d} = 0$. In this case, if we call $\bar{R}$ the curvature tensor of the Levi–Civita connection $\overline{\nabla}$ and R the curvature tensor of the induced connection ∇ on Σ, then we have the following Gauss-Codazzi equations.

$$\begin{aligned} g(\bar{R}_{UV}W, \xi) = {} & (\nabla_U B)(V, W) - (\nabla_V B)(U, W) \\ & + \tau(U)B(V, W) - \tau(V)B(U, W), \end{aligned} \tag{11}$$

$$\begin{aligned} g(\bar{R}_{UV}W, X) = {} & g(R_{UV}W, X) \\ & + B(U, W)C(V, X) - B(V, W)C(U, X), \end{aligned} \tag{12}$$

$$g(\bar{R}_{UV}W, N) = g(R_{UV}W, N), \tag{13}$$

$$\begin{aligned} g(\bar{R}_{UV}X, N) = {} & \left(\nabla^{**}_U C\right)(V, X) - \left(\nabla^{**}_V C\right)(U, X) \\ & + \tau(V)C(U, X) - \tau(U)C(V, X), \end{aligned} \tag{14}$$

$$g(\bar{R}_{UV}\xi, N) = C(V, A^*(U)) - C(U, A^*(V)) - d\tau(U, V) \tag{15}$$

for all $U, V, W \in \mathfrak{X}(\Sigma)$ and $X \in \Gamma(\mathcal{S})$. Here $\left(\nabla^{**}_U C\right)(V, X)$ means

$$\left(\nabla^{**}_U C\right)(V, X) = U(C(V, X)) - C(\nabla_U V, X) - C(V, \nabla^*_U X).$$

The induced connection ∇ is not the unique connection that appears naturally on a null submanifold once a rigging is chosen. Indeed,

$$\widetilde{g} = g + \omega \otimes \omega$$

is a semi-Riemannian metric on Σ called the rigged metric. We have that $\mathcal{S} \perp_{\widetilde{g}} \xi$, $\widetilde{g}(\xi, \xi) = 1$ and ω is the $\widetilde{g}$-metrically equivalent one-form to ξ. The Levi–Civita connection of $\widetilde{g}$ is denoted by $\widetilde{\nabla}$ and it is called the rigged connection. There are important relations between the semi-Riemannian manifold $(\Sigma, \widetilde{g})$ and the null submanifold Σ. For example, we have

$$\left(L_\xi \widetilde{g}\right)(X, Y) = -2B(X, Y) \tag{16}$$

for all $X, Y \in \Gamma(\mathcal{S})$.

The rigged metric is the main ingredient of the so-called rigging technique introduced in [6] (see also [8] for a review of the topic). It has been used as a key tool to prove some results on null hypersurfaces, [1, 9–11, 14, 19] and it has played an important role in some constructions on null hypersurfaces, [2].

3 Statistical connections on a semi-Riemannian manifold

In the first part of this section we state some basic facts about statistical structures. The main references are [13, 18]. We can also find useful short reviews on statistical structures in [20–22]. In the second part, we study some special types of statistical structures that will appear in the next section.

Given (M, g) a semi-Riemannian manifold and $\bar{\nabla}^s$ an affine connection on M, the tensor $\bar{\nabla}^s g$ is called the cubic form, which is symmetric in the last two arguments.

Definition 3 A symmetric connection $\bar{\nabla}^s$ on (M, g) is a statistical connection for g if the cubic form $\bar{\nabla}^s g$ is totally symmetric, i.e.,

$$\left(\bar{\nabla}^s_U g\right)(V, W) = \left(\bar{\nabla}^s_V g\right)(U, W) \tag{17}$$

for all $U, V, W \in \mathfrak{X}(M)$. We also say that $(g, \bar{\nabla}^s)$ is a statistical structure on M.

The dual or conjugate connection to $\bar{\nabla}^s$ is the unique connection $\bar{\nabla}^d$ such that

$$Ug(V, W) = g\left(\bar{\nabla}^s_U V, W\right) + g\left(V, \bar{\nabla}^d_U W\right) \tag{18}$$

for all $U, V, W \in \mathfrak{X}(M)$. If T^s and T^d are the torsions of $\bar{\nabla}^s$ and $\bar{\nabla}^d$ respectively, then a straightforward computation shows

$$g\left(T^d(U, V) - T^s(U, V), W\right) = \left(\bar{\nabla}^s_U g\right)(V, W) - \left(\bar{\nabla}^s_V g\right)(U, W).$$

Thus, $\bar{\nabla}^s$ is a statistical connection if and only if its dual connection $\bar{\nabla}^d$ is also a symmetric connection.

If $(g, \bar{\nabla}^s)$ is a statistical structure on M, then it is easy to check that the Levi–Civita connection $\overline{\nabla}$ of g is given by

$$\overline{\nabla} = \frac{1}{2}\left(\bar{\nabla}^s + \bar{\nabla}^d\right). \tag{19}$$

The difference tensor is defined by

$$\bar{K}(U, V) = \bar{\nabla}^s_U V - \overline{\nabla}_U V = \frac{1}{2}\left(\bar{\nabla}^s_U V - \bar{\nabla}^d_U V\right)$$

for all $U, V \in \mathfrak{X}(M)$, which is a symmetric $(1, 2)$-tensor. Moreover, using Eqs. (18) and (19), it is easy to show that

$$\left(\bar{\nabla}^s_U g\right)(V, W) = -2g\left(\bar{K}(U, V), W\right), \tag{20}$$

so the metrically equivalent $(0, 3)$-tensor to $\bar{K}$ is totally symmetric.

Conversely, if $\bar{K}$ is a $(1, 2)$-tensor whose metrically equivalent $(0, 3)$-tensor is totally symmetric, then $\bar{\nabla}^s = \overline{\nabla} + \bar{K}$ is a statistical connection with dual connection given by $\bar{\nabla}^d = \overline{\nabla} - \bar{K}$. Therefore, fixed a semi-Riemannian manifold, there is a bijective correspondence between statistical connections and totally symmetric $(0, 3)$-tensors. Moreover, if K_1 and K_2 are totally symmetric $(0, 3)$-tensors and $f, h \in C^\infty(M)$, then it has sense to refer to the statistical connection determined by $f K_1 + h K_2$.

The one-form $\overline{\alpha}(U) = \text{trace}\, \bar{K}_U$ is usually called the Tchebychev one-form and its metrically equivalent vector field is the Tchebychev vector field. This form is zero if and only if $\bar{\nabla}^s \Omega = 0$, where Ω is the (locally defined) volume form associated to g. In this case, the statistical structure is called trace-free. More generally, it is called locally equiaffine if around any point there is a locally defined $\bar{\nabla}^s$-parallel volume form. This condition is equivalent to $d\overline{\alpha} = 0$.

The curvature tensors $\bar{R}^s$ and $\bar{R}^d$ of the connections $\bar{\nabla}^s$ and $\bar{\nabla}^d$ respectively are defined in the usual way. They hold the first and second Bianchi identities and they are skew-symmetric in the first two arguments. Moreover, if $\bar{R}$ denotes the curvature tensor of the Levi–Civita connection $\overline{\nabla}$, then we have

$$g(\bar{R}^s_{UV} W, T) = -g(\bar{R}^d_{UV} T, W), \tag{21}$$

$$\bar{R}^s_{UV} W + \bar{R}^d_{UV} W = 2\bar{R}_{UV} W + 2\left[\bar{K}_U, \bar{K}_V\right] W, \tag{22}$$

$$\bar{R}^s_{UV} W - \bar{R}^d_{UV} W = 2\left(\left(\overline{\nabla}_U \bar{K}\right)(V, W) - \left(\overline{\nabla}_V \bar{K}\right)(U, W)\right). \tag{23}$$

If $\bar{R}^s = \bar{R}^d$, then the statistical structure is called conjugate symmetric. From Eq. (23), this is equivalent to $\overline{\nabla}\bar{K}$ being totally symmetric. Moreover, it implies that the Tchebychev one-form is closed and thus $\bar{\nabla}^s$ is locally equiaffine.

The statistical curvature is defined as

$$\bar{S}_{UV} W = \frac{1}{2}\left(\bar{R}^s_{UV} W + \bar{R}^d_{UV} W\right). \tag{24}$$

If $[\bar{K}_U, \bar{K}_V] = 0$, then the statistical curvature $\bar{S}$ coincides with the Riemannian curvature $\bar{R}$ and we say that the statistical structure is *statistical curvature symmetric*.

The Ricci tensor of $\bar{\nabla}^s$ is defined as $\overline{Ric}^s(U, V) = \text{trace}\left(w \mapsto \bar{R}^s_{wU} V\right)$. It follows that

$$\overline{Ric}^s(U, V) - \overline{Ric}^s(V, U) = -d\overline{\alpha}(U, V).$$

Therefore, $\overline{Ric}^s$ is symmetric if and only if the statistical structure is locally equiaffine. If the Tchebychev one-form is not closed, then $\overline{Ric}^s$ is not symmetric and so the statistical connection $\bar{\nabla}^s$ is not metric (it is not the Levi–Civita connection of some metric).

The most simple example of statistical connection on a semi-Riemannian manifold (M, g) is the one corresponding to $\bar{K} = 0$, which are called self-dual or trivial. This is nothing but the Levi–Civita connection of the metric. We can construct a nontrivial statistical structure from a fixed vector field $\xi \in \mathfrak{X}(M)$ taking

$$\bar{K}(U, V) = g(U, V)\xi + \omega(V)U + \omega(U)V, \tag{25}$$

where ω is the metrically equivalent one-form to ξ.

Another nontrivial statistical structure is obtained from

$$\bar{K} = \omega \otimes \omega\xi, \tag{26}$$

which we call ξ*-statistical structure.* We can also construct a statistical structure from two vector fields $\xi, E \in \mathfrak{X}(M)$, called (ξ, E)*-statistical structure*, taking

$$\begin{aligned}\bar{K} &= (\omega \otimes \tau + \tau \otimes \omega)\,\xi + \omega \otimes \omega E\\ &= \mathfrak{s}(\omega \otimes \tau)\xi + \omega \otimes \omega E,\end{aligned} \tag{27}$$

where ω and τ are the metrically equivalent one-form to ξ and E respectively. If E is proportional to ξ, then this statistical structure turns into a ξ'-statistical structure for a suitable choice of ξ'.

The statistical structures (25–27) appear naturally, under suitable conditions, when we deal with null submanifolds. Some aspects of the statistical structure (25) were studied in [15, 21]. Next, we study some properties of the ξ-statistical and (ξ, E)-statistical structures.

Proposition 1 *Let (M, g) be a semi-Riemannian manifold and $\xi \in \mathfrak{X}(M)$. Consider the ξ-statistical connection given by (26).*

1. *The Tchebychev one-form is $\omega(\xi)\omega$.*
2. *It is statistical curvature symmetric.*
3. *If $g(\xi, \xi)$ is a nonzero constant, then it is conjugate symmetric if and only if ξ is a parallel vector field. Moreover, in this case the ξ-statistical connection is locally a metric connection.*

Proof Points (1) and (2) are trivial. On the other hand, we have

$$\begin{aligned}\left(\overline{\nabla}_U \bar{K}\right)(V, W) - \left(\overline{\nabla}_V \bar{K}\right)(U, W) &= d\omega(U, V)\omega(W)\xi\\ &\quad + \left(\omega(V)g(\overline{\nabla}_U\xi, W) - \omega(U)g(\overline{\nabla}_V\xi, W)\right)\xi\\ &\quad + \omega(W)\left(\omega(V)\overline{\nabla}_U\xi - \omega(U)\overline{\nabla}_V\xi\right).\end{aligned} \tag{28}$$

Therefore, if ξ is parallel, then the ξ-statistical structure is conjugate symmetric. Now, suppose that the ξ-statistical structure is conjugate symmetric and $g(\xi, \xi)$ is a nonzero constant. Since being conjugate symmetric implies that the Tchebychev one form is closed, we have $d\omega = 0$ and $\overline{\nabla}_\xi \xi = 0$. Taking $U \perp \xi$ and $V = W = \xi$ in Eq. (28) we get $\overline{\nabla}_U \xi = 0$ for all $U \perp \xi$ and so ξ is parallel.

Since $g(\xi, \xi) = c$ is a nonzero constant, from the De Rham-Wu theorem we get that (M, g) is locally isometric to $\left(\mathbb{R} \times F, cdt^2 + g_F\right)$, where (F, g_F) is a semi-Riemannian manifold and ξ is identified with ∂t, [24]. Now, it is straightforward to check that the Levi–Civita connection of the metric $e^{2c^2 t}dt^2 + g_F$ coincides with the ξ-statistical connection. □

Proposition 2 *Let (M, g) be a semi-Riemannian manifold and $\xi, E \in \mathfrak{X}(M)$. Consider the (ξ, E)-statistical structure given by (27).*

1. *The Tchebychev one-form is $2\omega(E)\omega + \omega(\xi)\tau$.*
2. *It is statistical curvature symmetric if and only if for each point $g(\xi, \xi) = 0$ or ξ and E are proportional.*

Proof Point (1) is straightforward. For the second point, we have that

$$\begin{aligned}\bar{K}_U \bar{K}_V(W) = \big(&\mathfrak{s}(\omega \otimes \tau)(V, W)\mathfrak{s}(\omega \otimes \tau)(U, \xi) \\ &+ \mathfrak{s}(\omega \otimes \tau)(U, E)(\omega \otimes \omega)(V, W)\big)\xi \\ &+ \big((\omega \otimes \omega)(U, \xi)\mathfrak{s}(\omega \otimes \tau)(V, W) \\ &+ (\omega \otimes \omega)(U, E)(\omega \otimes \omega)(V, W)\big)E.\end{aligned}$$

Therefore, taking into account that $\omega(E) = \tau(\xi)$ we get

$$[\bar{K}_U, \bar{K}_V](W) = \omega(\xi)(\omega \wedge \tau)(U, V)\big(\omega(W)E - \tau(W)\xi\big)$$

and the conclusion follows. □

Theorem 1 *Let (M, g) be a semi-Riemannian manifold with $dim\ M \geq 3$ and $\xi, E \in \mathfrak{X}(M)$ such that $g(\xi, \xi) = c$, $g(\xi, E) = 0$ and $g(E, E) \neq 0$ at every point in M, where $c \neq 0$ is certain constant. If the (ξ, E)-statistical structure is conjugate symmetric, then the distributions $\mathcal{D} = span(\xi, E)$ and $\mathcal{D}^\perp$ are integrable, the leaves of $\mathcal{D}^\perp$ are totally geodesic and the leaves of $\mathcal{D}$ are totally umbilical with mean curvature vector given by*

$$\mathcal{P}_{\mathcal{D}^\perp}\left(\overline{\nabla} \ln \sqrt{|g(E, E)|}\right),$$

being $\mathcal{P}_{\mathcal{D}^\perp}$ the projection onto $\mathcal{D}^\perp$. Moreover, (M, g) decomposes locally as a twisted product

$$\left(L \times S, g|_L + h^2 g|_S\right),$$

where L is a leaf of $\mathcal{D}^\perp$, S is a leaf of $\mathcal{D}$ and $h \in C^\infty(L \times S)$.

Proof A long but easy computation gives us

$$\begin{aligned}&\left(\overline{\nabla}_U \bar{K}\right)(V, W) - \left(\overline{\nabla}_V \bar{K}\right)(U, W) \\ &\quad = (d\omega(U, V)\tau(W) + d\tau(U, V)\omega(W))\xi \\ &\qquad + d\omega(U, V)\omega(W)E + g(\omega(V)\overline{\nabla}_U \xi - \omega(U)\overline{\nabla}_V \xi, W)E \\ &\qquad + g(\omega(V)\overline{\nabla}_U E - \omega(U)\overline{\nabla}_V E, W)\xi + g(\tau(V)\overline{\nabla}_U \xi - \tau(U)\overline{\nabla}_V \xi, W)\xi \\ &\qquad + \tau(W)(\omega(V)\overline{\nabla}_U \xi - \omega(U)\overline{\nabla}_V \xi) + \omega(W)(\tau(V)\overline{\nabla}_U \xi - \tau(U)\overline{\nabla}_V \xi) \\ &\qquad + \omega(W)(\omega(V)\overline{\nabla}_U E - \omega(U)\overline{\nabla}_V E) \qquad (29)\end{aligned}$$

for all $U, V, W \in \mathfrak{X}(M)$. If the (ξ, E)-statistical structure is conjugate symmetric, then the above vanishes. Observe that $\mathcal{D}$ is a two-dimensional nondegenerate distribution and $\tau(\xi) = \omega(E) = 0$ since $g(\xi, E) = 0$.

If we take $U = \xi$ and $V, W \in \mathcal{D}^\perp$ in formula (29), then we get that

$$\begin{aligned}\overline{\nabla}_V \xi &= \mu(V)E, \\ \overline{\nabla}_V E &= -\frac{\tau(E)}{c}\mu(V)\xi + \beta(V)E\end{aligned}$$

for all $V \in \mathcal{D}^\perp$ and certain one-forms μ and β. If we choose $U = W = \xi$ and $V \in \mathcal{D}^\perp$, then we get

$$\mu(V) = -\frac{1}{3\tau(E)} g(\overline{\nabla}_\xi E, V),$$
$$\beta(V) = \frac{1}{c} g(\overline{\nabla}_\xi \xi, V) \tag{30}$$

for all $V \in \mathcal{D}^\perp$, but taking $U = W = E$ and $V \in \mathcal{D}^\perp$ we get that $\mu(V) = \frac{1}{2\tau(E)} g(\overline{\nabla}_E \xi, V)$. Therefore,

$$g(\overline{\nabla}_\xi E, V) = -\frac{3}{2} g(\overline{\nabla}_E \xi, V) \tag{31}$$

for all $V \in \mathcal{D}^\perp$.

If we take now $U = \xi$ and $V = W = E$ in formula (29), then we get that $\overline{\nabla}_E \xi \in \mathcal{D}$ and thus from the Eq. (31) we also have $\overline{\nabla}_\xi E \in \mathcal{D}$, which implies that $\mathcal{D}$ is integrable and $\mu(U) = 0$ for all $U \in \mathcal{D}^\perp$.

Next, we show that the leaves of $\mathcal{D}$ are totally umbilical. Taking $U = E$, $V \in \mathcal{D}^\perp$ and $W = \xi$ in formula (29), we also have that

$$\beta(V) = \frac{1}{\tau(E)} g(\overline{\nabla}_E E, V). \tag{32}$$

If we call $\mathbb{I}^{\mathcal{D}}$ the second fundamental form of the leaves of $\mathcal{D}$, then $\mathbb{I}^{\mathcal{D}}(\xi, E) = 0$ and Eqs. (30) and (32) show that

$$\frac{1}{g(\xi,\xi)} \mathbb{I}^{\mathcal{D}}(\xi,\xi) = \frac{1}{g(E,E)} \mathbb{I}^{\mathcal{D}}(E,E),$$

which implies that the leaves of $\mathcal{D}$ are totally umbilical. Call H the mean curvature vector field, i.e. $\mathbb{I}^{\mathcal{D}} = g \cdot H$. If we choose $U = \xi$, $V \in \mathcal{D}^\perp$ and $W = E$ in formula (29), then we get $g(\overline{\nabla}_\xi \xi, V)\tau(E) = c g(\overline{\nabla}_V E, E)$ or equivalently $g(H, V) = g(\overline{\nabla} \ln \sqrt{|g(E,E)|}, V)$, which means that $H = \mathcal{P}_{\mathcal{D}^\perp}\left(\overline{\nabla} \ln \sqrt{|g(E,E)|}\right)$.

Since $\overline{\nabla}_U \xi = 0$ and $\overline{\nabla}_U E = \beta(U) E$ for all $U \in \mathcal{D}^\perp$, it follows that $\mathcal{D}^\perp$ is also integrable with totally geodesic leaves. From [23] we get the local decomposition of the manifold as a twisted product. □

In a twisted product $\left(L \times S, g|_L + h^2 g|_S\right)$ the mean curvature vector field of the second canonical foliation is $\mathcal{P}_1\left(-\overline{\nabla} \ln h\right)$, where $\mathcal{P}_1$ is the projection onto the first factor, [5, 23]. Therefore, in the above theorem we have that

$$h = \frac{A}{\sqrt{|g(E,E)|}},$$

where $A \in C^\infty(S)$ is some function.

Corollary 1 *Let (M, g) be a semi-Riemannian manifold and $\xi, E \in \mathfrak{X}(M)$ such that $g(\xi, E) = 0$ and $g(E, E)$ and $g(\xi, \xi)$ are nonzero constants. If the (ξ, E)-statistical structure is conjugate symmetric, then ξ and E are parallel and (M, g) decomposes locally as a direct product $L \times \mathbb{R}^2$.*

Proof Suppose first that $dim\ M = 2$. Since $g(\xi, \xi)$ and $g(E, E)$ are constant, then $\overline{\nabla}_E E$ is proportional to ξ and $\overline{\nabla}_\xi \xi$ and $\overline{\nabla}_E \xi$ are proportional to E. If we take $U = \xi$, $V = W = E$ in formula (29), then we get

$$g(\overline{\nabla}_\xi \xi, E)g(E, E)\xi - g(\overline{\nabla}_E \xi, E)g(\xi, \xi)E = 0.$$

Since ξ and E are linearly independent, then $\overline{\nabla}_\xi \xi = \overline{\nabla}_E \xi = 0$.

On the other hand, if we take $U = \xi$, $V = E$ and $W = \xi$ in formula (29), then we deduce that $\overline{\nabla}_E E = 0$. Since $g(E, E)$ is constant and $g(E, \xi) = 0$ we also have that $\overline{\nabla}_\xi E = 0$ and therefore ξ and E are parallel and (M, g) is locally isometric to the semi-euclidean space $\mathbb{R}^2$.

If $dim\ M \geq 3$, then we know from the above theorem that the leaves of $\mathcal{D}$ are also totally geodesic. Therefore, (M, g) decomposes locally as a direct product $(L \times S, g|_L + g|_S)$, where S is a leaf of $\mathcal{D}$, [23, 24]. If we restrict the (ξ, E)-statistical structure of the ambient manifold to $(S, g|_S)$, then we get a (ξ, E)-structure on $(S, g|_S)$, which is also conjugate symmetric. Using the two-dimensional case we get the conclusion. □

In [21] some conditions for the statistical structure given by (25) to be complete are stated. We can also give some results about the completeness of a ξ-statistical and a (ξ, E)-statistical connection. For this, we need the following lemma.

Lemma 1 *Let (M, g) be a complete Riemannian manifold, $\bar{\nabla}^s$ a connection and $\gamma : (a, b) \to M$ a maximal $\bar{\nabla}^s$-geodesic. If there are $t_0, c \in \mathbb{R}$ such that $g(\gamma'(t), \gamma'(t)) \leq c^2$ for all $t \in [t_0, b)$, then $b = \infty$.*

Proof It is a standard argument, so we only sketch the proof. Suppose that $b < \infty$ and take a sequence $t_n < b$ converging to b. If d is the Riemannian distance associated to g, then

$$d(\gamma(t_n), \gamma(t_m)) \leq \left| \int_{t_n}^{t_m} \sqrt{g(\gamma'(t), \gamma'(t))} dt \right| \leq c|t_n - t_m|.$$

Therefore, $\gamma(t_n)$ is a Cauchy sequence and it follows that it exists $p = \lim_{t \to b^-} \gamma(t)$. Using a $\bar{\nabla}^s$-convex neighbourhood of p [12, p. 149], we can conclude that γ can be extended, which contradicts the maximality of γ. □

We say that a function $f : [0, b) \to \mathbb{R}$ is not oscillating when t approaches to b if there exists $t_0 < b$ such that $f'(t) \geq 0$ or $f'(t) \leq 0$ for all $t \in [t_0, b)$.

Proposition 3 *Let (M, g) be a complete Riemannian manifold and take $f \in C^\infty(M)$ and $\xi \in \mathfrak{X}(M)$ such that f and $g(\xi, \xi)$ are bounded on M. Take $\bar{\nabla}^s$ the (ξ, E)-statistical connection, being $E = \overline{\nabla} f$. If $\gamma : (a, b) \to M$ is a maximal $\bar{\nabla}^s$-geodesic such that $f(\gamma(t))$ is not oscillating when t approaches to b, then $b = \infty$.*

Proof Suppose that $|f(p)| \leq A$ and $g(\xi, \xi)_p \leq B$ for some $A, B \in \mathbb{R}$ and for all $p \in M$. If we call $y(t) = g(\gamma'(t), \gamma'(t))$, then

$$y' = -6\omega(\gamma'(t))^2 \tau(\gamma'(t)) \leq 6|\tau(\gamma'(t))| g(\xi, \xi) g(\gamma'(t), \gamma'(t)) \leq 6|\tau(\gamma'(t))| B y.$$

Since $\tau(\gamma'(t)) = \frac{d}{dt} f(\gamma(t))$ and $f(\gamma(t))$ is not oscillating, we can suppose that there exists t_0 such that $\tau(\gamma'(t)) \geq 0$ for all t with $t_0 \leq t < b$. Therefore,

$$y(t) \leq y(0) e^{6B(f(\gamma(t)) - f(\gamma(0)))} \leq y(0) e^{6B(A - f(\gamma(0)))}$$

for all $t \in [t_0, b)$. Applying the above lemma, $b = \infty$. □

Observe that in this context, if $f(\gamma(t))$ is oscillating, then the angle between $\overline{\nabla} f_{\gamma(t)}$ and $\gamma'(t)$ is oscillating around $\frac{\pi}{2}$.

Proposition 4 *Let (M, g) be a Riemannian manifold and $\xi \in \mathfrak{X}(M)$ a Killing vector field. Take $\bar{\nabla}^s$ the ξ-statistical connection and γ a $\bar{\nabla}^s$-geodesic.*

1. *If $g(\xi_{\gamma(0)}, \gamma'(0)) = 0$, then γ is also a g-geodesic orthogonal to ξ.*
2. *If $g(\xi_{\gamma(0)}, \gamma'(0)) \neq 0$ and $0 < A \leq g(\xi, \xi)_{\gamma(t)} \leq B$ for all t and certain $A, B \in \mathbb{R}$, then γ is incomplete.*
3. *If γ is periodic, then γ is a g-geodesic orthogonal to ξ.*

Proof The function $h(t) = \omega(\gamma'(t))$ holds $h' = -h^2 g(\xi, \xi)$. If $h(0) = 0$, then $h(t) = 0$ for all t and if $h(0) \neq 0$, then

$$h(t) = \frac{1}{\frac{1}{h(0)} + \int_0^t g(\xi, \xi)_{\gamma(s)} ds}.$$

In the first case, we have $\overline{\nabla}_{\gamma'}\gamma' = 0$ and γ is also a g-geodesic. In the second case, if γ is complete, then there is a value $t_0 \in \mathbb{R}$ with $\lim_{t \to t_0} |h(t)| = \infty$, which is a contradiction. If γ is periodic, then it is complete and we have necessarily the first case. □

4 Null submanifolds and statistical structures

If Σ is a nondegenerate submanifold in a semi-Riemannian manifold (M, g) furnished with a statistical connection $\bar{\nabla}^s$, then the induced connection on Σ from $\bar{\nabla}^s$ is a statistical connection. We obtain in this manner an induced statistical structure on Σ. Its dual connection is the induced connection on Σ from $\bar{\nabla}^d$ and the Tchebychev vector field is the projection onto Σ of the Tchebychev vector field of $\bar{\nabla}^s$. Moreover, if $\bar{\nabla}^s$ is trace-free or locally equiaffine, then so is the induced statistical structure on Σ.

However, the above does not work for a null submanifold. Furthermore, the definition of statistical structure in this case is distorted because the inherited metric is degenerate and so even the dual connection is not well-defined. Anyway, there are some interesting questions concerning null submanifolds and statistical structures that arise in a natural way.

For example, fixed a rigging for a null submanifold Σ, we can construct the induced connection ∇ and the rigged connection $\widetilde{\nabla}$ on Σ, so we can wonder under what conditions they coincide. A rigging with this property was called a preferred rigging in [16], where some obstructions for its existence were also given. Observe that if ∇ and $\widetilde{\nabla}$ coincide, then we can also say that $(\widetilde{g}, \nabla)$ is a self-dual or trivial statistical structure on Σ. Therefore, a more general question than before is: under what conditions $(\widetilde{g}, \nabla)$ is a (not necessarily trivial) statistical structure?

Since the Levi–Civita connection $\overline{\nabla}$ is a trivial statistical connection on M and ∇ is its induced connection on Σ, we can consider an even more general situation. Let (M, g) be a semi-Riemannian manifold, $(g, \bar{\nabla}^s)$ a statistical structure on M and Σ a null submanifold. We can project $\bar{\nabla}^s$ onto Σ as we did with the Levi–Civita connection $\overline{\nabla}$ in Sect. 2. We write

$$\begin{aligned} \bar{\nabla}^s_U V &= \nabla^s_U V + \mathfrak{h}^s(U, V) + B^s(U, V)N, \\ \nabla^s_U \xi &= -A^{*s}(U) - \tau^s(U)\xi, \\ \nabla^s_U X &= \nabla^{*s}_U X + C^s(U, X)\xi, \end{aligned} \tag{33}$$

for all $U, V \in \mathfrak{X}(\Sigma)$ and $X \in \Gamma(\mathcal{S})$, obtaining in this way the analogous geometric objects to those in Sect. 2. We also have in this case that B^s and $\mathfrak{h}^s$ are symmetric tensors and ∇^s is a connection without torsion on Σ, but now $B^s(\xi, U)$ and $A^{*s}(\xi)$ are not necessarily zero.

Now, the questions stated before are particular cases of the following one: under what conditions $(\widetilde{g}, \nabla^s)$ is a statistical structure on Σ? This will be answered in Theorem 2.

If we also project the dual connection $\bar{\nabla}^d$ onto Σ and we write

$$\begin{aligned}\bar{\nabla}^d_U V &= \nabla^d_U V + \mathfrak{h}^d(U, V) + B^d(U, V)N,\\ \nabla^d_U \xi &= -A^{*d}(U) - \tau^d(U)\xi,\\ \nabla^d_U X &= \nabla^{*d}_U X + C^d(U, X)\xi,\end{aligned}$$

then we obtain another torsion free connection ∇^d and the tensors B^d, C^d and τ^d. We can easily deduce that

$$B^s(U, V) = -g(\bar{\nabla}^d_U \xi, V) = g(A^{*d}(U), V) - B^d(\xi, U)\omega(V), \tag{34}$$

$$C^s(U, X) = -g(\bar{\nabla}^d_U N, X), \tag{35}$$

$$\tau^s(U) = g(\bar{\nabla}^d_U N, \xi). \tag{36}$$

We have similar equations for B^d, C^d and τ^d. Moreover, from Eq. (19) we immediately have

$$B = \frac{1}{2}\left(B^s + B^d\right), \tag{37}$$

$$C = \frac{1}{2}\left(C^s + C^d\right), \tag{38}$$

$$\tau = \frac{1}{2}\left(\tau^s + \tau^d\right), \tag{39}$$

$$A^* = \frac{1}{2}\left(A^{*d} + A^{*s}\right), \tag{40}$$

$$\nabla_U V = \frac{1}{2}\left(\nabla^s_U V + \nabla^d_U V\right). \tag{41}$$

Recall that Eq. (41) does not imply that ∇^s is a statistical connection. On the other hand, from Eq. (37) we get that $B^s(\xi, U) = -B^d(\xi, U)$ for all $U \in \mathfrak{X}(\Sigma)$. From the definition of $\bar{K}$ we also get

$$B^s(U, V) = B(U, V) + g(\bar{K}(U, V), \xi), \tag{42}$$

$$B^d(U, V) = B(U, V) - g(\bar{K}(U, V), \xi) \tag{43}$$

$$C^s(U, X) = C(U, X) + g(\bar{K}(U, X), N), \tag{44}$$

$$C^d(U, X) = C(U, X) - g(\bar{K}(U, X), N) \tag{45}$$

$$\tau^s(U) = \tau(U) - g(\bar{K}(U, \xi), N), \tag{46}$$

$$\tau^d(U) = \tau(U) + g(\bar{K}(U, \xi), N). \tag{47}$$

We can relate the tensors C^s, C^d, τ^s and τ^d as follows.

Lemma 2 *Let Σ be a null submanifold in a semi-Riemannian manifold (M, g) and ζ a rigging for it. If $\bar{\nabla}^s$ is a statistical connection on M, then*

1. *For all $U, V \in \mathfrak{X}(\Sigma)$ it holds*

$$C^d(U, \mathcal{P}_{\mathcal{S}}(V)) - C^d(V, \mathcal{P}_{\mathcal{S}}(U)) = \left(\tau^d(U) - \tau^s(U)\right)\omega(V) + \left(\tau^s(V) - \tau^d(V)\right)\omega(U) + C^s(U, \mathcal{P}_{\mathcal{S}}(V)) - C^s(V, \mathcal{P}_{\mathcal{S}}(U)).$$

2. *$\mathcal{S}$ is integrable if and only if C^s is symmetric for all $X, Y \in \Gamma(\mathcal{S})$ if and only if C^d is symmetric for all $X, Y \in \Gamma(\mathcal{S})$.*

Proof If we subtract Eqs. (44) and (45) , then

$$C^s(U, \mathcal{P}_{\mathcal{S}}(V)) - C^d(U, \mathcal{P}_{\mathcal{S}}(V)) = 2g(\bar{K}(U, \mathcal{P}_{\mathcal{S}}(V)), N).$$

Analogously, from Eqs. (46) and (47) we get

$$\tau^d(U) - \tau^s(U) = 2g(\bar{K}(U, \xi), N).$$

Therefore,

$$C^s(U, \mathcal{P}_{\mathcal{S}}(V)) - C^d(U, \mathcal{P}_{\mathcal{S}}(V)) + \omega(V)\left(\tau^d(U) - \tau^s(U)\right) = 2g(\bar{K}(U, V), N).$$

Since $\bar{K}$ is totally symmetric, we get point one. The second point follows directly from Eqs. (44) and (45). □

The following theorem gives us necessary and sufficient conditions for a null submanifold to inherit a statistical structure from the ambient respect to the rigged metric.

Theorem 2 *Let (M, g) be a semi-Riemannian manifold furnished with a statistical connection $\bar{\nabla}^s$ and Σ a null submanifold. Fixed a rigging for Σ, the connection ∇^s on Σ obtained in (33) is a statistical connection with respect to the rigged metric $\widetilde{g}$ if and only if*

$$B^s(X, Y) = C^s(X, Y), \tag{48}$$

$$B^s(\xi, X) + C^s(\xi, X) = -2\tau^s(X) \tag{49}$$

for all $X, Y \in \Gamma(\mathcal{S})$. In particular, if $(\widetilde{g}, \nabla^s)$ is a statistical structure on Σ, then the screen distribution is integrable.

Proof We have to check that $\nabla^s\widetilde{g} = \nabla^s g + \nabla^s\omega \otimes \omega$ is totally symmetric. Using Eqs. (19) and (33) we have that

$$\begin{aligned}\left(\nabla^s_U g\right)(V, W) &= Ug(V, W) - g(\nabla^s_U V, W) - g(V, \nabla^s_U W)\\ &= -2g(\bar{K}(U, W), V) + B^s(U, V)\omega(W) + B^s(U, W)\omega(V).\end{aligned}$$

On the other hand, from Eqs. (35) and (36), we get

$$\begin{aligned}\left(\nabla^s_U \omega\right)(V) &= Ug(V, N) - g(\nabla^s_U V, N) = g(\bar{\nabla}^d_U N, V)\\ &= -C^s(U, \mathcal{P}_{\mathcal{S}}(V)) + \tau^s(U)\omega(V).\end{aligned}$$

Now,

$$\left(\nabla^s_U \omega \otimes \omega\right)(V, W) = \left(\nabla^s_U \omega\right)(V)\omega(W) + \omega(V)\left(\nabla^s_U \omega\right)(W)$$
$$= -C^s(U, \mathcal{P}_{\mathcal{S}}(V))\omega(W) - C^s(U, \mathcal{P}_{\mathcal{S}}(W))\omega(V) + 2\tau^s(U)\omega(V)\omega(W). \tag{50}$$

Therefore, since B^s is symmetric and $\bar{K}$ is totally symmetric, we have

$$\begin{aligned}
&\left(\nabla^s_U \widetilde{g}\right)(V, W) - \left(\nabla^s_V \widetilde{g}\right)(U, W) \\
&\quad = \big(C^s(V, \mathcal{P}_{\mathcal{S}}(U)) - C^s(U, \mathcal{P}_{\mathcal{S}}(V)\big)\omega(W) \\
&\qquad + \big(B^s(U, W) - C^s(U, \mathcal{P}_{\mathcal{S}}(W))\big)\omega(V) - \big(B^s(V, W) - C^s(V, \mathcal{P}_{\mathcal{S}}(W))\big)\omega(U) \\
&\qquad + 2\left(\tau^s(U)\omega(V) - \tau^s(V)\omega(U)\right)\omega(W).
\end{aligned}$$

If we call $\phi(U, V, W) = \left(\nabla^s_U \widetilde{g}\right)(V, W) - \left(\nabla^s_V \widetilde{g}\right)(U, W)$, then it is clear that ϕ is skew-symmetric in the first two entries and $\mathfrak{C}(\phi) = 0$, where $\mathfrak{C}$ stands for the cyclic permutation. Therefore, $\nabla^s \widetilde{g}$ is totally symmetric, i.e., $\phi = 0$, if and only if $\phi(X, Y, Z) = \phi(\xi, Y, Z) = \phi(X, \xi, \xi) = 0$ for all $X, Y, Z \in \Gamma(\mathcal{S})$. These three conditions are equivalent to

$$B^s(X, Y) = C^s(X, Y),$$
$$B^s(\xi, X) = -2\tau^s(X) - C^s(\xi, X)$$

for all $X, Y \in \Gamma(\mathcal{S})$. Finally, since B^s is symmetric, from Lemma 2 we have that the screen distribution is integrable. □

Remark 1 From Lemma 2 we have $C^d(\xi, X) = C^s(\xi, X) + \tau^s(X) - \tau^d(X)$, so using Eqs. (37) and (39), we get that Eqs. (48) and (49) are equivalent to

$$B^s(X, Y) = C^s(X, Y),$$
$$C^d(\xi, X) - B^d(\xi, X) = -2\tau(X)$$

for all $X, Y \in \Gamma(\mathcal{S})$.

Definition 4 Let (M, g) be a semi-Riemannian manifold furnished with a statistical connection $\bar{\nabla}^s$. If ζ is a rigging for a null submanifold Σ such that

$$B^s(X, Y) = C^s(X, Y),$$
$$B^s(\xi, X) + C^s(\xi, X) = -2\tau^s(X)$$

for all $X, Y \in \Gamma(\mathcal{S})$, then we say that ζ is a $\bar{\nabla}^s$-statistical rigging for Σ.

If the screen distribution is integrable, then their leaves are nondegenerate submanifolds in (M, g) and thus they inherit the statistical structure of the ambient. More concretely, if L is a leaf of the screen $\mathcal{S}$, then $(g|_L, \nabla^{*s})$ is a statistical structure on L with dual connection ∇^{*d}. If there exists a $\bar{\nabla}^s$-statistical rigging, then the statistical structure $(g|_L, \nabla^{*s})$ on L coincides with the one inherited from $(\widetilde{g}, \nabla^s)$.

Proposition 5 *Let (M, g) be a semi-Riemannian manifold furnished with a statistical connection $\bar{\nabla}^s$. If ζ is a $\bar{\nabla}^s$-statistical rigging for a null submanifold Σ, then the difference tensor $K = \nabla^s - \widetilde{\nabla}$ is given by*

$$\begin{aligned}K(U,V) &= \mathcal{P}_{\mathcal{S}}(\bar{K}(U,V)) + \big(B^s(U,V) - B(U,V)\big)\xi \\ &\quad + \mathfrak{s}(\beta\otimes\omega)(U,V)\xi + \omega(U)\omega(V)A^{*s}(\xi) \\ &\quad + \big(2\tau^s(\xi) - B^s(\xi,\xi)\big)\,\omega(U)\omega(V)\xi \\ &\quad - \mathfrak{s}(\omega\otimes\tau^s)(U,V)\xi - \omega(U)\omega(V)E^s,\end{aligned}$$

for all $U, V \in \mathfrak{X}(\Sigma)$, where E^s is the $\widetilde{g}$-metrically equivalent vector field to τ^s and β is the $\widetilde{g}$-metrically equivalent one-form to $A^{*s}(\xi)$.

Proof We know that $-2\widetilde{g}(K(U,V),W) = (\nabla^s\widetilde{g})(U,V,W)$, but from the proof of the above theorem we have that

$$\begin{aligned}\left(\nabla^s\widetilde{g}\right)(U,V,W) &= -2g(\bar{K}(U,V),W) + B^s(U,V)\omega(W) + B^s(U,W)\omega(V) \\ &\quad - C^s(U,\mathcal{P}_{\mathcal{S}}(V))\omega(W) - C^s(U,\mathcal{P}_{\mathcal{S}}(W))\omega(V) + 2\tau^s(U)\omega(V)\omega(W).\end{aligned}$$

Using Eq. (48) we get

$$\begin{aligned}C^s(U,\mathcal{P}_{\mathcal{S}}(V)) &= B^s(U,V) - \omega(U)B^s(\xi,V) - \omega(V)B^s(\xi,U) \\ &\quad + \omega(U)\omega(V)B^s(\xi,\xi) + \omega(U)C^s(\xi,\mathcal{P}_{\mathcal{S}}(V)).\end{aligned}$$

But from Eq. (49) we have that

$$C^s(\xi,\mathcal{P}_{\mathcal{S}}(V)) = -B^s(\xi,V) + \omega(V)B^s(\xi,\xi) - 2\tau^s(V) + 2\tau^s(\xi)\omega(V)$$

and so

$$\begin{aligned}C^s(U,\mathcal{P}_{\mathcal{S}}(V)) &= B^s(U,V) - 2\omega(U)B^s(\xi,V) - \omega(V)B^s(\xi,U) \\ &\quad + 2\omega(U)\omega(V)B^s(\xi,\xi) - 2\omega(U)\tau^s(V) + 2\tau^s(\xi)\omega(U)\omega(V).\end{aligned}$$

We compute $C^s(U,\mathcal{P}_{\mathcal{S}}(W))$ in the same way and we get

$$\begin{aligned}\widetilde{g}(K(U,V),W) &= g(\bar{K}(U,V),W) + 2B^s(\xi,\xi)\omega(U)\omega(V)\omega(W) \\ &\quad - \omega(V)\omega(W)B^s(\xi,U) - B^s(\xi,V)\omega(U)\omega(W) - B^s(\xi,W)\omega(U)\omega(V) \\ &\quad + 2\tau^s(\xi)\omega(U)\omega(V)\omega(W) \\ &\quad - \omega(U)\omega(W)\tau^s(V) - \omega(U)\omega(V)\tau^s(W) - \tau^s(U)\omega(V)\omega(W).\end{aligned}$$

From Eq. (34) we have that

$$\begin{aligned}B^s(\xi,V)\omega(U) + B^s(\xi,U)\omega(V) &= g(A^{*d}(\xi),U)\omega(V) + g(A^{*d}(\xi),V)\omega(U) \\ &\quad - 2B^d(\xi,\xi)\omega(U)\omega(V) \\ &= -g(A^{*s}(\xi),U)\omega(V) - g(A^{*s}(\xi),V)\omega(U) \\ &\quad - 2B^d(\xi,\xi)\omega(U)\omega(V)\end{aligned}$$

and taking into account that $B^s(\xi,\xi) + B^d(\xi,\xi) = 0$ we can write

$$\begin{aligned}\widetilde{g}(K(U,V),W) &= g(\bar{K}(U,V),W) + \mathfrak{s}\left(\beta\otimes\omega\right)(U,V)\omega(W) \\ &\quad - B^s(\xi,W)\omega(U)\omega(V) + 2\tau^s(\xi)\omega(U)\omega(V)\omega(W) \\ &\quad - \mathfrak{s}(\tau^s\otimes\omega)(U,V)\omega(W) \\ &\quad - \omega(U)\omega(V)\tau^s(W),\end{aligned}$$

where β is the one form given by $\beta(U) = g(A^{*s}(\xi),U)$ for all $U \in \mathfrak{X}(\Sigma)$. Observe that β is the $\widetilde{g}$-metrically equivalent one form to $A^{*s}(\xi)$, since $A^{*s}(\xi) \in \Gamma(\mathcal{S})$.

Using again Eq. (34) we get

$$B^s(\xi, W) = \widetilde{g}(A^{*d}(\xi), W) - B^d(\xi, \xi)\omega(W) \\ - \widetilde{g}(A^{*s}(\xi), W) + B^s(\xi, \xi)\omega(W).$$

From the decompositions (2) and (3) we can write

$$g\left(\bar{K}(U, V), W\right) = \widetilde{g}(\mathcal{P}_{\mathcal{S}}(\bar{K}(U, V)), W) + g(\bar{K}(U, V), \xi)\omega(W) \\ = \widetilde{g}(\mathcal{P}_{\mathcal{S}}(\bar{K}(U, V)), W) + \left(B^s(U, V) - B(U, V)\right)\omega(W).$$

Replacing the expression for $B^s(\xi, W)$ and $g(\bar{K}(U, V), W)$ we get the result. □

Observe that in the statistical structure constructed in the null submanifold Σ, it appears a combination of a ξ-statistical structure, a (ξ, E^s)-statistical structure and a $(\xi, A^{*d}(\xi))$-statistical structure.

On the other hand, if ∇^s is a statistical connection respect to the rigged metric $\widetilde{g}$, then ∇^d is not in general its dual connection. In fact, if ζ is a $\bar{\nabla}^s$-statistical rigging, then ∇^d does not need to be a statistical connection respect to the rigged metric. However, the following relations hold.

Proposition 6 *Let (M, g) be a semi-Riemannian manifold furnished with a statistical connection $\bar{\nabla}^s$. Suppose that ζ is a $\bar{\nabla}^s$-rigging for a null submanifold Σ and call ∇^{ds} the dual connection of ∇^s respect to $\widetilde{g}$. Given $X, Y \in \Gamma(\mathcal{S})$ it holds*

1. $\nabla^{ds}_X Y = \nabla^d_X Y + 2\left(B(X, Y) - C(X, Y)\right)\xi$.
2. $\nabla^{ds}_X \xi = \nabla^d_X \xi + 2\tau(X)\xi$.
3. $\nabla^{ds}_\xi \xi = -\nabla^d_\xi \xi + 2\mathcal{P}_{\mathcal{S}}(E^s) + \left(\tau^s(\xi) - \tau(\xi)\right)\xi$, *where E^s is the $\widetilde{g}$-metrically equivalent vector field to τ^s.*

Proof Since ∇^s is a statistical connection on Σ respect to the rigged metric $\widetilde{g}$, then $\mathcal{S}$ is integrable. Moreover, since its leaves are nondegenerate submanifolds, then (g, ∇^{*s}) is a statistical structure when we restrict to a leaf. If we take $X, Y, Z \in \Gamma(\mathcal{S})$, then

$$g(\nabla^{*s}_X Z, Y) + g(Z, \nabla^{*d}_X Y) = Xg(Z, Y) \\ = X\widetilde{g}(Z, Y) = \widetilde{g}(\nabla^s_X Z, Y) + \widetilde{g}(Z, \nabla^{ds}_X Y).$$

But $g(\nabla^{*s}_X Y, Z) = \widetilde{g}(\nabla^s_X Y, Z)$ and so $\nabla^{ds}_X Y = \nabla^{*d}_X Y + \widetilde{g}(\nabla^{ds}_X Y, \xi)\xi$. Since

$$\widetilde{g}(\nabla^{ds}_X Y, \xi) = -\widetilde{g}(Y, \nabla^s_X \xi) = B^d(X, Y)$$

we get $\nabla^{ds}_X Y = \nabla^{*d}_X Y + B^d(X, Y)\xi = \nabla^d_X Y + \left(B^d(X, Y) - C^d(X, Y)\right)\xi$. From Eqs. (37), (38) and (48) we have

$$B^d(X, Y) - C^d(X, Y) = 2\left(B(X, Y) - C(X, Y)\right)$$

and we obtain the first point.

For the second one, we have in the same manner as above that

$$\widetilde{g}(\nabla^{ds}_X \xi, \xi) = -\widetilde{g}(\xi, \nabla^s_X \xi) = -g(N, \nabla^s_X \xi) = \tau^s(X).$$

On the other hand, using Eq. (48), we get

$$\widetilde{g}(\nabla^{ds}_X \xi, Y) = -\widetilde{g}(\xi, \nabla^s_X Y) = -g(N, \nabla^s_X Y) = -C^s(X, Y) = -g(A^{*d}(X), Y).$$

Therefore,

$$\nabla^{ds}_X \xi = -A^{*d}(X) + \tau^s(X)\xi = \nabla^d_X \xi + 2\tau(X)\xi.$$

Finally, using Eq. (49) we get

$$\begin{aligned}
\widetilde{g}(\nabla^{ds}_\xi \xi, \xi) &= -\widetilde{g}(\xi, \nabla^s_\xi \xi) = \tau^s(\xi),\\
\widetilde{g}(\nabla^{ds}_\xi \xi, X) &= -\widetilde{g}(\xi, \nabla^s_\xi X) = -C^s(\xi, X)\\
&= B^s(\xi, X) + 2\tau^s(X) = \widetilde{g}(A^{*d}(\xi), X) + 2\widetilde{g}(E^s, X)
\end{aligned}$$

and we obtain the last point. □

Proposition 7 *Let (M, g) be a semi-Riemannian manifold furnished with a statistical connection $\bar{\nabla}^s$. If ζ is a $\bar{\nabla}^s$-rigging for a null hypersurface Σ, then the Tchebychev one-form of ∇^s is given by*

$$\alpha(U) = \overline{\alpha}(U) - \tau^d(U)$$

for all $U \in \mathfrak{X}(\Sigma)$, where $\overline{\alpha}$ is the Tchebychev one-form of $\bar{\nabla}^s$.

Proof From Eqs. (4) and (33) we have that

$$\bar{K}(U, V) = K(U, V) - D(U, V) + (B^s - B)(U, V)N \tag{51}$$

for all $U, V \in \mathfrak{X}(\Sigma)$, where $D(U, V) = \nabla_U V - \widetilde{\nabla}_U V$. If we take $u \in T_x\Sigma$ for some $x \in \Sigma$ and $\{e_1, \ldots, e_{n-2}\}$ an orthonormal basis of $\mathcal{S}_x$, then

$$\overline{\alpha}(u) = 2g(\bar{K}(\xi, N), u) + \sum_{i=1}^{n-2} g(\bar{K}(u, e_i), e_i). \tag{52}$$

On the other hand, taking into account Eq. (51), we can write

$$\begin{aligned}
\alpha(u) &= \widetilde{g}(K(u, \xi), \xi) + \sum_{i=1}^{n-2} \widetilde{g}(K(e_i, u), e_i) = g(K(u, \xi), N) + \sum_{i=1}^{n-2} g(K(e_i, u), e_i)\\
&= g(\bar{K}(\xi, N), u) + g(D(u, \xi), N) + \sum_{i=1}^{n-2} g(\bar{K}(e_i, u), e_i) + g(D(e_i, u), e_i).
\end{aligned}$$

From [6, Corollary 3.6] we know that $g(D(e_i, u), e_i) = 0$ and from Eq. (6) we get

$$g(D(u, \xi), N) = g(\nabla_u \xi, N) - \widetilde{g}(\widetilde{\nabla}_u \xi, \xi) = -\tau(u).$$

Using Eq. (52) we arrive at

$$\alpha(u) = \overline{\alpha}(u) - g(\bar{K}(\xi, N), u) - \tau(u).$$

But $g(\bar{K}(\xi, N), u) = g(\bar{K}(u, \xi), N) = -\tau^s(u) + \tau(u)$ and from Eq. (39) we get the result. □

As a corollary of the above results, we obtain conditions for the induced connection ∇ from the ambient Levi–Civita connection $\overline{\nabla}$ to be a statistical structure respect to the rigged metric $\widetilde{g}$. Following the Definition 4, in this case the rigging should be called a $\overline{\nabla}$-statistical rigging, but we call it Levi–Civita statistical rigging to emphasize that we are dealing with the trivial statistical structure given by the ambient Levi–Civita connection.

Table 1 Conditions for $(\widetilde{g}, \nabla)$ to be a statistical structure

Conditions for all $X, Y \in \Gamma(\mathcal{S})$	The statistical structure $(\widetilde{g}, \nabla)$ on Σ is	The rigging ζ for Σ is called
$B(X,Y) = C(X,Y)$	Combination of (ξ, E)-statistical and ξ-statistical	Levi–Civita statistical rigging
$C(\xi, X) = -2\tau(X)$		
$B(X,Y) = C(X,Y)$	ξ'-statistical where $\xi' = \sqrt[3]{\tau(\xi)}\xi$	Strong Levi–Civita statistical rigging
$C(\xi, X) = \tau(X) = 0$		
$B(X,Y) = C(X,Y)$	Self-dual or trivial	Preferred rigging
$C(\xi, X) = \tau(X) = 0$		
$\tau(\xi) = 0$		

Corollary 2 *Let (M, g) be a semi-Riemannian manifold and Σ a null submanifold. A rigging ζ for Σ is a Levi–Civita statistical rigging if and only if*

$$B(X,Y) = C(X,Y), \tag{53}$$

$$C(\xi, X) = -2\tau(X) \tag{54}$$

for all $X, Y \in \Gamma(\mathcal{S})$. Moreover, in this case, the Tchebychev one-form is $-\tau$ and the difference tensor $K = \nabla - \widetilde{\nabla}$ is given by

$$K = 2\tau(\xi)\omega \otimes \omega\xi - \mathfrak{s}(\omega \otimes \tau)\xi - \omega \otimes \omega E, \tag{55}$$

where $E \in \mathfrak{X}(\Sigma)$ is the $\widetilde{g}$-metrically equivalent vector field to τ.

Proof It immediately follows taking into account that in this case $\bar{K} = 0$, $B^s = B^d = B$, $\tau^s = \tau$ and $A^{*s} = A^*$. □

Observe that the statistical structure obtained in the above corollary is a combination of a ξ-statistical structure and a (ξ, E)-statistical structure.

Definition 5 Let (M, g) be a semi-Riemannian manifold and Σ a null submanifold. If ζ is a rigging for Σ such that

$$B(X,Y) = C(X,Y),$$
$$C(\xi, X) = \tau(X) = 0$$

for all $X \in \Gamma(\mathcal{S})$, then we say that ζ is a strong Levi–Civita statistical rigging.

If ζ is a strong Levi–Civita statistical rigging, then $\tau = \tau(\xi)\omega$ and $E = \tau(\xi)\xi$. Thus, the statistical structure $(\widetilde{g}, \nabla)$ given in Corollary 2 turns into a ξ'-statistical structure, where $\xi' = \sqrt[3]{\tau(\xi)}\xi$. On the other hand, from [16, Theorem 4.2] if $B(U, X) = C(U, X)$ and $\tau(U) = 0$ for all $U \in \mathfrak{X}(\Sigma)$ and $X \in \Gamma(\mathcal{S})$, then the statistical structure $(\widetilde{g}, \nabla)$ is trivial, i.e. $\widetilde{\nabla} = \nabla$. In this case ζ was called in [16] a preferred rigging for Σ. We summarize all this in the Table 1.

Example 1 Let (M_0, g_0) be a Riemannian manifold and consider the Lorentzian manifold

$$(M, g) = \left(\mathbb{R}^2 \times M_0, 2e^{u-v}dudv + \left(e^u + e^v\right)^2 g_0\right).$$

For a fixed $u_0 \in \mathbb{R}$ we have that $\Sigma = \{(u_0, v, x) : v \in \mathbb{R}, x \in M_0\}$ is a null hypersurface and $\zeta = e^{v-u}\partial u$ is a rigging for it. The rigged vector field is $\xi = \partial v$ and the null transverse vector field is $N = \zeta$. Moreover, the screen distribution at a point (u_0, v, x) can be identified with $\mathcal{L}(TM_0)$, the lift of TM_0.

Using the formulas for the Levi–Civita connection of a warped product we can check that

$$\begin{aligned}\overline{\nabla}_X Y &= -\frac{1}{e^{u-v}+1} g(X,Y)N - \frac{1}{e^{u-v}+1} g(X,Y)\xi + \nabla^0_X Y,\\ \overline{\nabla}_\xi X &= \frac{e^v}{e^u+e^v} X\end{aligned}$$

where ∇^0 is the Levi–Civita connection in (M_0, g_0) and $X, Y \in \mathcal{L}(TM_0)$. Therefore,

$$\begin{aligned}B(X,Y) &= -\frac{1}{e^{u-v}+1} g(X,Y),\\ C(X,Y) &= -\frac{1}{e^{u-v}+1} g(X,Y),\\ C(\xi, X) &= 0,\\ \tau(X) &= 0\end{aligned}$$

for all $X, Y \in \Gamma(\mathcal{S})$. Moreover, $\overline{\nabla}_\xi \xi = -\xi$, so $\tau(\xi) = 1$. Thus, ζ is a strong Levi–Civita statistical rigging for Σ which is not a preferred rigging.

Example 2 Let (M_0, g_0) be a Riemannian manifold and $f \in C^\infty(M_0)$ a positive function with $g_0(\nabla^0 f, \nabla^0 f) = 1$. Consider the standard static space $(M, g) = \left(M_0 \times \mathbb{R}, g_0 - f^2(x)dt^2\right)$ and $\Sigma = \{(x, \ln f(x)) \in M_0 \times \mathbb{R} : x \in M_0\}$. We can check that Σ is a null hypersurface, $\zeta = \frac{\sqrt{2}}{f}\partial t$ is a rigging for it with associated rigged vector field

$$\xi = -\frac{1}{\sqrt{2}f}\partial t - \frac{1}{\sqrt{2}}\nabla^0 f$$

and the screen distribution is given by $\mathcal{S} = \{X \in TM_0 : X(f) = 0\}$.

On the other hand, we have that ∂t is Killing and orthogonally closed in (M, g). Thus, it holds

$$\overline{\nabla}_X \partial t = \frac{X(f)}{f}\partial t$$

for all $X \in \mathfrak{X}(M_0)$. From Eqs. (10) and (8) we get

$$\begin{aligned}C(X,Y) &= -g(\overline{\nabla}_X \zeta, Y) - \frac{1}{2} g(\zeta,\zeta) B(X,Y) = B(X,Y),\\ -2C(\xi, X) &= 2g(\overline{\nabla}_\xi \zeta, X) = 0,\\ \tau(X) &= 0\end{aligned}$$

for all $X \in \Gamma(\mathcal{S})$. Therefore, ζ is a strong Levi–Civita statistical rigging for Σ.

If we assume some additional conditions on a Levi–Civita statistical rigging, then we get that it is in fact a preferred or a strong Levi–Civita statistical rigging.

Corollary 3 *Let (M, g) be a semi-Riemannian manifold and Σ a null submanifold. Suppose that ζ is a Levi–Civita statistical rigging for Σ.*

1. *If ζ is closed, then it is a strong Levi–Civita statistical rigging.*

2. *If ζ is conformal, then it is a preferred rigging.*

Proof 1. From Eqs. (8) and (10) we get that

$$-2C(\xi, X) = 2g(\overline{\nabla}_{\xi}\zeta, X) = 2g(\overline{\nabla}_{X}\zeta, \xi) = 2\tau(X)$$

for all $X \in \Gamma(\mathcal{S})$. Since ζ is a Levi–Civita rigging, we have that $C(\xi, X) = -2\tau(X)$ and thus $C(\xi, X) = \tau(X) = 0$. This means that ζ is a strong Levi–Civita statistical rigging.

2. As before, from Eqs. (8) and (10) we have

$$-2C(\xi, X) = 2g(\overline{\nabla}_{\xi}, X) = -2g(\overline{\nabla}_{X}\zeta, \xi) = -2\tau(X)$$

for all $X \in \Gamma(\mathcal{S})$, but being ζ a Levi–Civita statistical rigging, it follows that $\tau(X) = C(\xi, X) = 0$. Moreover, $\tau(\xi) = g(\overline{\nabla}_{\xi}\zeta, \xi) = 0$. Thus ζ is a preferred rigging.

□

The following corollary shows a relationship between geometric conditions on a null submanifold and the statistical structure constructed from a Levi–Civita statistical rigging.

Corollary 4 *Let (M, g) be a semi-Riemannian manifold and Σ a null submanifold. Suppose that ζ is a Levi–Civita statistical rigging for Σ.*

1. *If $\tau(\xi) = 0$ but τ is not identically zero at any point and the statistical structure $(\widetilde{g}, \nabla)$ is conjugate symmetric, then the multiplicity of the 0-principal curvature is at least* $\dim \Sigma - 2$.
2. *If the metrically equivalent one form to ζ is closed, $\tau(\xi)$ is a nonzero constant and Σ is totally geodesic, then the statistical structure $(\widetilde{g}, \nabla)$ is conjugate symmetric.*
3. *If ζ is a strong Levi–Civita statistical rigging, $\tau(\xi)$ is a nonzero constant and the statistical structure $(\widetilde{g}, \nabla)$ is conjugate symmetric, then Σ is totally geodesic.*

Proof 1. Since $\tau(\xi) = 0$, the statistical structure in Corollary 2 is a (ξ, E)-statistical structure, being E the $\widetilde{g}$-metrically equivalent vector field to τ. Observe that $\tau(\xi) = 0$ implies that $\widetilde{g}(E, \xi) = 0$. Moreover, $\widetilde{g}(E, E) \neq 0$ because τ is not identically zero at any point. Theorem 1 ensures that $(\Sigma, \widetilde{g})$ locally decomposes as a twisted product $L \times_h S$, where S is a leaf of $\mathcal{D} = span(\xi, E)$ and L a leaf of $\mathcal{D}^{\perp_{\widetilde{g}}}$. Using formula (16) and the formulas for the Levi–Civita connection of a twisted product, we get

$$B(X, Y) = -\frac{1}{2}\left(L_{\xi}\widetilde{g}\right)(X, Y) = 0$$

for all $X, Y \in span(\xi, E)^{\perp_{\widetilde{g}}}$. Therefore, if we take an orthonormal basis

$$\left\{X_1, \ldots, X_{k-2}, \frac{1}{\sqrt{|\tau(E)|}}E\right\}$$

of $\mathcal{S}$, being $k = \dim \Sigma$, then the associated matrix of $A^* : \mathcal{S} \to \mathcal{S}$ is of the form

$$\begin{pmatrix} 0 & 0 & \ldots & \gamma_1 \\ 0 & 0 & \ldots & \gamma_2 \\ \vdots & \vdots & & \vdots \\ \gamma_1 & \gamma_2 & \ldots & \gamma_{k-1} \end{pmatrix}.$$

Thus, the multiplicity of the 0 eigenvalue of $A^* : T\Sigma \to T\Sigma$ is at least $k - 2$.

2. If Σ is totally geodesic, then the Eq. (16) gives us that $\left(L_\xi \widetilde{g}\right)(X, Y) = 0$ for all $X, Y \in \Gamma(\mathcal{S})$. Since the metrically equivalent one form to ζ is closed, we have that the rigged one form ω is also closed, $d\omega = 0$. Hence, ξ is $\widetilde{g}$-parallel. Moreover, from Corollary 3, ζ is a strong statistical rigging and the statistical structure $(\widetilde{g}, \nabla)$ is conjugate symmetric due to Proposition 1.
3. From Proposition 1 we know that ξ is $\widetilde{g}$-parallel. Equation (16) implies that Σ is totally geodesic.

□

As we said before, if we have a statistical connection $\bar{\nabla}^s$ on a semi-Riemannian manifold and a null submanifold Σ, then $(\widetilde{g}, \nabla^d)$ does not need to be a statistical structure on Σ, even if $(\widetilde{g}, \nabla^s)$ is a statistical structure on Σ. The following corollary gives us conditions for this to happen.

Corollary 5 *Let (M, g) be a semi-Riemannian manifold furnished with a statistical connection $\bar{\nabla}^s$ and Σ a null hypersurface. Fixed a rigging, the following properties are equivalent.*

- *ζ is a $\bar{\nabla}^s$-statistical and a $\bar{\nabla}^d$-statistical rigging simultaneously, i.e. both $(\widetilde{g}, \nabla^s)$ and $(\widetilde{g}, \nabla^d)$ are statistical structures on Σ.*
- *ζ is a Levi–Civita statistical rigging and $\bar{K}(U, \xi - N) \in span(\xi, N)$ for all $U \in \mathfrak{X}(\Sigma)$.*

Proof Suppose that $(\widetilde{g}, \nabla^s)$ and $(\widetilde{g}, \nabla^d)$ are statistical structures on Σ. From Theorem 2 it holds

$$B^s(X, Y) = C^s(X, Y), \tag{56}$$
$$B^s(\xi, X) + C^s(\xi, X) = -2\tau^s(X), \tag{57}$$
$$B^d(X, Y) = C^d(X, Y), \tag{58}$$
$$B^d(\xi, X) + C^d(\xi, X) = -2\tau^d(X). \tag{59}$$

for all $X, Y \in \Gamma(\mathcal{S})$. From Eqs. (56), (58) and (37) we get that $B(X, Y) = C(X, Y)$ and from Eqs. (57), (59), (38) and (39) we get that $C(\xi, X) = -2\tau(X)$. Applying Corollary 2 we have that ζ is a Levi–Civita statistical rigging for Σ. Moreover, from Eqs. (42), (44) and (46) the Eqs. (56) and (57) can be written as

$$g(\bar{K}(X, Y), \xi) = g(\bar{K}(X, Y), N),$$
$$g(\bar{K}(\xi, X), \xi) + g(\bar{K}(\xi, X), N) = 2g(\bar{K}(\xi, X), N)$$

for all $X \in \Gamma(\mathcal{S})$. In other words, $g(\bar{K}(U, \xi - N), X) = 0$ for all $U \in \mathfrak{X}(\Sigma)$ and $X \in \Gamma(\mathcal{S})$, which is equivalent to $\bar{K}(U, \xi - N) \in span(\xi, N)$ for all $U \in \mathfrak{X}(\Sigma)$.

Conversely, suppose that ζ is a Levi–Civita statistical rigging and $\bar{K}(U, \xi - N) \in span(\xi, N)$ for all $U \in \mathfrak{X}(\Sigma)$. Using Eqs. (42–47) we can easily check that Eqs. (56–59) hold and thus ζ is a $\bar{\nabla}^s$-statistical and a $\bar{\nabla}^d$-statistical rigging. □

Example 3 Let (M_0, g_0) be a Riemannian manifold and consider the Lorentzian direct product $(M, g) = \left(\mathbb{R} \times M_0, -dt^2 + g_0\right)$. From Eqs. (8) and (10) we have that $\zeta = \sqrt{2}\partial t$ is a preferred rigging for any null hypersurface Σ in M. In particular, it is also a Levi–Civita statistical rigging for any null hypersurface. Moreover, Σ is locally given by the graph of a function $h : \theta \subset M_0 \to \mathbb{R}$ with $||\nabla h||_{M_0} = 1$ and

$$\xi = -\frac{1}{\sqrt{2}}\partial t - \frac{1}{\sqrt{2}}\nabla^{M_0} h,$$

$$N = \frac{1}{\sqrt{2}}\partial t - \frac{1}{\sqrt{2}}\nabla^{M_0} h.$$

Suppose that $\bar{\nabla}^s$ is a statistical connection on M. Using the above corollary, the induced connections ∇^s and ∇^d on Σ are statistical connections on Σ respect to the rigged metric if and only if $\bar{K}(U, \partial t) \in span\left(\partial t, \nabla^{M_0} h\right)$ for all $U \in \mathfrak{X}(\Sigma)$.

5 Screen strongly convex null hypersurfaces and statistical structures

Strongly convex hypersurfaces in the Euclidean space carry a natural statistical structure. We recall some basic facts, [18]. Let M be a manifold and $\overline{\nabla}$ an affine connection without torsion on M. Given a hypersurface Σ, we fix a transversal vector field ζ, which gives us the decomposition $T_x M = T_x\Sigma \oplus span(\zeta_x)$ for all $x \in \Sigma$. Using this, we can decompose

$$\overline{\nabla}_U V = \nabla^{\zeta}_U V + h^{\zeta}(U, V)\zeta,$$
$$\overline{\nabla}_U \zeta = -S^{\zeta}(U) + \tau^{\zeta}(U)\zeta$$

for all $U, V \in \mathfrak{X}(\Sigma)$. We have that ∇^{ζ} is an affine connection without torsion on Σ, h^{ζ} is a symmetric tensor called affine second fundamental form and S^{ζ} is called affine shape operator. If we consider $\bar{R}$ the curvature tensor of $\overline{\nabla}$ and R^{ζ} the curvature tensor of ∇^{ζ}, then we can check that

$$\left(\bar{R}_{UV} W\right)^{T\Sigma} = R^{\zeta}_{UV} W + S^{\zeta}(V)h^{\zeta}(U, W) - S^{\zeta}(U)h^{\zeta}(V, W), \tag{60}$$

$$\left(\bar{R}_{UV} W\right)^{span(\zeta)} = \Big(\left(\nabla^{\zeta}_U h^{\zeta}\right)(V, W) - \left(\nabla^{\zeta}_V h^{\zeta}\right)(U, W) + \tau^{\zeta}(U)h^{\zeta}(V, W) - \tau^{\zeta}(V)h^{\zeta}(U, W)\Big)\zeta \tag{61}$$

for all $U, V, W \in \mathfrak{X}(\Sigma)$, where $\left(\bar{R}_{UV} W\right)^{T\Sigma}$ denotes the projection onto $T\Sigma$ and $\left(\bar{R}_{UV} W\right)^{span(\zeta)}$ the projection onto $span(\zeta)$.

We say that Σ is strongly convex if $h^{\zeta}(U, U) \neq 0$ for all $U \in \mathfrak{X}(\Sigma)$. This property does not depend on the chosen transversal vector field. Moreover, changing the sign of ζ if necessary, we can suppose that h^{ζ} defines a Riemannian metric on Σ, which is called the Blaschke metric. If $\overline{\nabla}_U \zeta$ is tangent to Σ for all $U \in \mathfrak{X}(\Sigma)$, then we say that ζ is equiaffine.

Now we particularize to the case $\left(\mathbb{R}^n, \overline{\nabla}\right)$, where $\overline{\nabla}$ is the standard flat connection on $\mathbb{R}^n$. If Σ is a strongly convex hypersurface and ζ is equiaffine, then from Eq. (61) we have that $\left(h^{\zeta}, \nabla^{\zeta}\right)$ is a statistical structure on Σ, which turns out to be always locally equiaffine. The classical Maschke–Pick–Berwald theorem asserts that if this statistical structure is trivial, then Σ is a hyperquadric in $\mathbb{R}^n$, [18, Theorem 4.5, p. 53]. Moreover, it can be shown that this statistical structure is conjugate symmetric if and only if Σ is an equiaffine sphere, [20, Lemma 12.5]. Observe that the unitary and normal vector field to Σ (respect to the standard Euclidean metric) is equiaffine.

Now, we consider the Lorentz–Minkowski space

$$\mathbb{L}^n = \left(\mathbb{R} \times \mathbb{R}^{n-1}, -dt^2 + dx_1^2 + \ldots + dx_{n-1}^2\right).$$

The same construction as before works for timelike or spacelike hypersurfaces in $\mathbb{L}^n$, providing a statistical structure on them. However, it does not work for a null hypersurface, because the null second fundamental form holds $B(\xi, U) = 0$ for all $U \in \mathfrak{X}(\Sigma)$, i.e. it is always degenerate.

Observe that the family of null hypersurfaces in $\mathbb{L}^n$ is large enough to deserve the effort in generalizing the above construction. If we have $\theta \subset \mathbb{R}^{n-1}$ an open set and $d : \theta \to \mathbb{R}$ a function with $||\overline{\nabla} d|| = 1$, then $\{(d(x), x) \in \mathbb{L}^n : x \in \theta\}$ is a null hypersurface in $\mathbb{L}^n$. Conversely, any null hypersurface in $\mathbb{L}^n$ can be locally described in this way. On the other hand, such a function d can be obtained as the signed distance function from an arbitrary hypersurface $S \subset \mathbb{R}^{n-1}$, [7, 17].

Nevertheless, despite of the degeneracy of B, we can do a similar construction for null hypersurfaces following the spirit of the rigging technique.

Definition 6 Let (M, g) be a semi-Riemannian manifold, Σ a null hypersurface and ζ a rigging for it. We say that Σ is screen strongly convex if $B(v, v) \neq 0$ for all $v \in \mathcal{S}_x$ with $v \neq 0$ and all $x \in \Sigma$.

The above definition does not depend on the chosen rigging. Indeed, if ζ' is another rigging for Σ, then $\xi' = \frac{1}{\Phi}\xi$ and $B' = \frac{1}{\Phi}B$, where $\Phi = g(\zeta', \xi)$. Therefore, if $w \in \mathcal{S}'_x$, then we can decompose it as $w = v + \omega(w)\xi$, where $v \in \mathcal{S}_x$ is nonzero, and thus $B'(w, w) = \frac{1}{\Phi}B(v, v) \neq 0$. Moreover, observe that being screen strongly convex trivially implies that the null mean curvature never vanishes.

Now we can construct a Riemannian metric from B as we did with g. Fix a rigging ζ for Σ and consider

$$B^s = B + \omega \otimes \omega.$$

Observe that B^s is the second fundamental form induced on Σ from the ζ-statistical connection $\bar{\nabla}^s = \overline{\nabla} + \eta \otimes \eta\zeta$, being η the metrically equivalent one form to ζ. If Σ is screen strongly convex, then, changing the sign of ζ if necessary, B^s defines a Riemannian metric on Σ, which will be called the rigged-Blaschke metric.

Theorem 3 *Let Σ be a screen strongly convex null hypersurface in the Lorentz–Minkowski space $\mathbb{L}^n$. Fix a rigging for Σ such that B^s is Riemannian and consider the induced connection ∇ on Σ from the standard flat connection on $\mathbb{L}^n$. We have that (B^s, ∇) is a statistical structure on Σ if and only if*

$$C(U, X) = \tau(\xi)B(U, X), \tag{62}$$

$$\tau(X) = 0 \tag{63}$$

for all $U \in \mathfrak{X}(\Sigma)$ and $X \in \Gamma(\mathcal{S})$.

Moreover, in this case, $d\tau = 0$, the screen distribution $\mathcal{S}$ is integrable, $X(\tau(\xi)) = 0$ for all $X \in \Gamma(\mathcal{S})$ and the Tchebychev one-form of the statistical structure (B^s, ∇) is

$$\alpha = -\frac{1}{2} d \ln |H| - \tau = -\frac{1}{2} d \ln |H| - \tau(\xi)\omega,$$

where H is the null mean curvature. In particular, the statistical structure (B^s, ∇) is locally equiaffine.

Proof From Eqs. (11) and (50) we have

$$\begin{aligned}\left(\nabla_U B^s\right)(V, W) - \left(\nabla_V B^s\right)(U, W) = &\ \tau(V)B(U, W) - \tau(U)B(V, W)\\ &+ \big(C(V, \mathcal{P}_{\mathcal{S}}(U)) - C(U, \mathcal{P}_{\mathcal{S}}(V))\big)\omega(W)\\ &+ 2\big(\tau(U)\omega(V) - \tau(V)\omega(U)\big)\omega(W)\\ &+ C(V, \mathcal{P}_{\mathcal{S}}(W))\omega(U) - C(U, \mathcal{P}_{\mathcal{S}}(W))\omega(V).\end{aligned} \tag{64}$$

Thus, if Eqs. (62) and (63) hold, then $(\nabla_U B^s)(V, W) = (\nabla_V B^s)(U, W)$ for all $U, V, W \in \mathfrak{X}(\Sigma)$ and thus (B^s, ∇) is a statistical structure on Σ.

Conversely, suppose that (B^s, ∇) is a statistical structure on Σ. Since Σ is screen strongly convex, then for each point $x \in \Sigma$ we can take $\{e_1, \ldots, e_{n-2}\}$ a basis of $\mathcal{S}_x$ such that $B(e_i, e_j) = \delta_{ij}$. If we take $U = e_i$, $V = e_j$, $W = e_j$ in Eq. (64), then we get $\tau(e_i) = 0$ and so $\tau(X) = 0$ for all $X \in \Gamma(\mathcal{S})$. On the other hand, if we take $U = \xi$ and $V = Y$, $W = Z \in \Gamma(\mathcal{S})$ in Eq. (64), then we get $C(Y, Z) = \tau(\xi)B(Y, Z)$. Finally, if we take $U = X \in \Gamma(\mathcal{S})$ and $V = W = \xi$, then we obtain that $C(\xi, X) = 0$ and therefore Eqs. (62) and (63) hold.

We prove the rest of the statements. From Eqs. (7), (15) and (62) it follows that $d\tau = 0$. From Eq. (62) it is clear that $\mathcal{S}$ is integrable. On the other hand, using Eq. (62) we have

$$\left(\nabla^{**}_U C\right)(V, X) = U(\tau(\xi))B(V, X) + \tau(\xi)\left(\nabla_U B\right)(V, X)$$

and so from Eqs. (11) and (14), we get

$$U(\tau(\xi))B(V, X) - V(\tau(\xi))B(U, X) = 2\tau(\xi)\left(\tau(U)B(V, X) - \tau(V)B(U, X)\right)$$

for all $U, V \in \mathfrak{X}(\Sigma)$ and $X \in \Gamma(\mathcal{S})$. If we take $V = X$ and $U = Y \in \Gamma(\mathcal{S})$ with $B(X, X) \neq 0$ and $B(X, Y) = 0$, then we get $Y(\tau(\xi)) = 0$.

Now we compute the Tchebychev one-form of the statistical structure (B^s, ∇). Fix a point $x \in \Sigma$ and a vector $u \in T_x\Sigma$. Since $A^* : \mathcal{S} \to \mathcal{S}$ is self-adjoint respect to g and $\mathcal{S}$ is spacelike, it exists a g-orthonormal basis $\{e_1, \ldots, e_{n-2}\}$ of $\mathcal{S}_x$ such that $A^*(e_i) = \lambda_i e_i$. Moreover, since we are assuming that B^s is Riemannian, then $\lambda_i > 0$. If we call $v_i = \frac{1}{\sqrt{\lambda_i}} e_i$, then $\{v_1, \ldots, v_{n-2}, \xi\}$ is a B^s-orthonormal basis of $T_x\Sigma$ and so

$$\alpha(u) = B^s(K(\xi, \xi), u) + \sum_{i=1}^{n-2} B^s(K(v_i, v_i), u),$$

where K is the difference tensor. But from Eq. (20) we can write

$$-2\alpha(u) = \left(\nabla_u B^s\right)(\xi, \xi) + \sum_{i=1}^{n-2} \left(\nabla_u B^s\right)(v_i, v_i).$$

Now, taking into account Eq. (6), we have

$$\left(\nabla_u B^s\right)(\xi, \xi) = u(B^s(\xi, \xi)) - 2B^s(\nabla_u \xi, \xi) = -2\omega(\nabla_u \xi) = 2\tau(u).$$

To compute the last term, we extend each v_i to a vector field V_i defined in a neighbourhood of x in Σ such that $V_i \in \Gamma(\mathcal{S})$, $g(V_i, V_j) = 0$ for $i \neq j$ and $g(V_i, V_i)$ is constant and equal to $\frac{1}{\lambda_i(x)}$. This can be done taking a local basis which spans $\mathcal{S}$ and applying Gram-Schmidt. We have

$$\begin{aligned}
\left(\nabla_u B^s\right)(v_i, v_i) &= u(B^s(V_i, V_i)) - 2B^s(\nabla_u V_i, v_i) \\
&= u(B(V_i, V_i)) - 2g(\nabla_u V_i, A^*(v_i)) \\
&= \frac{1}{\lambda_i(x)} u\left(B\left(\sqrt{\lambda_i(x)}V_i, \sqrt{\lambda_i(x)}V_i\right) - 2\lambda_i(x) g(\nabla_u V_i, v_i)\right).
\end{aligned}$$

Since $g(V_i, V_i)$ is constant, we have $g(\nabla_u V_i, v_i) = 0$. On the other hand, we know that $\{\sqrt{\lambda_1(x)}V_1, \ldots, \sqrt{\lambda_{n-2}}V_{n-2}\}$ is a g-orthonormal basis of $\mathcal{S}$. Hence the above is $\frac{u(H)}{H_x}$ (recall that x is fixed) and thus

$$\alpha(u) = -\frac{1}{2}\frac{u(H)}{H} - \tau(u).$$

Since $\tau(X) = 0$ for all $X \in \Gamma(\mathcal{S})$, we have that $\tau = \tau(\xi)\omega$ and we get the result. □

Remark 2 Under the conditions of the above theorem, since $\tau = \tau(\xi)\omega$ and $d(\tau(\xi)) = d(\tau(\xi))(\xi)\omega$, if there is $x \in \Sigma$ with $\tau(\xi)_x \neq 0$, then $d\omega_x = 0$. There are important consequences from the fact $d\omega = 0$. For example, an explicit expression for the rigged connection $\widetilde{\nabla}$ in terms of the induced connection ∇ and the tensors B, C and τ can be given in this case, [16, Proposition 2.10]. Moreover, if Σ is totally umbilical, then $(\Sigma, \widetilde{g})$ can be decomposed as a twisted product, [6, Theorem 5.3].

If we take a parallel null rigging for Σ (which always exists at least locally), then from Eqs. (8) and (10) we have $C = \tau = 0$ and the above theorem ensures that (B^s, ∇) is a statistical structure. We can prove that it also exists a timelike rigging such that (B^s, ∇) is a statistical structure with the additional property that $\tau(\xi)$ never vanishes.

Lemma 3 *Let Σ be a null hypersurface in the Lorentz–Minkowski space $\mathbb{L}^n$. For each point of Σ there is a locally defined timelike rigging for Σ such that*

$$C(U, X) = \tau(\xi)B(U, X),$$
$$\tau(X) = 0$$

for all $U \in \mathfrak{X}(\Sigma)$ and $X \in \Gamma(\mathcal{S})$ and $\tau(\xi)_x \neq 0$ for all $x \in \Sigma$. In particular, $d\omega = 0$.

Proof We can suppose without loss of generality that Σ passes through the origin of coordinates. Take $\zeta = \Phi\sqrt{2}\partial t$, where t is the timelike coordinate in $\mathbb{L}^n$ and Φ is the solution to the differential equation

$$\Phi'(t) = -\sqrt{2}\Phi(t)^4 \tag{65}$$

with $\Phi(0) = 1$.

From Eqs. (8) and (10) we have that $C(U, X) = \Phi^2 B(U, X)$ and $\tau(X) = 0$ for all $U \in \mathfrak{X}(\Sigma)$ and $X \in \Gamma(\mathcal{S})$. Moreover,

$$\tau(\xi) = \sqrt{2}\xi(\Phi)g(\partial t, \xi) = \frac{1}{\Phi}\xi(\Phi) = -\frac{1}{\sqrt{2}\Phi^2}\Phi' = \Phi^2.$$

Thus $\tau(\xi)_x \neq 0$ for all $x \in \Sigma$ and from Remark 2 we also have $d\omega = 0$. □

The solution to the differential Eq. (65) with the initial condition $\Phi(0) = c \neq 0$ is

$$\Phi(t) = \sqrt[3]{\frac{1}{\frac{1}{c^3} + 3\sqrt{2}t}},$$

which is not defined for $t = -\frac{1}{3\sqrt{2}c^3}$. Therefore, the rigging constructed in Lemma 3 is defined globally for null hypersurfaces in $\mathbb{L}^n$ which are lower or upper timelike bounded, i.e. they are contained in $\{(t, x_1, \ldots, x_{n-1}) \in \mathbb{L}^n : t < K\}$ or $\{(t, x_1, \ldots, x_{n-1}) \in \mathbb{L}^n : t > K\}$ for some constant K. Observe that being screen strongly convex does not ensure that the null hypersurface is lower or upper timelike bounded, as the following example shows.

Example 4 Consider Σ the parametrized surface $X : A \to \mathbb{L}^3$ given by

$$X(u, v) = \left(v, u - \frac{2uv}{\sqrt{1 + 4u^2}}, u^2 + \frac{v}{\sqrt{1 + 4u^2}}\right),$$

where $A = \{(u, v) \in \mathbb{R}^2 : 2v < \sqrt{(1+4u^2)^3}\}$. We can check that $g(X_u, X_v) = g(X_v, X_v) = 0$, so it is a null surface and $\xi = X_v$ is a null tangent vector field to it. Obviously is not upper neither lower timelike bounded. Moreover, it is screen strongly convex since

$$\begin{aligned} B(X_u, X_u) &= -g(\nabla_{X_u} X_v, X_u) = -\frac{1}{2}\frac{d}{dv} g(X_u, X_u) \\ &= \frac{2\left(\sqrt{(1+4u^2)^3} - 2v\right)}{(1+4u^2)^2} > 0. \end{aligned}$$

The most simple example of null hypersurfaces in $\mathbb{L}^n$ are the degenerate hyperplanes through the origin. These are in fact the unique totally geodesic null hypersurfaces in $\mathbb{L}^n$. On the other hand, the future null cone and the past null cone are given by

$$\mathcal{C}^+ = \left\{(t, x_1, \ldots, x_{n-1}) \in \mathbb{L}^n : t^2 = x_1^2 + \ldots + x_{n-1}^2, t > 0\right\}$$

and

$$\mathcal{C}^- = \left\{(t, x_1, \ldots, x_{n-1}) \in \mathbb{L}^n : t^2 = x_1^2 + \ldots + x_{n-1}^2, t < 0\right\}$$

respectively. Given $p_0 \in \mathbb{L}^n$ we call future null cone with vertex p_0 to $\mathcal{C}^+_{p_0} = p_0 + \mathcal{C}^+$ and analogously for $\mathcal{C}^-_{p_0}$. It is well-known that $\mathcal{C}^+_{p_0}$ and $\mathcal{C}^-_{p_0}$ are totally umbilical null hypersurfaces. Conversely, if $n \geq 4$, a totally umbilical null hypersurface in $\mathbb{L}^n$ is totally geodesic or it is contained in a null cone, see for example [7, Theorem 4.15]. In [10, 11] a characterization of totally umbilical null hypersurfaces as a null cone is given for more general ambient manifolds than the Lorentz–Minkowski space.

Proposition 8 *Let Σ be a screen strongly convex null hypersurface in the Minkowski space $\mathbb{L}^n$ and consider a rigging such that*

$$\begin{aligned} C(U, X) &= \tau(\xi) B(U, X), \\ \tau(X) &= 0, \\ \tau(\xi)_x &\neq 0 \end{aligned}$$

for all $U \in \mathfrak{X}(\Sigma)$, $X \in \Gamma(\mathcal{S})$ and $x \in \Sigma$.

1. *If (B^s, ∇) is conjugate symmetric, then Σ is contained in a null cone.*
2. *(B^s, ∇) is never trivial.*

Proof Suppose that the statistical structure (B^s, ∇) on Σ is conjugate symmetric. In this case, from Eq. (21) we get $B^s(R_{XY}Y, Y) = 0$. Using Eqs. (12) and (13) we have that

$$R_{UV}W = B(U, W)A(V) - B(V, W)A(U)$$

and so the above equation reads

$$B(X, Y)B(A(Y), Y) = B(Y, Y)B(A(X), Y)$$

for all $X, Y \in \Gamma(\mathcal{S})$. Take $\{e_1, \ldots, e_{n-2}\}$ a B^s-orthonormal basis of $\mathcal{S}_x$, i.e. $B(e_i, e_j) = \delta_{ij}$. If we set $X = e_i$ and $Y = e_j$ for $i \neq j$ in the above equation, then $B(A(e_i), e_j) = 0$ and so $A(e_i) = \alpha_i e_i$ for some $\alpha_i \in \mathbb{R}$. From this, it follows that $A(X) = \mu X$ for some $\mu \in C^\infty(\Sigma)$ and all $X \in \Gamma(\mathcal{S})$ and thus $B = \rho g$, being $\rho = \frac{\mu}{\tau(\xi)}$. Since Σ is screen strongly convex, then it can not be totally geodesic and thus, applying [7, Theorem 4.15], Σ is contained in a null cone.

Suppose now that (B^s, ∇) is trivial. In particular, it is conjugate symmetric and so Σ is totally umbilical. Moreover, it holds $\nabla B^s = 0$, but we have

$$\begin{aligned}\left(\nabla_U B^s\right)(V, W) &= U(\rho)g(V, W) + \rho\,(\nabla_U g)(V, W) + (\nabla_U \omega \otimes \omega)(V, W)\\ &= U(\rho)g(V, W) + \rho(\rho - \tau(\xi))\big(\omega(W)g(U, V) + \omega(V)g(U, W)\big)\\ &\quad + 2\tau(\xi)\omega(U)\omega(V)\omega(W)\end{aligned}$$

for all $U, V, W \in \mathfrak{X}(\Sigma)$. If we take $U = V = W = \xi$, then we have $\tau(\xi) = 0$, which is a contradiction. □

Example 5 The most simple screen strongly convex null hypersurfaces in $\mathbb{L}^n$ are the totally umbilical but no totally geodesic ones, i.e. the null cones. We show how looks the statistical structure (B^s, ∇) in this case.

Take a parallel null vector field ζ, which will be a rigging for a piece of Σ. We know that $C = \tau = 0$ and $B = \rho g$ for certain $\rho \in C^\infty(\Sigma)$. Therefore, from Eqs. (20) and (9) we have

$$-2B^s(K(U, V), W) = \frac{U(\rho)}{\rho}B(V, W) + \rho^2\left(g(U, V)\omega(W) + g(U, W)\omega(V)\right)$$

for all $U, V, W \in \mathfrak{X}(\Sigma)$. If we take $U = X, V = Y, W = Z \in \Gamma(\mathcal{S})$ in the above equation, then we get $B^s(K(X, Y), Z) = -\frac{X(\rho)}{2\rho}B^s(Y, Z)$ and if we take $U = X, V = Y \in \Gamma(\mathcal{S})$ and $W = \xi$, then we arrive to $B^s(K(X, Y), \xi) = -\frac{\rho^2}{2}g(X, Y) = -\frac{\rho}{2}B^s(X, Y)$. Therefore, since B^s is non-degenerate, we get

$$K(X, Y) = -\frac{X(\rho)}{2\rho}Y - \frac{\rho^2}{2}g(X, Y)\xi.$$

Since K is symmetric, it holds $X(\rho) = 0$ and thus

$$K(X, Y) = -\frac{\rho^2}{2}g(X, Y)\xi$$

for all $X, Y \in \Gamma(\mathcal{S})$. In an analogous way we can check that $K(X, \xi) = -\frac{\rho}{2}X$, $K(\xi, \xi) = 0$ and $\xi(\rho) = \rho^2$. Therefore,

$$K(U, V) = -\frac{\rho}{2}\left(B^s(U, V)\xi + \omega(U)V + \omega(V)U\right) + \frac{3\rho}{2}\omega \otimes \omega\xi,$$

which is a combination of a ξ-statistical structure and the statistical structure given in (25) (recall that here B^s plays the role of the metric and the B^s-metrically equivalent one-form to ξ is ω).

Now, suppose that we take a timelike rigging as in Lemma 3. We want to compute the difference tensor for the statistical structure (B^s, ∇) in this case. As before, we have

$$\begin{aligned}-2B^s(K(U, V), W) &= U(\rho)g(V, W) + \rho(\rho - \tau(\xi))\big(g(U, V)\omega(W)\\ &\quad + g(U, W)\omega(V)\big) + 2\tau(U)\omega(V)\omega(W)\end{aligned}$$

for all $U, V, W \in \mathfrak{X}(\Sigma)$. From this we deduce that $X(\rho) = 0$, $\xi(\rho) = \rho^2 - \tau(\xi)\rho$, $K(X, Y) = -\frac{\rho}{2}(\rho - \tau(\xi))g(X, Y)\xi$, $K(X, \xi) = -\frac{1}{2}(\rho - \tau(\xi))X$ and $K(\xi, \xi) = -\tau(\xi)\xi$ for all $X, Y \in \Gamma(\mathcal{S})$. Therefore,

$$\begin{aligned}K(U, V) &= -\frac{1}{2}(\rho - \tau(\xi))\big(B^s(U, V)\xi + \omega(U)V + \omega(V)U\big)\\ &\quad + \frac{3(\rho - \tau(\xi))}{2}\omega \otimes \omega\xi,\end{aligned}$$

which is again a combination of a ξ-statistical structure and the statistical structure given in (25).

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